

Fully Tunable Strong Spin Orbit Interactions in Light Hole Germanium Quantum Channels

Patrick Del Vecchio,^{1,2,*} Stefano Bosco,² Daniel Loss,³ and Oussama Moutanabbir^{1,†}

¹*Department of Engineering Physics, École Polytechnique de Montréal,
Montréal, C.P. 6079, Succ. Centre-Ville, Montréal, Québec, Canada H3C 3A7*

²*QuTech and Kavli Institute of Nanoscience, Delft University of Technology, Delft, Netherlands*

³*Department of Physics, University of Basel, Klingelbergstrasse 82, 4056 Basel, Switzerland*

Supplementary Note 1: General 6-band $k \cdot p$ Hamiltonian

The electronic band structure of the heterostructures are evaluated from 6-band $k \cdot p$ theory. We consider a two-dimensional hole gas (2DHG) under the influence of a gate electric field $\mathbf{F} = F_z \mathbf{e}_z$ applied along the z direction, which is perpendicular to the 2DHG plane, an externally applied uniform magnetic field

$$\mathbf{B} = B (\mathbf{e}_x \sin \theta \cos \phi + \mathbf{e}_y \sin \theta \sin \phi + \mathbf{e}_z \cos \theta), \quad (11)$$

and an in-plane confinement represented by an effective parabolic potential

$$V_{\parallel}(x, y) = -\alpha_0 \left(\frac{x^2}{l_x^4} + \frac{y^2}{l_y^4} \right), \quad (12)$$

characterized by the lengths l_x and l_y and where $\alpha_0 = \hbar^2/(2m_0)$. The minus sign in the potential comes from our electron-energy convention (a typical hole band has negative effective mass). Results provided in the main article are converted back into the hole perspective to agree with literature. The total Hamiltonian can be written as a sum of multiple contributions: a kinetic part $H_k(\mathbf{K})$, a spin-orbit part H_{so} , an epitaxial strain part H_{ε} and a potential energy part $V(\mathbf{r})$. We write the mechanical wavevector $\mathbf{K} = \mathbf{k} + e\mathbf{A}/\hbar$, where $\langle \mathbf{r} | \mathbf{k} = -i\nabla \langle \mathbf{r} |$ and $\mathbf{A}$ is the vector potential for the externally applied magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. The components of $\mathbf{K}$ have the following commutation rule: $\mathbf{K} \times \mathbf{K} = e\mathbf{B}/(i\hbar)$. We first write the Hamiltonian in the basis of the bulk Bloch states at the Γ -point in the cartesian representation:

$$\mathcal{B}_X = \{|p_x+\rangle, |p_y+\rangle, |p_z+\rangle, |p_x-\rangle, |p_y-\rangle, |p_z-\rangle\}. \quad (13)$$

The Hamiltonian is

$$H_{k \cdot p}(\mathbf{K}) = H_k(\mathbf{K}) + H_{so} + H_{\varepsilon} + V(\mathbf{r}), \quad (14)$$

where $V(\mathbf{r}) = \mathcal{E}_{\Gamma_5^+}(z) + eF_z z + V_{\parallel}(x, y)$, where $\mathcal{E}_{\Gamma_5^+}$ is the valence band edge energy at the Γ -point *without* spin-orbit interaction. The kinetic term is

$$H_k(\mathbf{K}) = \mathbb{1}_{2 \times 2} \otimes H_{vv}^k(\mathbf{K}) + H_B^k(\mathbf{K}) \otimes \mathbb{1}_{3 \times 3} + H_q(\mathbf{K}), \quad (15)$$

with

$$H_{vv}^k(\mathbf{K}) = \begin{bmatrix} K_x L K_x + K_y M K_y + K_z M K_z & K_x N_+ K_y + K_y N_- K_x & K_x N_+ K_z + K_z N_- K_x \\ K_y N_+ K_x + K_x N_- K_y & K_x M K_x + K_y L K_y + K_z M K_z & K_y N_+ K_z + K_z N_- K_y \\ K_z N_+ K_x + K_x N_- K_z & K_z N_+ K_y + K_y N_- K_z & K_x M K_x + K_y M K_y + K_z L K_z \end{bmatrix}, \quad (16)$$

* p.delvecchio@tudelft.nl

† oussama.moutanabbir@polymtl.ca

where L , M and $N_{\pm}$ are related to the Luttinger parameters $\gamma_{1,2,3}$ and κ :

$$\begin{bmatrix} L \\ M \\ N_+ + \alpha_0 \\ N_- - \alpha_0 \end{bmatrix} = -\alpha_0 \begin{bmatrix} 1 & 4 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \kappa \end{bmatrix}, \quad (17)$$

with

$$H_B^k(\mathbf{K}) = \frac{i\alpha_0 g_0}{2} \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} K_{\alpha} K_{\beta} \sigma_{\gamma}, \quad (18)$$

with $g_0 \approx 2$ and σ_{α} the Pauli matrices. The H_q term is

$$H_q(\mathbf{K}) = U_{JX}^{\dagger} \left[\left(-2i\alpha_0 \sum_{\alpha\beta\gamma} \epsilon_{\alpha\beta\gamma} K_{\alpha} q K_{\beta} J_{\gamma}^3 \right) \oplus \mathbb{0}_{2 \times 2} \right] U_{JX}, \quad (19)$$

where q is the cubic-in- J correction to the hole g -factor, J_{α} are the unitless spin 3/2 matrices ($[J_{\alpha}, J_{\beta}] = i\epsilon_{\alpha\beta\gamma} J_{\gamma}$) and where the unitary U_{JX} is defined further below. The spin-orbit term is proportional to the bulk split-off gap Δ_0 and given by

$$H_{so} = \frac{\Delta_0}{3} \sum_{\alpha} \sigma_{\alpha} \otimes \lambda_{\alpha}, \quad (110)$$

where

$$\lambda_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \lambda_y = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{bmatrix}, \quad \lambda_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (111)$$

H_{so} has eigenvalues 0 ($\times 2$), $-\Delta_0/3$ ($\times 2$) and $+\Delta_0/3$ ($\times 4$). Finally, the strain term $H_{\varepsilon} = \mathbb{1}_{2 \times 2} \otimes H_{vv}^{\varepsilon}$, where (assuming a diagonal strain tensor)

$$H_{vv}^{\varepsilon} = \begin{bmatrix} l\varepsilon_{xx} + m\varepsilon_{yy} + m\varepsilon_{zz} & 0 & 0 \\ 0 & m\varepsilon_{xx} + l\varepsilon_{yy} + m\varepsilon_{zz} & 0 \\ 0 & 0 & m\varepsilon_{xx} + m\varepsilon_{yy} + l\varepsilon_{zz} \end{bmatrix}, \quad (112)$$

where ε_{ij} are the components of the strain tensor and l and m are related to the a_v and b deformation potential constants:

$$\begin{bmatrix} l \\ m \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_v \\ b \end{bmatrix}. \quad (113)$$

Note that the ordering between material constants (e.g. γ_1 , a_v , $\mathcal{E}_{\Gamma_5^+}$, etc ...) and wavevector operator components K_i must be preserved as indicated. This is because (a) different components of K_i do not mutually commute (c.f. $\mathbf{K} \times \mathbf{K} = e\mathbf{B}/(i\hbar)$) and improper ordering would lead to incorrect Zeeman splittings and (b) the k_z operator does not commute with position-dependent material parameters. In fact, if $\gamma|z\rangle = |z\rangle\gamma(z)$ where $\gamma(z)$ is the value of γ at position z , then for any envelope function $\langle z|\psi\rangle = \psi(z)$: $\langle z|[\gamma, k_z]|\psi\rangle = i\gamma'(z)\psi(z)$ where $\gamma'(z) = \frac{d}{dz}\gamma(z)$.

