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"DYNAMIC MODELLING OF MECHANICAL SYSTEMS SUBJECT TO HOLONOMIC AND NONHOLONOMIC CONSTRAINTS"

par

Romano M. DeSantis

Département de génie électrique et génie informatique

Section automatique

École Polytechnique de Montréal

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DYNAMIC MODELING OF MECHANICAL SYSTEMS SUBJECT TO HOLONOMIC AND NONHOLONOMIC CONSTRAINTS

DeSantis, R.M., Professor, E & C Eng Dpt

Ecole Polytechnique de Montréal
2900, boul. Edouard-Montpetit, CP 6079-A
Montreal, Quebec, Canada, H3C 3A7

0. Summary

A conceptual procedure for modeling the dynamics of a system of rigid bodies is presented. The main idea is to first describe the system's kinematics (positions and orientations and linear and angular velocities), then to write down the Newton-Euler equations, and, finally, to impose the constraints concerning positions and orientations (holonomic constraints) and linear and angular velocities (nonholonomic constraints). This procedure is quite general and can be applied to the modeling of many of the mechanical systems encountered in the high-level automation of industrial operations, like, for example, autonomous vehicles, robotic manipulators and LHD (loading-hauling-dumping) mining vehicles.

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Contents

Symbols

n: number of rigid bodies in the mechanical system;

p_1: number of holonomic constraints;

ℓ_1: number of nonholonomic constraints;

p: dimension of the configuration vector (in general: $p=6n-p_1$);

ℓ: dimension of the generalized velocity (in general: $\ell=p-\ell_1$);

c_i: position in the work-space of the i-th body of the system;

ξ_i: orientation in the work-space of the i-th body of the system;

$\chi_i := [c_i' \ \xi_i']'$: configuration in the work-space of the i-th body of the system;

$\chi := [\chi_1' \ \chi_2' \ \dots \chi_n']'$; configuration of the system of the n bodies;

v_i: a vector describing the linear velocity of the i-th body with respect to the work-space;

Ω_i: a vector describing the angular velocity of the i-th body with respect to the work-space;

$\nu_i := [v_i' \ \Omega_i']'$: the twist of the i-th body;

$\nu := [\nu_1' \ \nu_2' \ \dots \ \nu_n']'$: the twist of the ensemble of the n bodies;

$S(\Omega_i)$: angular velocity matrix associated with Ω_i;

$$S(\Omega_i) := \begin{matrix} 0 & -\Omega_{iz} & \Omega_{ix} \\ \Omega_{iz} & 0 & -\Omega_{iy} \\ -\Omega_{ix} & \Omega_{iy} & 0 \end{matrix} \quad . \qquad (1)$$

W_i: extended angular velocity matrix

$$W_i := \begin{matrix} S(\Omega_i) & 0_3 \\ 0_3 & S(\Omega_i) \end{matrix} \ ;$$

$W := \mathrm{diag}(W_i)$, $i=1,\dots, n$;

q: vector of generalized coordinates, also configuration vector;

α: vector of generalized velocities;

h_i: a function expressing a holonomic constraint;

g_i': a vector representing a non-admissible velocity in the work-space;

J_h: a matrix, the columns of which represent a basis for the subspace of velocities in the work-space that are admissible in the presence of the holonomic constraints;

J_{nh}: a matrix, the columns of which represent a basis for the subspace of admissible velocities in the configuration space;

$J:=J_hJ_{nh}$: a matrix, the columns of which represent a basis for the subspace of velocities in the work-space that are admissible in the presence of the holonomic and nonholonomic constraints;

m_i: the mass of the i-th body;

j_i: inertia matrix of the i-th body with respect to its frame (the origin of the frame is located at the center of mass);

f_i: resultant of the forces applied to the i-th body;

n_i: resultant of the moment of the forces applied to the i-th body about its center of mass;

$\omega_i:=[f_i' \ n_i']'$: wrench applied to the i-th body;

M_i:= i-th body's extended mass-matrix

$$M_i := \begin{matrix} m_i I_3 & 0_3 \\ 0_3 & j_i \end{matrix} \;;$$

$M:=\text{diag}(M_i)$: system's mass-matrix;

$W:=\text{diag}(W_i)$: system's extended angular matrix;

$\omega:=[\omega_1' \ \omega_2' \ \ldots \ \omega_n']'$: force-torque applied to the system;

ω_c: component of ω that is required to impose the holonomic and nonholonomic constraints;

$D(q)$: inertia matrix of the system;

$C(q,\alpha)$: Coriolis and centripetal matrix;

τ_a: vector of generalized forces.

Figures

1. Introduction

The high-level automation of industrial machines, like, for example, robotic manipulators, overhead cranes, transport vehicles, and LHD (loading-hauling-dumping) vehicles, requires the solution of various problems such as motion- and force-control, path-tracking, task-planning, and other similar problems. An effective solution of these problems requires a full understanding of the relations between forces and torques on the one hand and accelerations, speeds, positions and orientations on the other (see, for example, references by DeSantis and al. 1988-1994).

The importance of these relations is widely recognized in the technical literature and ample discussions may be found in a number of introductory books (for example, Kane and Levinson (1983), Spong and Vidyasagar (1987), and Craig (1989)) as well as in specialized articles (for example, Kane and Levinson (1985) and Saha and Angeles (1991)). However, it is not always easy to extract from this literature an answer to a number of basic questions that arise in the automation of industrial machines . Why does the dynamics of a robotic manipulator enjoy the structural properties to which we have now become accustomed (such as a special form for the state model, the positive definite property of the inertia matrix, the special relation between the derivative of the inertia matrix and the matrix of Coriolis and centripetal forces)? To what extent are these properties shared by the dynamics of other mechanical systems of interest, such as overhead cranes, transport vehicles, and LHD vehicles? To what extent can the highly refined modeling procedures and control techniques that have been developed for robotic manipulators be applied to the other systems? With what modifications and what differences in computational efficiency, physical assumptions and conceptual tools?

The objective of the present report is to provide at least a partial answer to these questions by presenting a systematic and unified modeling procedure that is simultaneously applicable to a variety of mechanical systems (including those mentioned above). While inspired by the well-known Kane approach and influenced by the recent work of Saha and Angeles (1991), the steps of our procedure will be constructed so as to emphasize intuition and physical significance as well as numerical computations. In particular, we will rely on the notions of Jacobian matrices, admissible velocities in the work-space and in the configuration space, generalized forces and generalized displacements, holonomic and nonholonomic constraints. Applications to robotic manipulators, tractor-trailers and LHD vehicles will be discussed.

