POLYPUBLIE
Polytechnique Montréal

POLYTECHNIQUE MONTRÉAL
UNIVERSITÉ D'INGÉNIERIE

Titre: Title:	Lagrangian Derivation of Variable-Mass Equations of Motion using an Arbitrary Attitude Parameterization
Auteurs: Authors:	Charles Champagne Cossette, James Richard Forbes, & David Saussié
Date:	2020
Type:	Article de revue / Article
Référence: Citation:	Champagne Cossette, C., Forbes, J. R., & Saussié, D. (2020). Lagrangian Derivation of Variable-Mass Equations of Motion using an Arbitrary Attitude Parameterization. Journal of the Astronautical Sciences, 67(4), 1206-1219. https://doi.org/10.1007/s40295-020-00230-3

Document en libre accès dans PolyPublie
Open Access document in PolyPublie

URL de PolyPublie: PolyPublie URL:	https://publications.polymtl.ca/9265/
Version:	Version officielle de l'éditeur / Published version Révisé par les pairs / Refereed
Conditions d'utilisation: Terms of Use:	Creative Commons Attribution 4.0 International (CC BY)

Document publié chez l'éditeur officiel
Document issued by the official publisher

Titre de la revue: Journal Title:	Journal of the Astronautical Sciences (vol. 67, no. 4)
Maison d'édition: Publisher:	Springer Nature
URL officiel: Official URL:	https://doi.org/10.1007/s40295-020-00230-3
Mention légale: Legal notice:	

Ce fichier a été téléchargé à partir de PolyPublie, le dépôt institutionnel de Polytechnique Montréal
This file has been downloaded from PolyPublie, the institutional repository of Polytechnique Montréal

https://publications.polymtl.ca

The Journal of the Astronautical Sciences (2020) 67:1206–1219
https://doi.org/10.1007/s40295-020-00230-3

TECHNICAL NOTE

Lagrangian Derivation of Variable-Mass Equations of Motion using an Arbitrary Attitude Parameterization

Charles Champagne Cossette[1] · **James Richard Forbes**[1] · **David Saussié**[2]

Published online: 9 October 2020
© American Astronautical Society 2020

Abstract

Lagrange's equation is a popular method of deriving equations of motion due to the ability to choose a variety of generalized coordinates and implement constraints. When using a Lagrangian formulation, part of the generalized coordinates may describe the attitude. This paper presents a means of deriving the dynamics of variable-mass systems using Lagrange's equation while using an arbitrary constrained attitude parameterization. The equivalence to well-known forms of the equations of motion is shown.

Keywords Lagrange's equation · variable-mass dynamics · constrained attitude parameterizations

Introduction

Deriving the equations of motion of a dynamic system that expels mass is a complex problem with historical roots in rocketry [5]. Numerous technical reports and papers present different means to arrive at the now familiar equations of motion of a variable-mass system. A Newton-Euler approach is presented in [10, 15, 16, 22, 25], while

✉ Charles Champagne Cossette
charles.cossette@mail.mcgill.ca

James Richard Forbes
james.richard.forbes@mcgill.ca

David Saussié
d.saussie@polymtl.ca

1 Department of Mechanical Engineering, McGill University, 817 Sherbrooke St. W., Montréal, QC, H3A 0C3, Canada

2 Department of Electrical Engineering, Polytechnique Montréal, 2500 Chemin de Polytechnique, Montréal, QC, H3T 1J4, Canada

Kane's Equations are used in [4–6]. Hamilton's Principle is adapted to the variable-mass dynamics problem in [11, 14], and Lagrange's equation is commonly seen with an appended term to account for mass-variablility, as discussed in [9, 14, 18].

Lagrange's equation is a popular method of deriving equations of motion due to it's ability to accommodate different generalized coordinates, as well as its ease of handling constraints. Lagrange's equation can be applied to systems where a subset of the chosen generalized coordinates is an attitude parameterization. Attitude parameterizations will often have constraints, such as the quaternion possessing a unit-norm constraint. In [24], identities are derived for general attitude parameterizations that enable a streamlined application of Lagrange's equation, without having to resort to the use of the Boltzmann-Hamel equations [24]. These identities greatly enhance the tractability of the equations derived by maintaining the equations in matrix form. Similar identities appear in [3, 8, 17, 20, 23], when attitude is parameterized using Euler angles, a quaternion, or axis/angle parameters, whereas the identities in [24] collectively consider any attitude parameterization. The identities allow access to the analysis of new and more sophisticated systems using a Lagrangian framework, such as those seen in [2, 12, 26–28]. However, these identities have only been used to model constant-mass systems.

The contribution of this note is to show how the identities in [24] can be used to model variable-mass systems, thus even further expanding the category of systems that can be analysed using Lagrange's equation. An example is the equations of motion of a launch vehicle with coupled rigid-body, variable-mass, and flexible dynamics. Unlike other research articles, the presented formulation allows a Lagrangian approach for variable-mass systems, using any arbitrary attitude parameterization, while explicitly incorporating attitude constraints. This note shows how the well-known form of the variable-mass equations of motion can be directly retrieved from the presented formulation.

The upcoming derivation uses the popular approach of treating the variable-mass system as an equivalent constant-mass system [4, 19], and using Reynold's Transport Theorem to convert the result into a formulation appropriate for variable-mass systems. The "Preliminaries" section outlines the required preliminaries and notation used in this note, as well as the key identities that enable the use of arbitrary attitude parameterizations. The "Derivation of the Equations of Motion" section contains the derivation, and shows the equivalence to the well-known variable-mass equations of motion.

Nomenclature

$\underrightarrow{u}$ $\triangleq$ a physical vector

$\mathcal{F}_a$ $\triangleq$ a reference frame 'a' associated with the physical basis vectors $\underrightarrow{a}^i$, $i = 1, 2, 3$

$\underrightarrow{\mathcal{F}}_a \triangleq \begin{bmatrix} \underrightarrow{a}^1 \\ \underrightarrow{a}^2 \\ \underrightarrow{a}^3 \end{bmatrix}$ $\triangleq$ a vectrix, a column matrix of physical basis vectors $\underrightarrow{a}^i$, $i = 1, 2, 3$

$\mathbf{u}_a \triangleq$ the components of $\underrightarrow{u}$ resolved in $\mathcal{F}_a$ such that $\underrightarrow{u} = \underrightarrow{\mathcal{F}}_a^{\mathsf{T}}\mathbf{u}_a$

$\mathbf{C}_{ba} \triangleq$ the direction cosine matrix (DCM)

$(\cdot)^\times \triangleq$ the skew-symmetric cross-product matrix

$\underrightarrow{r}^{zw} \triangleq$ the position of z relative to w

$\underrightarrow{v}^{zw/a} = \underrightarrow{r}^{zw^{\cdot a}} \triangleq$ the velocity of z relative to w with respect to $\mathcal{F}_a$

$\mathbf{q}^{ba} \triangleq$ a column matrix parameterization of $\mathbf{C}_{ba}$

$\mathbf{q} \triangleq$ a column matrix of generalized coordinates

$\underrightarrow{\omega}^{ba} \triangleq$ the angular velocity of $\mathcal{F}_b$ relative to $\mathcal{F}_a$

$\underrightarrow{\omega}^{ba^{\cdot c}} \triangleq$ the angular acceleration of $\mathcal{F}_b$ relative to $\mathcal{F}_a$ with respect to $\mathcal{F}_c$

$m_{\mathcal{S}} \triangleq$ the mass of system $\mathcal{S}$

$\mathbf{c}_b^{\mathcal{S}z} \triangleq$ the first moment of mass of system $\mathcal{S}$ about point z, resolved in $\mathcal{F}_b$

$\mathbf{J}_b^{\mathcal{S}z} \triangleq$ the second moment of mass matrix of system $\mathcal{S}$ about point z, resolved in $\mathcal{F}_b$