The $k \cdot p$ Hamiltonian is instead often written in a basis that diagonalizes H_{so} , or that commutes with the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$. A somewhat conventional basis is the following [1]:

$$\mathcal{B}_J = \left\{ \left| \frac{3}{2}, \frac{3}{2} \right\rangle, \left| \frac{3}{2}, \frac{1}{2} \right\rangle, \left| \frac{3}{2}, -\frac{1}{2} \right\rangle, \left| \frac{3}{2}, -\frac{3}{2} \right\rangle, \left| \frac{1}{2}, \frac{1}{2} \right\rangle, \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \right\}, \quad (114)$$

where

$$\left| \frac{3}{2}, \frac{3}{2} \right\rangle \equiv |\text{HH}+\rangle = -\frac{1}{\sqrt{2}} (|p_x+\rangle + i|p_y+\rangle), \quad (115)$$

$$\left| \frac{3}{2}, \frac{1}{2} \right\rangle \equiv |\text{LH}+\rangle = \frac{1}{\sqrt{6}} (2|p_z+\rangle - |p_x-\rangle - i|p_y-\rangle), \quad (116)$$

$$\left| \frac{3}{2}, -\frac{1}{2} \right\rangle \equiv |\text{LH}-\rangle = \frac{1}{\sqrt{6}} (|p_x+\rangle - i|p_y+\rangle + 2|p_z-\rangle), \quad (117)$$

$$\left| \frac{3}{2}, -\frac{3}{2} \right\rangle \equiv |\text{HH}-\rangle = \frac{1}{\sqrt{2}} (|p_x-\rangle - i|p_y-\rangle), \quad (118)$$

$$\left| \frac{1}{2}, \frac{1}{2} \right\rangle \equiv |\text{SO}+\rangle = -\frac{1}{\sqrt{3}} (|p_z+\rangle + |p_x-\rangle + i|p_y-\rangle), \quad (119)$$

$$\left| \frac{1}{2}, -\frac{1}{2} \right\rangle \equiv |\text{SO}-\rangle = -\frac{1}{\sqrt{3}} (|p_x+\rangle - i|p_y+\rangle - |p_z-\rangle). \quad (120)$$

The unitary U_{JX} connecting the two bases is therefore

$$U_{JX} = \begin{bmatrix} -s_2 & is_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & s_{23} & -s_6 & is_6 & 0 \\ s_6 & is_6 & 0 & 0 & 0 & s_{23} \\ 0 & 0 & 0 & s_2 & is_2 & 0 \\ 0 & 0 & -s_3 & -s_3 & is_3 & 0 \\ -s_3 & -is_3 & 0 & 0 & 0 & s_3 \end{bmatrix}, \quad (121)$$

with $s_2 = 1/\sqrt{2}$, $s_3 = 1/\sqrt{3}$, $s_6 = 1/\sqrt{6}$ and $s_{23} = \sqrt{2/3}$.

Supplementary Note 2: First diagonalization of the $k \cdot p$ matrix at $B = 0$ and $\mathbf{k}_{\parallel} = 0$

The 6-band $k \cdot p$ matrix is first written down at $k_x = k_y = 0$ (and $B = 0$) and $V_{\parallel}(x, y) = 0$, whose eigenstates provide an orthonormal basis on which the full Hamiltonian at $\mathbf{K} \neq 0$ is projected. We do so in a basis similar to $\mathcal{B}_J$ but with a different ordering:

$$\mathcal{B}_{\mathcal{J}} = \{|\text{LH}+\rangle, |\text{SO}+\rangle, |\text{HH}+\rangle, |\text{LH}-\rangle, |\text{SO}-\rangle, |\text{HH}-\rangle\}. \quad (21)$$

Then, the Hamiltonian $H_0^{\text{qw}} \equiv H_{k \cdot p}(B = 0; k_x = k_y = 0)$ takes the following appearance:

$$H_0^{\text{qw}} = \begin{bmatrix} H_{\sigma=+} & 0 \\ 0 & H_{\sigma=-} \end{bmatrix} \quad (22)$$

$$H_{\sigma} = H_{\sigma}^k + H_{\sigma}^{\varepsilon} + V_z, \quad (23)$$

where

$$H_{\sigma}^k = \alpha_0 \begin{bmatrix} -k_z(\gamma_1 + 2\gamma_2)k_z & 2\sqrt{2}\sigma k_z \gamma_2 k_z & 0 \\ \dagger & -k_z \gamma_1 k_z & 0 \\ & & -k_z \gamma_- k_z \end{bmatrix}, \quad (24)$$

$$H_{\sigma}^{\varepsilon} = a_v \text{Tr } \varepsilon + b \cdot \delta \varepsilon \begin{bmatrix} -1 & \sqrt{2}\sigma & 0 \\ \dagger & 0 & 0 \\ & & 1 \end{bmatrix}, \quad (25)$$

$$V_z = \mathcal{E}_{\Gamma_5^+} + \frac{\Delta_0}{3} + eF_z z - \Delta_0 \begin{bmatrix} 0 & 0 & 0 \\ \dagger & 1 & 0 \\ & & 0 \end{bmatrix}. \quad (26)$$

Here, $\gamma_{\pm} = \gamma_1 + (1 \pm 3)\gamma_2$, $\delta\varepsilon = \varepsilon_{xx} - \varepsilon_{zz}$. We assume that all layers are pseudomorphic to each other, i.e., the in-plane lattice constant does not change along the growth direction and $\varepsilon_{xx} = \varepsilon_{yy}$. The energy band offsets between Ge and SiGe and the deformation potential constant b are obtained by linearly interpolating the values reported in reference [2]. The Luttinger parameters $\gamma_{1,2,3}$ and g -factor parameter κ are taken from reference [1]. The three $\gamma_{1,2,3}$ and κ are interpolated in the full composition range of SiGe according to the method outlined in reference [3]. We stress again that the ordering between k_z and material parameters must be preserved as it is shown.

The block-diagonal structure of (22) suggests that its eigenstates are of two types ($2 \times 2 = 4$ including pseudospin σ):

$$|\mathbf{H}l; \sigma\rangle = \left| \frac{3}{2}, \frac{3\sigma}{2} \right\rangle |f_l^h\rangle, \quad (27a)$$

$$|\mathbf{L}j; \sigma\rangle = \left| \frac{3}{2}, \frac{\sigma}{2} \right\rangle |f_j^\ell\rangle + \sigma \left| \frac{1}{2}, \frac{\sigma}{2} \right\rangle |f_j^s\rangle, \quad (27b)$$

where the first (H states) consists of pure HH subbands with associated envelope functions $\langle z | f_l^h \rangle$ and spin-independent energies E_l^H , and where the second (L states) consists of superpositions of pure LH and pure split-off holes with envelope functions $\langle z | f_j^\ell \rangle$ and $\langle z | f_j^s \rangle$, respectively with spin-independent energies E_j^L . l and j are subband indices for H and L subbands, respectively. To compute the envelopes and energies, we start from the spin-up ($\sigma = +$) block H_+ in (22), which we diagonalize by the finite difference method using a mesh-spacing $\delta z = 0.01$ nm. The eigenstates of H_- are the time-reversed conjugates of the eigenstates of H_+ .