2. Some kinematic concepts

Consider a mechanical system made up of n rigid bodies. To each of these bodies we attach a reference frame and describe the position and orientation of the i-th body with the vector $\chi_i := [c_i' \ \xi_i']$, where c_i gives the coordinates of the origin of the i-th frame with respect to the work-space and ξ_i gives the orientation of this frame. The velocity of the i-th body with respect to the work-space is described by a vector $\nu_i := [v_i' \ \Omega_i']'$, where v_i represents linear velocity and Ω_i angular velocity. To the angular velocity Ω_i, we associate the angular velocity matrix

$$S(\Omega_i) := \begin{matrix} 0 & -\Omega_{iz} & \Omega_{ix} \\ \Omega_{iz} & 0 & -\Omega_{iy} \\ -\Omega_{ix} & \Omega_{iy} & 0 \end{matrix} \quad . \qquad (1)$$

The ensemble of positions and orientations of the n bodies is described by the vector

$$\chi := [\chi_1' \ \chi_2' \ \dots \ \chi_n']'; \qquad (2)$$

the ensemble of velocities is described by the vector

$$\nu := [\nu_1' \ \nu_2' \ \dots \ \nu_n']'. \qquad (3)$$

In most applications the mechanical system must satisfy a set of position and orientation constraints (called holonomic constraints). From a mathematical point of view, these constraints may be modeled in terms of a set of scalar equations of the form

$$h_i(\chi)=0. \qquad i=1, \dots, p_1. \qquad (4)$$

It follows that the admissible values of χ can be described in terms of a vector of generalized coordinates (configuration vector) $q \epsilon R^p$, with p equal to $6n-p_1$:

$$\chi=f(q). \qquad (5)$$

Eqns 4 and 5 imply that the vectors ν_i may be expressed as

$$\nu_i=J_{hi}(q)\dot{q} \qquad i=1, \dots n, \qquad (6)$$

where $\dot{q}$ is referred to as the velocity in the configuration space; $J_{hi}(q)$ is a 6xp matrix (the Jacobian matrix of the i-th body), the columns of which span the subspace of velocities (in the work-space) that are compatible with the holonomic constraints. Introducing the notation

$$J_h := [J_{h1}' \ J_{h2}' \ \dots \ J_{hn}']' \tag{7}$$

we can rewrite eqn 6 in the more compact form:

$$v = J_h(q)\dot{q}. \tag{8}$$

In addition to the holonomic contraints, in some applications (for example, transport vehicles and overhead cranes) the vector v must also satisfy a set of constraints of the form

$$g_i(\chi)v = 0, \qquad i = 1, \dots, \ell_1 \tag{9}$$

(these are called nonholonomic constraints, see chapter 9 in Latombe (1991)). Taking into account eqns 5, 8 and 9 it follows

$$A(q)\dot{q} = 0 \tag{10}$$

where

$$A(q) := g(f(q))J_h(q). \tag{11}$$

This implies that $\dot{q}$ can then be represented by

$$\dot{q} = J_{nh}(q)\alpha \qquad \alpha \epsilon R^{\ell}, \ \ell = p - \ell_1, \tag{12}$$

where α is referred to as the generalized velocity and J_{nh} is a matrix, the structure of which is of the following type

$$J_{nh} := (I_n - A(A'A)^{-1}A')B(q). \tag{13}$$

By analogy to J_h, the columns of J_{nh} represent a basis for the subspace of velocities in the configuration space that are compatible with the nonholonomic constraints.

With the introduction of the matrix

$$J := J_h J_{nh} \tag{14}$$

from eqns 8 and 12, we obtain

$$\nu = J\alpha \tag{15}$$

and therefore

$$\dot{\nu} = \dot{J}\alpha + J\dot{\alpha}. \tag{16}$$

Note that the columns of the $6x\ell$ matrix J give a basis for the subspace of velocities in the work-space that are admissible under the simultaneous imposition of both the holonomic and nonholonomic constraints.

3. The Newton-Euler equations

To describe the dynamics of the i-th body, let v_i and $\dot{v}_i$ denote the velocity and the acceleration of its center of mass, and let Ω_i and $\dot{\Omega}_i$ represent angular velocity and acceleration. Further, denote by f_i the resultant of the forces applied to the i-th body, and by n_i the moment about the center of mass of these forces. Consider that all these vectors be measured with respect to the body's frame.

Assuming the work-space to be an inertial system, the motion of the i-th body subject to the application of f_i and n_i can be described by the Newton-Euler equations

$$f_i = m_i\dot{v}_i + m_i S(\Omega_i) v_i \tag{1}$$

$$n_i = S(\Omega_i) j_i \Omega_i + j_i \dot{\Omega}_i \tag{2}$$

where m_i is the mass of the body, and j_i is the inertia mass-matrix with respect to the body's frame translated to its center of mass.

Remark 1. Eqns 1 and 2 allow us to compute the force f_i and the torque n_i that are required to impose a specified position and orientation trajectory. They also make it possible to compute the position and orientation that result from the application of a specified time-function of f_i and n_i. In particular, with the vector c_i denoting the work-space coordinates of the origin of the body's frame and with $\xi_i = R_i$, where R_i is the cosine matrix of the body's frame relative to the work-space, we have

$$\dot{v}_i = \frac{f_i}{m_i} - S(\Omega_i)v_i \qquad (3)$$

$$\dot{\Omega}_i = j_i^{-1}\{n_i - S(\Omega_i)j_i\Omega_i\} \qquad (4)$$

$$\dot{R}_i = S(R_i\Omega_i)R_i \qquad (5)$$

$$\dot{c}_i = R_i v_i. \qquad (6)$$

Remark 2. Eqns 3 and 4 may be written in the more compact form

$$M_i\dot{\nu}_i = -W_i M_i \nu_i + \omega_i \qquad (7)$$

where M_i is the extended mass-matrix,

$$M_i := \begin{matrix} m_i I_3 & 0_3 \\ 0_3 & j_i \end{matrix} \; ; \qquad (8)$$

W_i is the extended angular velocity matrix

$$W_i := \begin{matrix} S(\Omega_i) & 0_3 \\ 0_3 & S(\Omega_i) \end{matrix} \; ; \qquad (9)$$

and vector ω_i, the force-torque applied to the body, is defined as

$$\omega_i := [f_i' \ n_i']'. \tag{10}$$

Remark 3. In writing eqns 1-6, we have taken into account that when the linear and angular velocities v_i and Ω_i are measured with respect to the i-th body frame, then the measures with respect to this frame of the linear and angular accelerations do not necessarily correspond to the derivatives of the measures of v_i and Ω_i.

