Characterization of Randomly Time-Variant Linear Channels

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Summary—**This paper is concerned with various aspects of the characterization of randomly time-variant linear channels. At the outset it is demonstrated that time-varying linear channels (or filters) may be characterized in an interesting symmetrical manner in time and frequency variables by arranging system functions in (time-frequency) dual pairs. Following this a statistical characterization of randomly time-variant linear channels is carried out in terms of correlation functions for the various system functions. These results are specialized by considering three classes of practically interesting channels. These are the wide-sense stationary (WSS) channel, the uncorrelated scattering (US) channel, and the wide-sense stationary uncorrelated scattering (WSSUS) channel. The WSS and US channels are shown to be (time-frequency) duals. Previous discussions of channel correlation functions and their relationships have dealt exclusively with the WSSUS channel. The point of view presented here of dealing with the dually related system functions and starting with the unrestricted linear channels is considerably more general and places in proper perspective previous results on the WSSUS channel. Some attention is given to the problem of characterizing radio channels. A model called the Quasi-WSSUS channel is presented to model the behavior of such channels.**

All real-life channels and signals have an essentially finite number of degrees of freedom due to restrictions on time duration and bandwidth. This fact may be used to derive useful canonical channel models with the aid of sampling theorems and power series expansions. Several new canonical channel models are derived in this paper, some of which are dual to those of Kailath.

I. Introduction

During recent years there has been an increasing amount of attention given to the study of randomly time-variant linear channels. This attention has been motivated to a large extent by the advent of troposcatter, ionoscatter, chaff and moon communication links and radar astronomy systems. The determination of optimum modulation and demodulation techniques and the analytical determination of the efficacy of optimum and suboptimum communication (or radar) techniques for such channels depends heavily upon a satisfactory characterization of the transmission channel. Thus, the characterization of randomly time-variant linear channels is of some interest.

The characterization of time-variant linear filters (whether random or not) in terms of system functions received its first general analytical treatment by Zadeh,[1] who introduced the Time-Variant Transfer Function and the Bi-Frequency Function as frequency domain methods of characterizing time-variant linear filters to complement the time-variant impulse response which is a time domain method of characterization. Further interesting work on the characterization of time-varying linear filters in terms of system functions has been done by Kailath,[2] who has pointed out that a third type of impulse response may be defined in addition to the two already used for time-variant linear filters. He has defined single and double Fourier transforms of these impulse responses in order to demonstrate that certain variables may be identified with frequencies at the filter input and output and certain variables may be identified with the rate of variation of the filter. However, (excepting the Time-Variant Transfer Function) only the impulse responses and their double Fourier transforms were demonstrated to be system functions; *i.e.*, filter input-output relations were derived which used only impulse responses and their double Fourier transforms.

In Section II we demonstrate that time-varying linear channels (or filters) may be characterized in an interesting symmetrical manner in time and frequency variables by arranging system functions in (time-frequency) dual pairs.[3] Most of these system functions (which include, among others, those introduced by Zadeh and Kailath) are shown to imply circuit model interpretations or representations of the time-varying linear channels. The relationship between these system functions is demonstrated in a simple way with the aid of a graph involving duality and Fourier transformations.

When the filter becomes randomly time-variant the various system functions become random processes. An exact statistical characterization of a randomly time-variant linear channel in terms of multidimensional probability density distributions for system functions, while necessary for some theoretical investigations, presupposes more knowledge than is likely to be available in physical situations. A less ambitious but more practical goal involves a statistical characterization in terms of correlation functions for the various system functions, since knowledge of these correlation functions allows a

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[1] L. A. Zadeh, "Frequency analysis of variable networks," Proc. IRE, vol. 38, pp. 291–299; March, 1950.

[2] T. Kailath, "Sampling Models for Linear Time-Variant Filters," M.I.T. Research Lab. of Electronics, Cambridge, Mass., Rept. No. 352; May 25, 1959.

[3] P. A. Bello, "Time-frequency duality," IEEE Trans. on Information Theory, vol. IT-10, pp. 18–33; January, 1964. Section V-D of this paper is essentially identical to Section II of the present one. *Note added in proof*: Since the present material has been accepted for publication the author has discovered that A. J. Gersho ["Characterization of time-varying linear systems," Proc. IEEE, (*Correspondence*), vol. 51, p. 238; January, 1963] also has determined a symmetrical formulation of system functions for time-variant linear channels. His formulation omits the Delay-Doppler Spread and Doppler-Delay Spread Functions and two Kernel System Functions.

determination of the autocorrelation function of the channel output.

In Section IV we define and determine relationships between the correlation functions of the various system functions for the general randomly time-variant linear channel. These results are specialized by considering three classes of practically interesting channels. These are the WSS channel, the US channel, and the WSSUS channel. The WSS and US channels are shown to be (time-frequency) duals.

Previous discussions[4–6] of channel correlation functions and their relationships have dealt exclusively with the WSSUS channel. Our point of view dealing with the dually related system functions and starting with the unrestricted linear channel is considerably more general and places in proper perspective previous results on the WSSUS channel.

Virtually all radio transmission media may be regarded as randomly time-variant linear channels. In the case of the transmission of digital signals over radio transmission media certain simplifications may be effected in channel characterization when the channel contains very slow fluctuations superimposed upon more rapid fluctuations, the latter of which exhibit an approximate statistical stationarity. In Section V we introduce the quasiwide-sense stationary uncorrelated scattering (QWSSUS) channel as a means for characterizing such channels.