Using the eigenstates of H_0^{qw} as the new basis for the Hilbert space, the full 2DHG Hamiltonian $H_{k,p}$ for $\mathbf{K}_{\parallel} \neq 0$ takes the following appearance (bold symbols indicate that we are in the eigenbasis of H_0^{qw}):

$$\begin{aligned} \mathbf{H}_{k,p}(\mathbf{K}_{\parallel}) = & \mathbf{E}_0^{\text{qw}} + \alpha_0 \left[\mathbf{M}_{\gamma} K_{\parallel}^2 + \frac{\cos \theta}{2l_B^2} \mathbf{M}_g + \frac{\sin \theta}{2l_B^2} \left(\frac{\sin \theta}{2l_B^2} \mathbf{N}_{\gamma} r_{\phi} - \tilde{\mathbf{N}}_{\gamma} \right) r_{\phi} \right] \\ & + \alpha_0 \left[i\mathbf{M}_1 K_- + \mathbf{M}_2 K_-^2 + \frac{\sin \theta}{2l_B^2} (e^{-i\phi} \mathbf{N}_g + i\mathbf{N}_1^+ r_{\phi} K_- + i\mathbf{N}_1^- K_- r_{\phi}) + \text{h.c.} \right] \\ & + V_{\parallel}(x, y), \end{aligned} \quad (28)$$

where $K_{\pm} = K_x \pm iK_y$, $K_{\parallel}^2 = K_x^2 + K_y^2 = \{K_-, K_+\}/2$, $\{A, B\} = AB + BA$, $r_{\phi} = x \sin \phi - y \cos \phi$ and $l_B = \sqrt{\hbar/(eB)}$. The gauge for the in-plane part of the B -field is set as $A_z = \mathbf{e}_z \cdot \mathbf{A} = -Br_{\phi} \sin \theta$, while the gauge for the out-of-plane part of the field is left arbitrary for now, provided it does not include any z operator. Regardless of the gauge the in-plane components of the wavenumber satisfy the commutation relation $[K_-, K_+]/2 = \cos \theta / l_B^2$. Hamiltonian (28) implicitly depends on strain and on the SOI strength Δ_0 through the envelopes $|f_j^{\ell, s, h}\rangle$ and the energies E_j^{τ} ($\tau = \{\text{H}, \text{L}\}$). Those terms do not explicitly appear in (28) because the $k \cdot p$ Hamiltonian does not contain terms such as $\Delta_0 k_{\pm}$ or $\delta\varepsilon \cdot k_{\pm}$. It also integrates out the z dimension through the matrices $\mathbf{M}$ and $\mathbf{N}$. We have

$$\mathbf{E}_0^{\text{qw}} = \begin{bmatrix} \mathbf{E}^H & 0 & 0 & 0 \\ 0 & \mathbf{E}^L & 0 & 0 \\ 0 & 0 & \mathbf{E}^L & 0 \\ 0 & 0 & 0 & \mathbf{E}^H \end{bmatrix}, \quad (29)$$

with $\mathbf{E}^{\tau} = \text{diag}\{E_1^{\tau}, E_2^{\tau}, \dots\}$ ($\tau = \{\text{H}, \text{L}\}$) are the energies of H_0^{qw} . The $\mathbf{M}$ -matrices are

$$\mathbf{M}_{\gamma} = \begin{bmatrix} \mathbf{\Gamma}_{\parallel}^H & 0 & 0 & 0 \\ 0 & \mathbf{\Gamma}_{\parallel}^L & 0 & 0 \\ 0 & 0 & \mathbf{\Gamma}_{\parallel}^L & 0 \\ 0 & 0 & 0 & \mathbf{\Gamma}_{\parallel}^H \end{bmatrix}, \quad \mathbf{M}_1 = \begin{bmatrix} 0 & \mathbf{T}^{\times} & 0 & 0 \\ 0 & 0 & \mathbf{T}^L & 0 \\ 0 & 0 & 0 & \mathbf{T}^{\times\dagger} \\ \mathbf{T}^H & 0 & 0 & 0 \end{bmatrix}, \quad (210)$$

$$\mathbf{M}_g = \begin{bmatrix} \mathbf{G}_{\perp}^H & 0 & 0 & 0 \\ 0 & \mathbf{G}_{\perp}^L & 0 & 0 \\ 0 & 0 & -\mathbf{G}_{\perp}^L & 0 \\ 0 & 0 & 0 & -\mathbf{G}_{\perp}^H \end{bmatrix}, \quad \mathbf{M}_2 = \begin{bmatrix} 0 & 0 & \boldsymbol{\mu} & 0 \\ 0 & 0 & \boldsymbol{\mu}^{\dagger} & 0 \\ \boldsymbol{\delta}^{\dagger} & 0 & 0 & 0 \\ 0 & \boldsymbol{\delta} & 0 & 0 \end{bmatrix}. \quad (211)$$

The non-zero blocks of these matrices are all explicitly expanded in terms of the eigenstates of H_0^{qw} :

$$\Gamma_{\parallel l, l'}^{\text{H}} = -\langle f_l^h | \gamma_1 + \gamma_2 | f_{l'}^h \rangle, \quad (212\text{a})$$

$$\Gamma_{\parallel j, j'}^{\text{L}} = -\langle f_j^\perp | \gamma_- | f_{j'}^\perp \rangle - \langle f_j^\circ | \gamma_1 + \gamma_2 | f_{j'}^\circ \rangle, \quad (212\text{b})$$

$$G_{\perp l, l'}^{\text{H}} = -\langle f_l^h | 6\kappa + \frac{27q}{2} | f_{l'}^h \rangle \quad (212\text{c})$$

$$G_{\perp j, j'}^{\text{L}} = -6\langle f_j^\circ | \kappa | f_{j'}^\circ \rangle + 2(\langle f_j^\perp | f_{j'}^\perp \rangle - 2\langle f_j^\circ | f_{j'}^\circ \rangle) - \frac{1}{2}\langle f_j^\ell | q | f_{j'}^\ell \rangle \quad (212\text{d})$$

$$T_{l, j}^\times = -\frac{i}{\sqrt{2}}\langle f_l^h | \left(3u_+ | f_j^\perp \rangle + \frac{7\sqrt{6}}{4}[q, k_z] | f_j^\ell \rangle \right), \quad (212\text{e})$$

$$T_{l, l'}^{\text{H}} = -\frac{3i}{2}\langle f_l^h | [q, k_z] | f_{l'}^h \rangle \quad (212\text{f})$$

$$T_{j, j'}^{\text{L}} = -\frac{3i}{\sqrt{2}}(\langle f_j^\circ | u_+ | f_{j'}^\perp \rangle - \langle f_j^\perp | u_- | f_{j'}^\circ \rangle) - 5i\langle f_j^\ell | [q, k_z] | f_{j'}^\ell \rangle, \quad (212\text{g})$$

$$\mu_{l, j} = \frac{3}{2}\langle f_l^h | \gamma_2 + \gamma_3 | f_j^\circ \rangle, \quad (212\text{h})$$

$$\delta_{l, j} = \frac{3}{2}\langle f_l^h | \gamma_2 - \gamma_3 | f_j^\circ \rangle, \quad (212\text{i})$$

where $u_\pm = \{\gamma_3, k_z\} \pm [\kappa, k_z]$ and

$$|f_j^\perp\rangle \equiv \frac{1}{\sqrt{3}}\left(\sqrt{2}|f_j^\ell\rangle - |f_j^s\rangle\right), \quad |f_j^\circ\rangle \equiv \frac{1}{\sqrt{3}}\left(|f_j^\ell\rangle + \sqrt{2}|f_j^s\rangle\right). \quad (213)$$

The $\mathbf{N}$ matrices are

$$\mathbf{N}_\gamma = 4 \begin{bmatrix} \Gamma_\perp^{\text{H}} & 0 & 0 & 0 \\ 0 & \Gamma_\perp^{\text{L}} & 0 & 0 \\ 0 & 0 & \Gamma_\perp^{\text{L}} & 0 \\ 0 & 0 & 0 & \Gamma_\perp^{\text{H}} \end{bmatrix}, \quad \tilde{\mathbf{N}}_\gamma = 4 \begin{bmatrix} \tilde{\Gamma}_\perp^{\text{H}} & 0 & 0 & 0 \\ 0 & \tilde{\Gamma}_\perp^{\text{L}} & 0 & 0 \\ 0 & 0 & \tilde{\Gamma}_\perp^{\text{L}} & 0 \\ 0 & 0 & 0 & \tilde{\Gamma}_\perp^{\text{H}} \end{bmatrix}, \quad (214)$$

$$\mathbf{N}_g = \begin{bmatrix} 0 & \mathbf{G}_\parallel^\times & 0 & 0 \\ 0 & 0 & \mathbf{G}_\parallel^{\text{L}} & 0 \\ 0 & 0 & 0 & \mathbf{G}_\parallel^{\times\dagger} \\ \mathbf{G}_\parallel^{\text{H}} & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}_1^\pm = \mathbf{N}_1 = \begin{bmatrix} 0 & \mathbf{R}^\times & 0 & 0 \\ 0 & 0 & \mathbf{R}^{\text{L}} & 0 \\ 0 & 0 & 0 & \mathbf{R}^{\times\dagger} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (215)$$

