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Geometrically Non-Linear Dynamics of Anisotropic Open Cylindrical Shells with a Refined Shell Theory

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Geometrically Non-Linear Dynamics of Anisotropic Open Cylindrical Shells with a Refined Shell Theory par :M.H. Toorani et A.A. Lakis
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CONTENTS

NOMENCLATURE

A, B, C, D, E	defined by equation (6)
$a_i , b_i , \ldots , t_i$	modal coefficients determined by equation (C-2)
$a_i^{(1)} , b_i^{(1)} , \ldots , t_i^{(1)}$	coefficients determined by equation (C-5)
$a_{ij}^{(1)} , a_{ij}^{(2)} , b_{ij}^{(1)} , \ldots , t_{ij}^{(1)}$	coefficients determined by equation (C-4)
$AA_{jk} , BB_{jk} , \ldots, TT_{jk}$	modal coefficients determined by equations (B-5 to B-8)
$AA_{ijk} , BB_{ijk} , \ldots, TT_{ijk}$	modal coefficients determined by equation (B-3)
$AA_{ijks} , BB_{ijks} , \ldots, TT_{ijks}$	modal coefficients determined by equation (B-4)
$AUX_{ijk}^{(1)} ,\ldots, AUX_{ijk}^{(58)}$	modal coefficients determined by equation (B-10)
$AUX_{ijks}^{(1)} ,\ldots, AUX_{ijks}^{(58)}$	modal coefficients determined by equation (B-11)
f_i (i=1 to 10)	coefficients of determinant of the matrix [H] equation (7)
GG (p,q)	defined by equation (23-a)
h_i (i=1,2):	Lamé's parameters equation (2)
k_{ij}^L	general element of linear stiffness matrix equation (21)
k_{ijk}^{NL2}	general element of second-order non-linear stiffness matrix equation (21)
k_{ijks}^{NL3}	general element of third-order non-linear stiffness matrix equation (21)
L	length of shell
L_i	equations of motion equation (5)
m	axial mode number

$\overline{m}$	defined by $\frac{m\pi}{L}$
m_{ij}	general element of mass matrix equation (15)
M_x, M_θ, $M_{x\theta}$, $M_{\theta x}$	the moment resultants
n	circumferential wave number
N_x, N_θ, $N_{x\theta}$, $N_{\theta x}$	the in-plane force resultants
P_{ij}	terms of elasticity matrix(i=1,… ,10 ; j=1,…, 10) equation (4)
Q_{xx}, $Q_{\theta\theta}$	the transverse force resultants equation (4)
R	mean radius of the shell
SA (p,q)	defined by equation (23-b)
t	thickness of the shell
u, v, w	the axial, circumferential and radial displacement respectively
U_m, V_m, W_m, β_{xm}, $\beta_{\theta m}$	amplitudes of u, v, w, β_x, and β_θ associated with m_{th} axial mode number
x	axial coordinate
α_i, β_i, γ_i and δ_i	defined by equation (8)
β_x and β_θ	the rotations of the normal about the coordinates of the reference surface
η_i	complex roots of the characteristic equation (7)
ε_{ij}	deformation vector components equation (A-1)
ε_{pq}	the term (p,q) of matrix $[A^{-1}][A^{-1}]^T$ equation (23-b)
ε_x^o *and* ε_θ^o	normal strains of the reference surface
γ_x^o *and* γ_θ^o	in-plane shearing strains of the reference surface
κ_x *and* κ_θ	change in the curvature of the reference surface

τ_x and τ_θ	torsion of the reference surface
μ_x^o and μ_θ^o	the shearing strains
θ	circumferential coordinate
ϕ_T	angle for the whole open shell
ρ	density of the shell material
Γ_i	vibration amplitude
ψ_i	function determined equation (30)

Liste of matrices:

$[A]_{(10\times10)}$	defined by equation (10)
$[B]_{(10\times10)}$	defined by equation (12)
$[EA]_{(10\times10)} = [A^{-1}]\,[A^{-1}]^T$	defined by equation (23-b)
$[G]_{(10\times10)}$	defined by equation (15)
$[H]_{(5\times5)}$	defined by equation (7)
$[k^{(L)}]$	linear stiffness matrix equation (15)
$[k^{(NL2)}]$	second-order stiffness matrix equation (22-a)
$[k^{(NL3)}]$	third-order stiffness matrix equation (22-b)
[m]	local mass matrix equation (15)
[N]	shape function matrix (11)
[P]	elasticity matrix equation (4)
[QQ]	defined by equation (12)
[R]	defined by equation (11)
[S]	defined by equation (15)
$[T_1]$	transformation matrix equation (9)

{C}	vector for arbitrary constants equation (9)
{q}	time-related vector equation (26)
$\{\delta_i\}$	degrees of freedom at node ***i***
$\{\delta_T\}$	degrees of freedom for total shell
$\{\varepsilon_L\}$ and $\{\varepsilon_{NL}\}$	linear and non-linear deformation vector equation (A-1)

Résumé

Dans cet article, une approche générale est presentée afin de prédire l'influence de la non-linéairité géométrique sur les fréquences naturelles des coques cylindriques élastiques et anisotropes laminés basée sur la théorie raffinée (la théorie de déformations de cisaillement du premier ordre) des coques, en incorporant les grands déplacements et rotations. Les effets de déformations de cisaillement et de l'inertie rotative sont pris en compte dans les équations du mouvement. La méthode utilisée est celle des éléments finis hybrides qui est la combinaison de la méthode des éléments finis et de la théorie des déformations de cisaillement. Là, où les équations des coques sont utilisées intégralement ce qui permet d'utiliser les équations entières d'équilibre pour dériver les fonctions du déplacement. La solution analytique est divisée en deux parties. En première partie, les fonctions de déplacements sont obtenues de la théorie raffinée des coques et par la suite, les matrices de masse et de rigidité linéaire sont obtenues par la procédure des éléments finis hybrides et l'integration analytique exacte. En deuxième partie, les coefficients modaux sont obtenus, en utilisant les relations exactes de déformation-déplacements de Green, pour les fonctions des déplacements nommées au-dessus. Les expressions décrivant les matrices de rigidités non-linéaires de deuxième et troisième ordre sont obtenues par l'intégration analytique précise et sont intégrées dans le système linéaire pour obtenir l'équation non-linéaire de mouvement.

Abstract

A general approach, based on shearable shell theory, to predict the influence of geometric non-linearities on the natural frequencies of an elastic anisotropic laminated cylindrical shell incorporating large displacements and rotations is presented in this paper. The effects of shear deformations and rotary inertia are taken into account in the equations of motion. The hybrid finite element approach and shearable shell theory are used to determine the shape function matrix. The analytical solution is divided into two parts. In part one, the displacement functions are obtained by the *exact* solution of the equilibrium equations of a cylindrical shell based on shearable shell theory instead of the usually used and more arbitrary interpolating polynomials. The mass and linear stiffness matrices are derived by *exact* analytical integration. In part two, the modal coefficients are obtained, using Green's exact strain-displacement relations, for these displacement functions. The second- and third-order non-linear stiffness matrices are then calculated by precise analytical integration and superimposed on the linear part of equations to establish the non-linear modal equations. Comparison with available results is satisfactorily good.

KEY WORDS: non-linear shells, dynamics, cylindrical shells, anisotropic, shear deformations

1) Introduction- Multilayered composite shells are being used extensively as structural elements in modern construction engineering, ship building, nuclear, space and aeronautical industries as well as the petroleum and petrochemical industries (pressure vessel, pipeline). It is very important to know the static and dynamic behavior of these structures subjected to different loads in order to avoid destructive effects during their industrial usage.

There are many situations where these structures are subjected to severe environment conditions, impact, collision or other intensive transient loads, which can cause large transient structural deformations and damage, or catastrophic failure. These phenomena may be attributed to changes in the equilibrium state characterizing the load-response mode. In addition, due to high ratio of tangential Young's modulus to transverse shear modulus in composite materials such as graphite-epoxy and boron-epoxy, (i.e., of order 25-40 instead of 2.6 for isotropic materials), the shear deformation effect on the non-linear behavior of anisotropic composite shells is more significant than that of isotropic ones. This effect plays a very important role in reducing the effective flexural stiffness of anisotropic composite plates and shells.

Therefore, response of these types of structures may be predicted only when one accounts for their geometric non-linear behavior. Accordingly, the need for accurate and efficient methods for structural analysis and design, especially for this category of large-deflection (geometrically non-linearity) and elastic-plastic (material non-linearity) dynamic response problems, has increased.

Several articles are available dealing with geometrical non-linearities, based on the classical shell theory, in isotropic shells of arbitrary shapes [1-5]. In case of anisotropic composite shells, the reader is referred to the following references [6-11] where geometrically non-linear behavior of shearable anisotropic composite plates and shells is analyzed. A refined theory accounting the small strains and moderate rotation for anisotropic shell has been developed by Librescu and Schmidt [8]. Successive approximations, as steps toward an estimate of exact shell strain displacement relations where displacements, large strains and rotations, were initially allowed, are presented for isotropic shells by Sanders [3] and for anisotropic shells by Librescu et al. [6-11].

Also, a number of theories for layered anisotropic shells exist in literature [12], which are developed for thin shells and are based on the Kirchhoff-Love hypotheses. The first paper to deal with non-linear vibrations of shells was the pioneering work of Reissner [13], who used the method of assumed mode shapes to investigate the vibration of curved cylindrical panels. Chu [14] first presented an analysis for circular isotropic cylindrical shells with the hardening type of non-linearity for the amplitude-frequency response. Nowinski [15] confirmed the results of Chu [14] by investing the non-linear vibration of orthotropic cylindrical shells. However, Evensen [16] pointed out that the mode shapes assumed by Chu did not satisfy the condition of continuity of the circumferential in-plane displacement. A more rigorous study of non-linear free flexural vibrations of circular cylindrical shells was conducted by Atluri [17] who compared his results with the available data and concluded the possibility of the softening type of non-linearity.

The non-linear behavior of antisymmetric cross-ply circular cylindrical shells, based on the von-Kármán Donnell kinematics assumptions, are discussed by Iu and Chia [18]. Reddy and Chandrashekhara [19] solved the laminated shell, both cylindrical and spherical, assuming Reissner-Mindlin (*RM*) theory and an intermediate non-linearity. The formulation and computational procedure are presented for the geometrically nonlinear analysis of laminated orthotropic and anisotropic composite shells based upon a modified incremental Hellinger-Reissner principle and the total Lagrangian description by Rothert and Di [20].

Noor and Peters [21] analyzed the non-linear response of an anisotropic cylindrical panel that included transverse shear deformation. Their formulations are based on the Rayleigh-Ritz technique and the Hu-Washizu mixed shallow shell finite element approach. A non-linear two-dimensional theory for laminated thick plates and shells, which can predict the in-plane stresses as well as transverse direct stresses and transverse shearing stresses, was made by Stein [22]. The equations derived were similar to those of [13], except that non-linear strain displacement relations were used and expanded into a series that contained all first and second degree terms by retaining only the first few terms. Tsai and Palazotto [23] developed a finite element formulation to geometric non-linear vibration analysis of cylindrical shells, based on a curved quadrilateral, with 36 degree of freedom, thin shell elements and total Lagrangian description.

Rotter and Jumikis [24] have presented a set of non-linear strain-displacement relations for axisymmetric thin shells, based on the Kirchhoff assumptions, subject to large displacements with moderate rotations by retaining more terms. Their work is based on the Kirchhoff assumptions. They have shown that nonlinear strains arising from products of in-plane strain terms, which were omitted in previous theories, may be important in certain buckling problems. The new relations are particularly important when branched shells are being studied and when the buckling mode may involve a translation of the branching joint. Modal approximations in deriving the equations of motion for the non-linear flexural vibrations of a cylindrical shell using Donnell's shallow shell theory was presented by Dowell and Ventres [25]. The purpose of their work was to satisfy, more accurately, the boundary and the continuity conditions and investigate their effects on the form of the modal equations.

Most of these approaches can include various degrees of nonlinearity in the strain displacement relations in representing the displacements and rotations. Considerable simplification was achieved in the Donnell equations by assuming that the nonlinear membrane strains derived only from out-of-plane rotations. For example, Donnell theory is not suitable for the analysis of shells in which the buckling mode involves fewer than three full waves around the circumference [24]. More accurate nonlinear shell equations are given by Sanders [3] and by Novozhilov[1], but these are somewhat more complex than the Donnell equations. More terms are retained because fewer assumptions are made about the relative magnitude of various terms in the nonlinear strain-displacement relations.

Chao and Reddy [26] have presented a first order shear deformation theory based on kinematics and geometric assumption of Sander's thin shell theory for geometrically non-linear analysis of doubly curved composite shells. The influence of large amplitudes on the free vibrations of finite circular cylindrical shells with linearly varying wall thickness, embedded in an incompressible fluid is made by Ramachandran [27]. Most of mentioned works have been based on Galerkin's approach [15,16,28], the small perturbation procedure [17,29], the Rayleigh-Ritz method [27], the Ritz-Galerkin approach [30], the modal expansion method [31, 32], the finite element method [23, 33, 34] and the hybrid finite element method [35 to 40]. All of these methods have their advantages and disadvantages. The best of any method is probably its general content and the capacity to predict, with precision, both the high and low frequencies of vibration.

These criteria were not met in Galerkin's and small perturbation methods. Adopting a perturbation technique, Chen and Babcock [29] also considered the large-amplitude vibration of a thin-walled cylindrical shell. Ramachandran [27] studied the non-linear vibration of cylindrical shells with varying thickness. On the other hand, the post-buckling behavior of a laminated cylindrical shell subjected to axial load and torsion based on von-Kármán -Donnell equations was studied by Khot [41]. The results obtained by Khot [41] show that, in general, composite shells are less imperfection sensitive than isotropic shells. The small perturbation method was used [29] to transform the non-linear equations to a linear system by expanding the unknown variables in a power series with respect to small parameters. The major advantage of this technique, compared to other methods requiring an initial hypothesis regarding the form of the vibration mode, is that the results are not preconceived.

By incorporating the modal expansion (assumed mode shapes) technique, Rawdan and Genin [31] eliminated some weakness and serious drawbacks of other theories, by using Sanders-Koiter [3,5] general non-linear theory. Their work is limited to a finite simply supported isotropic cylindrical shell. After adopting the finite element method, Raju and Rao [33] obtained the frequency variation of thin shells of revolution in conjunction with the maximum normal displacement for various boundary conditions. However, the modal expansion and finite element methods appear to be ideally suited to the analysis of complex shell structures from this viewpoint. Numerous general computer programs, based on the finite element method are available for industrial use for the linear and non-linear analysis where the displacement functions of the finite element used are assumed to be polynomial but precise prediction of both the high and the low frequencies requires the use of a great many elements in the classical finite element method.

The present study presents a general approach to the dynamics non-linear analysis, incorporating the large displacements and rotations, of anisotropic laminated open or closed cylindrical shells based on the refined shell theory in which the shear deformation and rotary inertia effects are taken into account. The method used, in this work, is a combination of finite element analysis and shearable shell theory. In this method the equations of cylindrical shells are used in full to obtain the pertinent displacement functions, instead of using the more common arbitrary polynomial forms. The finite element method employed is a cylindrical panel-segment finite element, Figure (3), rather than the more commonly used triangular or rectangular elements. The displacement functions over an element are derived by *exact* solution of the equilibrium equations of a cylindrical shell (in case of shearable shell theory), and the mass and stiffness matrices of each element are derived by *exact* analytical integration. In doing so, the accuracy of the formulation will be less affected as the number of elements used is decreased (thus reducing computation time) and as the dynamic characteristics of the shell are required at higher beam-mode (m) or higher shell-mode (n), a significant advantage over polynomial interpolation. Therefore, this method is more accurate than the more usual finite element methods.

The present method offers many advantages, some of which are:

a) Simple inclusion of thickness discontinuities, material property variations and differences in materials comprising the shell,

b) Arbitrary boundary conditions without changing the displacement functions in each case,

c) High and low frequencies may be obtained with high accuracy as shown in [35-40], *d)* This approach has also been applied, with satisfactory results, to the dynamic analysis of shells containing a flowing fluid or partially filled with liquid [36, 37, and 39].

The analytical solution involves two steps:

i) Using the strain-displacement relations expressed in an arbitrary orthogonal curvilinear coordinate system and Green's exact stress-strain relationships for anisotropic laminated materials. These relations are then inserted into equilibrium equations given in [38-40] including the transverse shear deformations and rotary inertia effects. The displacement functions are analytically determined by solving the linear equation system. The mass and linear stiffness matrices are then determined by exact analytical integration, for each element and are assembled for the complete shell [38-40]. The displacement functions presented in this work allow dynamic behavior analysis of open or closed cylindrical shells with arbitrary boundary conditions.

ii) Using the non-linear part of the general strain-displacement relations as the linear part [given in 42, 43] and then applying the approach proposed by Radwan and Genin [31] to obtain the coefficients of the modal equations from the displacement functions. The non-linear stiffness matrices of the second- and third-order are then calculated by precise analytical integration with respect to modal coefficients. These matrices, once calculated, are superimposed on the linear part of equations to establish the non-linear modal equations.

