Titre: Title:	Analyse par une méthode hybride d'éléments finis des coques cylindriques non-uniformes, anisotropes et contenant un liquide en écoulement
Auteur: Author:	Abbas Selmane
Date:	1996
Type:	Mémoire ou thèse / Dissertation or Thesis
Référence: Citation:	Selmane, A. (1996). Analyse par une méthode hybride d'éléments finis des coques cylindriques non-uniformes, anisotropes et contenant un liquide en écoulement [Thèse de doctorat, École Polytechnique de Montréal]. PolyPublie. https://publications.polymtl.ca/8937/

URL de PolyPublie: PolyPublie URL:	https://publications.polymtl.ca/8937/
Directeurs de recherche: Advisors:	Aouni A. Lakis
Programme: Program:	Non spécifié

UNIVERSITÉ DE MONTRÉAL

ANALYSE PAR UNE MÉTHODE HYBRIDE D'ÉLÉMENTS FINIS DES COQUES CYLINDRIQUES NON-UNIFORMES, ANISOTROPES ET CONTENANT UN LIQUIDE EN ÉCOULEMENT

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THÈSE PRÉSENTÉE EN VUE DE L'OBTENTION
DU DIPLÔME DE PHILOSOPHIAE DOCTOR (Ph.D.)
(GÉNIE MÉCANIQUE)
AVRIL 1996

0-612-26434-3

UNIVERSITÉ DE MONTRÉAL

ÉCOLE POLYTECHNIQUE DE MONTRÉAL

Cette thèse intitulée:

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en vue de l'obtention du diplôme de: Philosophiae Doctor

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À ma famille,

À mes amis,

À toute l'humanité.

REMERCIEMENTS

Je tiens à adresser mes remerciements les plus sincères à mon directeur de recherche, Monsieur A. Aouni Lakis, pour son apport et son soutien constant au cours de ce travail.

Je remercie vivement mes parents et mon épouse pour avoir souhaité cet instant et pour m'avoir supporté par leurs encouragements et leur confiance durant toutes ces années.

Je voudrais également remercier tous mes amis qui m'ont soutenu de près ou de loin, particulièrement Monsieur A. Meddad pour son aide précieuse à la réalisation de ce travail.

RÉSUMÉ

L'analyse des coques minces soumises à un fluide en écoulement a été le sujet de plusieurs recherches. La plupart de ces études traitent de l'analyse linéaire des coques cylindriques fermées avec ou sans interaction avec un fluide en écoulement. Peu de travaux ont été faits pour des coques cylindriques ouvertes, anisotropes, non uniformes et soumises à un fluide en écoulement.

Nous proposons de développer une méthode pour l'analyse linéaire et non-linéaire des coques minces, élastiques, anisotropes, ouvertes et soumises à un écoulement interne et externe. La stabilité dynamique des coques cylindriques fermées et le cas des coques partiellement ou complètement remplies de liquide sont aussi analysés. La méthode développée est une combinaison de la méthode des éléments finis, de la théorie des coques et de celle des fluides.

Les coques ouvertes sont simplement supportées selon leurs rives courbes et elles ont des conditions frontières arbitraires sur les rives droites. La structure peut être uniforme ou non uniforme dans la direction circonférencielle.

La première partie de ce travail traite de l'analyse linéaire des coques cylindriques vides ouvertes ou fermées. La coque est divisée en plusieurs éléments finis de type coque cylindrique ouverte et les fonctions de déplacement sont dérivées de la théorie des coques cylindriques minces de Sanders. Les expressions des matrices de masse et de rigidité sont déterminées par intégration analytique exacte. Les vibrations libres des coques cylindriques ouvertes et fermées sont analysées par cette méthode dans le cas isotrope et anisotrope, uniforme et non uniforme. Les résultats obtenus nous permettent de conclure qu'il y a une bonne concordance entre les fréquences calculées par cette méthode et celles obtenues par d'autres auteurs.

Dans la seconde partie de cette thèse, nous présentons une théorie pour l'analyse dynamique des coques cylindriques ouvertes, anisotropes et soumises à un fluide en écoulement interne et externe. L'équation du potentiel des vitesses et l'équation de Bernouilli de l'élément fini fluide nous permettent d'exprimer la pression exercée par le fluide comme une fonction des déplacements nodaux et de trois forces (inertie, centrifuge et Coriolis) du fluide en écoulement. L'intégration analytique de la pression nous donne trois matrices pour le fluide en écoulement (masse, rigidité et amortissement). Plusieurs exemples illustrent la théorie et le comportement dynamique des coques cylindriques ouvertes et fermées, soumises à un fluide en écoulement et à des coques partiellement ou complètement remplies de fluide. Une bonne concordance des résultats a été obtenue avec d'autres théories et expériences.

Dans la troisième partie de cette étude, nous présentons une approche générale pour prédire l'influence des non-linéarités géométriques des parois sur les fréquences naturelles des coques cylindriques ouvertes ou fermées, élastiques, minces, orthotropiques et non uniformes. Les coefficients modaux sont dérivés de la théorie non linéaire de Sanders-Koiter pour les coques cylindriques en termes de fonctions de déplacement développées dans la première partie. Les matrices de rigidité non linéaires du second et troisième ordre sont déterminées à partir de la méthode des éléments finis. Les équations non linéaires sont résolues par la méthode numérique de Runge-Kutta du quatrième ordre. Les fréquences linéaires et non linéaires sont alors déterminées en fonction de l'amplitude du mouvement de la coque pour plusieurs cas. Les résultats obtenus sont en bonne concordance avec ceux des autres auteurs.

La quatrième et dernière partie de cette recherche présente un modèle pour prédire l'influence des non-linéarités associées aux parois de la coque et au fluide en écoulement des coques ouvertes, élastiques, minces, orthotropiques, non-uniformes, submergées et soumises simultanément à un écoulement interne et externe. Avec les matrices de masse et de rigidités linéaires et non linéaires de la coque vide ainsi qu'avec les matrices de masse, de rigidité, et d'amortissement linéaires du fluide en mouvement, nous développons dans cette partie trois matrices non linéaires pour le fluide en écoulement. En dérivant une expression non linéaire pour la pression dynamique en fonction des déplacements nodaux, des forces d'inertie, centrifuges et Coriolis ainsi que de la combinaison des effets non-linéaires du fluide en écoulement, l'équation non linéaire obtenue est résolue par la méthode numérique de Runge-Kutta du quatrième ordre. Les fréquences linéaires et non linéaires sont alors déterminées en fonction de l'amplitude du mouvement de la coque pour plusieurs cas.

Cette méthode combine les avantages de la méthode des éléments finis qui traite des coques complexes et la précision de la formulation basée sur des fonctions de déplacement dérivées de la théorie des coques. Nous avons ici un modèle puissant pour prédire les caractéristiques vibratoires linéaires et non linéaires des coques cylindriques ouvertes ou fermées, non uniformes dans la direction circonférencielle et soumises à un fluide en écoulement.

ABSTRACT

The analysis of thin shells subjected to a flowing flow has been the focus of many investigations. Most of the research has involved analysis of linear thin closed cylindrical shells with and without interaction between the structure and the surrounding fluid medium. Very little is known concerning the linear and non-linear dynamics of non-uniform open cylindrical shells subjected to a flowing flow.

The purpose of this study is to present a method for the linear and non-linear dynamic analysis of thin, elastic, anisotropic open cylindrical shells submerged and subjected simultaneously to an internal and external flow. The dynamic stability of closed cylindrical shell and the case of an open or closed cylindrical shell partially or completely filled with liquid are also investigated. The method developed is a hybrid of finite element method, classical shell theory and fluid theory.

The open shells are assumed to be freely simply-supported along their curved edges and to have arbitrary straight edge boundary conditions, and the structure may be uniform or non uniform in the circumferential direction.

The first part of this work dealt with the linear analysis of an empty open or closed cylindrical shell. The shell is subdivided into cylindrical panel segment finite elements, the displacement functions are derived from exact solutions Sanders' equations for thin cylindrical shells. Expressions for the mass and stiffness matrices are determined by precise analytical integration. The linear free vibration of open and closed cylindrical shells are studied by this method as well as anisotropic shells and shells having circumferentially varying thicknesses. The results obtained reveal that the frequencies calculated by this method are in good agreement with those obtained by others.

In the second part of this thesis, a theory is presented for the determination of the linear effects of a flowing fluid on the vibration characteristics of an open, anisotropic cylindrical shell submerged and subjected simultaneously to an internal and external flow. The velocity potential and Bernoulli's equation for a liquid finite element yield an expression for fluid pressure as a function of the nodal displacements of the element and three forces (inertial, centrifugal and Coriolis) of the moving fluid. An analytical integration of the fluid pressure over the liquid element leads to three components: mass, stiffness and damping matrices. Calculations are given to illustrate the dynamic behaviour of open and closed cylindrical shells subjected to a flowing fluid and shells partially or completely filled with liquid. Reasonable agreement is found with other theories and experiments.

In the third part of this study, we present a general approach to predict the influence of geometric non-linearities on the free vibration of elastic, thin, orthotropic and non-uniform empty open cylindrical shells. The modal coefficients derived from the Sanders-Koiter non-linear theory of thin shells are obtained for the displacement functions developed in part one. Expressions for the second order and third order non-linear stiffness matrices are then determined through the finite element method. The non-linear equation of motion is solved by the fourth-order Runge-Kutta numerical method. The linear and non-linear natural frequency variations are determined as a function of shell amplitudes for different cases. The results obtained reveal that the frequencies calculated by this method are in good agreement with those obtained by other authors.

The fourth and last part of this research present a model to predict the influence of non-linearities associated with the wall of the shell and with the fluid flow on the dynamic of elastic, thin, orthotropic and non-uniform open cylindrical shells submerged and subjected simultaneously to an internal and external fluid. With the mass, linear and non-linear stiffness matrices for the empty shell and linear matrices for the moving fluid, we develop in this part three non-linear matrices for the fluid by deriving an expression for non-linear pressure as a function of nodal displacements, the inertial, centrifugal and Coriolis forces and a combination of non-linear effects of the fluid flow. The non-linear equation of motion is then solved by the fourth-order Runge-Kutta numerical method. The linear and non-linear natural frequency variations are determined as a function of shell amplitudes for different cases.

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This method combines the advantages of finite element analysis which deals with complex shells, and the precision of formulation which the use of displacement functions derived from shell and fluid theories contributes. We have here, a powerful model to predict linear and non-linear vibrationary characteristics of circumferentially non-uniform open and closed cylindrical shells empty or subjected to a flowing fluid.

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INTRODUCTION

1.1 GÉNÉRALITÉS

Les coques sont des structures utilisées dans différents domaines de l'ingénierie. Pour ne citer que quelques applications mentionnons: l'industrie aérospatiale et aéronautique (fuselages d'avions, fusées, turboréacteurs), l'industrie nucléaire (enceintes des réacteurs), l'industrie navale (composantes de sous-marins et de navires), le domaine pétrolier (réservoirs, pipelines), le génie civil (construction sous forme de dômes et de voiles minces). La connaissance des caractéristiques statiques et dynamiques de ces structures est importante aussi bien pour le chercheur désirant comprendre leur comportement que pour l'ingénieur soucieux d'éviter tout effet destructif lors de leurs utilisations industrielles.

Les coques sont le sujet de recherche par excellence de plusieurs travaux allant de la statique à la dynamique. De nombreuses théories ont été établies. Il est généralement convenu de classer les études pour ce type de structures en considérant des facteurs tels que la courbure, l'anisotropie, les contraintes résiduelles, la variation de l'épaisseur, les grands déplacements, l'inertie de rotation, l'effet du milieu environnant, la forme des bords de la coque, le type des conditions aux rives, etc.

La plupart de ces études ont porté sur l'analyse linéaire des coques minces de révolution fermées avec ou sans interaction entre cette structure et le milieu fluide environnant. On classe le milieu fluide environnant selon les caractéristiques suivantes: le type d'écoulement, la viscosité, la compressibilité, le mouvement de la surface libre, la linéarité ou la non-linéarité des équations gouvernant l'écoulement, etc.

Dans cette étude nous allons développer un nouveau modèle pour l'analyse dynamique et statique des coques cylindriques ouvertes, non uniformes dans la direction circonférentielle, anisotropes et soumises à un fluide en écoulement, dans le domaine linéaire et non linéaire.

Cette étude entre dans le cadre d'un projet de recherche dirigé par le professeur A.A. Lakis et dont le but est de développer un modèle numérique d'une coque quelconque soumise à un écoulement interne et/ou externe.

Les résultats de ces travaux seront utiles pour tout développement de réservoirs sous pression, échangeurs de chaleur, pipelines, fuselages d'avions, construction sous forme de dômes en génie civil. Ces résultats serviront aussi à l'analyse de l'influence des non-linéarités associées à la coque et au fluide en écoulement sur le comportement dynamique du système coque-fluide.

1.2 REVUE BIBLIOGRAPHIQUE

La première tentative pour élaborer une théorie des coques minces en flexion à partir des équations générales de l'élasticité a été réalisée par Aron en 1874. Love en 1888 développait une série d'équations fondamentales décrivant le comportement des coques minces élastiques. Ces équations et les hypothèses sur lesquelles elles sont bâties, sont souvent identifiées comme la première approximation de Love.

Depuis, la théorie des coques minces linéaires a été réexaminée à plusieurs reprises. Reissner (1941) et Knowles et Reissner (1957) ont fait une nouvelle dérivation des équations pour des coordonnées orthogonales. Certaines théories additionnelles ont été proposées: Sanders (1959, 1963), tout en conservant les hypothèses originales de Love, a construit un système d'équations induisant des déformations nulles provenant des mouvements rigides. Ces équations sont identiques à celle dérivées par Byrne (1944), Lu're (1959), Koiter (1960) et Flügge (1973) qui ont présenté, indépendamment les uns des autres, une théorie d'ordre supérieur dans laquelle l'hypothèse portant sur l'épaisseur de la coque est différée dans les formulations. Les effets des contraintes de cisaillement transversal et des déformations de cisaillement ont été examinés par Hildebrand, Reissner et Thomas (1949), Green et Zerna (1950), Reissner (1952), Naghdi (1956, 1957, 1961, 1963). Les résultats du travail de Naghdi (1957) incluent aussi l'effet des forces d'inertie.

Un certain nombre de travaux ont été faits sur l'analyse des coques cylindriques ouvertes, généralement pour un seul type de conditions frontières. Bogner et al. (1967) ont développé un élément fini cylindrique ouvert où les fonctions de déplacement sont de type Hermite de premier ordre. Des travaux similaires ont été réalisés par Cantin et Clough (1968). Olsen et Lindberg (1968) ont développé un élément fini cylindrique à 4 noeuds et 7 degrés de liberté par noeud, où le mouvement de corps rigide a été vérifié. Boyd (1969) analysa des coques ouvertes en solutionnant les équations de Donnels pour des conditions frontières de type simplement supportées. Kurt et Boyd (1971) ont analysé les vibrations libres des coques cylindriques ouvertes avec des conditions aux frontières de type simplement supportées, tout en exprimant les fonctions de déplacement en double série trigonométrique et polynomiale. Petyt (1971) obtient les fréquences naturelles d'une coque cylindrique ouverte ayant les quatre rives encastrées en utilisant les méthodes des éléments finis, de Rayleigh-Ritz et de Kantorivich. Srinivasan et Boby (1976) ont développé une méthode pour l'analyse des coques cylindriques ouvertes et encastrées en utilisant les fonctions de Green. Massalas et al. (1980) ont analysé des coques cylindriques non circulaires en assumant des fonctions de déplacement en double série de cosinus et sinus avec des conditions aux frontières de type simplement supportées.

Blevins (1981) simplifia les travaux de Sewall (1967) pour analyser les coques cylindriques ouvertes. Leissa et al. (1981) ont étudié les vibrations des coques cylindriques ouvertes encastrées sur une rive et libre sur les trois autres. En utilisant la méthode de Rayleigh-Ritz, Tonin et Bies (1979) et Suzuki et Leissa (1985, 1986) ont étudié des coques ouvertes avec une épaisseur variable dans la direction circonférencielle. Srinivasan et Krishnan (1987) ont analysé les coques ouvertes ayant des rives encastrées dans la direction latérale et libres dans les deux autres directions. Cheung et al. (1989) ont utilisé la méthode de "Spline finite strip" pour l'analyse des vibrations forcées des coques ouvertes.

Plus récemment, Kumar et Singh (1993) ont analysé les vibrations des coques cylindriques non circulaires. Cette analyse est basée sur la méthode de Ritz où les déplacements sont une combinaison des fonctions des vecteurs propres des poutres et des fonctions de Bezier. Jiang et Olsen (1994) ont développé un élément fini pour l'analyse des coques cylindriques orthotropiques. Ils ont utilisé des fonctions de déplacement polynomiales et analytiques. Leissa (1973) a regroupé dans une excellente référence le travail de plusieurs recherches.

Quand le déplacement de la paroi d'une coque est supérieur à son épaisseur, une théorie non linéaire est alors nécessaire pour l'étude du comportement de cette coque. Plusieurs théories traitant des non-linéarités géométriques pour des coques de forme arbitraire ont été développées. Reissner (1963) est considéré comme un pionnier dans l'analyse des effets de la non-linéarité géométrique sur la dynamique des coques cylindriques. Actuellement, plusieurs théories sont disponibles pour décrire la non-linéarité géométrique dans les coques . Citons celles développées par Naghdi et Nordgren (1963), Sanders (1963), Koiter (1966), Yokoo et Matsunaga (1974).

Beaucoup de méthodes ont été développées pour l'étude des vibrations non linéaires des coques cylindriques. Parmi celles-ci, la méthode de Galerkin [Nowinski (1963), Evensen (1967), Dowell et Ventres (1968), Leissa et Kadi (1971), Birman et Bert (1987), Raouf et Palazotto (1994) et Kobayashi et Leissa (1995)], la méthode des petites perturbations [Alturi (1972), Ginsberg (1972) et Chen et Babcock (1975)], la méthode d'expansion modale [Meirovitch (1967) et Radwan et Genin (1975)] et la méthode des éléments finis [Raju et Rao (1976), Basar et Ding (1990), Tsai et Palazotto (1991) et Jiang et Olsen (1991)]. La plupart de ces recherches ont été faites sur des coques isotropes.

Seulement Nowinski (1963), Raouf et Palazotto (1994) et Jiang et Olsen (1991) ont développé des modèles pour des coques orthotropes. Ambrasumyan (1961) a produit un important travail pour l'analyse de l'anisotropie dans les coques.

La réponse des coques due au fluide en écoulement a été étudiée par plusieurs chercheurs qui ont appliqué plusieurs techniques telles que les fonctions de Green avec Cottis et Jasonides (1964), les fonctions de Dirac Delta avec Nasser (1968), la méthode de Join-Acceptance avec Clinch (1970), la théorie de Timoshenko avec Magrab et Burroughs (1972), les fonctions de Transfert par Cottis (1968), la méthode de Rayleigh avec Dym (1970) et la simulation numérique et les équations de Fokker-Planck (Nash et Kahematsu, 1972).

Fung (1957) analysa le spectre de fréquence et les modes de vibration d'une coque cylindrique avec un écoulement interne. Kuleshov et al. (1971) ont proposé une méthode pour analyser les coques cylindriques avec fond plat rigide et partiellement remplies de liquide. Housner et al. (1958) ont présenté une série de représentations du comportement oscillatoire d'une coque avec un fluide incompressible non visqueux. Jain (1974) a étudié le comportement des vibrations des coques cylindriques orthotropes partiellement ou entièrement remplies de fluide incompressible et non visqueux.

L'influence de la vitesse d'écoulement sur les vibrations libres d'une coque cylindrique a été décrite par plusieurs chercheurs. Lindholm et al. (1963) ont étudié les vibrations d'une coque cylindrique avec un écoulement interne. Paidoussis et Denis (1972) et Weaver et Unny (1973) ont étudié l'instabilité dans des coques cylindriques avec un fluide en écoulement. Des travaux ont aussi été réalisés sur les vibrations des coques avec une excitation aléatoire [Townsend (1956), Corcos (1963), Lakis et Paidoussis (1972b), Mulhearn (1975)].

Plus récemment, la dynamique du système coque-fluide a été examinée par plusieurs auteurs. Brown (1982), Au-Yang (1986) et Paidoussis et Li (1993) ont fait une revue bibliographique élaborée dans ce domaine. Ces dernières années plusieurs articles sont parus: Citons ceux de Mistry et Menenzes (1995), Harari et al. (1994), Cheng (1994), Han et Liu (1994), Terhune et Karim-Panahi (1993), Brenneman et Au-Yang (1992), Endo et Tosaka (1989) et Goncalves et Batista (1987).

La solution analytique des équations de mouvement des coques minces est généralement difficile; seules les méthodes d'approximation sont utilisées. Parmi celles-ci il y a la méthode des différences finies, la méthode de Galerkin, la méthode de Rayleigh-Ritz, la méthode des matrices de transfert et celle des éléments finis. Toutes ces méthodes ont des avantages et des inconvénients; un des critères importants de succès d'une méthode est sa capacité de prédire aussi bien les hautes que les basses fréquences et les modes propres correspondants avec une bonne précision.

Dans la méthode des différences finies, on donne à priori des valeurs initiales de la fréquence. Cette procédure exige beaucoup de temps de calcul; d'autre part, elle ne détermine pas tout le spectre de vibration. De même, la méthode de Galerkin perd sa précision aux hautes fréquences de la coque. La méthode des matrices de transfert a fait ses preuves ces dernières années, ceci par les travaux de Dupuis et Rousselet (1985) et Tran Van (1987), mais présentement les résultats obtenus ne sont valables que pour des coques cylindriques. La méthode de Rayleigh-Ritz et la méthode des éléments finis répondent à ce critère. Elles ramènent le problème de vibrations à un problème symétrique de valeurs propres.

La méthode de Rayleigh-Ritz présente des inconvénients parmi lesquels on retrouve le choix des fonctions de déplacement qui doit tenir compte des conditions aux rives et la nécessité de retenir un grand nombre de termes pour l'expression des fonctions de déplacement. La méthode des éléments finis [Zienkiewicz (1977), Tinawi (1981), Datt et Touzot (1984), Gallagher (1985) et Yang (1986)] est, par contre, satisfaisante de ces points de vue. La coque est modélisée par un assemblage d'éléments finis. La précision de la méthode dépend de la nature de ces éléments et des degrés de liberté retenus pour stimuler le comportement de la coque.

De nombreux programmes généraux de calcul permettent d'utiliser industriellement la méthode des éléments finis, principalement dans le domaine de la mécanique des solides. Citons par exemple: MSCPAL, ABAQUS, ADINA et NASTRAN.

En général les éléments triangulaires et quadrilatères sont utilisés où les fonctions de déplacement sont polynomiales. Pour augmenter la précision, on a été amené à choisir des éléments courbes qui modélisent mieux la géométrie de l'enveloppe. La formulation analytique de ces éléments est complexe.

L'un des critères les plus importants pour déterminer la flexibilité d'une méthode est sa capacité à solutionner le problème d'interaction coque-fluide pour les hautes fréquences aussi bien pour que les basses, avec une bonne précision. Ce critère nécessite l'utilisation d'un très grand nombre d'éléments dans la méthode des éléments finis. Pour pallier à ce défaut l'équipe de recherche dirigée par le professeur Lakis a développé un nouveau type d'éléments finis. Ce sont des éléments hybrides où les fonctions de déplacement de la méthode des éléments finis sont dérivées de la théorie des coques. Cette méthode a été appliquée aux analyses statiques et dynamiques des coques fermées de révolution. Les coques cylindriques fermées ont fait l'objet de plusieurs études dans le domaine linéaire et non linéaire, isotrope et anisotrope, uniforme et axialement non uniforme, vide, partiellement ou complètement remplie de liquide, avec ou sans écoulement, liquide à une phase ou biphasique [Lakis et Paidoussis (1971, 1972a, 1972b, 1973), Lakis (1976a, 1976b), Lakis et Doré (1978), Lakis, Sami et Rousselet (1978), Lakis et Laveau (1991), Lakis et Sinno (1992)]. D'autres travaux ont été faits sur les coques coniques (Lakis, Van Dyke et Ouriche, 1992) et sphériques (Lakis, Tuy et Selmane, 1989), ainsi que sur des plaques circulaires et annulaires [Lakis et Selmane (1990a, 1990b)].

1.3 BUT DE LA RECHERCHE

Une synthèse de la revue bibliographique nous mène à dire que les coques ont fait l'objet de plusieurs travaux dans le domaine statique et dynamique, avec ou sans écoulement. De nombreuses théories ont été établies. Vu la difficulté de solutionner les équations différentielles des coques, les résultats généralement obtenus ne sont valables que pour des coques fermées de révolution. Peu de travaux ont été faits pour des coques ouvertes avec ou sans fluide en écoulement. Rares sont les méthodes capables de déterminer les hautes fréquences du système coque-fluide avec autant de précision que les basses fréquences, et ce dans le domaine linéaire et non linéaire.

Nous proposons de développer un modèle pour l'analyse linéaire et non linéaire des coques minces, élastiques, anisotropes, ouvertes et soumises à un écoulement interne et externe. La stabilité dynamique des coques cylindriques fermées et le cas des coques partiellement ou complètement remplies de liquide sont aussi analysés. La méthode développée est une combinaison de la méthode des éléments finis, de la théorie des coques et de celle des fluides.

Les coques ouvertes sont simplement supportées selon leurs rives courbes et elles ont des conditions frontières arbitraires sur les rives droites. La structure peut être uniforme ou non uniforme dans la direction circonférencielle.

L'équation du mouvement du système coque-fluide, en tenant compte des matrices linéaires et non linéaires associées à la coque et au fluide en écoulement, peut s'écrire de la façon suivante:

$$\begin{aligned} [[M_s] - [M_f^{(L)}]]\,\{\ddot{\delta}\} - [C_f^{(L)}]\,\{\dot{\delta}\} + [[K_s^{(L)}] - [K_f^{(L)}]]\,\{\delta\} \\ + [K_s^{(NL2)}]\,\{\delta^2\} + [K_s^{(NL3)}]\,\{\delta^3\} \\ - [C_f^{(NL)}]\,\{\dot{\delta}^2\} - [KC_f^{(NL)}]\,\{\delta\,\dot{\delta}\} - [K_f^{(NL)}]\,\{\delta^2\} = \{0\} \end{aligned} \tag{1.1}$$

où: $\{\delta\}$ est le vecteur déplacement; $[M_s]$, $[K_s^{(L)}]$ sont les matrices de masse et de rigidité linéaire de la coque vide; $[K_s^{(NL2)}]$, $[K_s^{(NL3)}]$ sont les matrices de rigidité non linéaire du second et troisième ordre de la coque vide; $[M_f^{(L)}]$, $[C_f^{(L)}]$ et $[K_f^{(L)}]$ sont respectivement les matrices linéaires associées aux forces d'inertie, de Coriolis et centrifuge dues au fluide; $[C_f^{(NL)}]$, $[KC_f^{(NL)}]$ et $[K_f^{(NL)}]$ sont les matrices non linéaires associées au fluide en écoulement.

Dans cette thèse, notre objectif est donc de trouver les matrices $[M_s]$, $[K_s^{(L)}]$, $[K_s^{(NL2)}]$, $[K_s^{(NL3)}]$, $[M_f^{(L)}]$, $[C_f^{(L)}]$, $[K_f^{(L)}]$, $[C_f^{(NL)}]$, $[KC_f^{(NL)}]$ et $[K_f^{(NL)}]$. Nous résoudrons dans cette étude l'équation du mouvement du système coque-fluide (1.1) afin de déterminer les modes de vibration et les fréquences naturelles du système dans le cas linéaire, de trouver l'influence du fluide sur une coque cylindrique ouverte ou fermée et de prédire, dans le cas non linéaire, l'influence des non-linéarités associées aux parois de la coque et au fluide en écoulement dans une coque ouverte, élastique, mince, orthotropique, non uniforme, submergée et soumise simultanément à un écoulement interne et externe.

Cette étude entre dans le cadre d'un large projet de recherche dirigé par le professeur A.A. Lakis, et ayant pour but d'analyser dynamiquement une coque quelconque avec ou sans fluide en écoulement.

1.4 PLAN DE LA THÈSE

Cette thèse est répartie en six principaux chapitres et est organisée de la manière suivante.

A la suite de cette introduction, le deuxième chapitre présente l'analyse linéaire des coques cylindriques vides. Cette étude est présentée sous la forme d'un article intitulé: **"Dynamic Analysis of Anisotropic Open Cylindrical Shells"** (Selmane et Lakis, 1995a). Cet article développe un nouvel élément fini pour la détermination des vibrations libres des coques cylindriques ouvertes et fermées.

Au troisième chapitre, nous développons un modèle pour l'analyse dynamique des coques cylindriques ouvertes soumises à un fluide en écoulement. Ce modèle est présenté dans l'article intitulé: **"Vibration Analysis of Anisotropic Open Cylindrical Shells Containing Flowing Fluid"** (Selmane et Lakis, 1995b). Cet article présente l'analyse de la stabilité dynamique des coques cylindriques ouvertes soumises simultanément à un écoulement interne et externe, ainsi que l'analyse des coques partiellement ou complètement remplies de liquide.

Le quatrième chapitre présente l'analyse non linéaire des coques cylindriques ouvertes vides. Cette analyse est présentée sous la forme d'un article intitulé: **"Influence of Geometric Non-Linearities on the Vibrations of Orthotropic Open Cylindrical Shells"** (Selmane et Lakis, 1996a).

Dans le chapitre cinq, nous résolvons le problème général de l'interaction coque-fluide en tenant compte des non-linéarités associées aux parois de la coque et au fluide en écoulement. Ce travail est présenté dans l'article intitulé: **"Non-Linear Dynamic Analysis of Orthotropic Open Cylindrical Shells Subjected to a Flowing Fluid"** (Selmane et Lakis, 1996b).

Finalement, nous conclurons et élaborerons sur les avenues à de recherche potentielle.

CHAPITRE II

ARTICLE I

DYNAMIC ANALYSIS OF ANISOTROPIC OPEN CYLINDRICAL SHELLS

2.1 ABSTRACT

This paper presents a method for the dynamic and static analysis of thin, elastic, anisotropic and non-uniform open cylindrical shells.

The open shells are assumed to be freely simply-supported along their curved edges and to have arbitrary straight-edge boundary conditions. The method is a hybrid of finite element method and classical shell theories.

The shell is subdivided into cylindrical panel segment finite elements, the displacement functions are derived from Sanders' equation for thin cylindrical shells. Expressions for the mass and stiffness are determined by precise analytical integration.

The free vibration of open and closed cylindrical shells are studied by this method as well as anisotropic shells and shells having circumferentially varying thicknesses. The results obtained reveal that the frequencies calculated by this method are in good agreement with those obtained by others.

2.2 INTRODUCTION

The analysis of thin shells under static or dynamic load has been the focus of many theories. Most of the research in this field has involved analysis of linear thin closed cylindrical shells. Very little is known concerning the dynamics of open cylindrical shells with circumferentially varying geometry and material properties.

This paper presents a method for the dynamic and static analysis of thin, elastic, anisotropic and circumferentially non-uniform open cylindrical shells.

The first attempt to formulate a bending theory of thin shells from the general equations of elasticity was made by Aron in 1874, and was followed by a successful approximate theory known as Love's first approximation [1]. Since then the theory of elastic shells has repeatedly been re-examined, [2]-[8].

Open cylindrical shells (panels) have been analyzed by a number of authors. In general, the finite element method was used for solving these problems [9]-[16]. Various types of finite elements were used and a polynomial displacement functions were assumed.

Boyd [17] analysed a simply supported open cylindrical shell by solving Donell equations. Kurt and Boyd [18] used a trigonometric and polynomials displacement function and solved the dynamics of simply supported cylindrical panels. Strinivasan and Bobby [19] developed a matrix method for analysis of clamped cylindrical shell panels by using a Green function. Massalas et al. [20] analyzed a non-circular cylindrical panel by choosing a double series of cosine and sine for the displacement functions.

Blevins [21] simplified the work of Sewall [22] by studying an open cylindrical shell. Leissa et al. [23] analyzed the vibration of cantilevered cylindrical panels by using the Ritz method, with algebraic polynomial trial functions for the displacements.

Tonin and Bies [24] used the Rayleigh-Ritz method; Suzuki and Leissa [25]-[26], analysed the free vibration of circular and non-circular cylindrical shells having circumferentially varing thickness. Srinivasan and Krishnan [27] calculated the natural frequencies of cylindrical panels with clamped edges in the lateral direction and free to move in the in-plane directions. Cheung et al. [28] applied the Spline finite strip method to the forced vibration analysis of a singly curved shell panel.

Recently, Kumar and Singh [29] analysed the vibration of non-circular cylindrical shells. This analysis is based on the Ritz method in which a combination of eigenfunctions for beams and quintic Bezier functions are used to represent the displacement. Jiang and Olsen [30] developed a finite element to analyse the vibration of orthogonally stiffened cylindrical shells and panels. They used a combination of polynomials and analytical functions to formulate the displacement functions.

Leissa [31] collected the work of several researchers into one excellent book. We find in different types of shells and panels, a particular study of Heki [32] in analytical and experimental analysis have been used to compare with our study. In that work the solution is developed for the Donnell-Mushtari theory neglecting tangential inertia, where the straight edges of the panel are free and the others edges are supported by shear diaphragms.

One of the most important criteria in determining the versatility of a method is the capacity to predict, with precision, both the high and the low frequencies. This criterion demands the use of a great many elements in the finite element method, and in order to meet it, our research group has developed a hybrid type of finite element, wherein the displacement functions in the finite element method are derived from Sanders' classical shell theory [5]. This method has been applied with satisfactory results to the dynamic linear and non-linear analysis of cylindrical [33]-[39], conical [40], spherical [41], isotropic and anisotropic, uniform and axially non-uniform shells, both empty and liquid-filled. This method has also been applied to the dynamic analysis of circular and annular plates [42], [43].

The purpose of this study is to explore the static and dynamic analysis of thin, elastic, anisotropic and non-uniform open cylindrical shells subjected to a flowing fluid. Here we consider the problem of panels which are freely simply-supported along their curved edges and have arbitrary straight edge boundary conditions. The effect of the flowing fluid on the natural frequencies of these panels will be the subject of a later work.

2.3 FUNDAMENTAL EQUATIONS FOR OPEN CYLINDRICAL SHELLS

Sanders' thin shell theory [5] is used in order to obtain the equations of motion. These equations are based on Love's first approximation [1] and give zero strain for small rigid-body motion, this is not the case with other theories. The geometry of the mean surface of the shell studied and the coordinates used are shown in Figure 2.1.

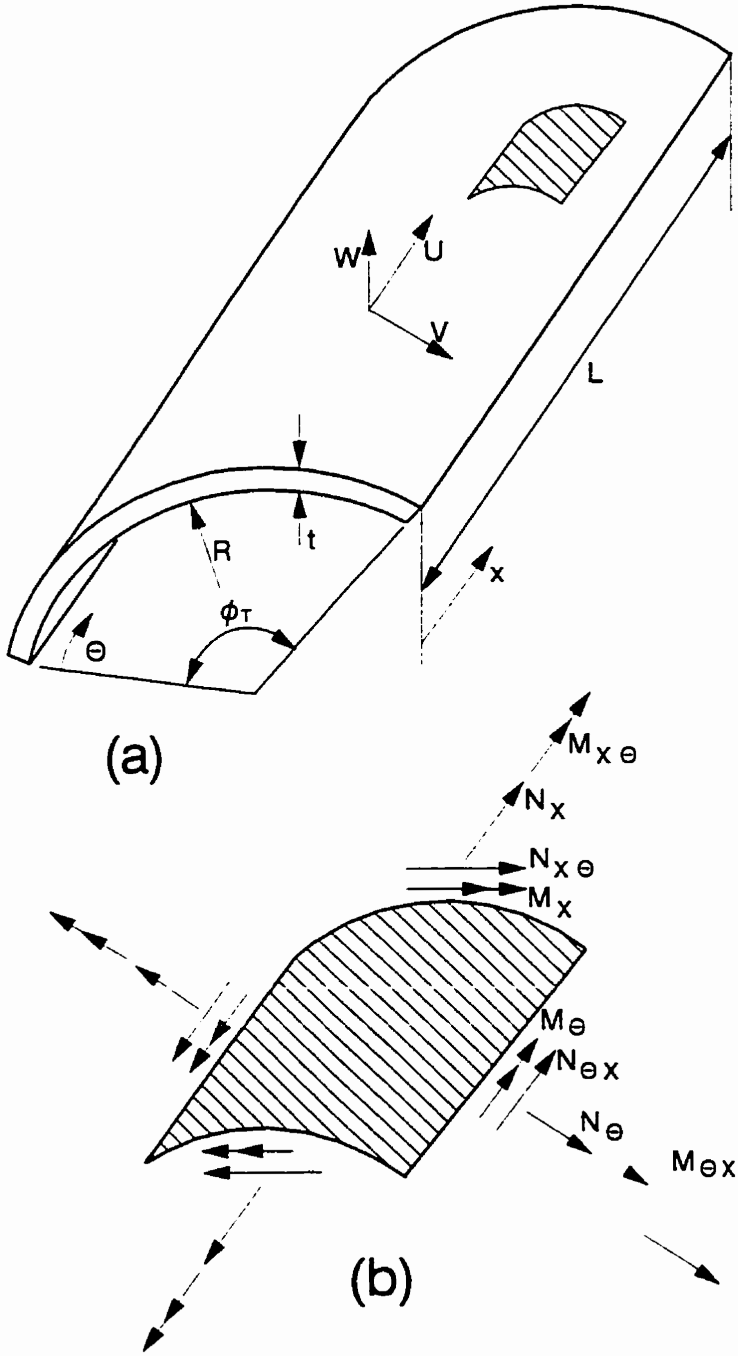

Figure 2.1 **(a) Open cylindrical shell geometry.**
(b) Differential elment for an open cylindrical shell.

The equilibrium equations of an open cylindrical shell may be written as follows:

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{R}\frac{\partial \bar{N}_{x\theta}}{\partial \theta} - \frac{1}{2R^2}\frac{\partial \bar{M}_{x\theta}}{\partial \theta} = 0$$

$$\frac{1}{R}\frac{\partial N_{\theta\theta}}{\partial \theta} + \frac{\partial \bar{N}_{x\theta}}{\partial x} + \frac{3}{2R}\frac{\partial \bar{M}_{x\theta}}{\partial x} + \frac{1}{R^2}\frac{\partial M_{\theta\theta}}{\partial \theta} = 0 \qquad (2.1)$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{2}{R}\frac{\partial^2 \bar{M}_{x\theta}}{\partial x\,\partial \theta} + \frac{1}{R^2}\frac{\partial^2 M_{\theta\theta}}{\partial \theta^2} - \frac{N_{\theta\theta}}{R} = 0$$

where N_x, N_θ, $\bar{N}_{x\theta}$, M_x, M_θ and $\bar{M}_{x\theta}$ are the stress components and x and θ are the coordinates of the shell.

The strain vector of the middle surface is:

$$\{\epsilon\} = \{\epsilon_x,\ \epsilon_\theta,\ 2\bar{\epsilon}_{x\theta},\ \kappa_x,\ \kappa_\theta,\ 2\bar{\kappa}_{x\theta}\}^T$$

where ϵ_x, ϵ_θ are the in-plane tensile or compressive strains, $2\bar{\epsilon}_{x\theta}$ is the in-plane shear, κ_x, κ_θ are the bending components and $2\bar{\kappa}_{x\theta}$ is the torsion of middle surface during deformation. For a linear elastic behaviour, the strain vector is related to the displacements through the following equation:

$$\{\epsilon\} = \left\{ \begin{array}{c} \dfrac{\partial U}{\partial x} \\ \dfrac{1}{R}\left(\dfrac{\partial V}{\partial \theta} + W\right) \\ \dfrac{\partial V}{\partial x} + \dfrac{1}{R}\dfrac{\partial U}{\partial \theta} \\ \dfrac{-\partial^2 W}{\partial x^2} \\ \dfrac{1}{R^2}\left(\dfrac{\partial^2 W}{\partial \theta^2} - \dfrac{\partial V}{\partial \theta}\right) \\ -\dfrac{2}{R}\dfrac{\partial^2 W}{\partial x\,\partial \theta} + \dfrac{3}{2R}\dfrac{\partial V}{\partial x} - \dfrac{1}{2R^2}\dfrac{\partial U}{\partial \theta} \end{array} \right\} \qquad (2.2)$$

where U, V, W are axial, tangential and radial displacements.

For an anisotropic and elastic material, the constitutive equation which links the stress vector to the strain vector is:

$$\{\sigma\} = [P]\{\epsilon\} \qquad (2.3)$$

where [P] the elasticity matrix may be given as follows:

$$[P] = \begin{bmatrix} P_{11} & P_{12} & 0 & P_{14} & P_{15} & 0 \\ P_{21} & P_{22} & 0 & P_{24} & P_{25} & 0 \\ 0 & 0 & P_{33} & 0 & 0 & P_{36} \\ P_{41} & P_{42} & 0 & P_{44} & P_{45} & 0 \\ P_{51} & P_{52} & 0 & P_{54} & P_{55} & 0 \\ 0 & 0 & P_{63} & 0 & 0 & P_{66} \end{bmatrix} \tag{2.4}$$

For isotropic material, the only non-vanishing terms are:

$$\begin{aligned} P_{11} &= P_{12} = D & P_{44} &= P_{55} = K \\ P_{12} &= P_{21} = \nu D & P_{45} &= P_{54} = \nu K \\ P_{33} &= \frac{(1-\nu)}{2} D & P_{66} &= \frac{(1-\nu)}{2} K \end{aligned} \tag{2.5}$$

where D, the membrane stiffness and K, the bending stiffness, are given by:

$$D = \frac{Et}{1-\nu^2} \qquad K = \frac{Et^3}{12(1-\nu^2)}$$

E being Young's modulus, ν Poisson's ratio and t the shell thickness.

The elements P_{ij} of [P] characterize the shell's anisotropy which depends on the mechanical properties of the material of the structure.

By substituting equations (2.2) and (2.3) in the equilibrium equations (2.1), we obtain new equations (2.6) in terms of axial, tangential and radial displacements (U, V, W) of the mean surface of the shell and in terms of the element P_{ij} of the matrix of elasticity [P], these equations are:

$$
\begin{aligned}
L_1\ (U,V,W,P_{ij}) &= 0 \\
L_2\ (U,V,W,P_{ij}) &= 0 \\
L_3\ (U,V,W,P_{ij}) &= 0
\end{aligned}
\tag{2.6}
$$

where L_k ($k = 1, 2, 3$) are three linear differential operators, the form of which is given in Appendix 2.1.

The solution of equations (2.6) will permit us to derive the displacement functions.

2.4 DISPLACEMENT FUNCTIONS

The finite element used in this theory, as shown in Figure 2.2, is a cylindrical panel segment defined by two nodal lines i and j. As stated in the introduction, in the present method, we employ the equilibrium equations of this cylindrical shell to obtain the pertinent displacement function, instead of using the more common arbitrary polynomial forms.

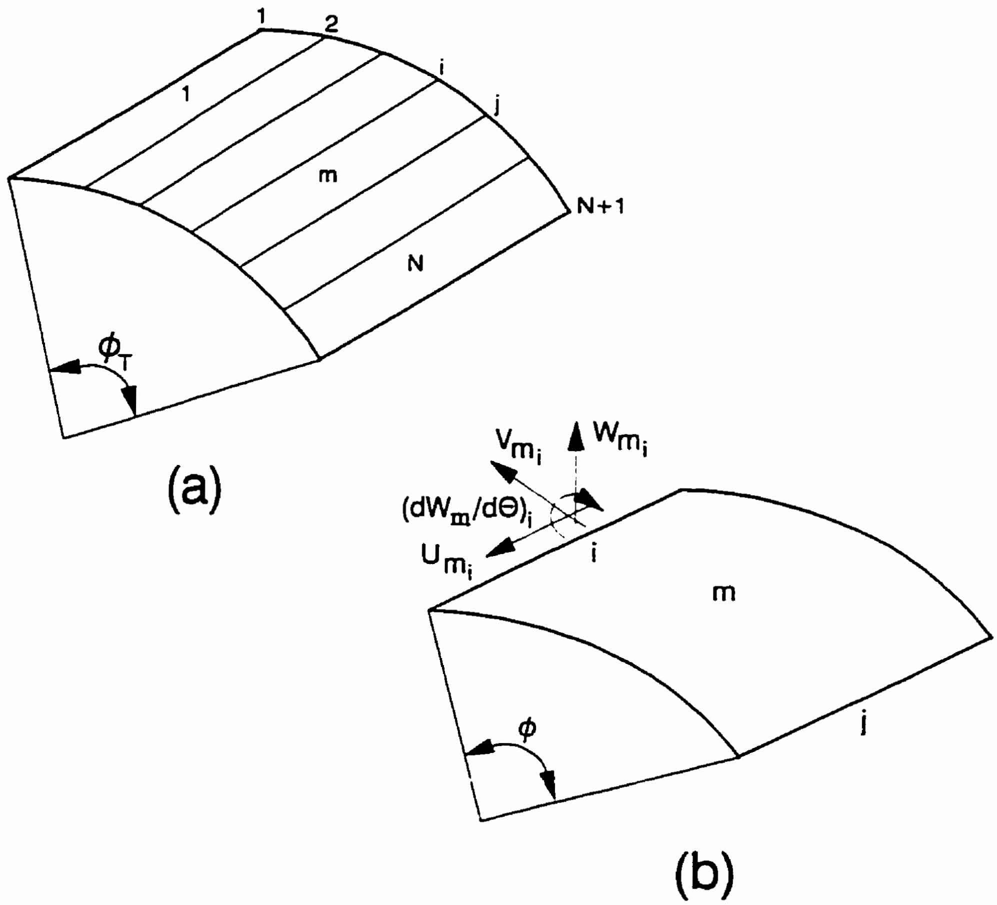

Figure 2.2 (a) Finite element idealization.
(b) Nodal displacements at node i for the finite element m.
N: number of finite elements.

By assuming that the panels are to be freely supported (V = W = 0) along their curved edges, the displacements are periodic functions of x, and therefore, they may be developed into a Fourier series as follows:

$$\{U(x,\theta),\ W(x,\theta),\ V(x,\theta)\}^T = \sum_{m=1}^{\infty} [T_m]\ \{U_m(\theta),\ W_m(\theta),\ V_m(\theta)\}^T \qquad (2.7)$$

where m is the axial wave number and $[T_m]$ is a 3 x 3 square diagonal matrix given in Appendix 2.2. U_m, W_m, V_m are the magnitudes of the deflections and depend on θ only.