All real-life channels and signals have an essentially finite number of degrees of freedom due to restrictions on time duration and bandwidth. This fact has been exploited by Kailath[2] to derive canonical channel models for the cases in which the channel band-limits signals at its input or output and in which the channel impulse response is time-limited. With the aid of the dual system functions derived in Section III we derive new canonic sampling models in Section V, some of which may be identified as dual to those of Kailath. As might be expected, these dual models are particularly useful under the dual time- frequency constraints, namely when the input or output time functions are time-limited or when the channel fading rate is band-limited. In addition we derive two new dually related canonic channel models, called f-power series and t-power series models. The f-power series model is of particular use in evaluating the effect of frequency selective fading on a signal whose bandwidth is less than the correlation bandwidth of a scatter channel. The t-power series model will be of use in the dual situation, *i.e.*, in evaluating the effect of (time-selective) fading on a pulse whose duration is less than the correlation time constant of a scatter channel.

II. Complex Envelopes

A process $x(t)$ whose spectral components cover a band of frequencies which is small compared to any frequency in the band may be expressed as

$$x(t) = Re\,\{\gamma(t)e^{j\omega_c t}\} \tag{1}$$

where $Re\{\ \}$ is the usual real part notation, ω_c is some (angular) frequency within the band and $\gamma(t)$ is the complex envelope of $x(t)$. This name for $\gamma(t)$ derives from the fact that the magnitude of $\gamma(t)$ is the conventional envelope of $x(t)$ while the angle of $\gamma(t)$ is the conventional phase of $x(t)$ measured with respect to carrier phase $\omega_c t$. The non-narrow-band case may be handled with the complex notation also by the use of Hilbert transforms.[7–10] However, the complex envelope will then no longer have the simple interpretation described above. Complex envelope notation will be used extensively for the remainder of this paper. However, it should be understood that there is always implied the existence of a center or reference frequency ω_c which via an equation such as (1) converts the complex time functions under discussion into physical narrow-band signals.

When dealing with problems in which there are wide-band filters (time-variant included) whose inputs and outputs are narrow-band (when expressed with reference to the same center frequency), it is possible to replace these filters with equivalent narrow-band filters which leave the input-output relations invariant. This fact becomes obvious when it is realized that by preceding and following a wide-band filter with narrow-band filters which have flat transfer functions over the range of input and output frequencies of interest, one produces a composite filter which is narrow-band and of course cannot change the input-output relations for the properly restricted class of input and output narrow-band signals. It is readily demonstrated that (except for an unimportant constant of one-half) the complex envelope of a narrow-band signal at the output of a narrow-band filter due to a narrow-band input may be obtained by passing the complex envelope of the input through an "equivalent" low-pass filter whose impulse response is just equal to the complex envelope of the narrow-band filter impulse response.

In defining the autocorrelation function of the complex envelope of a random process a certain difficulty appears that is not generally appreciated, namely, that *two* autocorrelation functions are needed in order to uniquely specify the autocorrelation function of the original real process. This fact is demonstrated by direct calculation

[4] T. Hagfors, "Some properties of radio waves reflected from the moon and their relation to the lunar surface," *J. Geophys. Res.*, vol. 66, pp. 777–785; March, 1961.

[5] R. Price and P. E. Green, Jr., "Signal Processing in Radar Astronomy," M.I.T. Lincoln Lab., Lexington, Mass., Rept. No. 234; October 6, 1960.

[6] P. E. Green, Jr., "Radar measurement of target characteristics," in "Radar Astronomy," J. V. Harrington and J. V. Evans, Eds., Chapter 9; to be published.

[7] P. M. Woodward, "Probability and Information Theory," McGraw-Hill Book Co., Inc., New York, N. Y.; 1953.

[8] J. Dugundji, "Envelopes and pre-envelopes of real waveforms," IRE Trans. on Information Theory, vol. IT-4, pp. 53–57; March, 1958.

[9] R. Arens, "Complex processes for envelopes of normal noise," IRE Trans. on Information Theory, vol. IT-3, pp. 204–207; September, 1957.

[10] D. Gabor, "Theory of communications," *J. IEE*, Part III, vol. 93, pp. 429–457; November, 1946.

of the autocorrelation function of $x(t)$, (1), as

$$\overline{x(t)x(s)} = \tfrac{1}{2} Re \{\overline{\gamma^*(t)\gamma(s)}\, e^{i\omega_c(s-t)}\} + \tfrac{1}{2} Re \{\overline{\gamma(t)\gamma(s)}\, e^{i\omega_c(s+t)}\}. \quad (2)$$

Thus the two autocorrelation functions

$$R_\gamma(t, s) = \overline{\gamma^*(t)\gamma(s)}$$
$$\approx{R}_\gamma(t, s) = \overline{\gamma(t)\gamma(s)} \quad (3)$$

are needed to specify the autocorrelation function of the real process. Fortunately, in most applications the narrow-band process is so constituted that

$$\tilde{\tilde{R}}_\gamma(t, s) = 0. \quad (4)$$

In fact, from (2), one may readily deduce that (4) is necessary if $x(t)$ is to be wide-sense stationary.