$\mathbf{1} \triangleq$ an appropriately dimensioned identity matrix

$\mathbf{0} \triangleq$ an appropriately dimensioned matrix of zeros

Preliminaries

Physical Vectors and Reference Frames

A physical vector $\underrightarrow{u} \in \mathbb{P}$ is an element of physical space, where physical space is denoted $\mathbb{P}$. Physical vectors often represent physical quantities, such as position, velocity, and acceleration. Consider the orthornormal, dextral, physical basis vectors $\underrightarrow{a}^1$, $\underrightarrow{a}^2$, and $\underrightarrow{a}^3$ that may be used to define a reference frame, denoted $\mathcal{F}_a$. A physical vector may be resolved in, for example, $\mathcal{F}_a$ as [13]

$$\underrightarrow{u} = u_{a1}\underrightarrow{a}^1 + u_{a2}\underrightarrow{a}^2 + u_{a3}\underrightarrow{a}^3 = \underrightarrow{\mathcal{F}}_a^{\mathsf{T}}\mathbf{u}_a,$$

where the vectrix $\underrightarrow{\mathcal{F}}_a \in \mathbb{P}^3$ and the column matrix $\mathbf{u}_a \in \mathbb{R}^3$ are defined as $\underrightarrow{\mathcal{F}}_a = \left[\underrightarrow{a}^1\ \underrightarrow{a}^2\ \underrightarrow{a}^3\right]^{\mathsf{T}}$ and $\mathbf{u}_a = [u_{a1}\ u_{a2}\ u_{a3}]^{\mathsf{T}}$, respectively. A physical vector may be resolved in any frame,

$$\underrightarrow{u} = \underrightarrow{\mathcal{F}}_a^{\mathsf{T}}\mathbf{u}_a = \underrightarrow{\mathcal{F}}_b^{\mathsf{T}}\mathbf{u}_b,$$

and the relationship between $\mathbf{u}_a$ and $\mathbf{u}_b$ is given by $\mathbf{u}_a = \mathbf{C}_{ab}\mathbf{u}_b$ where $\mathbf{C}_{ab} = \underrightarrow{\mathcal{F}}_a \cdot \underrightarrow{\mathcal{F}}_b^{\mathsf{T}}$ is the direction cosine matrix (DCM) [13]. DCMs are orthonormal, meaning that $\mathbf{C}_{ba} = \mathbf{C}_{ab}^{\mathsf{T}}$ and $\mathbf{C}_{ba}^{\mathsf{T}}\mathbf{C}_{ba} = \mathbf{1}$.

The skew-symmetric cross-product matrix is defined for any column matrix $\mathbf{v} = [v_1\ v_2\ v_3]^{\mathsf{T}} \in \mathbb{R}^3$ as

$$\mathbf{v}^\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix},$$

where $\underrightarrow{v} \times \underrightarrow{u} = \underrightarrow{\mathcal{F}}_a^{\mathsf{T}}\mathbf{v}_a^\times\mathbf{u}_a$ holds in any reference frame.

The position of point z relative to point w is denoted by $\underrightarrow{r}^{zw}$. The velocity of point z relative to point w with respect to $\mathcal{F}_a$ is denoted by $\underrightarrow{v}^{zw/a} = \underrightarrow{r}^{zw^{\cdot a}}$. A physical vector's time rate-of-change with respect to an arbitrary frame $\mathcal{F}_a$ can be related to its time rate-of-change with respect to another arbitrary frame $\mathcal{F}_b$ using the Kinematic Transport Theorem [13],

$$\underrightarrow{u}^{\cdot a} = \underrightarrow{u}^{\cdot b} + \underrightarrow{\omega}^{ba} \times \underrightarrow{u},$$

where $\underrightarrow{\omega}^{ba}$ is the angular velocity of $\mathcal{F}_b$ relative to $\mathcal{F}_a$.

The attitude of a body can be globally and uniquely described by a DCM, $\mathbf{C}_{ba}$, where $\mathcal{F}_b$ is a frame fixed to the body and $\mathcal{F}_a$ is a datum reference frame.

There are many ways to parameterize a DCM, such as quaternions, Euler angles, and Gibbs parameters [13]. An arbitrary attitude parameterization of $\mathbf{C}_{ba}$ will be denoted $\mathbf{q}^{ba}$, which can be the quaternion elements, a set of three Euler angles, or simply the columns of the DCM stacked in a 9×1 column matrix. The angular velocity differs from the attitude parameterization's time rate-of-change $\dot{\mathbf{q}}^{ba}$, yet they can be related by a mapping matrix $\mathbf{S}_b^{ba}(\mathbf{q}^{ba})$, where [13]

$$\boldsymbol{\omega}_b^{ba} = \mathbf{S}_b^{ba}(\mathbf{q}^{ba})\dot{\mathbf{q}}^{ba},$$

and the inverse mapping is given by

$$\dot{\mathbf{q}}^{ba} = \boldsymbol{\Gamma}_b^{ba}(\mathbf{q}^{ba})\boldsymbol{\omega}_b^{ba}.$$

The matrices $\mathbf{S}_b^{ba}$ and $\boldsymbol{\Gamma}_b^{ba}$ can be shown to be orthogonal complements, that is

$$\mathbf{S}_b^{ba}\boldsymbol{\Gamma}_b^{ba} = \mathbf{1}.$$

Certain attitude parameterizations also have constraints. For example, the unit-length constraint associated with a quaternion can be written in the form,

$$\boldsymbol{\Xi}^{ba}(\mathbf{q}^{ba})\dot{\mathbf{q}}^{ba} = \mathbf{0}, \tag{1}$$

where $\boldsymbol{\Xi}^{ba}(\mathbf{q}^{ba})$ is a constraint matrix associated with the attitude constraint. The columns of $\boldsymbol{\Gamma}_b^{ba}$ lie in the null-space of $\boldsymbol{\Xi}^{ba}(\mathbf{q}^{ba})$, that is

$$\boldsymbol{\Xi}^{ba}(\mathbf{q}^{ba})\boldsymbol{\Gamma}_b^{ba} = \mathbf{0}.$$

For any vector $\underrightarrow{u}$ that is not a function of $\mathbf{q}_{ba}$, the following three identities (and their transposes) [24],

$$\left(\dot{\mathbf{S}}_b^{ba} - \frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \mathbf{q}^{ba}}\right)\boldsymbol{\Gamma}_b^{ba} = -\boldsymbol{\omega}_b^{ba^\times} \iff \boldsymbol{\Gamma}_b^{ba^\mathsf{T}}\left(\dot{\mathbf{S}}_b^{ba^\mathsf{T}} - \left(\frac{\partial \boldsymbol{\omega}_b^{ba}}{\partial \mathbf{q}^{ba}}\right)^\top\right) = \boldsymbol{\omega}_b^{ba^\times}, \tag{2}$$

$$\frac{\partial(\mathbf{C}_{ab}\mathbf{u}_b)}{\partial \mathbf{q}^{ba}}\boldsymbol{\Gamma}_b^{ba} = -\mathbf{C}_{ab}\mathbf{u}_b^\times \iff \boldsymbol{\Gamma}_b^{ba^\mathsf{T}}\left(\frac{\partial(\mathbf{C}_{ab}\mathbf{u}_b)}{\partial \mathbf{q}^{ba}}\right)^\top = \mathbf{u}_b^\times\mathbf{C}_{ba}, \tag{3}$$

$$\frac{\partial(\mathbf{C}_{ba}\mathbf{u}_a)}{\partial \mathbf{q}^{ba}}\boldsymbol{\Gamma}_b^{ba} = (\mathbf{C}_{ba}\mathbf{u}_a)^\times \iff \boldsymbol{\Gamma}_b^{ba^\mathsf{T}}\left(\frac{\partial(\mathbf{C}_{ba}\mathbf{u}_a)}{\partial \mathbf{q}^{ba}}\right)^\top = -(\mathbf{C}_{ba}\mathbf{u}_a)^\times, \tag{4}$$

are of crucial importance in forthcoming derivations. Note that the argument $\mathbf{q}^{ba}$ has been suppressed in Eq. 2, Eq. 3, and Eq. 4, and will continue to be suppressed in forthcoming derivations.

Derivation of the Equations of Motion

Consider a constant-mass system $\mathcal{S}$. Consider also a region with volume $V(t)$ and boundary $B(t)$ defined such that it encloses the mass of $\mathcal{S}$ at all times. The volume $V(t)$ and boundary $B(t)$ are therefore time-varying quantities. The system $\mathcal{S}$ can be arbitrarily composed of mass that is rigid, and some that is not.

Next, consider an arbitrary time-varying mass system $\bar{\mathcal{S}}(t)$ that has a known, constant volume $\bar{V}$ with boundary $\bar{B}$. Note that system $\bar{\mathcal{S}}$ has time-varying mass but constant volume and boundary, whereas system $\mathcal{S}$ has constant mass but time-varying volume and boundary. Let $\mathcal{S}$ be defined such that at a specific instant $\bar{t}$ the system $\mathcal{S}$ conincides exactly with $\bar{\mathcal{S}}$, and consequently so does $V(\bar{t}) = \bar{V}$, $B(\bar{t}) = \bar{B}$. For any other instant, $\bar{V}$ will generally differ from V, and $\bar{B}$ will generally differ from B.