Upon substituting equation (2.7) into equation (2.6), we obtain three ordinary differential equations in U_m, W_m and V_m. Solutions of these equations have the general form [8]:

$$U_m(\theta) = \bar{A}\ e^{\eta\theta} \qquad V_m(\theta) = \bar{B}\ e^{\eta\theta} \qquad W_m(\theta) = \bar{C}\ e^{\eta\theta} \qquad (2.8)$$

where η is a complex number.

The substitution of equation (2.8) into equations (2.6) yield three ordinary linear equations in $\bar{A}$, $\bar{B}$ and $\bar{C}$ of the form:

$$[H] \begin{Bmatrix} \bar{A} \\ \bar{B} \\ \bar{C} \end{Bmatrix} = \{0\} \qquad (2.9)$$

For a non-trivial solution of (2.9), the determinant of [H] must vanish yielding the following characteristic equation:

$$h_8\,\eta^8 + h_6\,\eta^6 + h_4\,\eta^4 + h_2\,\eta^2 + h_0 = 0 \qquad (2.10)$$

The expressions for [H] and hi are given in Appendix 2.2.

Equation (2.10) provides for eight complex roots, the complete solution is a linear combination of these eight solutions:

$$\begin{aligned} U_m(\theta) &= \sum_{i=1}^{8} \bar{A}_i\, e^{\eta_i \theta} \\ V_m(\theta) &= \sum_{i=1}^{8} \bar{B}_i\, e^{\eta_i \theta} \\ W_m(\theta) &= \sum_{i=1}^{8} \bar{C}_i\, e^{\eta_i \theta} \end{aligned} \qquad (2.11)$$

A_i, B_i and C_i are not independent, we shall next express the A_i and B_i in terms of C_i as:

$$\bar{A}_i = \alpha_i\, \bar{C}_i \qquad \bar{B}_i = \beta_i \bar{C}_i \qquad i = 1,2,\dots\ 8 \qquad (2.12)$$

where α_i and β_i are complex. Substituting equation (2.12) into equation (2.9), we may now determine α_i and β_i by solving the simple Cramer system:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{Bmatrix} \alpha_i \\ \beta_i \end{Bmatrix} = \begin{Bmatrix} -H_{13} \\ -H_{23} \end{Bmatrix} \tag{2.13}$$

H_{ij} are the terms of matrix [H] given in Appendix 2.2.

The final form of U, V and W may be written as:

$$\begin{Bmatrix} U \\ W \\ V \end{Bmatrix} = [T_m]\,[R]\,\{\,C\,\} \tag{2.14}$$

where $[T_m]$ and [R] are shown in Appendix 2.2 and $\{C\} = \{C_1, \ldots, C_8\}^T$ is a set of constants. The C_i ($i = 1,8$) are the only free constants in our problem and must be determined from eight boundary conditions, four at each edge of constant θ.

We are now in position to specify the displacement function. At each node in Figure 2.2, the axial, circumferential and radial displacements, as well as a rotation, will be prescribed. The displacement of node i can thus be defined by the vector:

$$\{\delta_i\} = \left\{ U_{mi}, W_{mi}, \left(\frac{dW_m}{d\theta}\right)_i, V_{mi} \right\}^T \tag{2.15}$$

where all these components represent amplitudes of U,V, W and dW/dθ associated with the m th axial wave number. The element, having two nodes and eight degrees of freedom, will have the following nodal displacements:

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \begin{Bmatrix} U_{mi} \; W_{mi} \left(\frac{dW_n}{d\theta} \right)_i \; V_{mi} \; U_{mj} \; W_{mj} \left(\frac{dW_m}{d\theta} \right)_j \; V_{mj} \end{Bmatrix}^T = [A]\{C\} \qquad (2.16)$$

where [A] is given in Appendix 2.2, the terms of [A] being obtained from the terms of [R].

Now, pre-multiplying by $[A^{-1}]$, we obtain:

$$\{C\} = [A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (2.17)$$

and substituting into equation (2.14), we obtain:

$$\begin{Bmatrix} U \\ W \\ V \end{Bmatrix} = [T] \; [R] \; [A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (2.18)$$

The displacement function is defined by:

$$[N] = [T] \; [R] \; [A^{-1}] \qquad (2.19)$$

2.5 STRESS VECTOR

The strain vector may be found by using equations (2.2) and (2.18):

$$\{\epsilon\} = \begin{bmatrix} [T_m] & [0] \\ [0] & [T_m] \end{bmatrix} [Q]\,[A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{2.20}$$

where the matrices [T], [A] and [Q] are given in Appendix 2.2.

Referring to equation (2.3), the stress vector is given as:

$$\{\sigma\} = [P]\,\{\epsilon\} = [P]\,[B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{2.21}$$

2.6 MASS AND STIFFNESS MATRICES FOR ONE FINITE ELEMENT

Following the framework of the finite element approach [44], the mass and stiffness matrices may be expressed as:

$$[m] = \rho t \int_o^L \int_o^\phi [N]^T\,[N]\;dA \tag{2.22}$$

$$[k] = \int_0^L \int_0^\phi [B]^T \; [P] \; [B] \; dA \tag{2.23}$$

where dA = rdxdθ. Here, [N], [B] and [P] are defined in equations (2.19), (2.20) and (2.3). Using these equations in equations (2.22), (2.23) and integrating over x and θ, we obtain:

$$[m] = [A^{-1}]^T \; [S] \; [A^{-1}] \tag{2.24}$$

$$[k] = [A^{-1}]^T \; [G] \; [A^{-1}] \tag{2.25}$$

where [S] and [G] are defined by the above equations:

$$S(i,j) = \frac{RL}{2} \; \frac{(\alpha_i \; \alpha_j + \beta_i \; \beta_j + 1)}{(\eta_i + \eta_j)} \; (e^{(\eta_i+\eta_j)\phi} - 1) \quad \text{if } \eta_i + \eta_j \neq 0 \tag{2.26}$$

$$S(i,j) = \frac{RL \; \phi}{2} \; (\alpha_i \; \alpha_j + \beta_i \; \beta_j + 1) \qquad \text{if } \eta_i + \eta_j = 0 \tag{2.27}$$

$$G(i,j) = \frac{RL}{2} (P_{11} A_i A_j + P_{12} A_i B_j + P_{14} A_i D_j + P_{15} A_i E_j$$
$$+ P_{21} B_i A_j + P_{22} B_i B_j + P_{24} B_i D_j + P_{25} B_i E_j$$
$$+ P_{41} D_i A_j + P_{42} D_i B_j + P_{44} D_i D_j + P_{45} D_i E_j \qquad (2.28)$$
$$+ P_{51} E_i A_j + P_{52} E_i B_j + P_{54} E_i D_j + P_{55} E_i E_j$$
$$+ P_{33} C_i C_j + P_{36} C_i F_j + P_{63} F_i C_j + P_{66} F_i F_j)$$
$$\frac{(e^{(\eta_i + \eta_j)\phi} - 1}{(\eta_i + \eta_j)} \qquad \text{if } \eta_i + \eta_j \neq 0$$

$$G(i,j) = \frac{RL\ \phi}{2} (P_{11} A_i A_j + \ldots\ldots + P_{66} F_i F_j) \quad \text{if } \eta_i + \eta_j = 0 \qquad (2.29)$$

where η_i $(i=1,\ldots,8)$ are the complex roots of the characteristic equation (2.10), α_i and β_i are the solutions of system (2.13), R is the mean radius of the shell, L its length, ϕ is the angle for one finite element and P_{ij} are the terms of elasticity matrix.

The terms A_i, B_i, C_i, D_i, E_i and F_i $(i = 1, \ldots, 8)$ may be expressed as follows:

$$A_i = - \frac{m \pi \alpha_i}{L}, \tag{2.30}$$

$$\bar{B}_i = - \frac{\eta_i \beta_i + 1}{R}, \tag{2.31}$$

$$C_i = - \frac{m \pi \beta_i}{L} + \frac{\eta_i \alpha_i}{R} \tag{2.32}$$

$$D_i = - \frac{(m \pi)^2}{L^2}, \tag{2.33}$$

$$E_i = - \frac{\eta_i^2 + \eta_i \beta_i}{R^2} \tag{2.34}$$

and $$F_i = - \frac{2 m \pi \eta_i}{RL} + \frac{3 m \pi \beta_i}{2 RL} - \frac{\eta_i \alpha_i}{2R^2} \tag{2.35}$$

2.7 THE GLOBAL MASS AND STIFFNESS MATRICES

The complete shell or panel is divided into finite elements each of which is a cylindrical segment panel. The position of the nodal points (nodal lines) may be chosen arbitrarily. With the mass and stiffness matrices known of each element, the global mass and stiffness matrices for the whole structure, M and K, respectively, may be constructed by superposition in the normal manner. Each of these square matrices will be of order 4(N+1), where N is the total number of finite elements (see Figure 2.2).

If the panel has in the straight edges constraints such as simply-supported, clamped, etc., the appropriate lines and columns in [M] and [K] are deleted to satisfy these constraints. Consequently, matrices [M] and [K] reduce to square matrices of order 4(N+1)-J, where J is the number of constraints applied. Thus, for a closed cylindrical shell, free simply supported along its curved edges, no specification of boundary conditions need be made and J = 0. For this case we connect the last node of the structure to the first node with the total angle ϕ_T equal 360°. For a panel with two straight edges clamped we have J = 8.

2.8 ANALYSIS OF AN OPEN SHELLS SUBJECTED TO STATIC LOADS

The study of the static equilibrium is carried out in the following manner:

When: $\{F_A\}$ is the vector of the forces applied to the nodes of the shell

$\{F_B\}$ is the vector of unknown reactions

$\{\delta_A\}$ is the vector of unknown nodal displacements

$\{\delta_B\}$ is the vector of displacements defined by the boundary conditions

The static equilibrium equation $[K]\{\delta\} = \{F\}$ becomes

$$\begin{bmatrix} K_{AA} & K_{AB} \\ K_{BA} & K_{BB} \end{bmatrix} \begin{Bmatrix} \delta_A \\ \delta_B \end{Bmatrix} = \begin{Bmatrix} F_A \\ F_B \end{Bmatrix} \tag{2.36}$$

We have therefore

$$\begin{aligned} \{\delta_A\} &= [K_{AA}]^{-1} (\{F_A\} - [K_{AB}]\{\delta_B\}) \\ \{F_B\} &= [K_{BA}]\{\delta_A\} + [K_{BB}]\{\delta_B\} \end{aligned} \tag{2.37}$$

Finally, the stresses can then be found from the displacements by relation (2.21).

2.9 FREE VIBRATIONS

In the case of free vibrations, the equations of motion are:

$$[M]\{\ddot{\delta}\}_T + [K]\{\delta\}_T = \{0\} \tag{2.38}$$

where [M] and [K] are the global mass and stiffness matrices, $\{\delta_T\}$ is the vector for the global displacements of the whole shell.

$$\{\delta_T\} = \{\delta_1 , \delta_2 , ..., \delta_{N+1}\}^T$$

N being the number of finite elements.

By specifying:

$$\{\delta_T\} = \{\delta_o\}_T \quad \sin(\omega t + \psi) \qquad (2.39)$$

where ω is the natural angular frequency and ψ is the phase angle.

By introducing equation (2.39) in (2.38), we obtain

$$([K] - \omega^2[M]) \{\delta_0\}_T = 0 \qquad (2.40)$$

This relation holds only for certain values of ω where the determinant of the matrix in parentheses is zero. These values define the natural angular frequencies of the structure and give rise to a typical problem of eigenvalues and eigenvectors.

$$\det [[K] - \omega^2[M]] = 0 \qquad (2.41)$$

2.10 CALCULATIONS AND DISCUSSION

2.10.1 Convergence of the method

A first set of calculations was undertaken to determine the requisite number of finite elements for a precise determination of natural frequencies. Calculations were made for the same panel with the number of finite elements N = 2, 4, 6, 8, 10. The data for the panel are as follows : R = 2.286 m, t = 0.01143 m, L = 1.143 m, ϕ_T = 30°, E = 193.26 GPa, ν = 0.3 and ρ = 7933 kg/m^3, the boundary conditions are clamped at the straight edges and free simply-supported in the curved edges. The results for m = 2,10 and n = 1,2 are shown in Table 2.1. We conclude that the convergence of the system demands 6 finite elements for both the low and the high modes.

Table 2.1

Convergence study for increasing number of finite element (N) for m = 2, 10 and n = 1,2

N Frequency [Hz]	**2**	**4**	**6**	**8**	**10**
m = 2, n = 1	313.8	298.1	288.9	286.8	286.2
m = 2, n = 2	407.0	307.5	299.1	296.8	296.1
m = 10, n = 1	2310	2244	2133	2105	2098
m = 10, n = 2	3435	2305	2199	2166	2158

2.10.2 Calculations for uniform panels and shells

The eigenvalues of a uniform shell may unquestionably be calculated by simpler methods than these. Our main aim here is to test the correctness of the mass and stiffness matrices in their general form as developed in this paper.

(a) The first calculation involves the determination of the natural frequencies of a particular panel, having its straight edges free and the others free simply-supported.

The data of the panel are as follows : ϕ_T = 60°, L = 20 cm, R = 10 cm, t = 0.1 cm, E = 210 GPa, ν = 0.3 and ρ = 7800 kg/m^3.

As may be seen in Table 2.2, our results are in fairly good agreement with other theories and with experiments.

Table 2.2

Frequency (Hz) of cylindrical panel having its straight edges free and the others free simply-supported

(m,n)	Theory [32]	Experimental [32]	Present method
(1, 1)	299	300	286
(1, 2)	474	470	476
(1, 3)	1530	1490	1486
(2, 1)	860	870	859
(2, 2)	840	850	819
(3, 2)	1320	1330	1341
(3, 3)	1450	1460	1440

(b) The second calculation involves the determination of natural frequencies of a particular simply-supported closed shell which has been analysed by Michalopoulos and Muster [45], Baron and Bleich [46], Lakis and Paidoussis [33] and many others.

The data for the shell are as follows : R = 103.6 mm, t = 1.194 mm, L = 471 mm, ϕ_T = 360°, E = 207 GPa, ν = 0.3, ρ = 7790 kg/m^3.

The natural frequencies of this shell for n = 0 to 5 and m = 1 are shown in Table 2.3. The results obtained by our method were calculated using 10 equal finite elements. As may be seen, the results obtained by this method are in good agreement with those from other theories.

Table 2.3

Natural frequencies, in Hz, for a particular uniform closed shell, as calculated by various theories (m = 1)

n	Michalopoulos and Muster [45]	Baron and Bleich [46]	Lakis and Paidoussis [33]	Present method
0	3384	3540	3398*	3385
1	1775	1920	1790*	1777
2	750	760	752	750
3	436	435	436	435
4	467	463	468	468
5	675	670	678	675

* Lakis and Sinno [37]

2.10.3 Calculations for orthotropic shell and panel

This example illustrates that of the hybrid finite element method developed in this paper can be used with success for an orthotropic closed or open cylindrical shell.

The data for the shell are the same as for the panel except the total angle ϕ_T. $\phi_T = 360°$ for closed shell and $\phi_T = 90°$ for the panel described in Figures 2.3 and 2.4.

For n = 1 (beam bending mode) and long axial wave lengths, the frequency parameters are asymptotic to those of beams according to the Euler-Bernouilli theory [32]. This asymptotic behavior is shown in Figure 2.3 for the case when $E_\theta/E_x > 1$.

The results for an open cylindrical shell are given in Figure 2.4 for different axial and circumferential modes. This figure shows that the small axial wave length 'mR/L' has little effect on the frequency. This effect decreases when the circumferential mode increases.

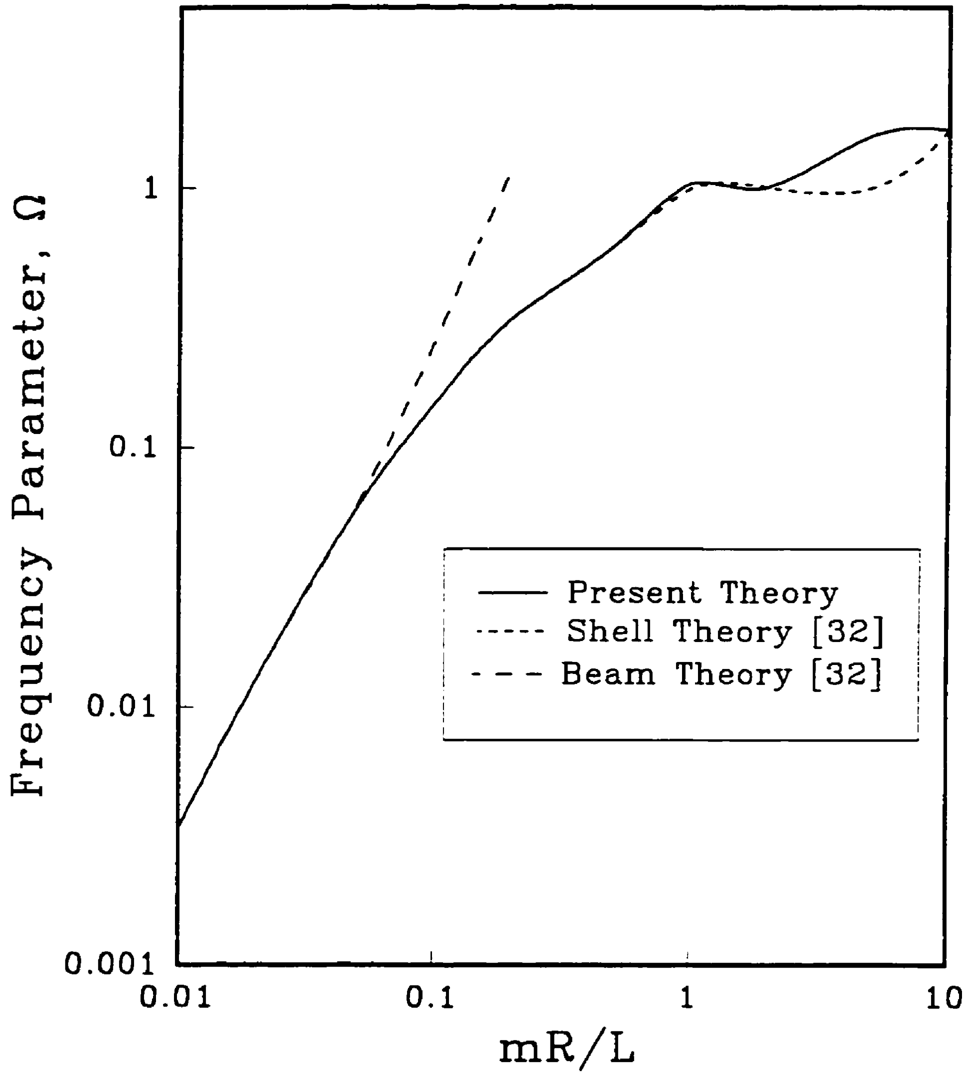

Figure 2.3 **Frequency parameters for the beam-type mode (n=1) of simply-supported orthotropic closed cylindrical shells.**

$R/t = 1000$, $E_\theta/E_x = 24.2$, $G/E_x = 0.527$, $\nu_\theta = 0.527$

$$\Omega = \omega R \sqrt{\rho\ (1-\nu_x \nu_\theta)/E_x}$$

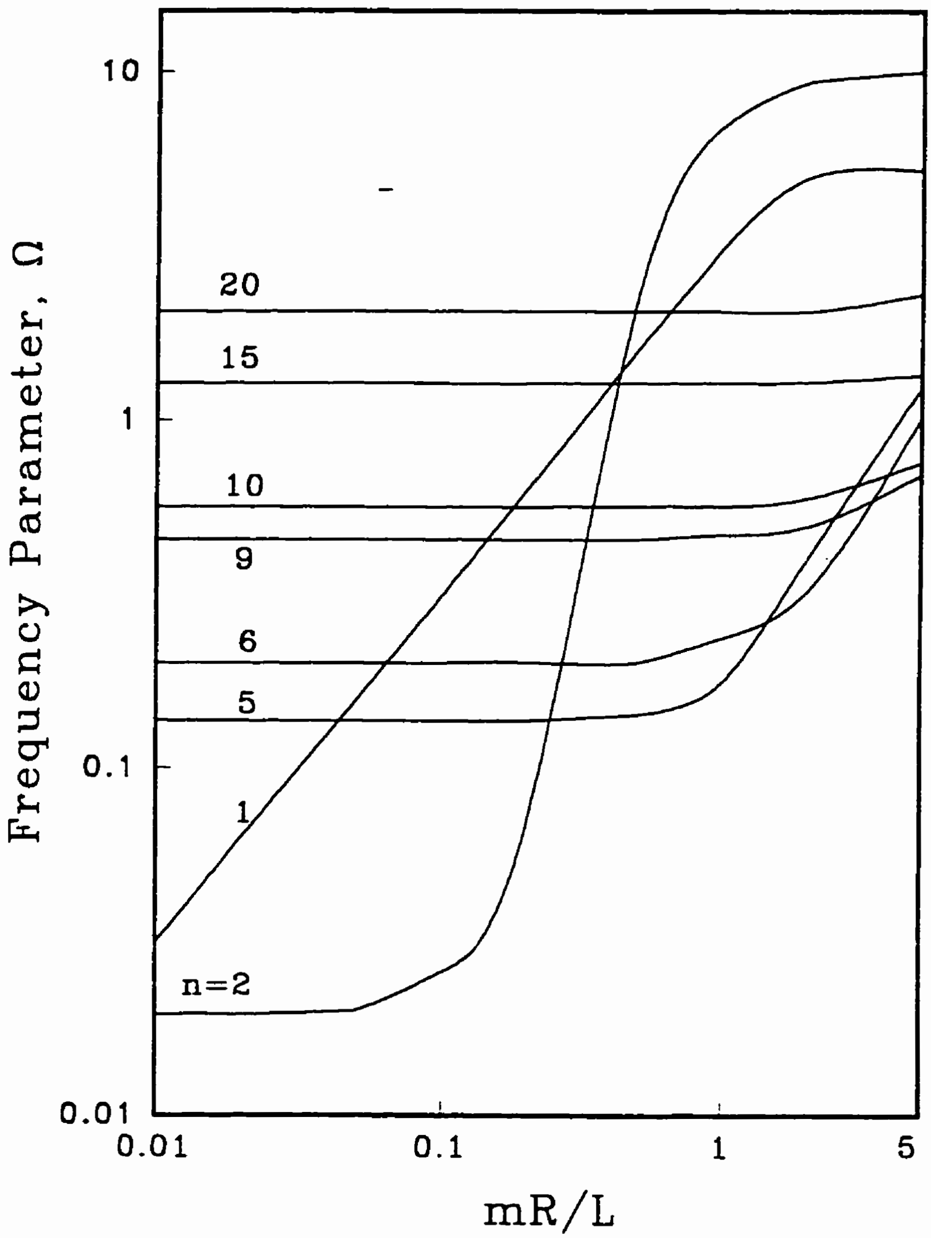

Figure 2.4 **Variation of frequency parameter with mR/L for an orthotropic open cylindrical shell; having its straight edges free and the others free simply-supported.**

$$\phi_T = 90° \, , \; R/h = 1000, \; E_\theta/E_x = 24.2, \; G/E_x = 0.527, \; \nu_\theta = 0.527, \; n \geq 1$$

$$\Omega = \omega R \sqrt{\rho \; (1-\nu_x \nu_\theta)/E_x}$$

2.10.4 Calculations for shells having circumferentially varying thickness

The present method has been applied to a cylinder whose inner bore is circular but non-concentric with circular outer surface (Figure 2.5). This case was studied by Tonin and Bies [24] using the Rayleigh-Ritz method.

The steel cylinder is free simply supported at both ends, and the data for this analysis are as follows :

$$a^- = 37.83 \text{ mm}, \; a^+ = 40.75 \text{ mm}, \; a = 39.29 \text{ mm}, \; L = 398.8 \text{ mm},$$

and the eccentricity e was studied for three values e = 0, 0.5 and 1 mm. The effect of the eccentricity on the calculated natural frequencies for various modes is detailed in Table 2.4. Note that the effect of increasing eccentricity is to lower the frequencies of the shell.

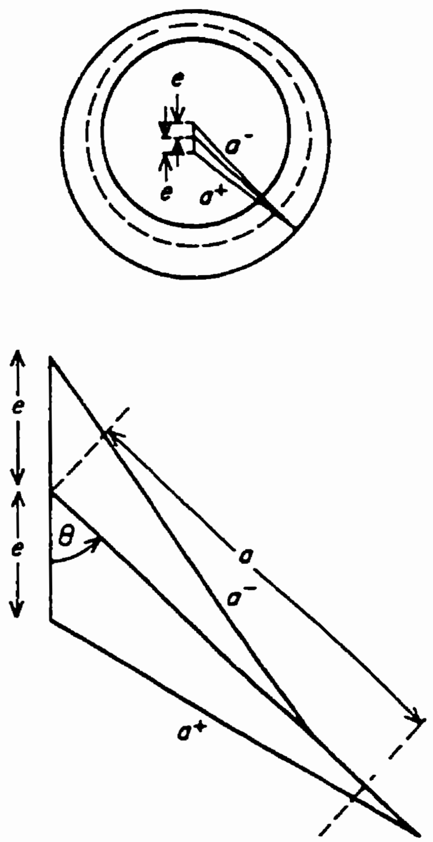

Figure 2.5 **Geometry of the distortion.**

Table 2.4

Variation of natural frequencies (Hz) of some modes with varying distortion

m, n	e = 0 mm		e = 0.5 mm			e = 1 mm	
	TONIN and BIES [24]	present method	[24]	Experimental [24]	present method	[24]	present method
1,2	1340	1341	1347	1330	1343	1302	1303
1,3	3553	3540	3420	3442	3410	3060	2949
1,4	6773	6758	6510	6495	6479	6177	5499
2,2	2105	2090	2071	2063	2062	1955	1954
2,3	3740	3728	3605	3627	3596	3243	3132
2,4	6905	6890	6638	6617	6607	6308	5618
3,2	3598	3568	3542	3463	3518	3302	3253
3,3	4204	4188	4083	4085	4071	3816	3743
3,4	7159	7144	6890	6861	6860	6575	5869

2.11 CONCLUSIONS

A method based on Sanders' equations for thin shells and making use of the finite element method has been formulated for the static and dynamic analysis of thin, elastic, anisotropic and non-uniform open cylindrical shells. The extensional and bending stiffnesses of the structures have been taken into account.

A new panel finite element was developed, making possible the derivation of the displacement functions from the equation of motion of the shell. Mass and stiffness matrices were also determined by analytical integration. The convergence of the method was established and the natural frequencies were obtained for different shells and panels. These were compared with the results of other investigations and satisfactory agreement was obtained.

This method combines the advanges of finite element analysis and the precision of formulation which the use of displacement functions derived from shell theory contributes.

Only a few cases have been presented here; a sufficient number, the authors believe, to illustrate the capabilities of the method. Several other cases could also have been tackled, but were not because of the volume of the paper.

A paper currently under preparation will deal with liquid-filled open and closed cylindrical shells. The dynamic stability of shells containing flowing fluid will also be analysed. Further work is under way to deal with the non-linear dynamic analysis of an open cylindrical shell containing flowing fluid.

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2.13 NOMENCLATURE

LIST OF SYMBOLS

A_i , B_i , C_i , D_i , E_i , F_i	Defined by equations (2.30) to (2.35)
$\bar{A}$, $\bar{B}$, $\bar{C}$	Defined by equation (2.8)
$\bar{A}_i$, $\bar{B}_i$, $\bar{C}_i$	Defined by equation (2.11)
D	Membrane Stiffness
E	Young's modulus for isotropic shell
E_x, E_θ	Young's modulus for orthotropic shell
e	Distortion (Figure 2.5)
f_i (i = 1, 12)	Defined in Appendix 2.2, Table 2.6
G	Shear modulus
h_i	Coefficients of the characteristic equation (2.10)
K	Bending stiffness
L	Length of the shell
M_x, M_θ, $\bar{M}_{x\theta}$	Bending moments
m	Axial mode number
$\bar{m}$	Defined by $m\pi/L$

N	Number of finite elements
N_x, N_θ, $\bar{N}_{x\theta}$	Stress components
n	Circumferential mode number
Pij	Terms of elasticity matrix
R	Mean radius of the shell
t	Thickness of the shell
U, V, W	Axial, tangential and radial displacements
U_m , V_m , W_m	Amplitudes of U, V, W associated with m th axial mode number
x	Axial coordinate
α_i, β_i	Defined by equation (2.12)
η_i	Complex roots of the characteristic equation (2.10)
ϵ_x, ϵ_θ, $\bar{\epsilon}_{x\theta}$	Deformation of reference surface
κ_x, κ_θ, $\bar{\kappa}_{x\theta}$	Changes in curvature and torsion of reference surface
θ	Circumferential coordinate
ν	Poisson's ratio for isotropic shell
ν_x, ν_θ	Poisson's ratio for orthotropic shell
Φ_T	Angle for the whole open shell

ω	Natural frequency (rad/s)
Ω	Nondimensional frequency, Figures 2.3 and 2.4
ρ	Density of the shell material

LIST OF MATRICES

[A]	Defined by equation (2.16)
[B]	Defined by equation (2.20)
{C}	Vector for arbitrary constants
[G]	Defined by equations (2.28) and (2.29)
[H]	Defined by equation (2.13)
[k]	Stiffness matrix for one finite element
[K]	Global stiffness matrix
[m]	Mass matrix for one finite element
[M]	Global mass matrix
[N]	Displacement function defined by equation (2.19)
[P]	Elasticity matrix
[Q]	Defined by equation (2.20)
[R]	Defined by equation (2.14)
[S]	Defined by equations (2.26) and (2.27)

$[T_m]$	Defined by equation (2.14)
$\{\epsilon\}$	Deformation vector
$\{\sigma\}$	Stress vector
$\{\delta_i\}$	Degrees of freedom at node i
$\{\delta_T\}$	Degrees of freedom for total shell

APPENDIX 2.1

EQUATIONS OF MOTION

This appendix contains the equations of motion for a thin cylindrical anisotropic shell.

$$L_1 (U,V,W,P_{ij}) = P_{11} \frac{\partial^2 U}{\partial x^2} + \frac{P_{12}}{R} (\frac{\partial^2 V}{\partial x \, \partial \theta} + \frac{\partial W}{\partial x}) - P_{14} \frac{\partial^3 W}{\partial x^3} +$$

$$\frac{P_{15}}{R^2} (\frac{\partial^3 W}{\partial x \, \partial \theta^2} + \frac{\partial^2 V}{\partial x \, \partial \theta}) + (\frac{P_{33}}{R} - \frac{P_{63}}{2R^2}) (\frac{\partial^2 V}{\partial x \, \partial \theta} + \frac{1}{R} \frac{\partial^2 U}{\partial \theta^2}) +$$

$$(\frac{P_{36}}{R^2} - \frac{P_{66}}{2R^3}) (- \frac{2\partial^3 W}{\partial x \, \partial \theta^2} + \frac{3}{2} \frac{\partial^2 V}{\partial x \, \partial \theta} - \frac{1}{2}R \frac{\partial^2 U}{\partial \theta^2})$$

$$L_2 (U,V,W,P_{ij}) = (\frac{P_{21}}{R} + \frac{P_{51}}{R^2}) (\frac{\partial^2 U}{\partial x \, \partial \theta}) + \frac{1}{R} (\frac{P_{22}}{R} + \frac{P_{52}}{R^2})$$

$$(\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial W}{\partial \theta}) - (\frac{P_{24}}{R} + \frac{P_{54}}{R^2}) (\frac{\partial^3 W}{\partial x^2 \, \partial \theta}) + \frac{1}{R^2} (\frac{P_{25}}{R} + \frac{P_{55}}{R^2})$$

$$(- \frac{\partial^3 W}{\partial \theta^3} + \frac{\partial^2 V}{\partial \theta^2}) + (P_{33} + \frac{3P_{63}}{2R}) (\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 U}{R\partial x \, \partial \theta}) +$$

$$\frac{1}{R} (P_{36} + \frac{3P_{66}}{2R}) (-2 \frac{\partial^3 W}{\partial x^2 \, \partial \theta} + \frac{3}{2} \frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 U}{2R \, \partial x \, \partial \theta})$$

$$L_3\,(U,V,W,P_{ij}) = P_{41}\frac{\partial^3 U}{\partial x^3} + \frac{P_{42}}{R}\left(\frac{\partial^3 V}{\partial x^2\,\partial\theta} + \frac{\partial^2 W}{\partial x^2}\right) - P_{44}\frac{\partial^4 W}{\partial x^4} +$$

$$\frac{P_{45}}{R^2}\left(-\frac{\partial^4 W}{\partial x^2\,\partial\theta^2} + \frac{\partial^3 V}{\partial x^2\,\partial\theta}\right) + \frac{2\,P_{63}}{R}\left(\frac{\partial^3 U}{R\,\partial x\,\partial\theta^2} + \frac{\partial^3 V}{\partial x^2\,\partial\theta}\right) + \left(\frac{2P_{66}}{R^2}\right)$$

$$\left(-2\frac{\partial^4 W}{\partial x^2\partial\theta^2} + \frac{3}{2}\frac{\partial^3 V}{\partial x^2\,\partial\theta} - \frac{\partial^3 U}{2R\partial x\,\partial\theta^2}\right) + \frac{P_{51}}{R^2}\frac{\partial^3 U}{\partial x\,\partial\theta^2} + \frac{P_{52}}{R^3}\left(\frac{\partial^3 V}{\partial\theta^3} +\right.$$

$$\left.\frac{\partial^2 W}{\partial\theta^2}\right) + \frac{P_{55}}{R^4}\left(-\frac{\partial^4 W}{\partial\theta^4} + \frac{\partial^3 V}{\partial\theta^3}\right) - \frac{P_{21}}{R}\frac{\partial U}{\partial x} - \frac{P_{54}}{R^2}\frac{\partial^4 W}{\partial x^2\,\partial\theta^2}$$

$$-\frac{P_{22}}{R^2}\left(\frac{\partial V}{\partial\theta} + W\right) + \frac{P_{24}}{R}\frac{\partial^2 W}{\partial\theta^2} - \frac{P_{25}}{R^3}\left(-\frac{\partial^2 W}{\partial\theta^2} + \frac{\partial V}{\partial\theta}\right)$$

APPENDIX 2.2

Appendix 2.2 contains the matrices referred to in the text which were too large to be included therein. The matrices are listed as follows.

[H]	(See Table 2.5)
hi (i=0,2,4,6,8)	(See Table 2.6)
$[T_m]$	(See Table 2.7)
[R]	(See Table 2.8)
[A]	(See Table 2.9)
[Q]	(See Table 2.10)

Table 2.5 : Matrix $[H]_{3\times 3}$

$$[H] \begin{Bmatrix} \bar{A} \\ \bar{B} \\ \bar{C} \end{Bmatrix} = \{0\} \qquad (2.9)$$

Where:

$$[H] = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}$$

$$H_{11} = -P_{11}\,\bar{m}^{2} + \frac{\eta^{2}}{R^{2}}\left[P_{33} - \frac{1}{R}P_{36} + \frac{1}{4R^{2}}P_{66}\right]$$

$$H_{12} = \bar{m}\ \eta\left[\frac{1}{R}(P_{12} + P_{33}) + \frac{1}{R^{2}}(P_{15} + P_{36}) - \frac{3}{4R^{2}}P_{66}\right]$$

$$H_{13} = \frac{P_{12}}{R}\bar{m} + P_{14}\,\bar{m}^{3} - \frac{\bar{m}}{R^{2}}\eta^{2}\left(P_{15} + 2P_{36} - \frac{1}{R}P_{66}\right)$$

$$H_{21} = H_{12}$$

$$H_{22} = \bar{m}^{2}\left(P_{33} + \frac{3}{R}P_{36} + \frac{9}{4R^{2}}P_{66}\right) - \frac{\eta^{2}}{R^{2}}\left(P_{22} + \frac{1}{R^{2}}P_{55} + \frac{2}{R}P_{25}\right)$$

$$H_{23} = -\frac{\eta}{R^{2}}\left(P_{22} + \frac{1}{R}P_{52}\right) + \frac{\eta^{3}}{R^{3}}\left(P_{25} + \frac{1}{R}P_{55}\right) - \frac{\eta\bar{m}^{2}}{R}\left(2P_{36} + P_{24} + \frac{3}{R}P_{66} + \frac{1}{R}P_{54}\right)$$

$$H_{31} = H_{13}$$

$$H_{32} = H_{23}$$

$$H_{33} = -\frac{1}{R^{4}}P_{55}\,\eta^{4} + \frac{\eta^{2}}{R^{2}}\left[\frac{2}{R}P_{25} + \bar{m}^{2}(2P_{45} + 4P_{66})\right]$$

and $\bar{m} = \dfrac{m\pi}{L}$

Table 2.6 : Characteristic Equation

$$h_8\,\eta^8 + h_6\,\eta^6 + h_4\,\eta^4 + h_2\,\eta^2 + h_0 = 0 \tag{2.10}$$

where

$$h_8 = f_1\,f_6\,f_{10} - f_1\,f_8^2$$

$$\begin{aligned} h_6 = {} & f_1\,f_6\,f_{11} + f_1\,f_7\,f_{10} - 2f_1\,f_8\,f_9 \\ & + f_2\,f_6\,f_{10} - f_2\,f_8^2 - f_3^2\,f_{10} \\ & + f_3\,f_8\,f_4 + f_4\,f_3\,f_8 - f_4^2\,f_6 \end{aligned}$$

$$\begin{aligned} h_4 = {} & f_1\,f_6\,f_{12} + f_1\,f_7\,f_{11} - f_1\,f_9^2 + f_2\,f_6\,f_{11} \\ & + f_2\,f_7\,f_{10} - 2f_2\,f_8\,f_9 - f_3^2\,f_{11} + f_3\,f_9\,f_4 \\ & + f_3\,f_8\,f_5 + f_4\,f_3\,f_9 - f_4^2\,f_7 - f_4\,f_6\,f_5 \\ & + f_5\,f_3\,f_8 - f_5\,f_6\,f_4 \end{aligned}$$

$$\begin{aligned} h_2 = {} & f_1\,f_7\,f_{12} + f_2\,f_6\,f_{12} + f_2\,f_7\,f_{11} - f_2\,f_9^2 \\ & - f_3^2\,f_{12} + f_3\,f_9\,f_5 - f_4\,f_7\,f_5 + f_5\,f_3\,f_9 \\ & - f_5\,f_7\,f_4 - f_5^2\,f_6 \end{aligned}$$

$$h_o = f_2\,f_7\,f_{12} - f_7\,f_5^2$$

The coefficients f_i (i = 1,12) are given by the above equations :

$$f_1 = \frac{1}{R}(P_{55} - \frac{1}{R} P_{36} + \frac{1}{4R^2} P_{66})$$

$$f_2 = - P_{11} \bar{m}^2$$

$$f_3 = \bar{m} \left[\frac{1}{R} (P_{12} + P_{13}) + \frac{1}{R^2} (P_{15} + P_{36}) - \frac{3}{4R^3} P_{66}\right]$$

$$f_4 = - \frac{\bar{m}}{R^2} (P_{15} + 2 P_{36} - \frac{1}{R} P_{66})$$

$$f_5 = \frac{P_{12}}{R} \bar{m} + P_{14} \bar{m}^3$$

$$f_6 = - \frac{1}{R^2} (P_{22} + \frac{1}{R^2} P_{55} + \frac{2}{R} P_{25})$$

$$f_7 = \bar{m} (P_{33} + \frac{3}{R} P_{36} + \frac{9}{4R^2} P_{66})$$

$$f_8 = \frac{1}{R^3} (P_{25} + \frac{1}{R} P_{55})$$

$$f_9 = - \frac{1}{R^2} (P_{22} + \frac{1}{R} P_{52}) - \frac{\bar{m}^2}{R} (2P_{36} + P_{24} + \frac{3}{R} P_{66} + \frac{1}{R} P_{54})$$

$$f_{10} = - \frac{1}{R^4} P_{55}$$

$$f_{11} = \frac{2}{R^3} P_{25} + \frac{\bar{m}}{R^2} (2P_{45} + 4P_{66})$$

$$f_{12} = - \frac{1}{R} P_{22} - \frac{2}{R} P_{24} \bar{m}^2 - P_{44} \bar{m}$$

and $\bar{m} = m \frac{\pi}{L}$

Table 2.7 : Matrix $[T_m]_{3\times3}$

$$[T_m] = \begin{bmatrix} \cos\frac{m\pi x}{L} & 0 & 0 \\ 0 & \sin\frac{m\pi x}{L} & 0 \\ 0 & 0 & \sin\frac{m\pi x}{L} \end{bmatrix}$$

Table 2.8 : Matrix $[R]_{3\times8}$

$$R(1,j) = \alpha_j\, e^{\eta_j\theta} \qquad j = 1,8$$
$$R(2,j) = e^{\eta_j\theta} \qquad j = 1,8$$
$$R(3,j) = \beta_j\, e^{\eta_j\theta} \qquad j = 1,8$$

Table 2.9 : Matrix $[A]_{8\times8}$

For $j = 1,8$

$$A(1,j) = \alpha_j$$
$$A(2,j) = 1$$
$$A(3,j) = \eta_j$$
$$A(4,j) = \beta_j$$
$$A(5,j) = \alpha_j\, e^{\eta_j\theta}$$
$$A(6,j) = e^{\eta_j\theta}$$
$$A(7,j) = \eta_j\, e^{\eta_j\theta}$$
$$A(8,j) = \beta_j\, e^{\eta_j\theta}$$

Table 2.10 : Matrix $[Q]_{6\times8}$

For $j = 1,8$

$$Q(1,j) = A_j \, e^{\eta_j \theta}$$

$$Q(2,j) = B_j \, e^{\eta_j \theta}$$

$$Q(3,j) = C_j \, e^{\eta_j \theta}$$

$$Q(4,j) = D_j \, e^{\eta_j \theta}$$

$$Q(5,j) = E_j \, e^{\eta_j \theta}$$

$$Q(6,j) = F_j \, e^{\eta_j \theta}$$

The terms A_j, B_j, C_j, D_j, E_j and F_j ($j = 1,8$) are given by equations (2.30) to (2.35).

CHAPITRE III

ARTICLE II

VIBRATION ANALYSIS OF ANISOTROPIC OPEN CYLINDRICAL SHELLS CONTAINING FLOWING FLUID

3.1 ABSTRACT

A theory is presented for the determination of the effects of a flowing fluid on the vibration characteristics of an open, anisotropic cylindrical shell submerged and subjected simultaneously to an internal and external flow. The case of an open shell partially or completely filled with liquid is also investigated. The structure may be uniform or non uniform in the circumferential direction. The formulation used is a combination of finite element method and classical shell theory. The displacement functions are derived from exact solutions of Sanders' shell equations.

The velocity potential and Bernoulli's equation for a liquid finite element yield an expression for fluid pressure as a function of the nodal displacements of the element and

three forces (inertial, centrifugal and Coriolis) of the moving fluid. An analytical integration of the fluid pressure over the liquid element leads to three components: mass, stiffness and damping matrices.

Calculations are given to illustrate the dynamic behaviour of open and closed cylindrical shells subjected to a flowing fluid and shells partially or completely filled with liquid. Reasonable agreement is found with other theories and experiments.

3.2 INTRODUCTION

Knowledge of the vibration characteristics of fluid-filled cylindrical shells and panels is of considerable practical interest, since cylindrical shells and panels are commonly used to contain or convey fluids. There are many ways in which the presence of the fluid may influence the dynamics of the structure. If the structure contains a stationary gas at low pressure, then the vibration of the shell differs only slightly from that of the same shell in vacuo. If the fluid is compressible, the compressibility of the fluid alters the effective stiffness of the system. Also, if the density of the fluid is relatively high, as in the case of a liquid, then the fluid exerts considerable inertial loading on the shell, and this results in a significant lowering of the resonant frequencies. Other effects of coupled fluid-shell motions occur when the fluid is flowing. Depending upon the boundary conditions, if the flow velocities are high, buckling or oscillatory flexural instabilities are possible.

The dynamics of coupled fluid-shells were reviewed extensively by Brown (1982), Au-Yang (1986), Paidoussis & Li (1993) and others (Mistry & Menenzes 1995; Harari, Sandman & Zaldonis 1994; Cheng 1994; Han & Liu 1994; Terhune & Karim-Panahi 1993; Brenneman & Au-Yang 1992; Endo & Tosaka 1989 and Goncalves & Batista 1987). There have been few analyses of closed cylindrical shells having axially varying thickness. Similarly, while there is extensive literature relevant to the vibration of empty open cylindrical shells (cylindrical panels), no analysis has been found of open cylindrical shells, circumferentially non-uniform, totally submerged and subjected simultaneously to an internal and external flow.

The purpose of this study is to present a method for the dynamic and static analysis of open, thin, anisotropic cylindrical shells containing flowing fluid (Figure 3.1). The structure may be uniform or non-uniform in the circumferential direction and we consider the problem of open cylindrical shells which are freely simply-supported along their curved edges and have arbitrary straight edge boundary conditions.

