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CONDITION BASED MAINTENANCE USING THE PROPORTIONAL HAZARD MODEL WITH IMPERFECT INFORMATION

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Ce mémoire intitulé :

CONDITION BASED MAINTENANCE
USING THE PROPORTIONAL HAZARD MODEL WITH
IMPERFECT INFORMATION

présenté par : GHASEMI Alireza
en vue de l'obtention du diplôme de: Maîtrise ès sciences appliquées
a été dûment accepté par le jury d'examen constitué de :

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Mme YACOUT Soumaya, D.Sc., membre et directrice de recherche
M. ADJENGUE Luc, Ph.D., membre

To those who have supported me by dedicating their loves throughout my life

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On a general level, I would like to see this thesis as a modest tribute to all the people that have had a formative influence on me over the years –teachers, professors and research assistants, and of course my parents, and friends. Each in their own way, has helped to unlock some of the mysteries of this world.

RÉSUMÉ

La maintenance conditionnelle (CBM), ou la maintenance prédictive, est basée sur l'observation à intervalles réguliers d'éléments indiquant l'état de l'équipement. L'objectif de ce mémoire est de déterminer la politique de remplacement optimale lorsque la détérioration de l'équipement n'est pas facilement observable. Un "Proportional Hazard Model (PHM)" est utilisé afin de modéliser le taux de défaillance de l'équipement. Dans la littérature, plusieurs auteurs présentent des politiques de remplacement optimales lorsqu'il existe une relation déterministe entre l'état de l'équipement et les indicateurs. Nous présentons une politique CBM optimale lorsque la relation est stochastique.

Un critère de décision nous menant à remplacer effectuer des remplacement optimaux est introduite. Ce critère est fonction de l'âge, de la distribution de probabilité conditionnelle de l'état de l'équipement et du coût moyen de remplacement à long terme. Une méthode récursive permettant de calculer le coût moyen est également introduite. Quelques exemples numériques sont résolus et les coûts moyens des problèmes parfaits et imparfaits sont comparés. Aussi, nous étudions le comportement du coût moyen à long terme lorsque les paramètres sont changeants.

Programmation Dynamique (DP), Processus de Markov Partiellement Observé (POMDP) et probabilités appliquées sont utilisés dans ce mémoire.

ABSTRACT

Condition Based Maintenance (CBM) or predictive maintenance is based on observing an indicator of the working state of the equipment, at different intervals of time. The objective of this thesis is to determine the optimal policy for the equipment's replacement while the deterioration of the equipment is not outwardly visible. The Proportional Hazard Model (PHM) is used to model the failure behavior of the equipment. In the literature, many papers presented optimal policies when the relation between the equipment state and the indicators is deterministic. We present an optimal CBM policy when a stochastic relation between the unknown equipment's working state and the measured value of the indicator, exists.

A decision criterion which leads to optimum replacement decision is introduced. This criterion is a function of the age, the probability distribution of working state, and the long-run average cost of the replacement system. A recursive method for calculating the average cost is also introduced. Some numerical examples are solved and the perfect and imperfect problems' long run average costs are compared. Also the behavior of the long-run average cost while the problem's parameters change are studied.

Dynamic programming (DP), Partially Observed Markov Decision Process (POMDP) and applied probabilities are used in this thesis.

Keywords: CBM, predictive maintenance, PHM, POMDP, Dynamic Programming.

CONDENSÉ EN FRANÇAIS

Introduction

Estimer la condition d'un équipement en observant un ou plusieurs indicateurs de sa détérioration forme la base de la maintenance conditionnelle (CBM), ou de la maintenance prédictive. Si ces observations donnent directement l'état de l'équipement, nous avons alors de l'information parfaite ou complète. Cependant, si ces observations sont liées de façon stochastique à l'état de l'équipement, on parle d'information imparfaite ou incomplète. Dans ce mémoire, nous présentons une politique de remplacement optimale d'un équipement dans le cas où nous sommes en présence d'information imparfaite. De plus, nous étudierons plus spécifiquement les cas ou le taux de défaillance suit un Proportional Hazard Model (PHM).

Description de modèle et suppositions

Considérons un équipement se détériorant avec le temps et sujet aux défaillances. L'état de œt équipement, S, peut être illustré par un série finie de nombre strictement positifs $S = \{1,2,\ldots,N\}$ Lorsque léquipement est neuf, il est à l'état 1 et au fur et à mesure qu'il se détériore, il s'approche de l'état N. Nous utilisons le PHM pour présenter le taux de défaillance qui dépend tout autant de l'âge ainsi que de l'état de détérioration. Pour plus de détails, consultez les travaux de Cox [1984].

On suppose que l'état de l'équipement suit un processus de Markov. La probabilité de transition d'un état à un autre est donnée par la matrice stochastique $P = \left[p_{ij} \right]$ $i, j \in S$, où p_{ij} est la probabilité de passage de l'état i à l'état j en une étape avec $i, j \in S$ c'est-à-dire $p_{ij} = \Pr\left(X(t+\Delta) = j \mid X(t) = i\right)$. La détérioration de l'état d'un équipement, en tant que telle, n'est pas observable mais un indicateur d'état, θ permet la caractérisation de cette détérioration à des instants de temps discrets $t = \Delta, 2\Delta, \ldots$ Δ est un intervalle constant entre deux observations consécutives. Cet indicateur peut prendre un nombre fini de valeurs dans un ensemble d'entiers naturels, i.e. $\theta \in \Theta = \{1,2,\ldots,M\}$. On suppose que la valeur de θ est observée avec une probabilité connue $q_{j\theta}$ quand l'état de l'équipement est j. On désigne par Q la matrice de ces dites probabilités, on a $Q = \left[q_{j\theta}\right]$, $j \in S, \theta \in \Theta$.

Une défaillance peut survenir n'importe quand. Si cela survient, l'équipement cesse de fonctionner. Les défaillances sont par conséquent observables. Lorsqu'une défaillance survient, la seule action pouvant être entreprise est le remplacement correctif. Cependant, après une inspection préventive, on peut décider de faire un remplacement préventif ou de ne rien faire d'ici la prochaine inspection. Les deux remplacements (correctif ou préventif) ramènent l'équipement à l'état 1. Le coût d'un remplacement préventif est C, alors qu'un remplacement correctif coûte $K + C, K, C > 0$. Après un remplacement correctif, la collecte d'observations continue selon l'horaire prévu originalement, peu importe le moment de la défaillance. Les deux actions, le remplacement correctif et le remplacement préventif, sont instantanées. L'objectif est de déterminer la politique de remplacement optimale qui permettra de minimiser à long terme le coût moyen par unité de temps.

Le problème du remplacement optimal

La distribution de probabilité conditionnelle de détérioration de l'équipement est estimée en utilisant l'historique du fonctionnement de l'équipement, les observations et les décisions prises, dans le temps jusqu'au point d'observation actuel. Le problème est formulé sous forme d'un processus de Markov partiellement observé POMDP, (Partially Observed Markov Decision Process). En résolvant ce problème, on détermine le critère de décision optimal qui est en fonction de l'âge de l'équipement, de la distribution de la probabilité de détérioration de l'état de l'équipement et d'une constante qui est le coût moyen à long terme du système de maintenance. Ce dit critère est utilisé à chaque observation pour décider de l'action à prendre, remplacer l'équipement ou ne rien faire jusqu'à la prochaine inspection. Le calcul du coût moyen à long terme du système est décrit dans la section suivante.

Le Processus de Markov Partiellement Observé

L'espace d'états

La distribution de probabilité conditionnelle de l'état de l'équipement, π^k est utilisée comme l'espace d'états du POMDP.

$$\pi^k = \left\{\pi_i^k; 0 \leq \pi_i^k \leq 1 \text{ for } i = 1,\ldots,N, \sum_{i=1}^{N} \pi_i^k = 1\right\}, k = 0,1,2,\ldots$$

$$\pi_i^0 = \begin{cases} 1 & i=1 \\ 0 & 1 < i \leq N \end{cases}$$

L'espace de décisions

L'espace de décisions du modèle est $\{0, \infty\}$ où 0 indique 'remplacement immédiat de l'équipement' et ∞ indique 'ne rien faire jusqu'à la prochaine inspection ou effectuer un remplacement correctif si l'équipement tombe en panne avant la prochaine inspection'.

Transition d'états

À l'observation $k+1$, après avoir constaté l'état θ, la distribution de probabilité conditionnelle de l'état de l'équipement π^k est mise à jour à $\pi_j^{k+1}(\theta)$ qui est calculé selon la formule de Bayes.

$$\pi_j^{k+1}(\theta) = \frac{\sum_{i=1}^{N} \pi_i^k p_{ij} q_{j\theta}}{\sum_{i=1}^{N}\sum_{l=1}^{N} \pi_i^k p_{il} q_{l\theta}} \quad \text{for } j = 1,\ldots,N$$

Cette mise à jour de la probabilité contient en elle tout l'historique des observations et des décisions prises depuis le dernier remplacement. À chaque remplacement, on remet à zéro le compteur k et la distribution de probabilité conditionnelle de l'état de l'équipement est remise à la valeur π^0.

Formulation en programmation dynamique

Soit $V(k,\pi^k)$ le coût minimum par période de renouvellement, sachant qu'à la période k la distribution conditionnelle mise à jour de l'état de détérioration d'équipement est π^k :

$$V(k,\pi^k) = \min\left\{ C + V(0,\pi^0), W(k,\pi^k,g) \right\}$$

Où $C+V(0,\pi^0)$ est le coût de remplacement préventif et $W(k,\pi^k,g)$ est le coût espéré du cas où l'équipement est laissé sans remplacement jusqu'à la prochaine inspection. $W(k,\pi^k,g)$ est défini comme suit:

$$\begin{aligned} W(k,\pi^k,g) = {} & \left[K + C + V(0,\pi^0) \right] \times \left[1 - \overline{R}(k,\pi^k,\Delta) \right] - g\overline{\tau}(k,\pi^k,\Delta) \\ & + \left[\sum_{\theta=1}^{M} V(k+1,\pi^{k+1}(\theta)) \Pr(\theta \mid k,\pi^k) \right] \times \overline{R}(k,\pi^k,\Delta) \end{aligned}$$

Où $\overline{R}(k,\pi^k,\Delta)$ et $\overline{\tau}(k,\pi^k,\Delta)$ sont respectivement la probabilité que l'équipement soit encore en marche à la fin de la période $k+1^{\text{ieme}}$ et la période moyenne de séjour de l'équipement pendant la $k+1^{\text{ieme}}$ période, quand la distribution conditionnelle de probabilité d'état d'équipement à la k^{ime} période, est connue. g est le coût d'entretien moyen par unité de temps sur un horizon infini. Aussi $\Pr(\theta \mid \pi^k)$ est la probabilité d'avoir l'état θ à la $k+1^{\text{ieme}}$ observation.

$$\Pr(\theta \mid \pi^k) = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i^{\,k} p_{ij} q_{j\theta}$$

En utilisant la même approche que Kurano [1985], on prouve que $V(k,\pi^k)$ est une fonction non décroissante de k et de π, en terme de rapport de vraisemblance [Rosenfield 1975]. Ce caractère monotone de $V(k,\pi^k)$ permet de trouver un temps d'arrêt T_g tel que:

$$T_g = \Delta.\inf\left\{n \geq 0 \ ; \ K*\left[1-\overline{R}\left(n,\pi^n,\Delta\right)\right] \geq g\overline{\tau}\left(n,\pi^n,\Delta\right)\right\}$$

Dans la section suivante, nous présentons la formulation qui permet d'atteindre l'optimum de g.

Le coût moyen optimal de longue durée

Sur une longue période, le coût espéré par unité de temps, ϕ^{T_g} avec T_g le temps d'arrêt, est donné par :

$$\phi^{T_g} = \frac{C + K\overline{\Pr}\left(T_g > T\right)}{\overline{E}_{\min}\left(T_g, T\right)}$$

Où T est le temps écoulé jusqu'à la panne, $\overline{\Pr}(T_g > T)$ est la probabilité d'un remplacement correctif (d'arrivée d'une panne) et $\overline{E}_{\min}(T_g, T)$ est le cycle de remplacement moyen espéré. Il est prouvé que le temps d'arrêt T_{g^*}, avec $g^* = \min \phi^{T_g}$, minimise ϕ^{T_g}. La valeur de g^* est l'unique solution de $g = \phi^{T_g}$ [Aven 1996].

Conclusion

Nous avons établi une politique optimale de remplacement pour un équipement qui se détériore. L'état de la détérioration de l'équipement est inconnu, mais la probabilité d'être dans un certain état peut être calculée en employant des indicateurs de l'état de détérioration de l'équipement. En employant la programmation dynamique, le POMDP (Partially Observed Markov Decision Process) et la probabilité appliquée, la politique optimale de remplacement et le coût moyen de longue durée optimal du remplacement sont obtenus. Quelques exemples numériques sont résolus et les résultats sont comparés à des cas connus. Les résultats sont consistants, le coût de longue durée moyen de l'équipement avec une information imparfaite est toujours plus haut que celui avec une information parfaite. En outre le comportement du coût moyen de longue durée du système de rechange est étudié quand les divers paramètres du système changent.

SUMMARY

Introduction

Condition Based Maintenance (CBM) or predictive maintenance is based on determining equipment's condition by observing one or more indicators of the equipment working state. If these observations directly indicate the equipment's working state, then we have perfect information. If the observations are stochastically related to the equipment's working state, then we have a case of imperfect information. In this thesis we present an optimal cost replacement policy for the later case while its failure rate follows a Proportional Hazard Model.

Model description and assumptions

We consider a piece of equipment that deteriorates with time and is subjected to failure at any time. The equipment's working state, S, is illustrated by a finite set of non-negative integers, i.e. $S=\{1,2,,...,N\}$. The equipment is in state 1 when it is new, and it can deteriorate until state N. We use the Proportional Hazard Model (PHM) to represent the failure rate. This means that the equipment's failure rate depends on its age as well as on its working state. For more details see [Makis and Jardine 1992]

The transition probability between one state and the other is given by the stochastic matrix $P=\left[p_{ij}\right]$ $i,j \in S$, where p_{ij} is the one step transition probability from state i to state j and $i,j \in S$ i.e. $p_{ij}=\Pr(X(t+\Delta)=j \mid X(t)=i)$. The working state of the equipment is not observable but an indicator of the working state, θ, is observed at

discrete-times, $t = \Delta, 2\Delta, \ldots$ Δ is the constant interval between two consecutive observations. This indicator can take any value in a finite set of non-negative integers, i.e. $\theta \in \Theta = \{1,2,\ldots,M\}$. It is supposed that the values of θ are observed with a known probability $q_{j\theta}$ when the state of the equipment is j. Q is the stochastic matrix which specifies these probabilities $Q = \left[q_{j\theta} \right]$, $j \in S, \theta \in \Theta$.

The equipment's failure can happen at any time. If it happens, it causes the equipment cease functioning, hence it is outwardly observable. When a failure happens, the only action performable on the equipment is *Failure Replacement*. At any observation point, we can decide whether to *Replace Preventively* or *Do-Nothing*. Both actions, *Failure Replacement* and *Replace preventively,* renew the equipment to state 1. The cost for preventive replacement is C, while a failure replacement costs $K + C, K, C > 0$. The equipment is observed at the first scheduled observation point, after a *Failure Replacement* has occurred. Both actions, *Failure Replacement* and *Preventive Replacement*, are instantaneous. The objective is to find the optimal replacement policy that minimizes the long run average cost per unit time for the replacement system.

Optimal Replacement Problem

The conditional probability distribution of the equipment working state is estimated by using the history of the equipment including all the observations and the performed decisions (actions) until the observation point of time. Then the optimization problem is formulated as a Partially Observed Markov Decision Process (POMDP). By solving this problem, the optimal decision criterion is derived which is a function of the equipment age, the probability distribution of the equipment working state and a constant, which is the long-run average cost of the maintenance system. Based on this decision criterion, we decide on whether to replace the equipment or to do nothing and let the equipment work until the next observation point. The long run average cost of the system is calculated in the subsequent part.

The Partially Observable Markov Decision Process

State space

The conditional probability distribution of the equipment's state, π^k, is used as the state space of the POMDP.

$$\pi^k = \left\{\pi_i^k; 0 \le \pi_i^k \le 1 \text{ for } i = 1,\dots,N, \sum_{i=1}^{N} \pi_i^k = 1\right\}, k = 0,1,2,\dots$$

$$\pi_i^0 = \begin{cases} 1 & i = 1 \\ 0 & 1 < i \le N \end{cases}$$

Decision space

The decision space of the model is $\{0, \infty\}$, where 0 means "Replace the equipment immediately", and ∞ means "Do-Nothing until the next observation point or replace at the failure time, if the failure takes place before the next observation point".

State transition

At the $k+1$ observation point, after having observed θ, the prior conditional distribution of the equipment, π^k, is updated to $\pi_j^{k+1}(\theta)$ which is calculated by using the Bayes' formula.

$$\pi_j^{k+1}(\theta)=\frac{\sum_{i=1}^{N}\pi_i^k p_{ij}q_{j\theta}}{\sum_{i=1}^{N}\sum_{l=1}^{N}\pi_i^k p_{il}q_{l\theta}} \text{ for } j=1,\ldots,N$$

This updated conditional distribution carries all the history of the observations and the actions since the last replacement. At any Replacement, the observation period's counter, k, will be reset to 0 and the conditional probability distribution of the equipment working state will be set to π^0.

Dynamic Programming Formulation

Let $V(k,\pi^k)$ denote the minimum cost from period k until next renewal point, where the updated conditional distribution of the equipment working state is π^k:

$$V\left(k,\pi^k\right)=\min\left\{C+V\left(0,\pi^0\right),W\left(k,\pi^k,g\right)\right\}$$

$C+V(0,\pi^0)$ is the cost of preventive replacement and $W(k,\pi^k,g)$ is the expected cost of leaving the equipment to work until the next observation point. $W(k,\pi^k,g)$ is defined as follows:

$$W\left(k,\pi^k,g\right)=\left[K+C+V\left(0,\pi^0\right)\right]\times\left[1-\overline{R}\left(k,\pi^k,\Delta\right)\right]-g\overline{\tau}\left(k,\pi^k,\Delta\right)$$
$$+\left[\sum_{\theta=1}^{M}V\left(k+1,\pi^{k+1}(\theta)\right)\Pr\left(\theta\,|\,k,\pi^k\right)\right]\times\overline{R}\left(k,\pi^k,\Delta\right)$$

where $\overline{R}(k,\pi^k,\Delta)$ and $\overline{\tau}(k,\pi^k,\Delta)$ are respectively the probability that the equipment is still working during the $k+1^{\text{st}}$ period and the mean sojourn time of the equipment during $k+1^{\text{st}}$ period, when the equipment state conditional probability distribution at the k-th period, π^k, is known. g is the average cost of maintenance per unit time over an infinite horizon. Also $\Pr(\theta\,|\,\pi^k)$ is the probability of observing θ at the $k+1^{\text{st}}$ observation epoch:

$$\Pr(\theta\,|\,\pi^k)=\sum_{i=1}^{N}\sum_{j=1}^{N}\pi_i^k p_{ij}q_{j\theta}$$

By using a similar approach to that of Kurano [1985], it is shown that $V(k,\pi^k)$ is a non-decreasing function of k and of π, in *Likelihood Ratio* [Rosenfield 1975]. This monotone behavior of $V(k,\pi^k)$ results in finding a stopping-time, T_g, such that:

$$T_g=\Delta.\inf\left\{n\geq 0\ :K*\left[1-\overline{R}\left(n,\pi^n,\Delta\right)\right]\geq g\overline{\tau}\left(n,\pi^n,\Delta\right)\right\}$$

In the next section, the formulation leading to the optimum g is presented.