A simple physical test of $x(t)$ (deterministic components removed) to determine whether (4) is satisfied is to multiply it by itself delayed and examine the sum frequency component for the presence of a deterministic component. According to (2) the complex amplitude of this component is $\frac{1}{2}\tilde{\tilde{R}}_\gamma(t, s)$, so that the presence of a deterministic component would mean that (4) is violated. In the subsequent discussion involving complex envelopes we shall deal only with that form of autocorrelation function which involves the conjugate under the expectation sign. It should be kept in mind, however, that an analogous discussion applies for $\tilde{\tilde{R}}_\gamma(t, s)$ in those cases where it is nonzero.

The above discussion of complex envelopes, equivalent noises and equivalent filters is supplied as a physical justification for our subsequent use of "low-pass" complex time functions, complex white noise and low-pass filters with complex impulse responses.

III. System Functions for Time-Variant Linear Filters

A. Dual Operators and Kernel System Functions

The concept of "time-frequency" duality is discussed at some length by Bello.[3] For the purposes of this section it will be sufficient to define the concept of dual operators.

A device which processes communication signals may be thought of in mathematical terms as an operator which transforms input signals into output signals. The inputs and outputs of such a device may be described in either the time or frequency domain according to convenience. Since either time or frequency domain descriptions may be used at the input and output, a two-terminal device (a single-input single-output device) may be described by any one of four operators. If we define time and frequency domain descriptions of processes as dual descriptions, then these four operators may be grouped into dual pairs with the aid of the following definition:

> Two operators associated with a particular two-terminal-pair device are defined as duals when dual descriptions are used for corresponding inputs and outputs.

If $z(t)$, $Z(f)$ denote the input time function and spectrum, and $w(t)$, $W(f)$ denote the output time function and spectrum of a device, then the four possible operators are described by the equations

$$w(t) = O_{tt}[z(t)] \qquad W(f) = O_{ff}[Z(f)]$$
$$w(t) = O_{tf}[Z(f)] \qquad W(f) = O_{ft}[z(t)] \quad (5)$$

where the operator pairs O_{tt}, O_{ff} and O_{tf}, O_{ft} individually consist of dual operators.

In the case of a linear device, such as a linear time-variant channel, the four equations in (5) may be formally expressed[11] as linear integral operators with associated kernels; *i.e.*,

$$w(t) = \int z(s)K_1(t, s)\, ds \qquad W(f) = \int Z(l)K_2(f, l)\, dl \quad (6)$$

$$w(t) = \int Z(f)K_3(t, f)\, df \qquad W(f) = \int z(t)K_4(f, t)\, dt. \quad (7)$$

These kernels are, in effect, system functions and we shall call them *kernel* system functions to distinguish them from other classes of system functions to be described. It is clear that the system function pairs K_1, K_2 and K_3, K_4 may be considered as dual system function pairs.

The system functions $K_1(t, s)$ and $K_2(f, l)$ may be recognized as the Time-Variant Impulse Response and the Bi-Frequency Function respectively, used by Zadeh.[1] The system functions $K_3(t, f)$ and $K_4(f, t)$ have not been defined previously. Without difficulty it may be established that $K_1(t, s)$ and $K_2(f, l)$, besides being duals, are double Fourier transform pairs, and similarly that the dual pairs $K_3(t, f)$ and $K_4(f, t)$ are double Fourier transform pairs. Also $K_1(t, s)$ and $K_3(t, f)$ are single Fourier transform pairs with t considered as a parameter, while $K_2(f, l)$ and $K_4(f, t)$ are single Fourier transform pairs with f considered as a parameter. It is worth noting that K_1, K_2 and K_3, K_4 are the only dual pairs of system functions among those to be presented which are related directly as double Fourier transform pairs.

The kernel system functions have simple physical interpretations in terms of the response of the channel to impulses and cissoids. Thus, it is readily determined that if the channel is excited with a unit impulse at $t = s$, the resulting channel output is the time function $K_1(t, s)$ with spectrum $K_4(f, s)$, while if the channel is excited with the cissoid $e^{i2\pi lt}$ (*i.e.*, frequency impulse at $f = l$), the resulting channel output is the time function $K_3(t, l)$ with spectrum $K_2(f, l)$.

The present discussion of kernel system functions has been included primarily in the interest of making our discussion of system functions as complete as possible, and in clarifying our subsequent discussion of system functions. We shall actually make little use of the kernel system functions in the remainder of this paper, primarily because we are interested in circuit model descrip-

[11] In order to include linear differential operators one must assume that the kernels may include singularity functions.

tions of the time-variant linear channel and the kernel system functions do not lend themselves readily to such phenomenological descriptions.

B. Delay-Spread and Doppler-Spread Functions

From a strictly mathematical point of view the kernel system functions are sufficient to describe the time-frequency input-output relations for a time-variant linear channel. From a physical intuitive point of view they are not as satisfactory, since they do not readily allow one to grasp by inspection the way in which the time-variant filter affects input signals to produce output signals. Section IIIC we will be concerned with system functions which, via circuit model analogies, provide a somewhat more physical interpretation of the action of the linear time-variant channel.