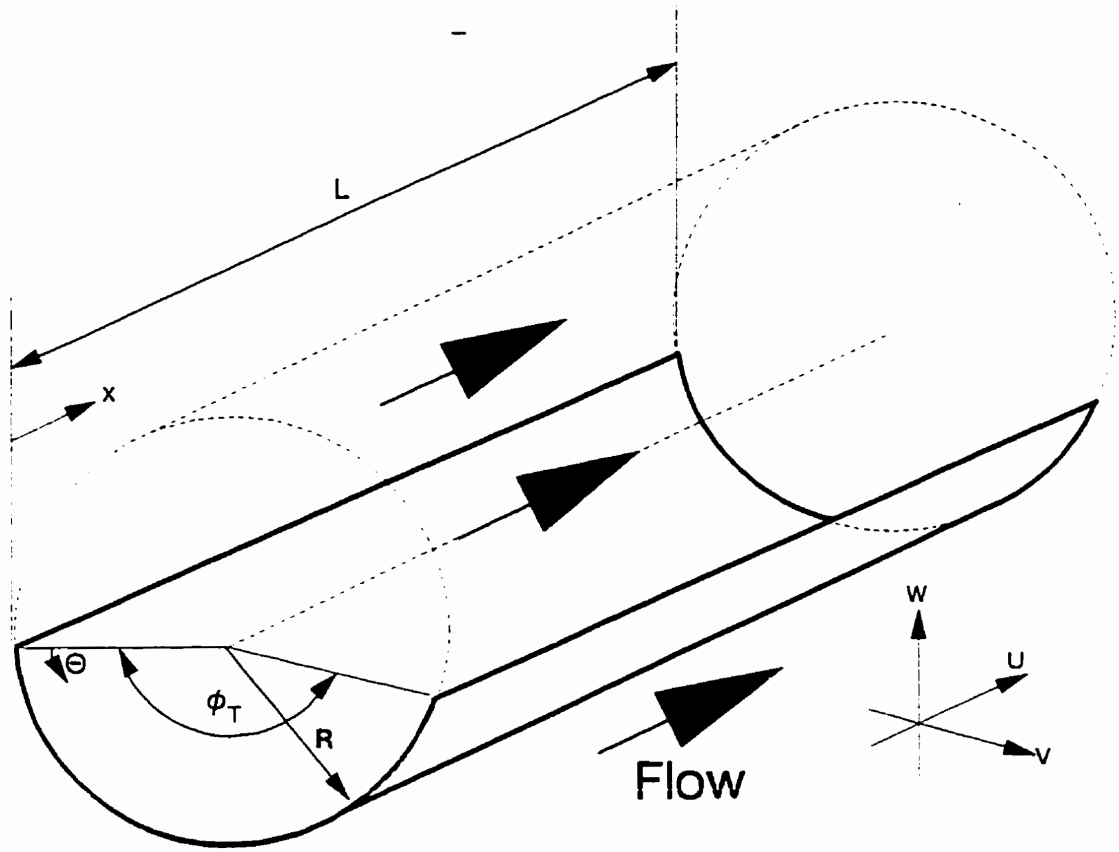

Figure 3.1 Open cylindrical shell geometry.

The method is a hybrid of finite element method, classical shell theories and fluid theories. The structure is subdivided into cylindrical panel segment finite elements. The displacement functions are derived from Sanders' (1959) equation of thin cylindrical shells. In this approach, it is possible to determine the mass and stiffness matrices of the individual finite elements by exact analytical integration. Accordingly, this method is more accurate than the more usual finite element methods based on polynomial displacement functions.

To account for the fluid effect on the structure, a panel finite fluid element bounded by two nodal lines was considered. By solving the equations of motion for the fluid element, an expression for fluid pressure as a function of the displacements of the element was obtained. Analytical integration for the pressure distribution along the element yielded three components: the mass, stiffness and damping matrices for a fluid element. Global matrices are, then, obtained by superimposing the individual matrices. The eigenvalue and eigenvector problem is solved by means of the equation reduction technique.

The hybrid approach (Finite element - Shell theory - Fluid theory) has been applied with satisfactory results to the dynamic linear and non-linear analysis of cylindrical (Lakis & Paidoussis 1971; Lakis & Paisoussis 1972; Lakis & Paidoussis 1973; Lakis 1976a; Lakis 1976b; Lakis, Sami & Rousselet 1978; Lakis & Laveau 1991 and Lakis & Sinno

1992), conical (Lakis, Van Dyke & Ouriche 1992), spherical (Lakis, Tuy & Selmane (1989), isotropic and anisotropic, uniform and axially non-uniform shells both empty and liquid filled. This method has been applied also to the dynamic analysis of circular and annular plates (Lakis & Selmane 1990a and Lakis & Selmane 1990b) and to an open anisotropic and circumferentially non-uniform cylindrical shell (Selmane & Lakis 1995). This study is an attempt to determine the vibration of a circumferentially non-uniform open cylindrical shell, subjected to a flowing fluid. The case of an open cylindrical shell partially or completely filled with liquid is also studied.

3.3 DETERMINATION OF THE DISPLACEMENT FUNCTIONS

Sanders' (1959) equations for thin, cylindrical shells, in terms of axial, tangential and radial displacements (U, V, W) of the mean surface of the shell (Figure 3.1) and in terms of element P_{ij} of the anisotropic matrix of elasticity [P] are:

$$\begin{aligned} L_1 (U,V,W,P_{ij}) &= 0 \\ L_2 (U,V,W,P_{ij}) &= 0 \\ L_3 (U,V,W,P_{ij}) &= 0 \end{aligned} \qquad \textbf{(3.1)}$$

where L_k (k = 1, 2, 3) are three linear differential operators, the form of which is fully explained in Selmane & Lakis (1995).

The strain-displacement relation is given by:

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_\theta \\ 2\bar{\epsilon}_{x\theta} \\ \kappa_x \\ \kappa_\theta \\ 2\bar{\kappa}_{x\theta} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{1}{R}\dfrac{\partial V}{\partial \theta} + \dfrac{W}{R} \\ \dfrac{\partial V}{\partial x} + \dfrac{1}{R}\dfrac{\partial U}{\partial \theta} \\ -\dfrac{\partial^2 W}{\partial x^2} \\ -\dfrac{1}{R^2}\dfrac{\partial^2 W}{\partial \theta^2} + \dfrac{1}{R^2}\dfrac{\partial V}{\partial \theta} \\ -\dfrac{2}{R}\dfrac{\partial^2 W}{\partial x \partial \theta} + \dfrac{3}{2R}\dfrac{\partial V}{\partial \theta} - \dfrac{1}{2R^2}\dfrac{\partial U}{\partial \theta} \end{Bmatrix} \quad (3.2)$$

The finite element used is shown in Figure 3.2. It is a cylindrical panel segment defined by two line nodes i and j. Each node has four degrees of freedom: three displacements (axial, circumferential and radial) and one rotation. The panels are assumed to be freely simply-supported along their curved edges and to have arbitrary straight edge boundary conditions.

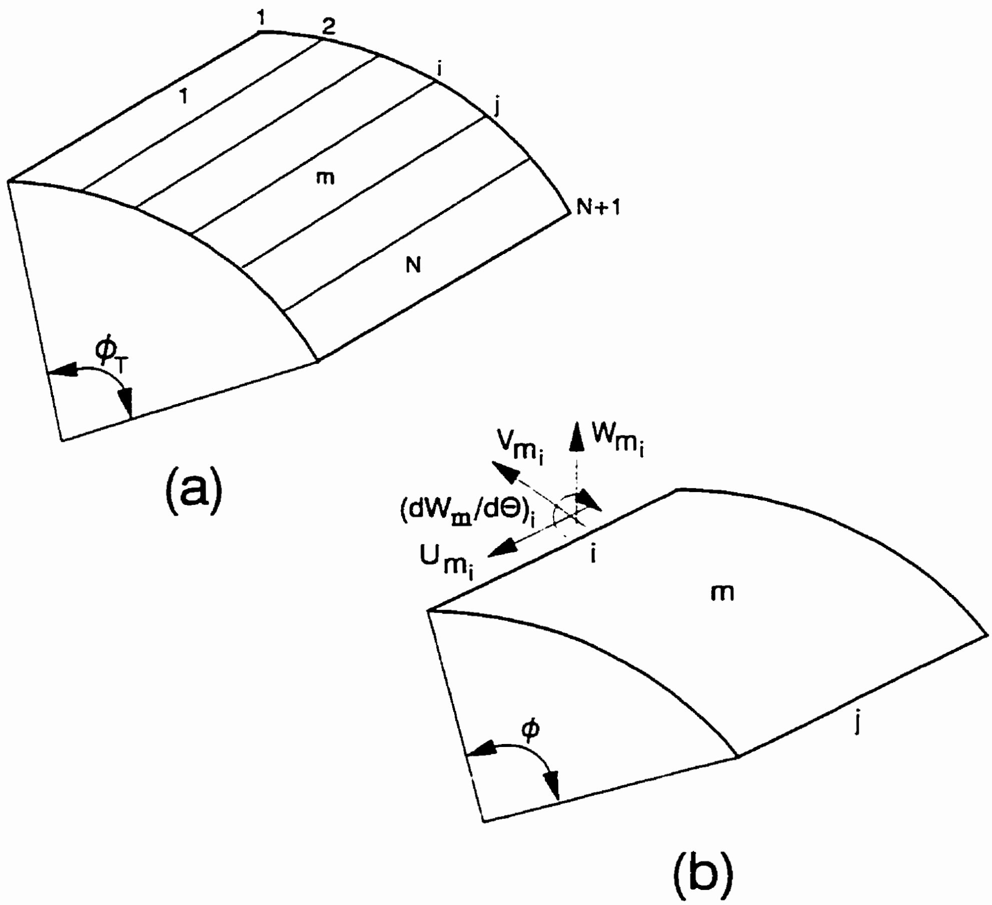

Figure 3.2 **(a) Finite element idealization.**
(b) Nodal displacements at node i.
N: Number of finite elements.

For motions associated with the m th axial wave number, we may write:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = \begin{bmatrix} \cos m\pi x/L & 0 & 0 \\ 0 & \sin m\pi x/L & 0 \\ 0 & 0 & \sin m\pi x/L \end{bmatrix} \begin{Bmatrix} U_m(\theta) \\ W_m(\theta) \\ V_m(\theta) \end{Bmatrix} = [T_m] \begin{Bmatrix} U_m(\theta) \\ W_m(\theta) \\ V_m(\theta) \end{Bmatrix} \quad (3.3)$$

By substituting equation (3.3) into equation (3.1) and letting

$$\begin{aligned} U_m(\theta) &= A\ e^{\eta\theta} \\ V_m(\theta) &= B\ e^{\eta\theta} \\ W_m(\theta) &= C\ e^{\eta\theta} \end{aligned} \quad (3.4)$$

we obtain

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T_m]\ [R]\ \{C\} \quad (3.5)$$

where [R] is a (3×8) matrix given by:

$$\begin{aligned} R(1,j) &= \alpha_j\ e^{\eta_j\theta} & j &= 1,\ldots,8 \\ R(2,j) &= e^{\eta_j\theta} & j &= 1,\ldots,8 \\ R(3,j) &= \beta_j\ e^{\eta_j\theta} & j &= 1,\ldots,8 \end{aligned} \quad (3.6)$$

η_j (j = 1,..., 8) are the roots of the characteristic equation of the empty panel. As A, B and C are not independent, we may write A = αC and B = βC, which determine α_j and β_j. {C} is a vector of eight constants which are linear combinations of the C_j. The eight C_j are the only free constants, which must be determined from eight boundary conditions, four at each straight edge of the finite element.

We now express the nodal displacement vectors as follows

$$\{\delta_i\} = \left\{ U_{mi},\ W_{mi},\ \left(\frac{dW_m}{d\theta}\right)_i,\ V_{mi} \right\}^T \tag{3.7}$$

Each $\{\delta_i\}$ may be determined from equation (3.5), where θ in [R] now has a definite value, $\theta = 0$ or $\theta = \phi$, as the case may be; hence we obtain

$$\left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} = [A]\ \{C\} \tag{3.8}$$

where the elements of matrix [A] are determined from those of matrix [R] and given by:

For $j = 1,\ldots,8$

$$
\begin{aligned}
A(1,j) &= \alpha_j & \qquad A(5,j) &= \alpha_j\, e^{\eta_j \phi} \\
A(2,j) &= 1 & \qquad A(6,j) &= e^{\eta_j \phi} \\
A(3,j) &= \eta_j & \qquad A(7,j) &= \eta_j\, e^{\eta_j \phi} \\
A(4,j) &= \beta_j & \qquad A(8,j) &= \beta_j\, e^{\eta_j \phi}
\end{aligned}
\tag{3.9}
$$

Finally, combining equation (3.5) and (3.8), we obtain:

$$
\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T_m]\ [R]\ [A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}
\tag{3.10}
$$

which defines the displacement functions.

3.4 MASS AND STIFFNESS MATRICES FOR EMPTY FINITE ELEMENTS

The strains are related to the displacements through equations (3.2); accordingly, we may now express $\{\epsilon\}$ in terms of δ_i and δ_j, and after lengthy manipulations we obtain:

$$
\{\epsilon\} = \begin{bmatrix} [T_m] & 0 \\ 0 & [T_m] \end{bmatrix} [Q]\ [A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}
\tag{3.11}
$$

where [Q] is a (6×8) matrix given in Selmane & Lakis (1995).

The corresponding stresses may be related to the strains by the elasticity matrix [P].

$$\{\sigma\} = [P]\{\epsilon\} = [P][B]\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix} \tag{3.12}$$

The matrix P of an anisotropic shell is given as follows :

$$[P] = \begin{bmatrix} P_{11} & P_{12} & 0 & P_{14} & P_{15} & 0 \\ P_{21} & P_{22} & 0 & P_{24} & P_{25} & 0 \\ 0 & 0 & P_{33} & 0 & 0 & P_{36} \\ P_{41} & P_{42} & 0 & P_{44} & P_{45} & 0 \\ P_{51} & P_{52} & 0 & P_{54} & P_{55} & 0 \\ 0 & 0 & P_{63} & 0 & 0 & P_{66} \end{bmatrix} \tag{3.13}$$

The elements P_{ij} of [P] characterize the shell's anisotropy which depends on the mechanical properties of the material of the structure.

The mass and stiffness matrices, $[m_s]$ and $[k_s]$ respectively, for one finite element may be written as follows:

$$[m_s] = \rho_s t \int_0^L \int_0^\phi [N]^T [N] \, dA \quad \text{and} \quad [k_s] = \int_0^L \int_0^\phi [B]^T [P] [B] \, dA \,, \qquad (3.14)$$

where ρ_s is the density of the shell, t its thickness, dA a surface element, [P] the elasticity matrix and the matrices [N] and [B] are derived from equations (3.10) and (3.11), respectively.

The matrices $[m_s]$ and $[k_s]$ were obtained analytically by carrying out the necessary matrix operations and integration over x and θ in equation (3.14). The global matrices $[M_s]$ and $[K_s]$ may be obtained, respectively, by superimposing the mass $[m_s]$ and stiffness $[k_s]$ matrices for each individual panel finite element. See (Selmane & Lakis 1995) for more details.

3.5 BEHAVIOUR OF THE FLUID-SHELL INTERACTION

3.5.1 Equations of motion

The dynamic behaviour of an open shell subjected to a pressure field can be represented by the following system:

$$\left[[M_s] - [M_f]\right] \{\ddot{\delta}\} - [C_f]\{\dot{\delta}\} + \left[[K_s] - [K_f]\right] \{\delta\} = \{F\} \qquad (3.15)$$

where $\{\delta\}$ is the displacement vector, $[M_s]$ and $[K_s]$ are, respectively, the mass and stiffness matrices of the system in vacuo; $[M_f]$ and $[C_f]$ and $[K_f]$ represent the inertial, Coriolis and centrifugal forces of the liquid flow and $\{F\}$ represents the external forces.

3.5.2 Assumptions

We assume here that the structure is subjected only to potential flow which induces inertial, Coriolis and centrifugal forces to participate in the vibration pattern. These forces are coupled with the elastic deformation of the shell.

The mathematical model which is developed is based on the following hypothesis:

(i) the fluid flow is potential ;

(ii) vibration is linear (small deformation) ;

(iii) pressure on the wall is purely lateral ;

(iv) the fluid mean velocity distribution is assumed to be constant across a shell section ;

and (v) the fluid is incompressible.

3.5.3 Mass, stiffness and damping matrices of the moving fluid

With the assumptions of section 3.5.2, the velocity potential must satisfy the Laplace equation. This relation is expressed in the cylindrical coordinate system by:

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial x^2} = 0 \tag{3.16}$$

Φ is the potential function that represents the velocity potential.

Therefore:

$$V_x = U_x + \frac{\partial \Phi}{\partial x} \; ; \quad V_\theta = \frac{1}{R}\frac{\partial \Phi}{\partial \theta} \; ; \quad V_r = \frac{\partial \Phi}{\partial r} \tag{3.17}$$

where U_x is the velocity of the liquid through the shell section; V_x, V_θ and V_r are respectively the axial, tangential and radial components of the fluid velocity.

The Bernouilli equation is given by:

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} V^2 + \frac{P}{\rho_f}\Big|_{r=\xi} = 0 \tag{3.18}$$

Introducing equation (3.17) into equation (3.18) and taking into account only the linear terms, we find the dynamic pressure P:

$$P_u = -\rho_{fu} \left\{ \frac{\partial \Phi_u}{\partial t} + U_{xu} \frac{\partial \Phi_u}{\partial x} \right\} \Big|_{r=\xi} \tag{3.19}$$

where u subscript represents "internal" or "external" fluid as the case may be:

$$\text{if} \quad u = i \qquad \text{then} \quad \xi = R_i = R - \frac{t}{2} \tag{3.20}$$

$$\text{if} \quad u = e \qquad \text{then} \quad \xi = R_e = R + \frac{t}{2} \tag{3.21}$$

A full definition of the flow requires that a condition be applied to the structure-fluid interface. The impermeability condition ensures contact between the shell and the fluid. This should be:

$$V_r \Big|_{r=R} = \frac{\partial \Phi}{\partial r} \Big|_{r=R} = \left(\frac{\partial W}{\partial t} + U_x \frac{\partial W}{\partial x} + \frac{U_x^2}{2} \frac{\partial^2 W}{\partial x^2} \right) \Big|_{r=R} \tag{3.22}$$

From the theory of shells (equation 3.5), we have:

$$W(x, \theta, t) = \sum_{j=1}^{8} C_j \, e^{\eta_j \theta} \sin \frac{m \pi x}{L} \, e^{i \omega t} \tag{3.23}$$

Assuming then,

$$\Phi(x,\theta,r,t) = \sum_{j=1}^{8} R_j(r)\, S_j(x,\theta,t) \tag{3.24}$$

and applying the impermeability condition (equation 3.22) with the radial displacement given by relation (3.23), we determine the function S_j (x, θ, t). Introducing this explicit term $S_j(x, \theta, t)$ into equation (3.24) and then into equation (3.19), we find a relation for the dynamic pressure as a function of the displacement W_j and the function $R_j(r)$:

$$P_u = -\rho_f \sum_{j=1}^{8} \frac{R_j(r)}{R_j'(R)} \left[\ddot{W}_j + 2U_{xu}\dot{W}'_j + \frac{U_{xu}^2}{2}\dot{W}''_j + U_{xu}^2 W''_j + \frac{U_{xu}^3}{2} W'''_j \right] \tag{3.25}$$

where $(')$, $(\cdot)$ and $(')$ represent $\frac{\partial(\ )}{\partial r}$, $\frac{\partial(\ )}{\partial t}$ and $\frac{\partial(\ )}{\partial x}$ respectively.

By using relation (3.16), we obtain the following differential Bessel equation:

$$r^2 \frac{d^2R_j(r)}{dr^2} + r\frac{dR_j(r)}{dr} + R_j(r)\left[\left(\frac{im\pi}{L}\right)^2 r^2 - (i\eta_j)^2\right] = 0 \tag{3.26}$$

where i is the complex number, $i^2 = -1$ and η_j is the complex solution of the characteristic equation.

The general solution of equation (3.26) is given by:

$$R_j(r) = A\,J_{i\eta_j}\left(\frac{im\pi}{L}r\right) + B\,Y_{i\eta_j}\left(\frac{im\pi}{L}r\right) \tag{3.27}$$

where $J_{i\eta_j}$ and $Y_{i\eta_j}$ are, respectively, the Bessel functions of the first and second kind of order "$i\eta_j$".

For inside flow, the solution (3.27) must be finite on the axis of the shell ($r = 0$); this means we have to set the constant 'B' equal to zero. For outside flow ($r \rightarrow \infty$); this means that the constant 'A' is equal to zero. When the shell is simultaneously subjected to internal and external flow, we have to take the complete solution (3.27).

Finally, we obtain the equation for the pressure on the wall as follows:

$$P_u = -\rho_u \sum_{j=1}^{8} Z_{uj}\left(\frac{im\pi R_u}{L}\right)\left[\ddot{W}_j + 2U_{xu}\dot{W}_j' + \frac{U_{xu}^2}{2}\dot{W}_j'' + U_{xu}^2 W_j'' + \frac{U_{xu}^3}{2}W_j'''\right] \tag{3.28}$$

where $(\cdot)$ and $(')$ represent $\frac{\partial(\ )}{\partial t}$ and $\frac{\partial(\ )}{\partial x}$ respectively, and

$$Z_{uj}\left(\frac{im\pi R_u}{L}\right) = \frac{R_u}{i\eta_j - \dfrac{im\pi R_u}{L}\dfrac{J_{i\eta_j+1}(im\pi R_u/L)}{J_{i\eta_j}(im\pi R_u/L)}} \quad \text{if } u = i \tag{3.29}$$

$$Z_{uj}\left(\frac{im\pi R_u}{L}\right) = \frac{R_u}{i\eta_j - \dfrac{im\pi R_u}{L}\dfrac{Y_{i\eta_j+1}(im\pi R_u/L)}{Y_{i\eta_j}(im\pi R_u/L)}} \quad \text{if } u = e \tag{3.30}$$

where η_j ($j = 1,\ldots, 8$) are the roots of the characteristic equation of the empty shell; $J_{i\eta j}$ and $Y_{i\eta j}$ are, respectively, the Bessel functions of the first and second kind of order "$i\eta_j$"; m is the axial mode number; R is the mean radius of the shell; L its length; the subscript "u" is equal to "i" for internal flow and is equal to "e" for external flow.

By introducing the displacement function (3.10), into the dynamic pressure expression (3.28) and performing the matrix operation required by the finite element method, the mass, damping and stiffness matrices for fluid are obtained by evaluating the following integral:

$$\int_A [N]^T \{P_u\}\, dA \tag{3.31}$$

we obtain:

$$[m_f] = [A^{-1}]^T [S_f] [A^{-1}] \tag{3.32}$$

$$[c_f] = [A^{-1}]^T [D_f] [A^{-1}] \tag{3.33}$$

$$[k_f] = [A^{-1}]^T [G_f] [A^{-1}] \tag{3.34}$$

The matrix [A] is given by equation (3.9) and the elements of $[S_f]$, $[D_f]$ and $[G_f]$ are given, as follows.

$$S_f(r,s) = -\frac{RL}{2} I_{rs} \left(\rho_i Z_{is} - \rho_e Z_{es}\right) \tag{3.35}$$

$$D_f(r,s) = \frac{Rm^2 \pi^2}{4L} I_{rs} \left(\rho_i U_{xi}^2 Z_{is} - \rho_e U_{xe}^2 Z_{es}\right) \tag{3.36}$$

$$G_f(r,s) = \frac{Rm^2 \pi^2}{2L} I_{rs} \left(\rho_i U_{xi}^2 Z_{is} - \rho_e U_{xe}^2 Z_{es}\right) \tag{3.37}$$

where r, s = 1, ..., 8; ρ is the density of the fluid; U_x is the velocity of the fluid; Z is defined by relations (3.29) and (3.30); the subscript "i" means internal flow and "e" means external flow and I_{rs} is defined by :

$$\begin{cases} I_{rs} = \dfrac{1}{(\eta_r + \eta_s)} \left[e^{(\eta_r+\eta_s)\phi} - 1 \right] & \text{for } \eta_r + \eta_s \neq 0 \\ I_{rs} = \phi & \text{for } \eta_r + \eta_s = 0 \end{cases} \tag{3.38}$$

where r, s=1,...,8; η is the root of the characteristic equation of the empty shell and ϕ is the angle for one finite element.

Finally, the global matrices $[M_f]$, $[C_f]$ and $[K_f]$ may be obtained, respectively, by superimposing the mass $[m_f]$, damping $[c_f]$ and stiffness $[k_f]$ matrices for each individual fluid finite element.

3.6 EIGENVALUE AND EIGENVECTOR PROBLEM

The eigenvalue and eigenvector problem is solved by means of the equation reduction technique. Equation (3.15) may be rewritten as follows:

$$\begin{bmatrix} [0] & \frac{1}{\xi_o}[M] \\ \frac{1}{\xi_o^2}[M] & \frac{1}{\xi_o}[C] \end{bmatrix} \begin{Bmatrix} \ddot{\delta} \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} -\frac{1}{\xi_o}[M] & [0] \\ [0] & [K] \end{bmatrix} \begin{Bmatrix} \dot{\delta} \\ \delta \end{Bmatrix} = \{0\} \tag{3.39}$$

where

$$\begin{aligned} [M] &= ([M_s] - [M_f])/\rho_{s1} t_1 R_1^2 \\ [K] &= ([K_s] - [K_f])/P_{11} \\ [C] &= -[C_f]/(P_{11}\rho_{s1} t_1 R_1^2)^{1/2} \end{aligned} \tag{3.40}$$

$[M_s]$ and $[K_s]$ are the global mass and stiffness matrices for the empty shell, $[M_f]$, $[C_f]$ and $[K_f]$ are the global mass, damping and stiffness matrices for the fluid.

$\xi_o^2 = P_{11}/\rho_{s1}\, t_1\, R_1^2$: where P_{11} is the first element of the elasticity matrix;

ρ_{s1}, t_1 and R_1 are respectively, the density, thickness and radius of the first element of the shell.

The problem for eigenvalues is given by:

$$| [DD] - \Lambda [I] | = 0 \tag{3.41}$$

where

$$[DD] = \begin{bmatrix} [0] & [I] \\ -\frac{1}{\xi_o^2} [K]^{-1} [M] & -\frac{1}{\xi_o} [K]^{-1} [C] \end{bmatrix} \tag{3.42}$$

and $\Lambda = \frac{1}{i\omega}$,

ω is the natural frequency of the system.

Particular case: If the velocity of the fluid ($U_x = 0$), the eigenvalue problem may be reduced to:

$$| \frac{1}{\omega_o^2} [K]^{-1} [M] - \Lambda [I] | = 0 \tag{3.43}$$

and ω (rad/s) $= (1/\Lambda)^{1/2}$.

Matrices [K], [M] and [C] are square matrices of order NDF (N+1)-J, where NDF is the number of degrees of freedom at each node, N is the number of finite elements in the structure and J is the number of constraints applied.

3.7 CALCULATIONS AND DISCUSSION

Calculations have already been conducted to test the theory in the case of EMPTY open and closed shells. The free vibrations of uniform and circumferentially non-uniform, isotropic and orthotropic open and closed shells were obtained for a variety of boundary conditions (Selmane & Lakis 1995). The computed natural frequencies were compared with those obtained by other theories and from experiments; the results were in agreement within a range of 5%.

Here we present some calculations to test the theory in the case of liquid-filled open and closed cylindrical shells. In the case when the shell is subjected to flowing fluid, the dynamic stability of this type of problem is analysed.

3.7.1 Free vibration of closed and open cylindrical shells partially or completely filled with liquid

3.7.1.1 Shell completely filled with liquid:

a) For the first set of calculations, we determine the frequency parameters (Ω) for different values of R/t and L/R for shells completely filled with liquid (internal).

The results obtained (10 elements) for n = 1 are given in Table 3.1 in the case of free simply-supported shells. We conclude that, as a result of the lateral pressure exerted by the liquid on the structure, the frequency parameters (Ω) depend both on L/R and R/t, in contrast to the case of the empty shell, where R/t ratio has only a slight effect upon the results.

TABLE 3.1 : Vibration parameter (Ω) of cylindrical shells simply-supported at both ends and filled with liquid. ($n = 1$, $m = 1$, $\nu = 0.3$, $\rho_l = 1000$ kg/m^3 and $\Omega = \omega R \sqrt{\rho(1-\nu^2)/E}$).

L/r \ R/t		20	50	100	200	Baron & Bleich all values of R/t
Empty	2.0	0.5775	0.5900	0.6067	0.5711	0.5728
Full		0.4196	0.3288	0.2629	0.1810	----
Empty	4.0	0.2572	0.2581	0.2594	0.2603	0.2569
Full		0.1809	0.1372	0.1065	0.07998	----
Empty	8.0	0.08744	0.08747	0.08752	0.08756	0.0874
Full		0.06020	0.04489	0.03424	0.02269	----
Empty	10.0	0.05911	0.05911	0.05913	0.05914	0.0592
Full		0.04044	0.03005	0.02283	0.01684	----

b) Next calculations were made for a steel cylindrical shell simply supported at both ends, empty or completely filled with liquid. The pertinent data are as follows:

$$R = 37.7 \text{ mm}, \quad t = 0.229 \text{ mm}, \quad L = 234 \text{ mm}, \quad \nu = 0.29, \quad \rho_i / \rho_s = 0.128.$$

The effects of the inertial force were calculated by this theory assuming $U_x = 0$ in equations (3.35) to (3.37). Table 3.2 shows some frequencies computed by the present method and compared with experimental results (Lindholm, Kana & Abramson 1962) in the case of a closed cylindrical shell both empty and completely filled with liquid. As may be seen the results obtained by the present method agree with experimental results to within 10%.

TABLE 3.2 : Natural Frequencies (Hz) of a simply-supported closed cylindrical shell, both when empty and when completely filled with liquid.

(m, n)	Empty		Full (inside fluid)	
	Present Method	Experimental (Lindholm et al.)	Present Method	Experimental (Lindholm et al.)
(1,2)	1133	1150	376	375
(1,3)	629	640	234	250
(1,4)	655	688	270	300
(1,5)	942	995	422	430
(1,6)	1353	1430	651	680
(1,7)	1853	1938	940	970
(2,3)	2067	2070	784	813
(2,4)	1368	1430	568	600
(2,5)	1248	1313	561	625
(2,6)	1489	1570	714	755
(2,7)	1927	2050	978	1000

3.7.1.2 Shells partially filled with liquid:

Here, we consider the case of a shell partially filled with liquid without taking into account the effects of the free surface. In the case of vertical cylindrical shells, partially filled with liquid, it has been determined, by comparing the results of Mistry & Menenzes (1995) with those based on the theory presented by Lakis & Paidoussis (1971), that these effects are negligible for low frequencies (less than 3%), but may go up to 30% for modes higher than seven ($m \geq 7$). In the case of horizontal shells partially filled with liquid, the results might be different due to a larger free surface. Nevertheless, our results presented here are an indication of the dynamic behaviour of such a system. A paper, under preparation, will consider in detail the effects of the free surface in the case of both horizontal and vertical shells partially filled with liquid.

a) In the case of a closed cylindrical shell, Figures 3.3 and 3.4 show some frequencies computed by the present method in which the liquid level was varied from zero to full in a cylinder with horizontal axis.

We see that, for some modes, the frequency decreases rapidly with increasing d_l/d in the range $0 < d_l/d < 1/4$ approximately and then decreases only slighly for higher fractional fillings. For other modes, however, the frequencies decrease appreciably with increasing d_l/d over the whole range of d_l/d, as might be expected.

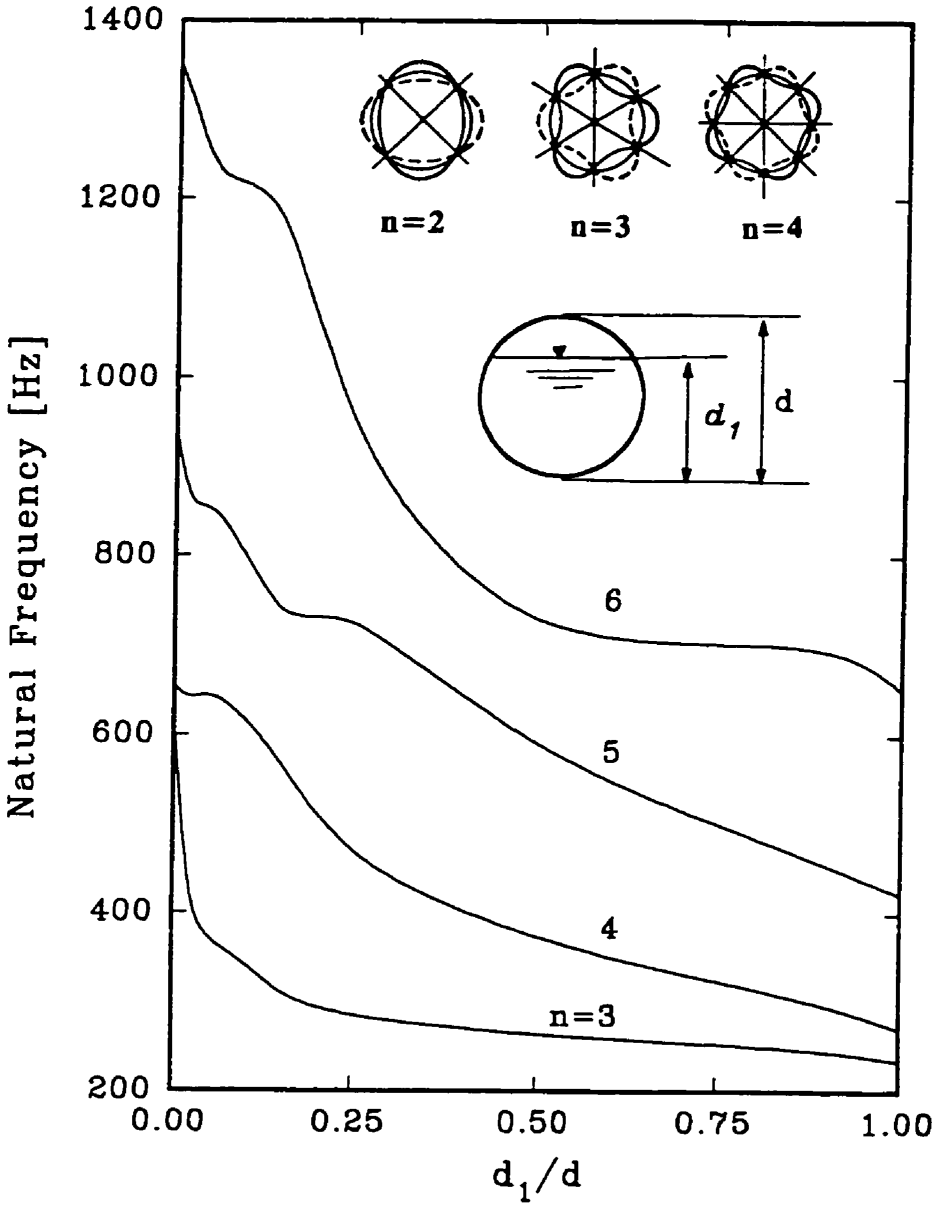

Figure 3.3 **Natural frequencies of a partially filled closed cylindrical shell supported at both ends as a function of liquid level, m=1. R = 37.7 mm, t = 0.229 mm, L = 234 mm, ν = 0.29, ρ_l/ρ_s = 0.128.**

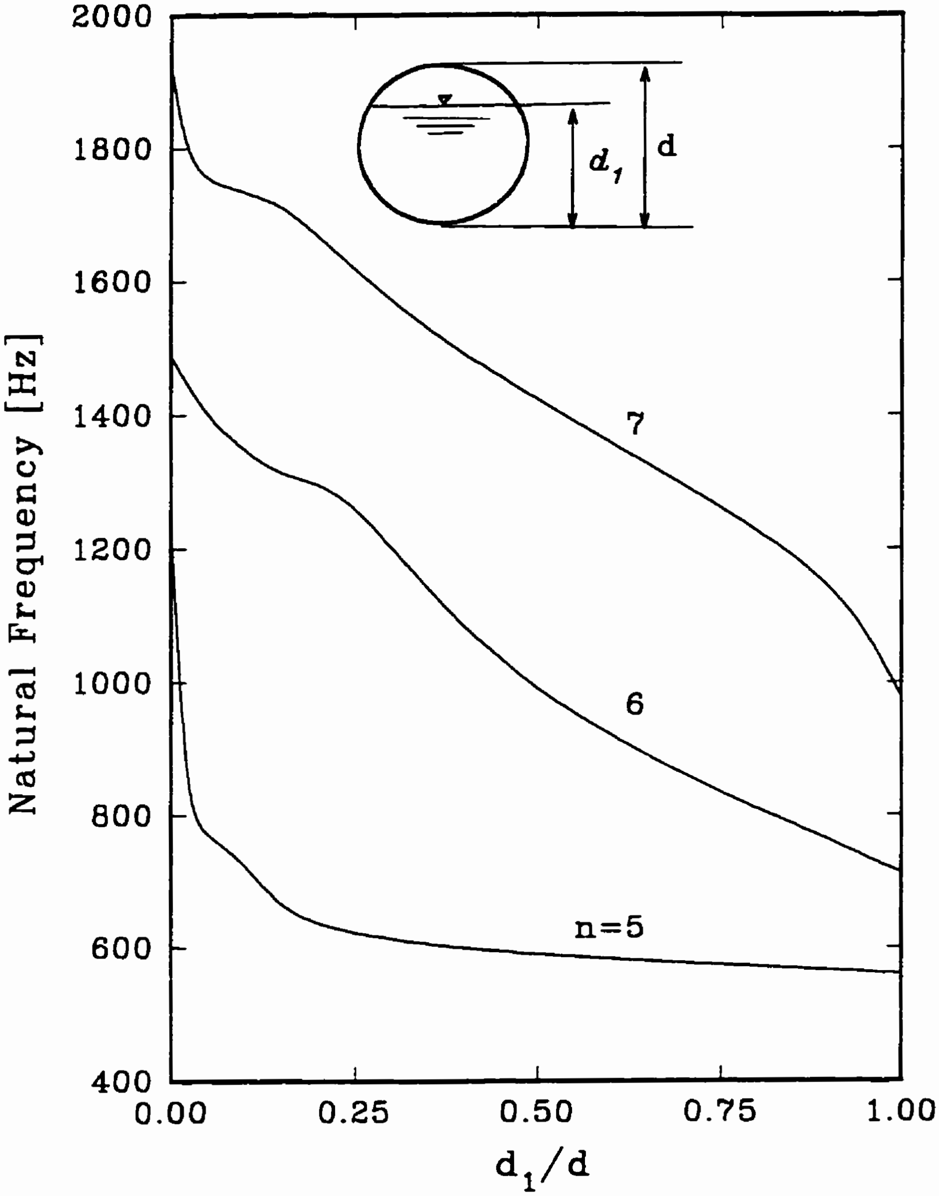

Figure 3.4 Natural frequencies of a partially filled closed cylindrical shell supported at both ends as a function of liquid level, m=2. R = 37.7 mm, t = 0.229 mm, L = 234 mm, ν = 0.29, ρ_l/ρ_s = 0.128.

b) Here, we present some results for an open cylindrical shell partially or completely filled with liquid. The open cylindrical shell is constructed of steel, is filled with water and is simply-supported at its four edges.

The pertinent data are as follows (see Figure 3.1):

$\phi_T = 180°$, $R = 37.7$ mm, $t = 0.229$ mm , $L = 234$ mm , $\nu = 0.29$, $\rho_i / \rho_s = 0.128$

In Figure 3.5, we see the behaviour of an open cylindrical shell empty and filled with liquid as a function of the number of circumferential modes. For a given *m*, the frequencies decrease to a minimum before they increase as the number of circumferential waves (*n*) is increased. This behaviour was first observed for a shell in vacuo by Arnold & Warburton (1953) , who were able to explain it by a consideration of the strain energy associated with bending and stretching of the reference surface. It may be concluded from their work, that at low *n* the bending strain energy is low and the stretching strain energy is high; while at the higher *n*, the relative contributions from the two types of strain energy are reversed. The interchange in the relative contributions of the bending and stretching strain energy as the circumferencial wave number *n* is increased explains the decrease and subsequent increase in the natural frequencies indicated in Figure 3.5. An open cylindrical shell partially or completely filled with liquid will behave in the same way.

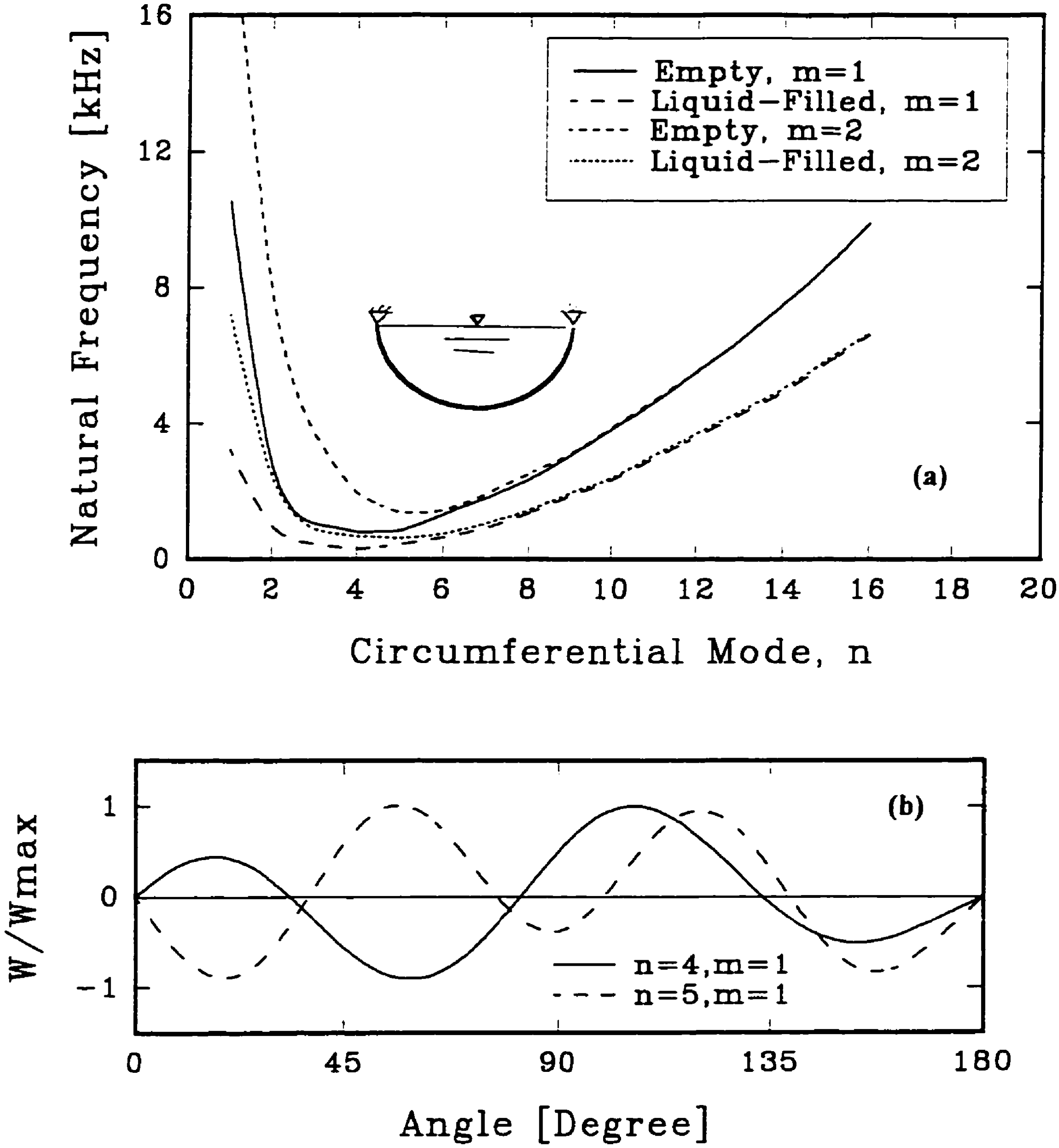

Figure 3.5 (a) Natural frequencies of an empty and liquid-filled open cylindrical shell with W=V=0 at the four edges as a function of circumferential mode number.

(b) The circumferential shapes of a liquid-filled open cylindrical shell for n=4, 5 and m=1

Figure 3.6 shows that when an open cylindrical shell is partially filled with liquid, the curves show a rapid decrease of the natural frequencies as a_1/a_2 increases from 0 to 3/4 approximately, and then decrease only slightly for higher fractional fillings.

To see the influence of the orientation of the shell, we present in Figure 3.7, the natural frequency as a function of the orientation of the shell and the free surface of the liquid, the liquid level $a_1 / a_2 = 0.64$ (see Figure 3.6).

We observe that the natural frequencies of the system decrease between the two extreme positions. The reduction is about 11% for the two modes ($m = 1$, $n = 2$) and ($m = 1$, $n = 7$).