Referring to Fig. 1, let w be an unforced particle and $\mathcal{F}_i$ be an inertial frame [1]. Let z be a reference point fixed to any rigid portion of $\mathcal{S}$, and $\mathcal{F}_b$ be a frame fixed to the same rigid portion of $\mathcal{S}$, as shown in Fig. 1. In order to properly define z and $\mathcal{F}_b$, there must exist some sort of reference rigid-body, and hence the requirement for $\mathcal{S}$ to possess at least some portion that is considered rigid. The enabling theorem in this derivation is Reynold's Transport Theorem, which states

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{V(t)} \underset{\rightarrow}{f}\,\mathrm{d}V\right)\bigg|_{\mathcal{F}_b} = \int_{V(t)} \underset{\rightarrow}{f}^{\cdot b}\mathrm{d}V + \int_{B(t)} \underset{\rightarrow}{f}\left(\underset{\rightarrow}{v}^{\mathrm{d}Sz/b} \cdot \underset{\rightarrow}{n}\right)\mathrm{d}S, \qquad (5)$$

where $\underset{\rightarrow}{f}$ is a scalar-, vector-, or tensor-valued property of a system of interest [4, 13, 15, 16, 19, 21, 29, 30]. The term $\underset{\rightarrow}{v}^{\mathrm{d}Sz/b}$ refers to the velocity of an area element $\mathrm{d}S$, which can be assumed to be equivalent to the velocity of a mass element at the boundary, $\underset{\rightarrow}{v}^{\mathrm{d}mz/b}$. The notation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\underset{\rightarrow}{u}\right)\bigg|_{\mathcal{F}_b} \triangleq \underset{\rightarrow}{u}^{\cdot b}$$

is an alternate way to write the time-derivative of $\underset{\rightarrow}{u}$ with respect to $\mathcal{F}_b$, in order to clarify the meaning of the time derivative on the left-hand-side of Eq. 5.

Generalized Coordinates

The generalized coordinates are $\mathbf{q} = \begin{bmatrix}\mathbf{r}_i^{zw^\mathsf{T}} & \mathbf{q}^{bi^\mathsf{T}}\end{bmatrix}^\mathsf{T}$ where $\mathbf{q}^{bi}$ is an arbitrary attitude parameterization describing the attitude of $\mathcal{F}_b$ relative to $\mathcal{F}_i$. The reader should be careful not to confuse the attitude parameterization $\mathbf{q}^{bi}$ with the generalized coordi-

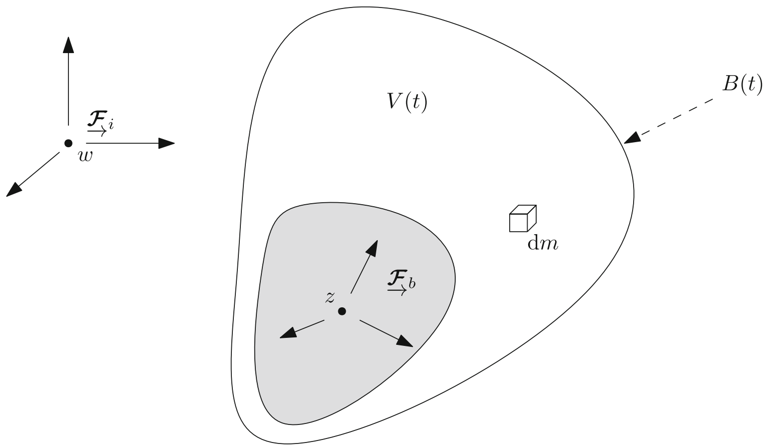

Fig. 1 Illustration of an arbitrary system $\bar{\mathcal{S}}$, with the grey area being rigid

nates $\mathbf{q}$, which also contains the position. The augmented velocity matrix is defined as $\boldsymbol{\nu} = \begin{bmatrix} \mathbf{v}_i^{zw/i^{\mathsf{T}}} & \boldsymbol{\omega}_b^{bi^{\mathsf{T}}} \end{bmatrix}^{\mathsf{T}}$, which is related to the generalized coordinates by

$$\boldsymbol{\nu} = \mathbf{S}\dot{\mathbf{q}},$$

where $\mathbf{S} = \mathrm{diag}(\mathbf{1}, \mathbf{S}_b^{bi})$. The inverse mapping is

$$\dot{\mathbf{q}} = \boldsymbol{\Gamma}\boldsymbol{\nu},$$

where $\boldsymbol{\Gamma} = \mathrm{diag}(\mathbf{1}, \boldsymbol{\Gamma}_b^{bi})$. Notice that the matrices $\mathbf{S}$ and $\boldsymbol{\Gamma}$ are orthogonal complements, that is, $\mathbf{S}\boldsymbol{\Gamma} = \mathbf{1}$.

Kinetic Energy and the Lagrangian

For simplicity, potential energy sources are not considered, and their effects can be included as external forces. The kinetic energy of the constant-mass system $\mathcal{S}$, relative to w, with respect to $\mathcal{F}_i$ is

$$T_{\mathcal{S}w/i} = \frac{1}{2}\int_{\mathcal{S}} \underrightarrow{v}^{\mathrm{d}mw/i} \cdot \underrightarrow{v}^{\mathrm{d}mw/i} \mathrm{d}m,$$

where, using the Kinematic Transport Theorem, the velocity of a mass element $\mathrm{d}m$ relative to w, with respect to $\mathcal{F}_i$ can be shown to be

$$\underrightarrow{v}^{\mathrm{d}mw/i} = \underrightarrow{v}^{\mathrm{d}mz/b} + \underrightarrow{\omega}^{bi} \times \underrightarrow{r}^{\mathrm{d}mz} + \underrightarrow{v}^{zw/i}.$$

Resolving in $\mathcal{F}_i$, the kinetic energy is

$$
\begin{aligned}
T_{\mathcal{S}w/i} &= \frac{1}{2}\int_{\mathcal{S}} (\mathbf{v}_i^{\mathrm{d}mz/b} + \mathbf{C}_{ib}\boldsymbol{\omega}_b^{bi^\times}\mathbf{r}_b^{\mathrm{d}mz} + \mathbf{v}_i^{zw/i})^\mathsf{T}(\mathbf{v}_i^{\mathrm{d}mz/b} + \mathbf{C}_{ib}\boldsymbol{\omega}_b^{bi^\times}\mathbf{r}_b^{\mathrm{d}mz} + \mathbf{v}_i^{zw/i})\mathrm{d}m, \\
&= \frac{1}{2}\begin{bmatrix}\mathbf{v}_i^{zw/i^\mathsf{T}} & \omega_b^{bi^\mathsf{T}}\end{bmatrix} \underbrace{\begin{bmatrix} m_{\mathcal{S}}\mathbf{1} & -\mathbf{C}_{ib}\mathbf{c}_b^{\mathcal{S}z^\times} \\ \mathbf{c}_b^{\mathcal{S}z^\times}\mathbf{C}_{bi} & \mathbf{J}_b^{\mathcal{S}z}\end{bmatrix}}_{\boldsymbol{M}} \begin{bmatrix}\mathbf{v}_i^{zw/i} \\ \boldsymbol{\omega}_b^{bi}\end{bmatrix} \\
&+ \begin{bmatrix}\mathbf{v}_i^{zw/i^\mathsf{T}} & \boldsymbol{\omega}_b^{bi^\mathsf{T}}\end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{C}_{ib}\int_{\mathcal{S}}\mathbf{v}_b^{\mathrm{d}mz/b}\mathrm{d}m \\ \int_{\mathcal{S}}\mathbf{r}_b^{\mathrm{d}mz^\times}\mathbf{v}_b^{\mathrm{d}mz/b}\mathrm{d}m\end{bmatrix}}_{\beta} \underbrace{+\frac{1}{2}\int_{\mathcal{S}}\mathbf{v}_i^{\mathrm{d}mz/b^\mathsf{T}}\mathbf{v}_i^{\mathrm{d}mz/b}\mathrm{d}m}_{T^0_{\mathcal{S}w/i}},
\end{aligned} \tag{6}
$$

where $m_{\mathcal{S}} = \int_{\mathcal{S}}\mathrm{d}m$, $\mathbf{c}_b^{\mathcal{S}z} = \int_{\mathcal{S}}\mathbf{r}_b^{\mathrm{d}mz}\mathrm{d}m$ and $\mathbf{J}_b^{\mathcal{S}z} = \int_{\mathcal{S}} -\mathbf{r}_b^{\mathrm{d}mz^\times}\mathbf{r}_b^{\mathrm{d}mz^\times}\mathrm{d}m$ are the zeroth, first, and second moments of mass of $\mathcal{S}$ about point z. Equation 6 can be written compactly as

$$
T_{\mathcal{S}w/i} = \frac{1}{2}\boldsymbol{\nu}^\mathsf{T}\mathbf{M}\boldsymbol{\nu} + \boldsymbol{\nu}^\mathsf{T}\beta + T^0_{\mathcal{S}w/i},
$$

which is alternatively written using the generalized coordinates,

$$
T_{\mathcal{S}w/i} = \frac{1}{2}\dot{\mathbf{q}}^\mathsf{T}\mathbf{S}^\mathsf{T}\mathbf{M}\mathbf{S}\dot{\mathbf{q}} + \dot{\mathbf{q}}^\mathsf{T}\mathbf{S}^\mathsf{T}\beta + T^0_{\mathcal{S}w/i}. \tag{7}
$$