Consider first the following input-output relationship for a linear time-variant channel obtained from the first equation in (6) by the transformation $s = t - \xi$:

$$w(t) = \int z(t - \xi)g(t, \xi)\, d\xi \tag{8}$$

where

$$g(t, \xi) = K_1(t, t - \xi). \tag{9}$$

Eq. (8) leads to a physical picture of the channel as a continuum of nonmoving scintillating scatterers, with with $g(t, \xi)d\xi$ equal to the (complex) modulation produced by hypothetical elemental "scatterers" that provide delays in the range $(\xi, \xi + d\xi)$. Fig. 1 illustrates such a physical picture with the aid of a densely tapped delay line. Note that the input signal is first delayed and then multiplied by the differential scattering gain. We shall call $g(t, \xi)$ the Input Delay-Spread Function to distinguish it from another system function called the Output Delay-Spread Function, to be described below, which leads to a channel representation similar to $g(t, \xi)$ except that the delay occurs on the output side of the channel (and the multiplication on the input).

If we consider $z(t)$ to be first multiplied by a differential gain function $h(t, \xi)d\xi$ and then delayed by ξ with a continuum of ξ values, we obtain the input-output relationship

$$w(t) = \int z(t - \xi)h(t - \xi, \xi)\, d\xi \tag{10}$$

and the circuit model representation of Fig. 2. By comparing (10) with (8) and (6) we quickly find that

$$h(t, \xi) = K_1(t + \xi, t) = g(t + \xi, \xi). \tag{11}$$

From the fact that $K_1(t, s)$ is the channel response at time t due to a unit impulse input at $t = s$, it is seen from (7) and (11) that $g(t, \xi)$ may be interpreted as the response at time t to a unit impulse input ξ seconds in the past, and $h(t, \xi)$ may be interpreted as the response ξ seconds in the future to a unit impulse input at time t. Since a physical channel (without internal sources) may not have an output before the input arrives, $K_1(t, s)$ must vanish for

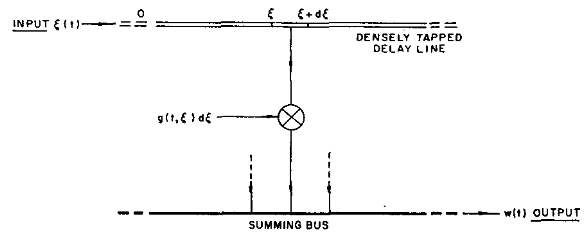

Fig. 1—A differential circuit model representation for linear time-variant channels using the Input Delay-Spread Function.

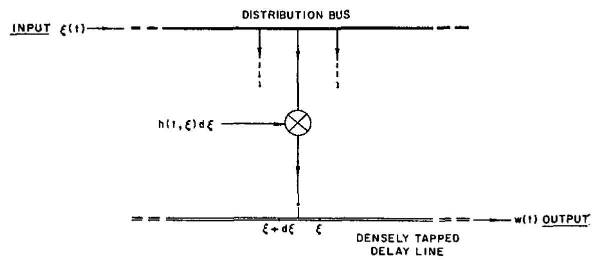

Fig. 2—A differential circuit model representation for linear time-variant channels using the Output Delay-Spread Function.

$t < s$ and $g(t, \xi)$, $h(t, \xi)$ must vanish for $\xi < 0$. These physical realizability conditions may be explicitly indicated by appropriate limits on the integrals defining input-output relations. However, for simplicity of presentation we will assume that the integral limits are $(-\infty, \infty)$, with the integrand being taken as zero in the appropriate intervals to assure physical realizability.

An entirely dual and just as general channel characterization exists in terms of frequency variables by employing the Input and Output Doppler-Spread Functions, the system functions which are the (time-frequency) duals of the Input and Output Delay-Spread Functions, respectively. Consider first the dual of the Input Delay-Spread Function. Such a system function must relate the channel output spectrum to the channel input spectrum in a manner identical in form to the way $g(t, \xi)$ relates the input and output time functions. This dual characterization involves a representation of the output spectrum $W(f)$ as a superposition of infinitesimal Doppler-shifted (the dual of delayed) and filtered (the dual of modulated) replicas of the input spectrum $Z(f)$. Thus we have

$$W(f) = \int Z(f - \nu)H(f, \nu)\, df \tag{12}$$

where $H(f, \nu)$ is the Input Doppler-Spread Function.

Eq. (12) may be interpreted physically with the aid of a model dual to that in Fig. 1. To construct such a dual it is necessary to note that the dual of a tapped delay line is a "frequency conversion chain," *i.e.*, a string of frequency converters arranged so that the output of one converter is not only the input to the next converter but is also the "local" frequency shifted output. Fig. 3 illustrates such an interpretation of (12) using a "dense" frequency conversion chain. Note that the quantity $H(f, \nu)d\nu$ is to be interpreted as the transfer function associated with hypothetical Doppler-shifting elements that provide frequency shifts in the range $(\nu, \nu + d\nu)$.

By comparing (12) and the last equation in (6) we find that the Input Doppler-Spread Function is related to Zadeh's Bi-Frequency Function by

$$H(f, \nu) = K_2(f, f - \nu) \tag{13}$$

which is an equation dual to (9).