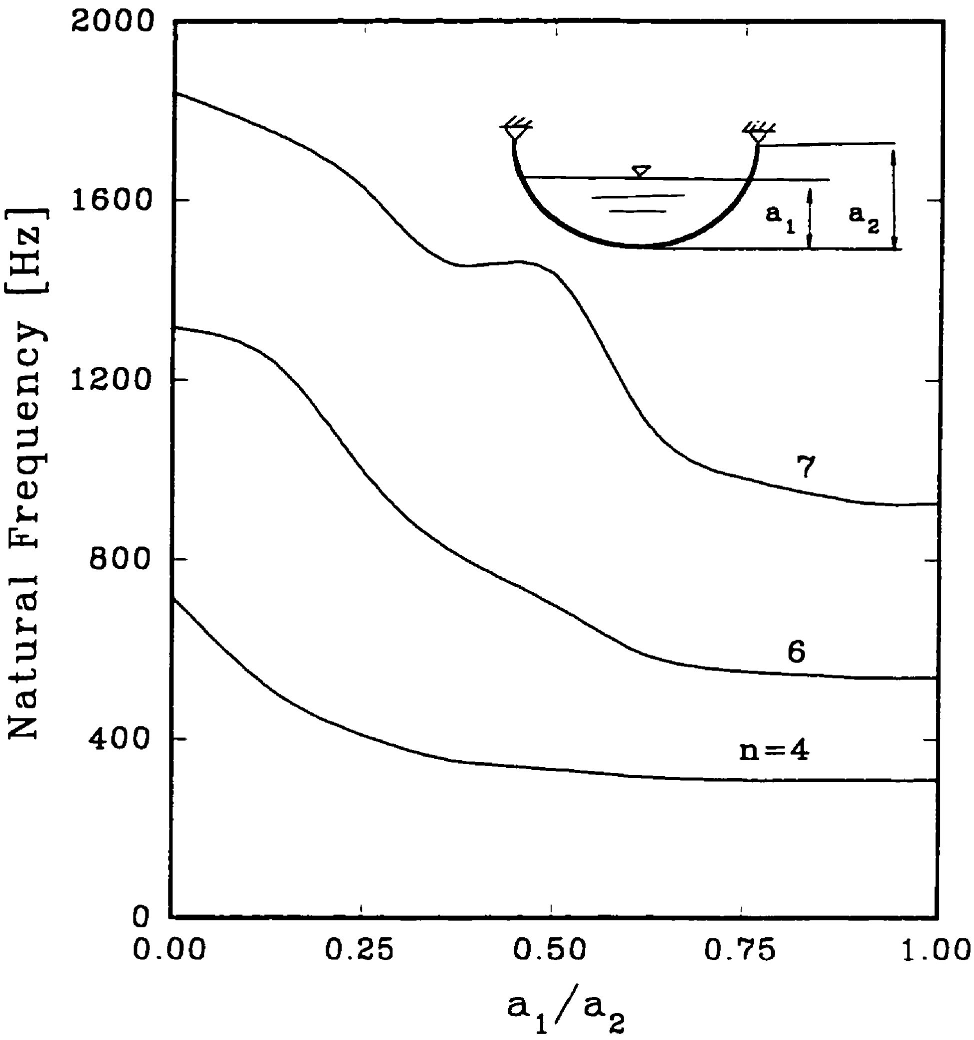

Figure 3.6 **Natural frequencies of an empty and liquid-filled open cylindrical shell with W=V=0 at the four edges as a function of liquid level, m = 1.**

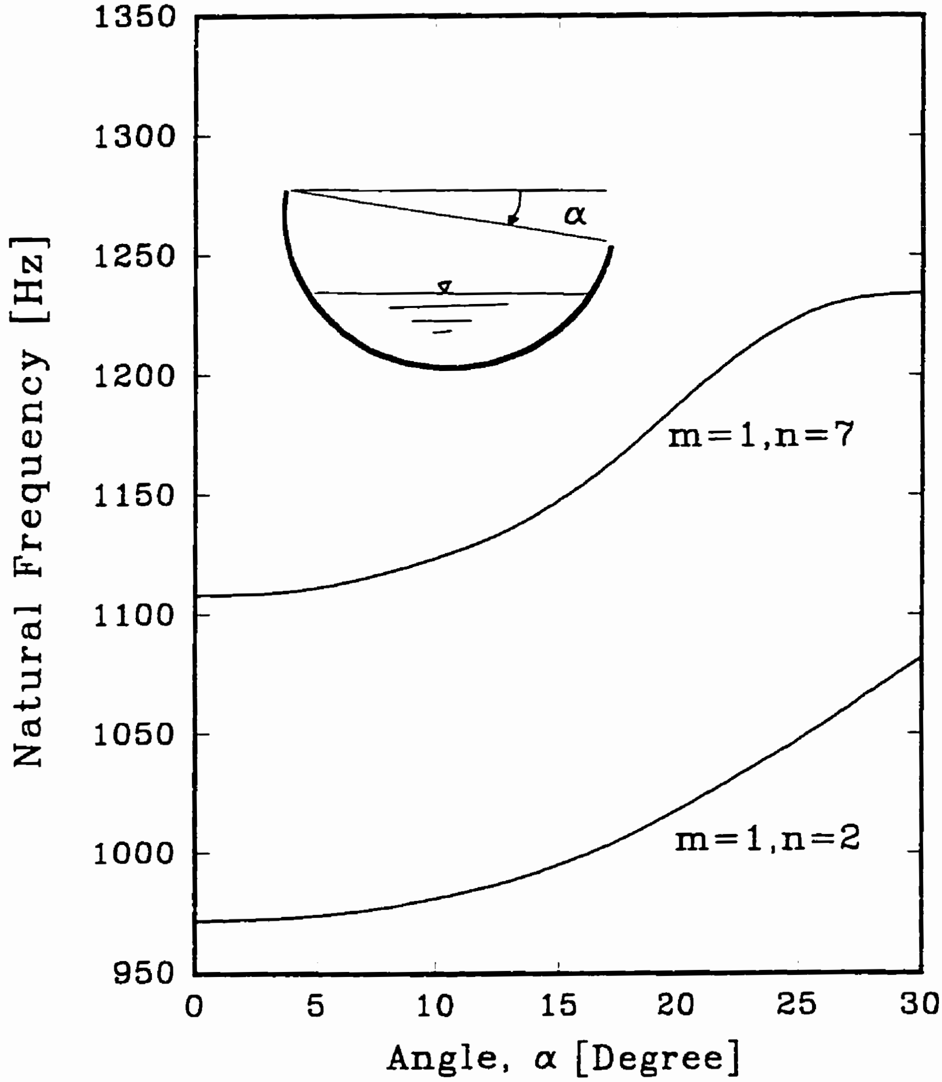

Figure 3.7 **Natural frequencies of an empty and liquid-filled open cylindrical shell with W=V=0 at the four edges as a function of the orientation of liquid level and the shell.**

3.7.2 Closed orthotropic cylindrical shells submerged in an incompressible fluid

In this calculation, we analyse the transverse vibration of isotropic and orthotropic cylindrical shells submerged in an incompressible fluid, simply-supported at both ends. This case was analysed by Ramachandran (1979) who use the Rayleigh-Ritz procedure. In Table 3.3, the values of the material properties used in the calculations are shown.

TABLE 3.3 : Material and physical properties of the shell

	E_x (x 10^{11}N/m^2)	E_θ (x 10^{11} N/m^2)	G (x 10^{11}N/m^2)	ν_x	ν_θ
Isotropy	21.981	21.981	0.8454	0.3	0.3
Orthotropy	1.0	0.5	0.1	0.05	0.025

$R = 0.235$ m, $t = 0.00235$ m. $\rho_s = 7850$ N/m^3, $\rho_f = 1000$ N/m^3

The natural frequencies of this shell-liquid system for $n = 4, 8$; $m = 1$; $L/R = 2, 4$ and different material properties of the shell are given in Table 3.4. Four cases were studied, when the shell is empty; when the fluid is inside or outside of the shell; and when the shell is submerged in a fluid.

If we compare our results with those of Ramachandran (1979), we find that there is agreement within 6% in the case of the empty isotropic shell. In the case of submerged isotropic shell (internal and external fluid), the agreement varies from 11% to 15%. But when the material of the shell is orthotropic, we find big differences between the two models (in the order of 98 %). On the other hand, in the case of the empty orthotropic cylindrical shell, our model has been tested (Selmane & Lakis 1995) and the results have been found to be within 5% of those of (Leissa 1973).

Our model combines the advantages of finite element method which deals with complex shell (variable thickness, non-uniform materials, various boundary conditions,...), and the precision of formulation which the use of dispacement functions derived from shell theory contributes (Lakis, Van Dyke & Ouriche 1992).

TABLE 3.4 : Frequency values (Hz) for simply-supported cylindrical shells, empty and filled with liquid.

Mat.	L/R	(n,m)	Theories	Empty	Inside and outside fluid (full)	Inside fluid	Outside fluid
Isotropy	4	(4,1)	Present Method	659	251.4	333.2	331.4
			Ramachandran (1979)	700	294.2	----	----
			Lakis (1976a)*	659	251.7	333.8	331.7
		(8,1)	Present Method	2187	1064	1361	1361
			Ramachandran (1979)	2200	944.1	----	----
			Lakis (1976a)*	2177	1073	1362	1360
Orthotropy	2	(4,1)	Present Method	240.1	92.2	121.9	121.6
			Ramachandran (1979)	----	183.1	----	----
			Lakis (1976a)*	238.8	92.4	121.7	121.9
		(8,1)	Present Method	327.3	158.5	203.3	200.2
			Ramachandran (1979)	----	248.5	----	----
			Lakis (1976a)*	324.1	160.7	203.2	203.9

* These results are computed from a computer program developed by A.A. Lakis & his co-workers and based on the theory presented in Lakis (1976a).

3.7.3 Dynamic stability of closed and open cylindrical shells subjected to a flowing fluid

3.7.3.1 Closed cylindrical shell containing flowing fluid:

When the fluid is flowing, the shell will be subjected to centrifugal, Coriolis and inertia forces. A simply-supported shell with the following characteristics:

$$L/R = 2, \quad t/R = 0.01, \quad \rho_i/\rho_s = 0.128, \quad n = 5$$

has been analysed, to see the influence of the speed of the flow U_{xi} on the frequencies (internal flow).

The dimensionless parameters of frequency and velocity are $\bar{\omega} = \omega/\omega_o$ and $\bar{U} = U/U_o$ where:

$$\omega_o = \frac{\pi^2}{L^2}(K/\rho_s t)^{1/2} \quad , \quad K = \frac{Et^3}{12(1-\nu^2)}$$

$$U_o = \frac{\pi^2}{L}(K/\rho_s t)^{1/2}$$

ω and U are respectively the natural frequency and the velocity of the flowing fluid.

The results are compared to a previous analysis by Weaver & Unny (1973) in Figure 3.8. We observe that the natural frequencies decrease with flow velocity. At zero flow velocity, the two methods give the same results but, as the flow velocity increases the two term Galerkin method used by Weaver & Unny (1973) generates significantly different results from those of the present hybrid finite element method. This is due to the limitations associated with the use of too few terms in the application of Galerkin's method.

One of the most important criteria in determining the versatility of a method is the capacity to predict, with precision, both the high and low frequencies. This criterion demands the use of a great many terms in Galerkin's method. The choice of the displacement functions which are derived from Sanders' (1959) classical shell theory enables our hybrid finite element model to give good low, as well as high frequencies, with a small number of finite elements.

Our results predict that the first mode frequency becomes negative imaginary at $\bar{U} = 3.1$, indicating static divergence instability in this mode. If the velocity is increased further, the first mode reappears and coalesces at $\bar{U} = 3.95$ with that of the second mode to produce coupled mode flutter.

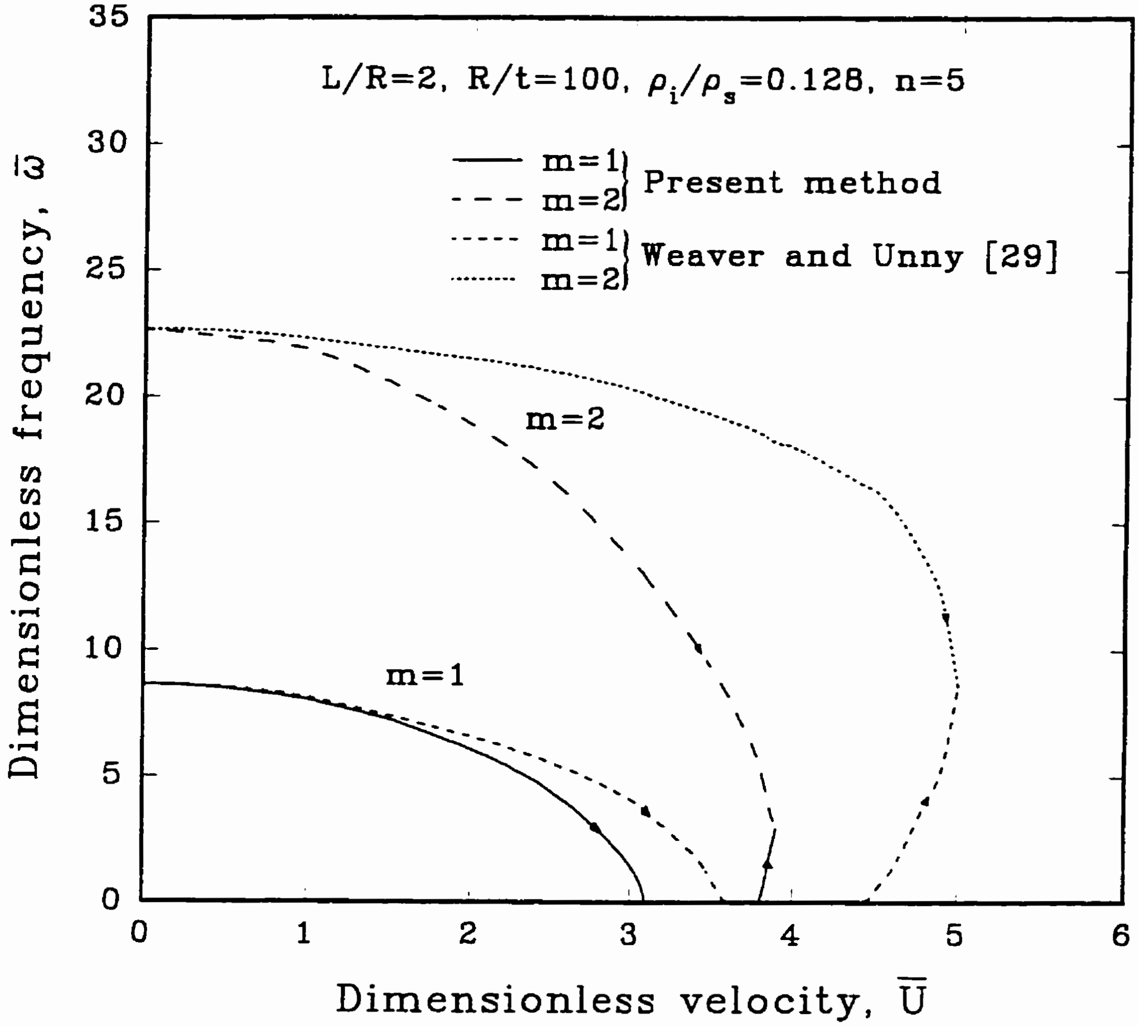

Figure 3.8 **Stability of a simply-supported closed cylindrical shell as a function of flow velocity. (internal flow).**

3.7.3.2 Totally submerged open cylindrical shell subjected simultaneously to an internal and external flow:

An open cylindrical shell subjected simultaneously to an internal and external flow has been analysed. In this case there are no effects of the free surface because the shell is totally submerged in the flowing liquid.

The data for the shell are as follows:

$$R/t = 165, \quad L/R = 6.2, \quad \phi_T = 180^\circ, \quad \rho_f / \rho_s = 0.128, \quad \nu = 0.29,$$

and the dimensionless parameters of frequency and velocity are $\bar{\omega} = \omega/\omega_o$ and $\bar{U} = U/U_o$ where

$$\omega_0 = \frac{\pi^2}{L^2}(K/\rho_s t)^{1/2} \quad , \quad K = \frac{Et^3}{12(1-\nu^2)}$$

$$U_0 = \frac{\pi^2}{L}(K/\rho_s t)^{1/2}$$

ω and U are respectively the natural frequency and the velocity of the flowing fluid.

We present here an examination of the natural frequencies of the system as functions of the flow velocity, and thereby a determination of the effect of flow on the dynamic behaviour of the system.

a) Simply-supported - simply-supported shell

A simply-supported open cylindrical shell containing flowing fluid (internal and external) has been analysed. Figure 3.9 shows the frequencies of the system as a function of the flow velocity. As the velocity increases from zero, the frequencies associated with all modes decrease, they remain real (the system being conservative) , until at sufficiently high velocities, they vanish, indicating the existence of buckling-type (divergence) instability. At higher flow velocity the frequencies become purely imaginary.

We predict the first loss of stability at a flow velocity equal to $\bar{U} = 7.75$ for the mode ($m = 1$, $n = 4$).

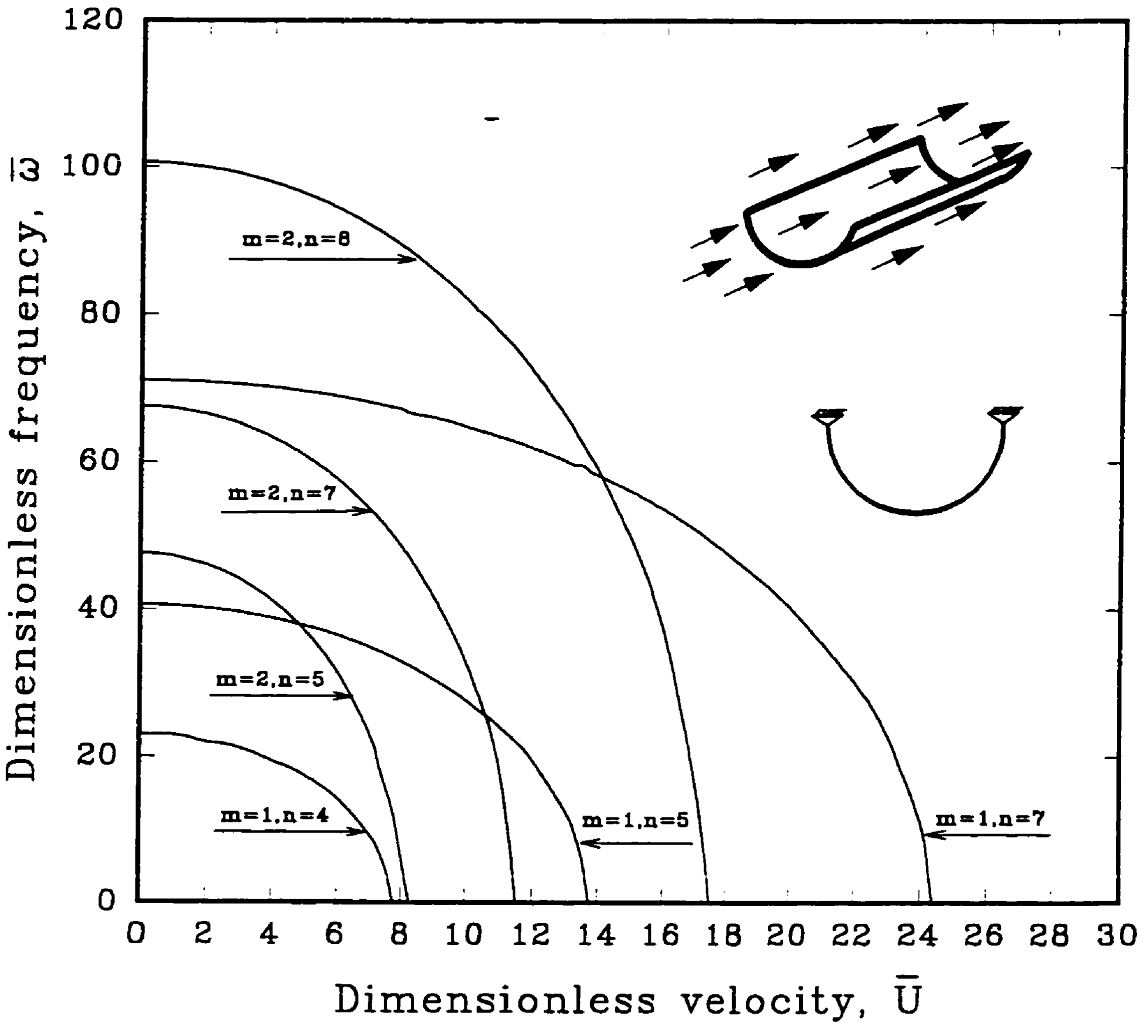

Figure 3.9 **Stability of a simply-supported submerged open cylindrical shell in a flowing fluid as a function of flow velocity.**

b) Free-Free Shell

The case of an open cylindrical shell having its straight edges free and the curved edges freely simply-supported has been studied by means of the present theory. Figure 3.10 shows that natural frequencies associated with all modes decrease with increasing flow velocity until at a value of $\bar{U} = 8.5$ (m = 1, n = 6) the system buckles.

c) Clamped-Clamped Shell

The calculations were performed for one open cylindrical shell having its straight edge clamped and the curved edges freely simply-supported. Here, we study the influence of the flow velocity on the dynamic stability of the open shell containing internal and external flow. We observe in Figure 3.11 that the frequencies associated with all modes decrease with increasing flow velocity, and similarly to the case of simply supported-simply supported and free-free open shells, the frequencies remain real until at a sufficiently high velocity, they vanish, indicating the instability. For the stipulated boundary conditions, we predict the first loss of instability at $\bar{U} = 8.25$ for the mode (m = 1, n = 4).

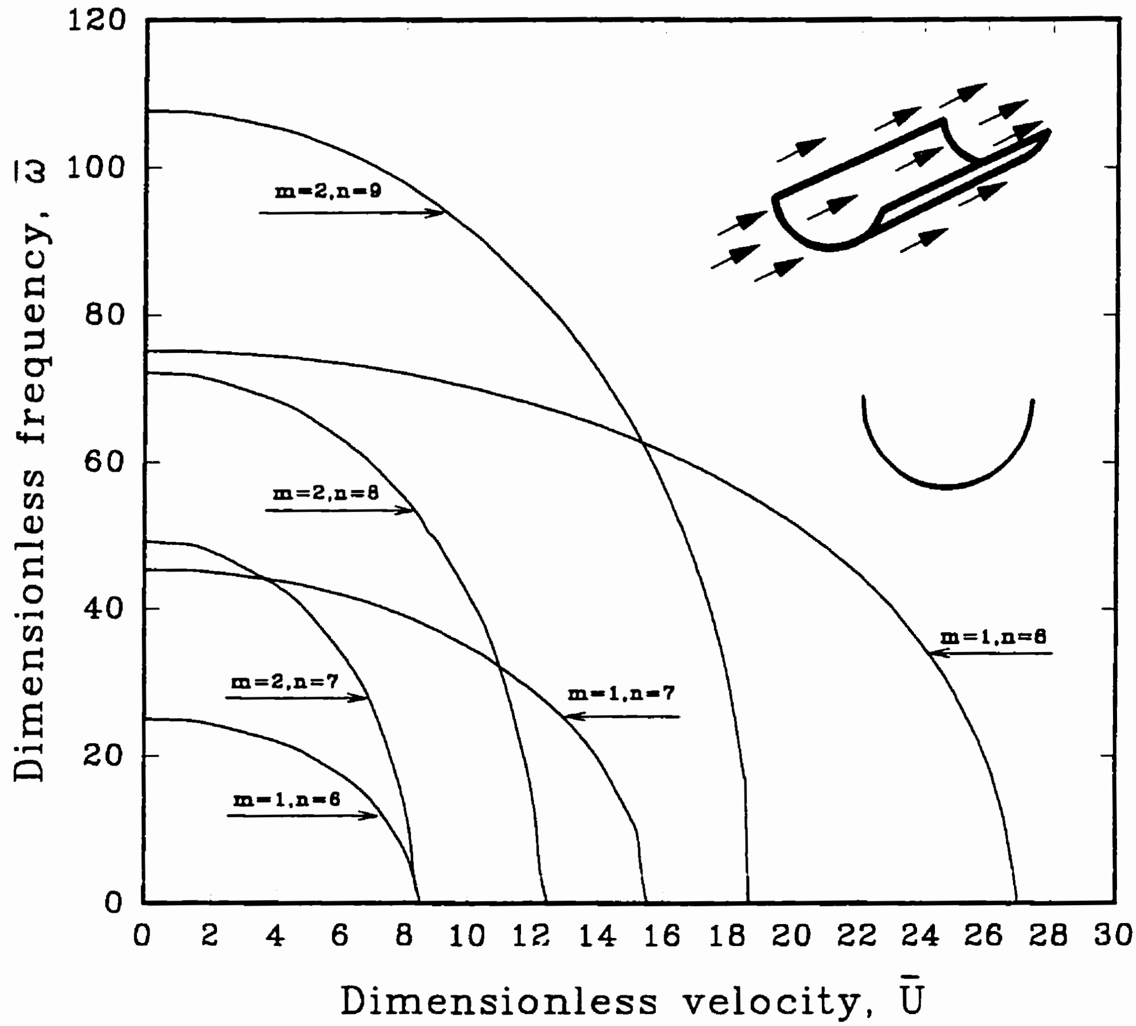

Figure 3.10 Stability of a free-free submerged open cylindrical shell in a flowing fluid as a function of flow velocity.

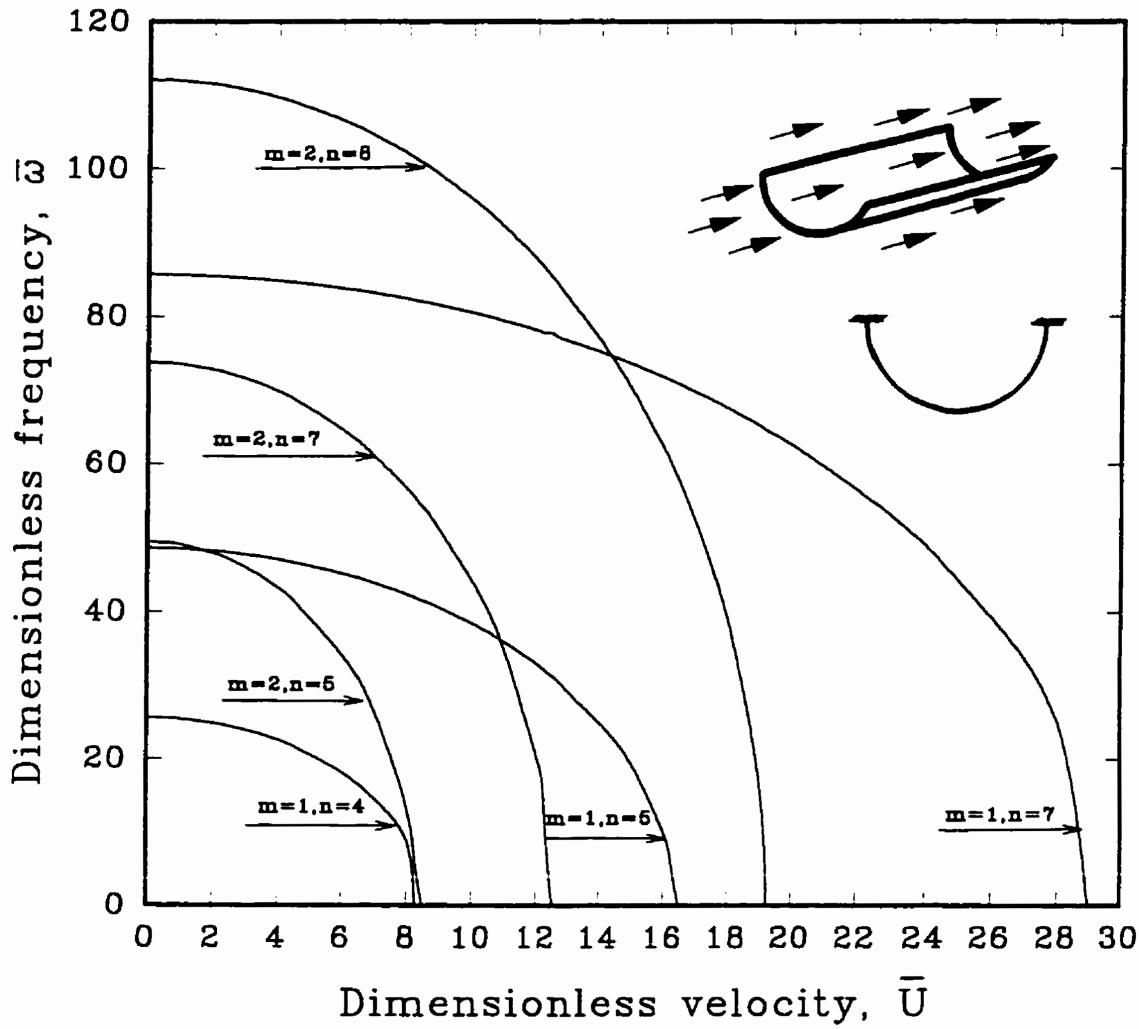

Figure 3.11 **Stability of a clamped-clamped submerged open cylindrical shell in a flowing fluid as a function of flow velocity.**

d) Comparison between the boundary conditions

In order to establish the effects of boundary conditions on the critical flow velocities which render the system dynamically unstable, we turn to Figure 3.12. We observe for the same mode and the same open shell with different boundary conditions, that the shell with free-free boundary conditions in its straight edges is the one which loses dynamic stability first.

For the mode (m = 1, n = 7) we have critical velocities as follows: Free-Free shell ($\bar{U}$ = 15.5), simply supported - simply supported shell ($\bar{U}$ = 24.4) and clamped-clamped shell ($\bar{U}$ = 29). For the mode (m = 2, n = 7), we have respectively $\bar{U}$ = 8.5; 11.5 and 12.5.

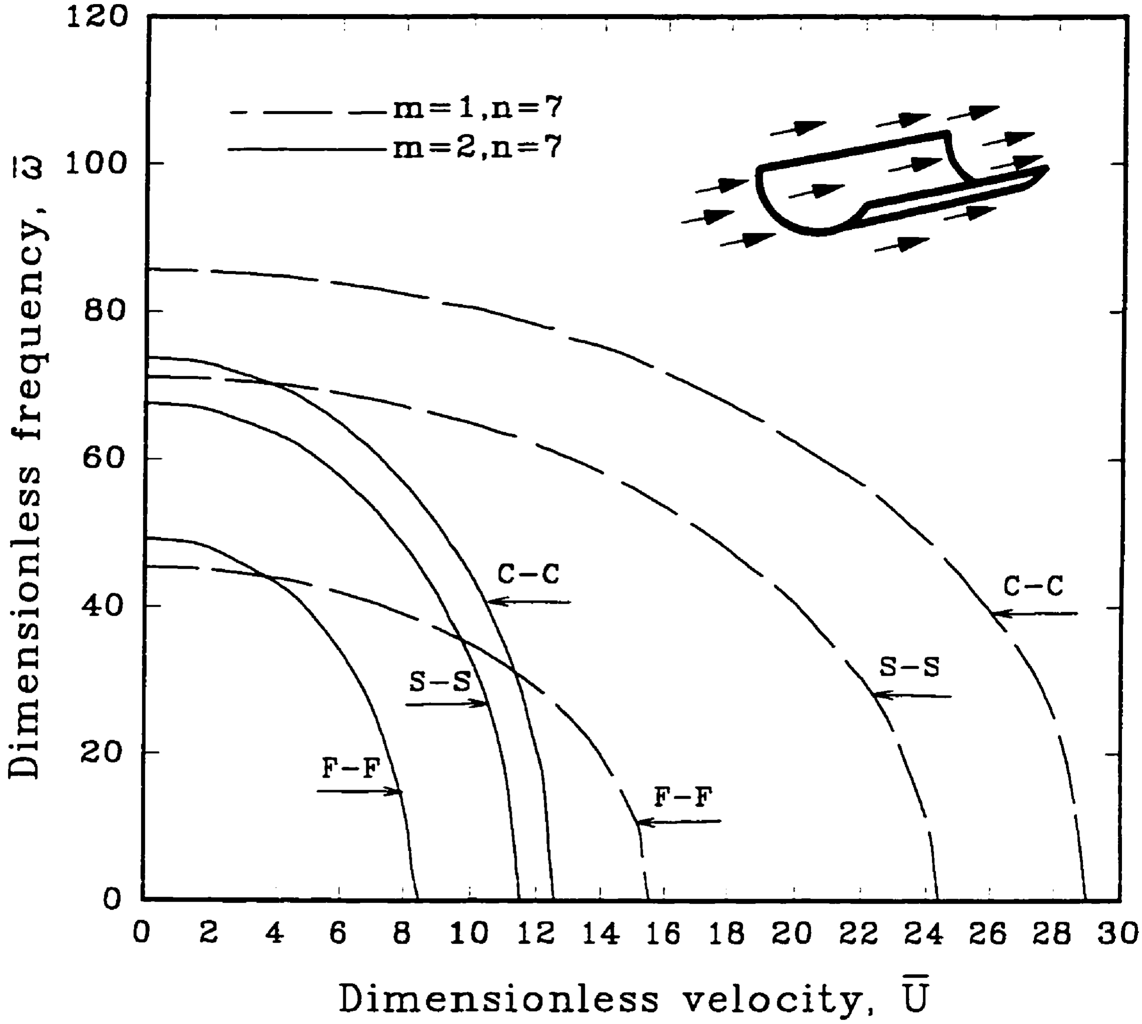

Figure 3.12 Effect of boundary conditions on the stability of an open cylindrical shell submerged in flowing fluid.

3.8 CONCLUSIONS

The theory developed in this paper is used to predict the effects of inertia, Coriolis and centrifugal forces on the vibration characteristics of totally submerged anisotropic open and closed cylindrical shells, subjected simultaneously to an internal and external flow.

A cylindrical panel finite element was developed, making possible the derivation of the displacement functions from the equations of motion of the shell. Mass and stiffness of each element were obtained by exact analytical integration.

The fluid pressure was derived from the velocity potential and from the linear impermeability and dynamic conditions applied to the shell-fluid interface. The finite element method was used to obtain the mass, stiffness and damping of fluid element. The results obtained by this method were compared with other investigations and satisfactory agreement was obtained. This method combines the advantages of finite element analysis which deals with complex shells, and the precision of formulation which the use of displacement functions derived from shell and fluid theories contributes.

This method enables us to predict the vibrationary characteristics of circumferentially non-uniform open and closed cylindrical shells subjected to a flowing fluid. In addition, this theory may be applied to a curved plate subjected to a flowing fluid in the case of large values of a shell's radius.

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3.10 NOMENCLATURE

LIST OF SYMBOLS

A, B, C	Constants in equations defining U, V, W respectively
a_1/a_2	Liquid level ratio for an open cylindrical shell
c	Velocity of sound in fluid
d_1/d	Liquid level ratio for closed cylindrical shell
E	Young's modulus
e	exponential
i	$i^2 = -1$
$J_{i\eta j}$	Bessel function of the first kind and of order $i\eta j$
K	Bending stiffness, $Et^3/12(1 - \nu^2)$
L	Length of the shell
m	Axial mode number
n	Circumferential mode number
P_u	Lateral pressure exerted on the shell, u = i for internal pressure and u = e for external pressure
P_{ij}	Terms of elasticity matrix (i= 1,...,6 ; j= 1, ..., 6)
R	Mean radius of the shell

R_j	Solution of Bessel equation (3.27)
S_j	Defined by equation (3.24)
t	Thickness of the shell
U, V, W	Axial, tangential and radial displacements
U_{xu}	Velocity of the liquid
U_o	Defined by $(\pi^2/L)\ (K/\rho_s t)^{1/2}$
$\bar{U}$	Dimensionless velocity, U_{xu}/U_o
V_x , V_θ , V_r	Axial, tangential and radial fluid velocity (3.17)
x	Axial coordinate
$Y_{i\eta j}$	Bessel function of the second kind and of order $i\eta j$
Z_{uj}	Defined by equation (3.29) for u=i and equation (3.30) for u=e
η_i	Complex roots of the characteristic equation
ϵ_x, ϵ_θ, $\bar{\epsilon}_{x\theta}$	Deformation of reference surface
κ_x, κ_θ, $\bar{\kappa}_{x\theta}$	Changes in curvature and torsion of reference surface
θ	Circumferential coordinate
ν	Poisson's ratio
ϕ	Angle for one finite element
ϕ_T	Angle for the whole open shell
Φ	Velocity potential
ρ_s	Density of the shell material
ρ_f	Density of fluid, f = i for internal fluid and f = e for external fluid

ω	Natural frequency (rad/s)
ω_0	Defined by $(\pi^2/L^2)\,(K/\rho_s t)^{1/2}$
$\bar{\omega}$	Dimensionless frequency, ω/ω_0

LIST OF MATRICES

[A]	Defined by equation (3.9)
[B]	Defined by equation (3.11)
$[c_f]$	Damping matrix for a fluid finite element
$[C_f]$	Damping matrix for the whole fluid
{C}	Vector of arbitrary constants
$[D_f]$	Defined by equation (3.36)
$[G_f]$	Defined by equation (3.37)
$[k_f]$	Stiffness matrix for a fluid finite element
$[k_s]$	Stiffness matrix for a shell finite element
$[K_f]$	Stiffness matrix for the whole fluid
$[K_s]$	Stiffness matrix for the whole shell
$[m_f]$	Mass matrix for a fluid finite element
$[m_s]$	Mass matrix for a shell finite element
$[M_f]$	Mass matrix for the whole fluid

$[M_s]$	Mass matrix for the whole shell
$[N]$	Displacement function defined by equation (3.10)
$[P]$	Elasticity matrix
$[Q]$	Defined by equation (3.11)
$[R]$	Defined by equation (3.6)
$[S_f]$	Defined by equation (3.35)
$[T_m]$	Defined by equation (3.3)
$\{\delta_i\}$	Degree of freedom at node i
$\{\epsilon\}$	Deformation vector
$\{\sigma\}$	Stress vector

CHAPITRE IV

ARTICLE III

INFLUENCE OF GEOMETRIC NON-LINEARITIES ON THE FREE VIBRATIONS OF ORTHOTROPIC OPEN CYLINDRICAL SHELLS

4.1 ABSTRACT

This paper presents a general approach to predict the influence of geometric non-linearities on the free vibration of elastic, thin, orthotropic and non-uniform open cylindrical shells. The open shells are assumed to be freely simply-supported along their curved edges and to have arbitrary straight edge boundary conditions. The method is a hybrid of finite element and classical thin shell theories.

The solution is divided into two parts. In part one, the displacement functions are obtained from Sanders' linear shell theory and the mass and linear stiffness matrices are obtained by the finite element procedure. In part two, the modal coefficients derived from the Sanders-Koiter non-linear theory of thin shells are obtained for these

displacement functions. Expressions for the second order and third order non-linear stiffness matrices are then determined through the finite element method.

The non-linear equation of motion is solved by the fourth-order Runge-Kutta numerical method. The linear and non-linear natural frequency variations are determined as a function of shell amplitudes for different cases. The results obtained reveal that the frequencies calculated by this method are in good agreement with those obtained by other authors.

4.2 INTRODUCTION

The analysis of thin shells under static or dynamic load has been the focus of many investigations. Most of the research in this field has involved analysis of linear thin shells. The results have proven to be satisfactory in cases where deflections of the shell were very small compared to the thickness of the shell itself. In several practical experiments, however, the linear analysis was not sufficiently accurate for satisfactory design. In those cases, a non-linear analysis was required.

The first paper to deal with non-linear vibrations of shells was the pioneering work of Reissner[1]. There are now several theories available dealing with geometric non-linearities in shells[2-5] and many others.

More specifically, several methods have been developed for the analysis of dynamic non-linear thin cylindrical shells. Among these were Galerkin's method[6-12], the small perturbation method[13-15], the modal expansion method[16] and the finite element method[17-20].

Most of the research done in references 6 to 20 was limited to studies of isotropic shells. Only Nowinski[6], Raouf & Palazotto[11] and Jiang & Olsen[20] made a generalization concerning orthotropic shell theory. Ambartsumyan[21] produced an important work involving a number of cases for anisotropic shells.

All of these methods have their advantages and disadvantages. The best test of any method is probably its general content and the capacity to predict, with precision, both the high and the low frequencies of vibration. These criteria were not met in Galerkin's small perturbation method, and studies in references 6 to 15 applied only to the particular case where the shell was simply-supported on both edges. Furthermore the analytical forms for the displacement components in the modal expansion[16] apply only to those cases where a uniform cylinder is supported at both ends.

The finite element method appears to be ideally suited to the analysis of complex shell structures. Numerous general computer programmes are available for industrial use in the linear and non-linear analysis, where the displacement functions of the finite elements used are assumed to be polynomial. To be able to predict with precision, both the high and the low frequencies, requires the use of a great many elements in the classical finite element method. In order to achieve this, the present paper presents a new finite element for the static or dynamic analysis of non-linear, elastic, thin, anisotropic and circumferentially non-uniform open cylindrical shells (Figure 4.1). The shells are assumed to be freely simply-supported along their curved edges and to have arbitrary straight edge boundary conditions. The finite element method is employed, but it is a hybrid, a combination of the finite element method and shell theory.

This choice allows us to use the complete equilibrium equations to determine the displacement functions and, further, the mass and stiffnesses matrices. This method proves to be more accurate than the usual finite element methods[22-29].

The dynamic behaviour of an empty open or closed cylindrical shell, in the absence of external loads, can be represented by the following equation:

$$[M]\,\{\ddot{\delta}\} + [K_L]\,\{\delta\} + [K_{NL2}]\,\{\delta^2\} + [K_{NL3}]\,\{\delta^3\} = \{0\} \qquad \textbf{(4.1)}$$

where $\{\delta\}$ is the displacement vector; $[M]$ the mass matrix, $[K_L]$ the linear stiffness matrix, $[K_{NL2}]$ the second order non-linear stiffess matrix and $[K_{NL3}]$ the third order non-linear stiffness matrix of the system.

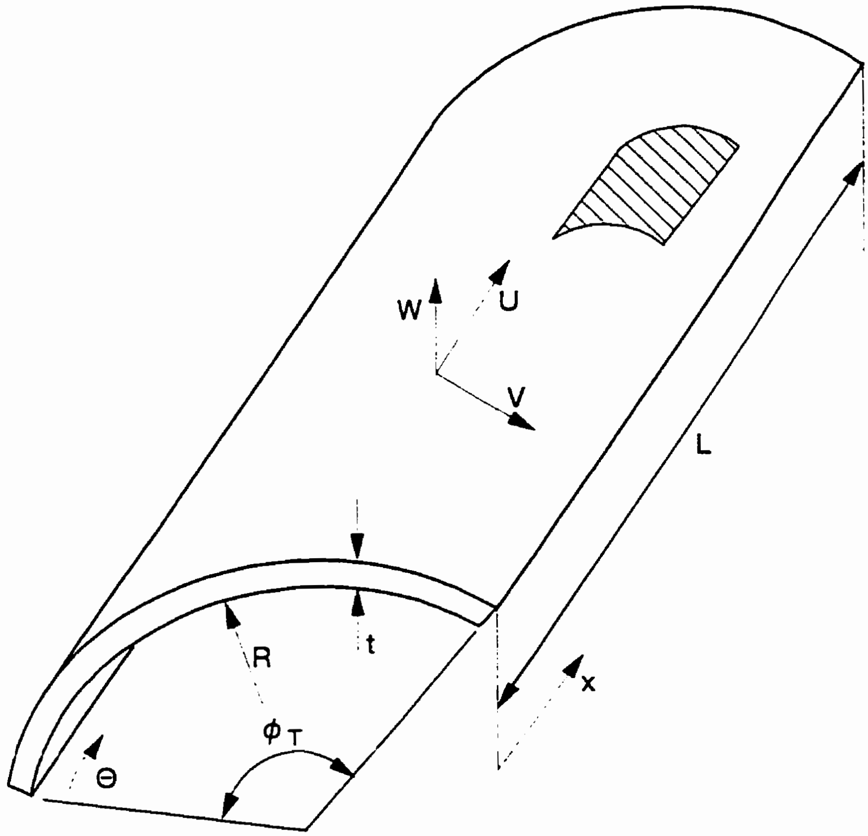

Figure 4.1 Open cylindrical shell geometry.

The analytical solution involves two steps:

a) Using the linear strain-displacement and stress-strain relationships which are inserted into Sanders' equations of equilibrium[30], we determine the displacement functions by solving the linear equation system. We then determine the mass and linear stiffness matrices for a finite element and assemble the matrices for the complete shell.

b) Using strain-displacement relationships from the Sanders-Koiter non linear theory[3,4], the modal coefficients are obtained from the displacements functions. The second and third order non-linear stiffness matrices for a finite element are then calculated by precise analytical integration with respect to modal coefficients[16].

The linear and non-linear natural vibration frequency ratio is then obtained by solving equation (4.1).

4.3 EQUATIONS OF MOTION

4.3.1 Hypotheses

Non-linear elastic thin shell theory is derived by approximation from the three-dimensional elasticity equation. As in the case of linear theory, it is based on Love's

"First Approximation" but the assumption concerning the order of magnitude of the bending has been modified.

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The non-linear theory is based on the following hypotheses:

a) Thickness (t) is infinitesimal in comparison with the minimum radius of curvature (R_{min}), ($R/t > 10$);

b) the displacement gradients are small and the squares of the rotation do not exceed reference surface deformation in order of magnitude, ($A/t < 2.5$);

c) the normal constraints, normal to the surface of reference, are negligible;

d) the normals to the surface of reference remain normal after deformation and are not subject to any elongation.

The theory based on these four hypotheses is known as the Sanders-Koiter non-linear theory[3,4]"; it has been used throughout this paper.

4.3.2 Strain-displacement and stress-strain relations

The non-linear Sanders-Koiter theory for thin shells postulates differences in the first and second fundamental forms between the reference surfaces, deformed and non deformed.

Generally, the deformation vector $\{\epsilon\}$ is written as:

$$\{\epsilon\} = \{\epsilon_L\} + \{\epsilon_{NL}\} = \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{\theta\theta} \\ 2\epsilon_{x\theta} \\ \kappa_{xx} \\ \kappa_{\theta\theta} \\ 2\kappa_{x\theta} \end{Bmatrix} \tag{4.2}$$

where subscripts "L" and "NL" mean "linear" and "non-linear", respectively.

For a cylindrical shell, the expressions for $\{\epsilon_L\}$ and $\{\epsilon_{NL}\}$ are given by :

$$\{\epsilon_L\} = \begin{Bmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{1}{R}\left[\dfrac{\partial V}{\partial \theta} + W\right] \\ \dfrac{\partial V}{\partial x} + \dfrac{1}{R}\dfrac{\partial U}{\partial \theta} \\ -\dfrac{\partial^2 W}{\partial x^2} \\ -\dfrac{1}{R^2}\left[\dfrac{\partial^2 W}{\partial \theta^2} - \dfrac{\partial V}{\partial \theta}\right] \\ -\dfrac{2}{R}\dfrac{\partial^2 W}{\partial x \partial \theta} + \dfrac{3}{2R}\dfrac{\partial V}{\partial x} - \dfrac{1}{2R^2}\dfrac{\partial U}{\partial \theta} \end{Bmatrix} \qquad \textbf{(4.3)}$$

and

$$\{\epsilon_{NL}\} = \begin{Bmatrix} \dfrac{1}{2}\left[\dfrac{\partial W}{\partial x}\right]^2 + \dfrac{1}{8}\left[\dfrac{\partial V}{\partial x} - \dfrac{1}{R}\dfrac{\partial U}{\partial \theta}\right]^2 \\ \dfrac{1}{2R^2}\left[V - \dfrac{\partial W}{\partial \theta}\right]^2 + \dfrac{1}{8}\left[\dfrac{\partial V}{\partial x} - \dfrac{1}{R}\dfrac{\partial U}{\partial \theta}\right]^2 \\ \dfrac{1}{2R}\left[\dfrac{\partial W}{\partial x}\dfrac{\partial W}{\partial \theta} - V\dfrac{\partial W}{\partial x}\right] \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \textbf{(4.4)}$$

where U, V and W are, respectively, the axial, tangential and radial displacements of the shell's surface of reference.

It is evident that in equations (4.3) and (4.4) the expressions for components κ_{xx}, $\kappa_{\theta\theta}$, $2\kappa_{x\theta}$ are linear. This fits in with hypothesis (b) from paragraph 4.3.1.

The constituent relations between the stress and deformation vectors of the surface of reference for anisotropic shells are given as follows:

$$\{\sigma\} = \{ N_{xx}\ N_{\theta\theta}\ \bar{N}_{x\theta}\ M_{xx}\ M_{\theta\theta}\ \bar{M}_{x\theta} \}^T = [P]\ \{\epsilon\} \qquad \textbf{(4.5)}$$

where [P] is the matrix of elasticity. The elements p_{ij} in [P] determine the orthotropy of the shell, which depends on the mechanical characteristics of the structure's material.