Optimal long-run average cost

The long run expected cost per unit of time, ϕ^{T_g}, where stopping-time is T_g, is given by:

$$\phi^{T_g}=\frac{C+K\overline{\Pr}\left(T_g>T\right)}{\overline{E}_{\min}\left(T_g,T\right)}$$

where T is the time to failure, $\overline{\Pr}(T_g > T)$ is the probability of a failure replacement, and $\overline{E}_{\min}(T_g, T)$ is the expected average length of a replacement cycle. It is shown that the stopping-time T_{g^*}, where $g^* = \min \phi^{T_g}$, minimizes ϕ^{T_g}. The value of g^* is the unique solution of $g = \phi^{T_g}$ [Aven 1996].

Conclusion

We derived an optimal replacement policy for equipment that deteriorates. The equipment's working state is unknown, but the probability of being in a certain state can be calculated by using observed indicators of the equipment working state. By using Dynamic Programming, Partially Observed Markov Decision Process and applied probability, the optimal replacement policy and the optimal long-run average cost of the replacement are obtained.

Some numerical examples are solved and the results are compared with perfect information cases. The results are consistent in that, the average long-run cost of the equipment with imperfect information is always higher than that with perfect information. Also the behavior of the long run average cost of the replacement system is studied while the various parameters of the system change.

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LIST OF ACRONYMS

CBM .... Condition Based Maintenance

cdf .... cumulative distribution function

DP .... Dynamic Programming

EM .... Expectation Minimization

IFR .... Increasing Failure Rate

LR .... Likelihood Ratio

MC .... Markov Chain

MDP .... Markov Decision Processes

MP .... Markov Process

MTTF .... Mean Time To Failure

pdf .... probability distribution function

PHM .... Proportional Hazard Model

POMDP .... Partially Observed Markov Decision Processes

ST .... Stochastically

TP2 .... Totally Positive of order 2

LIST OF APPENDICES

CHAPTER 1 :

INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

Depending on the specific industry, maintenance costs can represent between 15 and 40 per cent of the costs of goods produced. For example in food related industries, the average maintenance cost represents about 15 per cent of the cost of goods produced; while in iron and steel, pulp and paper and other heavy industries maintenance represents up to 40 percent of the total production costs [Mobley, 2002]. Since maintenance costs are a major part of the total operating costs of manufacturing or production, one of the tools in securing the productivity is to have a well functioning maintenance system in the company. The maintenance system in a company has the role of looking after the equipment and keeping track of it in order to secure the productivity. With no or a poor maintenance system, a company will lose money due to the cost of lost production capacity, the cost of keeping spare parts, the lack of quality, and the costs of late deliveries etc.

Most traditional maintenance systems are categorized either in *preventive approach* or *corrective approach*. The preventive maintenance, while keeps fixed maintenance periods, tries to prevent the components, sub-systems or systems from degrading by performing repair, service or component's replacement. The corrective maintenance is performed after a system failure or breakdown is occurred. While preventive maintenance is *Age Based*, i.e. the system's maintenance is based on its age, since few decades ago some industries have started to perform maintenance actions in a *Condition Based* or *predictive* approach. In the latter the equipment's condition is the key parameter to set the maintenance time and appropriate maintenance tasks. Maintenance based on predictive approach is called *Condition Based Maintenance* (CBM) and/or sometimes referred to as *predictive maintenance*. Figure 1.1 shows a schematic of different types of maintenance systems. The system's condition may be obtained through different levels of automation, from human visual inspection, to on-line highly sophisticated condition monitoring equipment.

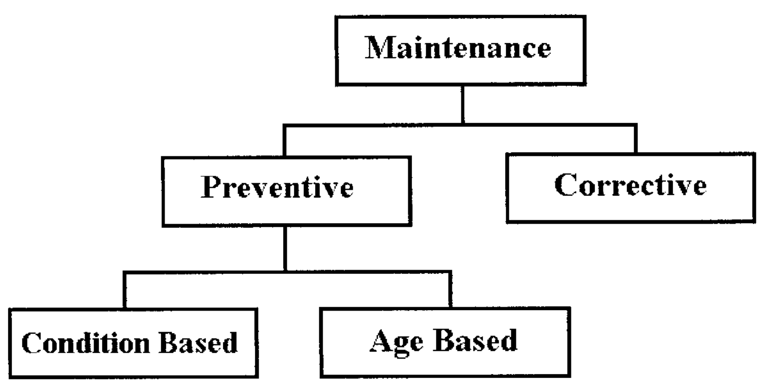

Figure 1.1: Schematic of different types of Maintenance

1.1.1 Condition Based Maintenance

Condition Based Maintenance has been defined as "Maintenance actions based on actual condition obtained from in-situ, non-invasive tests, operating and condition measurement." [Mitchell, 1998] or "CBM is a set of maintenance actions based on real-time or near-real time assessment of equipment state which is obtained from embedded sensors and/or external tests & measurements taken by portable equipment." [Butcher

2000]. In these definitions and many similar ones that can be found in the literature as well as in the internet, the common idea is that the maintenance actions are not considered until there is an obvious need. This will increase the availability of an equipment and decrease maintenance cost, including labor and spare parts costs.

The purpose of CBM is to eliminate breakdowns and protract the preventive maintenance intervals which will result in an increase in the availability of equipment. Using CBM technology, the condition monitoring data is analyzed in depth to determine whether the equipment is running at a normal operating condition or not. If the preset limits for normal condition are exceeded, the maintenance actions are performed. With this information, it is easier to plan the maintenance actions more effectively [Marcus *et al.* 2002].

CBM systems may need several components and level of automation in order to give the required information to make the right maintenance decision. Some companies may use hand-held devices out in the field and then make analysis of the data later in laboratories, while others use more complicated on-line system that give the results on site. In both cases, the way that the companies use the information determines whether they are having a CBM strategy or they are just inspecting their equipments. When having a CBM program, the results of the analysis are taken into account and the maintenance actions are planned accordingly.

A variety of technologies may be used as part of a CBM program. Since mechanical systems are parts of most industries' equipments, vibration monitoring is generally the most used technique in CBM programs. This technique is limited to monitoring the mechanical condition. For this reason a CBM program may include one or more of the following monitoring and diagnostic techniques:

- Vibration Monitoring
- Thermography
- Tribology
- Ultrasonic monitoring
- Other nondestructive testing techniques
- Process Parameters
- Visual Inspection

In the next part, a description of each of these techniques is provided [Mobley 2002].

Vibration Monitoring

Since most of the typical industry equipments are mechanical, this technique has the widest application. This technique uses the noise or vibration created by mechanical equipment to determine its actual condition. The degradation of the mechanical condition can be detected using vibration-monitoring techniques.

Thermography

Thermal anomalies, i.e. areas that are hotter or colder than they should be, can be used to monitor the operating conditions of the system. Thermography uses instrumentation designed to monitor the emission of infrared energy, i.e. temperature, by the equipment to determine its state. Infrared technology is based on the fact that; all the objects having a temperature above absolute zero emit energy or radiation.

Tribology

Tribology refers to design and operating dynamics of the bearing-lubrication rotor support structure of machinery. Several tribology techniques like: lubricating oils analysis, spectrographic analysis, and ferrography and wear particle analysis can be used for predictive maintenance. For instance, some forms of lubricating oil analysis will provide an accurate quantitative breakdown of individual chemical elements, both oil additive and contaminates, contained in the oil. A comparison of the amount of trace metals in successive oil samples can indicate wear patterns of oil wetted parts in the equipment and will provide indication of impending machine failure.

Ultrasonic Monitoring

This predictive maintenance technique uses principles similar to vibration analysis. Both monitor the noise generated by machines or systems to determine their actual working state. Unlike vibration monitoring, ultrasonic monitoring monitors the higher frequencies, i.e. ultrasound, produced by unique dynamics in process systems or machines. The normal monitoring range for vibration analysis is from less than 1 Hertz to 20,000 Hertz. Ultrasonic techniques monitor the frequency range between 20,000 Hertz and 100 kHz. This technique is ideal for detecting leaks in valves, steam traps, piping and similar process systems.

Process Parameters

Machinery that is not operating within acceptable efficiency parameters can severely limit the productivity of many systems. As an example of the importance of process parameters monitoring, consider a process pump that may be critical to industry operation. The pump can be operating at less than 50 percent efficiency and the predictive maintenance program which does not consider the efficiency, would not detect the problem.

Visual Inspection

Regular visual inspection of the machinery and systems in an industry is a necessary part of any predictive maintenance program. In many cases, visual inspection will detect potential problems that will be missed using the other predictive maintenance techniques. Routine visual inspection of critical systems will augment the other techniques and insure that potential problems are detected before serious damage can occur.

1.1.2 Proportional Hazard Model

Introduced by D. R. Cox, the Proportional Hazards Model (PHM) was developed in order to take into account the effects of system's condition that influences its times-to-failure. The model has been broadly used in the biomedical field [Leemis 1995] and recently there has been an increasing application in reliability engineering [Makis and Jardine 1992].

According to the PHM, the failure rate of a system is affected not only by its operating time, i.e. its age, but also by the working state under which it operates. It is clear that this factor affect the failure rate of the system. Equipment in good state has less chance to fail than a worn one even if they both have the same age. The proportional hazards model assumes that the failure rate of a system is the product of a baseline failure rate, $h_0(t)$, which is a function of age only and a positive function $\psi(X_t)$, that is independent of age and incorporates the effects of the system working state. The failure rate of a unit is then given by:

$$h(t, X_t) = h_0(t)\psi(X_t)$$

where, X_t is the random variable representing the working state of the system at time t.

PHM also assumes that the form of $\psi(X_t)$ is known and is the exponential form and is given by:

$$\psi(X_t) = e^{aX_t}$$

where a is a constant.

The Weibull distribution is one of the most commonly used distributions in reliability engineering because of the several shapes it attains for different values of its parameters. Hence it can model a great range of data and life characteristics. By considering a two parameters Weibull distribution to formulate the baseline failure rate, the parametric proportional hazard model is introduced. In this case, the baseline failure rate is given by: [Cox 1984]

$$h_0(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}$$

where α and β are the scale and shape parameters, respectively, of the Weibull distribution function. The PHM failure rate then becomes:

$$h(t, X_t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{aX_t}$$

More details about PHM and Weibull distribution are presented below.

1.1.3 CBM with Imperfect Observations

In many real cases, the working state of the system is not outwardly visible while the failure of the system is immediately obvious and causes the system to cease functioning. This system is known as a system with *obvious failures* as opposed to *silent failures*, which are not immediately discovered. Observations of a system with *Perfect Observations* reveal the states of the system with certainty, while the partially observed system can result from the probabilistic information, in contrary with deterministic information, related to the working state of the system. We refer to these cases as *Imperfect Observation* cases [Maillart 2004]. If the observations are taken in selected periods rather than all periods, the system is *Partly Observed*.

Table 1.1 shows these definitions in brief.

Table 1.1: Information Perfection

Observations	Periods	Named
Perfect (*Deterministic*)	All	***Completely Observed***
	Selected	***Partly Observed***
Imperfect (*Probabilistic*)	All	***completely Observed***
	Selected	***Partly Observed***
None	None	***Completely Unobserved***

1.2 Preliminary Notations

We define a continuous random variable T, as the time-to-failure of the system which can take any value in $[0,\infty]$.

The Probability and Cumulative Density Functions

The ***probability density function***, $f(t)$, and ***cumulative distribution function***, $F(t)$, of T are such that:

$$\Pr(a \le t \le b) = \int_a^b f(t)dt$$

and

$$F(t) = \Pr(T \le t) = \int_0^t f(s)ds = 1 - R(t)$$

The cdf is used to measure the probability that the system in question will fail before the associated time value, t, and is also called *unreliability* where $R(t)$ is the *reliability*.

The Conditional Reliability Function

Conditional reliability is the probability of successfully completing a future task following the successful completion of a previous one. The time of the previous task, t_0, and the time for the task to be accomplished, t, must be taken into account for conditional reliability calculations. The conditional reliability function is given by: [Ross 1997]

$$R(t \mid t_0) = \frac{R(t + t_0)}{R(t_0)}$$

The Failure Rate Function

The *failure rate function* enables the determination of the number of failures occurring per unit time. The failure rate function is mathematically calculated as:

$$h(t) = \frac{f(t)}{R(t)}$$

This gives the instantaneous failure rate. The cumulative of failure rate is called *hazard function*.

$$H(t) = \int_0^t h(s)\, ds$$

These functions are useful in characterizing the failure behavior of a system. [Ross 1997]

Mean Life (MTTF)

The *mean life* function, which provides a measure of the average time of operation to failure, is given by:

$$MTTF = \bar{T} = \int_0^\infty t.f(t)\, dt$$

This is the expected or average time-to-failure for a system with instantaneous replacement and is denoted as the *MTTF*, ***M****ean* ***T****ime* ***T****o* ***F****ailure*. The *MTTF*, even though an index of reliability performance, does not give much information on the failure distribution of the system in question when dealing with most probability distributions.

Weibull Distribution

The *Weibull distribution* is one of the most commonly used distributions in reliability engineering because of the several shapes it attains for different values of its parameters. Hence it can model a great range of data and life characteristics. The most general expression of the Weibull pdf is given by the *three-parameter Weibull distribution* expression, or:

$$f(t)=\frac{\beta}{\alpha}\left(\frac{t-\gamma}{\alpha}\right)^{\beta-1}e^{-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}}$$

Where, $f(t)\geq 0, t\geq\gamma$, $\beta>0$, $\alpha>0, -\infty<\gamma<\infty$ and:

- $\alpha =$ Scale parameter
- $\beta =$ Shape parameter or Slope.
- $\gamma =$ Location parameter

Usually, the location parameter is not used, and the value for this parameter is set to zero. When this is the case, the pdf expression reduces to that of the *two-parameter Weibull distribution*. In this thesis we use two-parameter Weibull distribution. The following plot shows the effect of different values of β value on the Weibull failure rate. As shown by the plot, Weibull distributions with $\beta<1$ have a failure rate that decreases with time, also known as *infantile* or *early-life* failures. Weibull distributions with β close to or equal to one have a fairly constant failure rate, indicative of *useful* *life* or *random* failures.

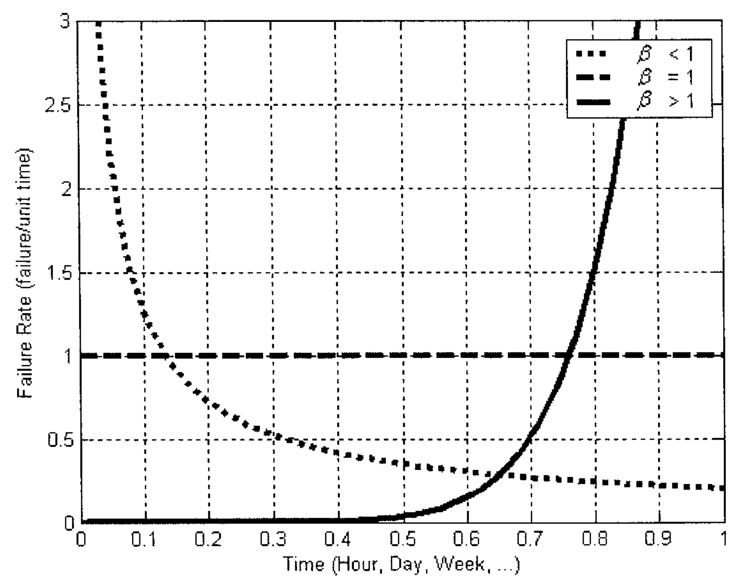

Figure 1.2: Weibull failure rate with $0 < \beta < 1, \beta = 1$ **and** $\beta > 1$

Weibull distributions with $\beta > 1$ have a failure rate that increases with time, also known as *wear-out* failures. These include the three sections of the *bathtub curve.* Figure 1.3 exhibits an example of a bathtub curve.