4. A system of n bodies

Introducing the notations

$$M := \mathrm{diag}(M_i) \tag{1}$$

$$W := \mathrm{diag}(W_i) \tag{2}$$

$$\omega := [\omega_1' \ \omega_2' \ \ldots \ \omega_n']'. \tag{3}$$

we can rewrite eqn 3.7 as

$$M\dot{v} = -WMv + \omega \tag{4}$$

and, using eqns 2.16,

$$MJ\dot{\alpha} = - M\dot{J}\alpha - WMJ\alpha + \omega. \tag{6}$$

Note that the force-torque ω can be decomposed into

$$\omega = \omega_c + \omega_a \tag{8}$$

where ω_c (the constraint force-torque) represents the resultants of the forces and torques that are required to impose the holonomic and the nonholonomic constraints; ω_a (the active force-torque) represents the resultants of forces and torques applied to the n bodies that are not generated by these constraints. Note also that under the assumption of workless constraints, the vector ω_c is orthogonal to the admissible values of the velocity vector ν, that is,

$$\nu'\omega_c = 0. \tag{9}$$

Using eqn 2.15, it follows that

$$\alpha' J' \omega_c = 0 \tag{10}$$

and therefore, since this last equation must hold for every $\alpha \epsilon R^{\ell}$,

$$J'\omega_c = 0. \tag{11}$$

By virtue of eqn 11, pre-multiplication by J' of both members of eqn 6, gives

$$J'MJ\dot{\alpha} = - J'M\dot{J}\alpha - J'WMJ\alpha + \tau_a \tag{12}$$

where τ_a:=$J'\omega_a$ is referred to as the vector of generalized forces. It follows that by introducing the inertia matrix

$$D(q):=J'MJ \tag{13}$$

and the Coriolis and centripetal matrix

$$C(q,\alpha):=J'(M\dot{J} + WMJ), \tag{14}$$

the dynamic model of the mechanical system may be rewritten as

$$\dot{\alpha} = D(q)^{-1}\{-C(q,\alpha)\alpha + \tau_a\} \tag{15}$$

$$\dot{q} = J_{nh}(q)\alpha \tag{16}$$

$$[c' \ \xi']' = f(q). \tag{17}$$

In a more synthetic form, we have

$$\dot{\alpha} = g_0 + g_{wa}\tau_a \tag{18}$$

$$\dot{q} = J_{nh}(q)\alpha \tag{19}$$

$$\chi = f(q) \tag{20}$$

$$g_0 := D(q)^{-1}C(q,\alpha)\alpha \tag{21}$$

$$g_{wa}:=D(q)^{-1} \tag{22}$$

$$D(q):=J'MJ \tag{23}$$

$$C(q,\alpha):=J'(M\dot{J} + WMJ) \tag{24}$$

$$\tau_a:=J'\omega_a. \tag{25}$$

Remark 1: By the principle of conservation of energy, we must have

$$\frac{d}{dt}\{\frac{1}{2}\alpha'D(q)\alpha\}=\alpha'\tau_a \tag{25}$$

hence

$$\alpha'D(q)\dot{\alpha} + \frac{1}{2}\alpha'\dot{D}(q)\alpha = \alpha'\{D(q)\dot{\alpha} + C(q,\alpha)\alpha\} \tag{26}$$

and therefore

$$\alpha'(\dot{D}(q) - 2\,C(q,\alpha))\alpha = 0. \tag{27}$$

This equation implies that the matrix $\dot{D}(q)-2C(q,\alpha)$ is skew-symmetric, a property that has an important implication in the control of a robotic manipulator (see section 7.f).

5. The modeling procedure

The modeling procedure presented in the previous section may be summarized as follows:

1. Attach a frame to each of the n bodies of the system; represent the position and orientation of this frame by selecting a suitable $\chi_i:=[c_i' \ \xi_i']$ vector;

2. With p the number of independent variables required to describe an admissible value of $\chi:=[\chi_1' \ \chi_2' \ \ldots\chi_n']$, select a meaningful configuration vector $q\epsilon R^p$ (this step takes into account the holonomic constraints);

3. Identify the (direct kinematics) function f(q) such that

$$\chi = f(q); \tag{1}$$

4. Determine the Jacobian matrices describing the work-space velocities in terms of the configuration space velocities

$$\nu_i := [v_i' \ \Omega_i']' = J_{hi}(q)\dot{q} \tag{2}$$

where ν_i is measured with respect to the i-th body's frame; express J_{hi} in terms of the body's frame translated into the body's center of gravity; set

$$J_h^c := [J_{h1}^{c\prime} \ J_{h2}^{c\prime} \dots \ J_{hn}^{c\prime}]'; \tag{3}$$

5. Select an appropriate generalized velocity vector $\alpha \epsilon R^{\ell}$, where ℓ is the number of independent variables required to specify an admissible value of the configuration-space velocity $\dot{q}$ (nonholonomic constraints come into play); express $\dot{q}$ in terms of the generalized velocity α:

$$\dot{q} = J_{nh}(q)\alpha; \tag{3}$$

6. Compute matrices

$$J = J_h^c J_{nh} \tag{4}$$

and

$$\dot{J} = \frac{d(J)}{dt}; \tag{5}$$

7. Compute matrices M and W

$$M := \mathrm{diag}(M_i) \tag{6}$$

$$W := \mathrm{diag}(W_i) \tag{7}$$

where

$$m_i I_3 \qquad 0_3$$

$$M_i := \begin{matrix} \\ 0_3 & j_i \end{matrix} \tag{8}$$

and

$$W_i := \begin{matrix} S(\Omega_i) & 0_3 \\ 0_3 & S(\Omega_i) \end{matrix} \; ; \tag{9}$$