From (10) we deduce that the dual of the Output Delay-Spread Function must provide the input spectrum-output spectrum relationship

$$W(f) = \int Z(f - \nu)G(f, \nu)\, d\nu \tag{14}$$

where $G(f, \nu)$ is defined as the Output Doppler-Spread Function. Whereas the Input Doppler-Spread Function leads to a cascaded Doppler shifter-filter realization as indicated in Fig. 3, the Output Doppler-Spread Function leads to a cascaded filter-Doppler shifter realization as shown in Fig. 4. The quantity $G(f, \nu)d\nu$ is the transfer function of a hypothetical differential filter at the input which is associated with a Doppler shift of ν cps at the channel output.

By comparing (14) with (13) and (6) we find that

$$G(f, \nu) = K_2(f + \nu, f) = H(f + \nu, \nu), \tag{15}$$

which is a set of equations dual to (11). Since $K_2(f, l)$ is the value of the spectral response of the channel at a frequency f due to a cissoidal excitation of frequency l cps, it is quickly seen from (13) and (15) that $H(f, \nu)$ may be interpreted as the spectral response of the channel at f cps due to a cissoidal input ν cycles below f, and $G(f, \nu)$ may be interpreted as the spectral response of the channel at a frequency ν cps above the cissoidal input at the frequency f cps.

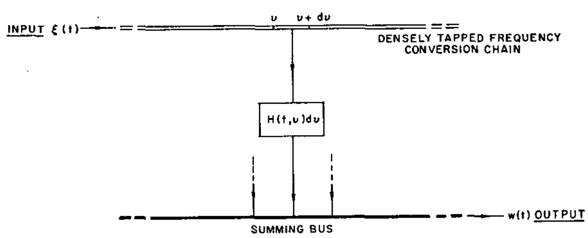

Fig. 3—A differential circuit model representation for linear time-variant channels using the Input Doppler-Spread Function.

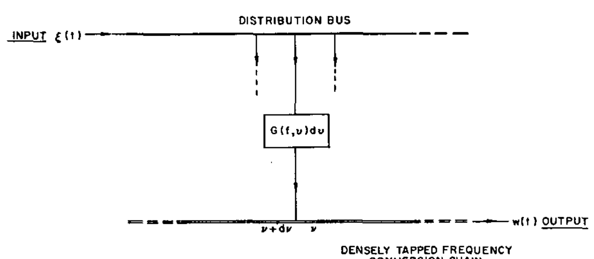

Fig. 4—A differential circuit model representation for linear time-variant channels using the Output Doppler-Spread Function.

C. Time-Variant Transfer Function and Frequency-Dependent Modulation Function

The characterizations of a time-variant channel in terms of the Delay-Spread Functions $g(t, \xi)$ and $h(t, \xi)$ and the Time-Variant Impulse Response $K_1(t, s)$ are strictly time domain approaches, while the characterizations in terms of the Doppler-Spread Functions $H(f, \nu)$ and $G(f, \nu)$ and the Bi-Frequency Function $K_2(f, l)$ are entirely frequency domain approaches. In the former cases the output time function is directly related to the input time function, while in the latter cases the output spectrum is directly related to the input spectrum. As discussed in Section III A and exemplified by the dual kernel system functions $K_3(t, f)$ and $K_4(f, t)$, two other approaches are possible. These involve an expression of the output time function directly in terms of the input spectrum in one case, and an expression of the output spectrum directly in terms of the input time function in the other. An example of the former approach was first introduced by Zadeh[1] with the aid of the Time-Variant Transfer Function.

In this section we will introduce a new system function called the Frequency-Dependent Modulation Function, which is the (time-frequency) dual of the Time-Variant Transfer Function. This system function relates the output spectrum to the input time function.

Assuming we have an input $z(t)$, which may be represented as a summation of infinitesimal cissoidal time functions, *i.e.*,

$$z(t) = \int Z(f)e^{j2\pi ft}\, df \tag{16}$$

where $Z(f)$ is the spectrum of $z(t)$, one may determine the channel output by superposing the separate responses to the infinitesimal cissoidal components. The response of the channel to the cissoidal time function exp $[j2\pi lt]$ (or spectral impulse $\delta(f - l)$) is given by [see (9)]

$$\int e^{j2\pi l(t-\xi)}g(t, \xi)\, d\xi = e^{j2\pi lt}T(l, t) \tag{17}$$

where

$$T(f, t) = \int e^{-j2\pi f\xi}g(t, \xi)\, d\xi \tag{18}$$

is the Fourier transform of the Input Delay-Spread Function with respect to the delay parameter. By superposition the network output is given by

$$w(t) = \int Z(f)T(f, t)e^{j2\pi ft}\, df. \tag{19}$$

Eq. (19) shows that even though the channel may be time-variant, one may determine the output by *exactly* the same frequency domain techniques as for time-variant (linear) channels. This involves, basically, a multiplication of the input spectrum by a system function followed by an inverse Fourier transformation with respect to the frequency variable. For time-variant channels, however, the system function is a function of the time variable. This explains use of the name Time-Variant Transfer Function to denote $T(f, t)$.

By using (14) to determine the spectrum of the response to the frequency impulse $\delta(f - l)$, and then inverse Fourier transforming to obtain the corresponding time response, it may be quickly determined that

$$T(f, t) = \int G(f, \nu)e^{i2\pi\nu t}\, d\nu, \tag{20}$$

i.e., that the Time-Variant Transfer Function is the inverse Fourier transform of the Output Doppler-Spread Function with respect to the Doppler-shift variable.