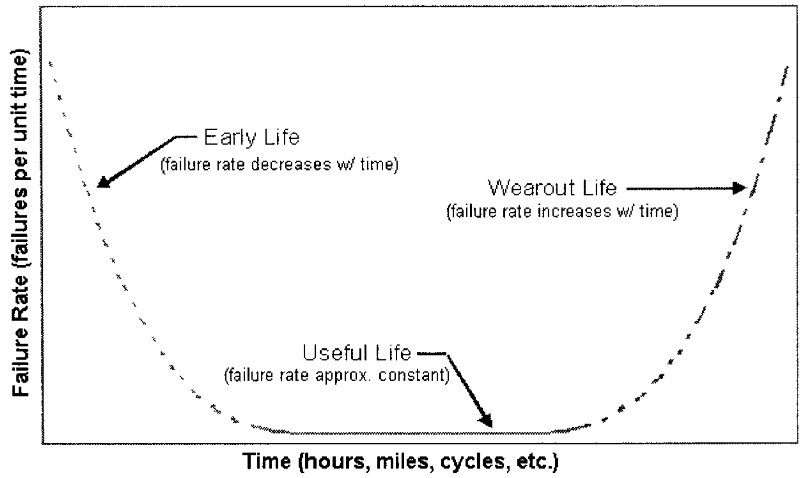

Figure 1.3: An example of a bathtub curve

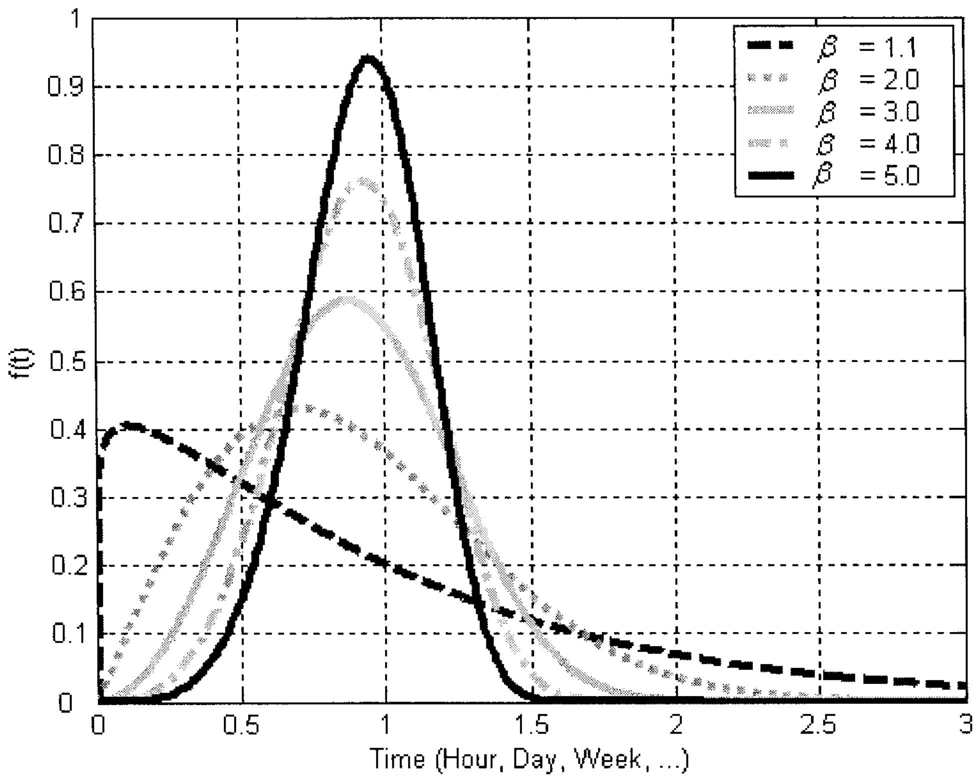

Figure 1.4: Weibull pdf plot with varying the value of β while $\beta > 1$

Increasing the value of β while $\beta > 1$ and holding α constant, has the effect of stretching in the pdf while moving the mass of pdf to the right. Since the area under a pdf curve is a constant value of one, the peak of the pdf curve will also increase with the increase of β, as indicated in Figure 1.4. If β is increased, while α is kept the same, the distribution gets stretched in to the right and its height increases. If β is decreased, while α is kept the same, the distribution mass gets pushed in toward the left i.e. toward 0, and its height decrease.

Increasing the value of α while holding β constant, has the effect of stretching out the pdf. Since the area under a pdf curve is a constant value of one, the peak of the pdf curve will also decrease with the increase of α, as indicated in Figure 1.5. If α is increased, while β is kept the same, the distribution gets stretched out to the right and its height decreases, while maintaining its shape and location. If α is decreased, while β is kept the same, the distribution gets pushed in toward the left i.e. toward 0, and its height increases. α has the same unit as t, such as hours.

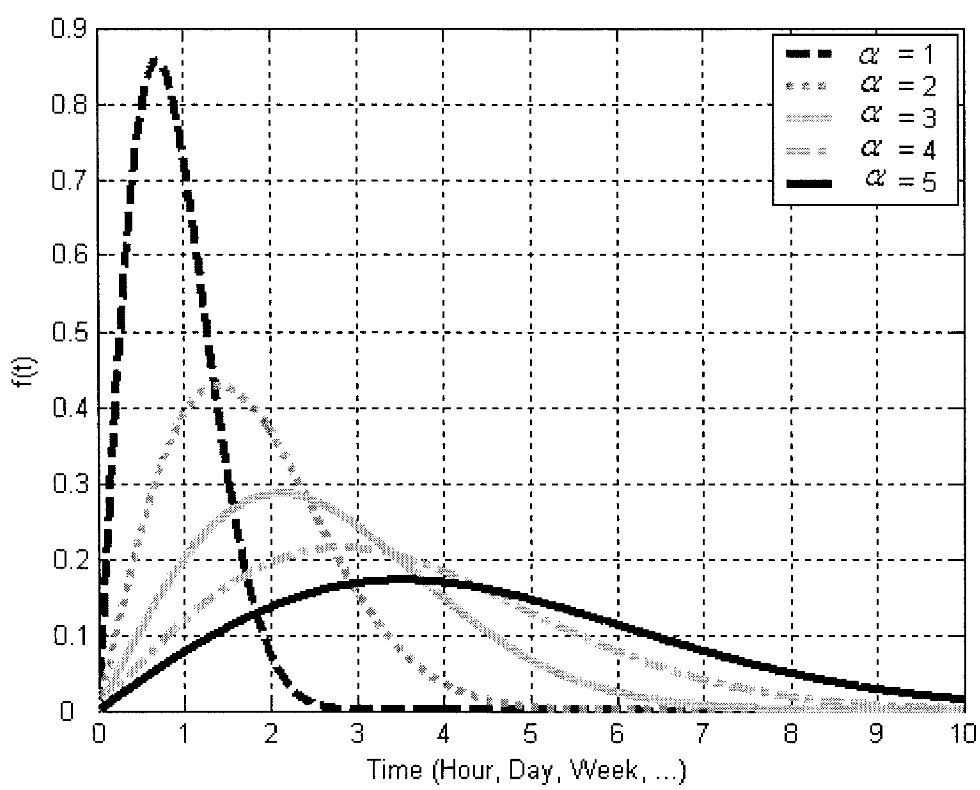

Figure 1.5: Weibull pdf plot with varying the value of α

Stochastic Processes

In many situations, we want to study the interaction of chance with time e.g. the number of failures in a certain period of time. To model this we need a family of random variables, all defined on the same probability space, $\{X(t);t \geq 0\}$ where $X(t)$ represents e.g. the working state of the system at time t. $\{X(t);t \geq 0\}$ is called a continuous time stochastic process or random process. For many studies, both theoretical and practical, we discretise time and replace the continuous interval $[0,\infty)$ with the discrete set $\mathbb{N}$ or sometimes $\mathbb{N} \cup \{0\}$. We then have a discrete time stochastic process, $\{X(n);n=0,1,2,\ldots\}$. $X(t)$ and X_t, and similarly $X(n)$ and X_n, are used interchangeably in the literature on this subject.

For stochastic processes, all the component random variables take values in a given set S, called the *state space*. Typically this will be a subset of $\mathbb{N}$, $\mathbb{N} \cup \{0\}$, $\mathbb{Z}$ or $\mathbb{R}$. When the random variable at time t takes an amount i from the state space, i.e. $X(t)=i$, we say that the *state* of the system at time t is i.[Ross 1997]

Markov Process

In general a stochastic process has the *Markov property* if, given the present state, the future state is conditionally independent of the past states. Many of the most popular stochastic processes used in both practical and theoretical work are supposed to have this property. If the states take on value in $\mathbb{R}$, we have a *Markov process*, if they take amount from $\mathbb{N}$ or $\mathbb{Z}$, we are dealing with a *Markov chain*. If we discretise time to be $\{0,1,2,\ldots\}$, we'll be working with *discrete time* Markov chains or process. For more details see [Ross 1997].

The probability that $X(t+1)=j$ given that $X(t)=i$ is called the *one-step transition probability* and we write:

$$p_{ij}^{t,t+1} = \Pr\left(X(t+1)=j \mid X(t)=i\right)$$

We have *stationary transition probabilities*, if $p_{ij}^{t,t+1}$ is the same for all t i.e. $p_{ij}^{t,t+1} = p_{ij}$ for any t, so the probability of going from i to j in one transition at any time is the same.

We can collect together all the transition probabilities into a matrix $P=\left[p_{ij}\right]$ called the *transition probability matrix* or sometimes *transition matrix* for short. [Ross 1997]

Dynamic Programming

An approach to solving dynamic optimization problems was pioneered by Richard Bellman in the late 1950s. This approach has been applied to problems in both continuous and discrete time. It is developed to solve sequential, or multi-stage, decision problems; hence, the name *dynamic* programming. [Bertsekas 1976] Dynamic Programming principle is given by Bellman [1920]:

> *"An optimal policy has the property that whatever the initial state and the initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."*

To show general idea of dynamic programming approach, consider the following diagram:

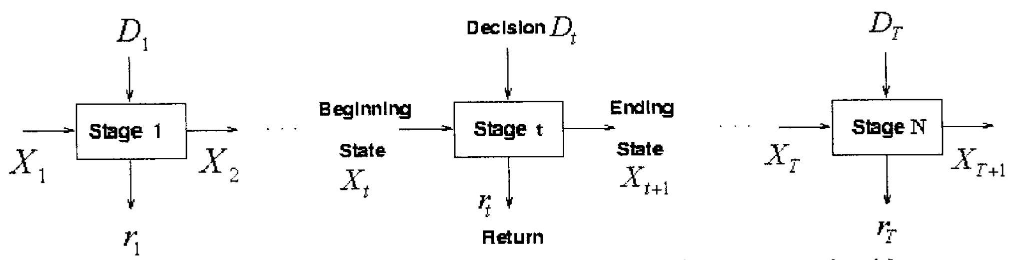

Figure 1.6: Schematic representation of dynamic programming idea

Knowing the state of the process at the beginning of a stage, say stage t, we make a decision which results in a specific return benefit/cost and changes the state to the ending state at the end of the stage. The objective is to maximize/minimize the total return over all the stages. The conceptual framework is as follows:

- We observe a system whose primary state, X_1, is known.
- We make a decision (action), D_1, which makes the system to change its state to a state X_2 by the transition function t_1 i.e. $X_2 = t_1(X_1, D_1)$. The transition's return is $r_1 = r_1(X_1, D_1)$.
- We make a second decision, D_2, upon which the system changes its state to $X_3 = t_2(X_2, D_2)$. The stage's return is $r_2 = r_2(X_2, D_2)$.

This process continues:

- After a number of iterations the system will be in state X_t and we make the t-th decision, D_t, by which the system will change its state to $X_{t+1} = t_t(X_t, D_t)$ and the stage returns $r_t = r_t(X_t, D_t)$.

There are finite number of possible states and decisions available. The set of all possible states is supposed to be S and the decisions come from the finite set of A. This would ultimately result in the fundamental deterministic recursion equations as follows: [Nemhauser 1966]

$$f_t(X_t) = \max_{D_t} Q_t(X_t, D_t), \quad t = 1,2,\ldots,T$$

$$Q_t(X_t, D_t) = \begin{cases} r_T(X_T, D_T) & t = T \\ r_t(X_t, D_t) + f_{t+1}(t_t(X_t, D_t)) & t = 1,\ldots,T-1 \end{cases}$$

Stochastic Dynamic Programming

An T-stage stochastic system is similar to an T-stage deterministic system except that at each stage there is a random variable, k_t, that affects the stage transformation and return.

$$r_t = r_t(X_t, D_t, k_t)$$

$$X_{t+1} = t_t(X_t, D_t, k_t), n = 1,\ldots,T$$

The random variables $k_1,\ldots,k_T$ are assumed to be independently distributed with probability distributions $p_1(k_1),\ldots,p_T(k_T)$ respectively. By defining the expected value return from stage t through T, $\overline{f}_t$, and applying probability rules e.g. see [Nemhauser 1966] fundamental stochastic recursion equations will be as follows:

$$\overline{f}_t(X_t) = \max_{D_t} \sum_{k_t} p_t(k_t) Q_t(X_t, D_t, k_t), \quad t = 1,2,\ldots,T$$

$$Q_t(X_t, D_t, k_t) = \begin{cases} r_T(X_T, D_T, k_T) & t = T \\ r_t(X_t, D_t, k_t) + \overline{f}_{t+1}(t_t(X_t, D_t, k_t)) & t = 1,\ldots,T-1 \end{cases}$$

Introduction of uncertainty causes no increase in amount of the state variables. Since Q_t is a function of only one random variable, k_t, some difficulties of optimizing functions with several random variables have been eliminated. The optimal decision policy, resulting from stochastic multistage optimization, is itself stochastic, except for the first optimal decision, $D_1^*(X_1)$. The rest, $D_2^*(X_2),\ldots,D_T^*(X_1)$, can not be expressed deterministically in terms of X_t, until the stochastic elements that precede them are revealed. This is not a deficiency of dynamic programming, but a property of stochastic multistage system. For more details see [Bertsekas 1976].

Markovian Decision Processes

Markovian Decision Processes represent a class of stochastic optimization problems. *MDP* is based on the Markov Process. It is assumed that there are a finite number of states at each stage and a finite number of stages. The states at stage t are denoted by i, $i=1,\ldots,N$ and $t=1,\ldots,T$. As explained above, the probability of transition from state i at stage t to state j at stage $t+1$ is denoted by p_{ij} and is independent from t. These probabilities can be represented by transition matrix P:

$$P=\begin{pmatrix} p_{11} & \cdots & p_{1j} & \cdots & p_{1N} \\ \vdots & & \vdots & & \vdots \\ p_{i1} & \cdots & p_{ij} & \cdots & p_{iN} \\ \vdots & & \vdots & & \vdots \\ p_{N1} & \cdots & p_{Nj} & \cdots & p_{NN} \end{pmatrix}$$

The probability of being in state j at stage $t+1$, denoted by π_j^{t+1}, is determined from:

$$\pi_j^{t+1} = \sum_{i=1}^{N} \pi_i^t p_{ij}, j = 1,\ldots, N, t = 1,\ldots,T$$

Corresponding to the transition matrix P, there is a return matrix R:

$$R = \begin{pmatrix} r_{11} & \cdots & r_{1j} & \cdots & r_{1N} \\ \vdots & & \vdots & & \vdots \\ r_{i1} & \cdots & r_{ij} & \cdots & r_{iN} \\ \vdots & & \vdots & & \vdots \\ r_{N1} & \cdots & r_{Nj} & \cdots & r_{NN} \end{pmatrix}$$

which gives the return r_{ij} from transition from state i to state j. A decision variable $d_t = k,\ k = 1,\ldots,K$ designates the choice of the k-th transition matrix and k-th return matrix at t-th stage, in other words, if the system is in state i, $d_t = k$ means that the relevant transition probabilities and returns at stage t are the i-th row of the k-th transition and return matrices. The probability of transition is denoted by $p_{ij}(d_t)$ and the return by $r_{ij}(d_t)$.

This is simply a variation of multistage stochastic optimization model given in previous section where $p_{ij}(d_t) = t_t(X_t, D_t, k_t)$ and $r_{ij}(d_t) = r_t(X_t, D_t, k_t)$. The state variable, X_t, is presented by i and decision variable, D_t, by d_t. The random variable k_t is hidden in the new notation. We denote the total excepted return from stage t through stage T, starting in state i, by $\overline{f}_t(i)$. The stochastic recursion equations will be as follows:

$$\bar{f}_t(i) = \max_{d_t=1,\ldots,K} \sum_{j=1}^{N} p_{ij}(d_t)\left[r_{ij}(d_t) + \bar{f}_{t+1}(j)\right], \quad t = 1,2,\ldots,T, i = 1,\ldots,N$$

The term $\sum_{j=1}^{N} p_{ij}(d_t) r_{ij}(d_t)$ is the expected return from stage t. We denote it by $q_i(d_t)$, so:

$$\bar{f}_t(i) = \max_{d_t=1,\ldots,K} \left[q_i(d_t) + \sum_{j=1}^{N} p_{ij}(d_t) \bar{f}_{t+1}(j) \right], \quad t = 1,2,\ldots,T, i = 1,\ldots,N$$

These stochastic recursion equations remain to be solved by the usual computational methods used in DP. For more details see [Howard 1960].

Infinite stage MDP

Problems containing an infinite number of decisions arise in two fundamental different ways. First are the cases that arise when there are a very large number of stages remaining and there is regularity in the stage returns and transformations in a way we expect the optimal decision to be independent of the particular stage number. In the second case, the horizon is infinite, but the time periods (stage) are very small and negligible in comparison with the horizon. In the limit, as the size of the time periods approaches zero, we assume that decisions are made continuously. The former is called a *discrete infinite-stage process* and the later a *continuous infinite-stage decision process.* Under certain circumstances, the solution to the infinite-stage problem is stated as:

$$f(X)=\lim_{T\to\infty} f_t(X_t)=\lim_{T\to\infty}\left\{\max_{D_t}\left[r_t(X_t,D_t)+f_{t+1}\left(t_t(X_t,D_t)\right)\right]\right\}$$
$$=\max_{D_t}\left[r(X,D)+f\left(t(X,D)\right)\right]$$

1.3 Literature Review

Under age replacement policies, a device is replaced or overhauled at failure or at a predetermined age. See [McCall 1965] and [Valdez-Flores and Feldman 1989] for some examples. Modeling the lifetime of a device whose failure depends upon the effects of time and usage has also received a great attention in the past decade. Scott *et al.* [2003] have considered a system whose age is measured in two time scales e.g. automobiles in the parallel scales of calendar time since purchase, and number of miles driven. Lawless *et al.* [1995], Murthy *et al.* [1995] and also Singpurwalla and Wilson [1998] have considered this case. Some literature reviews on maintenance optimization in general can be found in [McCall 1965], [Valdez-Flores 1989] and [Dekker 1996].

A condition-based maintenance policy performs generally better than an age based one, e.g. see [Rao 1996] and [Gertsbakh 2000]. Scarf [1997] believes that "the increase in the use of condition monitoring techniques within industry has been so extensive that it perhaps marks the beginning of a new era in maintenance management". Condition-based maintenance has been addressed in several papers, for some examples see [Christer and Wang 1992], [Scarf 1997], and [Wang 2000]. In most of the papers either the critical threshold for replacement or the inspection interval is a decision variable except for [Wang 2000] in which a result based on renewal theory is used to calculate the cost criterion in terms of both these decision variables. Also Dieulle *et al.* [2003], using a Gamma process, develop a model which allows to investigate the joint influence of the critical threshold value and the choice of the inspection dates on the total cost of the maintained system.

In general, many existing models of CBM policies are based on a continuous-time discrete-state Markovian deterioration process and focus on determining the states in which the system should be replaced to get the minimum expected cost. Mostly the inspection period and/or the critical states are optimized by applying the Markovian decision process. Coolen and Dekker [1995] optimize the interval between successive condition measurements (inspections), where measurements are expensive and cannot be made continuously. Lam and Yeh [1994] determine an optimal inspection & replacement policy such that the mean long-run average cost is minimized. For more instances on this approach see: [Mine and Kawai 1975], [Ohnishi *et al.* 1986], [Tijms and Schouten 1984], and [Wijnmalen and Hontelez 1992].