8. Consider the generalized force

$$\tau_a := J'[\omega_{a1}' \; \omega_{a2}' \; \ldots \; \omega_{an}']', \tag{10}$$

where ω_{ai} is the active force-torque applied to the i-th body

$$\omega_{ai} := [f_{ai}' \; n_{ai}']'; \tag{11}$$

9. Compute $D(q)$, $C(q,\alpha)$, g_0 and g_{wa}:

$$D(q) := J'MJ \tag{12}$$

$$C(q,\alpha) := J'(M\dot{J} + WMJ). \tag{13}$$

$$g_0 := D(q)^{-1}C(q,\alpha)\alpha \tag{14}$$

$$g_{wa} := D(q)^{-1'}; \tag{15}$$

10. The desired state model of the system dynamics is given by

$$\dot{\alpha} = g_0 + g_{wa}\tau_a \tag{16}$$

$$\dot{q} = J_{nh}(q)\alpha \tag{17}$$

$$\chi = f(q). \tag{18}$$

The vector $x := [\alpha' \; q']$ is called the state of the system.

Remark 1: As shown in Figure 1, the dynamic model of the system may be viewed as the cascade composition of a completely controllable dynamic component (eqn 16; a control τ_a may always be found so that $\dot{\alpha}$ has any desired value), followed by a kinematic component giving the system position and orientation as a function of these generalized velocities (eqns 17 and 18). This decomposition often suggests that the problem of having χ assume a desired value be approached in two steps (acceleration-resolved approach): first, determine appropriate values of $\dot{\alpha}(t)$ that make it possible to attain the intended objective; second, determine the control force required to impose such a value of $\dot{\alpha}(t)$.

6. Application to a tractor-trailer

Consider a tractor (equipped with two rear-drive wheels and two front-steering wheels) linked to a trailer (with two rear wheels) by means of a revolute vertical joint (Figure 2; see also DeSantis (1993a)). Assume the vehicle's motion to be planar, the geometric kinematic and dynamic properties of both tractor and trailer to be symmetrical with respect to their longitudinal axes, and the contact between tires and surface of motion to be point-wise. Assume further that the difference between the orientation of the tractor and that of its front wheels (steering angle) is sufficiently small that these wheels can be represented in terms of a "median" wheel located at the center of the front axle. The steps of the modeling procedure are applied as follows:

1. We attach a frame to the tractor and the trailer, as indicated in Figure 2; the positions and orientations of these frames are

represented by

$$\chi_1 := [x_1 \;\; y_1 \;\; \theta_1]' \quad \text{and} \quad \chi_2 := [x_2 \;\; y_2 \;\; \theta_2]' \qquad (1)$$

where the symbols have the meaning specified in the reference cited;

2. The relative motion between tractor and trailer is a rotation about an axis invariant with respect to each of the two bodies. Therefore, given the position and orientation of the tractor, only one additional parameter is needed to establish the position and orientation of the trailer. This means that the position and orientation of the tractor-trailer may be specified by a configuration vector $q := [x \;\; y \;\; \theta \;\; \Phi]' \epsilon R^4$ where x, y denote the coordinates of the origin of the tractor-frame with respect to an inertial system; θ is a measure of the orientation of the tractor; Φ is a measure of the orientation of the trailer with respect to the tractor;

3. The relation between the configuration vector and work-space positions and orientations of the tractor-trailer is as follows:

$$x_1 = x \qquad y_1 = y \qquad \theta_1 = \theta$$

and

$$x_2 = x - c\cos\theta - \ell_2\cos(\theta+\Phi) \qquad y_2 = y - c\sin\theta - \ell_2\sin(\theta+\Phi)$$

$$\theta_2 = \theta + \Phi; \qquad (2)$$

4. Measured with respect to their attached frames, the speeds of the tractor and trailer are given by the following vectors

$$v_1 := [v_{u1} \;\; v_{w1} \;\; \Omega_1]' \qquad v_2 := [v_{u2} \;\; v_{w2} \;\; \Omega_2]' \qquad (3)$$

where

$$\dot{x}_1 = \cos\theta v_{u1} - \sin\theta v_{w1}$$

$$\dot{y}_1 = \sin\theta v_{u1} + \cos\theta v_{w1}$$

and

$$\dot{\theta}_1 = \Omega_1 \tag{4}$$

$$\dot{x}_2 = \cos(\theta+\Phi)v_{u2} - \sin(\theta+\Phi)v_{w2}$$

$$\dot{y}_2 = \sin(\theta+\Phi)v_{u2} + \cos(\theta+\Phi)v_{w2}$$

$$\dot{\theta}_2 = \Omega_1 + \dot{\Phi}. \tag{5}$$

This implies that

$$v_{u1} = \cos\theta\dot{x} + \sin\theta\dot{y}$$

$$v_{w1} = -\sin\theta\dot{x} + \cos\theta\dot{y} \tag{6}$$

and

$$v_{u2} = \cos(\theta+\Phi)\dot{x} + \sin(\theta+\Phi)\dot{y} - c\Omega_1\sin\Phi \tag{7}$$

$$v_{w2} = -\sin(\theta+\Phi)\dot{x} + \cos(\theta+\Phi)\dot{y} - c\Omega_1\cos\Phi - \ell_2\Omega_1 - \ell_2\dot{\Phi}. \tag{8}$$

It follows that

$$J_{h1} := \begin{matrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix} \tag{9}$$

from which we obtain the Jacobian expressed in terms of the body's frame translated into the center of gravity

$$J_{h1}^{c} := \begin{matrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & b & 0 \\ 0 & 0 & 1 & 0 \end{matrix} . \tag{10}$$

Similarly,

$$J_{h2} := \begin{matrix} \cos(\theta+\Phi) & \sin(\theta+\Phi) & -c\sin\Phi & 0 \\ \sin(\theta+\Phi) & -\cos(\theta+\Phi) & -\ell_2 - c\cos\Phi & -\ell_2 \\ 0 & 0 & 1 & 1 \end{matrix} \qquad (11)$$

from which we obtain the Jacobian expressed in terms of the body's frame translated into the center of gravity

$$J_{h2}^{c} := \begin{matrix} \cos(\theta+\Phi) & \sin(\theta+\Phi) & -c\sin\Phi & 0 \\ \sin(\theta+\Phi) & -\cos(\theta+\Phi) & -e - c\cos\Phi & -e \\ 0 & 0 & -1 & 1. \end{matrix} \qquad (12)$$