where

$$\Gamma_{\perp l, l'}^{\text{H}} = -\langle f_l^h | \gamma_- | f_{l'}^h \rangle, \quad (216\text{a})$$

$$\tilde{\Gamma}_{\perp l, l'}^{\text{H}} = -\frac{1}{2}\langle f_l^h | \{\gamma_-, k_z\} | f_{l'}^h \rangle, \quad (216\text{b})$$

$$\Gamma_{\perp j, j'}^{\text{L}} = -\langle f_j^\circ | \gamma_- | f_{j'}^\circ \rangle - \langle f_j^\perp | \gamma_+ | f_{j'}^\perp \rangle, \quad (216\text{c})$$

$$\tilde{\Gamma}_{\perp j, j'}^{\text{L}} = -\frac{1}{2}\langle f_j^\circ | \{\gamma_-, k_z\} | f_{j'}^\circ \rangle - \frac{1}{2}\langle f_j^\perp | \{\gamma_+, k_z\} | f_{j'}^\perp \rangle, \quad (216\text{d})$$

$$G_{\parallel l, j}^\times = -\sqrt{2}\langle f_l^h | \left(3\kappa | f_j^\perp \rangle + \frac{7\sqrt{6}q}{4} | f_j^\ell \rangle - \sqrt{3} | f_j^s \rangle \right), \quad (216\text{e})$$

$$G_{\parallel l, l'}^{\text{H}} = -3\langle f_l^h | q | f_{l'}^h \rangle \quad (216\text{f})$$

$$G_{\parallel j, j'}^{\text{L}} = -\sqrt{2}\langle f_j^\circ | 3\kappa + 1 | f_{j'}^\perp \rangle - \sqrt{2}\langle f_j^\perp | 3\kappa + 1 | f_{j'}^\circ \rangle + 2\langle f_j^\perp | f_{j'}^\perp \rangle - 10\langle f_j^\ell | q | f_{j'}^\ell \rangle, \quad (216\text{g})$$

$$R_{l, j}^\times = 3\sqrt{2}i\langle f_l^h | \gamma_3 | f_j^\perp \rangle, \quad (216\text{h})$$

$$R_{j, j'}^{\text{L}} = 3\sqrt{2}i(\langle f_j^\circ | \gamma_3 | f_{j'}^\perp \rangle - \langle f_j^\perp | \gamma_3 | f_{j'}^\circ \rangle). \quad (216\text{i})$$

Supplementary Note 3: Fan-diagram calculations

The fan-diagram calculations of section 9 below are performed by diagonalizing (28) with $V_{\parallel}(x, y) = 0$ and in the axial approximation (which corresponds to $\delta = 0$ in $\mathbf{M}_2$), and by writing K_- in terms of the ladder operator

$$a = \frac{il_B K_-}{\sqrt{2}}. \quad (31)$$

Inserting this into (28) reveals that all eigenstates must be of the form

$$|\varphi\rangle = \begin{bmatrix} |m-1\rangle \mathbf{c}_{H+} \\ |m\rangle \mathbf{c}_{L+} \\ |m+1\rangle \mathbf{c}_{L-} \\ |m+2\rangle \mathbf{c}_{H-} \end{bmatrix}, \quad (32)$$

where m is an integer labeling Landau levels and $\mathbf{c}_{\tau\sigma}$ are vectors of coefficients. Following the approach outlined in Reference 4, replacing all the a 's within the Hamiltonian by the corresponding m , the fan-diagram is readily computed for as many Landau levels $m \geq -2$ as required.

Supplementary Note 4: Quantum channel system

The quantum channel Hamiltonian eigenstates $|\tilde{\sigma}; i, N\rangle$ with associated energy $E_{i,N}^c$ at $k_y = 0$ and $B = 0$ are calculated from (28) by setting $k_y = 0$ and $B = 0$, and by introducing a parabolic confinement potential $-\alpha_0 x^2/l_x^4$:

$$\mathbf{H}_0^{\text{qc}} = \mathbf{E}_0^{\text{qw}} + \alpha_0 \mathbf{M}_{\gamma} k_x^2 + \alpha_0 (i\mathbf{M}_1 k_x + \mathbf{M}_2 k_x^2 + \text{h.c.}) - \frac{\alpha_0}{l_x^4} x^2. \quad (41)$$

Projecting this Hamiltonian onto an in-plane basis of harmonic oscillator eigenstates $|n\rangle$ (with $a^\dagger a |n\rangle = |n\rangle n$, $a |n\rangle = |n-1\rangle \sqrt{n}$ and $a^\dagger |n\rangle = |n+1\rangle \sqrt{n+1}$), with

$$a = \frac{1}{\sqrt{2}} \left(\frac{x}{l_x} + il_x k_x \right), \quad (42)$$

provides an eigenbasis of quantum channel subband edges on which the full quantum channel Hamiltonian away from $k_y = 0$ and $B = 0$ is projected. The result is

$$\mathbf{H}^{\text{qc}} = \mathbf{E}_0^{\text{qc}} + \alpha_0 \left(\mathbf{L}_1 k_y + \mathbf{L}_{\gamma} k_y^2 + \frac{1}{2l_B^2} \mathbf{L}_g + \frac{1}{2l_B^2} \mathbf{L}_3 k_y + \frac{1}{4l_B^4} \mathbf{L}_4 \right), \quad (43)$$

where

$$\mathbf{E}_0^{\text{qc}} = \begin{bmatrix} \mathbf{E}^c & 0 \\ 0 & \mathbf{E}^c \end{bmatrix}, \quad (44)$$

and $\mathbf{E}^c = \text{diag}\{E_{1,0}^c, E_{1,1}^c, \dots, E_{2,0}^c, E_{2,1}^c, \dots\}$. Some $\mathbf{L}$ -matrices are further expanded into θ -independent $\mathbf{L}$ -matrices:

$$\mathbf{L}_g = \mathbf{L}_{g\perp} \cos \theta + \mathbf{L}_{g\parallel} \sin \theta, \quad (45a)$$

$$\mathbf{L}_3 = \mathbf{L}_{3\perp} \cos \theta + \mathbf{L}_{3\parallel} \sin \theta, \quad (45b)$$

$$\mathbf{L}_4 = \mathbf{L}_{4\perp} \cos^2 \theta + \mathbf{L}_{4\parallel} \sin^2 \theta + \mathbf{L}_{4\times} \sin \theta \cos \theta. \quad (45c)$$

Just like the $\mathbf{M}$ -matrices in the QW Hamiltonian, the $\mathbf{L}$ -matrices in the QC system are computed from the eigenstates of the QC Hamiltonian at $k_y = 0$ and $B = 0$:

$$\mathbf{L}_1 = \mathbf{T}^c \tilde{\sigma}_1 = \langle \tilde{\sigma}; i, N | \mathbf{m}_1 - 2\mathbf{m}_2 k_x | \tilde{\sigma}'; i', N' \rangle, \quad (46a)$$

$$\mathbf{L}_\gamma = \mathbf{\Gamma}^c \tilde{\sigma}_0 = \langle \tilde{\sigma}; i, N | \mathbf{m}_0 | \tilde{\sigma}'; i', N' \rangle, \quad (46b)$$

$$\mathbf{L}_{g\perp} = \mathbf{G}_\perp^c \tilde{\sigma}_3 = \langle \tilde{\sigma}; i, N | \mathbf{M}_g + 2\mathbf{m}_1 x - 2\mathbf{m}_2 \{x, k_x\} | \tilde{\sigma}'; i', N' \rangle, \quad (46c)$$

$$\mathbf{L}_{g\parallel} = \mathbf{G}_\parallel^c \tilde{\sigma}_2 = \langle \tilde{\sigma}; i, N | \mathbf{n}_g - \tilde{\mathbf{N}}_\gamma x + \tilde{\mathbf{n}}_1 \{x, k_x\} | \tilde{\sigma}'; i', N' \rangle, \quad (46d)$$

$$\mathbf{L}_{3\perp} = \mathbf{\Xi}_\perp \tilde{\sigma}_2 = 4 \langle \tilde{\sigma}; i, N | \mathbf{m}_0 x | \tilde{\sigma}'; i', N' \rangle, \quad (46e)$$