Second group of researches concern with continuous state processes. Hontelez *et al.* [1996] consider a continuous-time, continuous-state deterioration process. In the model, it is assumed that the relation between the condition of the system and its age is more or less known and, additionally, that the condition can be observed. Park [1998, 1998a] considers equipment that fails when it wears beyond a certain breakdown threshold and the wear accumulates continuously but difficult to monitor continuously. Chelbi and Ait-Kadi [1999] address the problem of generating optimal inspection strategies for equipment with the failure which is obvious only through inspection. A situation where it is possible to identify one parameter well correlated with the equipment deterioration process is considered. Generally, in this group, the aim is either to calculate the optimal inspection period while the critical threshold is given or to find the optimal threshold when the inspection period is prefixed. For more instance see [Hopp and Kuo 1998], [Christer and Wang 1992, 1995], [Barbera *et al.* 1996], [Wang 2000], [Wang and Christer 2000], [Christer *et al.* 1997] and [Aven 1996].

The more the information on the system is close to its real state, the more the policy is efficient e.g. see [Barros *et al.* 2002b]. The ideal case, widely studied in the literature, is when the system information is complete and perfect, i.e. the state of the system like running or failed, degradation level... is perfectly known, e.g. see [Cho and Parlar 1991]. Christer and Wang [1992, 1995] consider particular problems of directly monitored systems. Grall *et al.* [2002] have found the optimum threshold and inspection schedule jointly for a system releasing perfect information.

Condition based maintenance decisions in practice are based largely upon measures of the condition of the system obtained at monitoring time. These measures can likely contain noises and, in general, may not tell directly the exact condition of the monitored system. They are, however, assumed to be stochastically linked with the actual condition. This type of condition monitoring is called indirect condition monitoring which provides imperfect information or partial information in contrast to direct monitoring which provides perfect information or complete information. In many realistic situations, the system information is imperfect. For instance Rosqvist [2002] formulates a stopping time model, using experts' judgments on the residual operating time of the system. The judgment is based on an indication of the system state which releases imperfect information about system state. The objective is to maximize expected utility. Experts are asked to provide percentage information on residual lifetime of the system, given the indication of the system state.

There are two general approaches regarding the use of observations' information in Condition Based Maintenances. The first approach, considers just the current information obtained from the system observation. For instance, Christer *et al.* [1997] present a case study of furnace erosion prediction and replacement. A state space model is used to predict the erosion condition of the inductors in an induction furnace in which a measure of the conductance ratio is used to indirectly assess the relative condition of the inductors, and to guide replacement decisions. Campodonico and Singpurwalla [1994] use a Bayesian approach considering the vibration of the system as the covariant of the equipment working state. [Zilla 1993] considers the number of the defective items as the observable covariant of the system real state. The history of the process at each period contains the number of the defective items for each of the previous periods, and the decision made in each previous periods. Christer and Wang [1995] addressed the problem of condition monitoring of a component which has available wear as a measure condition. Supposing that the past measurements of the wear are available up to the present, and the component is still active (working), the decision problem is to choose an appropriate time for the next inspection based upon the condition information obtained to date. In other words, it generalizes the case of systems subject to preventive maintenance by using the entire history of monitoring and activities which we categorize as the second approach.

Proportional Hazards Model (PHM) was developed in order to take into account the effects of system condition influencing the times-to-failure of the system. The model has been broadly used in the biomedical field and recently there has been an increasing application in reliability engineering. Kumar and Westburg [1997] use PHM to identify the importance of monitored variables and estimate the reliability function using the values of monitored variables. Then the reliability function is used to estimate the optimum maintenance time interval or threshold values of monitored variables for the equipment. Jardine *et al.* [1987], and Makis and Jardine [1992] use the Proportional Hazard Model to model deterioration behavior of the system in condition based replacement problems and to find the optimal replacement policy in form of an recursive computational algorithm to minimize the expected average cost in long-run.

No significant work has been done on a Condition Based Maintenance with imperfect information while the deterioration of the system is modeled using the Proportional Hazards Model. In the next part, the general framework of the considered problem is explained.

1.4 Problem Statement

In this thesis we concentrate on a Condition Based Maintenance with Imperfect Observations for a piece of equipment which is operating continuously. Without intervention by the decision maker and before the occurrence of the failure, equipment's working state undergoes a stochastic multi-stage deterioration which forms a discrete homogeneous Markov chain with a finite state space. The Markov Chain has a known transition probability matrix. The working state of the equipment is not observable. An indicator of the equipment's working state is observed at discrete time epochs and a decision is made at the same time. The measured indicator at observation epoch is stochastically related to the working state of the equipment. A set of pre-known probabilities defines the stochastic relation between the working state and the observed indicator. Although equipment's working state is not outwardly observable, the failure state is observable. If the equipment fails, the failure causes the equipment to cease functioning. The failure may happen at any time

All the past history of the equipment's observations and the decisions (actions) performed are used in order to estimate the equipment working state. In our problem, the failure of the equipment depends on its working state as well as the age of the equipment i.e. the failure rate of the equipment follows the Proportional Hazard Model. Decision would be whether to replace the equipment preventively or leave the equipment to work until the next observation point or until the failure, whichever happens first.

The ultimate objective of this research is to find the optimal cost replacement policy and the long-run average cost of the maintenance system. Partially Observed Markov Decision Process, Dynamic Programming, and the Bayes' rule are the main tools used in this thesis.

The remainder of this thesis is organized as follows: In chapter 2, the problem is formulated and the optimal replacement policy is found. We also present a recursive method which is used to find the optimal average cost of the system over an infinite horizon. A numerical example along with the conclusions and some possible areas for future works are presented in Chapter 3. In the Appendices, the Matlab codes written in order to solve the examples are presented.

CHAPTER 2 :

PROBLEM FORMULATION AND OPTIMAL POLICY

2.1 Problem Formulation

We consider deteriorating equipment which is subject to random failure. The working state of the equipment is illustrated by a finite set of non-negative integers, i.e. a state space $S=\{1,2,,\ldots,N\}$. State 1 indicates the best possible state for the equipment. It means that the equipment is new or like new. The working state process $\{X(t)=1,2,\ldots,N\}$, is a discrete time homogeneous Markov chain with N unobservable states. Bigger the working state, worse the condition of the equipment. State 1 indicates the best possible state and state N is the indicator of the worst possible state of the equipment.

The working states of the equipment are not observable except for the time $t=0$ where the state of the equipment is certainly 1. An indicator, θ, of the equipment working state is observed at fixed interval, Δ, i.e. the equipment is observed in order to obtain the indicator at times; $t=\Delta,2\Delta,\ldots$. The information obtained can take a value in a finite set of non-negative integers, i.e. $\theta\in\Theta=\{1,2,\ldots,M\}$. It is supposed that a value of θ is observed with a known probability, $q_{j\theta}$, when the working state of the equipment is j. Q presents the stochastic matrix which specifies these probabilities, i.e. $Q=\left[q_{j\theta}\right]$, $j\in S$, $\theta\in\Theta$. In this thesis, equipment state and equipment working state are used interchangeably. Failure is not considered as a working state. It is a condition that causes the equipment to cease functioning and is outwardly obvious.

Since the information θ is available only at discrete points of time, $t = \Delta, 2\Delta, \ldots$, the equipment state $X(t)$ is estimated at these points, and a decision (an action) takes place at the same points. The transition probability of the state process is given by the stochastic matrix $P = \left[p_{ij}\right]$ $i, j \in S$, where p_{ij} is the one step transition probability from state i to state j, where $i, j \in S$. $p_{ij} = \Pr\left(X(t+\Delta) = j \mid X(t) = i\right)$, $t = 0, \Delta, 2\Delta, \ldots$. We assume that P is an Upper Triangular matrix, i.e. $p_{ij} = 0$ when $j < i$. This implies that the equipment will not improve by itself, which is the case in most practical problems.

The failure of the equipment which are immediately obvious and cause the system to cease functioning can happen at any time. This failure-characteristic is referred to as *obvious failures* as opposed to *silent failures*, which are not immediately obvious and must be discovered through some actions, e.g. see [Maillart 2004]. If the failure happens, it is immediately recognized and the only possible action is "Failure Replacement". At any observation point, we can decide whether to perform "Preventive Replacement" or "Do-Nothing". "Failure Replacement" and "Preventive Replacement" renew the equipment and return it to state 1. If the equipment is replaced preventively, it will cost C, while a failure replacement will cost $K + C$, $K, C > 0$.

If a failure replacement happens, the equipment will still be observed at the first next scheduled observation point. In both cases of failure replacement and preventive replacement, the replacements are instantly. Our objective is to find the optimal replacement policy to minimize the long run expected cost per unit time of maintenance system.

In this thesis we assume that the matrices Q and P are the model parameters and are known. Adjengue and Yacout [2005] have developed a recursive procedure for parameter estimation based on the Expectation–Minimization (EM) algorithm [Dempster 1976]. With this algorithm, the parameters are re-estimated when the new information on the equipment condition becomes available, and the updated estimates are used to re-calculate the posterior state estimates.

2.1.1 Formulation of the Optimal Replacement Policy

In this model, the failure rate of the equipment is considered to be following the Proportional Hazard Model. In the PHM the failure rate of the equipment is a product of two independent functions, $h_0(.)$ and $\psi(.)$, where $h_0(.)$ is a function of the equipment age and $\psi(.)$ is a function of the equipment state. This means that the failure rate of the equipment depends on the equipment age as well as the working state of the equipment and is given by:

$$h\left(t, X(t)\right) = h_0(t)\psi\left(X(t)\right), \; t = 0, \Delta, 2\Delta, \ldots \qquad \textbf{(2-1)}$$

Accordingly, the conditional reliability of the equipment is:

$$\begin{aligned} R(k, X_k, t) &= P\left(T > k\Delta + t \mid T > k\Delta, X_1, X_2, \ldots, X_k\right) \\ &= \exp\left(-\psi\left(X_k\right)\int_{k\Delta}^{k\Delta+t} h_0(s)ds\right) \end{aligned} \tag{2-2}$$

where $X_k = X(k\Delta)$. The conditional reliability of the equipment indicates the probability of not having the failure in time t after $k\Delta$, given that failure has not happened till time $k\Delta$ and the states of the equipment have been $X_1, X_2, \ldots, X_k$ at times; $\Delta, 2\Delta, \ldots k\Delta$. Also the conditional mean sojourn time of the equipment, if no action is performed while the equipment is in state X_k at period k, is: [Makis and Jardine 1992]

$$\tau\left(k, X_k, \Delta\right) = \int_0^{\Delta} R\left(k, X_k, t\right) dt \tag{2-3}$$

where T is the random variable indicating the failure time of the equipment.

Since the equipment's working state is unknown it is estimated from all past history of the equipment which includes performed actions and all observed values of the indicators. Using this estimation, the decision is taken whether to replace the equipment or to do nothing until the next observation point.

To make a decision, all the past information of the equipment is taken into account as well as the current observation. The conditional probability distribution of the equipment state is calculated. The problem is formulated as a Partially Observed Markov Decision Process (POMDP).

2.1.2 The formulation of the POMDP

State space:

We define π^k to be the conditional probability distribution of the equipment state at period k.

$$\pi^k = \left\{ \pi_i^k;\ \ 0 \le \pi_i^k \le 1 \text{ for } i = 1,\ldots,N, \sum_{i=1}^{N} \pi_i^k = 1 \right\},\ k = 0,1,2,\ldots \qquad \textbf{(2-4)}$$

where π_i^k represents the probability of being at state i at the k-th observation point. Also we define:

$$\pi_i^0 = \begin{cases} 1 & i = 1 \\ 0 & 1 < i \le N \end{cases} \qquad \textbf{(2-5)}$$

This would mean that new or renewed equipment will be at state 1.

Decision space:

$\{0, \infty\}$ is considered to be the decision space of the model, where 0 means to replace the equipment immediately and ∞ means to "Do-Nothing" till next observation point or to replace at the failure time, if it happens before next observation point.

State transition:

The probability of observing a certain value of θ at the $k+1^{\text{st}}$ observation epoch while the conditional probability distribution of the equipment state at the k-th period is π^k is calculated as follows:

$$\Pr(\theta \mid k, \pi^k) = \sum_{i=1}^{N} \sum_{j=1}^{N} \pi_i^k p_{ij} q_{j\theta} \qquad \textbf{(2-6)}$$

After an observation is collected, the prior conditional distribution of the equipment working state is update to $\pi_j^{k+1}(\theta)$. Using the Bayes' formula, and by knowing that the observation θ has occurred at the $k+1$ observation point, the $\pi_j^{k+1}(\theta)$ is calculated as follows:

$$\pi_j^{k+1}(\theta) = \frac{\sum_{i=1}^{N} \pi_i^k p_{ij} q_{j\theta}}{\sum_{i=1}^{N} \sum_{l=1}^{N} \pi_i^k p_{il} q_{l\theta}}, \; j = 1, \ldots, N \qquad \textbf{(2-7)}$$

The updated conditional distribution will be carrying all the observations and actions history from the last replacement point. After any preventive replacement or any failure replacement the period counter will be reset to zero and the conditional probability distribution of the equipment state will be set to π^0.

2.1.3 Dynamic programming formulation

Let $V(k, \pi^k)$ denote minimum cost of the *renewal period*, while the system is in k-th observation point and the conditional probability distribution is π^k. Renewal period is the time between two consequent replacements, whether failure or preventive replacement. $V(k, \pi^k)$ can be stated as:

$$V(k, \pi^k) = \min\left\{ C + V(0, \pi^0), W(k, \pi^k, g) \right\} \qquad \textbf{(2-8)}$$

where $C+V(0,\pi^0)$ is the cost of the preventive replacement and $W(k,\pi^k,g)$ is the expected cost of leaving the equipment to work till next observation point and is calculated as follows:

$$W\left(k,\pi^k,g\right)=\left[K+C+V\left(0,\pi^0\right)\right]\left[1-\overline{R}\left(k,\pi^k,\Delta\right)\right]-g\overline{\tau}\left(k,\pi^k,\Delta\right)$$
$$+\left[\sum_{\theta=1}^{M}V\left(k+1,\pi^{k+1}(\theta)\right)\Pr\left(\theta\mid k,\pi^k\right)\right]\overline{R}\left(k,\pi^k,\Delta\right) \tag{2-9}$$

$\left[K+C+V\left(0,\pi^0\right)\right]$ represents the cost of the failure replacement and $\left[1-\overline{R}\left(k,\pi^k,\Delta\right)\right]$ is the probability of having the failure during the k-th period, while the conditional distribution for the equipment state at period k is π^k. g is used to represent the average cost per unit of time over infinite horizon.

In addition, $\left[\sum_{\theta=1}^{M}V\left(k+1,\pi^{k+1}(\theta)\right)\Pr\left(\theta\mid k,\pi^k\right)\right]$ gives the expected future cost of the equipment at the $k+1$ observation point, provided that the failure has not happened during the k-th period. $\overline{R}(k,\pi^k,\Delta)$ and $\overline{\tau}(k,\pi^k,\Delta)$ are the probability that the equipment is still working at the beginning of the $k+1^{\text{st}}$ period and the mean sojourn time of the equipment at the $k+1^{\text{st}}$ period respectively, when the equipment state conditional probability distribution at the k-th period, π^k, is available. They can be calculated as follows:

$$\overline{R}(k,\pi^k,t)=\Pr\left(T>k\Delta+t\mid T>k\Delta,(k,\pi^k)\right)=\sum_{i=1}^{N}R(k,i,t)\pi_i^k \tag{2-10}$$

$$\overline{\tau}\left(k,\pi^k,a\right)=\int_0^a t\overline{F}\left(dt\mid k,\pi^k\right)dt+a\overline{R}\left(k,\pi^k,a\right)=\int_0^a\overline{R}\left(k,\pi^k,t\right)dt \tag{2-11}$$

and then:

$$\bar{\tau}(k,\pi^k,\Delta)=\int_0^\Delta \bar{R}(k,\pi^k,t)dt \qquad \textbf{(2-12)}$$

Also, since g represents the long-run average cost, $g\bar{\tau}(k,\pi^k,\Delta)$ is the expected cost of the overlapped time of two consecutive replacements of the equipment when the equipment has failed and replaced as shown in Figure 2.1.

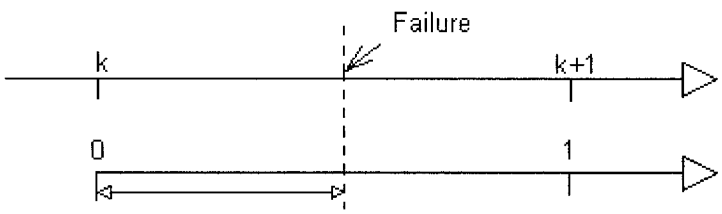

Figure 2.1: Observations after a failure replacement

To continue we need some monotonic behavior of the cost function introduced by (2-8), to be established. This is given in the following section.

2.1.4 Monotone Behavior

In this part, the condition under which the cost function introduced in the previous part has a monotonic behavior is established. Some definitions and propositions presented by [Ohnishi *et al.* 1994], [Rosenfield 1975], and [Kurano 1985] are used.

Definition 1: [Ohnishi 1994] An N-dimensional vector, x, is said to be *stochastically less* than another N-dimensional vector, y, if and only if:

$$\sum_{i=k}^{N} x_i \leq \sum_{i=k}^{N} y_i \text{ for any } k;\ 1 \leq k \leq N$$

and is denoted by $x \underset{ST}{\leq} y$.

Definition 2: [Ohnishi 1994] An N-dimensional vector, x, is said to be *less* than another N-dimensional vector, y, in *Likelihood ratio*, if and only if:

$$\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix} \geq 0 \text{ for } 1 \leq \mathrm{i} \leq \mathrm{j} \leq \mathrm{N} \text{ or equivalently } x_i y_j \geq x_j y_i \text{ for } 1 \leq \mathrm{i} \leq \mathrm{j} \leq \mathrm{N}$$

and is denoted by $x \underset{LR}{\leq} y$.

Definition 3: [Rosenfield 1995] An N-dimensional probability transition matrix P is said to be *Increasing Failure Rate, IFR,* if its rows are stochastically increasing i.e.:

$$\sum_{j=k}^{N} p_{i\,j} \leq \sum_{j=k}^{N} p_{\hat{i}\,j} \text{ for } 1 \leq k \leq N \text{ and } \hat{i} > i$$

In other words, the $\sum_{j=k}^{N} p_{i\,j}$ is non-decreasing in i for $1 \leq k \leq N$.