It follows that

$$J_h := [J_{h1}^{c\prime} J_{h2}^{c\prime}]'. \qquad (13)$$

5. With reference to Figure 2, under a slippage-free motion and corresponding to a steering angle equal to δ, we have:

$$v_{w1} = 0 \qquad (14)$$

$$\dot{\Omega}_1 = \frac{v_{u1}\tan\delta}{\ell_1} \qquad (15)$$

$$\dot{\Phi} = -\frac{v_{u1}}{\ell_1\ell_2}\{\ell_1\sin\Phi + (\ell_2 + c\cos\Phi)\tan\delta\}. \qquad (16)$$

From eqn 4 it follows that

$$\begin{matrix} \dot{x} \\ \dot{y} \\ \dot{\Omega} \\ \dot{\Phi} \end{matrix} = v_{u1} \begin{matrix} \cos\theta \\ \sin\theta \\ \dfrac{\tan\delta}{\ell_1} \\ -\dfrac{\ell_1\sin\theta + (\ell_2+c\cos\Phi)\tan\delta}{\ell_1\ell_2} \end{matrix} \qquad (17)$$

This implies that we can choose $\alpha := v_{u1} \in R$ as a vector of generalized velocity. With this choice we have

$$J_{nh} = \begin{bmatrix} \cos\theta + \dfrac{a\tan\delta\sin\theta}{\ell_1} \\ \sin\theta - \dfrac{a\tan\delta\cos\theta}{\ell_1} \\ \dfrac{\tan\delta}{\ell_1} \\ -\dfrac{\ell_1\sin\Phi + (\ell_2 + c\cos\Phi)\tan\delta}{\ell_1\ell_2} \end{bmatrix} ; \qquad (18)$$

6. With the above expressions we can compute

$$J = J_h J_{nh} \qquad (19)$$

and

$$\dot{J} = + \frac{d(J)}{dt} ; \qquad (20)$$

Using eqns 10-13 and eqn 18, we obtain

$$J' = [1 \; ; \frac{b\tan\delta}{\ell_1} \; ; \frac{\tan\delta}{\ell_1} \; ; \cos\Phi - \frac{c\sin\Phi\tan\delta}{\ell_1} \; ; -\frac{d\sin\Phi}{\ell_2} - \frac{cd\cos\Phi\tan\delta}{\ell_1\ell_2} \; ; - \frac{\ell_1\sin\Phi + c\cos\Phi\tan\delta}{\ell_1\ell_2}]' \qquad (21)$$

and

$$\dot{J}' = \frac{\dot{\delta}}{\cos^2\delta} [0 \; ; \frac{b}{\ell_1} \; ; \frac{1}{\ell_1} \; ; -\frac{c\sin\Phi}{\ell_1} \; ; -\frac{cd\cos\Phi}{\ell_1\ell_2} \; ; -\frac{c\cos\Phi}{\ell_1\ell_2}]'$$

$$+ \quad \dot{\Phi}\ [0\ ;0\quad ;0\quad ;-\ \frac{\ell_1\sin\Phi + c\cos\Phi\tan\delta}{\ell_1}\quad ;-\ \frac{d\ell_1\cos\Phi + cd\sin\Phi\tan\delta}{\ell_1\ell_2};$$

$$-\ \frac{\ell_1\cos\Phi + c\sin\Phi\tan\delta}{\ell_1\ell_2}\ ]', \tag{22}$$

where $\dot{\Phi}$ is as in eqn 16. For later convenience, we find it useful to express $\dot{J}$ as follows

$$\dot{J} = A(q,\delta,\alpha) + B(q,\delta,\alpha)\dot{\delta} \tag{23}$$

where

$$A(q,\delta,\alpha) := \dot{\Phi}\ [0\ ;0\ ;0\quad ;-\ \frac{\ell_1\sin\Phi + c\cos\Phi\tan\delta}{\ell_1}\quad ;-\ \frac{d\ell_1\cos\Phi + cd\sin\Phi\tan\delta}{\ell_1\ell_2};$$

$$-\ \frac{\ell_1\cos\Phi + c\sin\Phi\tan\delta}{\ell_1\ell_2}\ ]' \tag{24}$$

and

$$B(q,\delta,\alpha) := \frac{1}{\cos^2\delta}\ [0\quad ;\ \frac{b}{\ell_1}\ ;\ \frac{1}{\ell_1}\ ;-\ \frac{c\sin\Phi}{\ell_1}\quad ;-\ \frac{cd\cos\Phi}{\ell_1\ell_2}\quad ;-\ \frac{c\cos\Phi}{\ell_1\ell_2}\ ]'; \tag{25}$$

7. This step gives

$$M_1 = \begin{matrix} m_1 I_2 & 0 \\ 0_{12} & j_1 \end{matrix} \tag{26}$$

$$M_2 = \begin{bmatrix} m_2 I_2 & 0 \\ 0_{12} & j_2 \end{bmatrix} \tag{27}$$

$$W_1 = \begin{bmatrix} 0 & \Omega_1 & 0 \\ -\Omega_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 0 & \dot{\Phi}+\Omega_1 & 0 \\ -(\dot{\Phi}+\Omega_1) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \tag{28}$$

8. Denoting by F_{ui} and F_{wi} the longitudinal and lateral forces applied to the wheels (see Figure 3), we have

$$\omega_{a1} = \begin{bmatrix} F_{u2} + F_{u1}\cos\delta \\ -F_{u1}\sin\delta \\ 0 \end{bmatrix} \tag{29}$$

$$\omega_{a2} = \begin{bmatrix} F_{u3} \\ 0 \\ 0 \end{bmatrix} ; \tag{30}$$

9. Using the above expressions, we compute

$$D(q,\delta)=J'MJ \tag{31}$$

$$C_1:=J'MA(q,\delta,\alpha) + J'WMJ \tag{32}$$

$$C_2:=J'MB(q,\delta,\alpha) + J'WMJ \tag{33}$$

$$C(q,\delta,\alpha,\dot{\delta}) = C_1 + C_2\,\dot{\delta} \tag{34}$$

$$g_{01}:= D^{-1}C_1\alpha \tag{35}$$

$$g_{02} := D^{-1}C_2\alpha \tag{36}$$

$$g_0 = g_{01} + g_{02}\dot{\delta} \tag{37}$$

$$g_{wa} = D^{-1}; \tag{38}$$

10. The dynamic model of the system is as follows

$$\dot{\alpha} = g_{01} + g_{02}\dot{\delta} + g_{u1}F_{u1} + g_{u2}F_{u2} + g_{u3}F_{u3} \tag{39}$$