In general, this implies that:

$$[P] = \begin{bmatrix} p_{11} & p_{12} & 0 & p_{14} & p_{15} & 0 \\ p_{21} & p_{22} & 0 & p_{24} & p_{25} & 0 \\ 0 & 0 & p_{33} & 0 & 0 & p_{36} \\ p_{41} & p_{42} & 0 & p_{44} & p_{45} & 0 \\ p_{51} & p_{52} & 0 & p_{54} & p_{55} & 0 \\ 0 & 0 & p_{63} & 0 & 0 & p_{66} \end{bmatrix} \qquad \textbf{(4.6)}$$

4.3.3 Equations of equilibrium

By applying the virtual work principle to the infinitesimal element of the deformed surface of reference, the three equations of equilibrium, describing the non-linear behaviour of an arbitrarily formed shell, are obtained[3] (see Figure 4.2).

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{R}\frac{\partial \bar{N}_{x\theta}}{\partial \theta} - \frac{1}{2R^2}\frac{\partial \bar{M}_{x\theta}}{\partial \theta} - \frac{1}{2R}\frac{\partial}{\partial \theta}\left[\phi \left(N_{xx} + N_{\theta\theta}\right) \right] = 0 \tag{4.7}$$

$$\frac{1}{R}\frac{\partial N_{\theta\theta}}{\partial \theta} + \frac{\partial \bar{N}_{x\theta}}{\partial x} + \frac{1}{R^2}\frac{\partial M_{\theta\theta}}{\partial \theta} + \frac{3}{2R}\frac{\partial \bar{M}_{x\theta}}{\partial x} - \frac{1}{R}\left(\phi_x \bar{N}_{x\theta} + \phi_\theta N_{\theta\theta}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left[\phi \left(N_{xx} + N_{\theta\theta}\right) \right] = 0 \tag{4.8}$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{2}{R}\frac{\partial^2 \bar{M}_{x\theta}}{\partial x \partial \theta} + \frac{1}{R^2}\frac{\partial^2 M_{\theta\theta}}{\partial \theta^2} - \frac{1}{R} N_{\theta\theta} - \frac{\partial}{\partial x}\left[\phi_x N_{xx} + \phi_\theta \bar{N}_{x\theta} \right] - \frac{1}{R}\frac{\partial}{\partial \theta}\left[\phi_x \bar{N}_{x\theta} + \phi_\theta N_{\theta\theta} \right] = 0 \tag{4.9}$$

where

$$\phi = \frac{1}{2}\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta} \right], \quad \phi_x = -\frac{\partial W}{\partial x} \text{ and } \phi_\theta = -\frac{1}{R}\left[\frac{\partial W}{\partial \theta} - V \right] \tag{4.10}$$

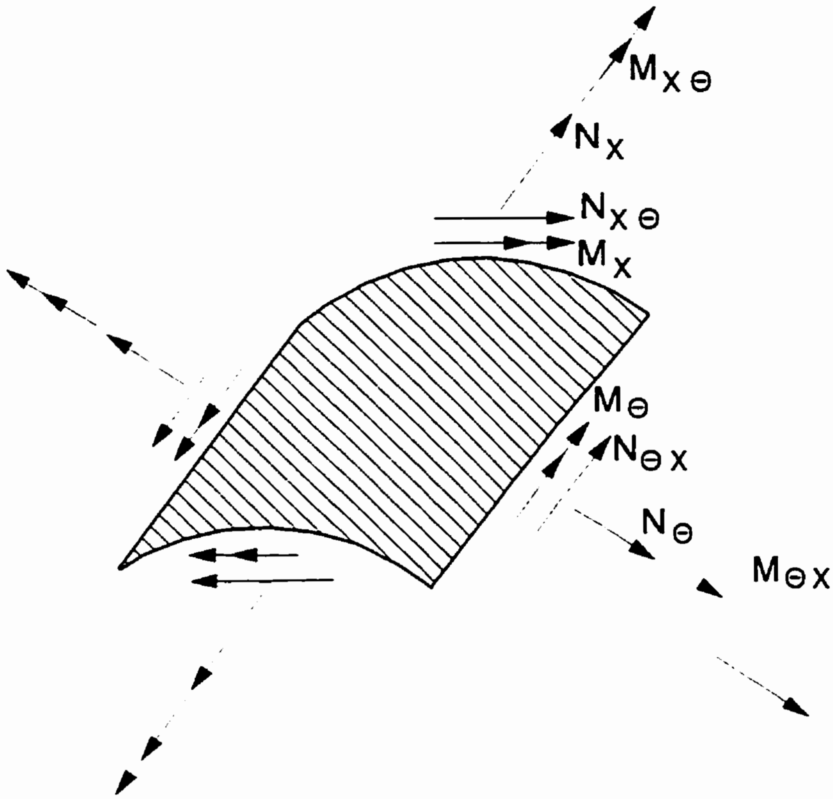


Figure 4.2 **Differential element for an open cylindrical shell.**

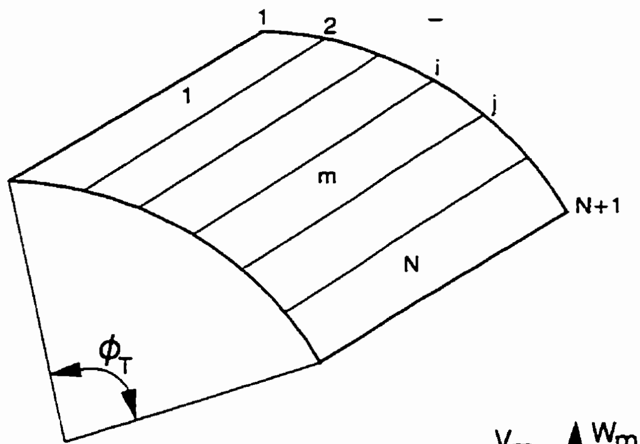
Substituting equations (4.2) to (4.6) for the equilibrium equations (4.7) to (4.10), we obtain equations as a function of elements p_{ij} in [P] and the axial, tangential and radial displacements U, V and W from a point on the shell surface of reference. The linear terms of these equations will be written as follows:

$$\begin{aligned} L_1\,(U, V, W, p_{ij}) &= 0 \\ L_2\,(U, V, W, p_{ij}) &= 0 \\ L_3\,(U, V, W, p_{ij}) &= 0 \end{aligned} \qquad \mathbf{(4.11)}$$

These equations are given in Appendix A4-1.

4.4 DISPLACEMENT FUNCTIONS

The shell is subdivided into several finite elements defined by two nodes i and j and by components U, V, W and $dW/d\theta$, representing axial, tangential, radial displacements and the rotation, respectively, from a point located on the shell's surface of reference (Figure 4.3).



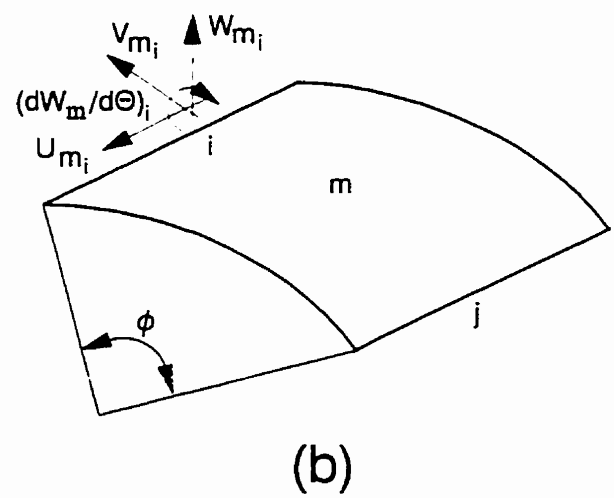


Figure 4.3 (a) Finite element idealization.
(b) Nodal displacements at node i.

The displacement functions are then assumed to be:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T_m] \begin{Bmatrix} U(\theta) \\ W(\theta) \\ V(\theta) \end{Bmatrix} \tag{4.12}$$

where m is the axial mode, $[T_m]$ is a (3 x 3) matrix in given in Appendix A4-3 and $U(\theta)$, $W(\theta)$ and $V(\theta)$ are functions of the θ coordinate and the shell's characteristics.

Assuming:

$$U(\theta) = Ae^{\eta\theta}\ , \quad V(\theta) = Be^{\eta\theta}\ , \quad W(\theta) = Ce^{\eta\theta} \tag{4.13}$$

Substituting (4.12) and (4.13) into the equations of motion (4.11), three homogeneous linear functions of constants A, B and C are obtained. For the solution to be non-trivial, the determinant of this system must be equal to zero. This brings us to the following characteristic equation:

$$\text{Det}\ ([H]) = h_8\eta^8 + h_6\eta^6 + h_4\eta^4 + h_2\eta^2 + h_0 = 0 \tag{4.14}$$

where h_0, h_2, h_4, h_6 and h_8 are listed in Appendix A4-2

Each root of this equation yields a solution to the equations of motion (4.11). The complete solution is obtained by adding the eight solutions independently with the constants A_p, B_p and C_p (p = 1, ..., 8), so that:

$$U(\theta) = \sum_{p=1}^{8} A_p e^{\eta_p \theta}, \quad V(\theta) = \sum_{p=1}^{8} B_p e^{\eta_p \theta}, \quad W(\theta) = \sum_{p=1}^{8} C_p e^{\eta_p \theta} \qquad \textbf{(4.15)}$$

The constants A_p, B_p and C_p are not independent. We can therefore express A_p and B_p as a function of C_p, for example:

$$A_p = \alpha_p C_p \quad \text{and} \quad B_p = \beta_p C_p \quad , \quad p = 1,\ldots,8 \qquad \textbf{(4.16)}$$

The values of α_p and β_p can be obtained from system (4.11) by introducing relations (4.16). Substituting expressions (4.15) and (4.16) into equations (4.12), the displacements $U(x,\theta)$, $V(x,\theta)$ and $W(x,\theta)$ can then be expressed in conjunction with the eight C_p constants only. We then have:

$$\{ U(x,\theta), W(x,\theta), V(x,\theta) \}^T = [T_m] [R] \{C\} \qquad \textbf{(4.17)}$$

where matrices $[T_m]$ and [R] are given in Appendix A4-3 and {C} is an 8^{th} order vector of the C_p constants:

$$\{C\} = \{C_1 C_2 \ldots C_8\}^T \tag{4.18}$$

To determine the eight C_p constants, it is necessary to formulate eight boundary conditions for the finite elements. The axial, tangential and radial displacements, as well as rotation, will be specified for each node. The degrees of freedom at node i can be defined by the vector:

$$\{\delta_i\} = \left\{ U_i, W_i, \left[\frac{dW}{d\theta}\right]_i, V_i \right\}^T \tag{4.19}$$

The elements which have two nodes and eight degrees of freedom will have $i(\theta = 0)$ and $j(\theta = \phi)$ as nodal displacements at the boundaries:

$$\left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} = \left\{ U_i, W_i, \left[\frac{dW}{d\theta}\right]_i, V_i, U_j, W_j, \left[\frac{dW}{d\theta}\right]_j, V_j \right\}^T = [A]\{C\} \tag{4.20}$$

where the terms of matrix [A], given in Appendix A4-3, are obtained from matrix [R] by successively setting $\theta = 0$ and $\theta = \phi$.

Multiplying equation (4.20) by $[A^{-1}]$ we obtain:

$$\{C\} = [A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.21}$$

Substituting equation (4.21) into equations (4.17), we obtain:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T_m]\ [R]\ [A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.22}$$

where the matrices $[T_m]$, [R] and [A] are given in Appendix A4-3. [N] represents the displacement functions matrix.

4.5 MASS AND LINEAR STIFFNESS MATRICES FOR AN ELEMENT

The linear deformation vector can be obtained from equations (4.3) and (4.22), therefore:

$$\{\epsilon_L\} = \begin{bmatrix} [T_m] & [O] \\ [O] & [T_m] \end{bmatrix} [Q]\ [A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.23}$$

where the matrices [A] and [Q] are given in Appendix A4-3.

Combining equations (4.5) and (4.23), the stress-strain relations can be written as:

$$\{\sigma_L\} = [P]\,[B] \left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} \tag{4.24}$$

The mass and linear stiffness matrices can then be expressed as:

$$\begin{aligned} [m] &= \rho t \int\int [N^T]\,[N]\,dA, \\ [k_L] &= \int\int [B^T]\,[P]\,[B]\,dA \end{aligned} \tag{4.25}$$

where $dA = Rdxd\theta$, ρ is the density of the shell, t its thickness, [P] the elasticity matrix and the matrices [N] and [B] are derived from equations (4.22) and (4.23) respectively.

The matrices [m] and $[k_L]$ were obtained analytically by carrying out the necessary matrix operations and integration over x and θ in equation (4.25). These matrices are also given in Appendix A4-3.

4.6 NON-LINEAR MATRIX CONSTRUCTION

The following approach, developed by Radwan and Genin[16], was used with particular attention to geometric non-linearities. The coefficients of the modal equations were obtained through the Lagrange method. Thus, the non-linear stiffness matrices, once calculated, were overlaid onto the linear system. Before we embark on matrix formulation, however, a brief summary of the method is in order.

(a) Shell displacements are expressed as generalized product coordinate sums and spatial functions;

(b) the deformation vector is written as a function of the generalized coordinates by separating the linear portion from the non-linear;

(c) these expressions are then introduced into the Lagrange equations up to and including the degree corresponding to the deformation energy;

(d) by substituting the expressions in a) into the strain-displacement relations in the Sanders-Koiter[3,4] non-linear theory, the generalized coordinate coefficients appearing in the equation derived under c) are determined in terms of spatial functions.

This approach give us the following non-linear modal equations:

$$\sum_r m_{pr}\ddot{\delta}_r + \sum_r k_{pr}^{(L)}\delta_r + \sum_r\sum_s k_{prs}^{(NL2)}\delta_r\delta_s + \sum_r\sum_s\sum_q k_{prsq}^{(NL3)}\delta_r\delta_s\delta_q = 0, \quad p=1,2,\ldots \tag{4.26}$$

where m_{pr}, $k_{pr}^{(L)}$ are the terms of mass and linear stiffness matrices given by equation (4.25); the terms $k_{prs}^{(NL2)}$ and $k_{prsq}^{(NL3)}$ which represent the second and third non-linear stiffnesses are given by the following integrals in the case of orthotropic open cylindrical shell.

$$k_{prs}^{(NL2)} = \int\int \{ p_{11}A_{prs} + p_{22}B_{prs} + p_{12}(D_{prs} + E_{prs}) + p_{33}C_{prs} \}\, dA \tag{4.27}$$

and

$$k_{prsq}^{(NL3)} = \int\int \{ p_{11}A_{prsq} + p_{22}B_{prsq} + p_{12}(D_{prsq} + E_{prsq}) + p_{33}C_{prsq} \}\, dA \tag{4.28}$$

where $dA = Rdxd\theta$, p_{ij} are the terms of the elasticity matrix [P], and the terms A_{prs}, B_{prs}, C_{prs}, D_{prs}, E_{prs} and A_{prsq}, B_{prsq}, C_{prsq}, D_{prsq}, E_{prsq} represent the coefficients of the modal equations mentioned in step d).

These coefficients are given by the following equation:

$$
\begin{aligned}
A_{prs} &= a_p A_{rs} + a_r A_{sp} + a_s A_{pr} & A_{prsq} &= 2 A_{pq} A_{rs} \\
B_{prs} &= b_p B_{rs} + b_r B_{sp} + b_s B_{pr} & B_{prsq} &= 2 B_{pq} B_{rs} \\
C_{prs} &= c_p C_{rs} + c_r C_{sp} + c_s C_{pr} & C_{prsq} &= 2 C_{pq} C_{rs} \\
D_{prs} &= a_r B_{sp} + a_s B_{pr} + b_p A_{rs} & D_{prsq} &= 2 A_{pq} B_{rs} \\
E_{prs} &= b_r A_{sp} + b_s A_{pr} + a_s B_{rs} & E_{prsq} &= 2 B_{pq} A_{rs}
\end{aligned} \tag{4.29}
$$

with

$$
a_p = U_{p,x}, \quad b_p = \frac{1}{R}\left(V_{p,\theta} + W_p\right), \quad c_p = \frac{1}{2}\left(\frac{U_{p,\theta}}{R} + V_{p,x}\right) \tag{4.30}
$$

$$
A_{pq} = \frac{1}{8R^2}\left(RV_{p,x} - U_{p,\theta}\right).\left(RV_{q,x} - U_{q,\theta}\right) + \frac{1}{2} W_{p,x} W_{q,x} \tag{4.31}
$$

$$
\begin{aligned}
B_{pq} = {} & \frac{1}{8R^2}\left(RV_{p,x} - U_{p,\theta}\right).\left(R\, V_{q,x} - U_{q,\theta}\right) \\
& + \frac{1}{2R^2}\left(W_{p,\theta} - V_p\right).\left(W_{q,\theta} - V_q\right)
\end{aligned} \tag{4.32}
$$

$$
C_{pq} = \frac{1}{4R}\left(W_{p,x} W_{q,\theta} - W_{q,x} W_{p,\theta}\right) - \frac{1}{4R}\left(V_p W_{q,x} + V_q W_{p,x}\right) \tag{4.33}
$$

where U, V and W are spatial functions determined by equation (4.17).

In equations (4.29) to (4.33), the subscripts 'p,q', 'p,r,s' and 'p,r,s,q' represent the coupling between two, three and four modes respectively.

Introducing equation (4.17) into equations (4.30), we obtain:

$$a_p = C_p a_p' e^{\eta_p \theta}, \quad a_p' = a_p^{(1)} \sin \bar{m}x, \quad a_p^{(1)} = -\bar{m}\alpha_p \tag{4.34}$$

$$b_p = C_p b_p' e^{\eta_p \theta}, \quad b_p' = b_p^{(1)} \sin \bar{m}x, \quad b_p^{(1)} = \frac{\eta_p \beta_p + 1}{R} \tag{4.35}$$

$$c_p = C_p c_p' e^{\eta_p \theta}, \quad c_p' = c_p^{(1)} \cos \bar{m}x, \quad c_p^{(1)} = \frac{\eta_p \alpha_p}{2R} + \frac{\bar{m}\beta_p}{2} \tag{4.36}$$

And introducing equation (4.17) into equations (4.31) to (4.33), we obtain:

$$\begin{aligned} A_{pq} &= C_p a_{pq}' e^{(\eta_p + \eta_q)\theta} C_q, \\ a_{pq}' &= a_{pq}^{(1)} \cos^2 \bar{m}x, \\ a_{pq}^{(1)} &= \frac{1}{8R^2} [R\bar{m}\beta_p - \alpha_p \eta_p][R\bar{m}\beta_q - \alpha_q \eta_q] + \frac{1}{2}\bar{m}^2 \end{aligned} \tag{4.37}$$

$$\begin{aligned} B_{pq} &= C_p b_{pq}' e^{(\eta_p + \eta_q)\theta} C_q, \\ b_{pq}' &= b_{pq}^{(1)} \cos^2 \bar{m}x + b_{pq}^{(2)} \sin^2 \bar{m}x , \\ b_{pq}^{(1)} &= \frac{1}{8R^2} [R\bar{m}\beta_p - \alpha_p \eta_p][R\bar{m}\beta_q - \alpha_q \eta_q] \\ b_{pq}^{(2)} &= \frac{1}{2R^2} [\eta_p - \beta_p][\eta_q - \beta_q] \end{aligned} \tag{4.38}$$

$$C_{pq} = C_p c_{pq}' e^{(\eta_p + \eta_q)\theta} C_q,$$
$$c_{pq}' = c_{pq}^{(1)} \cos\bar{m}x \ \sin\bar{m}x, \qquad \textbf{(4.39)}$$
$$c_{pq}^{(1)} = \frac{\bar{m}}{4R} [\eta_p + \eta_q - \beta_p - \beta_q]$$

η_p (p=1,..8) are the roots of characteristic equation (4.14); α_p and β_q are given by relation (4.16); R is the mean radius of the shell; $\bar{m} = m\pi/L$ where m is the axial wave number and L the length of the shell. The constants C_p (p=1,..,8) and C_q (q=1,..,8) may be obtained from equation (4.21).

Here we are limited to solving the equation of motion in the cases where the coupling between different modes is ignored. The fact nevertheless remains that the present theory constitutes a general approach to the dynamic study of non-linear cylindrical shells.

Assuming r=s in equation (4.27), replacing the terms of A_{prs}, B_{prs}, C_{prs}, D_{prs} and E_{prs} by their expressions (equation 4.29), using relations (4.34 - 4.39) and then integrating over x and θ, we obtain for the second order non-linear matrix for an empty element the following expression:

$$[k_s^{(NL2)}] = [A^{-1}]^T [J^{(NL2)}] [A^{-1}] \qquad \textbf{(4.40)}$$

where the (p,q) term in matrix $[J^{(NL2)}]$ is written as:

$$J^{(NL2)}(p,q) = \begin{cases} \displaystyle\sum_{k=1}^{8} \frac{R\ GG(p,q)}{(\eta_p + \eta_q + \eta_k)} \left[e^{(\eta_p + \eta_q + \eta_k)} - 1 \right] & \text{if } \eta_p + \eta_q + \eta_k \neq 0 \\ \displaystyle\sum_{k=1}^{8} R\ \ GG(p,q)\ \phi & \text{if } \eta_p + \eta_q + \eta_k = 0 \end{cases} \tag{4.41}$$

GG(p,q) is a coefficient in conjunction with α, β, η and element p_{ij} in matrix [P]. The general expression of GG(p,q) is:

$$\begin{aligned} GG(p,q) = \\ p_{11}I_1[\ a_p^{(1)}\ A_{pq}^{-1}\ a_{qk}^{(1)} + a_q^{(1)}\ A_{qk}^{-1}\ a_{kp}^{(1)} + a_k^{(1)}\ A_{kp}^{-1}\ a_{pq}^{(1)}\ ] + \\ p_{22}I_1[\ b_p^{(1)}\ A_{pq}^{-1}\ b_{qk}^{(1)} + b_q^{(1)}\ A_{qk}^{-1}\ b_{kp}^{(1)} + b_k^{(1)}\ A_{kp}^{-1}\ b_{pq}^{(1)}\ ] + \\ p_{22}I_2[\ b_p^{(1)}\ A_{pq}^{-1}\ b_{qk}^{(2)} + b_q^{(1)}\ A_{qk}^{-1}\ b_{kp}^{(2)} + b_k^{(1)}\ A_{kp}^{-1}\ b_{pq}^{(2)}\ ] + \\ p_{33}I_1[\ c_p^{(1)}\ A_{pq}^{-1}\ c_{qk}^{(1)} + c_q^{(1)}\ A_{qk}^{-1}\ c_{kq}^{(1)} + c_k^{(1)}\ A_{kp}^{-1}\ c_{pq}^{(1)}\ ] + \\ p_{12}I_1[\ a_q^{(1)}\ A_{qk}^{-1}\ b_{kp}^{(1)} + a_k^{(1)}\ A_{kp}^{-1}\ b_{pq}^{(1)} + b_p^{(1)}\ A_{pq}^{-1}\ a_{qk}^{(1)} + \\ b_q^{(1)}\ A_{qk}^{-1}\ a_{kp}^{(1)} + b_k^{(1)}\ A_{kp}^{-1}\ a_{pq}^{(1)} + a_p^{(1)}\ A_{pq}^{-1}\ b_{qk}^{(1)}\ ] + \\ p_{12}I_2[\ a_q^{(1)}\ A_{qk}^{-1}\ b_{kp}^{(2)} + a_k^{(1)}\ A_{kp}^{-1}\ b_{pq}^{(2)} + a_p^{(1)}\ A_{pq}^{-1}\ b_{qk}^{(2)}\ ] \end{aligned} \tag{4.42}$$

where:

$$I_1 = \frac{1}{3\bar{m}}[1-(-1)^m], \quad I_2 = 2I_1, \quad \bar{m} = m\pi/L \tag{4.43}$$

The terms $a_p^{(1)}$, $b_p^{(1)}$, $c_p^{(1)}$, $a_{pq}^{(1)}$, $b_{pq}^{(1)}$, $c_{pq}^{(1)}$ and $b_{pq}^{(2)}$ are terms appearing in expressions of coefficients a_p, b_p, c_p, A_{pq}, B_{pq} and C_{pq} [relations (4.34 - 4.39)] and A_{pq}^{-1} is the term (p,q) of matrix $[A^{-1}]$, where [A] is the matrix defined by relation (4.20).

Assuming r=s=q in equation (4.28), replacing the terms of A_{prsq}, B_{prsq}, C_{prsq}, D_{prsq} and E_{prsq} by their expressions (equation 4.29), using relations (4.37 - 4.39) and then integrating over x and θ, we obtain for the third order non-linear matrix for an empty element the following expression:

$$[k_s^{(NL3)}] = [A^{-1}]^T [J^{(NL3)}] [A^{-1}] \tag{4.44}$$

Where the (p,q) term in matrix $J^{(NL3)}$ is written as:

$$J^{(NL3)}(p,q) = \begin{cases} \displaystyle\sum_{k=1}^{8} \sum_{l=1}^{8} \frac{R\ L\ E(l,k)\ SS(p,q)}{8(\eta_p + \eta_q + \eta_k + \eta_l)} \left[e^{(\eta_p + \eta_q + \eta_k + \eta_l)} - 1 \right] & \text{if } \eta_p + \eta_q + \eta_k + \eta_l \neq 0 \\ \displaystyle\sum_{k=1}^{8} \sum_{l=1}^{8} \frac{1}{8} R\ L\ E(l,k)\ SS(p,q)\ \phi & \text{if } \eta_p + \eta_q + \eta_k + \eta_l = 0 \end{cases} \tag{4.45}$$

E(l,k) is the term (l,k) of matrix [E], where [E] represents a matrix of constants defined by $[E] = [A^{-1}]^T[A^{-1}]$, SS(p,q) is a coefficient in conjunction with α, β, η and element p_{ij} in matrix [P].

The general expression of SS(p,q) is:

$$\begin{aligned} SS(p,q) = \; & 3p_{11}\, a_{pl}^{(1)}\, a_{kq}^{(1)} + p_{33}\, c_{pl}^{(1)}\, c_{kq}^{(1)} \\ & + p_{22}\, (3b_{pl}^{(1)}\, b_{kq}^{(1)} + 3b_{pl}^{(2)}\, b_{kq}^{(2)} + b_{pl}^{(1)}\, b_{kq}^{(2)} + b_{pl}^{(2)}\, b_{kq}^{(1)}) \\ & + p_{12}\, (3a_{pl}^{(1)}\, b_{kq}^{(1)} + a_{pl}^{(1)}\, b_{kq}^{(2)} + 3b_{pl}^{(1)}\, a_{kq}^{(1)} + b_{pl}^{(2)}\, a_{kq}^{(1)}) \end{aligned} \tag{4.46}$$

where the terms $a_{pq}{}^{(1)}$, $b_{pq}{}^{(1)}$, $c_{pq}{}^{(1)}$ and $b_{pq}{}^{(2)}$ are coefficients given in relations (4.37 - 4.39).

4.7 THE INFLUENCE OF GEOMETRIC NON-LINEARITIES OF THE WALLS ON THE NATURAL FREQUENCIES OF AN OPEN CYLINDRICAL SHELL

The mass and stiffness matrices obtained apply to only one element. After the shell is subdivided into several open cylindrical elements (Figure 4.3), the global mass and stiffness matrices are determined by assembling the matrices for each element. Assembling is done in such a way that all the equations of motion and the continuity of displacements at each node are satisfied. These matrices are designated as [M], [K_L], [K_{NL2}] and [K_{NL3}] respectively. They are square matrices of order NDF * (N + 1), where N represents the number of finite elements and NDF represents the number of degrees of freedom at each node. In practice, very specific conditions are applied to the shell boundaries. Thus, matrices [M], $[K_L]$, $[K_{NL2}]$ and $[K_{NL3}]$ are reduced to square matrices of order NREDUC = NDF * (N + 1) - J, where J represents the number of constraints applied. These reduced matrices are written as $[M^{(r)}]$, $[K_{NL2}^{(r)}]$ and $[K_{NL3}^{(r)}]$. The superscript "r" means "reduced".

The system of equations (4.26) then becomes:

$$[M^{(r)}]\ \{\ddot{\delta}^{(r)}\} + [K_L^{(r)}]\ \{\delta^{(r)}\} + [K_{NL2}^{(r)}]\ \{\delta^{(r)^2}\} + [K_{NL3}^{(r)}]\ \{\delta^{(r)^3}\} = \{0\} \qquad \textbf{(4.47)}$$

Setting:

$$\{\delta^{(r)}\} = [\Phi]\ \{q\} \qquad \textbf{(4.48)}$$

where $[\Phi]$ represents the square matrix for the eigenvectors of the linear system and $\{q\}$ is a time-related vector.

Substituting equation (4.48) into system (4.47) and multiplying by $[\Phi^T]$, we obtain:

$$\begin{gathered}[\Phi^T][M^{(r)}][\Phi]\{\ddot{q}\} + [\Phi^T][K_L^{(r)}][\Phi]\{q\} + \\ [\Phi^T][K_{NL2}^{(r)}]([\Phi]\{q\})^2 + [\Phi^T][K_{NL3}^{(r)}]([\Phi]\{q\})^3 = \{0\}\end{gathered} \qquad \textbf{(4.49)}$$

The products of matrix ($[\Phi]^T\,[M^{(r)}]\,[\Phi]$) and ($[\Phi]^T\,[K_L^{(r)}]\,[\Phi]$) represent diagonal matrices, written as $[M^{(D)}]$ and $[K_L^{(D)}]$, respectively.

The (4.49) system of equations is written:

$$[M^{(D)}]\,\{\ddot{q}\} + [K_L^{(D)}]\,\{q\} + [\Phi^T]\,[K_{NL2}^{(r)}]\,([\Phi]\,\{q\})^2 + [\Phi^T]\,[K_{NL3}^{(r)}]\,([\Phi]\,\{q\})^3 = \{0\} \tag{4.50}$$

We saw how matrices contained in the linear part of the system (4.47) could be reduced to diagonal matrices. On the other hand, by neglecting the cross product terms in $([\Phi]\,\{q\})^2$ and $([\Phi]\,\{q\})^3$ of equation (4.50) we obtain:

$$m_{pp}\,\ddot{q}_p + k_{pp}^{(L)} q_p + \sum_{s=1}^{NREDUC} k_{ps}^{(NL2)} q_s^2 + \sum_{s=1}^{NREDUC} k_{ps}^{(NL3)} q_s^3 = 0 \tag{4.51}$$

where coefficients m_{pp} and $k_{pp}^{(L)}$, represent the p^{th} diagonal terms of matrices $[M^{(D)}]$ and $[K_L^{(D)}]$, respectively; $[K_{ps}^{(NL2)}]$ and $[K_{ps}^{(NL3)}]$ are the (p,s) term of the product ($[\Phi]^T\,[K_{NL2}^{(r)}]\,[\Phi^2]$) and ($[\Phi]^T\,[K_{NL3}^{(r)}]\,[\Phi^3]$).

Here we have "NREDUC" simultaneous equations of the form of (4.51). Numerical solution of such a system is difficult and costly. At first, we limit ourselves to solving equation (4.51) by taking into account only the diagonal terms of the product ($[\Phi]^T\,[K_{NL2}^{(r)}]\,[\Phi^2]$) and ($[\Phi]^T\,[K_{NL3}^{(r)}]\,[\Phi^3]$) and therefore equation (4.51) would be written as follows:

$$m_{pp}\,\ddot{q}_p + k_{pp}^{(L)} q_p + k_{pp}^{(NL2)} q_p^2 + k_{pp}^{(NL3)} q_p^3 = 0 \tag{4.52}$$

Setting:

$$q_p(\tau) = A_p f_p(\tau) \tag{4.53}$$

which satisfies the conditions:

$$f_p(0) = 1 \text{ and } \dot{f}_p(0) = 0 \tag{4.54}$$

Equation (4.52) becomes, after the A_p simplification:

$$m_{pp}\,\ddot{f}_p + k_{pp}^{(L)} f_p + k_{pp}^{(NL2)}\, t\,(A_p/t)\, f_p^2 + k_{pp}^{(NL3)}\, t^2 (A_p/t)^2 f_p^3 = 0 \tag{4.55}$$

where t represents shell thickness.

Dividing this last equation by m_{pp}, it becomes:

$$\ddot{f}_p + \omega_p^2 f_p + \Lambda_p^{(NL2)} (A_p/t)\, f_p^2 + \Lambda_p^{(NL3)} (A_p/t)^2 f_p^3 = 0 \tag{4.56}$$

where

$$\omega_p^2 = \frac{k_{pp}^{(L)}}{m_{pp}} \tag{4.57}$$

The coefficient " $k_{pp}^{(L)} / m_{pp}$ " represents the p^{th} linear vibration frequency of the shell. and

$$\Lambda_p^{(NL2)} = \frac{k_{pp}^{(NL2)}}{m_{pp}} t \tag{4.58}$$

$$\Lambda_p^{(NL3)} = \frac{k_{pp}^{(NL3)}}{m_{pp}} t^2 \tag{4.59}$$

The solution $f_p(\tau)$ of the non-linear differential equation (4.56) which satisfies the conditions in (4.54) is calculated by a fourth order Runge-Kutta numerical method. The linear and non linear natural frequencies are evaluated by a systematic search for the $f_p(\tau)$ roots as a function of time. The ω_{NL}/ω_L ratio of linear and non-linear frequency is expressed as a function of non-dimensional ratio (A_p/t) where A_p is the vibration amplitude.

4.8 CALCULATIONS AND DISCUSSION

The influence of the wall's geometric non-linearity on the open or closed cylindrical shell's free vibrations is expressed by equation (4.56). For a shell having the particular physical characteristics given, the ratio ω_{NL}/ω_L of linear and non-linear frequency have been graphically represented in Figures 4.4 to 4.9 with respect to the non-dimensional ratio, A_p/t. The straight horizontal line represents the linear vibration cases, where the frequency is independent of the motion's amplitude.

4.8.1 Non-linear free vibration of closed cylindrical shell

The first example of calculations to determine the influence of non-linearities in strain-displacement relations on the free vibrations of a simply-supported cylindrical shell is shown in the analyses in references 6 and 17. The shell has the following properties:

$$E = 200\,\text{GPa},\ \nu = 0.3,\ \rho = 7800\,\text{Kg/m}^3,$$

$$R = 2.54\,\text{cm},\ L = 40\,\text{cm},\ t = 0.0254\,\text{cm},\ \phi_T = 360°.$$

The variation in natural frequencies of this shell was calculated using the method we propose, and compared to the results Nowinski[6] and Raju and Rao[17] obtained for the case of n = 4 and m = 1 (Figure 4.4).

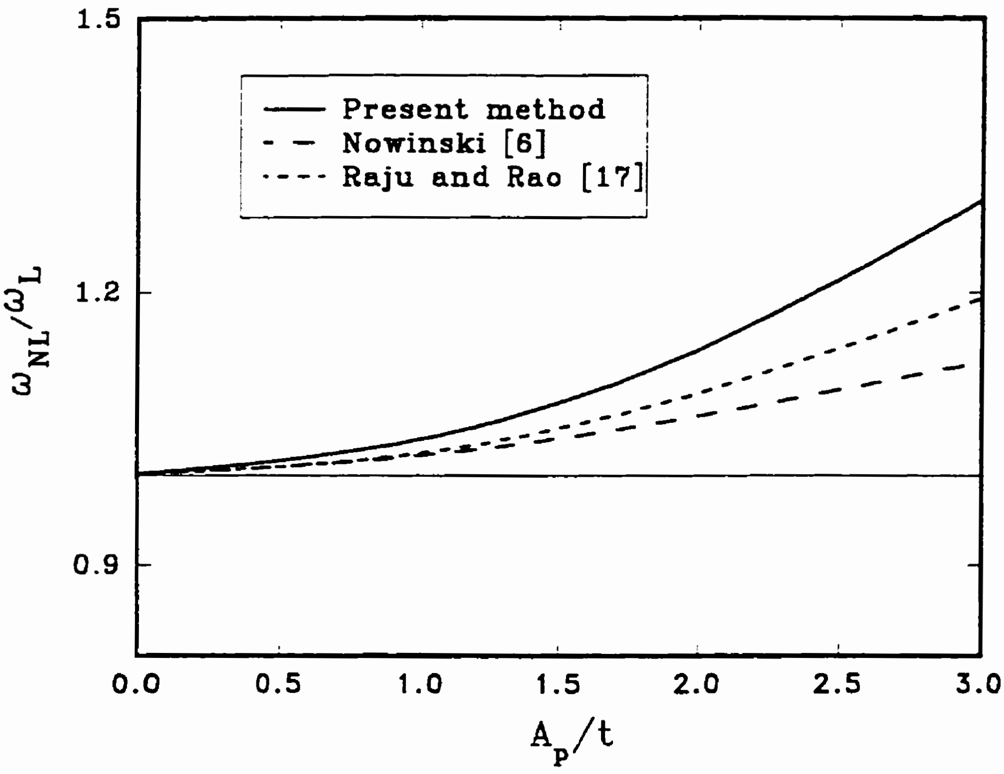


Figure 4.4 **Comparison of the effect of amplitude upon frequency for an empty simply-supported closed cylindrical shell, (m = 1, n = 4).**

Nowinski[6] based his analytical development on Donnell's simplified non-linear method. Only lateral displacement was considered. For their part, Raju and Rao[17], beginning with an energy formulation, used the finite element method.

In Figure 4.4, we observe that the variation ratio between the linear and non-linear frequency increases as ratio A/t increases. Non-linearity has a hardening effect. These variations are small for values A/t below 1.0. For values above 1.0, the variation is more pronounced than that which Nowinski[6] and Raju and Rao[17] obtained.

Its appears that these differences might be due to the fact that Nowinski[6] neglected in-plane inertia and took into account only lateral displacement. As for Raju and Rao[17], who used the Sanders-Koiter[3,4] non-linear theory, they expressed the displacement of components along the shell generator in polynomial form.

4.8.2 Non-linear free vibration of open cylindrical shell

One of the great advantages of the finite element method is the ease with which it can be applied to any geometry and any boundary condition. Thus, the second step of calculation is to study the non-linear dynamic characteristics of open cylindrical shells as a function of circumferential and axial modes for various boundary conditions.

4.8.2.1 Influence of the circumferential mode n

In Figure 4.5, we present the effect of large amplitude on the frequency of vibration for axial mode m = 1 and various circumferential mode n (1 to 12). The open shell is simply-supported at the four edges and the data are as follows :

$$E=200\text{GPa},\ \nu=0.3,\ \rho=7800\,\text{Kg/m}^3,$$
$$R=2.54\,\text{cm},\ L=40\,\text{cm},\ t=0.0254\,\text{cm},\ \phi_T=135°.$$

The Figure shows that the non-linearity is of the hardening type for circumferential mode n=1 and n > 9 and is of softening type for n between 2 and 9. We see also that the non-linear effect is more pronounced for the mode n = 5 and the variation is small for the case of n = 1.

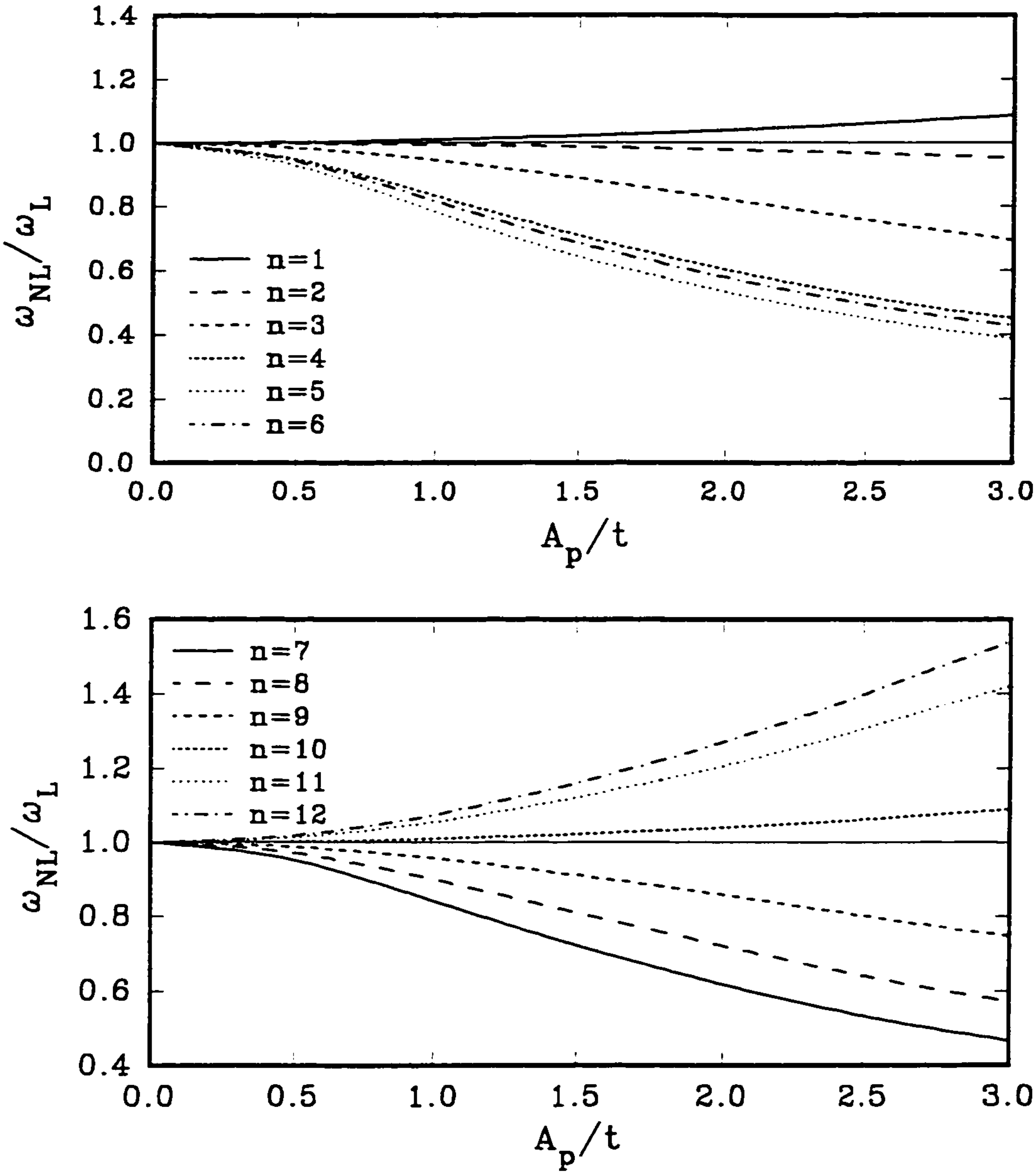


Figure 4.5 **Influence of large amplitude on natural frequency of simply-supported open cylindrical shell for various circumferential mode n and axial mode m = 1.**

Figure 4.6 shows the variation in frequency ratio as a function of A/t for axial mode m = 2 and circumferential mode n = 1 to 12. As in Figure 4.5, the same phenomena can be observed for this axial mode, the non-linearity is of the hardening type for circumferential mode n = 1, 2, 3 and n > 9 and is of softening type for n between 4 and 9. We see also that the non-linear effect is more pronounced for the mode n = 7 and the variation is small for the case of n = 1.

4.8.2.2 Influence of the boundary conditions and of the opening angle ϕ_T

In order to establish the effect of boundary condition on non-linear free vibration, we turn to Figure 4.7. We observe for the mode (n = 1, m = 2) and the same open shell with different boundary conditions, that the shell with the clamped - simply supported boundary conditions in its straight edges is the one on which the effect of non-linearity is the more pronounced, The effect is small for a panel with clamped - free boundary conditions. The steel panel analysis in Figure 4.7 has the following data:

$$R = 37.7 \text{ mm}, L = 234 \text{ mm}, t = 0.229 \text{ mm and } \phi_T = 180 \text{ deg.}$$

With the same data, Figure 4.8 shows the effect of the opening angle ϕ_T on the non-linear free vibration of the open cylindrical shell. It shows that the panel with opening angle ϕ_T = 135 deg. is the one which has the smaller effect on the non-linearity and the more pronounced effect is for ϕ_T = 10 deg.