Definition 4: [Rosenfield 1975] An N-dimensional probability transition matrix P, is said to be *Totally Positive of Order 2, TP2,* if its rows are increasing in likelihood ratio, i.e. [Ohnishi 1994]:

$$\begin{vmatrix} p_{im} & p_{in} \\ p_{jm} & p_{jn} \end{vmatrix} \geq 0 \text{ for } \begin{cases} 1 \leq \mathrm{i} \leq \mathrm{j} \leq \mathrm{N} \\ 1 \leq \mathrm{m} \leq \mathrm{n} \leq \mathrm{N} \end{cases}$$

In our problem, we suppose that $P=\left[p_{ij}\right]$ $i,j\in S$, the Markovian probability transition matrix, and $Q=\left[q_{j\theta}\right]$ $j\in S,\theta\in\Theta$ are *TP2.*

Proposition 1: [Ohnishi 1994] Having $[a_i, 1\le i\le N]$, an N-dimensional vector with non-decreasing elements, if $x\underset{ST}{\le}y$ then $\sum_{i=k}^{N}a_i x_i\le\sum_{i=k}^{N}a_i y_i$ for $1\le$ k$\le$ N.

Proposition 2: [Rosenfield 1976] If P is *TP2* then P is *IFR.*

Proposition 3: [Ohnishi 1994] $\pi\underset{LR}{\le}\hat{\pi}\Rightarrow\pi\underset{ST}{\le}\hat{\pi}$, where π, $\hat{\pi}$ are two N-dimensional vectors.

To prove the monotonic property of the cost function introduced in the previous part we establish some lemmas. In following parts, π^k and $\hat{\pi}^k$ are two imaginary equipment state conditional probability distributions at the k-th period.

Lemma 1: If $\pi^k\underset{ST}{\le}\hat{\pi}^k$ then $\overline{R}\left(k,\pi^k,a\right)\ge\overline{R}\left(k,\hat{\pi}^k,a\right)$ for any a.

Proof:

The reliability function, $R(k,i,a)$, is non-increasing in k and i [Makis and Jardine 1992], then $1-R(k,i,a)$ is non-decreasing. Multiplying by π_i^k, $\hat{\pi}_i^k$ and summing up on all possible states, using proposition 1 we get:

$$\sum_{i=1}^{N}\left[1-R(k,i,a)\right]\pi_i^k \le \sum_{i=1}^{N}\left[1-R(k,i,a)\right]\hat{\pi}_i^k$$

$$1-\overline{R}\left(k,\pi^k,a\right)\le 1-\overline{R}\left(k,\hat{\pi}^k,a\right)$$

This means that $\overline{R}\left(k,\pi^k,a\right)\ge \overline{R}\left(k,\hat{\pi}^k,a\right)$

Lemma 2: $\overline{R}\left(k,\pi^k,a\right)$ is non-increasing in k for any a.

Proof:

$R(k,i,a)$ is non-increasing in k and i [Makis and Jardine 1992], for $k_1 < k_2$:

$$R\left(k_1,i,a\right)\ge R\left(k_2,i,a\right)\Rightarrow R\left(k_1,i,a\right)\pi_i \ge R\left(k_2,i,a\right)\pi_i \Rightarrow.$$

$$\sum_{i=1}^{N}R\left(k_1,i,a\right)\pi_i \ge \sum_{i=1}^{N}R\left(k_2,i,a\right)\pi_i$$

This means that $\overline{R}\left(k_1,\pi,a\right)\ge \overline{R}\left(k_2,\pi,a\right)$

We optionally re-express Lemmas 1 and 2 together as: $\overline{R}\left(k,\pi^k,a\right)$ is non-increasing in $\left(k,\pi^k\right)$.

Lemma 3: If $\pi^k \underset{ST}{\le} \hat{\pi}^k$, then $\overline{\tau}\left(k,\pi^k,a\right)\ge \overline{\tau}\left(k,\hat{\pi}^k,a\right)$ for any a.

Proof:

By lemma 1, since $\pi^k \underset{ST}{\leq} \hat{\pi}^k \Rightarrow \overline{R}(k,\pi^k,s) \geq \overline{R}(k,\hat{\pi}^k,s)$ then $\int_0^a \overline{R}(k,\pi^k,s) \geq \int_0^a \overline{R}(k,\hat{\pi}^k,s)$ this implies that $\overline{\tau}(k,\pi^k,a) \geq \overline{\tau}(k,\hat{\pi}^k,a)$

Lemma 4: $\overline{\tau}(k,\pi^k,a)$ is non-increasing in k for any a.

Proof:

This results directly from lemma 2 and by using the definition of $\overline{\tau}$.

Lemmas 3 and 4 together can be stated as: $\overline{\tau}(k,\pi^k,a)$ is non-increasing in (k,π^k).

Theorem 1: Assuming that assumptions 1 through 5 as stated in [Makis and Jardine 1992] are satisfied, function V introduced by equation (2-8), defined on $\overline{S}$, where $\overline{S}$ is the set of all possible variations of the pair (k,π^k), with a constant $g \geq 0$, is a bounded measurable non-decreasing function.

Proof:

We consider the restricted action space to be defined as $A_\varepsilon = \{\varepsilon,\infty\}$ where ε means taking the action in a short time. Following their assumptions they have shown that:

$$0 < \inf \tau(k,i,a) \equiv m \leq \sup \tau(k,i,a) \equiv M < +\infty$$

and by definition:

$$\bar{\tau}\left(k,\pi^{k},a\right)=\int_{0}^{a}\bar{R}\left(k,\pi^{k},t\right)\,dt=\int_{0}^{a}\left(\sum_{i=1}^{N}R\left(k,i,t\right)\pi_{i}^{k}\right)dt$$

$$\bar{\tau}\left(k,\pi^{k},a\right)=\sum_{i=1}^{N}\left(\int_{1}^{a}R\left(k,i,t\right)dt\right)\,\pi_{i}^{k}=\sum_{i=1}^{N}\tau\left(k,i,a\right)\,\pi_{i}^{k}$$

It follows that:

$$0<\inf\tau\left(k,i,a\right)\pi_{i}^{k}\leq\sup\tau\left(k,i,a\right)\pi_{i}^{k}<+\infty$$

$$0<\sum_{i=1}^{N}\inf\tau\left(k,i,a\right)\pi_{i}^{k}\leq\sum_{i=1}^{N}\sup\tau\left(k,i,a\right)\pi_{i}^{k}<+\infty$$

It means that there exist $\bar{m}$ and $\bar{M}$ in a way that:

$$0<\inf\bar{\tau}\left(k,\pi^{k},a\right)\equiv\bar{m}\leq\sup\bar{\tau}\left(k,\pi^{k},a\right)\equiv\bar{M}<+\infty \qquad \textbf{(2-13)}$$

We suppose that D is a Borel subset of $\bar{S}$ while $D\subset\bar{S}$, and we define a measure γ on D such that:

$$\gamma(D)=\begin{cases}\dfrac{1-\alpha}{\bar{M}} & \text{if } \left\{0,\pi^{0}\right\}\in D\\ 0 & \text{if } \left\{0,\pi^{0}\right\}\notin D\end{cases} \qquad \textbf{(2-14)}$$

We also define β such that; $0<1-\dfrac{(1-\alpha)\bar{m}}{\bar{M}}<\beta<1$, where $\bar{m}$ and $\bar{M}$ can be obtained from (2-13) and we define Q on Borel subset of $\bar{S}$ such that:

$$Q\left(D\,|\left(k,\pi^{k}\right),a\right)=\Pr\left(\left(k+1,\pi^{k+1}\right)\in D\,|\left(k,\pi^{k}\right)\,,a\right) \qquad \textbf{(2-15)}$$

where a is the action taking place, which in our problem represents the time to preventive replacement, while the equipment at period k has the equipment state conditional probability distribution π^k. By this definition we can write:

$$Q\left(\left\{\left(0,\pi^0\right)\right\} \mid \left(k,\pi^k\right)\ ,a\right)=\begin{cases}1 & a=\varepsilon \\ 1-\overline{R}\left(k,\pi^k,\Delta\right) & a=\infty\end{cases} \qquad \textbf{(2-16)}$$

To continue with the proof, we need two following corollaries.

Corollary 1:

$$Q\left(D \mid \left(k,\pi^k\right)\ ,a\right)\geq \overline{\tau}\left(k,\pi^k,a\right)\gamma(D) \qquad \textbf{(2-17)}$$

Proof:

We consider two separate and complimentary cases:

a) $\left(0,\pi^0\right)\in D$

b) $\left(0,\pi^0\right)\notin D$

In the first case:

$$Q\left(D \mid \left(k,\pi^k\right)\ ,a\right)\rightarrow\begin{cases}=1 & a=\varepsilon \\ \geq 1-\overline{R}\left(k,\pi^k,\Delta\right) & a=\infty\end{cases} \qquad (2\text{-}18)$$

This implies that:

$$Q\left(D \mid \left(k,\pi^k\right)\ ,a\right)\geq 1-\overline{R}\left(k,\pi^k,\Delta\right)\geq 1-\overline{R}\left(k,\pi^k,a\right)\ ,a\in A_\varepsilon \qquad (2\text{-}19)$$

The first inequality follows from the definition of Q in this case and the second, from the fact that $a = \varepsilon < \Delta$. We also note that $\overline{R}(k, \pi^k, t)$ is non-increasing in t. From (2-13):

$$\sup \overline{\tau}\left(k, \pi^k, a\right) \equiv \overline{M}$$

so that for any k, π^k, we can write:

$$\frac{\overline{\tau}\left(k, \pi^k, a\right)}{\overline{M}} \leq 1$$

$$\frac{\overline{\tau}\left(k, \pi^k, a\right)\ (1-\alpha)}{\overline{M}} \leq 1-\alpha$$

$$\overline{\tau}\left(k, \pi^k, a\right)\frac{(1-\alpha)}{\overline{M}} \leq 1-\alpha$$

and by using equation (2-14) and considering that $(0, \pi^0) \in D$:

$$\overline{\tau}\left(k, \pi^k, a\right)\gamma(D) \leq 1-\alpha \qquad \textbf{(2-20)}$$

Also from assumption 2 in [Makis and Jardine 1992], and the definition of $\overline{R}\left(k, \pi^k, a\right)$, it follows that:

$$\overline{R}\left(k, \pi^k, a\right) \leq \alpha < 1 \qquad \textbf{(2-21)}$$

Combining equations (2-20) and (2-21) we get:

$$\overline{\tau}\left(k,\pi^{k},a\right)\gamma(D)\leq 1-\overline{R}\left(k,\pi^{k},a\right) \tag{2-22}$$

Equations (2-22) and (2-19) together mean that:

$$Q\left(D\,|\left(k,\pi^{k}\right),a\right)\geq\overline{\tau}\left(k,\pi^{k},a\right)\gamma(D) \tag{2-23}$$

For the second case, where $\left(0,\pi^{0}\right)\notin D$, and by the definition given in equation (2-14), $\gamma(D)=0$ then:

$$Q\left(D\,|\left(k,\pi^{k}\right),a\right)\geq\overline{\tau}\left(k,\pi^{k},a\right)\gamma(D)=0 \tag{2-24}$$

This is always true and finishes the proof of corollary1.

Corollary 2:

$$\gamma\left(\overline{S}\right)>\frac{1-\beta}{\overline{\tau}\left(k,\pi^{k},a\right)}$$

Proof: $\left(0,\pi^{0}\right)\in\overline{S}$ so that:

$$\gamma\left(\overline{S}\right)=\frac{1-\alpha}{\overline{M}}$$

$$\gamma\left(\overline{S}\right)\overline{m}=\frac{1-\alpha}{\overline{M}}\overline{m}$$

$$1-\gamma\left(\bar{S}\right)\bar{m}=1-\frac{1-\alpha}{\bar{M}}\bar{m}$$

By definition:

$$1-\gamma\left(\bar{S}\right)\bar{m}=1-\frac{1-\alpha}{\bar{M}}\bar{m}<\beta$$

$$-\gamma\left(\bar{S}\right)\bar{m}<\beta-1$$

$$\gamma(\bar{S})>\frac{1-\beta}{\bar{m}} \qquad \textbf{(2-25)}$$

Since $0<\inf\ \bar{\tau}\left(k,\pi^{k},a\right)\equiv\bar{m}$, we can write:

$$\frac{1-\beta}{\bar{\tau}\left(k,\pi^{k},a\right)}\leq\frac{1-\beta}{\bar{m}} \text{ for any } \left(k,\pi^{k}\right) \qquad \textbf{(2-26)}$$

Combining the equations (2-25) and (2-26) mean $\gamma\left(\bar{S}\right)>\dfrac{1-\beta}{\bar{\tau}\left(k,\pi^{k},a\right)}$ which finishes the proof of corollary 2.

Corollaries 1 and 2 together are the condition (**) given by [Kurano 1985]. He has shown that under this condition, there exist a non-negative real valued function $v_{\varepsilon}(k,\pi^{k})\in D$, for the restricted action space, A_{ε}, such that $v_{\varepsilon}\left(k,\pi^{k}\right)=U^{\varepsilon}v_{\varepsilon}\left(k,\pi^{k}\right)$. The map U^{ε} is defined as:

$$U^{\varepsilon}u\left(k,\pi^{k}\right)=\min\left\{\ C+U\left(0,\pi^{0},+\infty,u\right)\ ,\ U\left(k,\pi^{k},+\infty,u\right)\ \right\}\ ,\ u\in D$$

by letting $k=0$, since $C>0$:

$$\begin{aligned} U^{\varepsilon}u\left(0,\pi^{0}\right) &= \min\left\{ C+U\left(0,\pi^{0},+\infty,u\right),\ U\left(0,\pi^{0},+\infty,u\right) \right\} \\ &= U\left(0,\pi^{0},+\infty,u\right) \end{aligned}$$

and then we can write:

$$U^{\varepsilon}u\left(k,\pi^{k}\right)=\min\left\{ C+U^{\varepsilon}u\left(0,\pi^{k}\right),\ U\left(k,\pi^{k},+\infty,u\right)\right\} \tag{2-27}$$

where;

$$\begin{aligned} U\left(k,\pi^{k},a,u\right) &= \left[K+C+u\left(0,\pi^{0}\right)\right]-g\overline{\tau}\left(k,\pi^{k},a\right) \\ &+\sum_{\theta=1}^{M}\left(u\left(k+1,\pi^{k+1}(\theta)\right)-\left(K+C+u\left(0,\pi^{0}\right)\right)\right)\Pr\left(\theta \mid k,\pi^{k}\right)\overline{R}\left(k,\pi^{k},a\right) \end{aligned} \tag{2-28}$$

for each $u\in D$ and any constant $g>0$.

Corollary 3:

For any non-decreasing function $u(k,\pi)$, where $u(k,\pi)\le K+C+u(0,\pi^{0})$ for any (k,π), $U(k,\pi,\Delta,u)$ is non-decreasing in k i.e. $U(k,\pi,\Delta,u)\ge U(k',\pi,\Delta,u)$ where $k\le k'$.

Proof:

Using equation (2-28), we can write:

$$\begin{aligned} \frac{\delta U\left(k,\pi,\Delta,u\right)}{\delta k} &= -\left[K+C+u\left(0,\pi^{0}\right)\right]\frac{\delta\overline{R}(k,\pi,\Delta)}{\delta k}-g\frac{\delta\overline{\tau}\left(k,\pi,\Delta\right)}{\delta k} \\ &+\left[\sum_{\theta=1}^{M}u\left(k+1,\pi(\theta)\right)\Pr\left(\theta \mid k,\pi\right)\right]\frac{\delta\overline{R}(k,\pi,\Delta)}{\delta k} \\ &+\frac{\delta\left[\sum_{\theta=1}^{M}u\left(k+1,\pi(\theta)\right)\Pr\left(\theta \mid k,\pi\right)\right]}{\delta k}\overline{R}(k,\pi,\Delta) \end{aligned}$$

$$\frac{\delta U(k,\pi,\Delta,u)}{\delta k}=\underbrace{\left[\sum_{\theta=1}^{M}u\left(k+1,\pi(\theta)\right)\Pr(\theta\mid k,\pi)-K+C+u\left(0,\pi^{0}\right)\right]}_{1}$$
$$\times\frac{\delta\bar{R}(k,\pi,\Delta)}{\delta k}-g\frac{\delta\bar{\tau}(k,\pi,\Delta)}{\delta k} \tag{2-29}$$
$$+\underbrace{\frac{\delta\left[\sum_{\theta=1}^{M}u\left(k+1,\pi(\theta)\right)\Pr(\theta\mid k,\pi)\right]}{\delta k}}_{2}\bar{R}(k,\pi,\Delta)$$

Now we show that term 1 is negative. Following the assumption of the corollary which states that:

$$u\left(k+1,\pi^{k+1}\right)<K+C+u\left(0,\pi^{0}\right)$$

by multiplying both side with $\Pr\left(\theta\mid k,\pi^{k}\right)$ and summing up on all possible amounts of θ,we can write:

$$u\left(k+1,\pi^{k+1}\right)\Pr\left(\theta\mid k,\pi^{k}\right)<\left[K+C+u\left(0,\pi^{0}\right)\right]\Pr\left(\theta\mid k,\pi^{k}\right)$$

$$\sum_{\theta=1}^{M}u\left(k+1,\pi^{k+1}\right)\Pr\left(\theta\mid k,\pi^{k}\right)<\sum_{\theta=1}^{M}\left[K+C+u\left(0,\pi^{0}\right)\right]\Pr\left(\theta\mid k,\pi^{k}\right)$$

$$\sum_{\theta=1}^{M}u\left(k+1,\pi^{k+1}\right)\Pr\left(\theta\mid k,\pi^{k}\right)<K+C+u\left(0,\pi^{0}\right)$$

This means that:

$$\sum_{\theta=1}^{M}u\left(k+1,\pi^{k+1}\right)\Pr\left(\theta\mid k,\pi^{k}\right)-K-C-u\left(0,\pi^{0}\right)<0$$

Now consider term 2:

$$\frac{\delta\left[\sum_{\theta=1}^{M} u\left(k+1,\pi(\theta)\right)\Pr\left(\theta \mid k,\pi\right)\right]}{\delta k} = \frac{\sum_{\theta=1}^{M} \delta\left[u\left(k+1,\pi(\theta)\right)\Pr\left(\theta \mid k,\pi\right)\right]}{\delta k}$$
$$= \frac{\sum_{\theta=1}^{M} \Pr\left(\theta \mid k,\pi\right)\delta u\left(k+1,\pi(\theta)\right)}{\delta k}$$
$$= \sum_{\theta=1}^{M} \Pr\left(\theta \mid k,\pi\right)\frac{\delta u\left(k+1,\pi(\theta)\right)}{\delta k}$$

This results that the summation will be positive or zero when $u(k,\pi)$ is non-decreasing in k.