Also, either by noting, as discussed in Section III A, that $K_3(t, l)$ may be interpreted as the channel response at time t to an excitation $e^{i2\pi lt}$, or by comparing (7) and (15), it is readily seen that

$$K_3(t, f) = e^{i2\pi ft}T(f, t). \tag{21}$$

To develop the dual system function we assume that we have an input whose spectrum $Z(f)$ may be represented as a summation of infinitesimal cissoidal *frequency functions*; *i.e.*,

$$Z(f) = \int z(t)e^{-i2\pi ft}\, dt. \tag{22}$$

The spectrum of the response of the channel to the cissoidal frequency function exp $[-j2\pi fs]$ (*i.e.*, to the time function $\delta(t - s)$ whose spectrum is exp $[-j2\pi fs]$) is given by [see (12)]

$$\int e^{-i2\pi s(f-\nu)}H(f, \nu)\, d\nu = e^{-i2\pi fs}M(s, f) \tag{23}$$

where

$$M(t, f) = \int e^{i2\pi t\nu}H(f, \nu)\, d\nu \tag{24}$$

is the Fourier transform of the Input Doppler-Spread Function with respect to the Doppler-shift variable. By superposition the network output spectrum is given by

$$W(f) = \int z(t)M(t, f)e^{-i2\pi ft}\, dt. \tag{25}$$

Eq. (25) shows that, even though the channel may be a general time-variant linear filter, one may determine the output spectrum by exactly the same time domain techniques as for a channel which acts as a pure complex multiplier (or modulator). This involves, basically, a multiplication of the input time function by a complex time function characterizing the channel, followed by a Fourier transformation with respect to the time variable. For general time-variant channels, however, the complex multiplier is frequency-dependent. This explains our use of the name Frequency-Dependent Modulation Function to denote $M(t, f)$.

By using (6) to determine the time function response to the input $\delta(t - s)$ and then Fourier transforming to obtain the spectrum of the response, it may be quickly determined that

$$M(t, f) = \int e^{-i2\pi f\xi}h(t, \xi)\, d\xi; \tag{26}$$

i.e., that the Frequency-Dependent Modulation Function is the Fourier transform of the Output Delay-Spread Function with respect to the delay variable.

Also, either by noting, as discussed in Section III A, that $K_4(f, s)$ is the spectrum of the channel response to an excitation $\delta(t - s)$, or by comparing (25) and (7), it is readily seen that

$$K_4(f, t) = e^{-i2\pi ft}M(t, f). \tag{27}$$

In the case of time-invariant linear filters the transmission frequency characteristic of the filter can be determined by direct measurement as the cissoidal response or else indirectly as the spectrum of the impulse response. For time-variant linear filters these measurements procedures yield different results, as exemplified by the fact that $T(f, t)$, which corresponds to the cissoidal measurement, differs from $M(t, f)$ which corresponds to impulse response measurement followed by spectral analysis.

Moreover, as we have shown above, only $T(f, t)$ may properly be considered a transmission frequency characteristic, the proper interpretation of $M(t, f)$ being that of a channel "modulator."

D. Delay-Doppler-Spread and Doppler-Delay-Spread Functions

In Section III B it was demonstrated that any linear time-varying channel may be interpreted either as a continuum of nonmoving scintillating scatterers with the aid of the Delay-Spread Functions, or as a continuum of hypothetical Doppler-shifting elements with associated filters with the aid of the Doppler-Spread Functions. We demonstrate in this section that any linear time-varying channel may be represented as a continuum of elements which simultaneously provide both a corresponding delay and Doppler shift.

As in Section III B, we can consider system functions classified according to whether the corresponding phenomenological channel model has its delay operation or Doppler-shift operation at the channel input or output. Since delay and Doppler shift both occur in the model to be described, only two possibilities exist, *i.e.*, input-delay output-Doppler-shift and input-Doppler-shift output-delay, rather than the four possibilities of Section III B. To determine the system function corresponding to the input delay output Doppler-shift channel model, we express the Input Delay-Spread Function $g(t, \xi)$ as the inverse Fourier transform of its spectrum (where ξ is considered to be a fixed parameter), *i.e.*,

$$g(t, \xi) = \int U(\xi, \nu)e^{i2\pi\nu t}\, d\nu, \tag{28}$$

and then use (28) in (8) to obtain the following input-output relationship:

$$w(t) = \iint z(t - \xi)e^{j2\pi\nu t}U(\xi, \nu)\, d\nu\, d\xi. \tag{29}$$

Examination of (29) shows that the output is represented as a sum of delayed and then Doppler-shifted elements, the element providing delays in the interval $(\xi, \xi + d\xi)$ and Doppler shifts in the interval $(\nu, \nu + d\nu)$ having a differential scattering amplitude $U(\xi, \nu)d\nu d\xi$. For this reason we call $U(\xi, \nu)$ the Delay-Doppler-Spread Function.