Notice that the kinetic energy expression in Eq. 7 is not strictly quadratic in $\dot{\mathbf{q}}$, but includes first and zeroth order terms. In particular, the terms $\dot{\mathbf{q}}^\mathsf{T}\mathbf{S}^\mathsf{T}\beta$ and $T^0_{\mathcal{S}w/i}$ stem from the fact that $\mathbf{v}_b^{\mathrm{d}mz/b}$ is non-zero, which is not the case for rigid bodies. Since there is no potential energy considered, the Lagrangian simply reduces to the kinetic energy, $L_{\mathcal{S}w/i} = T_{\mathcal{S}w/i}$.

Lagrange's Equation

Lagrange's equation is

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathcal{S}w/i}}{\partial\dot{\mathbf{q}}}\right)^\mathsf{T} - \left(\frac{\partial L_{\mathcal{S}w/i}}{\partial\mathbf{q}}\right)^\mathsf{T} = \boldsymbol{f} + \boldsymbol{\Xi}^\mathsf{T}\boldsymbol{\lambda}, \tag{8}
$$

where $\boldsymbol{f}$ is the generalized forces and moments and $\boldsymbol{\Xi}$ is a constraint matrix such that $\boldsymbol{\Xi}\dot{\mathbf{q}} = \mathbf{0}$. Currently, the only constraint is the attitude constraint, and as such $\boldsymbol{\Xi} = \begin{bmatrix}\mathbf{0} & \boldsymbol{\Xi}^{bi}\end{bmatrix}$. Moreover, it is straightforward to confirm that $\boldsymbol{\Xi}\boldsymbol{\Gamma} = \mathbf{0}$. The terms of Lagrange's equation will now be evaluated one-by-one.

The First Term, $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathcal{S}w/i}}{\partial\dot{\mathbf{q}}}\right)^\mathsf{T}$

Using the Lagrangian in Eq. 7, it follows that

$$
\begin{aligned}
\frac{\partial L_{\mathcal{S}w/i}}{\partial\dot{\mathbf{q}}} &= \dot{\mathbf{q}}^\mathsf{T}\mathbf{S}^\mathsf{T}\mathbf{M}\mathbf{S} + \beta^\mathsf{T}\mathbf{S}, \\
&= \boldsymbol{\nu}^\mathsf{T}\mathbf{M}\mathbf{S} + \beta^\mathsf{T}\mathbf{S},
\end{aligned}
$$

owing to the fact that $\mathbf{M} = \mathbf{M}^\mathsf{T}$. It follows that

$$\left(\frac{\partial L_{\mathcal{S}w/i}}{\partial \dot{\mathbf{q}}}\right)^\mathsf{T} = \mathbf{S}^\mathsf{T}\mathbf{M}\boldsymbol{\nu} + \mathbf{S}^\mathsf{T}\boldsymbol{\beta},$$

and as such,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathcal{S}w/i}}{\partial \dot{\mathbf{q}}}\right)^\mathsf{T} = \dot{\mathbf{S}}^\mathsf{T}\mathbf{M}\boldsymbol{\nu} + \mathbf{S}^\mathsf{T}\dot{\mathbf{M}}\boldsymbol{\nu} + \mathbf{S}^\mathsf{T}\mathbf{M}\dot{\boldsymbol{\nu}} + \dot{\mathbf{S}}^\mathsf{T}\boldsymbol{\beta} + \mathbf{S}^\mathsf{T}\dot{\boldsymbol{\beta}}. \tag{9}$$

The terms $\dot{\mathbf{M}}$ and $\dot{\beta}$ are

$$\dot{\mathbf{M}} = \begin{bmatrix} 0 & -\left(\dot{\mathbf{C}}_{bi}^\mathsf{T}\mathbf{c}_b^{\mathcal{S}z^\times} + \mathbf{C}_{bi}^\mathsf{T}\dot{\mathbf{c}}_b^{\mathcal{S}z^\times}\right) \\ \dot{\mathbf{c}}_b^{\mathcal{S}z^\times}\mathbf{C}_{bi} + \mathbf{c}_b^{\mathcal{S}z^\times}\dot{\mathbf{C}}_{bi} & \dot{\mathbf{J}}_b^{\mathcal{S}z} \end{bmatrix},$$

$$\dot{\boldsymbol{\beta}} = \begin{bmatrix} \dot{\mathbf{C}}_{bi}^\mathsf{T}\int_\mathcal{S}\mathbf{v}_b^{\mathrm{d}mz/b}\mathrm{d}m + \mathbf{C}_{bi}^\mathsf{T}\int_\mathcal{S}\dot{\mathbf{v}}_b^{\mathrm{d}mz/b}\mathrm{d}m \\ \int_\mathcal{S}\dot{\mathbf{r}}_b^{\mathrm{d}mz^\times}\mathbf{v}_b^{\mathrm{d}mz/b}\mathrm{d}m + \int_\mathcal{S}\mathbf{r}_b^{\mathrm{d}mz^\times}\dot{\mathbf{v}}_b^{\mathrm{d}mz/b}\mathrm{d}m \end{bmatrix}.$$

Recall that the system $\mathcal{S}$ is a constant-mass system, but is not necessarily rigid. As such, $\dot{m}_\mathcal{S} = 0$ while $\dot{\mathbf{c}}_b^{\mathcal{S}z} \neq \mathbf{0}$ and $\dot{\mathbf{J}}_b^{\mathcal{S}z} \neq \mathbf{0}$.

The Second Term, $\left(\frac{\partial L_{\mathcal{S}w/i}}{\partial \mathbf{q}}\right)^\mathsf{T}$

First, write $\frac{\partial L_{\mathcal{S}w/i}}{\partial \mathbf{q}}$ as

$$\frac{\partial L_{\mathcal{S}w/i}}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial L_{\mathcal{S}w/i}}{\partial \mathbf{r}_i^{zw}} & \frac{\partial L_{\mathcal{S}w/i}}{\partial \mathbf{q}^{bi}} \end{bmatrix}, \tag{10}$$

where the second term can be expanded as

$$\frac{\partial L_{\mathcal{S}w/i}}{\partial \mathbf{q}^{bi}} = \frac{\partial L_{\mathcal{S}w/i}}{\partial \boldsymbol{\omega}_b^{bi}}\frac{\partial \boldsymbol{\omega}_b^{bi}}{\partial \mathbf{q}^{bi}} + \frac{\hat{\partial} L_{\mathcal{S}w/i}}{\hat{\partial} \mathbf{q}^{bi}}.$$