$$\dot{q} = J_{nh}(q,\delta)\alpha \tag{40}$$

$$\chi = f(q), \tag{41}$$

where J_{nh} and $f(q)$ are as in eqns 2 and 18, and

$$g_{u1} := D^{-1}J'[1\ 0\ 0\ 0\ 0\ 0]' \tag{42}$$

$$g_{u2} := D^{-1}J'[\cos\delta\ -\sin\delta\ 0\ 0\ 0\ 0]' \tag{43}$$

$$g_{u3} := D^{-1}J'[0\ 0\ 0\ 1\ 0\ 0]'. \tag{44}$$

In applications it is a common practice to set

$$\dot{\delta} = F_s \tag{45}$$

where F_s is the steering control. The propulsion control is usually applied to the rear-wheels of the tractor and is represented by

$$F_p = F_{u2}. \tag{46}$$

F_{u1} and F_{u3} are viewed as external perturbations.

7. Application to a robotic manipulator

A robotic manipulator is a mechanical system made up of a linearly ordered set of rigid bodies (links). In this system, the motion of two neighbouring bodies is either a rotation about or a translation along an axis (joint) that remains invariant with respect to each of the two bodies. When applied to a robotic manipulator, the procedure that we have presented in the previous section is characterized by the following features:

a) Since robotic manipulators are usually considered to be exempt from nonholonomic constraints, we have ℓ=p, and $J_{nh} = I_p$. It follows that the dynamic configuration-space model of the manipulator has the form

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} = \tau_a. \tag{1}$$

In open-chain manipulators, the configuration vector, q, has dimension, p, equal to the number of links, n. The entries of this vector are represented by either the angle or the distance (joint variable, generalized coordinate) that uniquely describes the relative displacement of two neighbouring links. The entries of the vector of generalized forces, τ_a, are usually represented by forces or torques applied along the joints' axes;

b) Denoting by v_n the velocity of the n-th body (manipulator's end-effector or tool) and by J_n its Jacobian matrix, we have

$$v_n = J_n \dot{q} \tag{2}$$

and

$$\dot{v}_n = \dot{J}_n \dot{q} + J_n \ddot{q}. \tag{3}$$

By assuming J_n to be invertible, it follows that

$$\dot{q} = J_n^{-1} v_n \tag{4}$$

$$\ddot{q} = J_n^{-1}(\dot{v}_n - \dot{J}_n \dot{q}) \tag{5}$$

hence

$$J_n'^{-1} D J_n^{-1} \dot{v}_n = J_n'^{-1}\{-C(q,\dot{q}) J_n^{-1} v_n - D J_n^{-1} \dot{J}_n J_n^{-1} v_n + \tau_a\}. \qquad (6)$$

By setting

$$M_\chi := J_n'^{-1} D J_n^{-1} \qquad (7)$$

$$V_\chi := -J_n'^{-1}\{C(q,\dot{q}) J_n^{-1} + D J_n^{-1} \dot{J}_n J_n^{-1}\} v_n \qquad (8)$$

$$F_\chi := J_n'^{-1} \tau_a, \qquad (9)$$

we can write

$$F_\chi := M_\chi \dot{v}_n + V_\chi. \qquad (10)$$

Eqns 7-10 represent what is usually called the manipulator's Cartesian model (see Craig 1989, p. 211). Using this model, the control of the n-th link of the manipulator (and therefore of the manipulator's tool) may be viewed as equivalent to the control of a rigid body with a time-variant extended mass matrix and with an appropriately defined Coriolis and centripetal matrix;

c) In the implementation of step 1 of the modeling procedure, the frames to be attached to each link are usually selected by following a set of well-established kinematic rules (Craig 1989, p. 77); the entries of vector c_i correspond to the coordinates of the origin of the i-th frame; the vector ξ_i usually corresponds to the rotational matrix describing the orientation of this frame with respect to the work-space;

d) The implementation of step 3 is greatly simplified because the direct kinematic function f(q) can be readily described in terms of link transformation matrices and link parameters (Denavit-

Hartenberg parameters) (Craig 1989, p.76);

e) The implementation of steps **4** and **7** is simplified by the property that if we choose the frames as indicated at point c), then J_{hi}=[0 0 1 0 0 0]' in the case of a prismatic joint, and J_{hi}=[0 0 0 0 0 1]' in the case of a rotational joint. An important role in the implementation of these steps is also played by the availability of link-velocity propagation formulas (Craig 1989, p. 166) and link-acceleration propagation formulas (Craig 1989, p. 197);

f) For a given value $q_D \epsilon R^n$, the application of the PD control law,

$$\tau_a = -K_p e - K_D \dot{q}, \tag{11}$$

where $e := q - q_D$ and K_p and K_D are positive definite matrices, leads to

$$\lim_{t \to \infty} q(t) = q_D. \tag{12}$$

To verify this result, consider the Lyapunov function

$$V := e'K_p e + \dot{q}' D \dot{q}, \tag{13}$$

and observe that

$$\frac{dV}{dt} = \dot{q}'K_p e + \dot{q}' D \ddot{q} + \frac{1}{2} \dot{q}' \dot{D} \dot{q} \tag{14}$$

hence

$$\frac{dV}{dt} = \dot{q}'K_p e - \dot{q}'K_p e - \dot{q}' K_D \dot{q} + \frac{1}{2} \dot{q}'(\dot{D} - C(q,\dot{q}))\dot{q}. \tag{15}$$

In view of eqn 3.27, this implies that

$$\frac{dV}{dt} = - \dot{q}' K_D \dot{q} \leq 0 \qquad (16)$$

and therefore, by the Lasalle theorem (Khalil (1992, p. 117)), we have

$$\lim_{t \to \infty} V(t) = 0 \, . \qquad (17)$$

Because K_P and K_D are positive-definite, this implies in turn that

$$\lim_{t \to \infty} q(t) = q_D . \qquad (18)$$

More detail may be found in Craig (1989, p. 352).