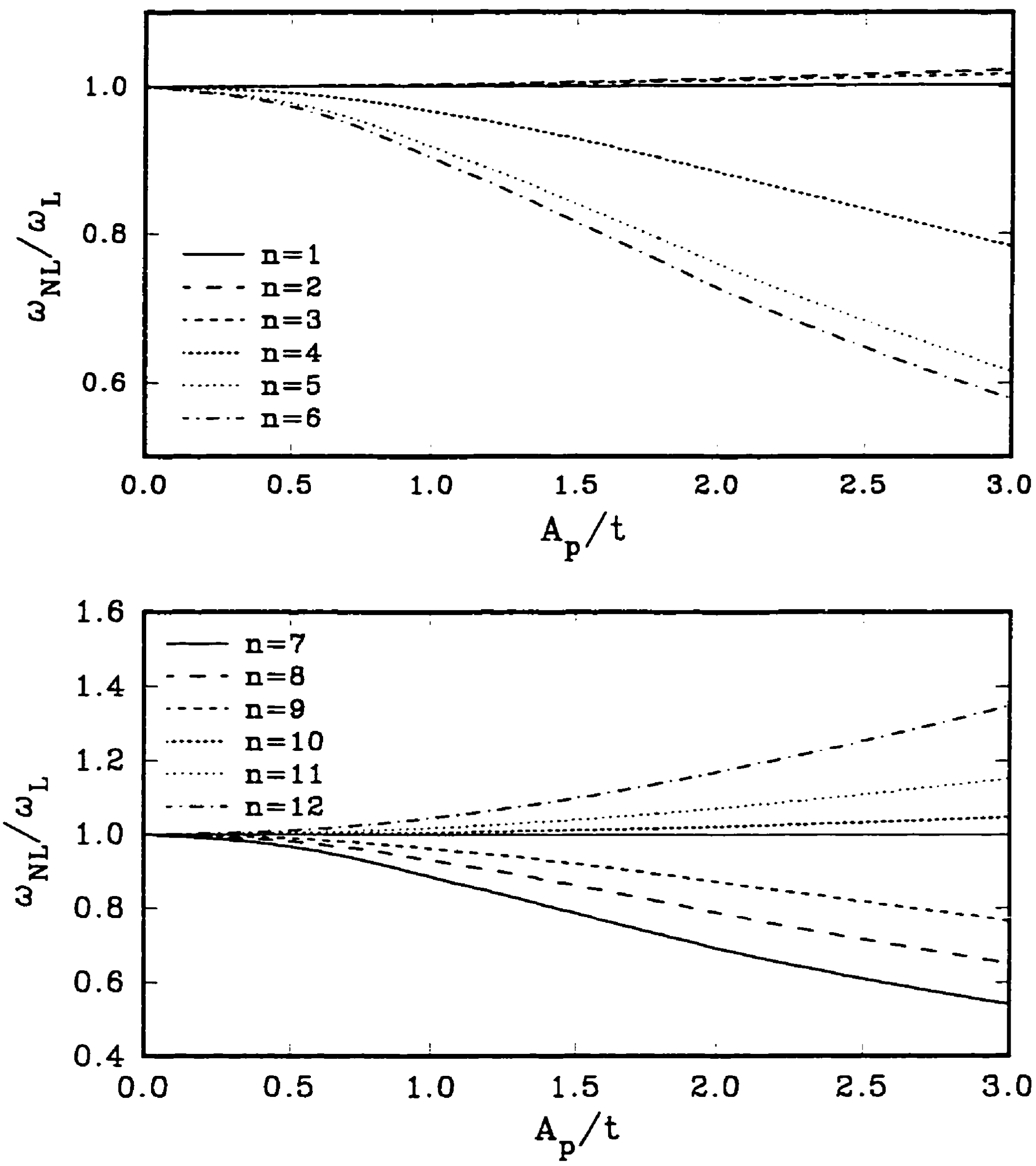


Figure 4.6 **Influence of large amplitude on natural frequency of simply-supported open cylindrical shell for various circumferential mode n and axial mode m = 2.**

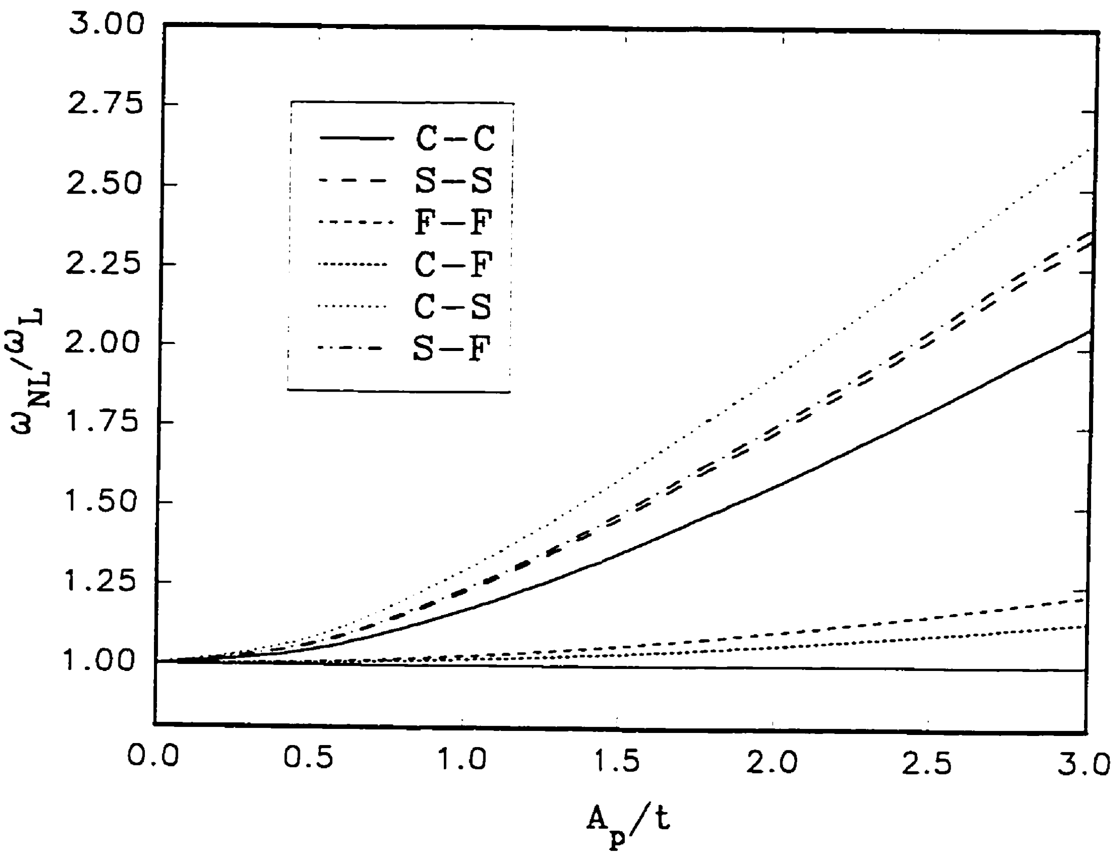


Figure 4.7 Influence of large amplitude on natural frequency of an open cylindrical shell for different boundary conditions, (n = 1, m = 2). (F: Free, S: Simply-supported, C: Clamped)

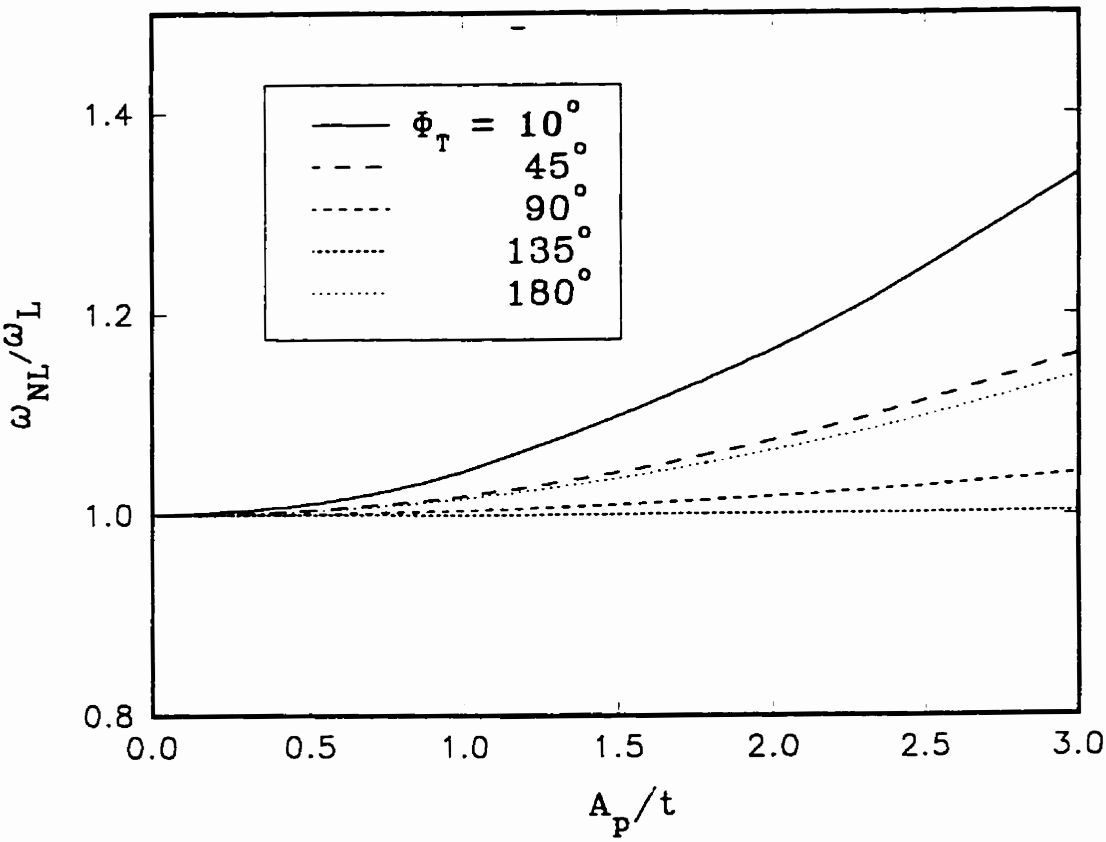


Figure 4.8 **Influence of large amplitude on natural frequency of clamped-free open cylindrical shell for different opening angle φ_T, (n = 1, m = 2).**

4.8.2.3 Non-linear free vibration of orthotropic cylindrical shell

The present method has been applied to the analysis of the non-linear vibration of an orthotropic open cylindrical shell, clamped along its straight edges and simply-supported along its curved edges.

The data for the open shell are as follows:

$$E_x = 1.0 \times 10^{11} N/m^2, \quad E_\theta = 0.05 \times E_x, \quad \nu_x = 0.2, \quad \nu_\theta = 0.05 \times \nu_x,$$
$$G_{x\theta} = 0.05 \times E_x, \; \rho_s = 7800 N/m^3,$$
$$R/L = 1, \; R/t = 50, \; \phi_T = 180°.$$

The results for the axial mode m = 2 and circumferential modes n = 1, 2, 3 and 4 are given in Figure 4.9. We observe that the non-linearity is of the hardening type for these modes. We see also that the non-linear effect is more pronounced for the mode n = 1. An exhaustive numerical investigation is needed in this matter in order to draw any concrete conclusion.

Our model combines the advantages of the finite element method which deals with complex shell (orthotropy, variable thickness,...) and the precision of formulation which the use of displacement functions derived from shell theory contributes.

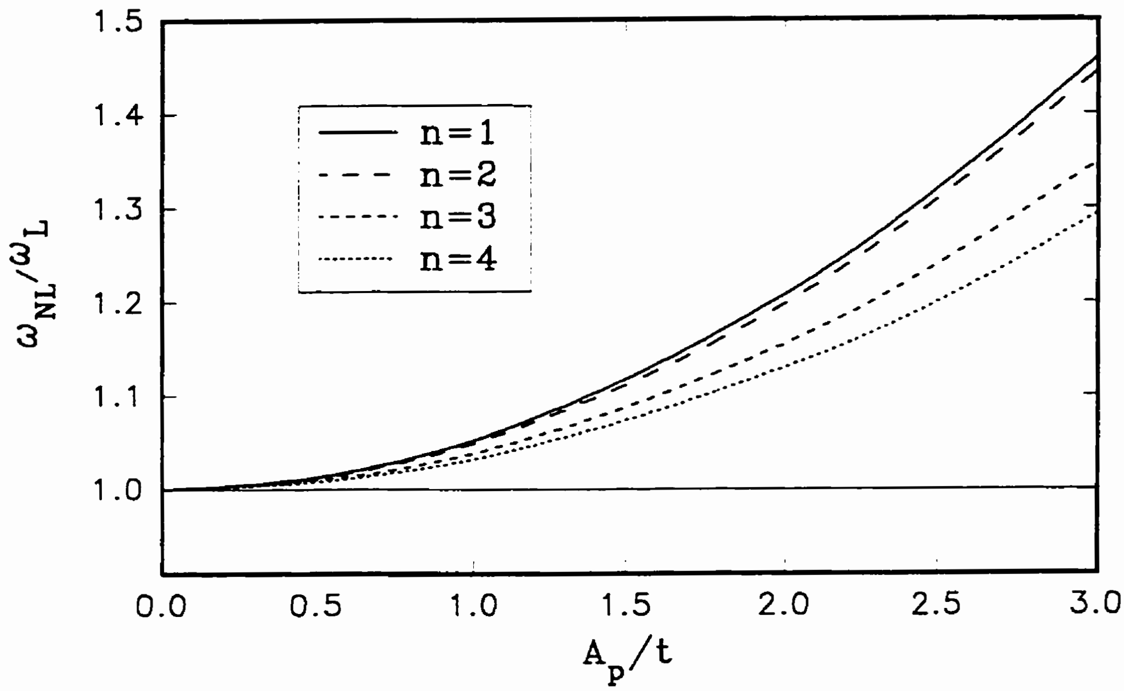


Figure 4.9 **Influence of large amplitude on natural frequency of clamped-clamped orthotropic open cylindrical shell for various circumferential mode n, m = 2.**

4.9 CONCLUSION

The method discussed in this paper demonstrates the influence of geometric non-linearities of the walls on the free vibrations of empty open or closed cylindrical shells. It is a hybrid method, based on a combination of thin shell theory and the finite element method.

An open cylindrical finite element was developed, so that the displacement functions could be derived directly from classical thin shell theory.

The solution was divided into two parts. In part one, the displacement functions were obtained from linear shell theory and the mass and linear stiffness matrices were determined by the finite element procedure. In part two, the modal coefficients corresponding to non-linearities in strain-displacement relations were obtained for the displacement functions. The second and third order non-linear stiffness matrices were then calculated using the finite element method.

With the help of a computer program, variations in the free vibration frequencies were determined in conjunction with motion amplitude for a closed or open cylindrical shell. Deviations in terms of linear vibrations were observed. The results obtained with this method are in agreement with other analytical and numerical methods.

A paper currently under preparation will deal with the non-linear free vibration analysis of liquid-filled open or closed cylindrical shells. The non-linear dynamic stability of shells containing flowing fluid will also be investigated.

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4.11 NOMENCLATURE

LIST OF SYMBOLS

A, B, C	Constants in equation (4.13) defining U, V, W respectively
A_i	Motion amplitude
a_p, b_p, c_p	Modal coefficients determined by equation (4.30)
$a_p^{(1)}$, $b_p^{(1)}$, $c_p^{(1)}$	Coefficient determined by equations (4.34-4.36)
$a_{rs}^{(1)}, b_{rs}^{(1)}$, $b_{rs}^{(2)}, c_{rs}^{(1)}$	Coefficient determined by equations (4.37-4.39)
aA_{prs}, bB_{prs}, cC_{prs}, aB_{prs}, bA_{prs}	Modal coefficients determined by equation (4.29)
A_{pq}, B_{pq}, C_{pq}	Modal coefficients determined by equations (4.31-4.33)
A_{prsq}, B_{prsq}, C_{prsq}, AB_{prsq}, BA_{prsq}	Modal coefficients determined by equation (4.29)
E	Young's modulus
e	Exponential
f_p	Function determined by equation (4.53)
GG(p,q)	Coefficient determined by equation (4.42)
L	Length of the shell
m	Axial mode number
N	Number of finite elements

n	Circumferential mode number
p_{ij}	Terms of elasticity matrix (i= 1,...,6 ; j= 1, ..., 6)
R	Mean radius of the shell
SS(p,q)	Coefficient determined by equation (4.46)
t	Thickness of the shell
U, V, W	Axial, tangential and radial displacements
x	Axial coordinate
η_i	Complex roots of the characteristic equation (4.14)
α_p, β_p	Determined by equation (4.16)
θ	Circumferential coordinate
ν	Poisson's ratio
ϕ	Opening angle for one finite element
ϕ_T	Opening angle for the whole open shell
ρ_s	Density of the shell material
ω_L	Linear frequency of free vibrations
ω_{NL}	Non-linear frequency of free vibrations
τ	Time related coordinates
ω_p	Coefficient determined by equation (4.57)
$\Lambda_p^{(NL2)}$, $\Lambda_p^{(NL3)}$	Coefficients determined by equations (4.58) and (4.59)

LIST OF MATRICES

[A]	Defined by equation (4.20)
[B]	Defined by equation (4.23)
{C}	Vector of arbitrary constants
$[k_s^{(L)}]$, $[k_s^{(NL2)}]$, $[k_s^{(NL3)}]$	Linear and non-linear stiffness matrices for a shell finite element
$[m_s]$	Mass matrix for a shell finite element
[N]	Displacement function defined by equation (4.22)
[P]	Elasticity matrix
[Q]	Defined by equation (4.23)
{q}	Time-related vector coordinates
[R]	Defined by equation (4.17)
$[T_m]$	Defined by equation (4.12)
$\{\delta_i\}$	Vector of degrees of freedom at node i
$\{\delta\}$	Vector of degrees of freedom for total shell
$\{\delta^{(r)}\}$	Reduced vector of degrees of freedom for total shell
$\{\sigma\}$	Stress vector
$\{\epsilon_L\}$ $\{\epsilon_{NL}\}$	Linear and non-linear components of the deformation vector
$[\Phi]$	Matrix of eigenvectors, equation (4.48)

APPENDIX A4-1

EQUATIONS OF MOTION

This appendix contains the equations of motion for a thin orthotropic cylindrical shell.

$$L_1(U,V,W,p_{ij}) = p_{11}\frac{\partial^2 U}{\partial x^2} + \frac{p_{12}}{R}\left(\frac{\partial^2 V}{\partial x\,\partial\theta} + \frac{\partial W}{\partial x}\right) - p_{14}\frac{\partial^3 W}{\partial x^3} +$$

$$\frac{p_{15}}{R^2}\left(\frac{\partial^3 W}{\partial x\,\partial\theta^2} + \frac{\partial^2 V}{\partial x\,\partial\theta}\right) + \left(\frac{p_{33}}{R} - \frac{p_{63}}{2R^2}\right)\left(\frac{\partial^2 V}{\partial x\,\partial\theta} + \frac{1}{R}\frac{\partial^2 U}{\partial\theta^2}\right) +$$

$$\left(\frac{p_{36}}{R^2} - \frac{p_{66}}{2R^3}\right)\left(-\frac{2\partial^3 W}{\partial x\,\partial\theta^2} + \frac{3}{2}\frac{\partial^2 V}{\partial x\,\partial\theta} - \frac{1}{2}R\frac{\partial^2 U}{\partial\theta^2}\right)$$

$$L_2(U,V,W,p_{ij}) = \left(\frac{p_{21}}{R} + \frac{p_{51}}{R^2}\right)\left(\frac{\partial^2 U}{\partial x\,\partial\theta}\right) + \frac{1}{R}\left(\frac{p_{22}}{R} + \frac{p_{52}}{R^2}\right)$$

$$\left(\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial W}{\partial\theta}\right) - \left(\frac{p_{24}}{R} + \frac{p_{54}}{R^2}\right)\left(\frac{\partial^3 W}{\partial x^2\,\partial\theta}\right) + \frac{1}{R^2}\left(\frac{p_{25}}{R} + \frac{p_{55}}{R^2}\right)$$

$$\left(-\frac{\partial^3 W}{\partial\theta^3} + \frac{\partial^2 V}{\partial\theta^2}\right) + \left(p_{33} + \frac{3p_{63}}{2R}\right)\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 U}{R\partial x\,\partial\theta}\right) +$$

$$\frac{1}{R}\left(p_{36} + \frac{3p_{66}}{2R}\right)\left(-2\frac{\partial^3 W}{\partial x^2\,\partial\theta} + \frac{3}{2}\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 U}{2R\,\partial x\,\partial\theta}\right)$$

$$L_3\,(U,V,W,p_{ij}) = p_{41}\frac{\partial^3 U}{\partial x^3} + \frac{p_{42}}{R}\left(\frac{\partial^3 V}{\partial x^2\,\partial\theta} + \frac{\partial^2 W}{\partial x^2}\right) - p_{44}\frac{\partial^4 W}{\partial x^4} +$$

$$\frac{p_{45}}{R^2}\left(-\frac{\partial^4 W}{\partial x^2\,\partial\theta^2} + \frac{\partial^3 V}{\partial x^2\,\partial\theta}\right) + \frac{2\,p_{63}}{R}\left(\frac{\partial^3 U}{R\,\partial x\,\partial\theta^2} + \frac{\partial^3 V}{\partial x^2\,\partial\theta}\right) + \left(\frac{2p_{66}}{R^2}\right)$$

$$\left(-2\frac{\partial^4 W}{\partial x^2\partial\theta^2} + \frac{3}{2}\frac{\partial^3 V}{\partial x^2\,\partial\theta} - \frac{\partial^3 U}{2R\partial x\,\partial\theta^2}\right) + \frac{p_{51}}{R^2}\frac{\partial^3 U}{\partial x\,\partial\theta^2} + \frac{p_{52}}{R^3}\left(\frac{\partial^3 V}{\partial\theta^3} +\right.$$

$$\left.\frac{\partial^2 W}{\partial\theta^2}\right) + \frac{p_{55}}{R^4}\left(-\frac{\partial^4 W}{\partial\theta^4} + \frac{\partial^3 V}{\partial\theta^3}\right) - \frac{p_{21}}{R}\frac{\partial U}{\partial x} - \frac{p_{54}}{R^2}\frac{\partial^4 W}{\partial x^2\,\partial\theta^2}$$

$$- \frac{p_{22}}{R^2}\left(\frac{\partial V}{\partial\theta} + W\right) + \frac{p_{24}}{R}\frac{\partial^2 W}{\partial\theta^2} - \frac{p_{25}}{R^3}\left(-\frac{\partial^2 W}{\partial\theta^2} + \frac{\partial V}{\partial\theta}\right)$$

APPENDIX A4-2

Characteristic Equation (4.14)

$$h_8 \eta^8 + h_6 \eta^6 + h_4 \eta^4 + h_2 \eta^2 + h_0 = 0$$

where

$$h_8 = f_1 f_6 f_{10} - f_1 f_8^2$$

$$h_6 = f_1 f_6 f_{11} + f_1 f_7 f_{10} - 2f_1 f_8 f_9 + f_2 f_6 f_{10} - f_2 f_8^2 - f_3^2 f_{10} + f_3 f_8 f_4 + f_4 f_3 f_8 - f_4^2 f_6$$

$$h_4 = f_1 f_6 f_{12} + f_1 f_7 f_{11} - f_1 f_9^2 + f_2 f_6 f_{11} + f_2 f_7 f_{10} - 2f_2 f_8 f_9 - f_3^2 f_{11} + f_3 f_9 f_4 + f_3 f_8 f_5 + f_4 f_3 f_9 - f_4^2 f_7 - f_4 f_6 f_5 + f_5 f_3 f_8 - f_5 f_6 f_4$$

$$h_2 = f_1 f_7 f_{12} + f_2 f_6 f_{12} + f_2 f_7 f_{11} - f_2 f_9^2 - f_3^2 f_{12} + f_3 f_9 f_5 - f_4 f_7 f_5 + f_5 f_3 f_9 - f_5 f_7 f_4 - f_5^2 f_6$$

$$h_o = f_2 f_7 f_{12} - f_7 f_5^2$$

The coefficients f_i (i = 1,...,12) are given by the above equations :

$$f_1 = \frac{1}{R}(P_{55} - \frac{1}{R} P_{36} + \frac{1}{4R^2} P_{66})$$

$$f_2 = - P_{11} \bar{m}^2$$

$$f_3 = \bar{m} \left[\frac{1}{R} (P_{12} + P_{13}) + \frac{1}{R^2} (P_{15} + P_{36}) - \frac{3}{4R^3} P_{66} \right]$$

$$f_4 = - \frac{\bar{m}}{R^2} (P_{15} + 2 P_{36} - \frac{1}{R} P_{66})$$

$$f_5 = \frac{P_{12}}{R} \bar{m} + P_{14} \bar{m}^3$$

$$f_6 = - \frac{1}{R^2} (P_{22} + \frac{1}{R^2} P_{55} + \frac{2}{R} P_{25})$$

$$f_7 = \bar{m} (P_{33} + \frac{3}{R} P_{36} + \frac{9}{4R^2} P_{66})$$

$$f_8 = \frac{1}{R^3} (P_{25} + \frac{1}{R} P_{55})$$

$$f_9 = - \frac{1}{R^2} (P_{22} + \frac{1}{R} P_{52}) - \frac{\bar{m}^2}{R} (2P_{36} + P_{24} + \frac{3}{R} P_{66} + \frac{1}{R} P_{54})$$

$$f_{10} = - \frac{1}{R^4} P_{55}$$

$$f_{11} = \frac{2}{R^3} P_{25} + \frac{\bar{m}}{R^2} (2P_{45} + 4P_{66})$$

$$f_{12} = - \frac{1}{R} P_{22} - \frac{2}{R} P_{24} \bar{m}^2 - P_{44} \bar{m}$$

and $\bar{m} = m\frac{\pi}{L}$

APPENDIX A4-3

MATRICES $[T_m]$, $[R]$, $[A]$, $[Q]$, $[m]$ and $[k_L]$

MATRIX $[T_m]_{(3x3)}$

$$[T_m] = \text{Diag}[\ \cos\bar{m}x,\ \sin\bar{m}x,\ \sin\bar{m}x\ ]$$
$$\bar{m} = m\pi/L$$

MATRIX $[R]_{(3x8)}$

$$R(1,j) = \alpha_j e^{\eta_j\theta}$$
$$R(2,j) = e^{\eta_j\theta}, \qquad j = 1,\ldots 8$$
$$R(3,j) = \beta_j e^{\eta_j\theta}$$

MATRIX $[A]_{(8x8)}$

$$A(1,j) = \alpha_j \qquad A(5,j) = \alpha_j e^{\eta_j\phi}$$
$$A(2,j) = 1 \qquad A(6,j) = e^{\eta_j\phi}$$
$$A(3,j) = \eta_j \qquad A(7,j) = \eta_j e^{\eta_j\phi}$$
$$A(4,j) = \beta_j \qquad A(8,j) = \beta_j e^{\eta_j\phi}$$

MATRIX $[Q]_{(6x8)}$

$$Q(1,j) = A_j e^{\eta_j \theta} \qquad Q(4,j) = D_j e^{\eta_j \theta}$$
$$Q(2,j) = B_j e^{\eta_j \theta} \qquad Q(5,j) = E_j e^{\eta_j \theta}$$
$$Q(3,j) = C_j e^{\eta_j \theta} \qquad Q(6,j) = F_j e^{\eta_j \theta}$$

The terms A_j, B_j, C_j, D_j, E_j and F_j (i = 1, ..., 8) may be expressed as follows:

$$A_j = - \frac{m \pi \alpha_j}{L},$$

$$B_j = - \frac{\eta_j \beta_j + 1}{R},$$

$$C_j = - \frac{m \pi \beta_j}{L} + \frac{\eta_j \alpha_j}{R}$$

$$D_j = - \frac{(m \pi)^2}{L^2},$$

$$E_j = - \frac{\eta_j^2 + \eta_j \beta_j}{R^2}$$

and $$F_j = - \frac{2 m \pi \eta_j}{RL} + \frac{3 m \pi \beta_j}{2 RL} - \frac{\eta_j \alpha_j}{2R^2}$$

MATRICES $[m]_{(8x8)}$ and $[k_L]_{(8x8)}$

$$[m] = [A^{-1}]^T\,[S]\,[A^{-1}], \qquad [k_L] = [A^{-1}]^T\,[G]\,[A^{-1}]$$

where [S] and [G] are defined by the above equations:

$$S(i,j) = \frac{RL}{2}\,\frac{(\alpha_i\,\alpha_j + \beta_i\,\beta_j + 1)}{(\eta_i + \eta_j)}\,(e^{(\eta_i+\eta_j)\phi} - 1) \qquad \text{if } \eta_i + \eta_j \neq 0$$

$$S(i,j) = \frac{RL\,\phi}{2}\,(\alpha_i\,\alpha_j + \beta_i\,\beta_j + 1) \qquad \text{if } \eta_i + \eta_j = 0$$

$$\begin{aligned} G(i,j) = \frac{RL}{2}\,(&p_{11}\,A_i\,A_j + p_{12}\,A_i\,B_j + p_{14}\,A_i\,D_j + p_{15}\,A_i\,E_j \\ &+ p_{21}\,B_i\,A_j + p_{22}\,B_i\,B_j + p_{24}\,B_i\,D_j + p_{25}\,B_i\,E_j \\ &+ p_{41}\,D_i\,A_j + p_{42}\,D_i\,B_j + p_{44}\,D_i\,D_j + p_{45}\,D_i\,E_j \\ &+ p_{51}\,E_i\,A_j + p_{52}\,E_i\,B_j + p_{54}\,E_i\,D_j + p_{55}\,E_i\,E_j \\ &+ p_{33}\,C_i\,C_j + p_{36}\,C_i\,F_j + p_{63}\,F_i\,C_j + p_{66}\,F_i\,F_j) \\ &\frac{(e^{(\eta_i+\eta_j)\phi} - 1)}{(\eta_i + \eta_j)} \qquad \text{if } \eta_i + \eta_j \neq 0 \end{aligned}$$

$$G(i,j) = \frac{RL\,\phi}{2}\,(p_{11}\,A_i\,A_j + \ldots\ldots + p_{66}\,F_i\,F_j) \qquad \text{if } \eta_i + \eta_j = 0$$

The terms A_i, B_i, C_i, D_i, E_i and F_i ($i = 1, \ldots, 8$) are listed with matrix [Q].

CHAPITRE V

ARTICLE IV

NON-LINEAR DYNAMIC ANALYSIS OF ORTHOTROPIC OPEN CYLINDRICAL SHELLS SUBJECTED TO A FLOWING FLUID

5.1 ABSTRACT

A theory is presented to predict the influence of non-linearities associated with the wall of the shell and with the fluid flow on the dynamic of elastic, thin, orthotropic and non-uniform open cylindrical shells submerged and subjected simultaneously to an internal and external fluid.

The open shells are assumed to be freely simply-supported along their curved edges and to have arbitrary straight edge boundary conditions. The method developed is a hybrid of thin shell theory, fluid theory and the finite element method.

The solution is divided into four parts. In part one, the displacement functions are obtained from Sanders' linear shell theory and the mass and linear stiffness matrices for

the empty shell are obtained by the finite element procedure. In part two, the modal coefficients derived from the Sanders-Koiter non-linear theory of thin shells are obtained for these displacement functions. Expressions for the second and third order non-linear stiffness matrices of the empty shell are then determined through the finite element method. In part three a fluid finite element is developed, the model requires the use of a linear operator for the velocity potential and a linear boundary condition of impermeability. With the non-linear dynamic pressure, we develop in the fourth part three non-linear matrices for the fluid.

The non-linear equation of motion is then solved by the fourth-order Runge-Kutta numerical method. The linear and non-linear natural frequency variations are determined as a function of shell amplitudes for different cases.

5.2 INTRODUCTION

The analysis of thin shells containing flowing fluid has been the focus of many investigations. Most of these studies have involved linear analyses of thin shells both with and without interaction between the structure and the surrounding fluid medium. Results proved to be satisfactory where wall deflections of the shell are very small compared to the wall thickness [1-11]. In several practical reports, however, the linear analysis was not sufficiently accurate for satisfactory design. In those cases, a non-linear analysis was required.

Several methods have been developed for the analysis of dynamic non-linear thin cylindrical shells. Among these were Galerkin's method [12-14], the small perturbation method [15-16], the Rayleigh-Ritz method [17], the modal expansion method [18-19], the finite element method [20-22] and the hydrid finite element method [23].

The finite element method appears to be ideally suited to the analysis of complex shell structures. Numerous general computer programmes are available for industrial use for the linear and non-linear analysis, where the displacement functions of the finite elements used are assumed to be polynomial. Precise prediction of both the high and the low frequencies requires the use of a great many elements in the classical finite element method.

In order to achieve this, the present study presents a general approach to the non-linear analysis of elastic, thin, orthotropic and circumferentially non-uniform open cylindrical shells submerged in liquid under flow or no-flow condition (Figure 5.1). We investigate the effect of non-linearities associated with the wall of the shell and with fluid flow on the natural frequencies of an interactive fluid-shell system.

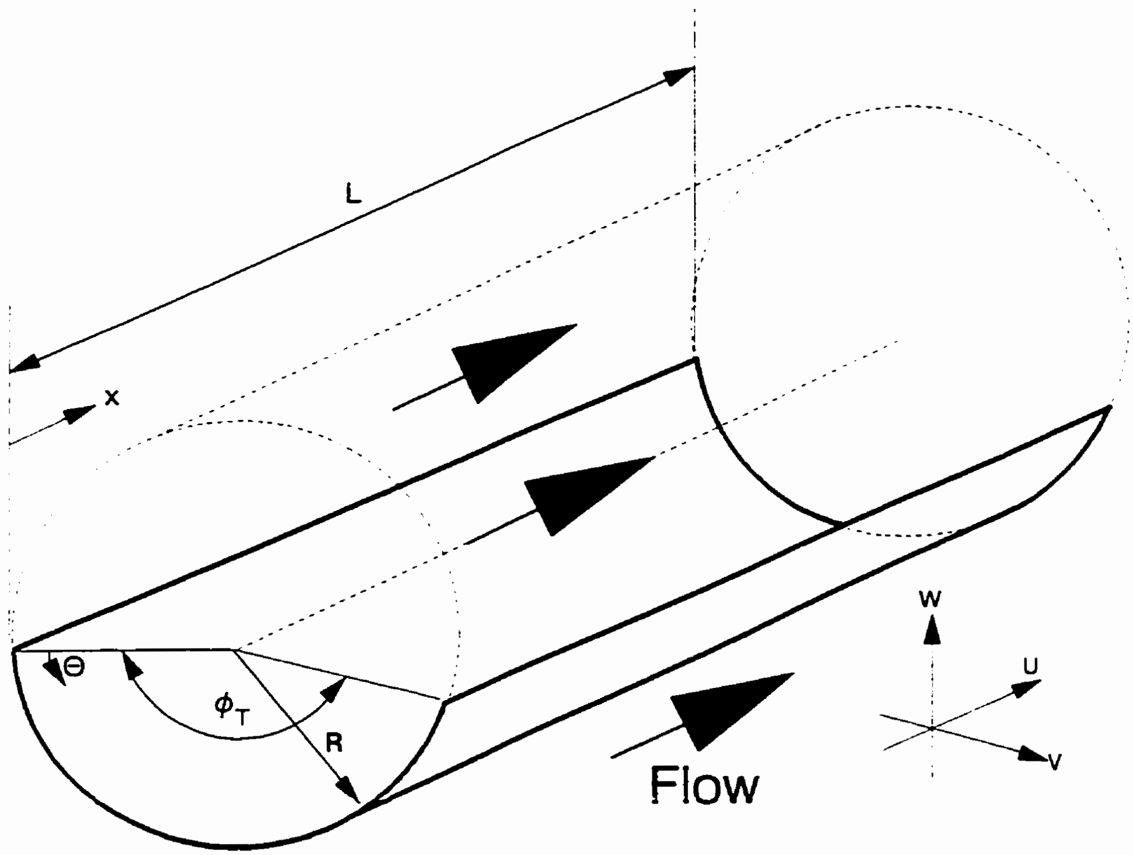

Figure 5.1 Open cylindrical shell geometry.

The shells are assumed to be freely simply-supported along their curved edges and to have arbitrary straight edge boundary conditions.

The finite element method is employed, but it is a hybrid, a combination of the finite element method, shell theory and fluid theory. This choice allows us to use the complete equilibrium equations to determine the displacement functions and, further, the mass, stiffness and damping matrices for the shell and the fluid element.

The analytical solution involves four steps:

a) Using the linear strain-displacement and stress-strain relationships which are inserted into Sanders' equations of equilibrium [24], we determine the displacement functions by solving the linear equation system. We then determine the mass and linear stiffness matrices for an empty finite element and assemble the matrices for the complete shell.

b) Using strain-displacement relationships from the Sanders-Koiter non linear theory [25-26], the modal coefficients are obtained from the displacement functions. The second and third order non-linear stiffness matrices for an empty finite element are then calculated by precise analytical integration with respect to modal coefficients.

c) To account for the effect of the fluid on the structure, a panel finite fluid element bounded by two nodal lines is considered. By solving the linear equations of motion for the fluid element, an expression for linear fluid pressure as a function of the displacement of the element is obtained. Analytical integration for the pressure distribution along the element yields three components: the mass, linear stiffness and linear damping matrices for a fluid element.

d) With the non-linear dynamic pressure, we develop in the fourth part three non-linear matrices for the fluid: stiffness, damping and combination of the two.

The linear and non-linear natural vibration frequency ratio is then obtained by solving the non-linear equations of motion.

5.3 DISPLACEMENT FUNCTIONS

Sanders' [24] linear equations for thin cylindrical shells, in terms of axial, tangential and radial displacements (U, V, W) of the mean surface of the shell (Figure 5.1) and in terms of elements p_{ij} of the orthotropic matrix of elasticity [P] are:

$$L_i \, (U, V, W, p_{ij}) = 0, \qquad i = 1 \text{ to } 3 \tag{5.1}$$

where L_i (i = 1, 2,3) are three linear differential operators. These equations are given in Appendix A5-1.

The shell is subdivided into several finite elements defined by two nodes i and j and by components U, V, W and $dW/d\theta$, representing axial, tangential, radial displacements and the rotation, respectively (Figure 5.2).

The displacement functions are assumed to be:

$$\begin{aligned} U(x,\theta) &= Ae^{\eta\theta} \cos m\pi x/L \\ V(x,\theta) &= Be^{\eta\theta} \sin m\pi x/L \\ W(x,\theta) &= Ce^{\eta\theta} \sin m\pi x/L \end{aligned} \qquad \textbf{(5.2)}$$

where m is the axial mode, and η is a complex number.

Substituting (5.2) into equations of motion (5.1), a system of three homogeneous linear functions of constants A, B and C are obtained. For the solution to be non-trivial, the determinant of this system must be equal to zero. This brings us to the following characteristic equation in η:

$$h_8\eta^8 + h_6\eta^6 + h_4\eta^4 + h_2\eta^2 + h_0 = 0 \qquad \textbf{(5.3)}$$

where h_0, h_2, h_4, h_6 and h_8 are listed in Appendix A5-2.

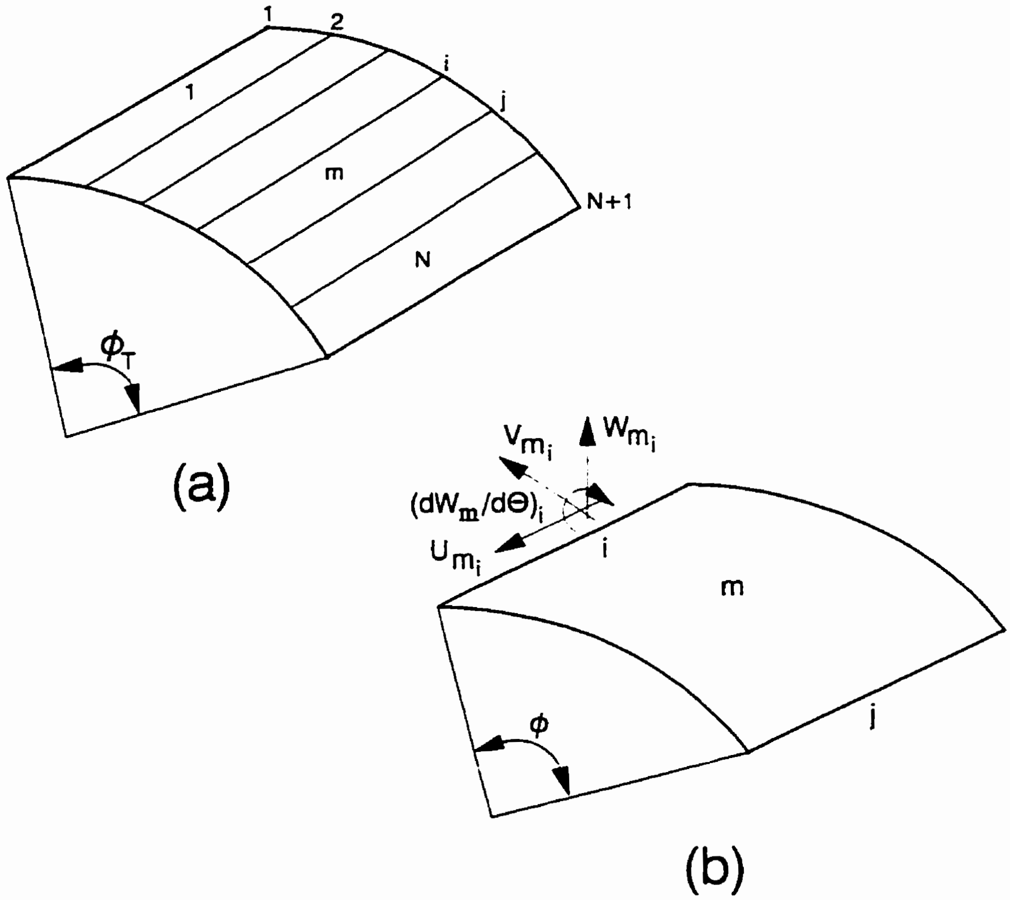

Figure 5.2 **(a) Finite element idealization.**
(b) Nodal displacements at node i.

Each root η of this equation yields a solution to the linear equations of motion (5.1). The complete solution is obtained by adding the eight solutions independently with the constants A_p, B_p and C_p ($p = 1,\ldots,8$). The constants A_p, B_p and C_p are not independent. We can therefore express A_p and B_p as a function of C_p, for example:

$$A_p = \alpha_p C_p \quad \text{and} \quad B_p = \beta_p C_p \quad , \quad p = 1,\ldots,8 \tag{5.4}$$

The values of α_p and β_p can be obtained from linear system (5.1) by introducing relations (5.4). Substituting expressions (5.4) into equations (5.2), the displacements $U(x,\theta)$, $V(x,\theta)$ and $W(x,\theta)$ can then be expressed in conjunction with the eight C_p constants.

We then have:

$$\{U(x,\theta),\ W(x,\theta),\ V(x,\theta)\}^T = [T_m]\ [R]\ \{C\} \tag{5.5}$$

where $[T_m]$ and $[R]$ are matrices given in Appendix A5-3 and $\{C\}$ is an 8^{th} order vector of the C_p constants:

To determine the eight C_p constants, it is necessary to formulate eight boundary conditions for the finite elements. The axial, tangential and radial displacements, as well as rotation, will be specified for each node. The elements which have two nodes and eight degrees of freedom will have i (θ = 0) and j (θ = ϕ) as nodal displacements at the boundaries:

$$\left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} = \left\{ U_i, W_i, \left[\frac{dW}{d\theta} \right]_i, V_i, U_j, W_j, \left[\frac{dW}{d\theta} \right]_j, V_j \right\}^T = [A]\{C\} \tag{5.6}$$

where the terms of matrix [A], given in Appendix A5-3, are obtained from matrix [R] by successively setting $\theta = 0$ and $\theta = \phi$.

Multiplying equation (5.6) by $[A^{-1}]$ and substituting into equations (5.5) we obtain:

$$\left\{ \begin{matrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{matrix} \right\} = [T_m]\ [R]\ [A^{-1}] \left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} = [N] \left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} \tag{5.7}$$

where the matrices $[T_m]$, [R] and [A] are given in Appendix A5-3. [N] represents the displacement functions matrix.

5.4 MASS AND LINEAR STIFFNESS MATRICES FOR AN EMPTY ELEMENT

Introducing the displacement functions (equation 5.7) into the linear deformation vector $\{\epsilon_L\}$ (Sanders [24]), we obtain:

$$\{\epsilon_L\} = \begin{bmatrix} [T_m] & [O] \\ [O] & [T_m] \end{bmatrix} [Q]\,[A^{-1}] \left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} = [B] \left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} \qquad \textbf{(5.8)}$$

where the matrices [A] and [Q] are given in Appendix A5-3.

For an orthotropic laminated material, the stress resultants may be expressed as follows:

$$\{\sigma\} = \{\, N_{xx}\; N_{\theta\theta}\; \bar{N}_{x\theta}\; M_{xx}\; M_{\theta\theta}\; \bar{M}_{x\theta} \,\}^T = [P]\,\{\epsilon_L\} \qquad \textbf{(5.9)}$$

where [P] is the elasticity matrix, in which the general term is designated by p_{ij}, may be written as follows:

$$[P] = \begin{bmatrix} p_{11} & p_{12} & 0 & p_{14} & p_{15} & 0 \\ p_{21} & p_{22} & 0 & p_{24} & p_{25} & 0 \\ 0 & 0 & p_{33} & 0 & 0 & p_{36} \\ p_{41} & p_{42} & 0 & p_{44} & p_{45} & 0 \\ p_{51} & p_{52} & 0 & p_{54} & p_{55} & 0 \\ 0 & 0 & p_{63} & 0 & 0 & p_{66} \end{bmatrix} \qquad \textbf{(5.10)}$$

Referring to equation (5.8), the stress vector (5.9) may be rewritten as follows:

$$\{\sigma\} = [P]\,[B]\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{5.11}$$

The mass and linear stiffness matrices can then be expressed as:

$$[m_s] = \rho t \int\int [N^T]\,[N]\,dA, \quad [k_s^{(L)}] = \int\int [B^T]\,[P]\,[B]\,dA \tag{5.12}$$

where $dA = Rdxd\theta$, ρ is the density of the shell, t its thickness, [P] the elasticity matrix given by equation (5.10) and the matrices [N] and [B] are derived from equations (5.7) and (5.8) respectively. The matrices $[m_s]$ and $[k_s^{(L)}]$ were obtained analytically by carrying out the necessary matrix operations and integration over x and θ in equation (5.12). These matrices are also given in Appendix A5-3.

5.5 NON-LINEAR STIFFNESS MATRICES FOR AN EMPTY ELEMENT

The non-linear Sanders-Koiter [25-26] theory for thin shells describes the behaviour of open cylindrical shells. This theory is derived by approximation from the three-dimensional elasticity equation. In common with linear theory, it is based on Love's "First Approximation" but the assumption concerning the order of magnitude of the bending has been modified. The displacement gradients are small and the squares of the rotation do not exceed the reference surface deformation in order of magnitude.

The following approach, developed by Radwan and Genin [18], is used with particular attention to geometric non-linearities. The coefficients of the modal equations are obtained through the Lagrange method. Thus, the non-linear stiffness matrices, once calculated, are overlaid onto the linear system. Before we embark on formulation, however, a brief summary of the method is in order.

(a) Shell displacements are expressed as generalized product coordinate sums and spatial functions;

$$\begin{aligned} u_1 &= \sum_i q_i(t)\, U(x,\theta) \\ u_2 &= \sum_i q_i(t)\, V(x,\theta) \\ w &= \sum_i q_i(t)\, W(x,\theta) \end{aligned} \qquad \textbf{(5.13)}$$

where the functions $q_i(t)$ are the generalized coordinates and the spatial functions U, V, W are given by equation (5.2)

(b) the deformation vector is written as a function of the generalized coordinates by separating the linear portion from the non-linear;

$$\{\epsilon\} = \{\epsilon_L\} + \{\epsilon_{NL}\} = \{\ \epsilon_{xx}\ \epsilon_{\theta\theta}\ 2\bar{\epsilon}_{x\theta}\ \kappa_{xx}\ \kappa_{\theta\theta}\ 2\bar{\kappa}_{x\theta}\ \}^T \tag{5.14}$$

where subscripts "L" and "NL" mean "linear" and "non-linear", respectively; ϵ_{xx}, $\epsilon_{\theta\theta}$, $\epsilon_{x\theta}$ represent the deformations of reference surface and κ_{xx}, $\kappa_{\theta\theta}$, $\kappa_{x\theta}$ are the changes in curvature and torsion of the reference surface. These expressions are given in ref. [25-26]

(c) these expressions are then introduced into the Lagrange equations up to and including the degree corresponding to the deformation energy;

(d) by substituting the expressions in a) into the strain-displacement relations in the Sanders-Koiter [25-26] non-linear theory, the generalized coordinate coefficients appearing in the equation derived under c) are determined in terms of spatial functions.