There are several reasons for undertaking the development of the present theory. The first is to develop a theory for either dynamic or stress analysis of anisotropic laminated open or closed cylindrical shells. The accurate prediction of the dynamic response or failure characteristics of these structures, made up from advanced composite materials, requires the use of non-linear refined shell theory where the shear deformation and geometrical non-linear effects as well as rotary inertia effect are taken into account.

The second deals with the numerical simulation of non-linear dynamic behavior of anisotropic laminated cylindrical shells based on the present theory. At the same time, the flowing fluid effect on the natural frequencies will be studied. One of the criteria of success of a method may be considered to be its capability of yielding the high and low natural frequencies and modal shapes with comparable high accuracy. The numerical method is based on a combination of hybrid finite element analysis [38-40] and non-linear refined shell theory (shearable shell theory). This allows us to use the shell equations in full for the determination of the displacement functions, and hence the mass, stiffness and stressresultant matrices, instead of the more usual polynomial displacement functions.

2) Hypotheses

The first order transverse shear deformation theory of shells and modal expansion approaches have been used to develop the non-linear dynamic equations of motion.

This theory is based on the following hypotheses:

a) It is assumed that the normal stress is negligible compared with stress tangential to the shell surface.

b) Linear elastic behavior of laminated anisotropic materials.

c) Use the Green exact strain-displacement relations expressed in arbitrary orthogonal curvilinear systems.

d) The first order transverse shear deformation theory of the shells is adopted so the transverse shear deformations and also the rotary inertia effects are taken into account to develop the governing equations.

e) The large displacements and rotations are incorporated into the theory.

3) Linear matrix construction

3-1) Strain-displacement and stress-strain relations- Consider an infinitesimal line segment ***MN*** of length ***ds*** embedded in a differential volume element ***B***, in the initial undeformed configuration, before transformation. These points are displaced, respectively, to ***M**** and ***N****, in the actual deformed configuration, by the displacement vector $\bar{u}$ as a result of the deformation (Figure 1). The change in length of the element ***MN*** can be expressed by:

$$(ds^*)^2 - (ds)^2 = 2\gamma_{ij}\,dy_i\,dy_j \tag{1}$$

where the quantity $\left[(ds^*)^2 - (ds)^2\right]$ is an invariant and $\gamma_{ij} = \gamma_{ji}$ is a symmetric tensor called Green's strain tensor, given in [42], and y_i is the orthogonal curvilinear coordinate of the undeformed system. The physical strains, ε_{ij}, are defined as below

[42, 43]:

$$\varepsilon_{ij} = \frac{\gamma_{ij}}{h_i h_j} \tag{2}$$

where, h_i are called the *Lamé's parameters* and defined by $G_{ij} = h_i^2$ (no sum), G_{ij} is a metric tensor which links two coordinate systems. The scale factors h_i are defined as bellow for cylindrical shell geometry:

$$h_1 = \sqrt{E^*}\left(1 - \xi/R_1\right),\quad h_2 = \sqrt{G^*}\left(1 - \xi/R_2\right),\quad h_3 = 1 \tag{3}$$

where E^* and G^* are the first fundamental magnitudes which are related to the elements of the surface metric [43]. For rigid body motion, the elongation ε_{ii} (no sum) and the shear $\varepsilon_{ij}(i \neq j)$ are identically zero, then there are no theoretical limitations. Based on these definitions, the deformation vector $\{\varepsilon\}$ for the cylindrical shells (Figure 2) is obtained and given in Appendix [A]. Unlike classical shell theory, the transverse shear strains do not vanish in the present theory and therefore β_i can not be expressed in terms of displacement components. The five degrees of freedom at each node, U, V, W, β_x and β_θ are function of the in-plane coordinates in which U, V and W are the axial, circumferential and radial displacements respectively. The β_x and β_θ are, respectively, rotations of the normal about the coordinates of the reference surface oriented along the parametric lines of middle surface.

The constitutive relations between the stress and deformation vectors of an anisotropic laminated cylindrical shell can be written as [42]:

$$\{N_{xx}, N_{x\theta}, Q_{xx}, N_{\theta\theta}, N_{\theta x}, Q_{\theta\theta}, M_{xx}, M_{x\theta}, M_{\theta\theta}, M_{\theta x}\}^T = [P]_{(10\times10)}\{\varepsilon\} \tag{4}$$

where *[P]* is the anisotropic matrix elasticity. The elements P_{ij}'s in [P] depend on the mechanical characteristics of the structure's materials and shell geometrical parameters. This matrix is given in [38].

The formulations of governing equations will be developed hereafter in terms of displacement measures only. There are other formulations in terms of stress resultants and stress couples, in terms of strain measures, as well, mixed formulation in terms of stress-resultants (or stress potential functions like the Airy functions) and the displacement quantities. In order to obtain such equations, the reader may follow the procedures described by Sanders [44] and Brull and Librescu [45].

In composite laminated plates and shells, the transverse shear stresses vary through layer thickness. This discrepancy is often corrected in computing the transverse shear force resultants (Q_{xx} and $Q_{\theta\theta}$) by considering shear correction factor. This factor is computed such that the strain energy due to transverse shear stresses equals the strain energy due to the true transverse stresses predicted by the three-dimensional elasticity theory. This factor depends, in general, on the lamination parameters such as number of layers, stacking sequence, degree of orthotropy and fiber orientation in each individual layer.

The equations of motion of cylindrical shells in terms of ***U, V*** and ***W***, β_x and β_θ and in terms of P_{ij}'s elements are written as follows:

$$L_m\left(U,V,W,\beta_x,\beta_\theta,P_{ij}\right)=0. \qquad m=1,...,5 \tag{5}$$

where L_m (m =1,2…5) are five linear differential operators are fully given in [38].

It is noted that the sixth equation of equilibrium is identically satisfied by the integral definitions of the shearing-stress resultants in terms of the shearing stress components. Two basic approaches that enable us to eliminate exactly this sixth equation of equilibrium have been developed:

i) Derivation of constitutive equations which fulfill identically this extra equation of equilibrium. For the classical theory of isotropic, such constitutive equations are known as Flügge-Lure-Byrne constitutive equations. In the case of composite shells, the reader is referred to the following monograph, [11, 46 and 47] by Librescu.

ii) Formulation of modified stress resultants and stress couple measures, satisfying identically this sixth extra equation of equilibrium. For the classical theory of isotropic shells, such modified stress resultants and stress couples have been defined by Sanders [3, 44]. For anisotropic shells, a similar stress resultants and stress couples were considered by Librescu [11, 46 and 47].

Neither 6th equation of equilibrium nor the five first ones are given in this paper, and the interested reader is referred to Reference [38] where all six equations are defined.

3-2) Displacement functions–The five equations of motion (5) are expressed in terms of displacement measures. Therefore, the five boundary conditions must be specified at each edge of the shell. The shell is subdivided into several finite elements defined by line-nodes, ***i*** and ***j***, and by components U, V, W, β_x and β_θ representing axial, tangential and radial displacements and two rotations, respectively (Figure 3) .The displacement functions associated with the axial wave number are assumed to be:

$$U(x,\theta) = ACos\frac{m\pi}{L}xe^{\eta\theta}; \qquad \beta_x(x,\theta) = DCos\frac{m\pi}{L}xe^{\eta\theta}$$
$$V(x,\theta) = BSin\frac{m\pi}{L}xe^{\eta\theta}; \qquad \beta_\theta(x,\theta) = ESin\frac{m\pi}{L}xe^{\eta\theta} \tag{6}$$
$$W(x,\theta) = CSin\frac{m\pi}{L}xe^{\eta\theta}$$

where ***m*** is the axial mode number and η is a complex number. Substituting definition (6) into equations of motion (5) and setting equal to zero the determinant of the coefficients in order to have a non-trivial solution, leads to a tenth order polynomial characteristic equation in terms of η:

$$Det([H]) = f_{10}\eta^{10} + f_8\eta^8 + f_6\eta^6 + f_4\eta^4 + f_2\eta^2 + f_0 \tag{7}$$

where f_i (i = 0 to 10) are the coefficients of the determinant of the matrix [H] given in [39]. Each root of the characteristic equation yields a solution of equation (5). The complete solution of equations of motion can be obtained by adding the ten independent solutions, obtaining for each root of the characteristic equation, involving the constants A_i, B_i, C_i, D_i and E_i (i =1 to 10). As these constants are not independents, A_i, B_i, D_i and E_i can be expressed as a function of C_i:

$$A_i = \alpha_i C_i, \qquad B_i = \beta_i C_i, \qquad D_i = \gamma_i C_i, \qquad \text{and} \quad E_i = \delta_i C_i \qquad (i = 1,2,...,10) \tag{8}$$

The values of $\alpha_i, \beta_i, \gamma_i$ and δ_i can be obtained from the linear system (5) by introducing relations (8). After carrying out the some manipulations, the displacements U(x, θ), V(x, θ) and W(x, θ) as well as β_x(x, θ) and β_θ(x,θ) can be expressed in conjunction with ten C_i constants only.

$$\begin{Bmatrix} U(x,\theta) \\ V(x,\theta) \\ W(x,\theta) \\ \beta_x(x,\theta) \\ \beta_\theta(x,\theta) \end{Bmatrix} = [T_1]_{(5\times5)}[R]_{(5\times10)}\{C\}_{(10\times1)} \tag{9}$$

where matrices $[T_1]_{(5\times5)}$ and $[R]_{(5\times10)}$ are given in [39], and {C} is a tenth order vector of the C_i constants. The C_i constants can be determined using ten boundary conditions for each finite element. The axial, tangential and radial displacements

(U, V, and W) as well as the rotations (β_x, β_θ) have to be specified for each node. Thus, the element displacement vector at the boundaries can be given by following relation:

$$\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix}=\left\{U_i\ V_i\ W_i\ \beta_{x_i}\ \beta_{\theta_i}\ U_j\ V_j\ W_j\ \beta_{x_j}\ \beta_{\theta_j}\right\}^T=[A]_{(10\times10)}\{C\}_{(10\times1)} \tag{10}$$

where the terms of matrix [A], given in [39], are obtained from those of [R] matrix by successively setting $\theta = 0$ and $\theta = \varphi$. Multiplying equation (10) by $[A^{-1}]$ and substituting that into equation (9), we obtain:

$$\begin{Bmatrix}U(x,\theta)\\ V(x,\theta)\\ W(x,\theta)\\ \beta_x(x,\theta)\\ \beta_\theta(x,\theta)\end{Bmatrix}=[T_1][R][A]^{-1}\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix}=[N]\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix} \tag{11}$$

where the matrices $[T_1]$, [R] and [A] are previously defined and [N] matrix represents the displacement function matrix.

***3-3) Mass and linear stiffness matrices for an element*-**From the expressions given in the linear part of deformation vector (given in Appendix A) and the expressions of the displacement functions (11), we obtain:

$$\{\varepsilon\}=\begin{bmatrix}[T_1] & 0\\ 0 & [T_1]\end{bmatrix}[QQ][A]^{-1}\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix}=[B]\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix} \tag{12}$$

where [QQ] is a (10×10) matrix given in [39]. The constitutive relations between the stress and deformation vectors of the anisotropic laminated cylindrical shells are given as:

$$\{N_{xx}, N_{x\theta}, Q_{xx}, N_{\theta\theta}, N_{\theta x}, Q_{\theta\theta}, M_{xx}, M_{x\theta}, M_{\theta\theta}, M_{\theta x}\}^T=[P]_{(10\times10)}\{\varepsilon\}=[P][B]\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix} \tag{13}$$

The P_{ij}'s elements describe the elasticity matrix of the anisotropic cylindrical shells, which depends on the mechanical properties of the material of the structure. Some coupling as in-plane extensional-shear, extensional-bending and bending-twisting can be present in anisotropic laminated composite shells due to anti-symmetry of scheme lamination or fiber orientation. The mass and linear stiffness matrices for one finite element can be expressed as:

$$[m]=\rho t\int_0^L\int_0^\varphi[N]^T[N]dA \qquad [K_L]=\int_0^L\int_0^\varphi[B]^T[P][B]dA \tag{14}$$

where $dA=Rdx\, d\theta$ and ρ is the density of the shell, *[P]* the elasticity matrix and the matrices *[N]* and *[B]* are before defined in equations (12 and 13), respectively. Substituting them in (14) and analytically integrating with respect to *x* and θ , we obtain the matrices *[m]* and *[k]*.

$$[m]=\rho\, t\left[A^{-1}\right]^{T}[S]\left[A^{-1}\right] \qquad [k]=\left[A^{-1}\right]^{T}[G]\left[A^{-1}\right] \tag{15}$$

where the S_{ij}'s and G_{ij}'s general elements are given in [39].

4 Non-Linear Part

4-1) Non-linear stiffness matrix construction for an element- The exact Green strain relations are used in order to describe the linear and non-linear, including large displacements and rotations, behavior of anisotropic open cylindrical shells. In common with linear theory, it is based on refined shell theory in which the shear deformations and rotary inertia effects are taken into account. The approach developed by Radwan and Genin [31] is used with particular attention to geometric non-linearities. The coefficients of the modal equations are obtained through the Lagrange method. Thus, the non-linear stiffness matrices of second- and third-order, once calculated, are superimposed on the linear part of equations to establish the non-linear modal equations. This section is limited to the relevant details of the method used to find the non-linear stiffness matrices. The main steps of this method are as follow:

a) Shell displacements are expressed as generalized product of coordinate sums and spatial functions:

$$\begin{aligned} u &= \sum_i q_i(t)\,U_i(x,\theta) & \beta_x &= \sum_i q_i(t)\,\beta_{xi}(x,\theta) \\ v &= \sum_i q_i(t)\,V_i(x,\theta) & \beta_\theta &= \sum_i q_i(t)\,\beta_{\theta i}(x,\theta) \\ & & w = \sum_i q_i(t)\,W_i(x,\theta) & \end{aligned} \tag{16}$$

where the $q_i\ (t)$'s functions are the generalized coordinates and the spatial functions U, V, W, β_x and β_θ are given by equation (6).

b) The deformation vector is written as a function of the generalised coordinates by separating the linear part from the non-linear one given in Appendix [A]:

$$\{\varepsilon\}=\{\varepsilon_L\}+\{\varepsilon_{NL}\}=\left\{\varepsilon_x^o,\gamma_x^o,\mu_x^o,\varepsilon_\theta^o,\gamma_\theta^o,\mu_\theta^o,\kappa_x,\tau_x,\kappa_\theta,\tau_\theta\right\}^T \tag{17}$$

where subscripts "L" and "NL" mean "linear" and "non-linear" respectively. The ε^o_i; γ^o_i; κ_i ; τ_i and μ^o_i are defined in Appendix [A]. In general, these terms can be expressed in the following form:

$$\varepsilon_x^o = \sum_j a_j q_j + \sum_j\sum_k AA_{jk} q_j q_k \qquad \mu_\theta^o = \sum_j g_j q_j + \sum_j\sum_k GG_{jk} q_j q_k$$

$$\gamma_x^o = \sum_j b_j q_j + \sum_j\sum_k BB_{jk} q_j q_k \qquad \kappa_x = \sum_j n_j q_j + \sum_j\sum_k NN_{jk} q_j q_k$$

$$\mu_x^o = \sum_j c_j q_j + \sum_j\sum_k CC_{jk} q_j q_k \qquad \tau_x = \sum_j p_j q_j + \sum_j\sum_k PP_{jk} q_j q_k \tag{18}$$

$$\varepsilon_\theta^o = \sum_j d_j q_j + \sum_j\sum_k DD_{jk} q_j q_k \qquad \kappa_\theta = \sum_j s_j q_j + \sum_j\sum_k SS_{jk} q_j q_k$$

$$\gamma_\theta^o = \sum_j e_j q_j + \sum_j\sum_k EE_{jk} q_j q_k \qquad \tau_\theta = \sum_j t_j q_j + \sum_j\sum_k TT_{jk} q_j q_k$$

Note: $AA_{ij}=AA_{ji}$, $BB_{ij}=BB_{ji}$ and etc.