8. Application to an LHD Vehicle

The geometrical structure of an LHD vehicle used in mining (loading-hauling-dumping) operations is similar to that of the tractor-trailer in section 6, except that a torque is now applied at the vertical joint and ℓ_1=0, c=ℓ_2 and δ=0. It follows that under the assumption of a planar motion, a longitudinal axis symmetry and a point-wise contact between tires and surface of motion, most of the modeling development of section 6 remains valid for this type of vehicle.

It should be noted, however, that, contrary to what happens in a tractor-trailer, in an LHD vehicle the wheels do not change orientation with respect to their axles. As a consequence the hypothesis of a slippage-free motion can no longer be made because this would imply the impossibility of changing the relative orientation of the vehicle's two components (see Appendix A). It follows that at step 5 we have to let $J_{nh} = I_4$ and at step 8 we have to modify eqns 6.24 and 6.25 as follows:

$$\omega_{a1} = \begin{matrix} F_{u2} \\ F_{w2} \\ \Gamma \end{matrix} \qquad (6.24')$$

$$\omega_{a2} = \begin{matrix} F_{u3} \\ F_{w3} \\ -\Gamma \end{matrix} \quad ; \qquad (6.25')$$

where F_{w2} and F_{w3} are the lateral forces exerted by the tires. These differences in dynamic models explain in part why the tractor-trailer path-planning and path-tracking results in Latombe (1991) and in DeSantis (1993b) are not applicable to an LHD vehicle.

Conclusions

The modeling procedure that we have presented is systematic and simultaneously applicable to a variety of mechanical systems that must be considered in industrial automation like, for example, robotic manipulators, mobile robots, transport vehicles, mechanical excavators, overhead cranes and similar machines. While similar to already available procedures (as, for example, the procedure based on the Kane approach (Kane Levinson, 1993) or that recently proposed by Saha and Angeles (1991)) its main advantage is in that it unfolds in steps that not only are meaningful from a numerical point of view, but that are also amenable to clear, intuitive and physical interpretations.

By following this procedure it becomes now easier to expose similarities and differences existing among the models that may be associated with a given system under alternative assumptions (as, for example, a tractor-trailer moving with a slippage-free motion or with a motion that is not slippage-free). It becomes also more efficient to compare structural and numerical properties of models of different mechanical systems operating under similar hypotheses (as, for example, a robotic manipulator, a tractor-trailer, a mechanical excavator or an LHD (loading-hauling-dumping) mining vehicle operating under workless constraints).

Remerciement

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Appendix A: Nonholonomic constraints and LHD vehicles

In the following, we show that under a slippage-free motion the relative orientation of an LHD vehicle's two components cannot vary (in the case of an LHD, a motion is slippage-free if the velocity of the point of contact between wheels and surface of motion has a null component along the wheels' axle). To discuss this question, it is helpful to first introduce the following notations (see Figure 4):

ℓ: distance between the wheels' axle and the vertical joint;

Ω_1, Ω_2: angular velocities of vehicle's components 1 and 2;

2δ: angle that gives the relative orientation of component 1 with respect to component 2;

r_1, r_2 : lines prolonging the axles of components 1 and 2;

P_1, P_2: points at the center of axles 1 and 2;

v_{u1}, v_{u2}: longitudinal velocity of P_1, P_2.

Under a slippage-free motion, the lateral velocities of P_1 and P_2 must be null. The instantaneous axis of rotation of component #1 must then belong to line r_1, and the instantaneous axis of rotation of component 2 must belong to line r_2. Moreover, the velocity of P_0 viewed as a point of component 1 must coincide with the velocity of P_0 viewed as a point of component 2. This implies that the instantaneous axes of rotation of components 1 and 2 must coincide with the intersection of line r_1 with r_2. Let us denote this intersection by the symbol O and let R denote the distance from O to P_1 (the same as the distance from O to P_2).

From these observations it follows that

$$\Omega_1 = -\frac{v_{u1}}{R}; \qquad \Omega_2 = -\frac{v_{u2}}{R}; \qquad \ell = R\tan\delta; \tag{1}$$

$$v_{u2} = v_{u1}\cos 2\delta - \Omega_1 \ell \sin 2\delta; \tag{2}$$

and therefore that

$$v_{u2} = v_{u1}(\cos 2\delta + \tan\delta \sin 2\delta). \tag{3}$$

Taking into account that

$$\cos 2\delta = \cos^2\delta - \sin^2\delta \tag{4}$$

and that

$$\sin 2\delta = 2\cos\delta \sin\delta, \tag{5}$$

it follows that

$$v_{u2} = v_{u1}(\cos^2\delta - \sin^2\delta + 2\sin^2\delta) = v_{u1} \tag{6}$$