Now again consider equation (2-29). From lemma 2 follows that $\frac{\delta \overline{R}(k,\pi,\Delta)}{\delta k} \le 0$, and from lemma 4 it follows that $\frac{\delta \overline{\tau}(k,\pi,\Delta)}{\delta k} \le 0$. So that $\frac{\delta U(k,\pi,\Delta,u)}{\delta k} \ge 0$ i.e. $U(k,\pi,\Delta,u)$ defined by equation (2-28) is non-decreasing in k.

Corollary 4:

For any non-decreasing function $u(k,\pi)$, where $u(k,\pi) \le K + C + u(0,\pi^0)$, for any (k,π), $U(k,\pi^k,\Delta,u)$ is non-decreasing in π, i.e. $U(k,\pi,\Delta,u) \ge U(k,\pi',\Delta,u)$ if $\pi \underset{LR}{\le} \pi'$.

Proof:

$$\frac{\delta U(k,\pi,\Delta,u)}{\delta\pi}=-\left[K+C+u\left(0,\pi^{0}\right)\right]\frac{\delta\overline{R}(k,\pi,\Delta)}{\delta\pi}-g\frac{\delta\overline{\tau}(k,\pi,\Delta)}{\delta\pi}$$
$$+\left[\sum_{\theta=1}^{M}u\left(k+1,\pi(\theta)\right)\Pr(\theta\mid k,\pi)\right]\frac{\delta\overline{R}(k,\pi,\Delta)}{\delta\pi}$$
$$+\frac{\delta\left[\sum_{\theta=1}^{M}u\left(k+1,\pi(\theta)\right)\Pr(\theta\mid k,\pi)\right]}{\delta\pi}\overline{R}(k,\pi,\Delta)$$

$$\frac{\delta U(k,\pi,\Delta,u)}{\delta\pi}=\left[\sum_{\theta=1}^{M}u\left(k+1,\pi(\theta)\right)\Pr(\theta\mid k,\pi)-K+C+u\left(0,\pi^{0}\right)\right]\frac{\delta\overline{R}(k,\pi,\Delta)}{\delta\pi}$$
$$-g\frac{\delta\overline{\tau}(k,\pi,\Delta)}{\delta\pi}+\frac{\delta\left[\sum_{\theta=1}^{M}u\left(k+1,\pi(\theta)\right)\Pr(\theta\mid k,\pi)\right]}{\delta\pi}\overline{R}(k,\pi,\Delta)$$

The rest of the proof is similar to same as for corollary 3, by using lemmas 1 and 3.

We suppose $u_0(k,\pi^k)=0$ for any (k,π^k) in (2-27), so u_0 is non-decreasing in (k,π^k). By corollaries 3 and 4, and also by considering that $u_n=U^{\varepsilon}u_{n-1}$, as given by [Kurano 1985], then u_1 is non-decreasing. By induction $u_n(k,\pi^k)$ is non-decreasing in (k,π^k) for any n.

Since $v_n\rightarrow v_{\varepsilon}$ when $n\rightarrow\infty$ [Kurano 1985], v_{ε} is non-decreasing as well. It is evident that $v_{\varepsilon}(k,\pi^k)\leq v_{\overline{\varepsilon}}(k,\pi^k)$ if $\varepsilon\leq\overline{\varepsilon}$. Suppose $v(k,\pi^k)=\lim_{\varepsilon\rightarrow 0}v_{\varepsilon}(k,\pi^k)$ for any (k,π^k). This implies that $v(k,\pi^k)$ is non-decreasing in (k,π^k). We note that $A_{\varepsilon}\rightarrow A$ while $\varepsilon\rightarrow 0$. Since map U is monotone [Kurano 1985], and by the monotone convergence theorem [Capinski 2004] we get:

$$\lim_{\varepsilon \to 0} U\left(k, \pi^k, +\infty, v_\varepsilon\right) = U\left(k, \pi^k, +\infty, v\right) \tag{2-30}$$

From equations (2-27) and (2-28), when $n \to \infty$ we get:

$$\lim_{n \to \infty} U^\varepsilon u_n\left(k, \pi^k\right) = \min\left\{ \lim_{n \to \infty}\left(C + U^\varepsilon u_n\left(0, \pi^k\right)\right) \ , \ \lim_{n \to \infty} U\left(k, \pi^k, +\infty, u_n\right)\right\}$$

$$U^\varepsilon v_\varepsilon\left(k, \pi^k\right) = \min\left\{ C + U^\varepsilon v_\varepsilon\left(0, \pi^k\right) \ , \ U\left(k, \pi^k, +\infty, v_\varepsilon\right)\right\}$$

and when $\varepsilon \to 0$ we get:

$$\lim_{\varepsilon \to \infty} U^\varepsilon v_\varepsilon\left(k, \pi^k\right) = \min\left\{ \lim_{\varepsilon \to \infty}\left(C + U^\varepsilon v_\varepsilon\left(0, \pi^k\right)\right) \ , \ \lim_{\varepsilon \to \infty} U\left(k, \pi^k, +\infty, v_\varepsilon\right)\right\}$$

$$v\left(k, \pi^k\right) = \min\left\{ C + v\left(0, \pi^k\right) \ , \ U\left(k, \pi^k, +\infty, v\right)\right\} \tag{2-31}$$

U and v defined by (2-30) and (2-31) respectively, represent W and V defined in (2-8) and (2-9) respectively. This finishes the proof of theorem 1.

2.2 Optimal Policy

In next part, we establish the decision criterion that helps to decide whether to replace the equipment preventively or leave it work till next observation point. This criterion is a function of the observed indicator, the age of the system, and a constant number g, the long-run average cost of the system. In the subsequent part, we introduce an iterative method to calculate the minimum long-run average cost of the system. The decision criterion and the minimum long-run average cost of the system together give the optimum decision criterion for the introduced problem.

2.2.1 Decision Criterion

From equation (2-9) we have:

$$\begin{aligned} &W\left(k,\pi^{k},g\right)-C-V\left(0,\pi^{0}\right)=K\left[1-\overline{R}\left(k,\pi^{k},\Delta\right)\right]-g\overline{\tau}\left(k,\pi^{k},\Delta\right) \\ &+\left[\sum_{\theta=1}^{M}V\left(k+1,\pi^{k+1}(\theta)\right)\Pr\left(\theta\mid k,\pi^{k}\right)-C-V\left(0,\pi^{0}\right)\right]\overline{R}\left(k,\pi^{k},\Delta\right) \end{aligned} \quad \textbf{(2-32)}$$

Since $V\left(k+1,\pi^{k+1}\right)$ is the minimum expected replacement cycle cost at the $k+1^{st}$ period, then:

$$V\left(k+1,\pi^{k+1}\right)<C+V\left(0,\pi^{0}\right)$$

$$V\left(k+1,\pi^{k+1}\right)\Pr\left(\theta\mid k,\pi^{k}\right)<\left[C+V\left(0,\pi^{0}\right)\right]\Pr\left(\theta\mid k,\pi^{k}\right)$$

$$\sum_{\theta=1}^{M}V\left(k+1,\pi^{k+1}\right)\Pr\left(\theta\mid k,\pi^{k}\right)<\sum_{\theta=1}^{M}\left[C+V\left(0,\pi^{0}\right)\right]\Pr\left(\theta\mid k,\pi^{k}\right)=C+V\left(0,\pi^{0}\right)$$

This means that:

$$\sum_{\theta=1}^{M}V\left(k+1,\pi^{k+1}\right)\Pr\left(\theta\mid k,\pi^{k}\right)-C-V\left(0,\pi^{0}\right)<0$$

$$\therefore\left[\sum_{\theta=1}^{M}V\left(k+1,\pi^{k+1}\right)\Pr\left(\theta\mid k,\pi^{k}\right)-C-V\left(0,\pi^{0}\right)\right]\overline{R}\left(k,\pi^{k},\Delta\right)<0 \quad \textbf{(2-33)}$$

From (2-32) and (2-33) it is obvious that if $K\left[1-\overline{R}\left(k,\pi^{k},\Delta\right)\right]<g\overline{\tau}\left(k,\pi^{k},\Delta\right)$ then $W\left(k,\pi^{k},g\right)-C-V\left(0,\pi^{0}\right)<0$, and consequently $V\left(k,\pi^{k}\right)=W\left(k,\pi^{k},g\right)$ from equation (2-8). It means that the optimal action is "Do-Nothing".

In what follows we will suppose that $K\left[1-\overline{R}\left(k,\pi^{k},\Delta\right)\right]\geq g\overline{\tau}\left(k,\pi^{k},\Delta\right)$ and at the same time, we suppose that the optimal action is "Do-Nothing", and we will show that there is a contradiction. If the statement is true, then from equation (2-8) we will have that

$$V\left(k,\pi^{k}\right)=W\left(k,\pi^{k},g\right)<C+V\left(0,\pi^{0}\right) \quad \textbf{(2-34)}$$

From equations (2-34) and (2-9) we can write:

$$\begin{aligned}V\left(k+1,\pi^{k}\right)-V\left(k,\pi^{k}\right)=&V\left(k+1,\pi^{k}\right)\overline{R}\left(k,\pi^{k},\Delta\right)\\&+V\left(k+1,\pi^{k}\right)\left[1-\overline{R}\left(k,\pi^{k},\Delta\right)\right]-V\left(k,\pi^{k}\right)\end{aligned}$$

$$\begin{aligned}V\left(k+1,\pi^{k}\right)-V\left(k,\pi^{k}\right)=&\left[-K-C-V\left(0,\pi^{0}\right)+V\left(k+1,\pi^{k}\right)\right]\left[1-\overline{R}\left(k,\pi^{k},\Delta\right)\right]\\&+g\overline{\tau}\left(k,\pi^{k},\Delta\right)\\&+\left[V\left(k+1,\pi^{k}\right)-\sum_{\theta=1}^{M}V\left(k+1,\pi^{k+1}(\theta)\right)\Pr\left(\theta\mid k,\pi^{k}\right)\right]\\&\times\overline{R}\left(k,\pi^{k},\Delta\right)\end{aligned}$$

$$
\begin{aligned}
V\left(k+1,\pi^{k}\right)-V\left(k,\pi^{k}\right) &= -K\left[1-\overline{R}\left(k,\pi^{k},\Delta\right)\right]+g\bar{\tau}\left(k,\pi^{k},\Delta\right) \\
&+\underbrace{\left[V\left(k+1,\pi^{k}\right)-C-V\left(0,\pi^{0}\right)\right]}_{1}\underbrace{\left[1-\overline{R}\left(k,\pi^{k},\Delta\right)\right]}_{2} \\
&+\underbrace{\left[\sum_{\theta=1}^{M}\left[V\left(k+1,\pi^{k}\right)-V\left(k+1,\pi^{k+1}(\theta)\right)\right]\Pr\left(\theta\mid k,\pi^{k}\right)\right]}_{3} \\
&\times\underbrace{\overline{R}\left(k,\pi^{k},\Delta\right)}_{4}
\end{aligned}
$$

Terms 1 and 3 are less than or equal to zero, because of the definition of $V\left(k+1,\pi^{k}\right)$ and considering that we proved it is non-decreasing. Terms 2 and 4 are bigger or equal to zero by definition. So:

$$V\left(k+1,\pi^{k}\right)-V\left(k,\pi^{k}\right)<-K\left[1-\overline{R}\left(k,\pi^{k},\Delta\right)\right]+g\bar{\tau}\left(k,\pi^{k},\Delta\right)\leq 0$$

$$\therefore V\left(k+1,\pi^{k}\right)-V\left(k,\pi^{k}\right)<0$$

Since in theorem 1 we have proved that $V\left(k,\pi^{k}\right)$ is non-decreasing in $\left(k,\pi^{k}\right)$, then $V\left(k+1,\pi^{k}\right)-V\left(k,\pi^{k}\right)\geq 0$, which is a contradiction, so equation (2-34) can not be true and $C+V\left(0,\pi^{0}\right)<W\left(k,\pi^{k},g\right)$. This means that the optimal action is "Preventive Replacement".

It can briefly be written as:

$$a\left(k,\pi^{k}\right)=\begin{cases}\infty & \text{if} \quad K\left[1-\overline{R}\left(k,\pi^{k},\Delta\right)\right]<g\overline{\tau}\left(k,\pi^{k},\Delta\right)\\ 0 & \text{if} \quad K\left[1-\overline{R}\left(k,\pi^{k},\Delta\right)\right]\geq g\overline{\tau}\left(k,\pi^{k},\Delta\right)\end{cases}$$

where $a\left(k,\pi^{k}\right)$ indicates the decision at period k while the state conditional probability distribution is π^{k}.

It is obvious that the optimal action is based on the long run expected cost per unit time of maintenance system i.e. g. In the next part, we use an iterative method to find this cost based on the problem definitions.

2.2.2 Minimum average Long-Run cost

In this part we introduce an iterative method to calculate the minimum long-run average cost of the maintenance system. Let C^{T_g} and P^{T_g} represent the expected cost and expected length of the replacement cycle associated with a replacement policy in which the time to replacement is T_g. The long run expected cost per unit of time is given as follows e.g. [Ross 1970]:

$$\phi^{T_g}=\frac{C^{T_g}}{P^{T_g}}=\frac{C+K\overline{\Pr}\left(T_g>T\right)}{\overline{E}_{\min}\left(T_g,T\right)} \qquad \textbf{(2-35)}$$

Remember that C and $C+K$ are the preventive replacement cost and failure replacement cost, respectively. $\overline{\Pr}\left(T_g>T\right)$ is the probability of a failure replacement and $\overline{E}_{\min}\left(T_g,T\right)$ is the expected average length of a replacement cycle. Above, we have proven that the time to replacement can be calculated by:

$$T_g = \Delta.\inf\left\{\ k \geq 0\ ;\ K\times\left[1-\overline{R}\left(k,\pi^k,\Delta\right)\right] \geq g\overline{\tau}\left(k,\pi^k,\Delta\right)\ \right\} \quad \textbf{(2-36)}$$

To determine the minimum average cost, g^*, instead of minimizing (2-35), we minimize (2-38) e.g. see [Aven and Bergman 1986] and [Aven 1992]:

$$C^{T_g} - gP^{T_g} = \left[C + K\overline{\Pr}\left(T_g > T\right)\right] - g\overline{E}_{\min}\left(T_g,T\right) \quad \textbf{(2-37)}$$

It has been shown by [Aven 1996] that the time to replacement T_g, where $g^* = \min\phi^{T_g}$, minimizes ϕ^{T_g}. The value of g^* is the unique solution of $g=\phi(g)$, where $\phi(g)=\phi^{T_g}$. To find T_g, we need to calculate the $\overline{\Pr}\left(T_g > T\right)$ and the $\overline{E}_{\min}\left(T_g,T\right)$. This is done next.

We define $W\left(j,\pi^j\right) = E\left\{\ \min\left(T,T_g\right) - j\Delta \,|\left(j,\pi^j\right)\right\}$ to indicate the residual time to replacement at period j i.e. at time $j\Delta$, when the conditional probability distribution of the equipment state is π^j, and with a given g, that forces the equipment to have T_g that satisfies equation (2-37).

Also we define $t_g(\pi) = \Delta\left\{\ r \in R^+ \,|\, K\left[1-\overline{R}(r,\pi,\Delta)\right] = g\overline{\tau}(r,\pi,\Delta)\ \right\}$ which calculates the real time of satisfaction of the cost condition in equation (2-37), for a given cost g, at a certain conditional probability distribution of the equipment state, π. To calculate $W(j,\pi^j)$, we consider k as shown in Figure 2.2, where:

$$(k-1)\ \Delta \le t_g\left(\pi^j\right) < k\Delta$$

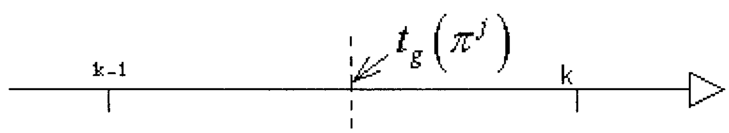

Figure 2.2: Real time of the satisfaction of the cost condition

We optionally call k "*the hazardous period*".

If $j \ge k$ (see Figure 2.3), according to the replacement criterion of equation (2-37), the equipment has to be replaced immediately, then $W(j,\pi^j) = 0$.