The a_j, b_j, …t_j and AA_{ij}, BB_{ij},…,TT_{ij} are given in Appendix [B].

c) Using equation (16) and Hamilton's principle leads to Lagrange's equations of motion in the generalized coordinates q_i (t):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \tag{19}$$

Where T is the total kinetic energy, V the total elastic strain energy of deformation and the Q_i's are the generalized forces. Assuming $\{\varepsilon_{NL}\} = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{10}\}^T$, the strain energy V can be defined as follow:

$$V = \frac{a}{2}\int_0^L\int_0^\phi \left(P_{ij}\varepsilon_i\varepsilon_j + P_{kl}\varepsilon_k\varepsilon_l\right) R\, dx\, d\theta \tag{20}$$

Where:

$a=1$ if $i=j$ or $k=l$ $(i, j=1,2,\ldots,10)$, $(k,l=3,6)$ and $i, j \neq 3,6$

$a=2$ if $i \neq j$ or $k \neq l$ $(i, j=1,2\ldots,10)$ $(k, l=3,6)$

d) After developing the total kinetic and strain energy, using definitions (18), and then substituting into the Lagrange equation (19) and carrying out a large number of the intermediate manipulations, that are not displayed here, the following non-linear modal equations are obtained. These non-linear modal equations are used to study the dynamic behavior of anisotropic cylindrical shells.

$$\sum_j m_{ij}\ddot{\delta}_j + \sum_j k_{ij}{}^{(L)}\delta_j + \sum_j\sum_k k_{ijk}{}^{(NL2)}\delta_j\delta_k + \sum_j\sum_k\sum_s k_{ijks}{}^{(NL3)}\delta_j\delta_k\delta_s = Q_i \qquad i=1,2,\ldots \quad (21)$$

Where m_{ij}, $k_{ij}^{(L)}$ are the terms of mass and linear stiffness matrices given by equation (15). The terms of $k_{ijk}^{(NL2)}$ and $k_{ijks}^{(NL3)}$, which represent the second- and third-order non-linear stiffness matrices. These terms, in the case of anisotropic laminated cylindrical shell, are given in Appendix [B].

Performing some intermediate manipulations, explained in Appendix [C], the following expression are obtained for the second- and third-order non-linear stiffness matrices.

$$\left[k_{ijk}\right] = \left[k^{(NL2)}\right] = \left[A^{-1}\right]^T\left[J^{(NL2)}\right]\left[A^{-1}\right] \qquad (22-a)$$

$$\left[k_{ijks}\right] = \left[k^{(NL3)}\right] = \left[A^{-1}\right]^T\left[J^{(NL3)}\right]\left[A^{-1}\right] \qquad (22-b)$$

Where the (i, j) term in matrices [J $^{(NL2)}$] and [J $^{(NL3)}$] are written as:

$$J^{(NL2)}_{(i,j)} = \begin{cases} \displaystyle\sum_{k=1}^{10} \frac{R\ GG(i,j)}{(\eta_i+\eta_j+\eta_k)}\left[e^{(\eta_i+\eta_j+\eta_k)\phi}-1\right] & if\ (\eta_i+\eta_j+\eta_k)\phi \neq 0. \\ \displaystyle\sum_{k=1}^{10} R\ GG(i,j)\phi & if\ (\eta_i+\eta_j+\eta_k)\phi = 0. \end{cases} \qquad (23-a)$$

$$J^{(NL3)}_{(i,s)} = \begin{cases} \displaystyle\sum_{q=1}^{10}\sum_{p=1}^{10} \frac{RL}{4}\frac{\varepsilon_{pq}SA(i,s)}{(\eta_i+\eta_s+\eta_p+\eta_q)}\left[e^{(\eta_i+\eta_s+\eta_p+\eta_q)\phi}-1\right] & if\ (\eta_i+\eta_s+\eta_p+\eta_q)\phi \neq 0. \\ \displaystyle\sum_{q=1}^{10}\sum_{p=1}^{10} \frac{RL}{4}\varepsilon_{pq}SA(i,s)\phi & if\ (\eta_i+\eta_s+\eta_p+\eta_q)\phi = 0. \end{cases} \qquad (23-b)$$

Where the $GG\ (i, j)$ and $SA\ (i, s)$ are coefficients in conjunction with α, β, γ, δ, η and P_{ij}'s elements in matrix [P]. The ε_{pq} is the term (p,q) of matrix [EA], where [EA] represents a matrix of constants defined by [EA]=[A^{-1}] [A^{-1}]^T . The general expression of $GG\ (i, j)$ and $SA\ (i,s)$ are given in Appendix [D].

5) The influence of geometric non-linearities on the natural frequencies of cylindrical shells- The mass and stiffness matrices, either linear or non-linear, given in equations (15 and 22) are only determined for one element. After subdividing the shell into several elements, (see Figure 3), the global mass and stiffness matrices are obtained by assembling the matrices for each element. Assembling is done in such way that all the equations of motion and the continuity of displacements at each node are satisfied. These matrices are designated as [M], [K^L], [K^{NL2}] and [K^{NL3}], respectively.

$$[M]\{\ddot{\delta}\}+[K^{(L)}]\{\delta\}+[K^{(NL2)}]\{\delta^2\}+[K^{(NL3)}]\{\delta^3\}=\{0\} \tag{24}$$

where $\{\delta\}$ is the displacement vector and [M], $[K^L]$, $[K^{NL2}]$ and $[K^{NL3}]$ are, respectively the mass, linear and second- and third-order non-linear stiffness matrices of the system. They are square matrices of order NDF*(N+1), where NDF represents the number of degrees of freedom, (=5 in the present theory), at each nodal line and N represents the number of finite elements. These matrices are then reduced to square matrices of order NREDUC=NDF*(N+1)-NC, where NC represents the number of constraints applied. The system of equation (24) then becomes:

$$[M^{(r)}]\{\ddot{\delta}^{(r)}\}+[K_L^{(r)}]\{\delta^{(r)}\}+[K_{NL2}^{(r)}]\{\delta^{(r)^2}\}+[K_{NL3}^{(r)}]\{\delta^{(r)^3}\}=\{0\} \tag{25}$$

where the superscript "r" means "reduced". The matrices contained in the linear part of equations (25) can be reduced to diagonal matrices. Setting:

$$\{\delta^{(r)}\}=\{\Phi\}\{q\} \tag{26}$$

where $[\Phi]$ represents the square matrix for eigenvectors of the linear system and $\{q\}$ is a time related vector. Substituting relation (26) into equation (25) and multiplying by $[\Phi]^T$, we obtain:

$$\begin{aligned}&[\Phi]^T[M^{(r)}][\Phi]\{\ddot{q}\}+[\Phi]^T[K_L^{(r)}][\Phi]\{q\}+\\&+[\Phi]^T[K_{NL2}^{(r)}][\Phi]([\Phi]\{q\})^2++[\Phi]^T[K_{NL3}^{(r)}]([\Phi]\{q\})^3=0\end{aligned} \tag{27}$$

The matrix product of $([\Phi]^T[\ ][\Phi])$ represents diagonal matrices, written as $[M^{(D)}]$ and $[K_L{}^{(D)}]$. By neglecting the cross-product terms in $([\Phi]\{q\})^2$ and $([\Phi]\{q\})^3$ of equation (27), we obtain:

$$m_{jj}\ddot{q}_j+k_{jj}^{(L)}q_j+\sum_{i=1}^{NREDUC}k_{ji}^{(NL2)}q_i^2+\sum_{i=1}^{NREDUC}k_{ji}^{(NL3)}q_i^3=0. \tag{28}$$

where coefficients m_{jj} and $k_{jj}^{(L)}$ represent the "j^{th}" diagonal terms of linear matrices. The $k_{ji}^{(NL2)}$ and $k_{ji}^{(NL3)}$ are the (j,i) term of the product $([\Phi]^T[K_{NL2}^{(r)}][\Phi^2])$ and $([\Phi]^T[K_{NL3}^{(r)}][\Phi^3])$, respectively. There are "NREDUC" simultaneous equations of the form of (28). Turn now to solution of equation (28) associated with the parametric function $q(t)$. For this purpose, a solution $q(t)$ of the equation (28) has to be found. It is expedient to introduce the following representation:

$$q_j(t)=\Gamma_j\psi_j(t) \tag{29}$$

which permits the use of the normalized initial conditions:

$$\psi_j(0) = 1. \qquad and \qquad \dot{\psi}_j(0) = 0. \tag{30}$$

Substituting definition (29) into equation (28) and dividing by m_{jj}, we obtain:

$$\ddot{\psi}_{jj} + \omega_j^2 \psi_j + \Lambda_j^{(NL2)} (\Gamma_j / t) \psi_j^2 + \Lambda_j^{(NL3)} (\Gamma_j / t)^2 \psi_j^3 = 0. \tag{31}$$

where " t" and $\omega_j^2 = k_{jj}^L / m_{jj}$ represent shell thickness and linear vibration frequency of the shell, respectively, and:

$$\Lambda_j^{(NL2)} = \frac{k_{jj}^{(NL2)}}{m_{jj}} t \qquad and \qquad \Lambda_j^{(NL3)} = \frac{k_{jj}^{(NL3)}}{m_{jj}} t^2 \tag{32}$$

The solution $\psi_j(t)$ of the non-linear differential equation (31), which satisfies condition (30), is calculated by a fourth-order Runge-Kutta numerical method. The linear and non-linear natural frequencies are evaluated by a symmetric search for the $\psi_j(t)$ roots as a function of time. The ω_{NL} / ω_L ratio of non-linear and linear frequency is expressed as a function of non-dimensional ratio (Γ_j / t) where Γ_j is the vibration amplitude.

6) Numerical Results and Discussion-This paper is focused on the application of the hybrid finite element method, based on shearable shell theory and modal expansion approach, to anisotropic laminated cylindrical shells to evaluate both transverse shear deformation and geometrically non-linear effects. Non linearity effects produce either hardening or softening behavior in circular cylindrical shells. The method has been developed to demonstrate the influence of the shear deformation effect and that of geometrical non-linearities on the free vibration of open or closed cylindrical shells. It is a hybrid finite element based on a combination of shearable shell theory and modal expansion approach such that the displacement functions could be derived directly from shearable shell theory. This method is capable of obtaining the high as well as low frequencies with high accuracy. The values of the shear correction factors used in calculations have been taken $\pi^2/12$.

As the first numerical example, Figure (4) shows the natural frequencies computed for a closed simply supported, circular cylindrical shell for *m=1 and 2* and compared with the experimental results given in [48]. As can be seen, there is good agreement between the present theoretical results and those of experimental. Dimensions and material properties are given as follow:

$$R = 0.175(m) \quad L = 0.664(m) \quad t = 1(mm) \quad E = 206(GPa) \quad \nu = 0.3 \quad \rho_s = 7680(kg/m^3)$$

The non-dimensional fundamental frequencies obtained from the present theory are shown in Figure (5) along with corresponding values given in Ref. [49], for a four layer cross-ply cylindrical shell to demonstrate the accuracy and range of applicability of the present theory. All layers, for Figure (5) and table (I), are assumed to have the same geometric and material parameters and the individual layer is assumed to be orthotropic with the following material properties:

$$E_1 = 25E_2 \quad G_{23} = 0.2E_2 \quad G_{13} = G_{12} = 0.5E_2 \quad \nu_{12} = 0.25 \qquad and \qquad \rho = 1$$

The effect of variation of the length–to-radius ratio on the frequency parameters of an open cylindrical shell having its straight edges clamped and the curved edges simply supported is shown in Figure (6) and compared with corresponding results based on Sanders' theory, obtained by author.

Figure (7) shows the influence of geometrical non-linearities effects on the free vibrations of a simply supported cylindrical shell, along with corresponding results given in References [15 and 33]. The given results in Ref. [15] were obtained based on Donnell's simplified non-linear method where only lateral displacement was considered. Raju and Rao [33] used the finite element method based on an energy formulation.

It is observed that the variation ratio between the non-linear and liner frequency ω_{NL} / ω_L increases as the ratio Γ_j / t increases although these variations are small for values Γ_j / t below 1.0. It can also be seen that the non-linearity has a hardening effect. The difference between results obtained by the present theory and those of [15 and 33] might be due to the fact that Nowinski [15] neglected in-plane inertia and took account only lateral displacement. Raju and Rao [33] expressed the displacement components along the shell generator in polynomial form.

The next step of calculations is the non-linear dynamics characteristics of an orthotropic open cylindrical shell, with clamped boundary conditions along its straight edges and simply supported along its curved edges, as a function of circumferential and axial mode numbers. The results are shown in Figure (8). As can be seen, the non-linearity is of the hardening type for these modes. It is also seen that the non-linear effect is more pronounced for the circumferential mode (n=1).

Table (I) presents a parametric study based on shell thickness to determine the effect of transverse shear deformation on the natural frequencies of cross-ply cylindrical shells, for various different ratios R/L and L/t. The natural frequencies for $L/t=10$ are less sensitive to L/R variations than those of thin shells, $L/t=100$. Classical shell theory overpredicts the frequency even for thin shells.

7) Conclusions- This paper deals with some of the problems that arise when considering geometric non linearities (incorporating the large displacements and rotations), shear deformation and rotary inertia effects in the study of static and dynamic behavior of elastic, anisotropic open or closed cylindrical shells. An efficient hybrid finite element method, modal expansion approach and shearable shell theory including shear deformation effects have been used to develop the non-linear dynamic equations. The shape functions are derived by analytical integration, from *exact* solution of shearable shell equations. It is believed that the refined shear deformation theory and the non-linearities effects presented here are essential for predicting an accurate response for anisotropic shell structures.

The shear deformation and rotary inertia effects are taken into account to develop the equations of motion. In case of rotary inertia, some preliminary results indicate that the effects of the rotary inertia are practically limited. On the other hand, Librescu [11] found that the inertia effect, in the case of anisotropic plate, was practically inexistant at the lowest branch of the frequency spectrum. Consequently, the rotary inertia effect is discarded in the remaining equations (after obtaining the equations of motion), so the present study focuses only on the shear deformation and geometrically non-linearity effects.

From the results obtained based on this theory, the following can be concluded:

1) The hybrid finite element model is confirmed as a very efficient tools to analysis the anisotropic laminated shells.

2) Refined shell theory, taking into account the effects of shear deformation, plays a fundamental role in vibration analysis of anisotropic composite structures.

3) The assumed mode shapes (modal expansion) appears to be the ideal tool for the non-linear analysis of complex shell structures, which could eliminate some weaknesses and serious drawbacks of other theories such as Galerkin's and perturbation's methods.

4) A good description of geometrical non-linear effects on the vibration behavior of shells could be presented when the coupling between different modes is taken into account by considering the cross-product terms of non-linear stiffness matrices. The non-linear strain terms arising from products of in-

plane strain terms may be important in buckling problems.

The equations obtained in this work are shown to be suitable for practical applications involving large static or dynamic deformations of anisotropic open or closed cylindrical shells for different parameter variations such as arbitrary boundary conditions and effect of materials properties. The non-linear modal equations derived here can render possible complete post-buckling investigations for cylindrical shells. The next work, under preparation, deals with the stability analysis of anisotropic cylindrical shells conveying fluid to show the reliability and effectiveness of the present formulations.

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APPENDICES

These appendices contain the various equations for non-linear dynamic analysis of anisotropic laminated cylindrical shells in which the shear deformation and rotary inertia effects are taken into account in the formulations.

APPENDIX A- The strain vector for an anisotropic cylindrical shell (equation 4)

$$\{\varepsilon\}=\{\varepsilon_L\}+\{\varepsilon_{NL}\}=\begin{Bmatrix}\varepsilon_x^o\\ \gamma_x^o\\ \mu_x^o\\ \varepsilon_\theta^o\\ \gamma_\theta^o\\ \mu_\theta^o\\ \kappa_x\\ \tau_x\\ \kappa_\theta\\ \tau_\theta\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial U}{\partial x}\\ \dfrac{\partial V}{\partial x}\\ \dfrac{\partial W}{\partial x}+\beta_x\\ \dfrac{1}{R}\dfrac{\partial V}{\partial \theta}+\dfrac{W}{R}\\ \dfrac{1}{R}\dfrac{\partial U}{\partial \theta}\\ \dfrac{1}{R}\dfrac{\partial W}{\partial \theta}-\dfrac{V}{R}+\beta_\theta\\ \dfrac{\partial \beta_x}{\partial x}\\ \dfrac{\partial \beta_\theta}{\partial x}+\dfrac{1}{2R}\dfrac{\partial V}{\partial x}\\ \dfrac{1}{R}\dfrac{\partial \beta_\theta}{\partial \theta}\\ \dfrac{1}{R}\dfrac{\partial \beta_x}{\partial \theta}-\dfrac{1}{2R^2}\dfrac{\partial U}{\partial \theta}\end{Bmatrix}_{L(10\times1)}+\begin{Bmatrix}\dfrac{1}{2}\left[\left(\dfrac{\partial U}{\partial x}\right)^2+\left(\dfrac{\partial V}{\partial x}\right)^2+\left(\dfrac{\partial W}{\partial x}\right)^2\right]\\ \dfrac{1}{R}\left[\dfrac{\partial V}{\partial x}\dfrac{\partial V}{\partial \theta}+W\dfrac{\partial V}{\partial x}\right]\\ \beta_x\dfrac{\partial U}{\partial x}+\beta_\theta\dfrac{\partial V}{\partial x}\\ \dfrac{1}{2R^2}\left[\left(\dfrac{\partial U}{\partial \theta}\right)^2+\left(V-\dfrac{\partial W}{\partial \theta}\right)^2+\left(W+\dfrac{\partial V}{\partial \theta}\right)^2\right]\\ \dfrac{1}{R}\left[\dfrac{\partial U}{\partial x}\dfrac{\partial U}{\partial \theta}+\dfrac{\partial W}{\partial x}\dfrac{\partial W}{\partial \theta}-V\dfrac{\partial W}{\partial x}\right]\\ \dfrac{1}{R}\left[\beta_\theta\dfrac{\partial V}{\partial \theta}+W\beta_\theta+\beta_x\dfrac{\partial U}{\partial \theta}\right]\\ \dfrac{\partial U}{\partial x}\dfrac{\partial \beta_x}{\partial x}+\dfrac{\partial V}{\partial x}\dfrac{\partial \beta_\theta}{\partial x}\\ \dfrac{1}{R}\left[\dfrac{\partial V}{\partial x}\dfrac{\partial \beta_\theta}{\partial \theta}+\dfrac{\partial \beta_\theta}{\partial x}\dfrac{\partial V}{\partial \theta}+W\dfrac{\partial \beta_\theta}{\partial x}\right]\\ \dfrac{1}{R^2}\left[\dfrac{\partial U}{\partial \theta}\dfrac{\partial \beta_x}{\partial \theta}+\dfrac{\partial V}{\partial \theta}\dfrac{\partial \beta_\theta}{\partial \theta}+V\beta_\theta+W\dfrac{\partial \beta_\theta}{\partial \theta}-\beta_\theta\dfrac{\partial W}{\partial \theta}\right]\\ \dfrac{1}{R}\left[\dfrac{\partial U}{\partial x}\dfrac{\partial \beta_x}{\partial \theta}+\dfrac{\partial \beta_x}{\partial x}\dfrac{\partial U}{\partial \theta}-\beta_\theta\dfrac{\partial W}{\partial x}\right]\end{Bmatrix}_{NL(10\times1)} \qquad (A-1)$$

where $\varepsilon^o{}_i$; $\gamma^o{}_i$; κ_i; τ_i and $\mu^o{}_i$ are normal and in-plane shearing strain, change in the curvature and torsion of the reference surface and the shearing strain components, respectively, and subscripts 'L' and 'NL' mean 'linear' and 'non-linear', respectively.