In an entirely analogous way, in order to determine the dual system function, *i.e.*, that corresponding to the input Doppler-shift output-delay channel model, we express the Input Doppler-Spread Function as a Fourier transform

$$H(f, \nu) = \int V(\nu, \xi)e^{-j2\pi\xi f}\, d\xi, \tag{30}$$

and then use (30) in (12) to obtain the input-output relationship

$$W(f) = \iint Z(f - \nu)e^{-j2\pi\xi f}V(\nu, \xi)\, d\xi\, d\nu. \tag{31}$$

Examination of (31) shows that the output is represented as a sum of Doppler-shifted and then delayed elements, the element providing Doppler shifts in the interval $(\nu, \nu + d\nu)$ and delays in the interval $(\xi, \xi + d\xi)$ having a differential scattering amplitude $V(\nu, \xi)d\xi\, d\nu$. For this reason we call $V(\nu, \xi)$ the Doppler-Delay-Spread Function.

If we Fourier transform both sides of (29) with respect to t and inverse Fourier transform both sides of (31) with respect to f we obtain the equations

$$W(f) = \iint Z(f - \nu)e^{-j2\pi\xi(f-\nu)}U(\xi, \nu)\, d\nu\, d\xi \tag{32}$$

and

$$w(t) = \iint z(t - \xi)e^{j2\pi(t-\xi)}V(\nu, \xi)\, d\xi\, d\nu. \tag{33}$$

A comparison of (31) and (32) or (29) and (33) reveals that $U(\xi, \nu)$ and $V(\nu, \xi)$ are simply related; *i.e.*,

$$U(\xi, \nu) = e^{-j2\pi\nu\xi}V(\nu, \xi). \tag{34}$$

If the integration with respect to ξ is carried out in (28) and the integration with respect to ν is carried out in (33), one finds that

$$h(t, \xi) = \int V(\nu, \xi)e^{-j2\pi\nu t}\, d\nu \tag{35}$$

and

$$G(f, \nu) = \int U(\xi, \nu)e^{-j2\pi\xi f}\, d\xi. \tag{36}$$

E. Relationship Between System Functions

At this point the reader may be somewhat bewildered by the variety of system functions that have been introduced. In addition to the four kernel system functions, we have discussed eight other system functions. The relationships between the kernel system functions are rather clearly outlined in Section III A. The relationships between the other eight system functions can be simply portrayed by grouping them according to duality and Fourier transform relationships. This grouping is illustrated in Fig. 5, in which the dashed line labeled D signifies that the system functions occupying mirror image positions with respect to the dashed line are duals, and the line labeled F signifies that the system functions at the terminals of the line are related by single Fourier transforms. Since the system functions involve two variables, any two system functions connected by an F must have a common variable which should be regarded as a fixed parameter in employing the Fourier transform relationships involving the other two variables. Note that one of these latter two variables is a time variable and the other is a frequency variable. To make the F notation unique we have employed the convention that in transforming from a time to a frequency variable a negative exponential is used in the Fourier integral, while in transforming from a frequency to a time variable a positive exponential is used.

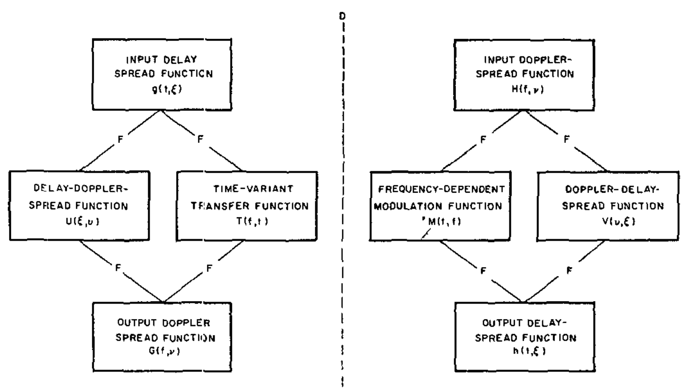

Fig. 5—Relationships between system functions for time-variant linear channels.

Examination of Fig. 5 reveals that the Time-Variant Transfer Function $T(f, t)$ and the Delay-Doppler-Spread Functions are double Fourier transforms; *i.e.*,

$$T(f, t) = \iint U(\xi, \nu)e^{-j2\pi\xi f}e^{j2\pi\nu t}\, d\xi\, d\nu. \tag{37}$$

Also the dual relationship exists. The Frequency-Dependent Modulation Function $M(t, f)$ and the Doppler-Delay-Spread Function are double Fourier transforms; *i.e.*,

$$M(t, f) = \iint V(\nu, \xi)e^{-j2\pi\xi f}e^{j2\pi\nu t}\, d\xi\, d\nu. \tag{38}$$

Since $T(f, t)$ and $M(t, f)$ are double Fourier transforms of system functions which differ only by the simple exponential factor exp $[-\, j2\pi\nu\xi]$ [see (34)] it might be supposed that they also are related in a similar simple fashion. However, this is not the case. The analytic relationship between $T(f, t)$ and $M(t, f)$ is quickly obtained from (21) and (27) with the aid of the fact that $K_3(t, f)$ and $K_4(f, t)$ are double Fourier transform pairs. This relationship is

$$M(t, f) = \iint T(f', t')e^{i2\pi(f-f')(t-t')}\, df'\, dt'$$
$$T(f, t) = \iint M(t', f')e^{-i2\pi(f-f')(t-t')}\, df'\, dt'. \tag{39}$$

$$\overline{g^*(t, \xi)g(s, \eta)} = R_g(t, s; \xi, \eta)$$
$$\overline{T^*(f, t)T(l, s)} = R_T(f, l; t, s)$$
$$\overline{G^*(f, \nu)G(l, \mu)} = R_G(f, l; \nu, \mu)$$
$$\overline{U^*(\xi, \nu)U(\eta, \mu)} = R_U(\xi, \eta; \nu, \mu)$$
$$\overline{H^*(f, \nu)H(l, \mu)} = R_H(f, l; \nu, \mu)$$
$$\overline{M^*(t, f)M(s, l)} = R_M(t, s; f, l)$$
$$\overline{h^*(t, \xi)h(s, \eta)} = R_h(t, s; \xi, \eta)$$
$$\overline{V^*(\nu, \xi)V(\mu, \eta)} = R_V(\nu, \mu; \xi, \eta) \tag{40}$$