The term $\frac{\hat{\partial} L_{\mathcal{S}w/i}}{\hat{\partial} \mathbf{q}^{bi}}$ is the partial derivative of $L_{\mathcal{S}w/i}$ with respect to $\mathbf{q}^{bi}$ neglecting the dependence of $\boldsymbol{\omega}_b^{bi}$ on $\mathbf{q}^{bi}$. Equation 10 can now be written as

$$\frac{\partial L_{\mathcal{S}w/i}}{\partial \mathbf{q}} = \begin{bmatrix} \frac{\partial L_{\mathcal{S}w/i}}{\partial \mathbf{r}_i^{zw}} & \mathbf{0} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial L_{\mathcal{S}w/i}}{\partial \mathbf{v}_i^{zw/i}} & \frac{\partial L_{\mathcal{S}w/i}}{\partial \boldsymbol{\omega}_b^{bi}} \end{bmatrix}}_{\frac{\partial L_{\mathcal{S}w/i}}{\partial \boldsymbol{\nu}}}\boldsymbol{\Omega}^{bi} + \begin{bmatrix} \mathbf{0} & \frac{\hat{\partial} L_{\mathcal{S}w/i}}{\hat{\partial} \mathbf{q}^{bi}} \end{bmatrix},$$

where $\boldsymbol{\Omega}^{bi} = \text{diag}(\mathbf{0}, \frac{\partial \boldsymbol{\omega}_b^{bi}}{\partial \mathbf{q}^{bi}})$. The term $\frac{\partial L_{\mathcal{S}w/i}}{\partial \mathbf{r}_i^{zw}}$ is zero, whilst $\frac{\partial L_{\mathcal{S}w/i}}{\partial \boldsymbol{\nu}} = \boldsymbol{\nu}^\mathsf{T}\mathbf{M} + \upbeta^\mathsf{T}$. To compute $\frac{\hat{\partial} L_{\mathcal{S}w/i}}{\hat{\partial} \mathbf{q}^{bi}}$, first note that the Lagrangian is

$$L_{\mathcal{S}w/i} = \frac{1}{2} m_{\mathcal{S}} \mathbf{v}_i^{zw/i^\mathsf{T}} \mathbf{v}_i^{zw/i} - \mathbf{v}_i^{zw/i^\mathsf{T}} \mathbf{C}_{bi}^\mathsf{T} \mathbf{c}_b^{\mathcal{S}z^\times} \boldsymbol{\omega}_b^{bi} + \frac{1}{2} \boldsymbol{\omega}_b^{bi^\mathsf{T}} \mathbf{J}_b^{\mathcal{S}z} \boldsymbol{\omega}_b^{bi}$$
$$+ \mathbf{v}_i^{zw/i^\mathsf{T}} \mathbf{C}_{bi}^\mathsf{T} \int_{\mathcal{S}} \mathbf{v}_b^{\mathrm{d}mz/b} \mathrm{d}m + \boldsymbol{\omega}_b^{bi^\mathsf{T}} \int_{\mathcal{S}} \mathbf{r}_b^{\mathrm{d}mz^\times} \mathbf{v}_b^{\mathrm{d}mz/b} \mathrm{d}m + T^0_{\mathcal{S}w/i}.$$

Therefore,

$$\begin{aligned}
\frac{\hat{\partial} L_{\mathcal{S}w/i}}{\hat{\partial} \mathbf{q}^{bi}} &= \frac{\hat{\partial}}{\hat{\partial} \mathbf{q}^{bi}} \left(-\mathbf{v}_i^{zw/i^\mathsf{T}} \mathbf{C}_{bi}^\mathsf{T} \mathbf{c}_b^{\mathcal{S}z^\times} \boldsymbol{\omega}_b^{bi} + \mathbf{v}_i^{zw/i^\top} \mathbf{C}_{bi}^\mathsf{T} \int_{\mathcal{S}} \mathbf{v}_b^{\mathrm{d}mz/b} \mathrm{d}m \right), \\
&= \frac{\hat{\partial}}{\hat{\partial} \mathbf{q}^{bi}} \left(\boldsymbol{\omega}_b^{bi^\mathsf{T}} \mathbf{c}_b^{\mathcal{S}z^\times} \mathbf{C}_{bi} \mathbf{v}_i^{zw/i} + \int_{\mathcal{S}} \mathbf{v}_b^{\mathrm{d}mz/b^\top} \mathrm{d}m \mathbf{C}_{bi} \mathbf{v}_i^{zw/i} \right), \\
&= \boldsymbol{\omega}_b^{bi^\mathsf{T}} \mathbf{c}_b^{\mathcal{S}z^\times} \frac{\partial(\mathbf{C}_{bi} \mathbf{v}_i^{zw/i})}{\partial \mathbf{q}^{bi}} + \int_{\mathcal{S}} \mathbf{v}_b^{\mathrm{d}mz/b^\top} \mathrm{d}m \frac{\partial(\mathbf{C}_{bi} \mathbf{v}_i^{zw/i})}{\partial \mathbf{q}^{bi}}.
\end{aligned}$$

Finally, the second term of Lagrange's equation is

$$\begin{aligned}
\frac{\partial L_{\mathcal{S}w/i}}{\partial \mathbf{q}} &= \left(\boldsymbol{\nu}^\mathsf{T}\mathbf{M} + \upbeta^\mathsf{T} \right) \boldsymbol{\Omega}^{bi} \\
&\quad + \left[\mathbf{0} \;\; \left(\boldsymbol{\omega}_b^{bi^\mathsf{T}} \mathbf{c}_b^{\mathcal{S}z^\times} \frac{\partial(\mathbf{C}_{bi} \mathbf{v}_i^{zw/i})}{\partial \mathbf{q}^{bi}} + \int_{\mathcal{S}} \mathbf{v}_b^{\mathrm{d}mz/b^\mathsf{T}} \mathrm{d}m \frac{\partial(\mathbf{C}_{bi} \mathbf{v}_i^{zw/i})}{\partial \mathbf{q}^{bi}} \right) \right],
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial L_{\mathcal{S}w/i}}{\partial \mathbf{q}} \right)^\mathsf{T} &= \boldsymbol{\Omega}^{bi^\mathsf{T}} \mathbf{M} \boldsymbol{\nu} + \boldsymbol{\Omega}^{bi^\mathsf{T}} \upbeta \\
&\quad + \underbrace{\begin{bmatrix} \mathbf{0} \\ \left(-\frac{\partial(\mathbf{C}_{bi} \mathbf{v}_i^{zw/i})^\mathsf{T}}{\partial \mathbf{q}^{bi}} \mathbf{c}_b^{\mathcal{S}z^\times} \boldsymbol{\omega}_b^{bi} + \frac{\partial(\mathbf{C}_{bi} \mathbf{v}_i^{zw/i})^\mathsf{T}}{\partial \mathbf{q}^{bi}} \int_{\mathcal{S}} \mathbf{v}_b^{\mathrm{d}mz/b} \mathrm{d}m \right) \end{bmatrix}}_{-\boldsymbol{a}^{\text{non}}}. \qquad (11)
\end{aligned}$$

Virtual Work and the Generalized Forces

For simplicity, only discrete forces acting at specific points will be considered. Consider a discrete force $\underrightarrow{f}^{\,p}$ acting at point p. The position of p relative to w resolved in $\mathcal{F}_i$ is

$$\mathbf{r}_i^{pw} = \mathbf{C}_{ib} \mathbf{r}_b^{pz} + \mathbf{r}_i^{zw}.$$

A virtual displacement associated with $\underrightarrow{r}^{\,pw}$ is [24]

$$\delta \mathbf{r}_i^{pw} = \delta \mathbf{r}_i^{zw} + \frac{\partial(\mathbf{C}_{bi}^\mathsf{T} \mathbf{r}_i^{pz})}{\partial \mathbf{q}^{bi}} \delta \mathbf{q}^{bi},$$

and thus the virtual work done by $\underset{\rightarrow}{f}^{\,p}$ is

$$
\begin{aligned}
\delta W^{\mathcal{S}w,p} &= \underset{\rightarrow}{f}^{\,p} \cdot \delta \underset{\rightarrow}{r}^{\,pw} \\
&= \mathbf{f}_i^{p^{\mathsf{T}}} \delta \mathbf{r}_i^{zw} + \mathbf{f}_i^{p^{\mathsf{T}}} \frac{\partial (\mathbf{C}_{bi}^{\mathsf{T}} \mathbf{r}^{pz})}{\partial \mathbf{q}^{bi}} \delta \mathbf{q}^{bi} \\
&= \underbrace{\begin{bmatrix} \delta \mathbf{r}_i^{zw^{\mathsf{T}}} & \delta \mathbf{q}^{bi^{\mathsf{T}}} \end{bmatrix}}_{\delta \mathbf{q}^{\mathsf{T}}} \underbrace{\begin{bmatrix} \mathbf{f}_i^p \\ \frac{\partial (\mathbf{C}_{bi}^{\mathsf{T}} \mathbf{r}^{pz})}{\partial \mathbf{q}^{bi}}^{\mathsf{T}} \mathbf{f}_i^p \end{bmatrix}}_{\boldsymbol{f}^p}.
\end{aligned} \tag{12}
$$

Since the total virtual work done on $\mathcal{S}$ is the sum of virtual work done by each force, the sum of generalized forces and moments would be a summation of terms identical to $\boldsymbol{f}^p$. That is, $\boldsymbol{f} = \boldsymbol{f}^{p_1} + \boldsymbol{f}^{p_2} + \cdots + \boldsymbol{f}^{p_N}$, and thus this is the term that is substituted into Lagrange's equation.