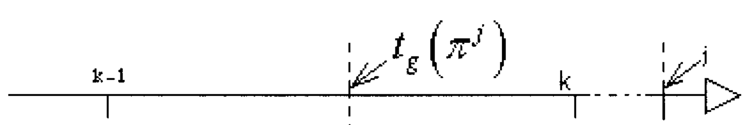

Figure 2.3 - The equipment current time has passed the "hazardous period"

In the case where $j = k-1$ (see Figure 2.4), then:

$$W\left(j,\pi^j\right) = \int_0^{t_g\left(\pi^j\right)-j\Delta} \overline{R}\left(j,\pi^j,s\right)ds$$

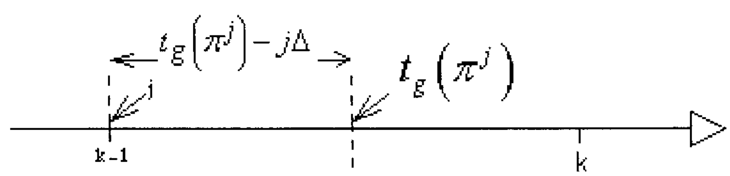

Figure 2.4 - The equipment current time is one period before the Hazardous period

And finally, if $j < k-1$ (see Figure 2.5), by conditioning the length of the replacement cycle on the failure time, we can write:

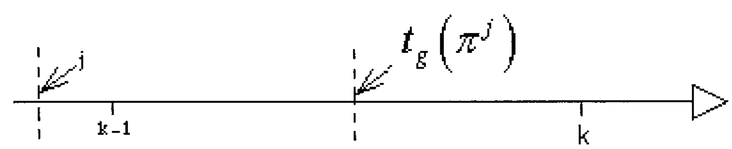

Figure 2.5 – The equipment current time is more than one period smaller than the "hazardous period"

$$W(j,\pi^j) = \underbrace{\int_{j\Delta}^{(j+1)\Delta} E\left(\min(T,T_g) \mid (j,\pi^j)\ ,\ T = s\right) d\bar{F}\left(s - j\Delta \mid (j,\pi^j)\right) +}_{1}$$
$$\underbrace{\sum_{\theta=1}^{M} E\left(\min(T,T_g) \mid T > (j+1)\Delta\ ,\theta,(j,\pi^j)\right) \Pr\left(T > (j+1)\Delta\ ,\ \theta \mid (j,\pi^j)\right)}_{2}$$
$$-j\Delta$$

where $\bar{F}\left(t \mid (k,\pi^k)\right) = 1 - \bar{R}\left(k,\pi^k,t\right)$. Term 1 indicates the expected length of the replacement cycle of the equipment in case that the failure happens between periods j and $j+1$. Term 2 is the expected length of the replacement cycle of the equipment if the failure does not happen before period $j+1$. Both terms are considering that the current conditional probability distribution of the equipment state is π^j. It is obvious that $E\left(\min\left(T,T_g\right)|\left(j,\pi^j\right),T=s\right)=s$, then:

$$\begin{aligned}W\left(j,\pi^j\right) &= \int_{j\Delta}^{(j+1)\Delta} sd\overline{F}\left(s-j\Delta\,|\left(j,\pi^j\right)\right)\\ &+\sum_{\theta=1}^{M} E\left(\min\left(T,T_g\right)|\,T>\left(j+1\right)\Delta,\theta,\left(j,\pi^j\right)\right)\\ &\times\Pr\left(T>\left(j+1\right)\Delta\,|\,\theta,\left(j,\pi^j\right)\right)\Pr\left(\theta\,|\left(j,\pi^j\right)\right)\\ &-j\Delta\end{aligned}$$

$$\begin{aligned}W\left(j,\pi^j\right) &= \int_{j\Delta}^{(j+1)\Delta} \left(s-j\Delta\right)d\overline{F}\left(s-j\Delta|\left(j,\pi^j\right)\right)+j\Delta\left(1-\overline{R}\left(j,\pi^j,\Delta\right)\right)\\ &+\sum_{\theta=1}^{M} E\left(\min\left(T,T_g\right)-\left(j+1\right)\Delta\,|\,T>\left(j+1\right)\Delta,\theta,\left(j,\pi^j\right)\right)\\ &\times\Pr\left(T>\left(j+1\right)\Delta\,|\,\theta,\left(j,\pi^j\right)\right)\Pr\left(\theta|\left(j,\pi^j\right)\right)+\left(j+1\right)\Delta\overline{R}\left(j,\pi^j,\Delta\right)\\ &-j\Delta\end{aligned}$$

$$\begin{aligned}W\left(j,\pi^j\right)=&\int_0^{\Delta} sd\overline{F}(s\,|\,(j,\pi^j))+\sum_{\theta=1}^{M} W\left(j+1,\pi^{j+1}(\theta)\right)\overline{R}\left(j,\pi^j,\Delta\right)\Pr\left(\theta\,|\,j,\pi^j\right)\\ &-j\Delta+j\Delta\left(1-\overline{R}\left(j,\pi^j,\Delta\right)\right)+\left(j+1\right)\Delta\overline{R}\left(j,\pi^j,\Delta\right)\end{aligned}$$

$$W\left(j,\pi^j\right)=\int_0^{\Delta} sd\overline{F}(s|\,(j,\pi^j))+\Delta\overline{R}(j,\pi^j,\Delta)+\sum_{\theta=1}^{M} W(j+1,\pi^{j+1}(\theta))\overline{R}(j,\pi^j,\Delta)\Pr(\theta\,|\,j,\pi^j)$$

by equation (2-11):

$$W\left(j,\pi^j\right)=\int_0^{\Delta} \overline{R}\left(j,\pi^j,s\right)ds+\sum_{\theta=1}^{M} W\left(j+1,\pi^{j+1}\left(\theta\right)\right)\overline{R}\left(j,\pi^j,\Delta\right)\Pr\left(\theta\,|\,j,\pi^j\right)$$

The expected minimum replacement cycle time can be calculated by:

$$\overline{E}_{\min}\left(T_g,T\right)=E\left\{\min\left(T,T_g\right)\mid\left(0,\pi^0\right)\right\}=W\left(0,\pi^0\right) \qquad \textbf{(2-38)}$$

Similar calculation leads to the following equations for $Q\left(j,\pi^j\right)=\Pr\left(T_g\geq T\mid\left(j,\pi^j\right)\right)$:

$$Q(j,\pi^j)=\begin{cases}0 & j\geq k\\ 1-\overline{R}(j,\pi^j,t_g\left(\pi^j\right)-j\Delta) & j=k-1\\ 1-\overline{R}(j,\pi^j,\Delta)+\sum_{\theta=1}^{M}Q(j+1,\pi^{j+1}(\theta))\Pr(\theta\mid j,\pi^j)\overline{R}(j,\pi^j,\Delta) & j<k-1\end{cases}$$

And the expected probability of having the failure before planned replacement can be calculated as follows:

$$\overline{\Pr}\left(T_g>T\right)=Q\left(0,\pi^0\right) \qquad \textbf{(2-39)}$$

CHAPTER 3 :

NUMERICAL EXAMPLE AND CONCLUSION

3.1 Numerical Example

We solve the example presented by Makis and Jardine [1992] after adapting it to our case of unobservable state of the equipment. In this example, it is assumed that equipment has a two parameter Weibull like behavior with baseline distribution hazard function with the following parameters.

$$h_0(t) = \frac{\beta t^{\beta-1}}{\alpha^{\beta}}, t \geq 0 \ , \alpha = 1, \beta = 2$$

and $\psi(X_t) = e^{0.5(X_t - 1)}$.

The hazard function of the equipment and the conditional reliability of the equipment, with $\Delta = 1$, will be as follows, consecutively:

$$h(t, X_t) = 2te^{0.5(X_t - 1)}$$

$$R(k, X_t, t) = \exp[-(t^2 + 2tk)e^{0.5(X_t - 1)}]$$

$\{X_t, \mathrm{t} \geq 0\}$ is a homogeneous Markov chain having two possible states $\{1,2\}$ with the transition probability matrix, P as follows:

$$P = \begin{bmatrix} 0.4 & 0.6 \\ 0 & 1 \end{bmatrix}$$

θ, the value of the equipment indicator, can take three possible values. The indicator and the equipment state are related by the probability distribution, Q, as follows:

$$Q=\begin{bmatrix}0.6 & 0.3 & 0.1\\ 0.2 & 0.4 & 0.4\end{bmatrix}$$

We also assume that $C=5$ and $K=2$.

To solve the problem in the perfect and imperfect cases, we have prepared two applications in Matlab 6.5. The codes for the perfect information and imperfect information can be found in Appendices A and B, respectively.

To find the optimal time to replacement, we begin with $g_1=5$ and by using equations (2-35), (2-38) and (2-39) iteratively we get:

$$\phi^{T_g}=[5+2*0.5964]/0.7285=8.5005$$

We let $g_2=8.5005$ and we continue until g stabilizes, the following results are obtained:

Table 3.1: Numerical Example solution steps

i	g	$t_g(\pi^0)$	k	$W(0,\pi^0)$	$Q(0,\pi^0)$	$\phi(g)$
1	5	0.9525	1	0.7285	0.5964	8.5005
2	8.5005	1.9146	2	0.8269	0.8733	8.1587
3	8.1587	1.8225	2	0.8174	0.8395	8.1709
4	8.1709	1.8257	2	0.8178	0.8408	8.1704
5	8.1704	1.8256	2	0.8178	0.8408	8.1704

This means that the optimal time to replacement is calculated using expected conditional reliability and expected mean sojourn time of the equipment at the observation point k

as follows:

$$T_{g^*} = \inf\left\{\ k \geq 0\ ;\ \ 2\times\left[1-\bar{R}\left(k,\pi^k,\Delta\right)\right] \geq 8.1704\times\bar{\tau}\left(k,\pi^k,\Delta\right)\ \right\}$$

Practically at period k, an observation is obtained from the equipment and the inequality is evaluated. If it is true, the preventive replacement will take place, otherwise the equipment will be left to work to the next observation point i.e. $(k+1)\Delta$ or until the failure, whichever happens first.

The long-time optimal average cost in this example is equal to 8.1704. With perfect data this cost is 8.16. There is an increase of approximately 0.01. Since we do not know the exact state of the equipment at investigation times, the cost of the optimal maintenance policy is increased as much as 0.01 per period. This is the value of the perfect information.

3.2 Analysis

To study the effect of changing the parameters' values on the long-run average cost and the difference between the cost of the optimal maintenance policy with perfect and with imperfect information, we performed a parametric analysis. We solved the problem with different values of the parameters; C, K, Δ, β, α, and Q.

Considering all problems solved with different parameters we can conclude that the results are always consistent in that optimizing with imperfect information costs more than optimizing with perfect information, e.g. see Table 3.2, Table 3.3, and Table 3.4.

It is also noted that smaller observation's interval results in bigger value of the perfect information, e.g. see Table 3.2. Significant values of the perfect information are expected to appear with smaller amount of Δ i.e. more frequent observation of the system indicator.

Table 3.2: Results for different observation intervals

Δ	Difference
1	0.009
0.8	0.010
0.7	0.012
0.5	0.013

The small difference of the long-run cost of the system between the perfect case and imperfect cases in the example can be explained by Figure 3.1.

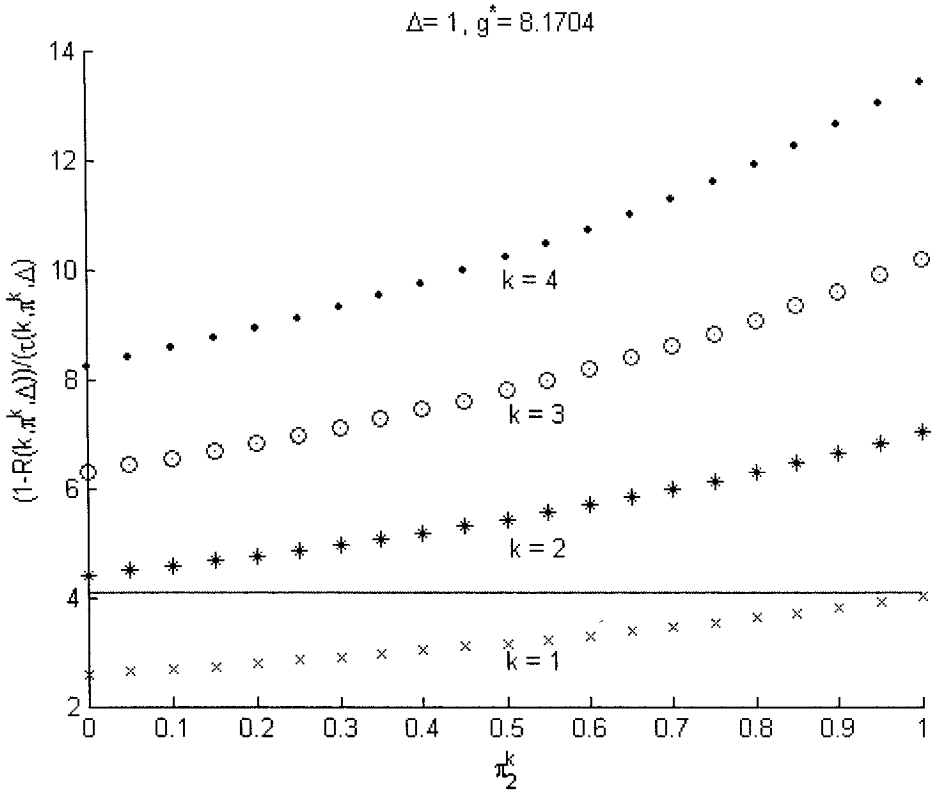

Figure 3.1: Decision Criterion behavior for $\Delta = 1$

The straight line indicates the amount of the $\frac{g^*}{K}$. It can be seen that for any possible value of π_2^1 the amount of the $\frac{1-\bar{R}(k,\pi^k,\Delta)}{\bar{\tau}(k,\pi^k,\Delta)}$ is less than the threshold $\frac{g^*}{K}$ so that the system will never be preventively replaced at the first observation point. Similar explanation shows that at $k = 2$, i.e. the second observation point, independent of value of the π_2^2, the system will be preventively replaced. In other words it means we have to replace the system at every second observation point regardless of its working state.

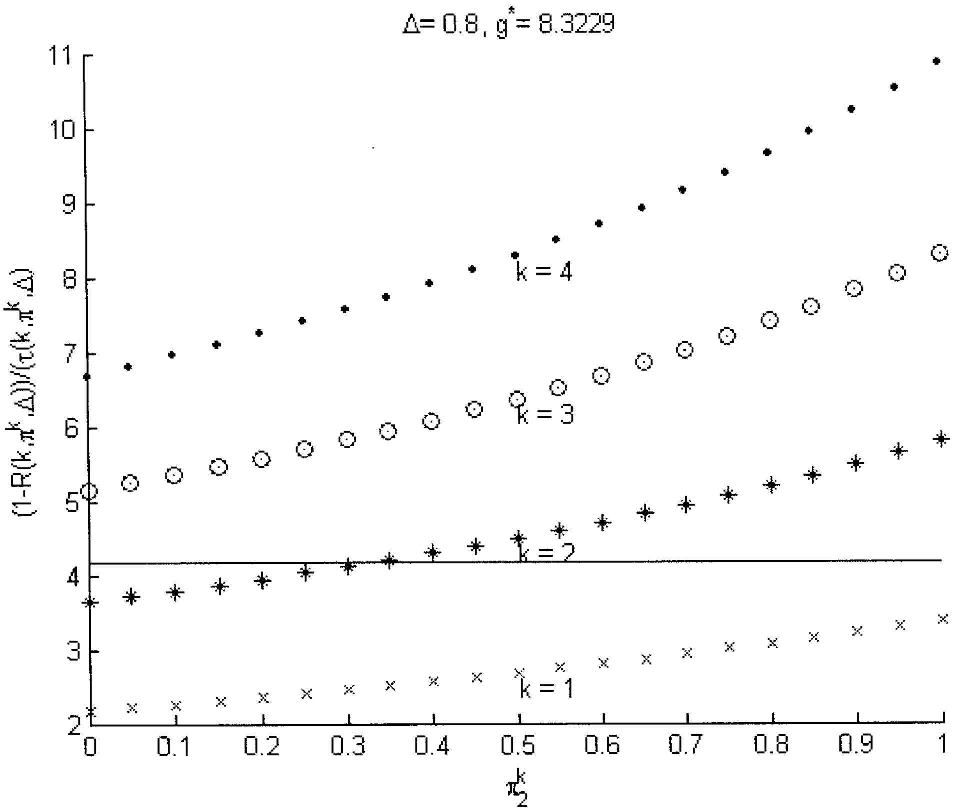

Figure 3.2: Decision Criterion behavior for $\Delta = 0.8$

This independency to the working state of the system does not exist any more when the observation period is reduced to 0.8 as shown in Figure 3.2. Here also the system will not be ever replaced preventively at the first observation point. In the second observation point, the system will be replaced preventively only if the probability of being in working state 2, π_2^2, is equal or higher than 0.35. At the third observation point the system will be replaced definitely.

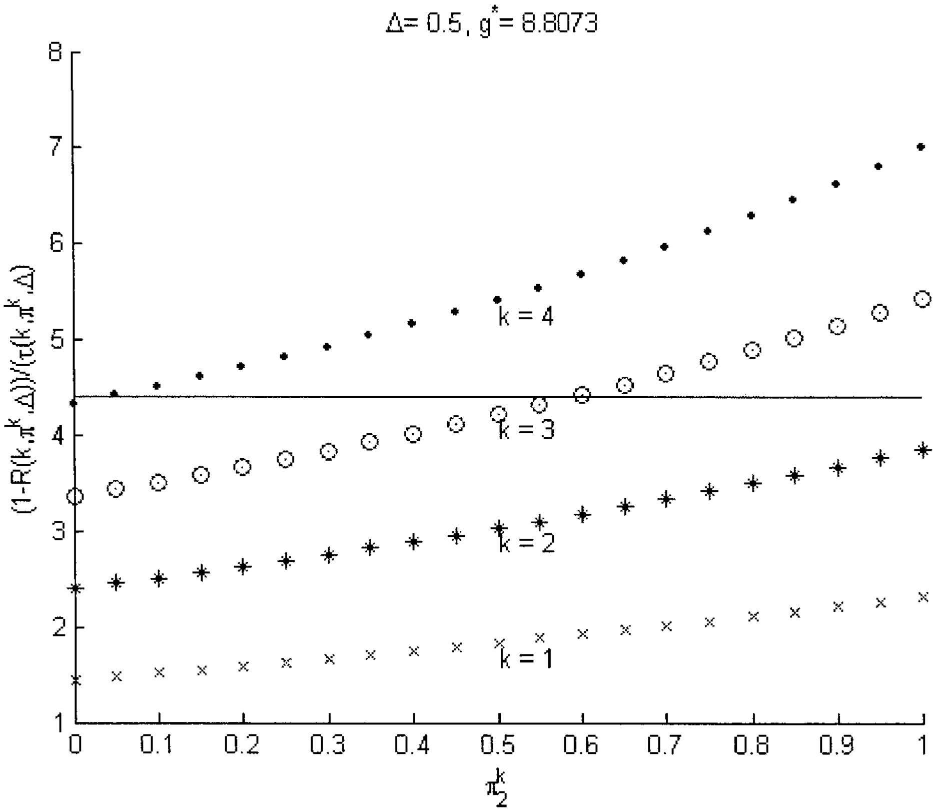

Figure 3.3: Decision Criterion behavior for $\Delta = 0.5$

Figure 3.3 shows the same pattern for $\Delta = 0.5$ where the replacement does not take place at two first observation points. The equipment will be replaced preventively on the third or the forth observation point regarding the value of π_2^k at corresponding period.

Since the method for solving the problem is an iterative method, the iteration counts augment exponentially with decreasing the inspection interval, Δ. Considering the fact that the time for solving the problem for small amount of Δ, with available hardware, was so long, we solved all the following problems with $\Delta = 1$. Here also, more significant differences between perfect and imperfect information costs with different parameters are expected in smaller inspection periods. But the behavior of the long-run average cost of the maintenance system can be explained based on these consistent differences, although the differences are small.

The values of the perfect information for different amounts of α, the scale parameter, are shown in Table 3.3. To explain the increment in long-run cost of the maintenance system while α decreases, we consider Figure 1.5, Weibull pdf plot with varying the value of α. We notice that when α decreases, while β is constant, the distribution gets pushed in toward the left i.e. toward 0, and its height increases; it means that the MTTF of the equipment is decreased.

Table 3.3: Long-Run average cost and cost difference for perfect and imperfect data with different α

α	Imperfect Cost	Difference
2.0	4.3984	0.008
1.2	6.9211	0.008
1.0	8.1704	0.009
0.8	10.0143	0.030
0.6	13.1732	0.040

This behavior can also be explained in terms of Mean and Mode equations for Weibull distribution. The Mean and the Mode of a Weibull distribution are determined by the following equations, consecutively:

$$\text{Mean} = \frac{\alpha}{\beta}\Gamma\left(\frac{1}{\beta}\right)$$

$$\text{Mode} = \alpha\left(\frac{\beta-1}{\beta}\right)^{1/\beta}$$

By decreasing α, both the Mean and the Mode of the Weibull distribution decrease. That is, more frequent failure and then shorter renewal periods. This results higher long-run average cost of the maintenance system.