APPENDIX B; Non-Linear stiffness matrices (equation 21)

$$k_{ijk}^{(NL2)} = \iint \begin{bmatrix} \left(P_{11}AA_{ijk} + P_{22}BB_{ijk} + P_{33}CC_{ijk} + P_{44}DD_{ijk} + P_{55}EE_{ijk}\right) + \\ \left(P_{66}GG_{ijk} + P_{77}NN_{ijk} + P_{88}PP_{ijk} + P_{99}SS_{ijk} + P_{1010}TT_{ijk}\right) + \\ \left(P_{mn}\left(AUX_{ijk}^{\ I} + AUX_{ijk}^{\ J}\right)\right) + \left(P_{36}\left(AUX_{ijk}^{\ 57} + AUX_{ijk}^{\ 58}\right)\right) \end{bmatrix} dA \qquad \text{(B-1)}$$

and

$$k_{ijks}^{(NL3)} = \iint \begin{bmatrix} \left(P_{11}AA_{ijks} + P_{22}BB_{ijks} + P_{33}CC_{ijks} + P_{44}DD_{ijks} + P_{55}EE_{ijks}\right) + \\ \left(P_{66}GG_{ijks} + P_{77}NN_{ijks} + P_{88}PP_{ijks} + P_{99}SS_{ijks} + P_{1010}TT_{ijks}\right) + \\ \left(P_{mn}\left(AUX_{ijks}^{\ I} + AUX_{ijks}^{\ J}\right)\right) + \left(P_{36}\left(AUX_{ijks}^{\ 57} + AUX_{ijks}^{\ 58}\right)\right) \end{bmatrix} dA \qquad \text{(B-2)}$$

Where $dA=R\ dx\ d\theta$ and:

$m = 1,2,\ldots,9$ $\qquad I=1,3,5,\ldots,55$ $\qquad$ *and* $\qquad m,\ n\neq 3,6$

$n=m+1$ *to* 10 $\qquad J=I+1$

The P_{ij}'s are the terms of the elasticity matrix [P] and the terms AA_{ijk}, BB_{ijk}, ..., AUX_{ijk}^{58} and AA_{ijks}, BB_{ijks},..., AUX_{ijks}^{58} represent the coefficients of the modal equations. The following relations define these definitions:

$$\begin{aligned} AA_{ijk} &= a_i AA_{jk} + a_j AA_{ki} + a_k AA_{ij} & GG_{ijk} &= g_i GG_{jk} + g_j GG_{ki} + g_k GG_{ij} \\ BB_{ijk} &= b_i BB_{jk} + b_j BB_{ki} + b_k BB_{ij} & NN_{ijk} &= n_i NN_{jk} + n_j NN_{ki} + n_k NN_{ij} \\ CC_{ijk} &= c_i CC_{jk} + c_j CC_{ki} + c_k CC_{ij} & PP_{ijk} &= p_i PP_{jk} + p_j PP_{ki} + p_k PP_{ij} \\ DD_{ijk} &= d_i DD_{jk} + d_j DD_{ki} + d_k DD_{ij} & SS_{ijk} &= s_i SS_{jk} + s_j SS_{ki} + s_k SS_{ij} \\ EE_{ijk} &= e_i EE_{jk} + e_j EE_{ki} + e_k EE_{ij} & TT_{ijk} &= t_i TT_{jk} + t_j TT_{ki} + t_k TT_{ij} \end{aligned} \qquad \text{(B-3)}$$

and

$$\begin{aligned} AA_{ijks} &= 2AA_{is}AA_{jk} & GG_{ijks} &= 2GG_{is}GG_{jk} \\ BB_{ijks} &= 2BB_{is}BB_{jk} & NN_{ijks} &= 2NN_{is}NN_{jk} \\ CC_{ijks} &= 2CC_{is}CC_{jk} & PP_{ijks} &= 2PP_{is}PP_{jk} \\ DD_{ijks} &= 2DD_{is}DD_{jk} & SS_{ijks} &= 2SS_{is}SS_{jk} \\ EE_{ijks} &= 2EE_{is}EE_{jk} & TT_{ijks} &= 2TT_{is}TT_{jk} \end{aligned} \qquad \text{(B-4)}$$

and AUX_{ijk}^{I} and AUX_{ijks}^{I} ($I = 1$ to 58) are defined by (B-10 and B-11).

With:

$$AA_{ij} = \frac{1}{2}\left[\frac{\partial U_i}{\partial x}\frac{\partial U_j}{\partial x} + \frac{\partial V_i}{\partial x}\frac{\partial V_j}{\partial x} + \frac{\partial W_i}{\partial x}\frac{\partial W_j}{\partial x}\right]$$

$$BB_{ij} = \frac{1}{2R}\left[\frac{\partial V_i}{\partial x}\frac{\partial V_j}{\partial \theta} + \frac{\partial V_j}{\partial x}\frac{\partial V_i}{\partial \theta}\right] + \frac{1}{2R}\left[W_i\frac{\partial V_j}{\partial x} + W_j\frac{\partial V_i}{\partial x}\right] \qquad \text{(B-5)}$$

$$CC_{ij} = \frac{1}{2}\left[\beta_{xi}\frac{\partial U_j}{\partial x} + \beta_{xj}\frac{\partial U_i}{\partial x}\right] + \frac{1}{2}\left[\beta_{\theta i}\frac{\partial V_j}{\partial x} + \beta_{\theta j}\frac{\partial V_i}{\partial x}\right]$$

$$DD_{ij} = \frac{1}{2R^2}\left[\frac{\partial U_i}{\partial \theta}\frac{\partial U_j}{\partial \theta}\right] + \frac{1}{2R^2}\left[\left(V_i - \frac{\partial W_i}{\partial \theta}\right)\left(V_j - \frac{\partial W_j}{\partial \theta}\right)\right] + \frac{1}{2R^2}\left[\left(W_i + \frac{\partial V_i}{\partial \theta}\right)\left(W_j + \frac{\partial V_j}{\partial \theta}\right)\right]$$

$$EE_{ij} = \frac{1}{2R}\left[\frac{\partial U_i}{\partial x}\frac{\partial U_j}{\partial \theta} + \frac{\partial U_j}{\partial x}\frac{\partial U_i}{\partial \theta}\right] + \frac{1}{2R}\left[\frac{\partial W_i}{\partial x}\frac{\partial W_j}{\partial \theta} + \frac{\partial W_j}{\partial x}\frac{\partial W_i}{\partial \theta}\right] - \frac{1}{2R}\left[V_i\frac{\partial W_j}{\partial x} + V_j\frac{\partial W_i}{\partial x}\right] \qquad \text{(B-6)}$$

$$GG_{ij} = \frac{1}{2R}\left[\beta_{\theta i}\frac{\partial V_j}{\partial \theta} + \beta_{\theta j}\frac{\partial V_i}{\partial \theta}\right] + \frac{1}{2R}\left[W_i\beta_{\theta_j} + W_j\beta_{\theta_i}\right] + \frac{1}{2R}\left[\beta_{x_i}\frac{\partial U_j}{\partial \theta} + \beta_{x_j}\frac{\partial U_i}{\partial \theta}\right]$$

$$NN_{ij} = \frac{1}{2}\left[\frac{\partial U_i}{\partial x}\frac{\partial \beta_{x_j}}{\partial x} + \frac{\partial U_j}{\partial x}\frac{\partial \beta_{x_i}}{\partial x}\right] + \frac{1}{2}\left[\frac{\partial V_i}{\partial x}\frac{\partial \beta_{\theta_j}}{\partial x} + \frac{\partial V_j}{\partial x}\frac{\partial \beta_{\theta_i}}{\partial x}\right]$$

$$PP_{ij} = \frac{1}{2R}\left[\frac{\partial V_i}{\partial x}\frac{\partial \beta_{\theta_j}}{\partial \theta} + \frac{\partial V_j}{\partial x}\frac{\partial \beta_{\theta_i}}{\partial \theta}\right] + \frac{1}{2R}\left[\frac{\partial \beta_{\theta_i}}{\partial x}\frac{\partial V_j}{\partial \theta} + \frac{\partial \beta_{\theta_j}}{\partial x}\frac{\partial V_i}{\partial \theta}\right] + \frac{1}{2R}\left[W_i\frac{\partial \beta_{\theta_j}}{\partial x} + W_j\frac{\partial \beta_{\theta_i}}{\partial x}\right] \qquad \text{(B-7)}$$

$$SS_{ij} = \frac{1}{2R^2}\left[\frac{\partial U_i}{\partial \theta}\frac{\partial \beta_{x_j}}{\partial \theta} + \frac{\partial U_j}{\partial \theta}\frac{\partial \beta_{x_i}}{\partial \theta}\right] + \frac{1}{2R^2}\left[\frac{\partial V_i}{\partial \theta}\frac{\partial \beta_{\theta_j}}{\partial \theta} + \frac{\partial V_j}{\partial \theta}\frac{\partial \beta_{\theta_i}}{\partial \theta}\right] +$$

$$+ \frac{1}{2R^2}\left[V_i\beta_{\theta_j} + V_j\beta_{\theta_i}\right] + \frac{1}{2R^2}\left[W_i\frac{\partial \beta_{\theta_j}}{\partial \theta} + W_j\frac{\partial \beta_{\theta_i}}{\partial \theta}\right] - \frac{1}{2R^2}\left[\beta_{\theta_i}\frac{\partial W_j}{\partial \theta} + \beta_{\theta_j}\frac{\partial W_i}{\partial \theta}\right] \qquad \text{(B-8)}$$

$$TT_{ij} = \frac{1}{2R}\left[\frac{\partial U_i}{\partial x}\frac{\partial \beta_{x_j}}{\partial \theta} + \frac{\partial U_j}{\partial x}\frac{\partial \beta_{x_i}}{\partial \theta}\right] + \frac{1}{2R}\left[\frac{\partial \beta_{x_i}}{\partial x}\frac{\partial U_j}{\partial \theta} + \frac{\partial \beta_{x_j}}{\partial x}\frac{\partial U_i}{\partial \theta}\right] - \frac{1}{2R}\left[\beta_{\theta_i}\frac{\partial W_j}{\partial x} + \beta_{\theta_j}\frac{\partial W_i}{\partial x}\right]$$

and

$$\begin{aligned}
a_i &= \frac{\partial U_i}{\partial x}; & g_i &= \frac{1}{R}\frac{\partial W_i}{\partial \theta} - \frac{V_i}{R} + \beta_{\theta_i} \\
b_i &= \frac{\partial V_i}{\partial x}; & n_i &= \frac{\partial \beta_{x_i}}{\partial x} \\
c_i &= \frac{\partial W_i}{\partial x} + \beta_{x_i}; & p_i &= \frac{\partial \beta_{\theta_i}}{\partial x} + \frac{1}{2R}\frac{\partial V_i}{\partial x} \\
d_i &= \frac{1}{R}\left(\frac{\partial V_i}{\partial \theta} + W_i\right); & s_i &= \frac{1}{R}\frac{\partial \beta_{\theta_i}}{\partial \theta} \\
e_i &= \frac{1}{R}\frac{\partial U_i}{\partial \theta}; & t_i &= \frac{1}{R}\frac{\partial \beta_{x_i}}{\partial \theta} - \frac{1}{2R^2}\frac{\partial U_i}{\partial \theta}
\end{aligned} \qquad \text{(B-9)}$$

where U, V, W, β_x and β_θ are the spatial functions determined by equation (6) of the text. In equations (B-3 to B-8), the subscripts " i, j " and " i, j, k, s " represent the coupling between two and four distinct modes, respectively.

The AUX $_{ijk}^{I}$'s elements (equation B-1):

$$AUX_{ijk}^{1} = a_j BB_{ki} + a_k BB_{ij} + b_i AA_{jk}; \qquad AUX_{ijk}^{2} = b_j AA_{ki} + b_k AA_{ij} + a_i BB_{jk}$$

$$AUX_{ijk}^{3} = a_j DD_{ki} + a_k DD_{ij} + d_i AA_{jk}; \qquad AUX_{ijk}^{4} = d_j AA_{ki} + d_k AA_{ij} + a_i DD_{jk}$$

$$AUX_{ijk}^{5} = a_j EE_{ki} + a_k EE_{ij} + e_i AA_{jk}; \qquad AUX_{ijk}^{6} = e_j AA_{ki} + e_k AA_{ij} + a_i EE_{jk}$$

$$AUX_{ijk}^{7} = a_j NN_{ki} + a_k NN_{ij} + n_i AA_{jk}; \qquad AUX_{ijk}^{8} = n_j AA_{ki} + n_k AA_{ij} + a_i NN_{jk}$$

$$AUX_{ijk}^{9} = a_j PP_{ki} + a_k PP_{ij} + p_i AA_{jk}; \qquad AUX_{ijk}^{10} = p_j AA_{ki} + p_k AA_{ij} + a_i PP_{jk}$$

$$AUX_{ijk}^{11} = a_j SS_{ki} + a_k SS_{ij} + s_i AA_{jk}; \qquad AUX_{ijk}^{12} = s_j AA_{ki} + s_k AA_{ij} + a_i SS_{jk}$$

$$AUX_{ijk}^{13} = a_j TT_{ki} + a_k TT_{ij} + t_i AA_{jk}; \qquad AUX_{ijk}^{14} = t_j AA_{ki} + t_k AA_{ij} + a_i TT_{jk}$$

$$AUX_{ijk}^{15} = d_j BB_{ki} + d_k BB_{ij} + b_i DD_{jk}; \qquad AUX_{ijk}^{16} = b_j DD_{ki} + b_k DD_{ij} + d_i BB_{jk}$$

$$AUX_{ijk}^{17} = e_j BB_{ki} + e_k BB_{ij} + b_i EE_{jk}; \qquad AUX_{ijk}^{18} = b_j EE_{ki} + b_k EE_{ij} + e_i BB_{jk}$$

$$AUX_{ijk}^{19} = n_j BB_{ki} + n_k BB_{ij} + b_i NN_{jk}; \qquad AUX_{ijk}^{20} = b_j NN_{ki} + b_k NN_{ij} + n_i BB_{jk}$$

$$AUX_{ijk}^{21} = p_j BB_{ki} + p_k BB_{ij} + b_i PP_{jk}; \qquad AUX_{ijk}^{22} = b_j PP_{ki} + b_k PP_{ij} + p_i BB_{jk}$$

$$AUX_{ijk}^{23} = s_j BB_{ki} + s_k BB_{ij} + b_i SS_{jk}; \qquad AUX_{ijk}^{24} = b_j SS_{ki} + b_k SS_{ij} + s_i BB_{jk}$$

$$AUX_{ijk}^{25} = t_j BB_{ki} + t_k BB_{ij} + b_i TT_{jk}; \qquad AUX_{ijk}^{26} = b_j TT_{ki} + b_k TT_{ij} + t_i BB_{jk}$$

$$AUX_{ijk}^{27} = d_j EE_{ki} + d_k EE_{ij} + e_i DD_{jk}; \qquad AUX_{ijk}^{28} = e_j DD_{ki} + e_k DD_{ij} + d_i EE_{jk}$$

$$AUX_{ijk}^{29} = d_j NN_{ki} + d_k NN_{ij} + n_i DD_{jk}; \qquad AUX_{ijk}^{30} = n_j DD_{ki} + n_k DD_{ij} + d_i NN_{jk}$$