and therefore that

$$\Omega_2 - \Omega_1 = 0 \tag{7}$$

that is,

$$\dot{\delta} = 0. \tag{8}$$

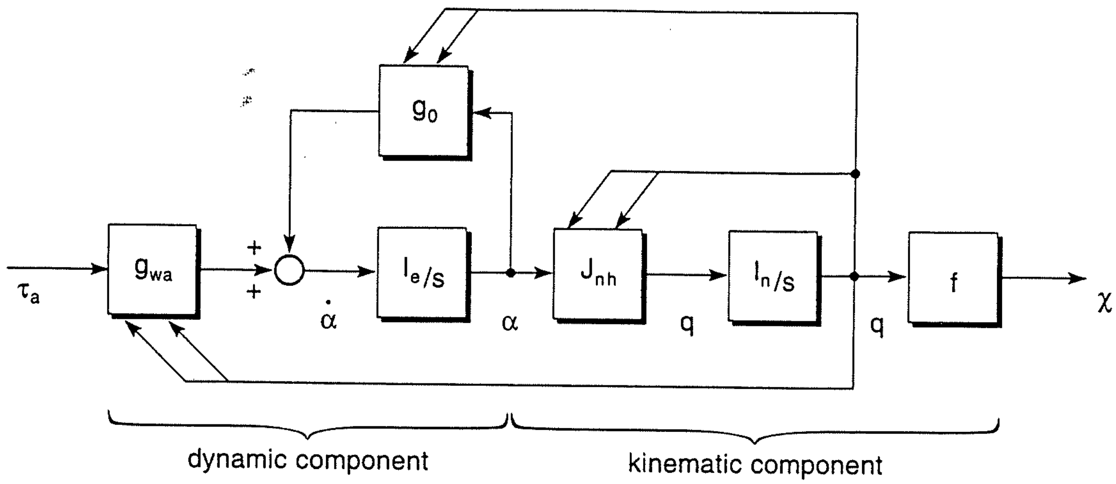

Figure 1: The structure of the dynamic model

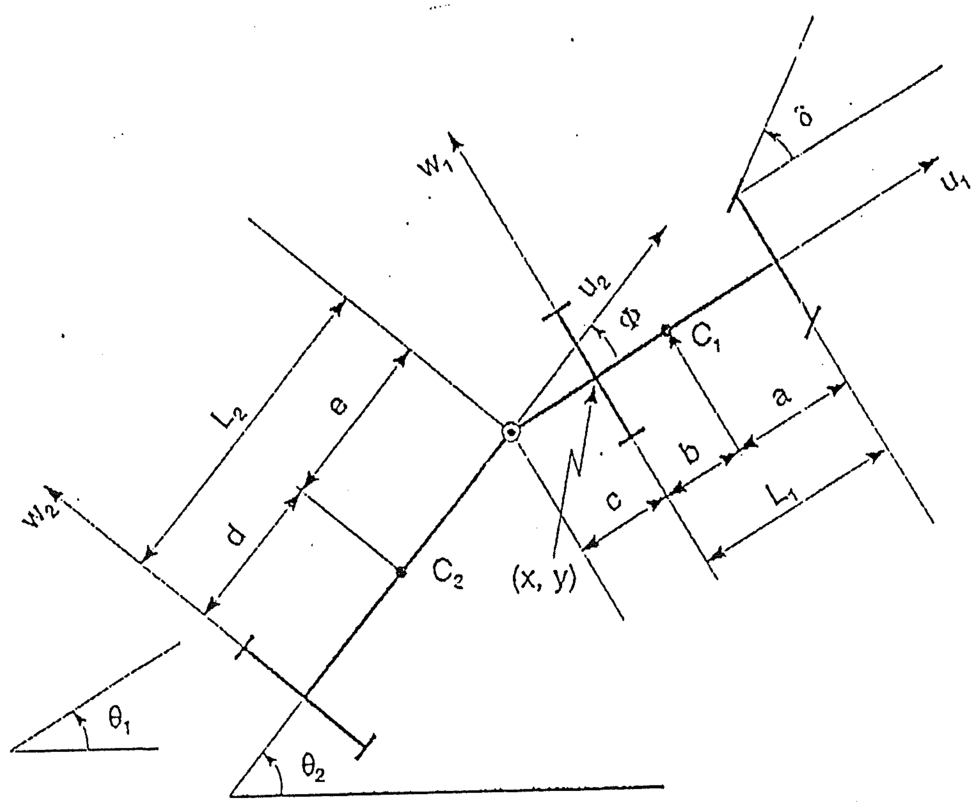

Figure 2: Geometry of a tractor-trailer

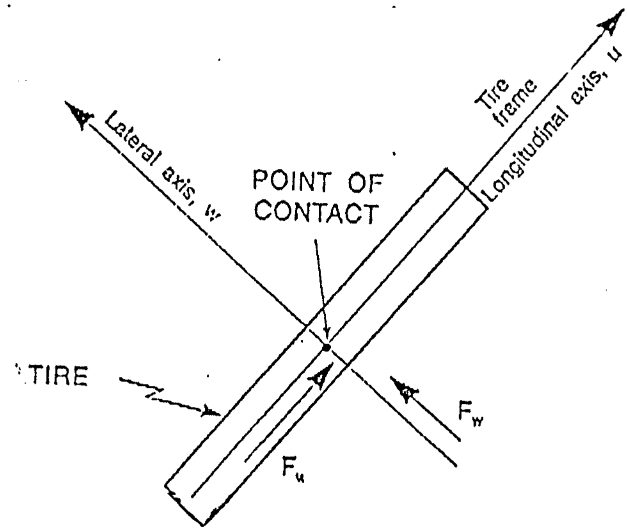

Figure 3: Longitudinal and lateral forces exerted by the tires

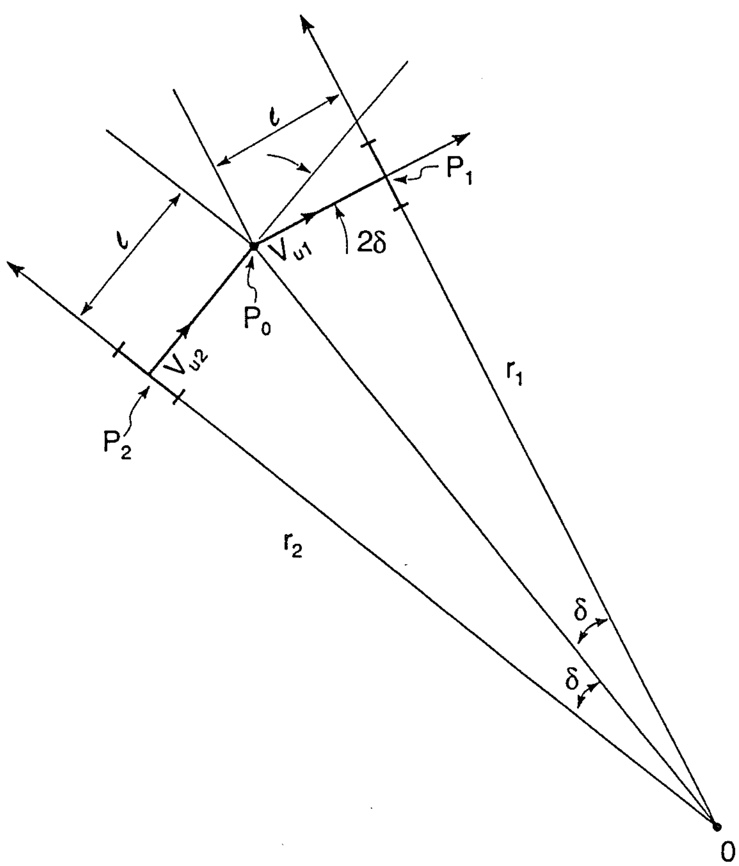

Figure 4: Geometry of an LHD vehicle