$$\mathbf{L}_{3\parallel} = \mathbf{\Xi}_\parallel \tilde{\sigma}_3 = 2 \langle \tilde{\sigma}; i, N | \mathbf{n}_1 x | \tilde{\sigma}'; i', N' \rangle, \quad (46f)$$

$$\mathbf{L}_{4\perp} = \mathbf{Z}_\perp \tilde{\sigma}_0 = 4 \langle \tilde{\sigma}; i, N | \mathbf{m}_0 x^2 | \tilde{\sigma}'; i', N' \rangle, \quad (46g)$$

$$\mathbf{L}_{4\parallel} = \mathbf{Z}_\parallel \tilde{\sigma}_0 = \langle \tilde{\sigma}; i, N | \mathbf{N}_\gamma x^2 | \tilde{\sigma}'; i', N' \rangle, \quad (46h)$$

$$\mathbf{L}_{4\times} = \mathbf{Z}_\times \tilde{\sigma}_1 = 4 \langle \tilde{\sigma}; i, N | \mathbf{n}_1 x^2 | \tilde{\sigma}'; i', N' \rangle, \quad (46i)$$

with $\tilde{\sigma}_0$ the 2×2 identity matrix and $\tilde{\sigma}_{1,2,3}$ the Pauli matrices. The $\mathbf{m}$ and $\mathbf{n}$ matrices are

$$\mathbf{m}_0 = \mathbf{M}_\gamma - \mathbf{M}_2 - \mathbf{M}_2^\dagger, \quad (47a)$$

$$\mathbf{m}_1 = \mathbf{M}_1 + \mathbf{M}_1^\dagger, \quad (47b)$$

$$\mathbf{m}_2 = i(\mathbf{M}_2 - \mathbf{M}_2^\dagger), \quad (47c)$$

$$\mathbf{n}_g = -i(\mathbf{N}_g - \mathbf{N}_g^\dagger), \quad (47d)$$

$$\mathbf{n}_1 = \mathbf{N}_1 + \mathbf{N}_1^\dagger, \quad (47e)$$

$$\tilde{\mathbf{n}}_1 = i(\mathbf{N}_1 - \mathbf{N}_1^\dagger). \quad (47f)$$

Supplementary Note 5: Quantum dot system

The quantum dot system eigenstates $|\tilde{\sigma}; i, N_1, N_2\rangle$, with associated energy E_{i,N_1,N_2}^d , are calculated at $B = 0$ similarly to those of the QC system, i.e., by introducing into (28) an in-plane parabolic potential $-\alpha_0 (x^2/l_x^4 + y^2/l_y^4)$:

$$\mathbf{H}_0^{\text{qd}} = \mathbf{E}_0^{\text{qw}} + \alpha_0 \left[\mathbf{M}_\gamma k_\parallel^2 + (i\mathbf{M}_1 k_- + \mathbf{M}_2 k_-^2 + \text{h.c.}) \right] - \alpha_0 \left(\frac{x^2}{l_x^4} + \frac{y^2}{l_y^4} \right), \quad (51)$$

and by projecting the in-plane degrees of freedom onto a harmonic oscillator basis, acted upon by the following two ladder operators:

$$a_1 = \frac{1}{\sqrt{2}} \left(\frac{x}{l_x} + il_x k_x \right), \quad a_2 = \frac{1}{\sqrt{2}} \left(\frac{y}{l_y} + il_y k_y \right). \quad (52)$$

The a_i 's satisfy $[a_i, a_j^\dagger] = \delta_{i,j}$. Eigenstates of (51) provide an orthonormal basis on which the full $k \cdot p$ Hamiltonian with $B > 0$ can be projected. The result is (with B lying in the y - z plane)

$$\begin{aligned} \mathbf{H}^{\text{qd}} = \mathbf{E}_0^{\text{qd}} + \frac{\alpha_0}{2l_B^2} & \left\{ \cos \theta \mathbf{L}_{2\perp} + \frac{\cos^2 \theta}{2l_B^2} \mathbf{L}_{4\perp} + \frac{\sin^2 \theta}{2l_B^2} \mathbf{L}_{4\parallel} \right. \\ & \left. - \sin \theta \left[i\mathbf{L}_{2\parallel} + \frac{\sin \theta}{2l_B^2} \tilde{\mathbf{L}}_{4\parallel} + i \frac{\cos \theta}{2l_B^2} \mathbf{L}_{4\times} + \text{h.c.} \right] \right\}, \end{aligned} \quad (53)$$

where

$$\mathbf{E}_0^{\text{qd}} = \begin{bmatrix} \mathbf{E}^d & 0 \\ 0 & \mathbf{E}^d \end{bmatrix} \quad (54)$$

with $\mathbf{E}^d = \text{diag}\{E_{1,0,0}^d, E_{1,0,1}^d, \dots\}$ are the eigenvalues of (51) and the $\mathbf{L}$ -matrices are expanded in terms of the eigenstates of (51):

$$\begin{aligned} \mathbf{L}_{2\perp} &= \mathbf{G}_{\perp+}^d \tilde{\sigma}_+ \tilde{\sigma}_- + \mathbf{G}_{\perp-}^d \tilde{\sigma}_- \tilde{\sigma}_+ \\ &= \langle \tilde{\sigma}; i, N_1, N_2 | \mathbf{M}_g + [i\mathbf{M}_\gamma \{\zeta_+, k_-\} + 2\mathbf{M}_1 \zeta_- - 4i\mathbf{M}_2 \zeta_- k_- + \text{h.c.}] | \tilde{\sigma}'; i', N'_1, N'_2 \rangle, \end{aligned} \quad (55a)$$

$$\begin{aligned} \mathbf{L}_{2\parallel} &= \mathbf{G}_{\parallel+}^d \tilde{\sigma}_+ + \mathbf{G}_{\parallel-}^d \tilde{\sigma}_- \\ &= \langle \tilde{\sigma}; i, N_1, N_2 | \mathbf{N}_g - i\tilde{\mathbf{N}}_\gamma \zeta_+ - \mathbf{N}_1 \{\zeta_+, k_-\} + 2\mathbf{N}_1^\dagger \zeta_+ k_+ | \tilde{\sigma}'; i', N'_1, N'_2 \rangle, \end{aligned} \quad (55b)$$

$$\mathbf{L}_{4\perp} = \langle \tilde{\sigma}; i, N_1, N_2 | 4\mathbf{M}_\gamma \zeta_+ \zeta_- - 4[\mathbf{M}_2 \zeta_-^2 + \text{h.c.}] | \tilde{\sigma}'; i', N'_1, N'_2 \rangle \quad (55c)$$

$$\mathbf{L}_{4\parallel} = \langle \tilde{\sigma}; i, N_1, N_2 | 2\mathbf{N}_\gamma \zeta_+ \zeta_- | \tilde{\sigma}'; i', N'_1, N'_2 \rangle \quad (55d)$$

$$\tilde{\mathbf{L}}_{4\parallel} = -\langle \tilde{\sigma}; i, N_1, N_2 | \mathbf{N}_\gamma \zeta_+^2 | \tilde{\sigma}'; i', N'_1, N'_2 \rangle \quad (55e)$$

$$\mathbf{L}_{4\times} = 4i \langle \tilde{\sigma}; i, N_1, N_2 | \mathbf{N}_1 \zeta_+ \zeta_- + \mathbf{N}_1^\dagger \zeta_+^2 | \tilde{\sigma}'; i', N'_1, N'_2 \rangle \quad (55f)$$

with $\tilde{\sigma}_\pm = (\tilde{\sigma}_1 \pm i\tilde{\sigma}_2)/2$ and $\zeta_\pm = (x \pm iy)/2$. The ground state two-level system with energy $\hbar\omega$ is computed by diagonalizing (53) with $B = 0.05$ T.