$$AUX_{ijk}^{31} = d_j PP_{ki} + d_k PP_{ij} + p_i DD_{jk}; \qquad AUX_{ijk}^{32} = p_j DD_{ki} + p_k DD_{ij} + d_i PP_{jk}$$

$$AUX_{ijk}^{33} = d_j SS_{ki} + d_k SS_{ij} + s_i DD_{jk}; \qquad AUX_{ijk}^{34} = s_j DD_{ki} + s_k DD_{ij} + d_i SS_{jk}$$

$$AUX_{ijk}^{35} = d_j TT_{ki} + d_k TT_{ij} + t_i DD_{jk}; \qquad AUX_{ijk}^{36} = t_j DD_{ki} + t_k DD_{ij} + d_i TT_{jk}$$

$$AUX_{ijk}^{37} = e_j NN_{ki} + e_k NN_{ij} + n_i EE_{jk}; \qquad AUX_{ijk}^{38} = n_j EE_{ki} + n_k EE_{ij} + e_i NN_{jk}$$

$$AUX_{ijk}^{39} = e_j PP_{ki} + e_k PP_{ij} + p_i EE_{jk}; \qquad AUX_{ijk}^{40} = p_j EE_{ki} + p_k EE_{ij} + e_i PP_{jk}$$

$$AUX_{ijk}^{41} = e_j SS_{ki} + e_k SS_{ij} + s_i EE_{jk}; \qquad AUX_{ijk}^{42} = s_j EE_{ki} + s_k EE_{ij} + e_i SS_{jk}$$

$$AUX_{ijk}^{43} = e_j TT_{ki} + e_k TT_{ij} + t_i EE_{jk}; \qquad AUX_{ijk}^{44} = t_j EE_{ki} + t_k EE_{ij} + e_i TT_{jk}$$

$$AUX_{ijk}^{45} = n_j PP_{ki} + n_k PP_{ij} + p_i NN_{jk}; \qquad AUX_{ijk}^{46} = p_j NN_{ki} + p_k NN_{ij} + n_i PP_{jk}$$

$$AUX_{ijk}^{47} = n_j SS_{ki} + n_k SS_{ij} + s_i NN_{jk}; \qquad AUX_{ijk}^{48} = s_j NN_{ki} + s_k NN_{ij} + n_i SS_{jk}$$

$$AUX_{ijk}^{49} = n_j TT_{ki} + n_k TT_{ij} + t_i NN_{jk}; \qquad AUX_{ijk}^{50} = t_j NN_{ki} + t_k NN_{ij} + n_i TT_{jk}$$

$$AUX_{ijk}^{51} = p_j SS_{ki} + p_k SS_{ij} + s_i PP_{jk}; \qquad AUX_{ijk}^{52} = s_j PP_{ki} + s_k PP_{ij} + p_i SS_{jk}$$

$$AUX_{ijk}^{53} = p_j TT_{ki} + p_k TT_{ij} + t_i PP_{jk}; \qquad AUX_{ijk}^{54} = t_j PP_{ki} + t_k PP_{ij} + p_i TT_{jk}$$

$$AUX_{ijk}^{55} = s_j TT_{ki} + s_k TT_{ij} + t_i SS_{jk}; \qquad AUX_{ijk}^{56} = t_j SS_{ki} + t_k SS_{ij} + s_i TT_{jk} \qquad \text{(B-10)}$$

$$AUX_{ijk}^{57} = c_j GG_{ki} + c_k GG_{ij} + g_i CC_{jk}; \qquad AUX_{ijk}^{58} = g_j CC_{ki} + g_k CC_{ij} + c_i GG_{jk}$$

The AUX $_{ijks}^{I}$'s elements (equation B-2):

$AUX^{1}_{ijks} = 2AA_{jk}BB_{is}$; $AUX^{2}_{ijks} = 2BB_{jk}AA_{is}$; $AUX^{3}_{ijks} = 2AA_{jk}DD_{is}$; $AUX^{4}_{ijks} = 2DD_{jk}AA_{is}$

$AUX^{5}_{ijks} = 2AA_{jk}EE_{is}$; $AUX^{6}_{ijks} = 2EE_{jk}AA_{is}$; $AUX^{7}_{ijks} = 2AA_{jk}NN_{is}$; $AUX^{8}_{ijks} = 2NN_{jk}AA_{is}$

$AUX^{9}_{ijks} = 2AA_{jk}PP_{is}$; $AUX^{10}_{ijks} = 2PP_{jk}AA_{is}$; $AUX^{11}_{ijks} = 2AA_{jk}SS_{is}$; $AUX^{12}_{ijks} = 2SS_{jk}AA_{is}$

$AUX^{13}_{ijks} = 2AA_{jk}TT_{is}$; $AUX^{14}_{ijks} = 2TT_{jk}AA_{is}$; $AUX^{15}_{ijks} = 2BB_{jk}DD_{is}$; $AUX^{16}_{ijks} = 2DD_{jk}BB_{is}$

$AUX^{17}_{ijks} = 2BB_{jk}EE_{is}$; $AUX^{18}_{ijks} = 2EE_{jk}BB_{is}$; $AUX^{19}_{ijks} = 2BB_{jk}NN_{is}$; $AUX^{20}_{ijks} = 2NN_{jk}BB_{is}$

$AUX^{21}_{ijks} = 2BB_{jk}PP_{is}$; $AUX^{22}_{ijks} = 2PP_{jk}BB_{is}$; $AUX^{23}_{ijks} = 2BB_{jk}SS_{is}$; $AUX^{24}_{ijks} = 2SS_{jk}BB_{is}$

$AUX^{25}_{ijks} = 2BB_{jk}TT_{is}$; $AUX^{26}_{ijks} = 2TT_{jk}BB_{is}$; $AUX^{27}_{ijks} = 2DD_{jk}EE_{is}$; $AUX^{28}_{ijks} = 2EE_{jk}DD_{is}$

$AUX^{29}_{ijks} = 2DD_{jk}NN_{is}$; $AUX^{30}_{ijks} = 2NN_{jk}DD_{is}$; $AUX^{31}_{ijks} = 2DD_{jk}PP_{is}$; $AUX^{32}_{ijks} = 2PP_{jk}DD_{is}$

$AUX^{33}_{ijks} = 2DD_{jk}SS_{is}$; $AUX^{34}_{ijks} = 2SS_{jk}DD_{is}$; $AUX^{35}_{ijks} = 2DD_{jk}TT_{is}$; $AUX^{36}_{ijks} = 2TT_{jk}DD_{is}$

$AUX^{37}_{ijks} = 2EE_{jk}NN_{is}$; $AUX^{38}_{ijks} = 2NN_{jk}EE_{is}$; $AUX^{39}_{ijks} = 2EE_{jk}PP_{is}$; $AUX^{40}_{ijks} = 2PP_{jk}EE_{is}$

$AUX^{41}_{ijks} = 2EE_{jk}SS_{is}$; $AUX^{42}_{ijks} = 2SS_{jk}EE_{is}$; $AUX^{43}_{ijks} = 2EE_{jk}TT_{is}$; $AUX^{44}_{ijks} = 2TT_{jk}EE_{is}$

$AUX^{45}_{ijks} = 2NN_{jk}PP_{is}$; $AUX^{46}_{ijks} = 2PP_{jk}NN_{is}$; $AUX^{47}_{ijks} = 2NN_{jk}SS_{is}$; $AUX^{48}_{ijks} = 2SS_{jk}NN_{is}$

$AUX^{49}_{ijks} = 2NN_{jk}TT_{is}$; $AUX^{50}_{ijks} = 2TT_{jk}NN_{is}$; $AUX^{51}_{ijks} = 2PP_{jk}SS_{is}$; $AUX^{52}_{ijks} = 2SS_{jk}PP_{is}$

$AUX^{53}_{ijks} = 2PP_{jk}TT_{is}$; $AUX^{54}_{ijks} = 2TT_{jk}PP_{is}$; $AUX^{55}_{ijks} = 2SS_{jk}TT_{is}$; $AUX^{56}_{ijks} = 2TT_{jk}SS_{is}$ ***(B-11)***

$AUX^{57}_{ijks} = 2CC_{jk}GG_{is}$; $AUX^{58}_{ijks} = 2GG_{jk}CC_{is}$

APPENDIX C-

Substituting equation (6), of the text, into equations (B-5 to B-9), we obtain:

$$
\begin{aligned}
&AA_{ij} = C_i\left(a_{ij}^{(1)} Sin^2\overline{m}x + a_{ij}^{(2)} Cos^2\overline{m}x\right) C_j e^{(\eta_i+\eta_j)\theta}; &&GG_{ij} = C_i\left(g_{ij}^{(1)} Sin^2\overline{m}x + g_{ij}^{(2)} Cos^2\overline{m}x\right) C_j e^{(\eta_i+\eta_j)\theta}\\
&BB_{ij} = C_i\left(b_{ij}^{(1)} Sin\overline{m}x Cos\overline{m}x\right) C_j e^{(\eta_i+\eta_j)\theta}; &&NN_{ij} = C_i\left(n_{ij}^{(1)} Sin^2\overline{m}x + n_{ij}^{(2)} Cos^2\overline{m}x\right) C_j e^{(\eta_i+\eta_j)\theta}\\
&CC_{ij} = C_i\left(c_{ij}^{(1)} Sin\overline{m}x Cos\overline{m}x\right) C_j e^{(\eta_i+\eta_j)\theta}; &&PP_{ij} = C_i\left(p_{ij}^{(1)} Sin\overline{m}x Cos\overline{m}x\right) C_j e^{(\eta_i+\eta_j)\theta}\\
&DD_{ij} = C_i\left(d_{ij}^{(1)} Cos^2\overline{m}x + d_{ij}^{(2)} Sin^2\overline{m}x\right) C_j e^{(\eta_i+\eta_j)\theta}; &&SS_{ij} = C_i\left(s_{ij}^{(1)} Cos^2\overline{m}x + s_{ij}^{(2)} Sin^2\overline{m}x\right) C_j e^{(\eta_i+\eta_j)\theta}\\
&EE_{ij} = C_i\left(e_{ij}^{(1)} Sin\overline{m}x Cos\overline{m}x\right) . C_j e^{(\eta_i+\eta_j)\theta}; &&TT_{ij} = C_i\left(t_{ij}^{(1)} Sin\overline{m}x Cos\overline{m}x\right) C_j e^{(\eta_i+\eta_j)\theta}
\end{aligned}
\tag{C-1}
$$

and

$$
\begin{aligned}
&a_i = C_i a_i^{(1)} Sin\overline{m}x\, e^{\eta_i\theta}; &&g_i = C_i g_i^{(1)} Sin\overline{m}x\, e^{\eta_i\theta}\\
&b_i = C_i b_i^{(1)} Cos\overline{m}x\, e^{\eta_i\theta}; &&n_i = C_i n_i^{(1)} Sin\overline{m}x\, e^{\eta_i\theta}\\
&c_i = C_i c_i^{(1)} Cos\overline{m}x\, e^{\eta_i\theta}; &&p_i = C_i p_i^{(1)} Cos\overline{m}x\, e^{\eta_i\theta}\\
&d_i = C_i d_i^{(1)} Sin\overline{m}x\, e^{\eta_i\theta}; &&s_i = C_i s_i^{(1)} Sin\overline{m}x\, e^{\eta_i\theta}\\
&e_i = C_i e_i^{(1)} Cos\overline{m}x\, e^{\eta_i\theta}; &&t_i = C_i t_i^{(1)} Cos\overline{m}x\, e^{\eta_i\theta}
\end{aligned}
\tag{C-2}
$$

Where $a_i^{(1)}$, $b_i^{(1)}$, ..., $t_i^{(1)}$ and $a_{ij}^{(1)}$, $a_{ij}^{(2)}$, ..., $t_{ij}^{(1)}$ are given in equations (C-4 and C-5). The η_i (i =1,...,10) are the roots of characteristic equation (7), of the text, and $\overline{m} = \frac{m\pi}{L} x$ in which m is the axial mode number. The constants $C_i(i=1,\ldots 10)$ and $C_j(j=1,\ldots 10)$ can be obtained from equation (10), of the text, as follows:

$$
\{C\} = \left[A^{-1}\right] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}
\tag{C-3}
$$

The matrix $[A^{-1}]$ is the inverse of [A] given in [39].

The $a_{ij}^{(1)}$ to $t_{ij}^{(1)}$ elements (equation C-2):

$$a_{ij}^{(1)} = \bar{m}^2 \alpha_i \alpha_j; \qquad a_{ij}^{(2)} = \bar{m}^2 \left(1+\beta_i+\beta_j\right); \qquad b_{ij}^{(1)} = \frac{\bar{m}}{2R}\left[\beta_i\beta_j\left(\eta_i+\eta_j\right)+\left(\beta_i+\beta_j\right)\right]$$

$$c_{ij}^{(1)} = \frac{\bar{m}}{2R}\left[\delta_i\beta_j+\delta_j\beta_i-\left(\gamma_i\alpha_j+\gamma_j\alpha_i\right)\right]; \quad d_{ij}^{(1)} = \frac{1}{2R^2}\left(\alpha_i\alpha_j\eta_i\eta_j\right)$$

$$d_{ij}^{(2)} = \frac{1}{2R^2}\left[\left(\beta_i-\eta_i\right)\left(\beta_j-\eta_j\right)\left(1+\beta_i\eta_i\right)\left(1+\beta_j\eta_j\right)\right]; \quad e_{ij}^{(1)} = \frac{\bar{m}}{2R}\left[\left(1-\alpha_i\alpha_j\right)\left(\eta_i+\eta_j\right)-\left(\beta_i+\beta_j\right)\right]$$

$$g_{ij}^{(1)} = \frac{1}{2R}\left[\delta_i\left(1+\beta_j\eta_j\right)+\delta_j\left(1+\beta_i\eta_i\right)\right]; \quad g_{ij}^{(2)} = \frac{1}{2R}\left[\gamma_i\alpha_j\eta_j+\gamma_j\alpha_i\eta_i\right] \qquad \text{(C-4)}$$

$$n_{ij}^{(1)} = \frac{\bar{m}^2}{2}\left[\alpha_i\gamma_j+\alpha_j\gamma_i\right]; \quad n_{ij}^{(2)} = \frac{\bar{m}^2}{2}\left[\beta_i\delta_j+\beta_j\delta_i\right]; \quad p_{ij}^{(1)} = \delta_i\left(1+\beta_j\left(\eta_i+\eta_j\right)\right)+\delta_j\left(1+\beta_i\left(\eta_i+\eta_j\right)\right)$$

$$s_{ij}^{(1)} = \frac{1}{2R^2}\left[\eta_i\eta_j\left(\alpha_i\gamma_j+\alpha_j\gamma_i\right)\right]; \quad s_{ij}^{(2)} = \frac{1}{2R^2}\left[\left(\beta_i\delta_j+\beta_j\delta_i\right)\left(1+\eta_i\eta_j\right)+\left(\delta_i-\delta_j\right)\left(\eta_i-\eta_j\right)\right]$$

$$t_{ij}^{(1)} = -\frac{\bar{m}}{2R}\left[\left(\alpha_i\gamma_j+\alpha_j\gamma_i\right)\left(\eta_i+\eta_j\right)+\left(\delta_i+\delta_j\right)\right]$$

The $a_i^{(1)}$ to $t_i^{(1)}$ elements (equation C-3):

$$a_j^{(1)} = -\alpha_j\bar{m}; \qquad g_j^{(1)} = \left[\frac{1}{R}\left(\eta_j-\beta_j\right)+\delta_j\right]$$

$$b_j^{(1)} = \beta_j\bar{m}; \qquad n_j^{(1)} = -\gamma_j\bar{m}$$

$$c_j^{(1)} = (\gamma_j+\bar{m}); \qquad p_j^{(1)} = \left[\delta_j\bar{m}+\frac{\bar{m}}{2R}\beta_j\right] \qquad j=1,2\ldots,10 \qquad \text{(C-5)}$$

$$d_j^{(1)} = \frac{1}{R}(1+\eta_j\beta_j); \qquad s_j^{(1)} = \frac{1}{R}\eta_j\delta_j$$

$$e_j^{(1)} = \frac{1}{R}(\eta_j\alpha_j); \qquad t_j^{(1)} = \left[\frac{1}{R}\eta_j\gamma_j-\frac{1}{2R^2}\eta_j\alpha_j\right]$$

$$\text{where} \qquad \bar{m} = \frac{m\pi}{L}$$

Using the relations (C-1 and C-2) for AA_{ijk}, BB_{ijk}, ..., TT_{ijk} and replacing these terms by their expressions (equation B-3) into equation (B-1) and then integrating over x and θ, we obtain the expression (22-a) for the second-order non-linear matrix for an element.

Substituting the terms of AA_{ijks}, BB_{ijks}, ..., TT_{ijks} by their expressions (B-4) into equation (B-2), using the definitions (C-1 and C-2) and then integrating over x and θ, we obtain the expression (22-b) that defines the third-order non-linear matrix for an element of anisotropic cylindrical shell.