Similar explanation can justify the increment of the long-run average cost of the maintenance while β decreases e.g. see Table 3.4.

Table 3.4: Long-Run average cost and cost difference for perfect and imperfect data with different β

β	Imperfect Cost	Difference
5	7.6237	0.000
2	8.1704	0.009
1.9	8.1802	0.008
1.6	8.1932	0.006

Consider the reliability function of the Weibull distribution:

$$R(t) = e^{-\left(\frac{t}{\alpha}\right)^{\beta}}$$

By decreasing β, while $\beta > 1$, the reliability of the system decreases, e.g. see Figure 3.4. This means more frequent failures and shorter renewal periods which results in increment of the long-run average cost of the maintenance system.

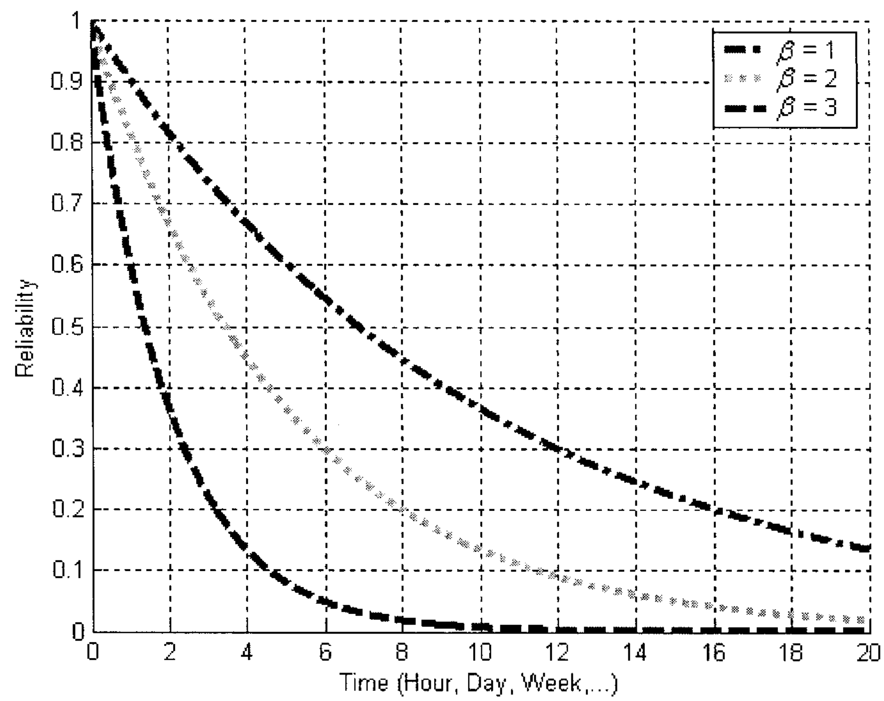

Figure 3.4: Reliability of Weibull distribution wi th different values of β.

Results of the several problems solved with different C and K is consistent in that if the both C and K increase/decrease with a specific ratio, the final minimum cost will also increase/decrease with the same ratio. Table 3.5 shows the result of these different cost parameters.

Table 3.5: Results for different value of cost parameters

C	K	Long-Run average Cost
2	5	8.1704
1	2.5	4.0852
4	10	16.3408
2	2	4.4744
5	5	11.1861
5	2	7.5606
1	5	7.0004

In a specific case, where $K = 0$, equation (2-36) would be :

$$T_g = \Delta.\inf\left\{ \; k \geq 0 \; ; \; 0 \geq g\bar{\tau}\left(k, \pi^k, \Delta\right) \; \right\}$$

Since g and $\bar{\tau}\left(k, \pi^k, \Delta\right)$ are always positive terms, this inequality will never be satisfied. This means that; there would be no preventive replacement for the equipment, which is reasonable. In absence of the failure cost, the best decision would always be to leave the system work until the failure and then replace it. In this case the long-run average cost of the system can be calculated by:

$$\phi^T = \frac{C^T}{P^T} = \frac{C}{E(T)} = \frac{C}{MTTF}$$

Where MTTF is the Mean Time To Failure of the Weibull distribution.

The numerical example was also solved for the following two values of the matrix Q:

$$Q_1 = \begin{bmatrix} 0.9999 & 0.0005 & 0.0005 \\ 0.0005 & 0.0005 & 0.9999 \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} 0.3333 & 0.3333 & 0.3333 \\ 0.3333 & 0.3333 & 0.3333 \end{bmatrix}$$

We note that the values of the probabilities in Q_1 mean that the observation information is close to the perfect case and the values of the probabilities in Q_2 mean that the data does not give any meaningful information about the working state of the equipment i.e. the equipment working state is completely unobservable. With Q_1, the optimal average long-run cost of the replacement policy is 8.16, which is the same as perfect information cost, and with Q_2, the cost is 8.18. These results are consistent in that the more the information is imperfect the more the cost of the optimal replacement policy increases. Any other matrix valid as Q, would result a cost between these two numbers e.g. $Q_2 = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$ resulted in 8.1752 as the long-run average cost.

3.3 Conclusions and Future Research

In this thesis, we derived an optimal replacement policy with condition monitoring. We determined the optimum decision to minimize the long-run average cost of the maintenance system while the main characteristics of the system are:

- The working state of the system is not observable.
- The equipment's working state undergoes a discrete homogeneous Markov chain with a finite state space.
- The Markov Chain has a pre-known transition probability matrix.
- The failure state is completely obvious and causes the system to cease functioning.
- An indicator of the equipment is observed at discrete time epochs.
- The measured indicator at observation epoch is stochastically related to the working state of the equipment. The stochastic relation matrix is pre-known.
- The failure rate of the equipment follows the Proportional Hazard Model.
- Decisions are made at the observation times. Decision would be whether to replace the equipment preventively or leave the equipment to work until the next observation point or until the failure, whichever happens first.

The probability distribution of being in the working states is estimated by Bayes' rule using all the past history of the equipment's observations and the decisions performed. A partially observable Markov decision process is formulated and a dynamic programming problem is solved. This ended up with finding the optimal replacement criterion for the equipment. The long-run average cost of the maintenance system is a parameter in the optimal criterion. Then, we introduced a recursive method to find the long-run average cost of the maintenance system.

Generally, cost of an optimal policy based on Condition Based Maintenance is less than that of an aged based maintenance. This follows the fact that in age based maintenance; the condition in which the equipment has been used is not taken in to consideration. This can result in replacing the equipment sooner than needed, or on the other hand, a piece of worn equipment may be trusted based on its age and this can result in an unexpected failure which is costly. Also the results of the numerical examples consistently show that the optimal cost of a replacement policy with imperfect information is higher than that, with perfect information i.e. the *value of perfect information* is always a positive amount. We define the value of perfect information as the difference between the average long run cost of a problem with imperfect information and the average long run cost of the problem with perfect information. This means that the quality of information is an important factor that affects the cost. It is also concluded that the value of perfect information increases consistently with decrement in observation interval. In other words, if a company wants to introduce a CBM program, it should begin with collecting good quality information that is representative of the condition of the equipment. The effects of the various parameters of the model on the optimal policy were also studied by solving several numerical examples.".

Several areas of future work and research can follow this thesis. An important area of further research would be to extend the model to find the optimal fixed inspection intervals. In reliability engineering applications, companies usually want to plan their maintenance activities in advance. In our problem we assumed that the inspections for obtaining the indicator are inexpensive and the inspection interval, which is fixed, is predefined based on company's policies. Considering a costly inspection would lead to optimization of the inspection fixed interval and replacement criteria at the same time.

Another area of future study is the case while using fixed inspection interval is not a restriction for the company. In this case, we need to set the next inspection time as well as replacement threshold to minimize the maintenance cost. In this case at each inspection point, we would decide whether to replace the system or leave it work until next observation point. This would be based on the threshold which has been set in the previous inspection. Also the next observation point would be set. The time between inspections in this case might be non-equal.

A case in which, a decision can be made among several repair possibilities, can be considered as another realistic case of study. In many practical cases, different partial repairs for the equipments which cause different costs may take place. Each specific partial repair would changes the working state of the system from a current state to a pre-known possible state. Another area of further research would be on how to develop a prediction model in order to benefit from the CBM data and at the same time plan the maintenance intervention a priori.

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APPENDIX A: CODES FOR CBM WITH PERFECT INFORMATION

Main Function (CBM.m):

```
clear all;
clc;

%Problem Parameters setting
global N K C Delta P g k t

P = zeros([N N]);                    %Transaction Markov Matrix
k = zeros([1 N]);                    %
t = zeros([1 N]);                    %

% Problem Data Entry
P=[0.4 0.6;0.0 1.0];                 % One step Transition  Matrix
N=2;                                 % Number of states
K=2;                                 % Failure Cost
C=5;                                 % Replacement Cost
Delta=1;                             % Inspection Interval

g1=9999;
g=(C+K)/Delta;                       % Beginning cost

% Beginning of calculations
while abs(g-g1) >= 0.0001            % Comparing the old and new cost
  for i= 1 : N;
    t(i)=Root(i);                    % Finding the roots
    k(i)=fix(t(i)/Delta)+1;          % Finding the related k
  end
  t
  k
  g1=g;                              % Saving Old cost
  Pr=eval(QQ(0,0))                   % Calling QQ function
  Emin=eval(W(0,0))                  % Calling W function
  g=eval((C+K*QQ(0,0))/W(0,0))       % Calculating the long-run cost
end                                  % Iteration with new cost obtained
```

$\psi(X_t)$ Function (Psi.m):

```
function Si=Psi(x)
Si=exp(0.5*x);
```

$h_0(t)$ Function (h0.m):

```
function Bh=h0(t)
Beta=2;
Alfa=1;
```

```
Bh=(Beta*t^(Beta-1))/(Alfa^Beta);
```

$h(t, X_t)$ Function (h.m):

```
function h=h(t,i)
h=h0(t)*Psi(i);
```

$R(k, X_k, \Delta)$ Function (R.m):

```
function R=R(l,y,u)
global N M K C Delta P g k t
syms q
R=exp(-Psi(y)*int(h0(q),q,l*Delta,l*Delta+u));
```

$\tau(k, X_k, \Delta)$ Function (Tu.m):

```
function Tu=Tu(k,y,u)
syms x;
Tu=int(R(k,y,x),x,0,u);
```

Root Finder Function (Root.m):

```
function Root=Root(iTemp)
global N K C Delta P g k t

BPoint=0;
EPoint=10;
Precision=0.0001;

myBegin=BPoint;
myEnd=EPoint;
myPace=1;

a1=BPoint;
f1=eval(K*(1-R(BPoint,iTemp-1,Delta))-g*Tu(BPoint,iTemp-1,Delta));
while myPace>Precision
   for x=myBegin+myPace:myPace:myEnd
      a2=x;
      f2=eval(K*(1-R(a2,iTemp-1,Delta))-g*Tu(a2,iTemp-1,Delta));
      if f1*f2>0
         f1=f2;
         a1=a2;
      else
         myBegin=a1;
         myEnd=a2;
         myPace=myPace/3;
         break
      end
```

```
  end
end
Root=(a1+a2)/2;
```

$W(j,i)$ Function (W.m):

```
function Emin = W(j,i)
global N M K C Delta P g k t

syms s
if j>= k(i+1)                          % Replacement has to happened
  Emin=0;
end
if j==k(i+1)-1                         % One period before the Hazardous period
  temp=char(R(k(i+1)-1,i,s));
  Emin=int(temp,s,0,t(i+1)-(k(i+1)-1)*Delta);
end
if j<k(i+1)-1                          % More than one period smaller than the hazardous period
  temp=char(R(j,i,s));
  Emin=int(temp,s,0,Delta);
  for r=i : N-1
    Emin=Emin+W(j+1,r)*P(i+1,r+1)*R(j,i,Delta);
  end
end
```

$Q(j,i)$ Function (QQ.m):

```
function Pr=QQ(j,i)
global N M K C Delta P g k t

if j>= k(i+1)                          % Replacement has to happened
  Pr=0;
end
if j==k(i+1)-1                         % One period before the Hazardous period
  Pr=1-R(k(i+1)-1,i,t(i+1)-(k(i+1)-1)*Delta);
end
if j<k(i+1)-1                          % More than one period smaller than the hazardous period
  Pr=1-R(j,i,Delta);
  for r=i : N-1
    Pr=Pr+QQ(j+1,r)*P(i+1,r+1)*R(j,i,Delta);
  end
end
```

APPENDIX B: CODES FOR CBM WITH IMPERFECT INFORMATION

Main Function (CBM.m):

```
clear all;
clc;
% Problem Parameters setting
global N M K C Delta P Q PI g


P = zeros([N N]);              % Transaction Markov Matrix
Q = zeros([M N]);              % Observation-State Relation
PI= zeros([1 N]);              % Conditional State Distribiution

% Problem Data Entry
N=2;                           % Number of states
M=3;                           % Number of Observations

K=2;                           % Failure Cost
C=5;                           % Replacement Cost
Delta=1.0;                     % Inspection Interval
P=[0.4 0.6;0.0 1.0];           %One step transition matrix

Q=[0.6 0.3 0.1; 0.1 0.3 0.6];  %Indicator and state relation Matrix

PI=[1 0];

g1=9999;
g=(K+C)/Delta                  % Beginning cost
g=8.1704
while abs(g-g1) >= 0.0001      % Comparing old and new cost
   g1=g;
   g=cost(0,PI);               % Calling Cost function
end
Finish='Finished'
```

$\psi(X_t)$ Function (Psi.m):

```
function Si=Psi(x)
Si=exp(0.5*x);
```

$h_0(t)$ Function (h0.m):

```
function Bh=h0(t)
Beta=2;
Alfa=1;
Bh=(Beta*t^(Beta-1))/(Alfa^Beta);
```

$h(t, X_t)$ Function (h.m):

```
function h=h(t,i)
h=h0(t)*Psi(i);
```

$R(k, X_k, \Delta)$ Function (R.m):

```
function R=R(l,y,u)
global N M K C Delta P g k t
syms q
R=exp(-Psi(y)*int(h0(q),q,l*Delta,l*Delta+u));
```

$\tau(k, X_k, \Delta)$ Function (Tu.m):

```
function Tu=Tu(k,y,u)
syms x;
Tu=int(R(k,y,x),x,0,u);
```

$\overline{R}\left(k, \pi^k, t\right)$ Function (Rb.m):

```
function Rb=Rb(k,Pi,u)
global N M K C Delta P Q PI
Rb=0;
for i=1:N;
   Rb=Rb+Pi(i)*R(k,i,u);
end
```

$\overline{\tau}\left(k, \pi^k, t\right)$ Function (Tub.m):

```
function Tu=Tu(k,y,u)
global N M K C Delta P Q PI
syms x;
Tu=eval(int(R(k,y,x),x,0,u));
```

Root Finder Function (Root.m):

```
function Root=Root(PiTemp)
global N M K C Delta P Q PI g

BPoint=0;
EPoint=10;
Precision=0.0001;

myBegin=BPoint;
myEnd=EPoint;
```

```
myPace=1;

a1=BPoint;
f1=eval(K*(1-Rb(BPoint,PiTemp,Delta))-g*Tub(BPoint,PiTemp,Delta));

while myPace>Precision
  for x=myBegin+myPace:myPace:myEnd
    a2=x;
    f2=eval(K*(1-Rb(a2,PiTemp,Delta))-g*Tub(a2,PiTemp,Delta));
    if f1*f2>0
      f1=f2;
      a1=a2;
    else
      myBegin=a1;
      myEnd=a2;
      myPace=myPace/3;
      break
    end
  end
end
Root=(a1+a2)/2;
```

$\pi_j^{k+1}(\theta)$ Function (PiNew.m):

```
function  PiNew=PiNew(PiOld,j,Theta)                % Calculates the j element of the new conditioned
probability while observation is Theta
global N M K C Delta P Q PI
temp1=0;
temp2=0;
for i=1:N;
    temp1=temp1+PiOld(i)*P(i,j)*Q(j,Theta);
end
for i=1:N;
  for l=1:N;
    temp2=temp2+PiOld(i)*P(i,l)*Q(l,Theta);
  end
end
PiNew=temp1/temp2;
```

$\Pr(\theta \mid k, \pi^k)$ Function (PrTheta.m):

```
function PrTheta=PrTheta(Pi,Theta)   % Calculates the probability of having observation i while Pi
global N M K C Delta P Q PI
PrTheta=0;
for i=1:N;
  for j=1:N;
    PrTheta=PrTheta+Pi(i)*P(i,j)*Q(j,Theta);
  end
end
```

$W\left(j,\pi^{j}\right)$ Function (W.m):

```
function Emin = W(j,PItemp)
global N M K C Delta P Q PI g t0 k0

if PItemp==PI
   t=t0;
   k=k0;
else
   t=Root(PItemp);
   k=fix(t/Delta)+1;
end

syms s
if j>= k                                    % Replacement has happend
   Emin=0;
elseif j==k-1                               % One period before the Hazardous period
   Emin=quad(inline(Rb(j,PItemp,s)),0,t-j*Delta);
elseif j<k-1                                % More than one period smaller than the hazardous period
   Emin=quad(inline(Rb(j,PItemp,s)),0,Delta);
   for r=1 : M     % Possible Observation
      for l= 1 : N
         PiFuture(l)=PiNew(PItemp,l,r);
      end
      Emin=Emin+W(j+1,PiFuture)*PrTheta(PItemp,r)*Rb(j,PItemp,Delta);
   end
end
Emin=double(Emin);
```

$Q(j,\pi^{j})$ Function (QQ.m):

```
function Pr=Qu(j,PItemp)
global N M K C Delta P Q PI g t0 k0


if PItemp==PI
   t=t0;
   k=k0;
else
   t=Root(PItemp);
   k=fix(t/Delta)+1;
end


if j>= k                                    % Replacement has to happened
   prTemp=0;
end
if j==k-1                                   % One period before the Hazardous period
   prTemp=1-Rb(j,PItemp,t-j*Delta);
```

```
end
if j<k-1                              % More than one period smaller than the hazardous period
  prTemp=1-Rb(j,PItemp,Delta);
  for r=1 : M
    for l= 1 : N
      PiFuture(l)=PiNew(PItemp,l,r);
    end
    prTemp=prTemp+Qu(j+1,PiFuture)*PrTheta(PItemp,r)*Rb(j,PItemp,Delta);
  end
end
Pr=prTemp;
```

Cost Function (Cost.m):

```
function cost=cost(j,PItemp)
global N M K C Delta P Q PI g t0 k0

t0=Root(PItemp)
k0=fix(t0/Delta)+1
Pr=eval(Qu(0,PI))
Emin=W(0,PI)
cost=(C+K*Pr)/Emin
```