Supplementary Note 6: Perturbative framework

Away from $\mathbf{K}_{\parallel} = 0$, Hamiltonian (28) becomes non-diagonal, and folding it down onto any given set of subbands, labeled as the $\mathcal{A}$ set, can be performed by means of a Schrieffer-Wolff transformation with $\mathbf{K}_{\parallel}$ as the perturbation parameter. The contributions from all remote subbands, the $\mathcal{B}$ set, are eliminated up to the desired perturbation order. More explicitly, the unperturbed Hamiltonian is $\mathbf{E}_0^{\text{qw}}$ ($\mathbf{K}_{\parallel}$ -independent), while the perturbation $\mathbf{W}(\mathbf{K}_{\parallel}) = \mathbf{W}_1(\mathbf{K}_{\parallel}) + \mathbf{W}_2(\mathbf{K}_{\parallel})$ is at least linear in $\mathbf{K}_{\parallel}$:

$$\mathbf{W}_1(\mathbf{K}_{\parallel}) = -\frac{\alpha_0 \sin \theta}{2l_B^2} \tilde{\mathbf{N}}_{\gamma} r_{\phi} + (i\alpha_0 \mathbf{M}_1 K_{-} + \text{h.c.}), \quad (61a)$$

$$\mathbf{W}_2(\mathbf{K}_{\parallel}) = \alpha_0 \left\{ \mathbf{M}_{\gamma} K_{\parallel}^2 + \frac{\cos \theta}{2l_B^2} \mathbf{M}_g + \frac{\sin^2 \theta}{4l_B^4} \mathbf{N}_{\gamma} r_{\phi}^2 + \left[\mathbf{M}_2 K_{-}^2 + \frac{\sin \theta}{2l_B^2} (e^{-i\phi} \mathbf{N}_g + i\mathbf{N}_1^{+} r_{\phi} K_{-} + i\mathbf{N}_1^{-} K_{-} r_{\phi}) + \text{h.c.} \right] \right\}. \quad (61b)$$

We use the following convention:

$i, j, k, \dots :$	indices that run through the whole Hilbert space
$\alpha, \beta, \gamma, \delta, \dots :$	indices that run through the $\mathcal{A}$ set
$\lambda, \mu, \nu, \dots :$	indices that run through the $\mathcal{B}$ set

We also define the following projection operators [5]:

$$P = \sum_{\alpha \in \mathcal{A}} |\varphi_{\alpha}\rangle \langle \varphi_{\alpha}|, \quad (62)$$

$$Q = \sum_{\lambda \in \mathcal{B}} |\varphi_{\lambda}\rangle \langle \varphi_{\lambda}|, \quad (63)$$

and superoperators:

$$\mathcal{P}[X] = PXP, \quad (64)$$

$$\mathcal{Q}[X] = QXQ, \quad (65)$$

$$\mathcal{D}[X] = PXP + QXQ, \quad (66)$$

$$\mathcal{O}[X] = PXQ + QXP, \quad (67)$$

$$\mathcal{L}[X] = \sum_{i,j} |\varphi_i\rangle \frac{\langle \varphi_i | \mathcal{O}[X] | \varphi_j \rangle}{E_i - E_j} \langle \varphi_j|. \quad (68)$$

where E_i is the energy of subband i of the unperturbed Hamiltonian (or at $\mathbf{K}_{\parallel} = 0$). It is clear from the definition of $\mathcal{L}[X]$ that $\mathcal{O}[\mathcal{L}[X]] = \mathcal{L}[X]$ and $\mathcal{D}[\mathcal{L}[X]] = 0$. Up to 3rd order (and only keeping terms up to k^3), the effective Hamiltonian for the $\mathcal{A}$ set is given by

$$\begin{aligned} \mathcal{H}_{\text{eff}}(\mathbf{K}_{\parallel}) = & \mathcal{P}[\mathbf{E}_0^{\text{qw}}] + \mathcal{P}[\mathbf{W}_1] + \mathcal{P}[\mathbf{W}_2] \\ & + \mathcal{P}[\text{sw}_2(\mathbf{W}_1, \mathbf{W}_1)] + \mathcal{P}[\text{sw}_2(\mathbf{W}_1, \mathbf{W}_2)] + \mathcal{P}[\text{sw}_2(\mathbf{W}_2, \mathbf{W}_1)] + \mathcal{P}[\text{sw}_2(\mathbf{W}_2, \mathbf{W}_2)] \\ & + \mathcal{P}[\text{sw}_3(\mathbf{W}_1, \mathbf{W}_1, \mathbf{W}_1)] + \mathcal{P}[\text{sw}_3(\mathbf{W}_1, \mathbf{W}_1, \mathbf{W}_2)] + \mathcal{P}[\text{sw}_3(\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_1)] + \mathcal{P}[\text{sw}_3(\mathbf{W}_2, \mathbf{W}_1, \mathbf{W}_1)], \end{aligned} \quad (69)$$

where

$$\text{sw}_2(X, Y) = \frac{1}{2} (\mathcal{L}[X]Y - X\mathcal{L}[Y]), \quad (610)$$

$$\text{sw}_3(X, Y, Z) = \frac{1}{2} (\mathcal{L}[\mathcal{L}[X]Y]Z + X\mathcal{L}[Y\mathcal{L}[Z]] + \mathcal{L}[X\mathcal{P}\mathcal{L}[Y]]Z + X\mathcal{L}[\mathcal{L}[Y]\mathcal{P}Z]). \quad (611)$$

In the special case where $\mathbf{B} \parallel \mathbf{e}_z$ and the $\mathcal{A}$ set consists of the 2-fold LH ground state, $\mathcal{A} = \{|\text{L1}; \pm\rangle\}$, one obtains the following effective Hamiltonian (assuming $E_1^{\text{L}} = 0$):

$$\mathcal{H}_{\text{eff}} = \alpha_0 \boldsymbol{\gamma} K_{\parallel}^2 + \frac{\alpha_0}{2l_B^2} \mathbf{g}_{\perp} + [i\boldsymbol{\beta}_1 K_- + i\boldsymbol{\beta}_2 K_-^3 + i\boldsymbol{\beta}_3 K_- K_+ K_- + \text{h.c.}], \quad (612)$$

where the bold symbols are 2×2 matrices acting in the $\mathcal{A}$ subspace. In terms of the $\mathbf{M}$ -matrices:

$$\begin{aligned} \boldsymbol{\gamma} &= \mathcal{P} [\mathbf{M}_{\gamma}] + \alpha_0 \mathcal{P} \left[\text{sw}_2 \left(\mathbf{M}_1, \mathbf{M}_1^{\dagger} \right) + \text{sw}_2 \left(\mathbf{M}_1^{\dagger}, \mathbf{M}_1 \right) \right] \\ &= \gamma \mathbb{1}_{2 \times 2}, \\ \mathbf{g}_{\perp} &= \mathcal{P} [\mathbf{M}_g] + 2\alpha_0 \mathcal{P} \left[\text{sw}_2 \left(\mathbf{M}_1, \mathbf{M}_1^{\dagger} \right) - \text{sw}_2 \left(\mathbf{M}_1^{\dagger}, \mathbf{M}_1 \right) \right] \\ &= g_{\perp} \sigma_z, \\ \boldsymbol{\beta}_1 &= \alpha_0 \mathcal{P} [\mathbf{M}_1] \\ &= \beta_1 \sigma_+, \\ \boldsymbol{\beta}_2 &= \alpha_0^2 \mathcal{P} [\text{sw}_2 (\mathbf{M}_1, \mathbf{M}_2) + \text{sw}_2 (\mathbf{M}_2, \mathbf{M}_1)] - \alpha_0^3 \mathcal{P} [\text{sw}_3 (\mathbf{M}_1, \mathbf{M}_1, \mathbf{M}_1)] \\ &= \beta_2 \sigma_-, \\ \boldsymbol{\beta}_3 &= \alpha_0^2 \mathcal{P} \left[\text{sw}_2 (\mathbf{M}_{\gamma}, \mathbf{M}_1) + \text{sw}_2 (\mathbf{M}_1, \mathbf{M}_{\gamma}) - \text{sw}_2 \left(\mathbf{M}_1^{\dagger}, \mathbf{M}_2 \right) - \text{sw}_2 \left(\mathbf{M}_2, \mathbf{M}_1^{\dagger} \right) \right] + \alpha_0^3 \mathcal{P} \left[\text{sw}_3 \left(\mathbf{M}_1, \mathbf{M}_1, \mathbf{M}_1^{\dagger} \right) + \text{c.p.} \right] \\ &= \beta_3 \sigma_+, \end{aligned}$$

where c.p. means cyclic permutations. Inserting this back into (612) gives

$$\mathcal{H}_{\text{eff}} = \alpha_0 \gamma K_{\parallel}^2 + \frac{\alpha_0}{2l_B^2} g_{\perp} \sigma_z + [(i\beta_1 K_- - i\beta_2 K_+^3 + i\beta_3 K_- K_+ K_-) \sigma_+ + \text{h.c.}]. \quad (613)$$