Lagrange's Equation

Substituting Eq. 9, Eq. 11, and the generalized forces described by Eq. 12 into Eq. 8 gives

$$
\mathbf{S}^{\mathsf{T}} \mathbf{M} \dot{\boldsymbol{\nu}} + \mathbf{S}^{\mathsf{T}} \dot{\mathbf{M}} \boldsymbol{\nu} + (\dot{\mathbf{S}}^{\mathsf{T}} - \boldsymbol{\Omega}^{bi^{\mathsf{T}}}) \mathbf{M} \boldsymbol{\nu} + \mathbf{S}^{\mathsf{T}} \dot{\upbeta} + (\dot{\mathbf{S}}^{\mathsf{T}} - \boldsymbol{\Omega}^{bi^{\mathsf{T}}}) \upbeta + \boldsymbol{a}^{\text{non}} = \boldsymbol{f} + \boldsymbol{\Xi}^{\mathsf{T}} \boldsymbol{\lambda}, \tag{13}
$$

where $\boldsymbol{a}^{\text{non}}$ has been defined in Eq. 11. Recalling the identities Eq. 2, Eq. 3, Eq. 4, as well as the fact that $\boldsymbol{\Gamma}^{\mathsf{T}} \mathbf{S}^{\mathsf{T}} = \mathbf{1}$ and $\boldsymbol{\Gamma}^{\mathsf{T}} \boldsymbol{\Xi}^{\mathsf{T}} = \mathbf{0}$, pre-multiplying Eq. 13 by $\boldsymbol{\Gamma}^{\mathsf{T}}$ and simplifying yields

$$
\mathbf{M} \dot{\boldsymbol{\nu}} + \dot{\mathbf{M}} \boldsymbol{\nu} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}_b^{bi^{\times}} \end{bmatrix} \mathbf{M} \boldsymbol{\nu} + \dot{\upbeta} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\omega}_b^{bi^{\times}} \end{bmatrix} \upbeta - \boldsymbol{\Gamma}^{\mathsf{T}} \boldsymbol{a}^{\text{non}} = \boldsymbol{\Gamma}^{\mathsf{T}} \boldsymbol{f}. \tag{14}
$$

The terms $\boldsymbol{\Gamma}^{\mathsf{T}} \boldsymbol{a}^{\text{non}}$ and $\boldsymbol{\Gamma}^{\mathsf{T}} \boldsymbol{f}$ are

$$
\begin{aligned}
\boldsymbol{\Gamma}^{\top} \boldsymbol{a}^{non} &= \begin{bmatrix} 0 \\ -(\mathbf{C}_{bi} \mathbf{v}_i^{zw/i})^{\times} \mathbf{c}_b^{\mathcal{S}z^{\times}} \boldsymbol{\omega}_b^{bi} + (\mathbf{C}_{bi} \mathbf{v}_i^{zw/i})^{\times} \int_{\mathcal{S}} \mathbf{v}_b^{\mathrm{d}mz/b} \mathrm{d}m \end{bmatrix}, \\
\boldsymbol{\Gamma}^{\top} \boldsymbol{f} &= \begin{bmatrix} \sum_{n=1}^{N} \mathbf{f}_i^{p_n} \\ \sum_{n=1}^{N} \mathbf{r}_b^{p_n z^{\times}} \mathbf{C}_{bi} \mathbf{f}_i^{p_n} \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{f}_i^{\mathcal{S}} \\ \mathbf{m}_b^{\mathcal{S}z} \end{bmatrix},
\end{aligned}
$$

where $\mathbf{f}_i^{\mathcal{S}}$ is the sum of external forces acting on $\mathcal{S}$ and $\mathbf{m}_b^{\mathcal{S}z}$ is the sum of external moments acting on $\mathcal{S}$ about point z. The mass matrix $\mathbf{M}$ in Eq. 14 is symmetric positive definite, regardless of the attitude parameterization, and is therefore invertible. Equation 14 constitutes the equation of motion of the constant-mass system $\mathcal{S}$. To convert this to variable-mass equations of motion, Eq. 14 will be expanded into its constituent translational and rotational dynamics and Reynold's Transport Theorem will be invoked. This will also make the equivalence to traditional forms clear.

Translational Dynamics

Expanding the first three rows of Eq. 14, whilst staying in matrix form, yields

$$m_{\mathcal{S}}\dot{\mathbf{v}}_i^{zw/i} - \mathbf{C}_{bi}^{\mathsf{T}}\mathbf{c}_b^{\mathcal{S}z^\times}\dot{\boldsymbol{\omega}}_b^{bi} - \left((-\boldsymbol{\omega}_b^{bi^\times}\mathbf{C}_{bi})^{\mathsf{T}}\mathbf{c}_b^{\mathcal{S}z^\times} + \mathbf{C}_{bi}^{\mathsf{T}}\dot{c}_b^{\mathcal{S}z^\times}\right)\boldsymbol{\omega}_b^{bi} + (-\boldsymbol{\omega}_b^{bi^\times}\mathbf{C}_{bi})^{\mathsf{T}}\int_{\mathcal{S}}\mathbf{v}_b^{\mathrm{d}mz/b}\mathrm{d}m + \mathbf{C}_{bi}^{\mathsf{T}}\int_{\mathcal{S}}\dot{\mathbf{v}}_b^{\mathrm{d}mz/b}\mathrm{d}m = \mathbf{f}_i^{\mathcal{S}}. \quad (15)$$

Recalling that

$$\mathbf{c}_b^{\mathcal{S}z^\times} = \int_{\mathcal{S}}\mathbf{r}_b^{\mathrm{d}mz^\times}\mathrm{d}m = m_{\mathcal{S}}\mathbf{r}_b^{\mathcal{S}z^\times},$$
$$\dot{\mathbf{c}}_b^{\mathcal{S}z^\times} = \int_{\mathcal{S}}\mathbf{v}_b^{\mathrm{d}mz/b^\times}\mathrm{d}m,$$

where $\mathbf{r}_b^{\mathcal{S}z}$ is the instantaneous center of mass of $\mathcal{S}$, Eq. 15 becomes

$$m_{\mathcal{S}}\left[\dot{\mathbf{v}}_i^{zw/i} + \mathbf{C}_{ib}\dot{\boldsymbol{\omega}}_b^{bi^\times}\mathbf{r}_b^{\mathcal{S}z} + \mathbf{C}_{ib}\boldsymbol{\omega}_b^{bi^\times}\boldsymbol{\omega}_b^{bi^\times}\mathbf{r}_b^{\mathcal{S}z}\right] + 2\mathbf{C}_{ib}\int_{\mathcal{S}}\boldsymbol{\omega}_b^{bi^\times}\mathbf{v}_b^{\mathrm{d}mz/b}\mathrm{d}m + \mathbf{C}_{ib}\int_{\mathcal{S}}\dot{\mathbf{v}}_b^{\mathrm{d}mz/b}\mathrm{d}m = \mathbf{f}_i^{\mathcal{S}}.$$

Using Reynold's Transport Theorem given in Eq. 5, it can be shown that

$$\int_{\mathcal{S}}\dot{\mathbf{v}}_b^{\mathrm{d}mz/b}\mathrm{d}m = \int_{V(t)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\mathbf{v}_b^{\mathrm{d}mz/b}\right)\Big|_{\mathcal{F}_b}\mathrm{d}V + \int_{B(t)}\rho\mathbf{v}_b^{\mathrm{d}mz/b}(\mathbf{v}_b^{\mathrm{d}mz/b^{\mathsf{T}}}\mathbf{n}_b)\mathrm{d}S,$$

where $\mathbf{n}_b$ is the components of an outwards-pointing normal unit vector to the boundary B, resolved in $\mathcal{F}_b$, and ρ is the mass density of the element dm. By defining,

$$\mathbf{f}_i^C = -2\mathbf{C}_{ib}\int_{V(t)}\rho\boldsymbol{\omega}_b^{bi^\times}\mathbf{v}_b^{\mathrm{d}mz/b}\mathrm{d}V,$$
$$\mathbf{f}_i^U = -\mathbf{C}_{ib}\int_{V(t)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\mathbf{v}_b^{\mathrm{d}mz/b}\right)\Big|_{\mathcal{F}_b}\mathrm{d}V,$$
$$\mathbf{f}_i^R = -\mathbf{C}_{ib}\int_{B(t)}\rho\mathbf{v}_b^{\mathrm{d}mz/b}(\mathbf{v}_b^{\mathrm{d}mz/b^{\mathsf{T}}}\mathbf{n}_b)\mathrm{d}S,$$

the well-known variable-mass translational equations reported in [4–7, 15, 16, 19] emerge, that being

$$m_{\mathcal{S}}\left[\dot{\mathbf{v}}_i^{zw/i} + \dot{\boldsymbol{\omega}}_i^{bi^\times}\mathbf{r}_i^{\mathcal{S}z} + \boldsymbol{\omega}_i^{bi^\times}\boldsymbol{\omega}_i^{bi^\times}\mathbf{r}_i^{\mathcal{S}z}\right] = \mathbf{f}_i^{\mathcal{S}} + \mathbf{f}_i^C + \mathbf{f}_i^U + \mathbf{f}_i^R. \quad (16)$$