The dynamic behaviour of an empty cylindrical shell, in the absence of external loads, can be represented by the following non-linear modal equations [18]:

$$\sum_r m_{pr}\ddot{\delta}_r + \sum_r k_{pr}^{(L)}\delta_r + \sum_r\sum_s k_{prs}^{(NL2)}\delta_r\delta_s + \sum_r\sum_s\sum_q k_{prsq}^{(NL3)}\delta_r\delta_s\delta_q = 0, \quad p=1,2,.. \tag{5.15}$$

where m_{pr}, $k_{pr}^{(L)}$ are the terms of mass and linear stiffness matrices given by equation (5.12); the terms $k_{prs}^{(NL2)}$ and $k_{prsq}^{(NL3)}$ which represent the second and third non-linear stiffnesses may be obtained by the following integrals in the case of the laminated orthotropic open cylindrical shell.

$$k_{prs}^{(NL2)} = \int\int \{ p_{11}A_{prs} + p_{22}B_{prs} + p_{12}(D_{prs} + E_{prs}) + p_{33}C_{prs} \} \, dA \quad (5.16)$$

and

$$k_{prsq}^{(NL3)} = \int\int \{ p_{11}A_{prsq} + p_{22}B_{prsq} + p_{12}(D_{prsq} + E_{prsq}) + p_{33}C_{prsq} \} \, dA \quad (5.17)$$

where $dA = Rdxd\theta$, p_{ij} are the terms of the elasticity matrix [P], and the terms A_{prs}, B_{prs}, C_{prs}, D_{prs}, E_{prs} and A_{prsq}, B_{prsq}, C_{prsq}, D_{prsq}, E_{prsq} represent the coefficients of the modal equations mentioned in step d). These coefficients are given by the following equation:

$$\begin{aligned}
A_{prs} &= a_p A_{rs} + a_r A_{sp} + a_s A_{pr} & \qquad A_{prsq} &= 2A_{pq}A_{rs} \\
B_{prs} &= b_p B_{rs} + b_r B_{sp} + b_s B_{pr} & \qquad B_{prsq} &= 2B_{pq}B_{rs} \\
C_{prs} &= c_p C_{rs} + c_r C_{sp} + c_s C_{pr} & \qquad C_{prsq} &= 2C_{pq}C_{rs} \\
D_{prs} &= a_r B_{sp} + a_s B_{pr} + b_p A_{rs} & \qquad D_{prsq} &= 2A_{pq}B_{rs} \\
E_{prs} &= b_r A_{sp} + b_s A_{pr} + a_s B_{rs} & \qquad E_{prsq} &= 2B_{pq}A_{rs}
\end{aligned} \quad (5.18)$$

with

$$a_p = U_{p,x}, \qquad b_p = \frac{1}{R}(V_{p,\theta} + W_p), \qquad c_p = \frac{1}{2}\left(\frac{U_{p,\theta}}{R} + V_{p,x}\right) \quad (5.19)$$

$$A_{pq} = \frac{1}{8R^2}(RV_{p,x} - U_{p,\theta}).(RV_{q,x} - U_{q,\theta}) + \frac{1}{2} W_{p,x}W_{q,x} \tag{5.20}$$

$$\begin{aligned} B_{pq} &= \frac{1}{8R^2}(RV_{p,x} - U_{p,\theta}).(R\,V_{q,x} - U_{q,\theta}) \\ &+ \frac{1}{2R^2}(W_{p,\theta} - V_p).(W_{q,\theta} - V_q\,) \end{aligned} \tag{5.21}$$

$$C_{pq} = \frac{1}{4R}(W_{p,x}W_{q,\theta} - W_{q,x}W_{p,\theta}) - \frac{1}{4R}(V_pW_{q,x} + V_qW_{p,x}) \tag{5.22}$$

where U, V and W are spatial functions determined by equation (5.5):

In equations (5.18) to (5.22), the subscripts 'p,q', 'p,r,s' and 'p,r,s,q' represent the coupling between two, three and four modes respectively.

Introducing equation (5.5) into equation (5.19), we obtain:

$$a_p = C_p a_p' e^{\eta_p\theta}, \quad a_p' = a_p^{(1)} \sin \bar{m}x, \quad a_p^{(1)} = -\bar{m}\alpha_p \tag{5.23}$$

$$b_p = C_p b_p' e^{\eta_p\theta}, \quad b_p' = b_p^{(1)} \sin \bar{m}x, \quad b_p^{(1)} = \frac{\eta_p\beta_p + 1}{R} \tag{5.24}$$

$$c_p = C_p c_p' e^{\eta_p\theta}, \quad c_p' = c_p^{(1)} \cos \bar{m}x, \quad c_p^{(1)} = \frac{\eta_p\alpha_p}{2R} + \frac{\bar{m}\beta_p}{2} \tag{5.25}$$

And introducing equation (5.5) into equations (5.20) to (5.22), we obtain:

$$A_{pq} = C_p a_{pq}' e^{(\eta_p + \eta_q)\theta} C_q, \quad a_{pq}' = a_{pq}^{(1)} \cos^2 \bar{m}x,$$

$$a_{pq}^{(1)} = \frac{1}{8R^2} [R\bar{m}\beta_p - \alpha_p \eta_p][R\bar{m}\beta_q - \alpha_q \eta_q] + \frac{1}{2}\bar{m}^2 \tag{5.26}$$

$$B_{pq} = C_p b_{pq}' e^{(\eta_p + \eta_q)\theta} C_q, \quad b_{pq}' = b_{pq}^{(1)} \cos^2 \bar{m}x + b_{pq}^{(2)} \sin^2 \bar{m}x ,$$

$$b_{pq}^{(1)} = \frac{1}{8R^2} [R\bar{m}\beta_p - \alpha_p \eta_p][R\bar{m}\beta_q - \alpha_q \eta_q] \tag{5.27}$$

$$b_{pq}^{(2)} = \frac{1}{2R^2}[\eta_p - \beta_p][\eta_q - \beta_q]$$

$$C_{pq} = C_p c_{pq}' e^{(\eta_p + \eta_q)\theta} C_q, \quad c_{pq}' = c_{pq}^{(1)} \cos\bar{m}x \sin\bar{m}x ,$$

$$c_{pq}^{(1)} = \frac{\bar{m}}{4R} [\eta_p + \eta_q - \beta_p - \beta_q] \tag{5.28}$$

η_p (p=1,..8) are the roots of characteristic equation (5.3); α_p and β_q are given by relation (5.4); R is the mean radius of the shell; $\bar{m} = m\pi/L$ where m is the axial wave number and L the length of the shell.

The constants C_p (p=1,..,8) and C_q (q=1,..,8) may be obtained from equation (5.6) as follows:

$$\{C\} = [A^{-1}]\left\{\begin{matrix}\delta_i \\ \delta_j\end{matrix}\right\} \tag{5.29}$$

The matrix $[A^{-1}]$ is the inverse of [A], where [A] is given by equation (5.6) and listed in Appendix A5-3.

Here we are limited to solving the equation of motion in the cases where the coupling between different modes is ignored. The fact nevertheless remains that the present theory constitutes a general approach to the dynamic study of non-linear cylindrical shells.

Assuming r=s in equation (5.16), replacing the terms of A_{prs}, B_{prs}, C_{prs}, D_{prs} and E_{prs} by their expressions (equation 5.18), using relations (5.23 - 5.28) and then integrating over x and θ, we obtain for the second order non-linear matrix for an empty element the following expression:

$$[k_s^{(NL2)}] = [A^{-1}]^T[J^{(NL2)}][A^{-1}] \tag{5.30}$$

where the (p,q) term in matrix $[J^{(NL2)}]$ is written as:

$$J^{(NL2)}(p,q) = \begin{cases} \sum_{k=1}^{8} \dfrac{R\ GG(p,q)}{(\eta_p + \eta_q + \eta_k)} \left[e^{(\eta_p + \eta_q + \eta_k)} - 1 \right] & \text{if } \eta_p + \eta_q + \eta_k \neq 0 \\ \sum_{k=1}^{8} R\ GG(p,q)\ \phi & \text{if } \eta_p + \eta_q + \eta_k = 0 \end{cases} \qquad \textbf{(5.31)}$$

GG(p,q) is a coefficient in conjunction with α, β, η and element p_{ij} in matrix [P]. The general expression of GG(p,q) is:

$$\begin{aligned} GG(p,q) = {} & p_{11}I_1[\ a_p^{(1)} A_{pq}^{-1} a_{qk}^{(1)} + a_q^{(1)} A_{qk}^{-1} a_{kp}^{(1)} + a_k^{(1)} A_{kp}^{-1} a_{pq}^{(1)}\ ] + \\ & p_{22}I_1[\ b_p^{(1)} A_{pq}^{-1} b_{qk}^{(1)} + b_q^{(1)} A_{qk}^{-1} b_{kp}^{(1)} + b_k^{(1)} A_{kp}^{-1} b_{pq}^{(1)}\ ] + \\ & p_{22}I_2[\ b_p^{(1)} A_{pq}^{-1} b_{qk}^{(2)} + b_q^{(1)} A_{qk}^{-1} b_{kp}^{(2)} + b_k^{(1)} A_{kp}^{-1} b_{pq}^{(2)}\ ] + \\ & p_{33}I_1[\ c_p^{(1)} A_{pq}^{-1} c_{qk}^{(1)} + c_q^{(1)} A_{qk}^{-1} c_{kq}^{(1)} + c_k^{(1)} A_{kp}^{-1} c_{pq}^{(1)}\ ] + \\ & p_{12}I_1[\ a_q^{(1)} A_{qk}^{-1} b_{kp}^{(1)} + a_k^{(1)} A_{kp}^{-1} b_{pq}^{(1)} + b_p^{(1)} A_{pq}^{-1} a_{qk}^{(1)} + \\ & \quad b_q^{(1)} A_{qk}^{-1} a_{kp}^{(1)} + b_k^{(1)} A_{kp}^{-1} a_{pq}^{(1)} + a_p^{(1)} A_{pq}^{-1} b_{qk}^{(1)}\ ] + \\ & p_{12}I_2[\ a_q^{(1)} A_{qk}^{-1} b_{kp}^{(2)} + a_k^{(1)} A_{kp}^{-1} b_{pq}^{(2)} + a_p^{(1)} A_{pq}^{-1} b_{qk}^{(2)}\ ] \end{aligned} \qquad \textbf{(5.32)}$$

where:

$$I_1 = \frac{1}{3\bar{m}}[1-(-1)^m], \quad I_2 = 2I_1, \quad \bar{m} = m\pi/L \qquad \textbf{(5.33)}$$

The terms $a_p^{(1)}$, $b_p^{(1)}$, $c_p^{(1)}$, $a_{pq}^{(1)}$, $b_{pq}^{(1)}$, $c_{pq}^{(1)}$ and $b_{pq}^{(2)}$ are terms appearing in expressions of coefficients a_p, b_p, c_p, A_{pq}, B_{pq} and C_{pq} [relations (5.23 - 5.28)] and A_{pq}^{-1} is the term (p,q) of matrix $[A^{-1}]$, where [A] is the matrix defined by relation (5.6).

Assuming r=s=q in equation (5.17), replacing the terms of A_{prsq}, B_{prsq}, C_{prsq}, D_{prsq} and E_{prsq} by their expressions (equation 5.18), using relations (5.26 - 5.28) and then integrating over x and θ, we obtain for the third order non-linear matrix for an empty element the following expression:

$$[k_s^{(NL3)}] = [A^{-1}]^T [J^{(NL3)}] [A^{-1}] \tag{5.34}$$

Where the (p,q) term in matrix $J^{(NL3)}$ is written as:

$$J^{(NL3)}(p,q) = \begin{cases} \displaystyle\sum_{k=1}^{8} \sum_{l=1}^{8} \frac{R\ L\ E(l,k)\ SS(p,q)}{8(\eta_p + \eta_q + \eta_k + \eta_l)} \left[e^{(\eta_p + \eta_q + \eta_k + \eta_l)} - 1 \right] & \text{if } \eta_p + \eta_q + \eta_k + \eta_l \neq 0 \\ \displaystyle\sum_{k=1}^{8} \sum_{l=1}^{8} \frac{1}{8} R\ L\ E(l,k)\ SS(p,q)\ \phi & \text{if } \eta_p + \eta_q + \eta_k + \eta_l = 0 \end{cases} \tag{5.35}$$

E(l,k) is the term (l,k) of matrix [E], where [E] represents a matrix of constants defined by $[E] = [A^{-1}]^T[A^{-1}]$, SS(p,q) is a coefficient in conjunction with α, β, η and element p_{ij} in matrix [P]. The general expression of SS(p,q) is:

$$SS(p,q) = 3p_{11}\, a^{(1)}_{pl}\, a^{(1)}_{kq} + p_{22}\,(3b^{(1)}_{pl}\, b^{(1)}_{kq} + 3b^{(2)}_{pl}\, b^{(2)}_{kq} + b^{(1)}_{pl}\, b^{(2)}_{kq} + b^{(2)}_{pl}\, b^{(1)}_{kq}) + p_{33}\, c^{(1)}_{pl}\, c^{(1)}_{kq} + p_{12}\,(3a^{(1)}_{pl}\, b^{(1)}_{kq} + a^{(1)}_{pl}\, b^{(2)}_{kq} + 3b^{(1)}_{pl}\, a^{(1)}_{kq} + b^{(2)}_{pl}\, a^{(1)}_{kq}) \tag{5.36}$$

where the terms $a_{pq}^{(1)}$, $b_{pq}^{(1)}$, $c_{pq}^{(1)}$ and $b_{pq}^{(2)}$ are coefficients given in relations (5.26 -5.28).

5.6 DYNAMIC BEHAVIOUR OF THE FLUID-SHELL INTERACTION

The pressure exerted by the fluid is given by using a non-linear development of the Bernoulli equation. From the solution of the potential equation we derive an expression of non-linear pressure as a function of 1) the nodal displacements of the fluid element, 2) the inertial, centrifugal and Coriolis forces and 3) a combination of non-linear effects. Through the usual finite element procedure, we obtain the linear mass, damping and stiffness matrices for the fluid as well as the non-linear matrices for damping and stiffness and a combination of the two.

The mathematical model which is developed is based on the following hypothesis: (i) the fluid flow is potential; (ii) vibration is non-linear; (iii) pressure on the wall is purely lateral; (iv) the fluid mean velocity distribution is assumed to be constant across a shell section and (v) the fluid is incompressible and non-viscous.

5.6.1 Dynamic pressure

With the previous hypothesis, the potential function must satisfy the Laplace equation. This relation is expressed in the cylindrical coordinate system by:

$$\nabla^2 \varphi = \frac{1}{r}(r\varphi_{,r})_{,r} + \frac{\varphi_{,\theta\theta}}{r^2} + \varphi_{,xx} = 0 \qquad \textbf{(5.37)}$$

φ is the potential function that represents the velocity potential.

Therefore:

$$V_x = U_{xu} + \varphi_{,x}\ ; \qquad V_\theta = \frac{\varphi_{,\theta}}{R}\ ; \qquad V_r = \varphi_{,r} \qquad \textbf{(5.38)}$$

where V_x, V_θ and V_r are respectively the axial, tangential and radial components of the fluid velocity; U_{xu} is the velocity of the liquid through the shell section

The Bernouilli equation is given by:

$$\varphi_{,t} + \frac{1}{2} V^2 + \frac{P_u}{\rho_{fu}} \Big|_{r=\xi} = 0 \tag{5.39}$$

Introducing equation (5.38) into equation (5.39) and taking into account the linear and non-linear terms, we find the dynamic pressure P_u:

$$P_u = -\rho_{fu} \left\{ \varphi_{,t} + U_{xu}\, \varphi_{,x} + \frac{1}{2} \left[(\varphi_{,x})^2 + \frac{(\varphi_{,\theta})^2}{r^2} + (\varphi_{,r})^2 \right] \right\} \Big|_{r=\xi} \tag{5.40}$$

where u subscript represents "i: internal" or "e: external" fluid as the case may be:

$$\text{if } u = i \quad \text{then} \quad \xi = R_i = R - \frac{t}{2} \tag{5.41}$$

$$\text{if } u = e \quad \text{then} \quad \xi = R_e = R + \frac{t}{2} \tag{5.42}$$

A full definition of the flow requires that a condition be applied to the structure-fluid interface. The impermeability condition ensures contact between the shell and the fluid. This should be:

$$V_r \big|_{r=R} = \varphi_{,r} \big|_{r=R} = W_{,t} + U_{xu} W_{,x} + \frac{U_{xu}^2}{2} W_{,xx} \big|_{r=R} \tag{5.43}$$

From the theory of shells (equation 5.5), we have:

$$W(x,\theta,t) = \sum_{j=1}^{8} C_j \, e^{\eta_j \theta} \sin \frac{m\pi x}{L} \, e^{i\omega t} \tag{5.44}$$

Assuming then,

$$\varphi(x,\theta,r,t) = \sum_{j=1}^{8} R_j(r) \; S_j(x,\theta,t) \tag{5.45}$$

and applying the impermeability condition (equation 5.43) with the radial displacement given by relation (5.44), we determine the function S_j (x, θ, t) explicitly. Using equation (5.37), we find the following differential Bessel equation:

$$r^2 \frac{d^2 R_j(r)}{dr^2} + r \frac{dR_j(r)}{dr} + R_j(r) \left[\left(\frac{im\pi}{L} \right)^2 r^2 - \left(i\eta_j \right)^2 \right] = 0 \tag{5.46}$$

where i is the complex number, $i^2 = -1$ and η_j is the complex solution of the characteristic equation for the empty shell (relation 5.3).

The general solution of equation (5.46) is given by:

$$R_j(r) = A J_{i\eta_j}\left(\frac{im\pi}{L} r\right) + B Y_{i\eta_j}\left(\frac{im\pi}{L} r\right) \tag{5.47}$$

where $J_{i\eta j}$ and $Y_{i\eta j}$ are, respectively, the Bessel functions of the first and second kind of complex order "$i\eta_j$".

For inside flow, the solution (5.47) must be finite on the axis of the shell (r=0); this means we have to set the constant 'B' equal to zero. For outside flow ($r \rightarrow \infty$); this means that the constant 'A' is equal to zero. When the shell is simultaneously subjected to internal and external flow, we have to take the complete solution (5.47).

We carry the Bessel equation solution back into (5.45) to obtain the final expression of velocity potential evaluated at the shell wall:

$$\varphi_u(r,\theta,x,t)_j = Z_{uj}\left(\frac{im\pi R_u}{L}\right)\left[W_{j,t} + U_{xu} W_{j,x} + \frac{U_{xu}^2}{2} W_{j,xx}\right] \tag{5.48}$$

where

$$Z_{uj}\left(\frac{im\pi R_u}{L}\right) = \frac{R_u}{i\eta_j - \dfrac{im\pi R_u}{L}\dfrac{J_{i\eta_j+1}(im\pi R_u/L)}{J_{i\eta_j}(im\pi R_u/L)}} \quad \text{if } u = i \tag{5.49}$$

$$Z_{uj}\left(\frac{im\pi R_u}{L}\right) = \frac{R_u}{i\eta_j - \dfrac{im\pi R_u}{L}\dfrac{Y_{i\eta_j+1}(im\pi R_u/L)}{Y_{i\eta_j}(im\pi R_u/L)}} \quad \text{if } u = e \tag{5.50}$$

where η_j ($j = 1, \ldots, 8$) are the roots of the characteristic equation of the empty shell; $J_{i\eta j}$ and $Y_{i\eta j}$ are, respectively, the Bessel functions of the first and second kind of order "$i\eta_j$"; m is the axial mode number; R is the mean radius of the shell; L its length; the subscript "u" is equal to "i" for internal flow and is equal to "e" for external flow.

Substituting relation (5.48) into the non-linear condition (5.40), we obtain the equation for the pressure on the shell wall. It is useful to separate the total pressure into its linear and non-linear terms:

$$P_u = P_{uL} + P_{uNL} \tag{5.51}$$

where

$$P_{uL}=-\rho_{fu}\sum_{j=1}^{8} Z_{uj}\left[W_{j,tt}+2U_{xu}W_{j,tx}+\frac{U_{xu}^2}{2}W_{j,txx}+U_{xu}^2W_{j,xx}+\frac{U_{xu}^3}{2}W_{j,xxx}\right] \tag{5.52}$$

and

$$\begin{aligned} P_{uNL} = -\frac{\rho_{fu}}{2}\sum_{j=1}^{8}\sum_{k=1}^{8} & Z_{uj}Z_{uk}\,[\,W_{j,tx}W_{k,tx}+U_{xu}^2W_{j,xx}W_{k,xx}+\frac{U_{xu}^4}{4}W_{j,xxx}W_{k,xxx}+ \\ & 2U_{xu}W_{j,tx}W_{k,xx}+U_{xu}^3W_{j,xx}W_{k,xxx}+U_{xu}^2W_{j,tx}W_{k,xxx}\,]+ \\ & (\frac{\eta_j\eta_k}{R^2}Z_{uj}Z_{uk}+1)[\,W_{j,t}W_{k,t}+U_{xu}^2W_{j,x}W_{k,x}+\frac{U_{xu}^4}{4}W_{j,xx}W_{k,xx}+ \\ & 2U_{xu}W_{j,t}W_{k,x}+U_{xu}^3W_{j,x}W_{k,xx}+U_{xu}^2W_{j,t}W_{k,xx}\,] \end{aligned} \tag{5.53}$$

5.6.2 Linear matrices for the moving fluid

By introducing the displacement function (5.44), into the dynamic pressure expression (5.52) and performing the matrix operation required by the finite element method, the mass, damping and stiffness matrices for fluid are obtained by evaluating the following integral:

$$\int [N]^T\{P_{uL}\}dA. \tag{5.54}$$

we obtain:

$$[m_f^{(L)}] = [A^{-1}]^T [S_f^{(L)}] [A^{-1}]$$
$$[c_f^{(L)}] = [A^{-1}]^T [D_f^{(L)}] [A^{-1}] \quad (5.55)$$
$$[k_f^{(L)}] = [A^{-1}]^T [G_f^{(L)}] [A^{-1}]$$

The matrix [A] is given by equation (5.6) and the elements of $[S_f^{(L)}]$, $[D_f^{(L)}]$ and $[G_f^{(L)}]$ are given, as follows:

$$S_f^{(L)}(r,s) = -\frac{RL}{2} I_{rs} \rho_{fu} Z_{us} \quad (5.56)$$

$$D_f^{(L)}(r,s) = \frac{Rm^2 \pi^2}{4L} I_{rs} \rho_{fu} U_{xu}^2 Z_{us} \quad (5.57)$$

$$G_f^{(L)}(r,s) = \frac{Rm^2 \pi^2}{2L} I_{rs} \rho_{fu} U_{xu}^2 Z_{us} \quad (5.58)$$

where r, s = 1,...,8; ρ_{fu} is the density of the fluid; U_{xu} is the velocity of the fluid; Z_{us} is defined by relations (5.49) and (5.50); the subscript "u" is equal to "i" for internal flow and is equal to "e" for external flow and I_{rs} is defined by :

$$\begin{cases} I_{rs} = \dfrac{1}{(\eta_r + \eta_s)} \left[e^{(\eta_r + \eta_s)\phi} - 1 \right] & \text{for } \eta_r + \eta_s \neq 0 \\ I_{rs} = \phi & \text{for } \eta_r + \eta_s = 0 \end{cases} \tag{5.59}$$

where r, s = 1,...,8; η_r are the roots of the characteristic equation of the empty shell and ϕ is the angle for one finite element.

Finally, the global matrices $[M_f^{(L)}]$, $[C_f^{(L)}]$ and $[K_f^{(L)}]$ may be obtained, respectively, by superimposing the mass $[m_f^{(L)}]$, damping $[c_f^{(L)}]$ and stiffness $[k_f^{(L)}]$ matrices for each individual fluid finite element.

5.6.3 Non-linear matrices for the moving fluid

We use the procedure outlined in the previous section, ignoring the cross products in the non-linear dynamic pressure expression (5.53). We obtain the following matrices for the non-linear effects:

$$\begin{aligned} [c_f^{(NL)}] &= [A^{-1}]^T [D_f^{(NL)}] [A^{-1}] \\ [kc_f^{(NL)}] &= [A^{-1}]^T [GD_f^{(NL)}] [A^{-1}] \\ [k_f^{(NL)}] &= [A^{-1}]^T [G_f^{(NL)}] [A^{-1}] \end{aligned} \tag{5.60}$$

The matrix [A] is given by equation (5.6) and the elements of $[D_f^{(NL)}]$, $[GD_f^{(NL)}]$ and $[G_f^{(NL)}]$ are given, as follows:

$$D_f^{(NL)}(r,s) = -\frac{\rho_{fu}}{2}\Pi_{rs}\left[(\frac{m\pi}{L})^2 Z_{us}^2 I_{SC2} + \frac{\eta_s^2}{R^2} Z_{us}^2 I_{S3} + I_{S3} \right] \tag{5.61}$$

$$G_f^{(NL)}(r,s) = -\frac{\rho_{fu}}{2}\Pi_{rs}\left\{ U_{xu}^2 \left[(\frac{m\pi}{L})^4 Z_{us}^2 I_{S3} + \frac{\eta_s^2}{R^2}(\frac{m\pi}{L})^2 Z_{us}^2 I_{SC2} + (\frac{m\pi}{L})^2 I_{SC2} \right] + \frac{U_{xu}^4}{4}\left[(\frac{m\pi}{L})^6 Z_{us}^2 I_{SC2} + \frac{\eta_s^2}{R^2}(\frac{m\pi}{L})^4 Z_{us}^2 I_{S3} + (\frac{m\pi}{L})^4 I_{S3} \right] \right\} \tag{5.62}$$

$$GD_f^{(NL)}(r,s) = -\frac{\rho_{fu}}{2}\Pi_{rs} U_{xu}^2 \left[-(\frac{m\pi}{L})^4 Z_{us}^2 I_{SC2} - \frac{\eta_s^2}{R^2}(\frac{m\pi}{L})^2 Z_{us}^2 I_{S3} - (\frac{m\pi}{L})^2 I_{S3} \right] \tag{5.63}$$

where r, s = 1,...,8; ρ_{fu} is the density of the fluid; U_x is the velocity of the fluid; Z_{us} is defined by relations (5.49) and (5.50); the subscript "u" is equal to "i" for internal flow and is equal to "e" for external flow; Π_{rs} is defined by:

$$\begin{cases} \Pi_{rs} = \dfrac{1}{(\eta_r + 2\eta_s)}\left[e^{(\eta_r + 2\eta_s)\phi} - 1\right] & \text{for } \eta_r + 2\eta_s \neq 0 \\ \Pi_{rs} = \phi & \text{for } \eta_r + 2\eta_s = 0 \end{cases} \tag{5.64}$$

where $r, s = 1,\ldots,8$; η_r are the roots of the characteristic equation of the empty shell and ϕ is the angle for one finite element, and I_{SC2} and I_{S3} are defined by:

$$I_{SC2} = \frac{L}{3m\pi}\left[\, 1 - (-1)^{3m} \,\right]$$
$$I_{S3} = \frac{L}{3m\pi}\left[\, (-1)^{3m} - 3(-1)^{m} \,\right] \tag{5.65}$$

where m is the axial mode number and L is the length of the shell.

Finally, the global matrices $[C_f^{(NL)}]$, $[KC_f^{(NL)}]$ and $[K_f^{(NL)}]$ may be obtained, respectively, by superimposing the non-linear damping $[c_f^{(NL)}]$, non-linear combination of damping and stiffness $[kc_f^{(NL)}]$ and non-linear stiffness $[k_f^{(NL)}]$ matrices for each individual fluid finite element.

5.7 INFLUENCE OF THE NON-LINEARITIES ON THE NATURAL FREQUENCIES

Taking into account the linear and non-linear matrices of the shell and of the fluid and in the case where the coupling between different modes is ignored, the dynamic behaviour of the open or closed cylindrical shell containing flowing fluid can be represented by the following system:

$$[M_t^{(L)}]\{\ddot{\delta}\} - [C_f^{(L)}]\{\dot{\delta}\} + [K_t^{(L)}]\{\delta\} + [K_s^{(NL2)}]\{\delta^2\} + [K_s^{(NL3)}]\{\delta^3\} - [C_f^{(NL)}]\{\dot{\delta}^2\} - [KC_f^{(NL)}]\{\delta\dot{\delta}\} - [K_f^{(NL)}]\{\delta^2\} = \{0\} \tag{5.66}$$

where: $[M_t^{(L)}] = [M_s] - [M_f^{(L)}]$; $[K_t^{(L)}] = [K_s^{(L)}] - [K_f^{(L)}]$;

$\{\delta\}$ is the displacement vector; $[M_s]$, $[K_s^{(L)}]$ are the global mass and linear stiffness matrices for the shell in vacuo; $[K_s^{(NL2)}]$, $[K_s^{(NL3)}]$ are the global second and the third order non-linear stiffness matrices of the shell in vacuo; $[M_f^{(L)}]$, $[C_f^{(L)}]$ and $[K_f^{(L)}]$ are the global linear mass, damping and stiffness matrices for the fluid; $[C_f^{(NL)}]$, $[KC_f^{(NL)}]$ and $[K_f^{(NL)}]$ are the global non-linear matrices for the fluid.

These matrices are square matrices of order 4(N+1), where N represents the number of finite elements. In practice, very specific conditions are applied to the shell boundaries. Thus, matrices are reduced to square matrices of order NREDUC = 4(N+1)-J, where J represents the number of constraints applied.

Setting:

$$\{\delta\} = [\Phi]\{q\} \tag{5.67}$$

where $[\Phi]$ represents the square matrix for the eigenvectors of the linear system and $\{q\}$ is a time-related vector.

Substituting equation (5.67) into system (5.66) and multiplying by $[\Phi]^T$, we obtain:

$$
\begin{gathered}
[M_t^{(L)}]^D \{\ddot{q}\} - [C_f^{(L)}]^D \{\dot{q}\} + [K_t^{(L)}]^D \{q\} + \\
[\Phi^T][K_s^{(NL2)}]([\Phi]\ \{q\})^2 + [\Phi^T][K_s^{(NL3)}]([\Phi]\ \{q\})^3 - \\
[\Phi^T][C_f^{(NL)}]([\Phi]\ \{\dot{q}\})^2 - [\Phi^T][KC_f^{(NL)}][\Phi]\ \{q\}[\Phi]\ \{\dot{q}\} - \\
[\Phi^T][K_f^{(NL)}]([\Phi]\ \{q\})^2 = \{0\}
\end{gathered}
\quad \textbf{(5.68)}
$$

where:

$$
\begin{aligned}
[M_t^{(L)}]^D &= [\Phi^T][M_t^{(L)}][\Phi] \\
[C_f^{(L)}]^D &= [\Phi^T]\ [C_f^{(L)}][\Phi] \\
[K_t^{(L)}]^D &= [\Phi^T]\ [K_t^{(L)}][\Phi]
\end{aligned}
\quad \textbf{(5.69)}
$$

where D stands for diagonal, the matrices quantifying the fluid contribution to the matrix equations of motions are non-symmetric. To facilitate the analysis, therefore, we consider only the symmetric portion of the matrices. This simplifying hypothesis is valid, since the original and simplified systems have comparable dynamic behaviour, the maximum variance between the natural frequencies obtained for the two systems is in order of 20%. (see ref. [23] for more justification).

We saw how matrices contained in the linear part of system (5.66) could be reduced to diagonal matrices. On the other hand, by neglectiong the cross product in $([\Phi]\{q\})^2, \ldots$ of equation (5.68) we obtain:

$$m_{ii}\ddot{q}_i - c_{ii}^{(L)}\dot{q}_i + k_{ii}^{(L)}q_i + \sum_{j=1}^{NREDUC} \Big(k_{ij}^{(NL2)}q_j^2 + k_{ij}^{(NL3)}q_j^3 - C_{ij}^{(NL)}\dot{q}_j^2 - KC_{ij}^{(NL)}q_j\dot{q}_j - K_{ij}^{(NL)}q_j^2 \Big) = 0 \qquad \textbf{(5.70)}$$

where coefficients m_{ii}, $c_{ii}^{(L)}$ and $k_{ii}^{(L)}$ represent the i^{th} diagonal terms of linear matrices $[M_t^{(L)}]^D$, $[C_f^{(L)}]^D$ and $[K_t^{(L)}]^D$, respectively; $k_{ij}^{(NL2)}$ and $k_{ij}^{(NL3)}$ are the (i,j) terms of the products $([\Phi]^T\,[K_s^{NL2}]\,[\Phi]^2)$ and $([\Phi]^T\,[K_s^{NL3}]\,[\Phi]^3)$; $C_{ij}^{(NL)}$, $KC_{ij}^{(NL)}$ and $K_{ij}^{(NL)}$ are the (i,j) terms of the products $([\Phi]^T]\,[C_f^{NL}]\,[\Phi]^2)$, $([\Phi]^T]\,[KC_f^{NL}]\,[\Phi]^2)$ and $([\Phi]^T]\,[K_f^{NL}]\,[\Phi]^2)$, respectively.

Here we have "NREDUC" simultaneous equations of the form of (5.70). Numerical solution of such a system is difficult and costly. At first, we limit ourselves to solving equation (5.70) by taking into account only the diagonal terms of the products $([\Phi]^T]\,[K^{NL2}]\,[\Phi]^2), \ldots$ and therefore equation (5.70) is written as follows:

$$m_{ii}\ddot{q}_i - c_{ii}^{(L)}\dot{q}_i + k_{ii}^{(L)}q_i + k_{ii}^{(NL2)}q_i^2 + k_{ii}^{(NL3)}q_i^3 - C_{ii}^{(NL)}\dot{q}_i^2 - KC_{ii}^{(NL)}q_i\dot{q}_i - K_{ii}^{(NL)}q_i^2 = 0 \qquad \textbf{(5.71)}$$

Setting:

$$q_i(\tau) = A_i f_i(\tau) \quad \text{with} \quad f_i(0) = 1 \quad \text{and} \quad \dot{f}_i(0) = 0 \tag{5.72}$$

Equation (5.71) becomes, after the A_p simplification and dividing by m_{ii} :

$$\begin{aligned} &\ddot{f}_i - \kappa_i \dot{f}_i + \omega_i^2 f_i + \lambda_i (A_i/t) f_i^2 + \sigma_i (A_i/t)^2 f_i^3 \\ &- (A_i/t)\left[\zeta_i \dot{f}_i^2 + \xi_i f_i \dot{f}_i + \gamma_i f_i^2\right] = 0 \end{aligned} \tag{5.73}$$

where

$$\omega_i^2 = \frac{k_{ii}^{(L)}}{m_{ii}} \; ; \qquad \kappa_i = \frac{c_{ii}^{(L)}}{m_{ii}} \tag{5.74}$$

$$\lambda_i = \frac{k_{ii}^{(NL2)}}{m_{ii}} t \; ; \qquad \sigma_i = \frac{k_{ii}^{(NL3)}}{m_{ii}} t^2 \tag{5.75}$$

$$\zeta_i = \frac{C_{ii}^{(NL)}}{m_{ii}} t \; ; \quad \xi_i = \frac{KC_{ii}^{(NL)}}{m_{ii}} t \; ; \quad \gamma_i = \frac{K_{ii}^{(NL)}}{m_{ii}} t \tag{5.76}$$

where t represents shell thickness, the coefficient $[k_{ii}^{(L)}/m_{ii}]$ represents the i^{th} linear vibration frequency of the system.

The solution $f_i(\tau)$ of the non-linear differential equation (5.73) which satisfies the conditions in (5.72) is calculated by a fourth order Runge-Kutta numerical method. The linear and non linear natural frequencies are evaluated by a systematic search for the $f_i(\tau)$ roots as a function of time. The ω_{NL}/ω_L ratio of linear and non-linear frequency is expressed as a function of non-dimensional ratio (A_i/t) where A_i is the vibration amplitude.

5.8 CALCULATIONS AND DISCUSSION

The influence of non-linearities associated with the wall of the shell and with the fluid on the open or closed cylindrical shell's free vibrations is expressed by equation (5.73). For a shell of given physical characteristics, we first present the results for the convergence of the model and, second, those obtained by the present method in the case of linear vibration. Then the ratio ω_{NL}/ω_L of linear and non-linear frequency is graphically represented in Figures 5.5 to 5.10 with respect to the non-dimensional ratio, A_p/t. The straight horizontal line represents the linear vibration cases, where the frequency is independent of the motion's amplitude.

5.8.1 Convergence of the method

A first set of calculations was undertaken to determine the required number of finite elements for a precise determination of natural frequencies. Calculations were made for the same closed cylindrical shell completely filled with internal fluid for the number of finite elements N = 2, 4, 6, 8, 10, 15 and 20. This steel shell is simply supported at both ends and has the following data:

$$R=37.7\text{mm},\ t=0.229\text{mm},\ L=234\text{mm},\ \nu=0.3,\ \rho_{fi}/\rho_s=0.128$$

The results for m = 1 and for n = 2, 3, 4, 5 and 6 are shown in Figure 5.3. We conclude that the convergence of the shell-fluid system demands ten elements for both the low and the high modes.

5.8.2 Linear free vibration of closed cylindrical shell

We present a calculation to test the method incorporating linear analysis which is developed in this paper. The closed cylindrical shell is simply supported at both ends and has the same physical properties as those given in the previous section. This shell was studied by Goncalves and Batista [3], who used the Rayleigh-Ritz technique to obtain the natural frequencies of the shell-fluid system. Figure 5.4 shows the linear natural frequencies as a function of the circumferential mode number n for the axial mode m=1.

As may be seen the results obtained by the present method are in good agreement with those of Goncalves & Batista [3]. For the case of empty or liquid-filled shells, there is the well- know dip in the frequency curve as the shell makes a transition through the lower values of n. This phenomenon can be explained by the interchange in the relative contributions of the bending and stretching strain energies of the shell.

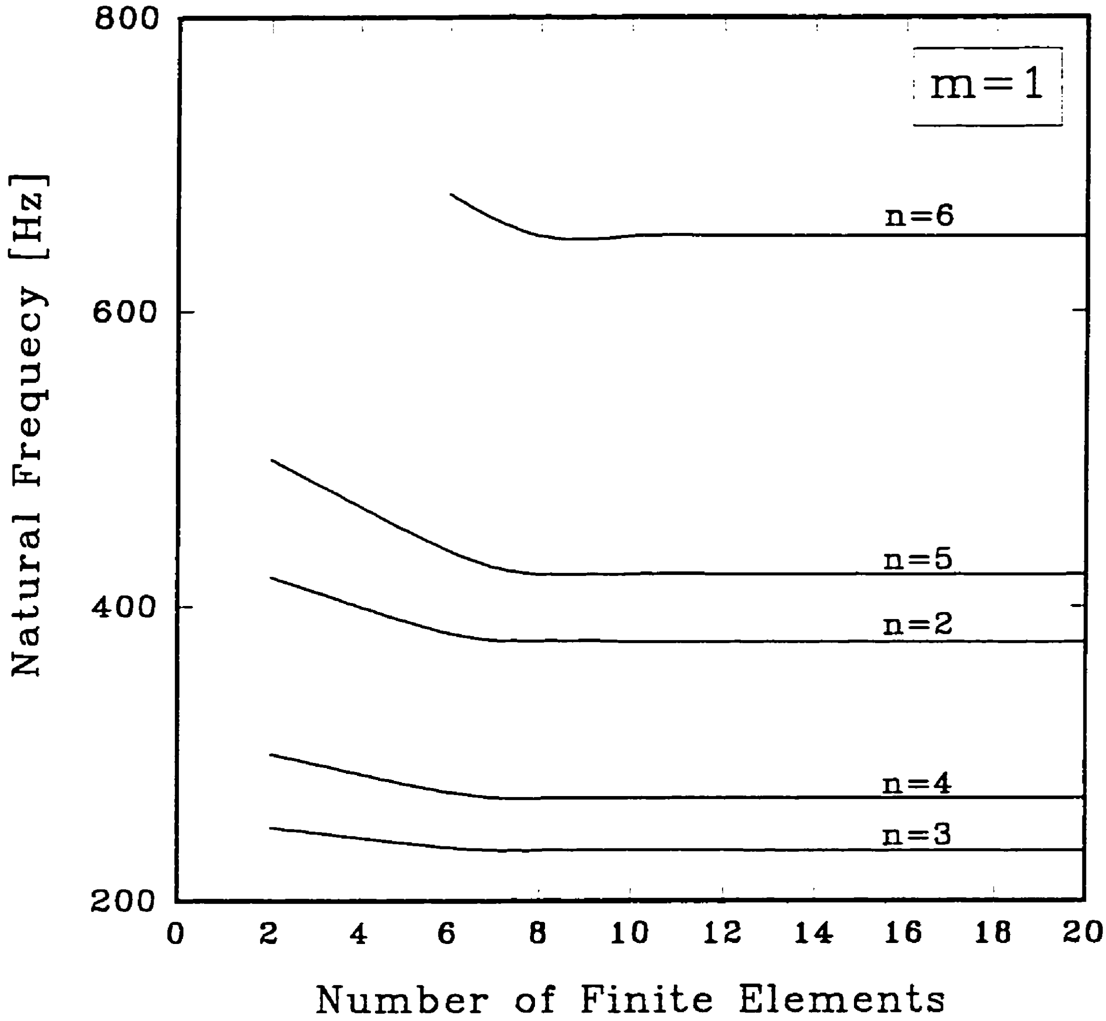

Figure 5.3 **Linear natural frequency for a simply-supported closed cylindrical shell completely filled with internal fluid as a function of the number of finite elements;**
n is the number of circumferential mode,
m is the number of axial mode.

R = 37.7 mm , t = 0.229 mm , L = 234 mm , ν = 0.3, ρ_{fl}/ρ_s = 0.128

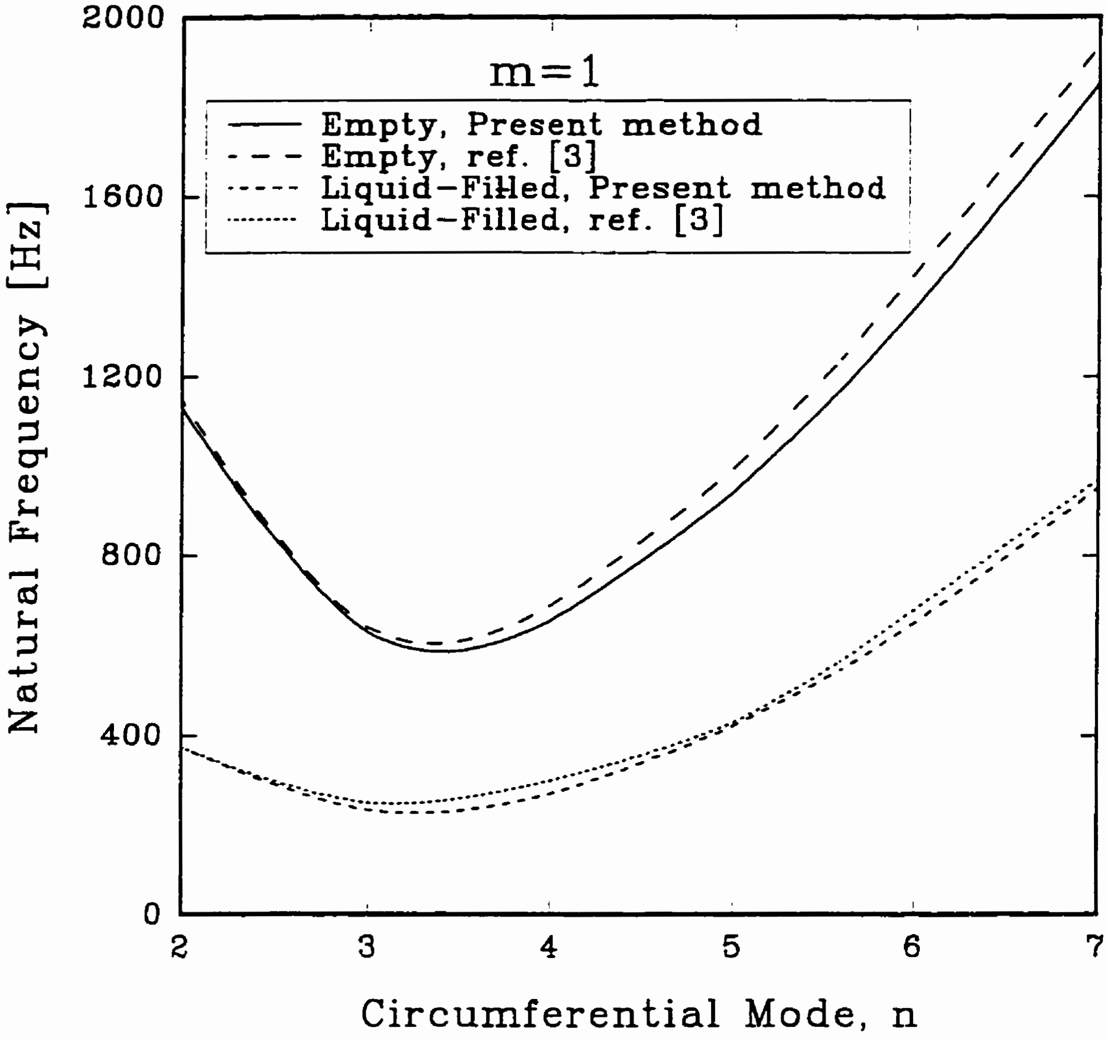

Figure 5.4 **Linear natural frequency for an empty and liquid-filled closed simply-supported cylindrical shell as a function of the number of circumferential mode n; (m = 1).**

R = 37.7 mm , t = 0.229 mm , L = 234 mm , ν = 0.3, ρ_{fl}/ρ_s = 0.128

5.8.3 Non-linear free vibration of closed cylindrical shell

5.8.3.1 Empty shell

This set of calculations is designed to determine the influence of geometric non-linearities in strain-displacement relations on the free vibrations of an empty isotropic cylindrical shell, simply-supported at both ends. The shell has the following properties:

$$\zeta = \pi Rm/nL = 2, \quad \epsilon = (n^2 t/R)^2 = 1 \quad \text{and} \quad \nu = 0.3$$

The variations in frequency ratio as a function of A/t for this shell (Figure 5.5) were calculated using the present method, and compared to the results of Evensen [13] and Atluri [15]. Evensen's analysis involved a two modes approximation and his equation was obtained using the Galerkin procedure. The work of Atluri is based on Donnell's equations, a modal expansion was used for displacements and the Galerkin technique was used to reduce the problem to a non-linear ordinary differential equation for the modal amplitudes.