APPENDIX D- The GG (i,j) 's elements (equation 23-a):

$$
GG(i,j) = I_I \left[\begin{array}{l}
P_{11}\left(a_i^{(1)}A_{ij}^{-1}a_{jk}^{(2)} + a_j^{(1)}A_{jk}^{-1}a_{ki}^{(2)} + a_k^{(1)}A_{ki}^{-1}a_{ij}^{(2)}\right) + P_{22}\left(b_i^{(1)}A_{ij}^{-1}b_{jk}^{(1)} + b_j^{(1)}A_{jk}^{-1}b_{ki}^{(1)} + b_k^{(1)}A_{ki}^{-1}b_{ij}^{(1)}\right) + \\
P_{33}\left(c_i^{(1)}A_{ij}^{-1}c_{jk}^{(1)} + c_j^{(1)}A_{jk}^{-1}c_{ki}^{(1)} + c_k^{(1)}A_{ki}^{-1}c_{ij}^{(1)}\right) + P_{44}\left(d_i^{(1)}A_{ij}^{-1}d_{jk}^{(1)} + d_j^{(1)}A_{jk}^{-1}d_{ki}^{(1)} + d_k^{(1)}A_{ki}^{-1}d_{ij}^{(1)}\right) + \\
P_{55}\left(e_i^{(1)}A_{ij}^{-1}e_{jk}^{(1)} + e_j^{(1)}A_{jk}^{-1}e_{ki}^{(1)} + e_k^{(1)}A_{ki}^{-1}e_{ij}^{(1)}\right) + P_{66}\left(g_i^{(1)}A_{ij}^{-1}g_{jk}^{(2)} + g_j^{(1)}A_{jk}^{-1}g_{ki}^{(2)} + g_k^{(1)}A_{ki}^{-1}g_{ij}^{(2)}\right) + \\
P_{77}\left(n_i^{(1)}A_{ij}^{-1}n_{jk}^{(2)} + n_j^{(1)}A_{jk}^{-1}n_{ki}^{(2)} + n_k^{(1)}A_{ki}^{-1}n_{ij}^{(2)}\right) + P_{88}\left(p_i^{(1)}A_{ij}^{-1}p_{jk}^{(1)} + p_j^{(1)}A_{jk}^{-1}p_{ki}^{(1)} + p_k^{(1)}A_{ki}^{-1}p_{ij}^{(1)}\right) + \\
P_{99}\left(s_i^{(1)}A_{ij}^{-1}s_{jk}^{(1)} + s_j^{(1)}A_{jk}^{-1}s_{ki}^{(1)} + s_k^{(1)}A_{ki}^{-1}s_{ij}^{(1)}\right) + P_{1010}\left(t_i^{(1)}A_{ij}^{-1}t_{jk}^{(1)} + t_j^{(1)}A_{jk}^{-1}t_{ki}^{(1)} + t_k^{(1)}A_{ki}^{-1}t_{ij}^{(1)}\right) + \\
P_{14}\left(a_i^{(1)}A_{ij}^{-1}d_{jk}^{(1)} + a_j^{(1)}A_{jk}^{-1}d_{ki}^{(1)} + a_k^{(1)}A_{ki}^{-1}d_{ij}^{(1)}\right) + P_{14}\left(d_i^{(1)}A_{ij}^{-1}a_{jk}^{(2)} + d_j^{(1)}A_{jk}^{-1}a_{ki}^{(2)} + d_k^{(1)}A_{ki}^{-1}a_{ij}^{(2)}\right) + \\
P_{17}\left(a_i^{(1)}A_{ij}^{-1}n_{jk}^{(2)} + a_j^{(1)}A_{jk}^{-1}n_{ki}^{(2)} + a_k^{(1)}A_{ki}^{-1}n_{ij}^{(2)}\right) + P_{17}\left(n_i^{(1)}A_{ij}^{-1}a_{jk}^{(2)} + n_j^{(1)}A_{jk}^{-1}a_{ki}^{(2)} + n_k^{(1)}A_{ki}^{-1}a_{ij}^{(2)}\right) + \\
P_{19}\left(a_i^{(1)}A_{ij}^{-1}s_{jk}^{(1)} + a_j^{(1)}A_{jk}^{-1}s_{ki}^{(1)} + a_k^{(1)}A_{ki}^{-1}s_{ij}^{(1)}\right) + P_{19}\left(s_i^{(1)}A_{ij}^{-1}a_{jk}^{(2)} + s_j^{(1)}A_{jk}^{-1}a_{ki}^{(2)} + s_k^{(1)}A_{ki}^{-1}a_{ij}^{(2)}\right) + \\
P_{25}\left(b_i^{(1)}A_{ij}^{-1}e_{jk}^{(1)} + b_j^{(1)}A_{jk}^{-1}e_{ki}^{(1)} + b_k^{(1)}A_{ki}^{-1}e_{ij}^{(1)}\right) + P_{25}\left(e_i^{(1)}A_{ij}^{-1}b_{jk}^{(1)} + e_j^{(1)}A_{jk}^{-1}b_{ki}^{(1)} + e_k^{(1)}A_{ki}^{-1}b_{ij}^{(1)}\right) + \\
P_{28}\left(b_i^{(1)}A_{ij}^{-1}p_{jk}^{(1)} + b_j^{(1)}A_{jk}^{-1}p_{ki}^{(1)} + b_k^{(1)}A_{ki}^{-1}p_{ij}^{(1)}\right) + P_{28}\left(p_i^{(1)}A_{ij}^{-1}b_{jk}^{(1)} + p_j^{(1)}A_{jk}^{-1}b_{ki}^{(1)} + p_k^{(1)}A_{ki}^{-1}b_{ij}^{(1)}\right) + \\
P_{210}\left(b_i^{(1)}A_{ij}^{-1}t_{jk}^{(1)} + b_j^{(1)}A_{jk}^{-1}t_{ki}^{(1)} + b_k^{(1)}A_{ki}^{-1}t_{ij}^{(1)}\right) + P_{210}\left(t_i^{(1)}A_{ij}^{-1}b_{jk}^{(1)} + t_j^{(1)}A_{jk}^{-1}b_{ki}^{(1)} + t_k^{(1)}A_{ki}^{-1}b_{ij}^{(1)}\right) + \\
P_{47}\left(d_i^{(1)}A_{ij}^{-1}n_{jk}^{(2)} + d_j^{(1)}A_{jk}^{-1}n_{ki}^{(2)} + d_k^{(1)}A_{ki}^{-1}n_{ij}^{(2)}\right) + P_{47}\left(n_i^{(1)}A_{ij}^{-1}d_{jk}^{(1)} + n_j^{(1)}A_{jk}^{-1}d_{ki}^{(1)} + n_k^{(1)}A_{ki}^{-1}d_{ij}^{(1)}\right) + \\
P_{49}\left(d_i^{(1)}A_{ij}^{-1}s_{jk}^{(1)} + d_j^{(1)}A_{jk}^{-1}s_{ki}^{(1)} + d_k^{(1)}A_{ki}^{-1}s_{ij}^{(1)}\right) + P_{49}\left(s_i^{(1)}A_{ij}^{-1}d_{jk}^{(1)} + s_j^{(1)}A_{jk}^{-1}d_{ki}^{(1)} + s_k^{(1)}A_{ki}^{-1}d_{ij}^{(1)}\right) + \\
P_{58}\left(e_i^{(1)}A_{ij}^{-1}p_{jk}^{(1)} + e_j^{(1)}A_{jk}^{-1}p_{ki}^{(1)} + e_k^{(1)}A_{ki}^{-1}p_{ij}^{(1)}\right) + P_{58}\left(p_i^{(1)}A_{ij}^{-1}e_{jk}^{(1)} + p_j^{(1)}A_{jk}^{-1}e_{ki}^{(1)} + p_k^{(1)}A_{ki}^{-1}e_{ij}^{(1)}\right) + \\
P_{510}\left(e_i^{(1)}A_{ij}^{-1}t_{jk}^{(1)} + e_j^{(1)}A_{jk}^{-1}t_{ki}^{(1)} + e_k^{(1)}A_{ki}^{-1}t_{ij}^{(1)}\right) + P_{510}\left(t_i^{(1)}A_{ij}^{-1}e_{jk}^{(1)} + t_j^{(1)}A_{jk}^{-1}e_{ki}^{(1)} + t_k^{(1)}A_{ki}^{-1}e_{ij}^{(1)}\right) + \\
P_{79}\left(n_i^{(1)}A_{ij}^{-1}s_{jk}^{(1)} + n_j^{(1)}A_{jk}^{-1}s_{ki}^{(1)} + n_k^{(1)}A_{ki}^{-1}s_{ij}^{(1)}\right) + P_{79}\left(s_i^{(1)}A_{ij}^{-1}n_{jk}^{(2)} + s_j^{(1)}A_{jk}^{-1}n_{ki}^{(2)} + s_k^{(1)}A_{ki}^{-1}n_{ij}^{(2)}\right) + \\
P_{810}\left(p_i^{(1)}A_{ij}^{-1}t_{jk}^{(1)} + p_j^{(1)}A_{jk}^{-1}t_{ki}^{(1)} + p_k^{(1)}A_{ki}^{-1}t_{ij}^{(1)}\right) + P_{810}\left(t_i^{(1)}A_{ij}^{-1}p_{jk}^{(1)} + t_j^{(1)}A_{jk}^{-1}p_{ki}^{(1)} + t_k^{(1)}A_{ki}^{-1}p_{ij}^{(1)}\right)
\end{array}\right] +
$$

$$
+ I_{II} \left[\begin{array}{l}
P_{11}\left(a_i^{(1)}A_{ij}^{-1}a_{jk}^{(1)} + a_j^{(1)}A_{jk}^{-1}a_{ki}^{(1)} + a_k^{(1)}A_{ki}^{-1}a_{ij}^{(1)}\right) + P_{44}\left(d_i^{(1)}A_{ij}^{-1}d_{jk}^{(2)} + d_j^{(1)}A_{jk}^{-1}d_{ki}^{(2)} + d_k^{(1)}A_{ki}^{-1}d_{ij}^{(2)}\right) + \\
P_{66}\left(g_i^{(1)}A_{ij}^{-1}g_{jk}^{(1)} + g_j^{(1)}A_{jk}^{-1}g_{ki}^{(1)} + g_k^{(1)}A_{ki}^{-1}g_{ij}^{(1)}\right) + P_{77}\left(n_i^{(1)}A_{ij}^{-1}n_{jk}^{(1)} + n_j^{(1)}A_{jk}^{-1}n_{ki}^{(1)} + n_k^{(1)}A_{ki}^{-1}n_{ij}^{(1)}\right) + \\
P_{99}\left(s_i^{(1)}A_{ij}^{-1}s_{jk}^{(2)} + s_j^{(1)}A_{jk}^{-1}s_{ki}^{(2)} + s_k^{(1)}A_{ki}^{-1}s_{ij}^{(2)}\right) + P_{14}\left(a_i^{(1)}A_{ij}^{-1}d_{jk}^{(2)} + a_j^{(1)}A_{jk}^{-1}d_{ki}^{(2)} + a_k^{(1)}A_{ki}^{-1}d_{ij}^{(2)}\right) + \\
P_{14}\left(d_i^{(1)}A_{ij}^{-1}a_{jk}^{(1)} + d_j^{(1)}A_{jk}^{-1}a_{ki}^{(1)} + d_k^{(1)}A_{ki}^{-1}a_{ij}^{(1)}\right) + P_{17}\left(a_i^{(1)}A_{ij}^{-1}n_{jk}^{(1)} + a_j^{(1)}A_{jk}^{-1}n_{ki}^{(1)} + a_k^{(1)}A_{ki}^{-1}n_{ij}^{(1)}\right) + \\
P_{17}\left(n_i^{(1)}A_{ij}^{-1}a_{jk}^{(1)} + n_j^{(1)}A_{jk}^{-1}a_{ki}^{(1)} + n_k^{(1)}A_{ki}^{-1}a_{ij}^{(1)}\right) + P_{19}\left(a_i^{(1)}A_{ij}^{-1}s_{jk}^{(2)} + a_j^{(1)}A_{jk}^{-1}s_{ki}^{(2)} + a_k^{(1)}A_{ki}^{-1}s_{ij}^{(2)}\right) + \\
P_{19}\left(s_i^{(1)}A_{ij}^{-1}a_{jk}^{(1)} + s_j^{(1)}A_{jk}^{-1}a_{ki}^{(1)} + s_k^{(1)}A_{ki}^{-1}a_{ij}^{(1)}\right) + P_{47}\left(d_i^{(1)}A_{ij}^{-1}n_{jk}^{(1)} + d_j^{(1)}A_{jk}^{-1}n_{ki}^{(1)} + d_k^{(1)}A_{ki}^{-1}n_{ij}^{(1)}\right) + \\
P_{47}\left(n_i^{(1)}A_{ij}^{-1}d_{jk}^{(2)} + n_j^{(1)}A_{jk}^{-1}d_{ki}^{(2)} + n_k^{(1)}A_{ki}^{-1}d_{ij}^{(2)}\right) + P_{49}\left(d_i^{(1)}A_{ij}^{-1}s_{jk}^{(2)} + d_j^{(1)}A_{jk}^{-1}s_{ki}^{(2)} + d_k^{(1)}A_{ki}^{-1}s_{ij}^{(2)}\right) + \\
P_{49}\left(s_i^{(1)}A_{ij}^{-1}d_{jk}^{(2)} + s_j^{(1)}A_{jk}^{-1}d_{ki}^{(2)} + s_k^{(1)}A_{ki}^{-1}d_{ij}^{(2)}\right) + P_{79}\left(n_i^{(1)}A_{ij}^{-1}s_{jk}^{(2)} + n_j^{(1)}A_{jk}^{-1}s_{ki}^{(2)} + n_k^{(1)}A_{ki}^{-1}s_{ij}^{(2)}\right) + \\
P_{79}\left(s_i^{(1)}A_{ij}^{-1}n_{jk}^{(1)} + s_j^{(1)}A_{jk}^{-1}n_{ki}^{(1)} + s_k^{(1)}A_{ki}^{-1}n_{ij}^{(1)}\right)
\end{array}\right]
$$

$$
\text{where: } I_I = \frac{1}{3\overline{m}}\left[1-(-1)^m\right]; \qquad I_{II} = 2I_I \qquad \text{and} \qquad \overline{m} = \frac{m\pi}{L}
$$

(D-1)

The SA (i,j) 's elements (equation 23-b):