Recall that, in general, neither $V(t)$ or $B(t)$ are known since they correspond to the constant-mass system, and the state of the mass that has left the primary volume of interest is unknown. Only at the specific time $t = \bar{t}$ where the systems $\mathcal{S}$ and $\bar{\mathcal{S}}$ coincide exactly, are $\bar{V}$ and $\bar{B}$ known, where by design $V(\bar{t}) = \bar{V}$ and $B(\bar{t}) = \bar{B}$. Since Eq. 16 has the same form for any general instant $\bar{t}$, the bounds of integration may now be replaced with $\bar{\mathcal{S}}$, $\bar{V}$, $\bar{B}$. This is the exact result seen in [4–7, 15, 16, 19], amongst others, when resolved in $\mathcal{F}_i$. Although Eq. 16 is written in a form that can be recognized, it is not necessarily convenient to resolve $\underrightarrow{\omega}^{bi}$ and $\underrightarrow{r}^{\mathcal{S}z}$ in $\mathcal{F}_i$. Terms in Eq. 16 can be resolved in any frame with an appropriate use of DCMs.

Rotational Dynamics

Expanding the last three rows of Eq. 14, whilst staying in matrix form, yields

$$\begin{aligned}&\mathbf{c}_b^{\mathcal{S}z^\times}\mathbf{C}_{bi}\dot{\mathbf{v}}_i^{zw/i}+\mathbf{J}_b^{\mathcal{S}z}\dot{\boldsymbol{\omega}}_b^{bi}+\left(\dot{\mathbf{c}}_b^{\mathcal{S}z^\times}\mathbf{C}_{bi}+\mathbf{c}_b^{\mathcal{S}z^\times}(-\boldsymbol{\omega}_b^{bi^\times}\mathbf{C}_{bi})\right)\mathbf{v}_i^{zw/i}+\dot{J}_b^{\mathcal{S}z}\boldsymbol{\omega}_b^{bi}\\&+\boldsymbol{\omega}_b^{bi^\times}\left(\dot{\mathbf{c}}_b^{\mathcal{S}z^\times}\mathbf{C}_{bi}\mathbf{v}_i^{zw/i}+\mathbf{J}_b^{\mathcal{S}z}\boldsymbol{\omega}_b^{bi}\right)+\int_{\mathcal{S}}\dot{\mathbf{r}}_b^{\mathrm{d}mz^\times}\mathbf{v}_b^{\mathrm{d}mz/b}\mathrm{d}m+\int_{\mathcal{S}}\mathbf{r}_b^{\mathrm{d}mz^\times}\dot{\mathbf{v}}_b^{\mathrm{d}mz/b}\mathrm{d}m\\&+\boldsymbol{\omega}_b^{bi^\times}\int_{\mathcal{S}}\mathbf{r}_b^{\mathrm{d}mz^\times}\mathbf{v}_b^{\mathrm{d}mz/b}-(\mathbf{C}_{bi}\mathbf{v}_i^{zw/i})^\times\mathbf{c}_b^{\mathcal{S}z^\times}\boldsymbol{\omega}_b^{bi}+(\mathbf{C}_{bi}\mathbf{v}_i^{zw/i})^\times\int_{\mathcal{S}}\mathbf{v}_b^{\mathrm{d}m/b}\mathrm{d}m=\mathbf{m}_b^{\mathcal{S}z}.\end{aligned}\tag{17}$$

The terms $-(\mathbf{C}_{bi}\mathbf{v}_i^{zw/i})^\times\mathbf{c}_b^{\mathcal{S}z^\times}\boldsymbol{\omega}_b^{bi}$ and $(\mathbf{C}_{bi}\mathbf{v}_i^{zw/i})^\times\int_{\mathcal{S}}\mathbf{v}_b^{\mathrm{d}m/b}\mathrm{d}m$ can be simplified. Specifically,

$$\begin{aligned}-(\mathbf{C}_{bi}\mathbf{v}_i^{zw/i})^\times\mathbf{c}_b^{\mathcal{S}z^\times}\boldsymbol{\omega}_b^{bi}&=(\mathbf{c}_b^{\mathcal{S}z^\times}\boldsymbol{\omega}_b^{bi})^\times\mathbf{C}_{bi}\mathbf{v}_i^{zw/i}\\&=(\mathbf{c}_b^{\mathcal{S}z^\times}\boldsymbol{\omega}_b^{bi^\times}-\boldsymbol{\omega}_b^{bi^\times}\mathbf{c}_b^{\mathcal{S}z^\times})\mathbf{C}_{bi}\mathbf{v}_i^{zw/i}\\&=\mathbf{c}_b^{\mathcal{S}z^\times}\boldsymbol{\omega}_b^{bi^\times}\mathbf{C}_{bi}\mathbf{v}_i^{zw/i}-\boldsymbol{\omega}_b^{bi^\times}\mathbf{c}_b^{\mathcal{S}z^\times}\mathbf{C}_{bi}\mathbf{v}_i^{zw/i},\end{aligned}$$

$$\begin{aligned}(\mathbf{C}_{bi}\mathbf{v}_i^{zw/i})^\times\int_{\mathcal{S}}\mathbf{v}_b^{\mathrm{d}m/b}\mathrm{d}m&=-\int_{\mathcal{S}}\mathbf{v}_b^{\mathrm{d}m/b^\times}\mathrm{d}m\mathbf{C}_{bi}\mathbf{v}_i^{zw/i}\\&=-\dot{\mathbf{c}}_b^{\mathcal{S}z^\times}\mathbf{C}_{bi}\mathbf{v}_i^{zw/i},\end{aligned}$$

resulting in multiple terms in Eq. 17 to cancel out. Equation 17 now becomes,

$$\begin{aligned}&\mathbf{c}_b^{\mathcal{S}z^\times}\mathbf{C}_{bi}\dot{\mathbf{v}}_i^{zw/i}+\mathbf{J}_b^{\mathcal{S}z}\dot{\boldsymbol{\omega}}_b^{bi}+\dot{\mathbf{J}}_b^{\mathcal{S}z}\boldsymbol{\omega}_b^{bi}+\boldsymbol{\omega}_b^{bi^\times}\mathbf{J}_b^{\mathcal{S}z}\boldsymbol{\omega}_b^{bi}\\&+\int_{\mathcal{S}}\mathbf{r}_b^{\mathrm{d}mz^\times}\dot{\mathbf{v}}_b^{\mathrm{d}mz/b}\mathrm{d}m+\boldsymbol{\omega}_b^{bi^\times}\int_{\mathcal{S}}\mathbf{r}_b^{\mathrm{d}mz^\times}\mathbf{v}_b^{\mathrm{d}mz/b}\mathrm{d}m=\mathbf{m}_b^{\mathcal{S}z}.\end{aligned}$$

It can be shown by use of Reynold's Transport Theorem that

$$\begin{aligned}\int_{\mathcal{S}}\mathbf{r}_b^{\mathrm{d}mz^\times}\dot{\mathbf{v}}_b^{\mathrm{d}mz/b}\mathrm{d}m&=\int_V\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\mathbf{r}_b^{\mathrm{d}mz^\times}\mathbf{v}_b^{\mathrm{d}mz/b}\right)\Big|_{\mathcal{F}_b}\mathrm{d}V\\&+\int_B\rho\mathbf{r}_b^{\mathrm{d}mz^\times}\mathbf{v}_b^{\mathrm{d}mz/b}(\mathbf{v}_b^{\mathrm{d}mz/b^\top}\mathbf{n}_b)\mathrm{d}S,\end{aligned}$$

and that

$$\begin{aligned}\boldsymbol{\omega}_b^{bi^\times}\int_{\mathcal{S}}\mathbf{r}_b^{\mathrm{d}mz^\times}\mathbf{v}_b^{\mathrm{d}mz/b}\mathrm{d}m&=\int_V\rho\boldsymbol{\omega}_b^{bi^\times}\mathbf{r}_b^{\mathrm{d}mz^\times}\mathbf{v}_b^{\mathrm{d}mz/b}\mathrm{d}V\\&+\int_B\rho\mathbf{r}_b^{\mathrm{d}mz^\times}\boldsymbol{\omega}_b^{bi^\times}\mathbf{r}_b^{\mathrm{d}mz}(\mathbf{v}_b^{\mathrm{d}mz/b^\top}\mathbf{n}_b)\mathrm{d}S.\end{aligned}$$