As may be seen, the results obtained by the present method are in satisfactory agreement with those of other authors.

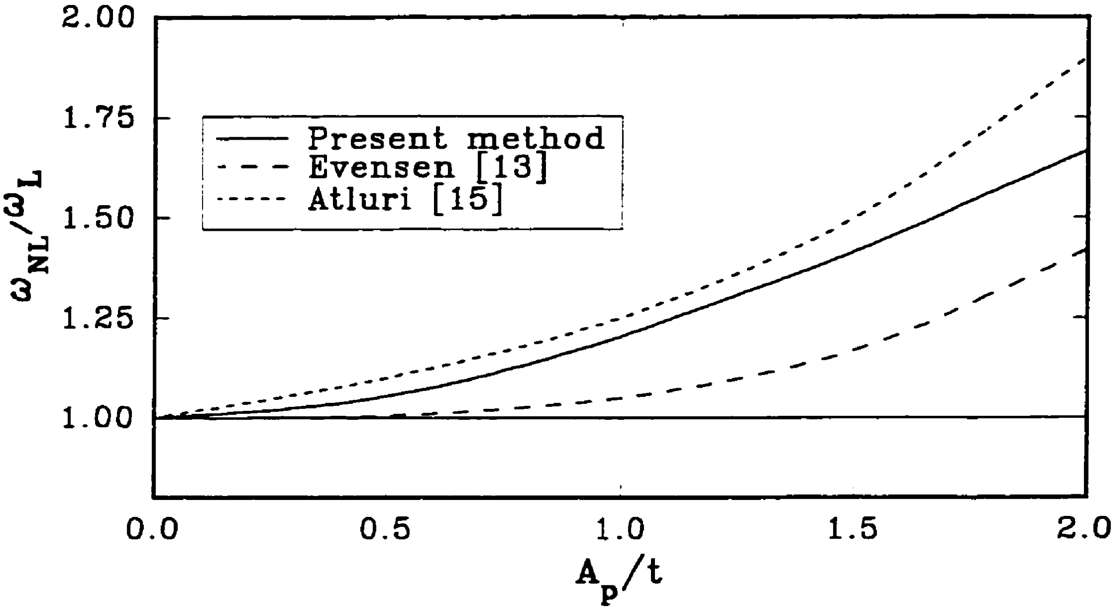

Figure 5.5 **Comparison of the effect of amplitude upon frequency for an empty simply-supported closed cylindrical shell.**

$\zeta=\pi Rm/nL=2$, $\chi=(n^2t/R)^2=1$, $\nu=0.3$

5.8.3.2 Submerged shell

The second comparative example is shown in Figure 5.6, the closed cylindrical shell is simply-supported at both ends and completely submerged in liquid. The pertinent data are as follows:

$$E = 21.981 \times 10^{11} N/m^2, \ \nu = 0.3, \ \rho_f/\rho_s = 0.128$$
$$R = 0.235 m, \ t = 0.00235 m, \ \phi_T = 360°.$$

This case was previously analysed by Ramachandran [17] who used the Rayleigh-Ritz procedure. In his study, he took into account only the influence of non-linearities associated with the shell and neglected the effect of non-linearities associated with the fluid. In addition, only lateral displacements were considered for the non-linear analysis.

In Figure 5.6, we present a comparison between the present work and that of Ramachandran [17], and show results for differents modes and geometry. For ratio L/R = 4 and the mode (n = 4, m = 1), we observe that the ratio between linear and non-linear natural frequency decreas as ratio A/t increase. The variations are small for values A/t below 1.0. For the value of A/t = 2, the variation calculated by the present method is more pronounced than that of Ramanchadran [17], the results obtained are in agreement within a range of 5%.

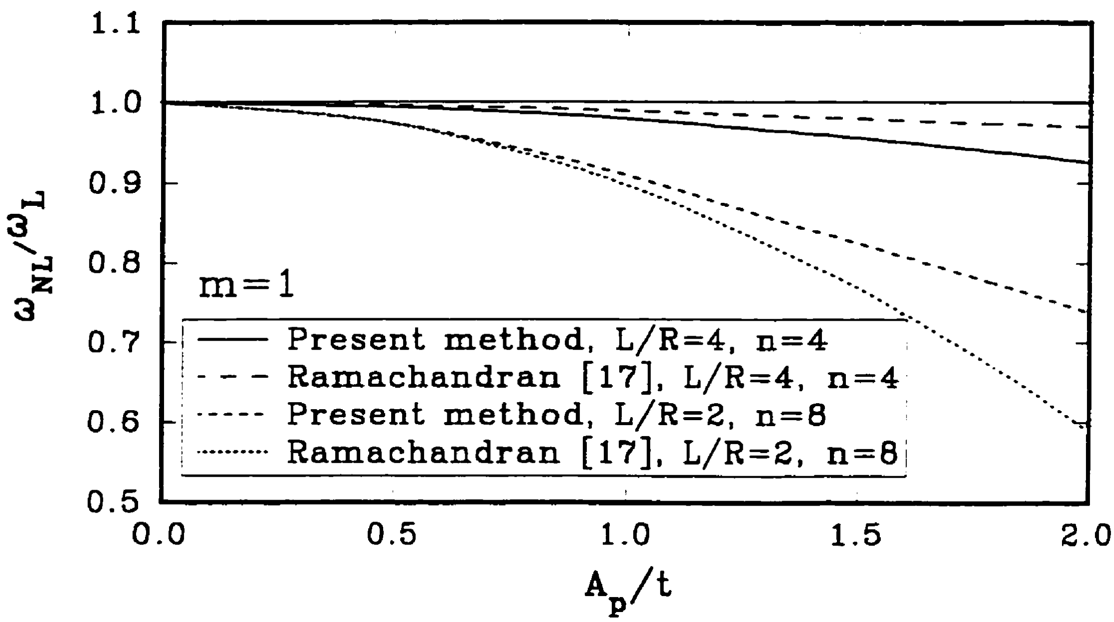

Figure 5.6 Comparison of the effect of amplitude upon frequency for a submerged simply-supported closed cylindrical shell.

$E = 21.981 \times 10^{11} N/m^2$, $\nu = 0.3$, $\rho_f/\rho_s = 0.128$
$R = 235$ mm, $t = 2.35$ mm, $\phi_T = 360°$.

For ratio L/R = 2 and the mode (n = 8, m = 1), we observe that the ratio between linear and non-linear natural frequency decrease and is more pronounced than the previous results. For the value A/t = 2, the variation calculated by the present method is less pronounced than that of Ramanchadran [17], the difference between the two results is in order of 25 %.

5.8.4 Non-linear free vibration of an open cylindrical shell totally submerged in liquid and subjected simultaneously to an internal and external fluid

One of the great advantages of the finite element method is the ease with which it can be applied to any geometry and any boundary condition. Thus, this step of calculation is to study the non-linear dynamic characteristics of an open cylindrical shell totally submerged in liquid as a function of flow velocity, circumferential and axial modes, boundary conditions, material properties, etc...

5.8.4.1 Influence of non-linearities associated with the wall of the shell versus non-linearities associated with the fluid

In this first calculation, we analyse the influence of non-linearities associated with the wall of the shell versus the non-linearities associated with the fluid at rest. The study is made on an open cylindrical shell totally submerged in fluid. Calculations have been made by solving equation (5.73), both when the non-linearity associated with the fluid is taken into account and when it is not taken into account.

The steel open shell is simply-supported at the four edges and has the following data:

$$R = 450\,\mathrm{mm},\; t = 1.5\,\mathrm{mm},\; L = 1350\,\mathrm{mm},\; \phi_T = 100°,\; \rho_f/\rho_s = 0.128$$

The results for this analysis are presented in Figure 5.7. We observe that the influence of non-linearities associated with fluid on the dynamic behaviour of the shell-fluid structure is negligible.

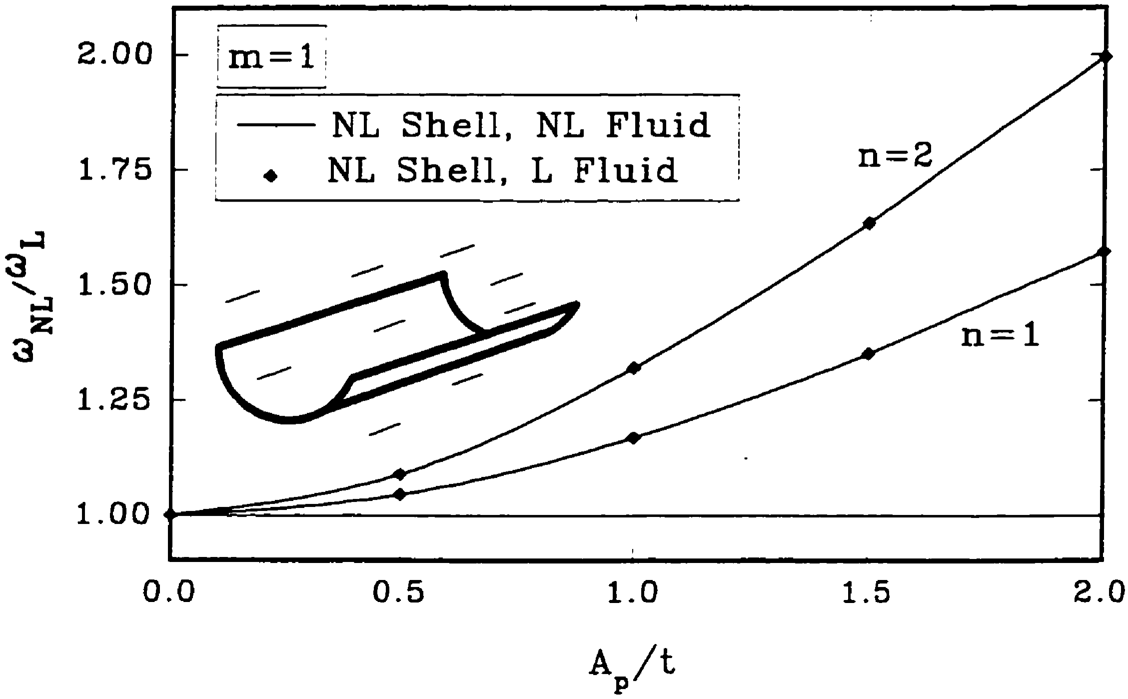

Figure 5.7 **Influence of non-linearities associated with the wall of the shell versus non-linearities associated with the fluid at rest for a simply-supported open cylindrical shell.**

$R = 450$ mm , $t = 1.5$ mm , $L = 1350$ mm , $\phi_T = 100$ °, $\rho_f/\rho_s = 0.128$

5.8.4.2 Effects of flow velocity

In order to establish the effect of the flow velocity on the non-linear free vibration, we turn to Figure 5.8.

The parameters of the investigation are as follows: m = 1, n = 9 and 10; Reynolds number, R_N =0.0 and 1.0 10^6, with $R_N = 2\ U_x\ R\ \rho_f / \nu_f$, where U_x is the mean velocity of the flow, R is the average radius of the open cylindrical shell and ρ_f and ν_f are respectively the density and viscosity of the flowing fluid.

The other data are:

$$R = 225\,\text{mm},\ t = 1.5\,\text{mm},\ L = 1350\,\text{mm},\ \phi_T = 120°,\ \rho_f/\rho_s = 0.128$$

The Figure shows that the non-linearity is of the softening type for circumferential mode n = 9 and is of hardening type for n = 10 for both flow and no-flow condition. We see also that the non-linear effect is more pronounced for the shell-fluid system when the fluid is moving. The difference between the the cases is of the order of 10%.

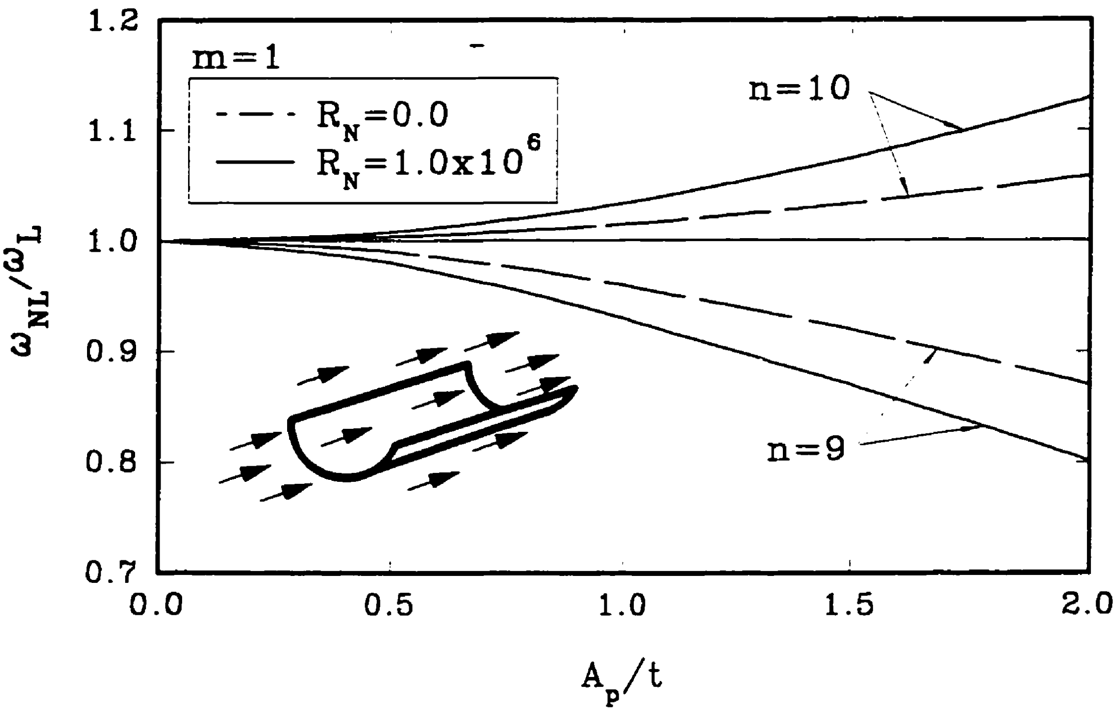

Figure 5.8 **Influence of large amplitude on the natural frequency of a submerged clamped-clamped open cylindrical shell for different Reynolds numbers.**

$R = 225$ mm , $t = 1.5$ mm , $L = 1350$ mm , $\phi_T = 120$ °, $\rho_f / \rho_s = 0.128$

5.8.4.3 Effects of material properties

With the same geometric data, Figure 5.9 shows the effect of non-linearities upon frequency for different material properties. When the open shell is simply-supported at the four edges and is completely submerged in water, the data are as follows :

$$R = 450\,mm,\ t = 1.5\,mm,\ L = 1350\,mm,\ \phi_T = 180°.$$

The materials chosen are steel, acrylic, rubber and an orthotropic material where the physical properties are:

$$E_x = 1.0 \times 10^{11}\,N/m^2,\quad E_\theta = 0.05 \times E_x,\quad \nu_x = 0.2,\quad \nu_\theta = 0.05 \times \nu_x,$$
$$G_{x\theta} = 0.05 \times E_x,\ \rho_s = 7800\,N/m^3,$$

We observe for the mode (m = 2, n = 3) that the steel shell is the one on which the effect of non-linearity is more pronounced, the orthotropic shell is the one on which the effect of non-linearity is less pronounced.

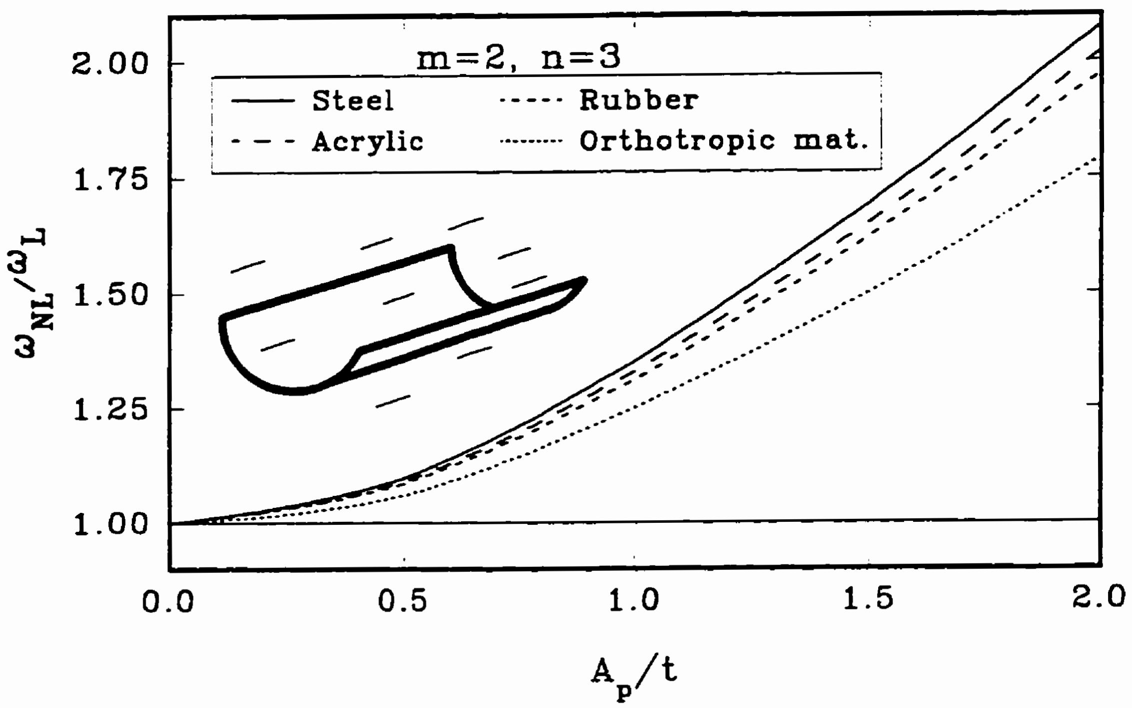

Figure 5.9 **Influence of large amplitude on the natural frequency of a submerged simply-supported open cylindrical shell for different material properties, (m = 2, n = 3).**

R = 450 mm , t = 1.5 mm , L = 1350 mm , ϕ_T = 100 °, ρ_f = 1000 Kg / m ^{3}

5.8.4.4 Effects of the circumferential mode n

In Figure 5.10, we present the effect of large amplitude on the frequency ratio as a function of A/t for axial mode m=1 and circumferential mode n = 6 to 12. The open shell is clamped along the straight edges and simply-supported along its curved edges. The data for the steel shell are:

$$R = 225\,mm,\ t = 1.5\,mm,\ L = 1350\,mm,\ \phi_T = 120°,\ \rho_f/\rho_s = 0.128$$

The Figure shows that the non-linearity is of the hardening type for circumferential mode n = 10, 11 and 12 and is of softening type for n between 6 and 9. We see also that the non-linear effect is more pronounced for the mode n = 6 and the variation is small for the case of n = 9.

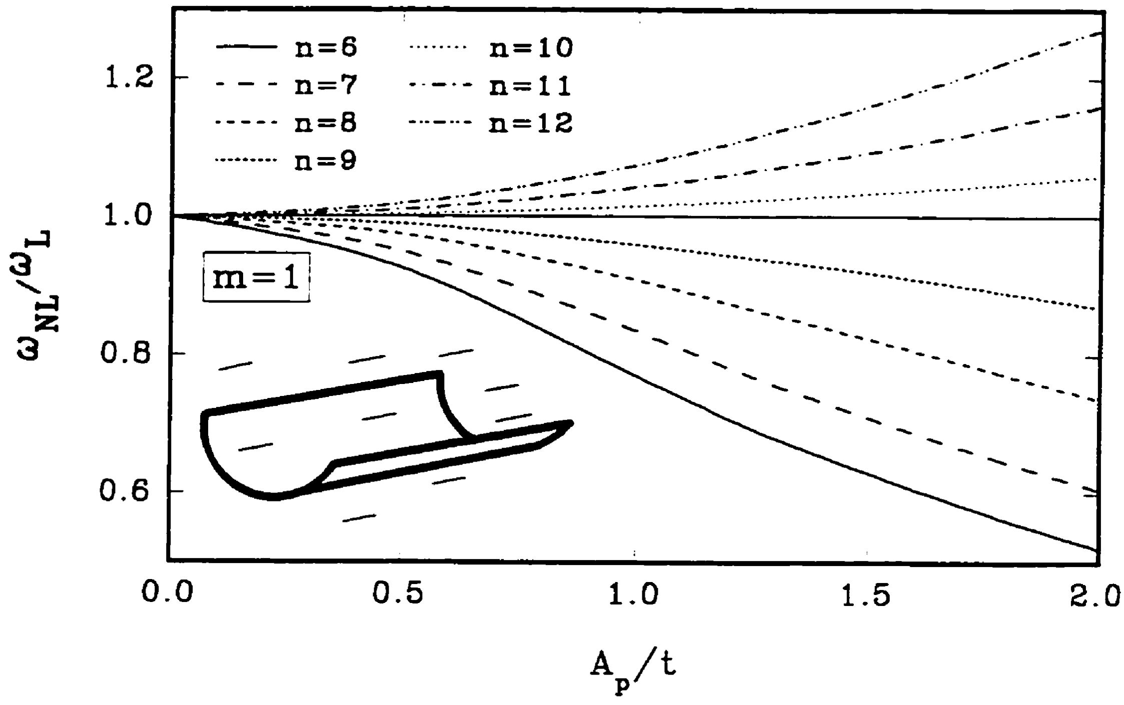

Figure 5.10 **Influence of large amplitude on the natural frequency of a submerged clamped-clamped open cylindrical shell for various circumferential mode n and axial mode m = 1.**

R = 225 mm , t = 1.5 mm , L = 1350 mm , ϕ_T = 120 °, ρ_f/ρ_s = 0.128

5.9 CONCLUSIONS

The method developed in this paper demonstrates the influence of the non-linearities associated with the wall of the shell and with the fluid flow on the free vibrations of totally submerged open or closed cylindrical shells, subjected simultaneously to an internal and external flow. It is a hybrid method, based on a combination of non-linear thin shell theory, non-linear fluid theory and the finite element method

An open cylindrical finite element is developed, in order that the displacement functions can be derived directly from classical thin shell theory. Mass and linear stiffness matrices are then obtained for the empty shell by the finite element method. With the modal coefficients derived from the Sanders-Koiter non-linear theory of thin shells and corresponding to non-linearities in strain-displacement relations, the second and third order non-linear stiffness matrices are then calculated using the finite element method.

The pressure exerted by the fluid is given using a non-linear development of Bernoulli's equation. From the solution of the potential equation we derive an expression of linear and non-linear pressure as a function of the nodal displacements of the fluid element, the inertial, centrifugal, Coriolis forces and a combination of non-linear effects. Through the finite element procedure, we obtain the linear mass, damping and stiffness matrices for the fluid as well as the non-linear matrices for damping and stiffness and a combination of the two.

The non-linear equations of motion are solved by the fourth-order Runge-Kutta numerical method. Variations in the free vibration frequencies are determined in conjunction with motion amplitude for a closed or open cylindrical shell, empty or submerged in flowing fluid. Deviations in terms of linear vibrations are observed.

This method combines the advantages of finite element analysis which deals with complex shells, and the precision of formulation which the use of displacement functions derived from shell and fluid theories contributes.

This area of investigation is still wide open and there is very little on the subject in the literature. We are unable, therefore, to confirm whether, in the context of a dynamic analysis, we are justified in completely neglecting the influence of non-linearities associated with fluid flow. On the other hand, the effect of geometric non-linearities associated with the walls is not negligible and should be taken into account in calculating the dynamic behaviour of shell-fluid interactions when the amplitude of vibration is greater than the thickness of the shell.

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5.11 NOMENCLATURE

LIST OF SYMBOLS

A, B, C	Constants in equation (5.2) defining U, V, W respectively
A_i	Motion amplitude
a_p, b_p, c_p	Modal coefficients determined by equation (5.19)
$a_p^{(1)}$, $b_p^{(1)}$, $c_p^{(1)}$	Coefficient determined by equations (5.23 - 5.25)
$a_{rs}^{(1)}, b_{rs}^{(1)}, b_{rs}^{(2)}, c_{rs}^{(1)}$	Coefficient determined by equations (5.26 - 5.28)
aA_{prs}, bB_{prs}, cC_{prs}, aB_{prs}, bA_{prs}	Modal coefficients determined by equation (5.18)
A_{pq}, B_{pq}, C_{pq}	Modal coefficients determined by equations (5.20 - 5.22)
A_{prsq}, B_{prsq}, C_{prsq}, AB_{prsq}, BA_{prsq}	Modal coefficients determined by equation (5.18)
E	Young's modulus
e	Exponential
f_i	Function determined by equation (5.72)
GG(p,q)	Coefficient determined by equation (5.32)
i	$i^2 = -1$
$J_{i\eta j}$	Bessel function of the first kind and of order $i\eta_j$
L	Length of the shell

m	Axial mode number
N	Number of finite elements
n	Circumferential mode number
P_u	Lateral pressure exerted on the shell, u=i for internal pressure and u=e for external pressure
P_{ij}	Terms of elasticity matrix (i= 1,...,6 ; j= 1, ..., 6)
R	Mean radius of the shell
R_j	Solution of Bessel equation (5.47)
S_j	Defined by equation (5.45)
$SS(p,q)$	Coefficient determined by equation (5.36)
t	Thickness of the shell
U, V, W	Axial, tangential and radial displacements
U_{xu}	Velocity of the liquid
V_x , V_θ , V_r	Axial, tangential and radial fluid velocity (5.38)
x	Axial coordinate
$Y_{i\eta j}$	Bessel function of the second kind and of order $i\eta_j$
Z_{uj}	Defined by equation (5.49) for u=i and equation (5.50) for u=e
η_i	Complex roots of the characteristic equation (5.3)
α_p, β_p	Determined by equation (5.4)
θ	Circumferential coordinate
ν	Poisson's ratio

ϕ	Opening angle for one finite element
ϕ_T	Opening angle for the whole open shell
Φ	Velocity potential
ρ_s	Density of the shell material
ρ_{fu}	Density of fluid, u=i for internal fluid and f = e for external fluid
ω_L	Linear frequency of free vibrations
ω_{NL}	Non-linear frequency of free vibrations
τ	Time related coordinates
ω_i, κ_i	Coefficient determined by equation (5.74)
λ_i, σ_i	Coefficient determined by equation (5.75)
ζ_i, ξ_i, γ_i	Coefficient determined by equation (5.76)

LIST OF MATRICES

[A]	Defined by equation (5.6)
[B]	Defined by equation (5.8)
$[c_f^{(L)}]$,$[c_f^{(NL)}]$	Linear and non-linear damping matrices for a fluid finite element
{C}	Vector of arbitrary constants

$[D_f^{(L)}]$	Defined by equation (5.57)
$[D_f^{(NL)}]$	Defined by equation (5.61)
$[G_f^{(L)}]$	Defined by equation (5.58)
$[G_f^{(NL)}]$	Defined by equation (5.62)
$[GD_f^{(NL)}]$	Defined by equation (5.63)
$[k_f^{(L)}]$,$[k_f^{(NL)}]$	Linear and non-linear stiffness matrix for a fluid finite element
$[k_s^{(L)}]$, $[k_s^{(NL2)}]$, $[k_s^{(NL3)}]$	Linear and non-linear stiffness matrices for a shell finite element
$[kc_f^{(NL)}]$	Defined by equation (5.60)
$[m_f^{(L)}]$	Mass matrix for a fluid finite element
$[m_s]$	Mass matrix for a shell finite element
[N]	Displacement function defined by equation (5.7)
[P]	Elasticity matrix
[Q]	Defined by equation (5.8)
{q}	Time-related vector coordinates
[R]	Defined by equation (5.5)
$[S_f^{(L)}]$	Defined by equation (5.56)
$[T_m]$	Defined by equation (5.5)
$\{\delta_i\}$	Vector of degrees of freedom at node i
$\{\delta\}$	Vector of degrees of freedom for total shell
$\{\sigma\}$	Stress vector

$\{\epsilon_L\}$ $\{\epsilon_{NL}\}$	Linear and non-linear components of the deformation vector, respectively
$[\Phi]$	Matrix of eigenvectors, equation (5.67)

APPENDIX A5-1

EQUATIONS OF MOTION

This appendix contains the equations of motion for a thin orthotropic cylindrical shell.

$$L_1\ (U,V,W,p_{ij}) = p_{11}\ \frac{\partial^2 U}{\partial\ x^2} + \frac{p_{12}}{R}\ (\frac{\partial^2 V}{\partial x\ \partial\theta} + \frac{\partial W}{\partial x}) - p_{14}\ \frac{\partial^3 W}{\partial x^3} +$$

$$\frac{p_{15}}{R^2}\ (\frac{\partial^3 W}{\partial x\ \partial\theta^2} + \frac{\partial^2 V}{\partial x\ \partial\theta}) + (\frac{p_{33}}{R} - \frac{p_{63}}{2R^2})\ (\frac{\partial^2 V}{\partial x\ \partial\theta} + \frac{1}{R}\ \frac{\partial^2 U}{\partial\theta^2}) +$$

$$(\frac{p_{36}}{R^2} - \frac{p_{66}}{2R^3})\ (- \frac{2\partial^3 W}{\partial x\ \partial\theta^2} + \frac{3}{2}\ \frac{\partial^2 V}{\partial x\ \partial\theta} - \frac{1}{2}R\ \frac{\partial^2 U}{\partial\theta^2})$$

$$L_2\ (U,V,W,p_{ij}) = (\frac{p_{21}}{R} + \frac{p_{51}}{R^2})\ (\frac{\partial^2 U}{\partial x\ \partial\theta}) + \frac{1}{R}\ (\frac{p_{22}}{R} + \frac{p_{52}}{R^2})$$

$$(\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial\ W}{\partial\theta}) - (\frac{p_{24}}{R} + \frac{p_{54}}{R^2})\ (\frac{\partial^3 W}{\partial x^2\ \partial\theta}) + \frac{1}{R^2}\ (\frac{p_{25}}{R} + \frac{p_{55}}{R^2})$$

$$(- \frac{\partial^3 W}{\partial\theta^3} + \frac{\partial^2 V}{\partial\theta^2}) + (p_{33} + \frac{3p_{63}}{2R})\ (\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 U}{R\partial x\ \partial\theta}) +$$

$$\frac{1}{R}\ (p_{36} + \frac{3p_{66}}{2R})\ (-2\ \frac{\partial^3 W}{\partial x^2\ \partial\theta} + \frac{3}{2}\ \frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 U}{2R\ \partial x\ \partial\theta})$$

$$L_3\,(U,V,W,p_{ij}) = p_{41}\frac{\partial^3 U}{\partial x^3} + \frac{p_{42}}{R}\left(\frac{\partial^3 V}{\partial x^2\,\partial\theta} + \frac{\partial^2 W}{\partial x^2}\right) - p_{44}\frac{\partial^4 W}{\partial x^4} +$$

$$\frac{p_{45}}{R^2}\left(-\frac{\partial^4 W}{\partial x^2\,\partial\theta^2} + \frac{\partial^3 V}{\partial x^2\,\partial\theta}\right) + \frac{2\,p_{63}}{R}\left(\frac{\partial^3 U}{R\,\partial x\,\partial\theta^2} + \frac{\partial^3 V}{\partial x^2\,\partial\theta}\right) + \left(\frac{2p_{66}}{R^2}\right)$$

$$\left(-2\frac{\partial^4 W}{\partial x^2\partial\theta^2} + \frac{3}{2}\frac{\partial^3 V}{\partial x^2\,\partial\theta} - \frac{\partial^3 U}{2R\partial x\,\partial\theta^2}\right) + \frac{p_{51}}{R^2}\frac{\partial^3 U}{\partial x\,\partial\theta^2} + \frac{p_{52}}{R^3}\left(\frac{\partial^3 V}{\partial\theta^3} +\right.$$

$$\left.\frac{\partial^2 W}{\partial\theta^2}\right) + \frac{p_{55}}{R^4}\left(-\frac{\partial^4 W}{\partial\theta^4} + \frac{\partial^3 V}{\partial\theta^3}\right) - \frac{p_{21}}{R}\frac{\partial U}{\partial x} - \frac{p_{54}}{R^2}\frac{\partial^4 W}{\partial x^2\,\partial\theta^2}$$

$$-\frac{p_{22}}{R^2}\left(\frac{\partial V}{\partial\theta} + W\right) + \frac{p_{24}}{R}\frac{\partial^2 W}{\partial\theta^2} - \frac{p_{25}}{R^3}\left(-\frac{\partial^2 W}{\partial\theta^2} + \frac{\partial V}{\partial\theta}\right)$$

APPENDIX A5-2

Characteristic Equation (5.3)

$$h_8\ \eta^8 + h_6\ \eta^6 + h_4\ \eta^4 + h_2\ \eta^2 + h_0 = 0$$

where

$$h_8 = f_1\ f_6\ f_{10} - f_1\ f_8^2$$

$$h_6 = f_1\ f_6\ f_{11} + f_1\ f_7\ f_{10} - 2f_1\ f_8\ f_9 + f_2\ f_6\ f_{10} - f_2\ f_8^2 - f_3^2\ f_{10} + f_3\ f_8\ f_4 + f_4\ f_3\ f_8 - f_4^2\ f_6$$

$$h_4 = f_1\ f_6\ f_{12} + f_1\ f_7\ f_{11} - f_1\ f_9^2 + f_2\ f_6\ f_{11} + f_2\ f_7\ f_{10} - 2f_2\ f_8\ f_9 - f_3^2\ f_{11} + f_3\ f_9\ f_4 + f_3\ f_8\ f_5 + f_4\ f_3\ f_9 - f_4^2\ f_7 - f_4\ f_6\ f_5 + f_5\ f_3\ f_8 - f_5\ f_6\ f_4$$

$$h_2 = f_1\ f_7\ f_{12} + f_2\ f_6\ f_{12} + f_2\ f_7\ f_{11} - f_2\ f_9^2 - f_3^2\ f_{12} + f_3\ f_9\ f_5 - f_4\ f_7\ f_5 + f_5\ f_3\ f_9 - f_5\ f_7\ f_4 - f_5^2\ f_6$$

$$h_o = f_2\ f_7\ f_{12} - f_7\ f_5^2$$

The coefficients f_i (i = 1,...,12) are given by the above equations :

$$f_1 = \frac{1}{R}(P_{55} - \frac{1}{R} P_{36} + \frac{1}{4R^2} P_{66})$$

$$f_2 = - P_{11} \bar{m}^2$$

$$f_3 = \bar{m}\left[\frac{1}{R} (P_{12} + P_{13}) + \frac{1}{R^2} (P_{15} + P_{36}) - \frac{3}{4R^3} P_{66}\right]$$

$$f_4 = - \frac{\bar{m}}{R^2} (P_{15} + 2 P_{36} - \frac{1}{R} P_{66})$$

$$f_5 = \frac{P_{12}}{R} \bar{m} + P_{14} \bar{m}^3$$

$$f_6 = - \frac{1}{R^2} (P_{22} + \frac{1}{R^2} P_{55} + \frac{2}{R} P_{25})$$

$$f_7 = \bar{m} (P_{33} + \frac{3}{R} P_{36} + \frac{9}{4R^2} P_{66})$$

$$f_8 = \frac{1}{R^3} (P_{25} + \frac{1}{R} P_{55})$$

$$f_9 = - \frac{1}{R^2} (P_{22} + \frac{1}{R} P_{52}) - \frac{\bar{m}^2}{R} (2P_{36} + P_{24} + \frac{3}{R} P_{66} + \frac{1}{R} P_{54})$$

$$f_{10} = - \frac{1}{R^4} P_{55}$$

$$f_{11} = \frac{2}{R^3} P_{25} + \frac{\bar{m}}{R^2} (2P_{45} + 4P_{66})$$

$$f_{12} = - \frac{1}{R} P_{22} - \frac{2}{R} P_{24} \bar{m}^2 - P_{44} \bar{m}$$

and $\bar{m} = m\frac{\pi}{L}$

APPENDIX A5-3

MATRICES $[T_m]$, $[R]$, $[A]$, $[Q]$, $[m_S]$ and $[k_S^{(L)}]$

MATRIX $[T_m]_{(3x3)}$

$$[T_m] = \text{Diag}[\ \cos\bar{m}x,\ \sin\bar{m}x,\ \sin\bar{m}x\ ]$$
$$\bar{m} = m\pi/L$$

MATRIX $[R]_{(3x8)}$

$$R(1,j) = \alpha_j e^{\eta_j\theta}$$
$$R(2,j) = e^{\eta_j\theta}, \qquad j = 1,\ldots 8$$
$$R(3,j) = \beta_j e^{\eta_j\theta}$$

MATRIX $[A]_{(8x8)}$

$$A(1,j) = \alpha_j \qquad A(5,j) = \alpha_j e^{\eta_j\phi}$$
$$A(2,j) = 1 \qquad A(6,j) = e^{\eta_j\phi}$$
$$A(3,j) = \eta_j \qquad A(7,j) = \eta_j e^{\eta_j\phi}$$
$$A(4,j) = \beta_j \qquad A(8,j) = \beta_j e^{\eta_j\phi}$$

MATRIX $[Q]_{(6x8)}$

$$Q(1,j) = A_j e^{\eta_j \theta} \qquad Q(4,j) = D_j e^{\eta_j \theta}$$
$$Q(2,j) = B_j e^{\eta_j \theta} \qquad Q(5,j) = E_j e^{\eta_j \theta}$$
$$Q(3,j) = C_j e^{\eta_j \theta} \qquad Q(6,j) = F_j e^{\eta_j \theta}$$

The terms A_j, B_j, C_j, D_j, E_j and F_j ($j = 1, \ldots, 8$) may be expressed as follows:

$$A_j = - \frac{m \pi \alpha_j}{L},$$

$$B_j = - \frac{\eta_j \beta_j + 1}{R},$$

$$C_j = - \frac{m \pi \beta_j}{L} + \frac{\eta_j \alpha_j}{R}$$

$$D_j = - \frac{(m \pi)^2}{L^2},$$

$$E_j = - \frac{\eta_j^2 + \eta_j \beta_j}{R^2}$$

and $$F_j = - \frac{2 m \pi \eta_j}{RL} + \frac{3 m \pi \beta_j}{2 RL} - \frac{\eta_j \alpha_j}{2R^2}$$

MATRICES $[m]_{(8x8)}$ and $[k^L]_{(8x8)}$

$$[m_S] = [A^{-1}]^T [S] [A^{-1}], \qquad [k_S^{(L)}] = [A^{-1}]^T [G] [A^{-1}]$$

where [S] and [G] are defined by the above equations:

$$S(i,j) = \frac{RL}{2} \frac{(\alpha_i \alpha_j + \beta_i \beta_j + 1)}{(\eta_i + \eta_j)} (e^{(\eta_i + \eta_j)\phi} - 1) \quad \text{if } \eta_i + \eta_j \neq 0$$

$$S(i,j) = \frac{RL\,\phi}{2} (\alpha_i \alpha_j + \beta_i \beta_j + 1) \quad \text{if } \eta_i + \eta_j = 0$$

$$G(i,j) = \frac{RL}{2} (p_{11} A_i A_j + p_{12} A_i B_j + p_{14} A_i D_j + p_{15} A_i E_j$$

$$+ p_{21} B_i A_j + p_{22} B_i B_j + p_{24} B_i D_j + p_{25} B_i E_j$$

$$+ p_{41} D_i A_j + p_{42} D_i B_j + p_{44} D_i D_j + p_{45} D_i E_j$$

$$+ p_{51} E_i A_j + p_{52} E_i B_j + p_{54} E_i D_j + p_{55} E_i E_j$$

$$+ p_{33} C_i C_j + p_{36} C_i F_j + p_{63} F_i C_j + p_{66} F_i F_j)$$

$$\frac{(e^{(\eta_i + \eta_j)\phi} - 1)}{(\eta_i + \eta_j)} \quad \text{if } \eta_i + \eta_j \neq 0$$

$$G(i,j) = \frac{RL\,\phi}{2} (p_{11} A_i A_j + \ldots\ldots + p_{66} F_i F_j) \quad \text{if } \eta_i + \eta_j = 0$$

The terms A_i, B_i, C_i, D_i, E_i and F_i ($i = 1, \ldots, 8$) are listed with matrix [Q].

CHAPITRE VI

INNOVATIONS, CONCLUSIONS ET RECOMMANDATIONS

Cette thèse avait pour but de développer une méthode pour l'analyse linéaire et non linéaire des coques cylindriques ouvertes, non uniformes, minces, élastiques, anisotropes et soumises à un écoulement interne et externe. La stabilité dynamique des coques cylindriques et le cas des coques partiellement ou complètement remplies de liquide ont été aussi analysés.

La méthode est basée sur la théorie des coques, la mécanique des fluides et la méthode des éléments finis. Le modèle développé nous permet de déterminer les fréquences naturelles des coques cylindriques ouvertes et fermées, vides, partiellement ou complètement remplies de liquide en régime stagnant ou en écoulement. De plus, il nous permet de prédire l'influence des non-linéarités géométriques associées aux parois de la coque et des non-linéarités associées à la définition du fluide en écoulement sur le comportement dynamique du système coque-fluide.

En premier lieu, nous avons développé un nouvel élément fini, qui est de type coque cylindrique ouverte, où les fonctions de déplacement ne sont pas polynomiales comme dans le cas de la méthode des éléments finis classique, mais où elles sont dérivées de la théorie des coques cylindriques minces. Les matrices de masse et de rigidité linéaire sont déterminées par intégration analytique exacte.

À partir de l'équation du potentiel des vitesses et de l'équation de Bernouilli, nous avons développé un nouvel élément fini fluide. La pression exercée par le fluide a été exprimée comme une fonction de déplacements nodaux et de trois forces (inertie, centrifuge et Coriolis) du fluide en écoulement. L'intégration analytique de cette pression nous a donné trois matrices pour le fluide en écoulement (masse, rigidité et amortissement).

Pour prédire l'influence des non-linéarités géométriques des parois sur les fréquences naturelles des coques cylindriques ouvertes ou fermées, un modèle basé sur les coefficients modaux dérivés de la théorie non linéaire des coques cylindriques a été développé, ce qui nous a permis de déterminer les matrices de rigidité non linéaires du second et troisième ordre à partir de la méthode des éléments finis.

La non-linéarité associée au fluide en écoulement nous a permis de développer une expression pour la pression dynamique non linéaire. À partir de cette pression nous avons développé trois matrices non linéaires pour le fluide en écoulement.

La convergence de la méthode étant établie, les fréquences naturelles sont obtenues pour différentes conditions aux rives et pour différents modes circonférenciels et axiaux, pour des coques cylindriques ouvertes et fermées, isotropes et anisotropes, uniformes et non uniformes, partiellement ou complètement remplies de fluide, soumises à un écoulement interne ou/et externe. La stabilité dynamique des coques soumises simultanément à un écoulement interne et externe est aussi analysée. Une bonne concordance des résultats a été obtenue avec d'autres théories et expériences. Nous avons présenté beaucoup de nouveaux résultats pour des coques cylindriques ouvertes soumises à un fluide en écoulement.

L'influence des non-linéarités géométriques des parois sur les fréquences naturelles des coques cylindriques ouvertes ou fermées a été représentée en fonction du rapport entre l'amplitude de vibration et l'épaisseur de la coque. Des déviations par rapport aux fréquences linéaires ont été observées. Les tendances des non-linéarités sont du type hardening et softening, dépendamment des conditions aux rives et des modes de vibration. Ce même phénomène est observé pour des coques avec un fluide en écoulement. Par contre, l'influence de la non-linéarité associée au fluide en écoulement est complètement négligeable.

Cette méthode combine les avantages de la méthode des éléments finis qui traite des coques complexes (épaisseur variable, matériaux anisotropes et non uniformes, différentes conditions aux rives, ect.) et la précision de la formulation en utilisant des fonctions de déplacement dérivées de la théorie des coques. Ce modèle prédit le comportement dynamique dans le domaine linéaire et non linéaire des coques ouvertes et fermées soumises à un fluide en écoulement.

Cette méthode pourra permettre de compléter le peu de résultats disponibles quant aux fréquences naturelles élevées associées aux modes circonférenciels et axiaux élevés pour des coques avec ou sans fluide en écoulement, ainsi que de déterminer l'influence des non-linéarités géométriques sur le comportement dynamique du système coque-fluide.

Toutefois, ce modèle ne peut pas s'appliquer à des coques cylindriques épaisses ou à des coques soumises à un écoulement turbulent.

Nous pouvons donc considérer que nous disposons d'une méthode adéquate pour prédire les caractéristiques vibratoires linéaires et non linéaires des coques cylindriques ouvertes ou fermées, non uniformes dans la direction circonférencielle et soumises à un fluide en écoulement. Les coques sont simplement supportées selon leurs rives courbes et elles ont des conditions frontières arbitraires sur les rives droites.

Les travaux effectués dans notre groupe de recherche ont pour but de développer un modèle numérique d'une coque vide, partiellement ou totalement remplie de liquide, soumise ou non à un fluide en écoulement. Pour atteindre ce but, le groupe de recherche a déjà développé un élément cylindrique fermé, conique, sphérique, une plaque circulaire et annulaire, et un élément cylindrique ouvert (cette thèse).

De même, la suite logique de cette étude serait l'analyse des coques cylindriques ouvertes ayant des conditions frontières autre que simplement supportées sur les rives courbes. L'étude de l'effet de la surface libre du fluide sur le comportement dynamique des coques horizontales serait également nécessaire pour des coques partiellement remplies de liquide. D'autre part, il serait intéressant d'inclure dans ce travail l'étude des vibrations forcées d'une coque cylindrique soumise à un chargement dynamique. L'étude des excitations dues à un écoulement turbulent serait particulièrement intéressant.

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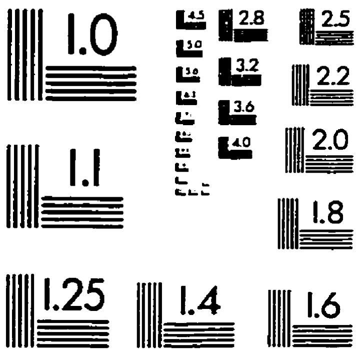

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