The linear Rashba parameter β_1 of the QW ground state L1 is exactly described by a first order SWT. It is given by

$$\beta_1^p = \alpha_0 T_{1,1}^{\text{L}}, \quad (614)$$

where $T_{1,1}^{\text{L}}$ belongs to the $\mathbf{M}_1$ matrix described in Section 2. The g -factor component $g_{\perp}^p$ is given exactly by a 2nd order SWT. It is given by

$$g_{\perp}^p = G_{\perp;1,1}^{\text{L}} + 2\alpha_0 \left[\sum_{j \neq 1} \frac{T_{1,j}^{\text{L}} T_{j,1}^{\text{L}}}{E_1^{\text{L}} - E_j^{\text{L}}} - \sum_l \frac{T_{1,l}^{\times \dagger} T_{l,1}^{\times}}{E_1^{\text{L}} - E_l^{\text{H}}} \right]. \quad (615)$$

An effective Hamiltonian for the quantum channel ground state is found using a similar approach but with the unperturbed Hamiltonian given by $\mathbf{E}_0^{\text{qc}}$ and the perturbation being (c.f. (43))

$$\mathbf{W}(k_y) = \alpha_0 \left(\mathbf{L}_1 k_y + \mathbf{L}_{\gamma} k_y^2 + \frac{1}{2l_B^2} \mathbf{L}_g + \frac{1}{2l_B^2} \mathbf{L}_3 k_y + \frac{1}{4l_B^4} \mathbf{L}_4 \right). \quad (616)$$

For the QC effective parameters $g_{\perp,\parallel}^c$ and β_1^c , only a 1st order SWT is required for an exact description. They are given directly by the corresponding matrix elements in the $\mathbf{L}$ matrices described above (with $i = 1$ and $N = 0$):

$$g_{\perp}^c = G_{\perp;1,0}^c \quad (617)$$

$$g_{\parallel}^c = G_{\parallel;1,0}^c \quad (618)$$

$$\beta_1^c = \alpha_0 T_{1,0}^c. \quad (619)$$

Supplementary Note 7: Additional ground state parameters of the QW system

Similarly to β_1^p , the other two cubic Rashba parameters $\beta_{2,3}^p$ can be evaluated as a function of l_z and F_z . The result is shown in Fig. S1. Note that in contrast to HH systems, $\gamma_3^p > \gamma_2^p$ for LH levels.

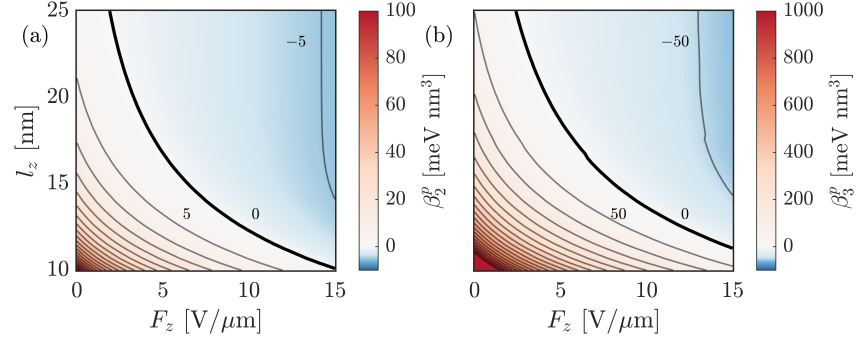


FIG. S1. Cubic Rashba parameters (a) β_2^p and (b) β_3^p as a function of l_z and F_z , with Sn content $x = 16\%$.

Supplementary Note 8: Derivation of the estimation for $g_\perp^c$

Starting from the effective Hamiltonian for the QW ground state L1, and only considering the effective mass term, the Zeeman splitting term, the linear Rashba term and the parabolic confinement along x , one gets

$$H_{\text{eff}}^p = H_0 + W, \quad (81)$$

where

$$H_0 = \alpha_0 \gamma_\parallel^p K_\parallel^2 + \frac{\alpha_0}{2l_B^2} g_\perp^p \sigma_z - \alpha_0 \frac{x^2}{l_x^4} \quad (82)$$

$$W = i\beta_1^p K_- \sigma_+ + \text{h.c.} \quad (83)$$

The part H_0 without linear Rashba is easily diagonalized by writing x and the wavevector components in terms of ladder operators

$$a = \frac{1}{\sqrt{2}} \left(\frac{x}{\lambda_x} + i\lambda_x k_x \right), \quad (84)$$

with $\lambda_x^4 = -\gamma_\parallel^p l_x^4$ chosen such that the Hamiltonian is readily diagonalized. With $\mathbf{B} = B\mathbf{e}_z$ and $k_y = 0$, the mechanical wavevector components become

$$K_x = k_x = \frac{1}{\sqrt{2}i\lambda_x} (a - a^\dagger) \quad (85)$$

$$K_y = 0 + \frac{x}{l_B^2} = \frac{\lambda_x}{\sqrt{2}l_B^2} (a + a^\dagger). \quad (86)$$

Neglecting quadratic-in- B terms, the eigenvalues of H_0 (denoted by $E_{n\sigma}^{(0)}$) are

$$E_{n\sigma}^{(0)} = -\frac{\alpha_0 \sqrt{-\gamma_\parallel^p}}{l_x^2} (2n+1) + \frac{\alpha_0}{2l_B^2} g_\perp^p \sigma \quad (87)$$

with $\sigma = \pm 1$ and $n \geq 0$ being the eigenvalue of $a^\dagger a |n\rangle = |n\rangle n$. Contributions to the energy from the linear Rashba term W are treated by simple perturbation theory. While 1st order perturbation gives no contributions, 2nd order perturbation gives

$$E_{0+}^{(2)} = \sum_{n'\sigma'} \frac{|\langle 0 + | W | n'\sigma' \rangle|^2}{E_{0+}^{(0)} - E_{n'\sigma'}^{(0)}} \approx \frac{(\beta_1^p l_x)^2}{4\alpha_0 \sqrt{-\gamma_\parallel^p}} \left(\frac{1}{\lambda_x^2} + \frac{2}{l_B^2} \right), \quad (88)$$

$$E_{0-}^{(2)} = \sum_{n'\sigma'} \frac{|\langle 0 - | W | n'\sigma' \rangle|^2}{E_{0-}^{(0)} - E_{n'\sigma'}^{(0)}} \approx \frac{(\beta_1^p l_x)^2}{4\alpha_0 \sqrt{-\gamma_\parallel^p}} \left(\frac{1}{\lambda_x^2} - \frac{2}{l_B^2} \right). \quad (89)$$

Subtracting $E_{0-} = E_{0-}^{(0)} + E_{0-}^{(2)}$ from $E_{0+} = E_{0+}^{(0)} + E_{0+}^{(2)}$ gives an approximation for the Zeeman splitting energy of the QC ground state. Multiplying by l_B^2/α_0 to get the g -factor,

$$g_\perp^c \approx \frac{l_B^2}{\alpha_0} (E_{0+} - E_{0-}) \quad (810)$$

$$= g_\perp^p + \frac{1}{\sqrt{-\gamma_\parallel^p}} \left(\frac{\beta_1^p l_x}{\alpha_0} \right)^2. \quad (811)$$

Supplementary Note 9: Consistency of the theory for wide channels

The behavior of $g_{\perp}^c$ and β_1^c is inconsistent with the principle that the quantum well values should be recovered as $l_x \rightarrow \infty$. Fig. 2c and 2d of the main Article show that $g_{\perp}^c$ and β_1^c tend to zero if $l_x \gg \alpha_0 \left| g_{\perp}^p / [(\gamma_{\parallel}^p)^{1/4} \beta_1^p] \right|$, and that only if $\beta_1^p = 0$ does $g_{\perp}^c$ tends to its respective quantum well value. In this section, we show that the theory employed for the quantum channel *is* in fact coherent, despite what Fig. 2 might suggest.