Therefore,

$$\mathbf{c}_b^{\mathcal{S}z^\times}\mathbf{C}_{bi}\dot{\mathbf{v}}_i^{zw/i} + \mathbf{J}_b^{\mathcal{S}z}\dot{\boldsymbol{\omega}}_b^{bi} + \dot{\mathbf{J}}_b^{\mathcal{S}z}\boldsymbol{\omega}_b^{bi} + \boldsymbol{\omega}_b^{bi^\times}\mathbf{J}_b^{\mathcal{S}z}\boldsymbol{\omega}_b^{bi} + \int_V \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\mathbf{r}_b^{\mathrm{d}mz^\times}\mathbf{v}_b^{\mathrm{d}mz/b}\right)\Big|_{\mathcal{F}_b}\mathrm{d}V$$
$$+ \int_B \rho\mathbf{r}_b^{\mathrm{d}mz^\times}\mathbf{v}_b^{dmz/b}(\mathbf{v}_b^{dmz/b^\top}\mathbf{n}_b)\mathrm{d}S + \int_V \rho\boldsymbol{\omega}_b^{bi^\times}\mathbf{r}_b^{dmz^\times}\mathbf{v}_b^{dmz/b}\mathrm{d}V$$
$$+ \int_B \rho\mathbf{r}_b^{\mathrm{d}mz^\times}\boldsymbol{\omega}_b^{bi^\times}\mathbf{r}_b^{\mathrm{d}mz}(\mathbf{v}_b^{\mathrm{d}mz/b^\top}\mathbf{n}_b)\mathrm{d}S = \mathbf{m}_b^{\mathcal{S}z}, \tag{18}$$

which is again the exact result obtained in [4–7, 15, 16, 19], amongst others, resolved in $\mathcal{F}_b$. As before, since the system $\mathcal{S}$ coincides exactly with $\bar{S}$ at the general instant $t = \bar{t}$, and since the form of Eq. 18 is identical for all $\bar{t}$, the bounds of integration can be changed to the known $\bar{V}$, $\bar{B}$.

Conclusion

This note presents a Lagrangian approach to deriving the translational and rotational dynamics of variable-mass systems. The advantage of the approach presented is that any arbitrary attitude parameterization can be used to derive equations of motion using the standard form of Lagrange's equation. Moreover, the equations of motion are maintained in matrix form, leading to concise representations of the equations of motion.

Acknowledgements This project was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

Compliance with Ethical Standards

Conflict of interests The authors declare that they have no conflict of interest.

References

1. Bernstein, D.S.: Newton's frames. IEEE Control. Syst. Mag. Febuary, 17–18. (2008)
2. Caverly, R.J., Forbes, J.R.: Dynamic modeling, trajectory optimization, and control of a flexible kiteplane. IEEE Trans. Control Syst. Technol. **25**(4), 1297–1306 (2017)
3. D'Eleuterio, G.M., Barfoot, T.D.: A Discrete Quasi-coordinate Formulation for the Dynamics of Elastic bodies. Journal of Applied Mechanics Transactions ASME **74**(2), 231–239 (2007)
4. Eke, F.O.: Dynamics of Variable Mass Systems. National Aeronautics and Space Administration, Tech. rep. (1998)
5. Eke, F.O., Mao, T.C.: On the dynamics of variable mass systems. Int. J. Mech. Eng. Educ. **30**(2), 123–137 (2000)
6. Eke, F.O., Wang, S.M.: Equations of motion of Two-Phase variable mass systems with solid base. J. Appl. Mech. **61**(4), 855–860 (1993)
7. Eke, F.O., Wang, S.M.: Attitude behavior of a variable mass cylinder. J. Appl. Mech. **62**(4), 935–940 (1995)

8. Forbes, J.R.: Identities for deriving equations of motion using constrained attitude parameterizations. Journal of Guidance, Control, and Dynamics **37**(4), 1283–1289 (2014)
9. Greene, R.L., Matese, J.J.: Lagrangian mechanics for open systems. European journal of physics **103**, 2 (1981)
10. Grubin, C.: Mechanics of variable mass systems. J. Franklin Inst. **276**(4), 305–312 (1963)
11. Guttner, W.C., Pesce, C.P.: On hamilton's principle for discrete systems of variable mass and the corresponding lagrange's equations. J. Braz. Soc. Mech. Sci. Eng. **36**(6), 1–8 (2016)
12. Hogan, F.R., Forbes, J.R.: Modeling of spherical robots rolling on generic surfaces. Multibody System Dynamics **35**(1), 91–109 (2015)
13. Hughes, P.C.: Spacecraft Attitude Dynamics second edn Dover Publications Inc., Mineola New York (2004)
14. Hurtado, J.E.: Hamilton's principle for Variable-Mass systems. Journal of guidance Control, and Dynamics **41**(12), 1–4 (2018)
15. Meirovitch, L.: General Motion of a Variable-Mass Flexible Rocket with Internal Flow. Tech. rep., Langley Research Center, National Aeronautics and Space Administration, Cincinnati Ohio (1970)
16. Meirovitch, L.: Methods of Analytical Dynamics. McGraw Hill Book Company, Inc., Cincinnati Ohio (1970)
17. Meirovitch, L., Stemple, T.: Hybrid Equations of Motion for Flexible Multibody Systems using Quasi-coordinates. Guidance, Navigation and Control Conference, 1993 **18**(4), 619–632 (1993)
18. Musicki, D.: Extended lagrangian formalism for rheonomic systems with variable mass. Theoretical and Applied Mechanics **44**(1), 115–132 (2017)
19. Nanjangud, A., Eke, F.O.: Lagrange's equations for Rocket-Type variable mass systems. International Review of Aerospace Engineering (IREASE) **5**(5), 256–260 (2018)
20. Nikravesh, P.E., Wehage, R.A., Kwon, O.K.: Euler parameters in computational kinematics and dynamics: part 1 and 2x. Journal of guidance Transmissions, and Automation in Design **107**(1), 358–369 (1985)
21. Pritchard, P.J.: Introduction to Fluid Mechanics eighth edn John Wiley & Sons Inc (2011)
22. Pritchard, P.J.: Momentum Equation for Control Volume with Arbitrary Acceleration Introduction to Fluid Mechanics, Eighth Edn., Chap. 4.6 W6-W9 John Wiley & Sons Inc (2011)
23. Quinn, R.D.: Equations of Motion for Structures in Terms of Quasi-coordinates. Journal of Applied Mechanics Transactions ASME **57**(3), 745 (1990)
24. de Ruiter, A.H.J., Forbes, J.R.: General identities for parameterizations of SO(3) with applications. Journal of Applied Mechanics **81**(7), 071–001 (2014)
25. Thomson, W.T.: Equations of motion for the variable mass system. AIAA J. **4**(4), 766–768 (1966)
26. Walsh, A., Forbes, J.R.: Modeling and control of a wind energy harvesting kite with flexible cables. In: Proceedings of the American Control Conference, vol. 2015-July, pp. 2383–2388. IEEE, Chicago, IL (2015)
27. Walsh, A., Forbes, J.R.: Modeling and control of flexible telescoping manipulators. IEEE Trans. Robot. **31**(4), 936–947 (2015)
28. Walsh, A., Zlotnik, D.E., Forbes, J.R.: Modelling of Spacecraft with N Reaction Wheels Using Arbitrary Attitude Parameterizations. In: Advances in the Astronautical Sciences, Vol. 157, Pp. 945–956 (2016)
29. Wang, S.M., Eke, F.O.: Rotational dynamics of axisymmetric variable mass systems. J. Appl. Mech. **62**(4), 970–974 (2008)
30. White, F.M.: Fluid Mechanics, seventh edn McGraw-Hill (2011)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.