$$
\begin{aligned}
SA(i,s) = {} & 3P_{11}a_{ip}^{(1)}a_{qs}^{(1)} + P_{11}\left(a_{ip}^{(1)}a_{qs}^{(2)} + a_{ip}^{(2)}a_{qs}^{(1)}\right) + 3P_{11}a_{ip}^{(2)}a_{qs}^{(2)} + \\
& + P_{22}b_{ip}^{(1)}b_{qs}^{(1)} + P_{33}c_{ip}^{(1)}c_{qs}^{(1)} + P_{44}\left[3d_{ip}^{(1)}d_{qs}^{(1)} + \left(d_{ip}^{(1)}d_{qs}^{(2)} + d_{ip}^{(2)}d_{qs}^{(1)}\right) + 3d_{ip}^{(2)}d_{qs}^{(2)}\right] + \\
& + P_{55}e_{ip}^{(1)}e_{qs}^{(1)} + P_{66}\left[3g_{ip}^{(1)}g_{qs}^{(1)} + \left(g_{ip}^{(1)}g_{qs}^{(2)} + g_{ip}^{(2)}g_{qs}^{(1)}\right) + 3g_{ip}^{(2)}g_{qs}^{(2)}\right] + \\
& + P_{77}\left[3n_{ip}^{(1)}n_{qs}^{(1)} + \left(n_{ip}^{(1)}n_{qs}^{(2)} + n_{ip}^{(2)}n_{qs}^{(1)}\right) + 3n_{ip}^{(2)}n_{qs}^{(2)}\right] + P_{88}p_{ip}^{(1)}p_{qs}^{(1)} + \\
& + P_{99}\left[3s_{ip}^{(1)}s_{qs}^{(1)} + \left(s_{ip}^{(1)}s_{qs}^{(2)} + s_{ip}^{(2)}s_{qs}^{(1)}\right) + 3s_{ip}^{(2)}s_{qs}^{(2)}\right] + P_{1010}t_{ip}^{(1)}t_{qs}^{(1)} + \\
& + P_{14}\left[\left(a_{ip}^{(1)}d_{qs}^{(1)} + a_{ip}^{(2)}d_{qs}^{(2)} + d_{ip}^{(1)}a_{qs}^{(1)} + d_{ip}^{(2)}a_{qs}^{(2)}\right) + 3\left(a_{ip}^{(1)}d_{qs}^{(2)} + a_{ip}^{(2)}d_{qs}^{(1)} + d_{ip}^{(1)}a_{qs}^{(2)} + d_{ip}^{(2)}a_{qs}^{(1)}\right)\right] \\
& + P_{17}\left[\left(a_{ip}^{(1)}n_{qs}^{(2)} + a_{ip}^{(2)}n_{qs}^{(1)} + n_{ip}^{(1)}a_{qs}^{(2)} + n_{ip}^{(2)}a_{qs}^{(1)}\right) + 3\left(a_{ip}^{(1)}n_{qs}^{(1)} + a_{ip}^{(2)}n_{qs}^{(2)} + n_{ip}^{(1)}a_{qs}^{(1)} + n_{ip}^{(2)}a_{qs}^{(2)}\right)\right] + \\
& + P_{19}\left[\left(a_{ip}^{(1)}s_{qs}^{(1)} + a_{ip}^{(2)}s_{qs}^{(2)} + s_{ip}^{(1)}a_{qs}^{(1)} + s_{ip}^{(2)}a_{qs}^{(2)}\right) + 3\left(a_{ip}^{(1)}s_{qs}^{(2)} + a_{ip}^{(2)}s_{qs}^{(1)} + s_{ip}^{(1)}a_{qs}^{(2)} + s_{ip}^{(2)}a_{qs}^{(1)}\right)\right] + \\
& + P_{25}\left[b_{ip}^{(1)}e_{qs}^{(1)} + e_{ip}^{(1)}b_{qs}^{(1)}\right] + P_{28}\left[b_{ip}^{(1)}p_{qs}^{(1)} + p_{ip}^{(1)}b_{qs}^{(1)}\right] + P_{210}\left[b_{ip}^{(1)}t_{qs}^{(1)} + t_{ip}^{(1)}b_{qs}^{(1)}\right] + \\
& + P_{47}\left[\left(d_{ip}^{(1)}n_{qs}^{(1)} + d_{ip}^{(2)}n_{qs}^{(2)} + n_{ip}^{(1)}d_{qs}^{(1)} + n_{ip}^{(2)}d_{qs}^{(2)}\right) + 3\left(d_{ip}^{(1)}n_{qs}^{(2)} + d_{ip}^{(2)}n_{qs}^{(1)} + n_{ip}^{(1)}d_{qs}^{(2)} + n_{ip}^{(2)}d_{qs}^{(1)}\right)\right] + \\
& + P_{49}\left[\left(d_{ip}^{(1)}s_{qs}^{(2)} + d_{ip}^{(2)}s_{qs}^{(1)} + s_{ip}^{(1)}d_{qs}^{(2)} + s_{ip}^{(2)}d_{qs}^{(1)}\right) + 3\left(d_{ip}^{(1)}s_{qs}^{(1)} + d_{ip}^{(2)}s_{qs}^{(2)} + s_{ip}^{(1)}d_{qs}^{(1)} + s_{ip}^{(2)}d_{qs}^{(2)}\right)\right] + \\
& + P_{58}\left[e_{ip}^{(1)}p_{qs}^{(1)} + p_{ip}^{(1)}e_{qs}^{(1)}\right] + P_{510}\left[e_{ip}^{(1)}t_{qs}^{(1)} + t_{ip}^{(1)}e_{qs}^{(1)}\right] + \\
& + P_{79}\left[\left(n_{ip}^{(1)}s_{qs}^{(1)} + n_{ip}^{(2)}s_{qs}^{(2)} + s_{ip}^{(1)}n_{qs}^{(1)} + s_{ip}^{(2)}n_{qs}^{(2)}\right) + 3\left(n_{ip}^{(1)}s_{qs}^{(2)} + n_{ip}^{(2)}s_{qs}^{(1)} + s_{ip}^{(1)}n_{qs}^{(2)} + s_{ip}^{(2)}n_{qs}^{(1)}\right)\right] + \\
& + P_{810}\left[p_{ip}^{(1)}t_{qs}^{(1)} + t_{ip}^{(1)}p_{qs}^{(1)}\right]
\end{aligned}
\qquad \text{(D-2)}
$$

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List of Tables

Table I: Non-dimensional fundamental frequencies, $\Omega=\omega_o L^2/(\rho/E_2)^{1/2}/t$ of simply-supported cylindrical shell with symmetric cross-ply 0°/90°/0°.

R/L	L/t=100			L/t=10		
	Sanders' Theory*	Reddy, J.N. [50]	Present	Sanders' Theory*	Reddy, J.N. [50]	Present
1	73.05	66.583	63.50	15.811	13.172	13.135
2	38.77	36.770	33.51	14.794	12.438	12.129
3	28.36	27.116	24.52	14.726	12.287	11.716
4	23.73	22.709	20.96	14.688	12.233	11.542
5	22.15	20.232	19.18	14.675	12.207	11.461
6	21.05	-	18.17	14.638	-	11.367
7	20.28	-	17.25	14.612	-	11.356
8	19.69	-	16.96	14.575	-	11.350
9	19.23	-	16.79	14.568	-	11.341
10	18.58	16.625	16.60	14.537	12.173	11.332

* These results have been obtained by author for Sanders' theory, not considering shear deformation effect.

List of Figures

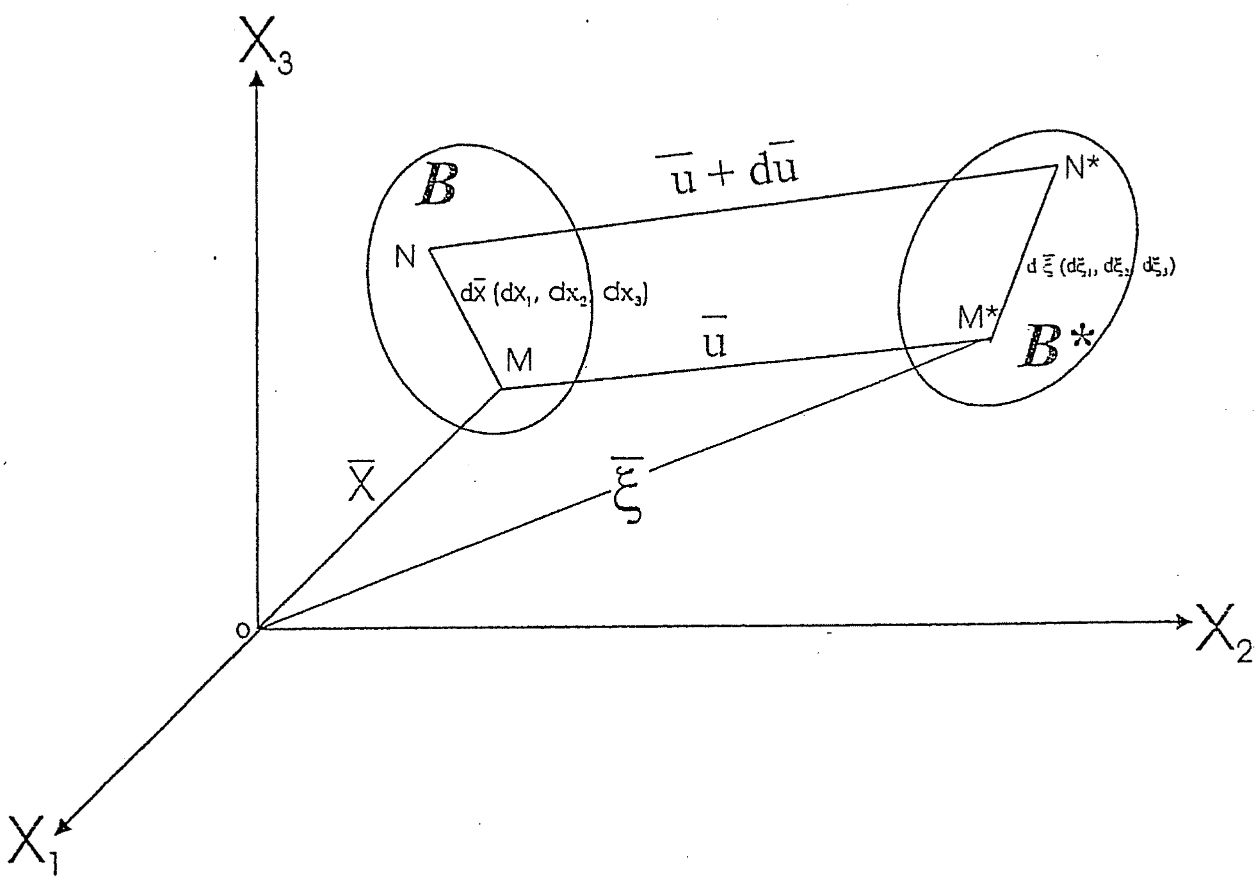
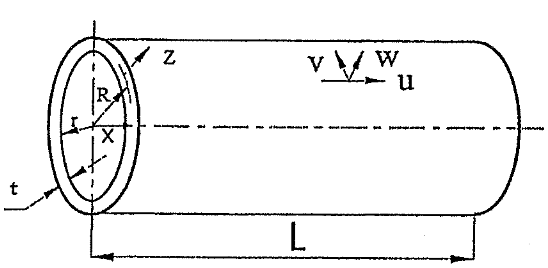
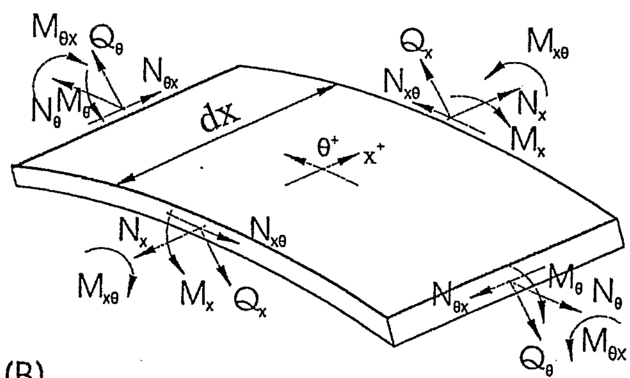


FIGURE 1

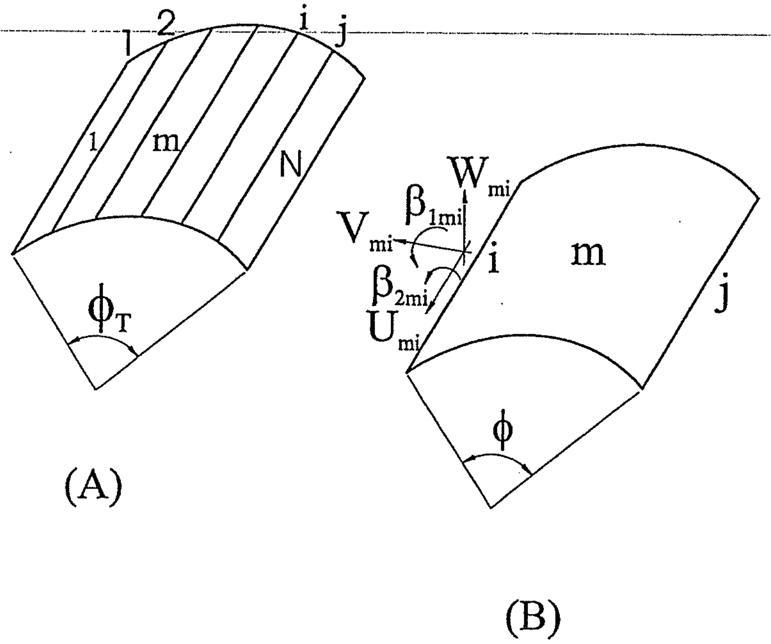

z
R
r
x
t
v
w
u
L

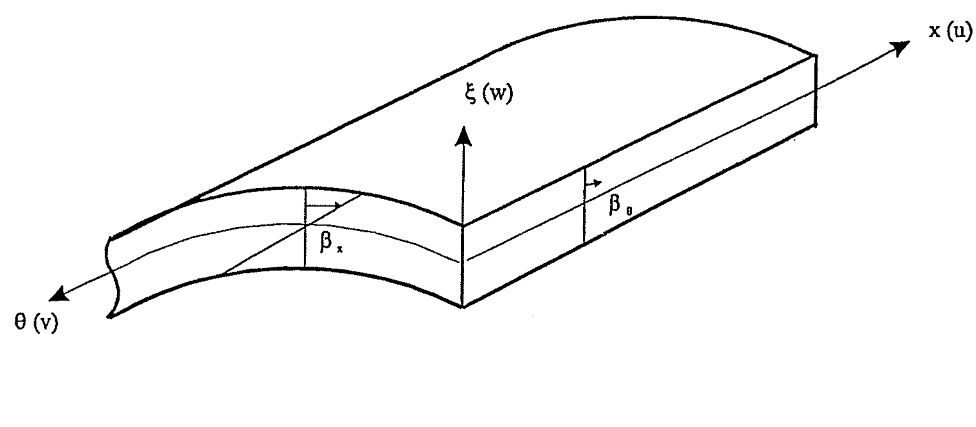
(A)


$M_{\theta x}$
Q_θ
$N_{\theta x}$
N_θ
M_θ
dx
Q_x
$M_{x\theta}$
$N_{x\theta}$
N_x
M_x
θ^+
x^+
N_x
$N_{x\theta}$
$M_{x\theta}$
M_x
Q_x
$N_{\theta x}$
M_θ
N_θ
Q_θ
$M_{\theta x}$

(B)

FIGURE 2

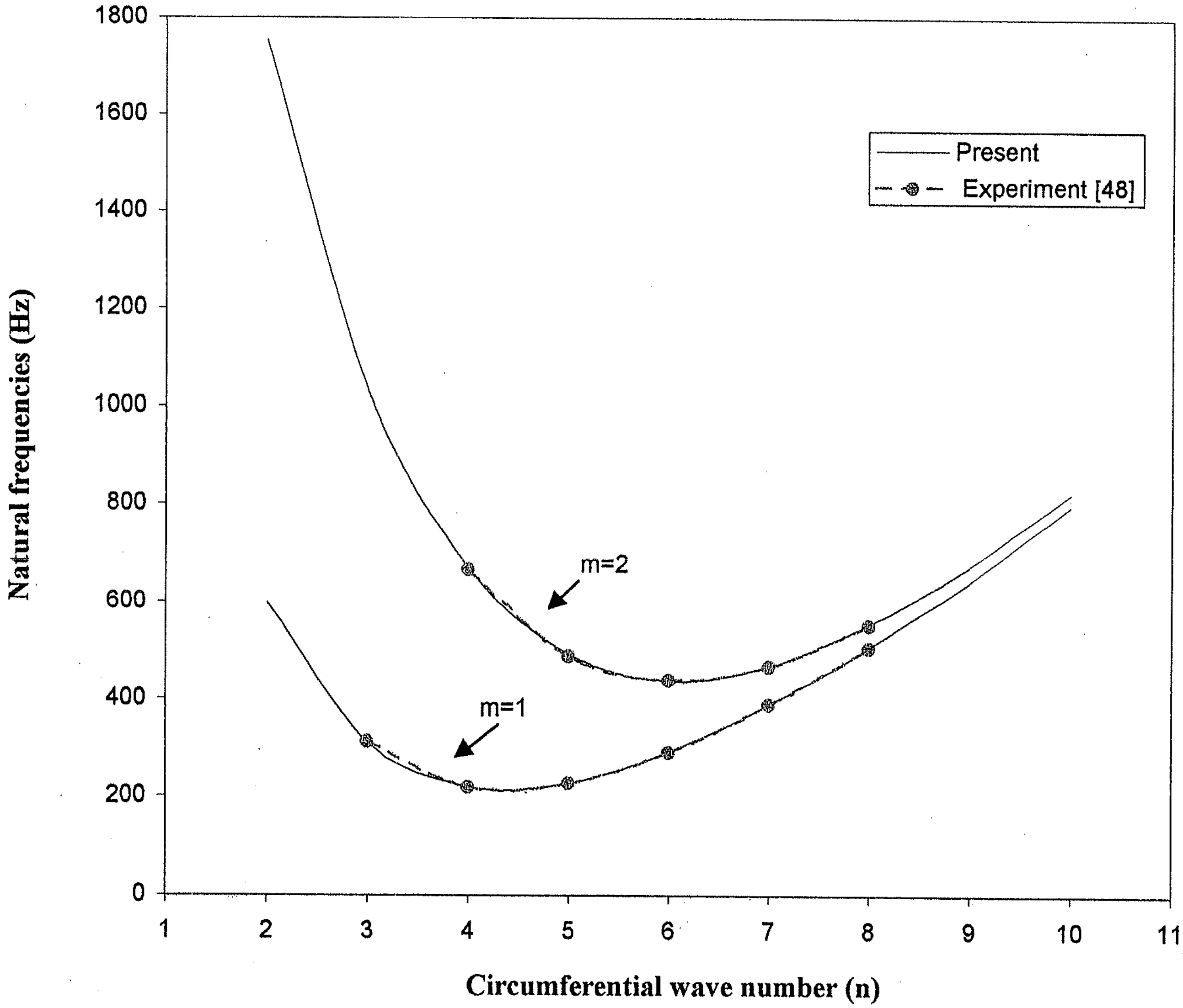

1
2
i
j
1
m
N
ϕ_T
(A)
W_{mi}
β_{1mi}
V_{mi}
i
m
β_{2mi}
j
U_{mi}
ϕ
(B)


x (u)
ξ (w)
β_θ
β_x
θ (v)

(C)