Here we demonstrate that the energy dispersion of the channel orbitals versus B converge to the fan-diagram of the QW system as $l_x \rightarrow \infty$. We select the following heterostructure parameters: $l_z = 15$ nm, $x = 16\%$, and $F_z = 6$ V/ μ m, such that $\beta_1^p \neq 0$. The fan-diagram of the QW ground state is computed with the method outlined at the end of Section 1 above, and is shown in Fig. S2 (bottom right). In contrast to HH levels, where the fan-diagram opens up linearly with B , a LH fan-diagram opens as $\sqrt{B}$ near $B = 0$ if $\beta_1^p \neq 0$.

This fan-diagram is shown alongside the dispersions obtained from (43) for finite l_x . As l_x increases, the energy gap between adjacent QC orbitals decreases as expected. Coupling between orbitals cause the g -factor of the lowest QC orbital to become very small: this is noticeable already from $l_x \geq 58$ nm (see Fig. S2 below, first panel), as the Zeeman energy splitting for the $N = 0$ orbital is much smaller than that of $N \geq 1$ orbitals. For very large l_x , orbitals gather closely near $B = 0$, but remain well separated at larger fields. As $l_x \rightarrow \infty$, the orbitals spectacularly reorder themselves such that the fan-diagram of the last panel is roughly recovered. The level reordering implies that beyond some value of l_x , $g_{\perp}^c$ does not represent the g -factor of the QW ground state anymore.

The energy level structure computed from (43) does not show all spin-split pairs like the fan-diagram does. This is merely an artifact of the eigenvalue sorting method. In the former, all levels closest to a given energy guess are shown, whereas in the former, all *Landau* levels closest to an energy guess are shown.

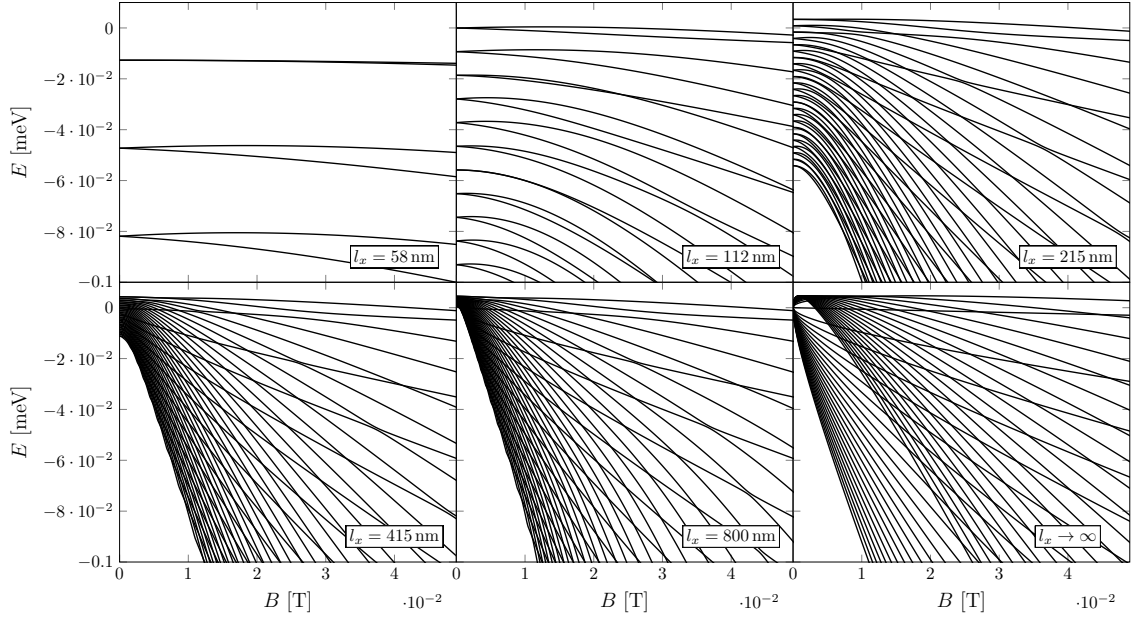


FIG. S2. Quantum channel energy dispersion as a function of the out-of-plane magnetic field up to 0.05 T for increasing widths l_x . The bottom right panel labeled $l_x \rightarrow \infty$ is the fan-diagram of the QW system. All other panels are computed from (43) with $k_y = 0$. QW parameters in all cases are $l_z = 15$ nm, $x = 16\%$ and $F_z = 6$ V/ μ m ($\beta_1^p \approx 1.21$ meV nm). The zero of energy is placed on the QW L1 subband ($E_1^L = 0$).

Supplementary Note 10: $\text{Ge}_{1-x}\text{Sn}_x$ Material parametrization

Most of the material parameters of bulk $\text{Ge}_{1-x}\text{Sn}_x$ (with the exception of the three Luttinger parameters $\gamma_{1,2,3}$) are calculated on the full composition range $x \in [0, 1]$ according to a Vegard relation between the parameters of pure Ge and pure Sn. For instance, the parameter $P(x)$ is given by

$$P(x) = (1 - x)P^{\text{Ge}} + xP^{\text{Sn}} - x(1 - x)b_P, \quad (101)$$

where P^{Ge} and P^{Sn} refer to the parameter of pure Ge and pure Sn, respectively, and b_P is a bowing for parameter P , if available.

The three Luttinger parameters $\gamma_{1,2,3}$ are interpolated within the composition range $x \in [0, 0.2]$ by fitting a Vegard relation on the parameters reported in Reference 6. The Luttinger parameters are therefore given by

$$\gamma_i(x) = \left(1 - \frac{x}{0.2}\right) \gamma_i^{\text{Ge}} + \frac{x}{0.2} \gamma_i^{(0.2)} - \frac{x}{0.2} \left(1 - \frac{x}{0.2}\right) b_{\gamma_i}, \quad (102)$$

where $\gamma_1^{(0.2)} = 29.2108$, $\gamma_2^{(0.2)} = 12.2413$ and $\gamma_3^{(0.2)} = 13.7387$ are the values at $x = 0.2$. The bowing parameters are $b_{\gamma_1} = 20.3391$, $b_{\gamma_2} = 9.6609$ and $b_{\gamma_3} = 9.8187$. All other parameters used in the $k \cdot p$ framework are shown in Table S1.

TABLE S1. Material parameters and bowings of the 6-band $k \cdot p$ framework.

	Ge	$\text{Ge}_{1-x}\text{Sn}_x$	Sn
Lattice constant			
a_0 [Å]	5.652357 ^a	-0.083 ^g	6.480117 ^b
Bulk band energies			
$\mathcal{E}_{\Gamma_5^+}$ [eV]	0		0.69 ^{h*}
Δ_0 [eV]	0.290 ^b	-0.100 ^g	0.770 ^f
Elastic constants			
c_{12}/c_{11}	0.333 ^b		0.425 ^b
Deformation potentials			
a_v [eV]	1.24 ^c		1.58 ⁱ
b [eV]	-2.86 ^d		-2.7 ^j
Effective mass and spin parameters			
γ_1	13.38 ^e	†	†
γ_2	4.24 ^e	†	†
γ_3	5.69 ^e	†	†
κ	3.41 ^f		-11.84 ^f
q	0.06 ^e		0.30 ^f

References: ^a:[7], ^b:[8], ^c:[9], ^d:[2],
^e:[1], ^f:[10], ^g:[11], ^h:[12], ⁱ:[13], ^j:[14]

* Relative to Ge

† See (102)

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