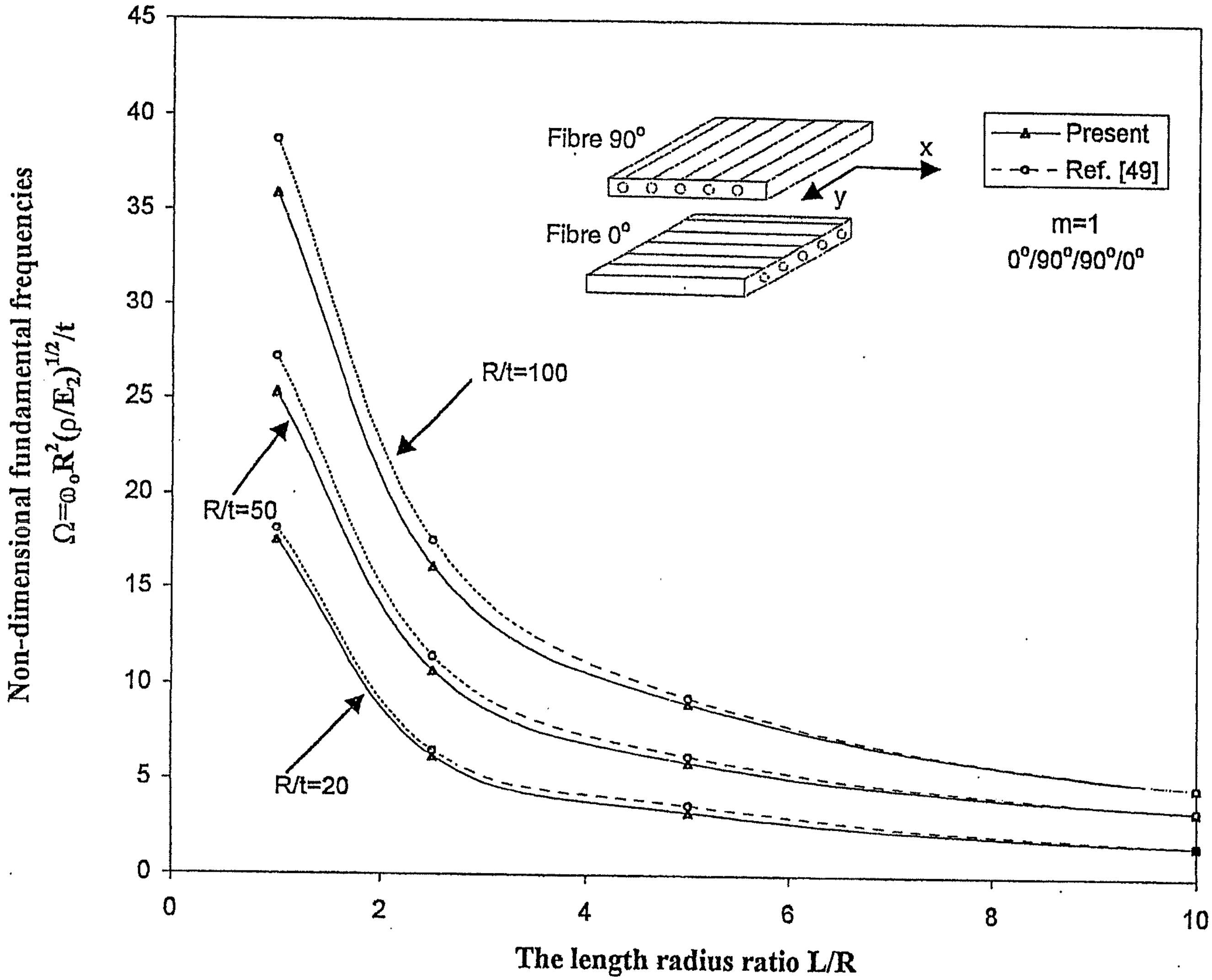
FIGURE 3

FIGURE 4



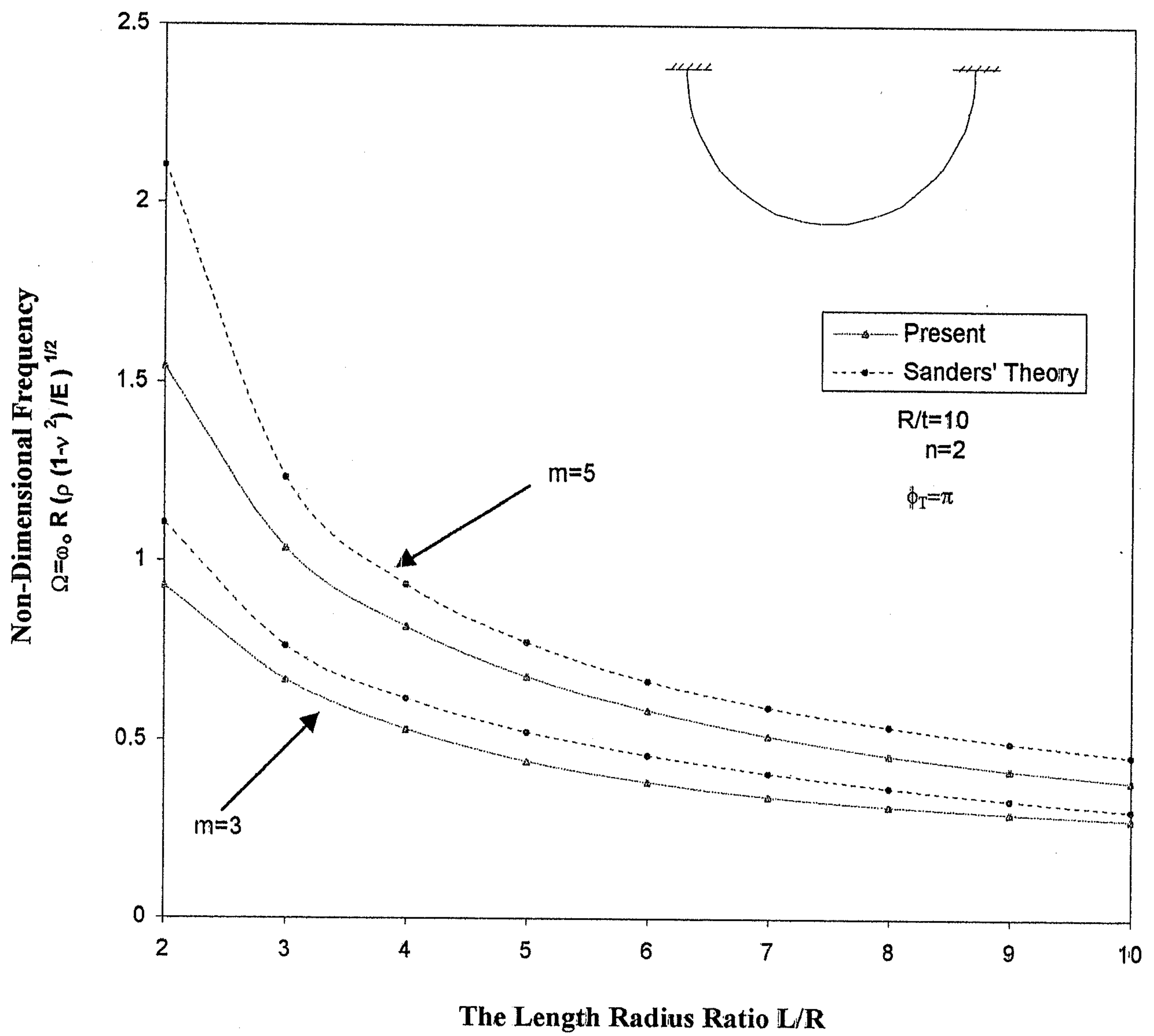
1800
1600
1400
1200
1000
800
600
400
200
0
1
2
3
4
5
6
7
8
9
10
11
Natural frequencies (Hz)
Circumferential wave number (n)
Present
Experiment [48]
m=2
m=1

FIGURE 5



Non-dimensional fundamental frequencies
$\Omega=\omega_o R^2(\rho/E_2)^{1/2}/t$
The length radius ratio L/R
Present
Ref. [49]
m=1
0°/90°/90°/0°
Fibre 90°
Fibre 0°
x
y
R/t=100
R/t=50
R/t=20
0
2
4
6
8
10
0
5
10
15
20
25
30
35
40
45

FIGURE 6



Non-Dimensional Frequency
Ω=ω₀ R (ρ (1-ν 2) /E) 1/2
2.5
2
1.5
1
0.5
0
2
3
4
5
6
7
8
9
10
The Length Radius Ratio L/R
Present
Sanders' Theory
R/t=10
n=2
ϕT=π
m=5
m=3

FIGURE 7

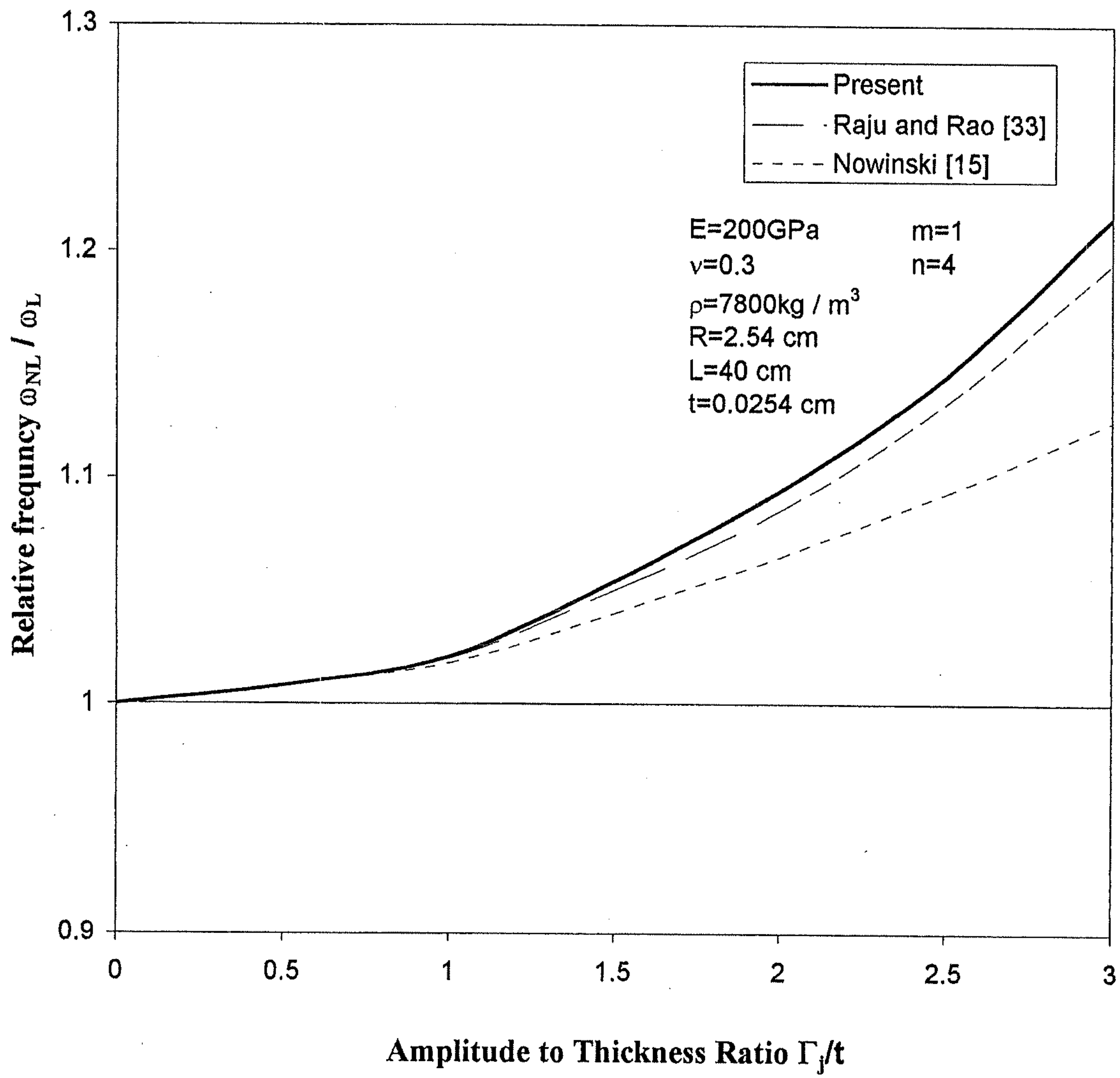
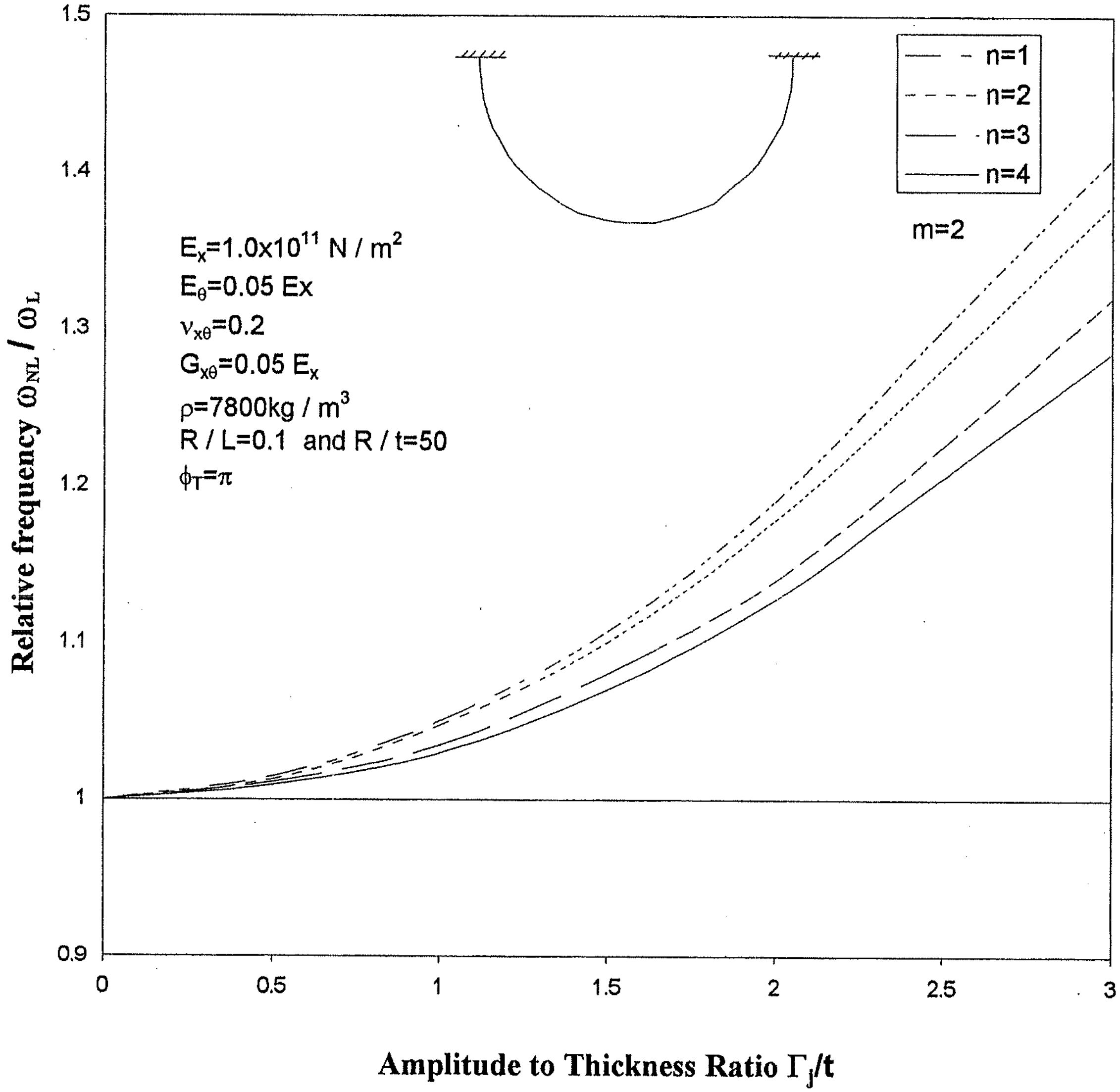

Present
Raju and Rao [33]
Nowinski [15]
E=200GPa
m=1
ν=0.3
n=4
ρ=7800kg / m³
R=2.54 cm
L=40 cm
t=0.0254 cm
1.3
1.2
1.1
1
0.9
0
0.5
1
1.5
2
2.5
3
Relative frequncy ω_NL / ω_L
Amplitude to Thickness Ratio Γj/t

FIGURE 8



n=1
n=2
n=3
n=4
m=2
E_x=1.0x10^{11} N / m^2
E_θ=0.05 Ex
$\nu_{x\theta}$=0.2
$G_{x\theta}$=0.05 E_x
ρ=7800kg / m^3
R / L=0.1 and R / t=50
ϕ_T=π
1.5
1.4
1.3
1.2
1.1
1
0.9
0
0.5
1
1.5
2
2.5
3
Relative frequency ω_{NL} / ω_L
Amplitude to Thickness Ratio Γ_j/t

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