We also note from Fig. 5 that the Input Delay-Spread Function $g(t, \xi)$ and the Output Doppler-Spread Function $G(f, \nu)$ are double Fourier transforms. Also, the dual relationship exists; *i.e.*, the Output Delay-Spread Function $h(t, \xi)$ and the Input Doppler-Spread Function $H(f, \nu)$ are double Fourier transform pairs. Other analytic relationships between system functions are readily obtained by using (11) and (15) and the Fourier transform relationships indicated in Fig. 5.

IV. Channel Correlation Functions

When the channel is randomly time-variant the system functions discussed in Section III become stochastic processes. An exact statistical characterization of a randomly time-variant linear channel in terms of multidimensional probability density distributions for system functions, while necessary for some theoretical investigations, presupposes more knowledge than is likely to be available in physical situations. A less ambitious but more practical goal involves a statistical characterization in terms of correlation functions for the various system functions since (as will be shown below) these correlation functions allow a determination of the autocorrelation function of the channel output. In this section we will be concerned with defining correlation function for these system functions and showing their interrelationships. Special attention will be given to simplifications that result for certain classes of channels of practical interest.

In general, a randomly time-variant channel has a mixed deterministic and random behavior. Thus, for example, the Input Delay-Spread Function $g(t, \xi)$ may separated into the sum of a purely random part and a deterministic part [equal to the ensemble average of $g(t, \xi)$]. This separation implies a representation of the channel as the parallel combination of a deterministic channel and a purely random channel. For simplicity of discussion, we shall only be concerned in this paper with the correlation properties associated with the purely random part of the channel. Thus it should be understood in subsequent discussions of correlation functions that each of the system functions has a zero ensemble average.

A. General Case

We shall confine our discussion of channel correlation functions to the eight system functions shown in Fig. 5, since it is felt that these system functions provide a better picture of the operation of a time-varying linear channel than the kernel system functions.

The correlation functions for the system functions in Fig. 5 will be defined as follows: (40)

where correlation functions for dual system functions have been placed in the same row and correlation functions for Fourier-transform-related system functions have been placed in the same column.

It is readily appreciated that the relationships between correlation functions in any column are double and quadruple Fourier transform relationships since the corresponding system functions are related by single and double Fourier transforms, respectively. As an illustration, consider the derivation of the relationship between $R_g(t, s; \xi, \eta)$ and $R_T(f, l; t, s)$. The Fourier transform relationship between $g(t, \xi)$ and $T(f, t)$ is shown explicitly in (18). Using this equation we find that

$$T^*(f, t)T(l, s) = \iint g^*(t, \xi)g(s, \eta)e^{i2\pi(\xi f-\eta l)}\, d\xi\, d\eta. \tag{41}$$

Then, taking the ensemble average of both sides of (41) (and assuming the validity of interchanging the order of integration and ensemble averging), we find that

$$R_T(f, l; t, s) = \iint R_g(t, s; \xi, \eta)e^{i2\pi(\xi f-\eta l)}\, d\xi\, d\eta \tag{42}$$

and by inverting the Fourier transform relationship

$$R_g(t, s; \xi, \eta) = \iint R_T(f, l; t, s)e^{-i2\pi(\xi f-\eta l)}\, dfdl. \tag{43}$$

If an identical procedure is followed to determine the other Fourier transform relationships between channel correlation functions, one finds that these relationships may be portrayed as shown in Fig. 6, wherein a double line labeled F indicates a double Fourier transform relationship between the connected correlation functions. The meaning of the dashed line labeled D is similar to the corresponding dashed line in Fig. 5, namely, the correlation functions which occupy mirror image positions with respect to the dashed line are correlation functions of dual system functions. Since the channel correlation functions involve four variables, any two correlation functions connected by an F must have two common variables [such as t, s in (42) and (43)] which should be regarded as fixed parameters in employing the double Fourier transform relationship involving the remaining variables. Note that the four variables of a correlation function are divided into two pairs separated by a semicolon. One of these pairs is involved directly in the Fourier transform relationship while the other pair is fixed. Note also that the double Fourier transform relationship always connects a pair of time (or delay) variables and a pair of frequency (or Doppler-shift) variables *e.g.*, pairs ξ, η and f, l in (42) and (43)]. In order to make the Fourier transform symbolism in Fig. 6 unique we have employed the convention that in Fourier transforming from a pair of time variables to a pair of frequency variables a positive exponential is to be used to connect the first variables in each pair and a negative exponential to connect the second variables [*e.g.*, $\exp[j2\pi\xi f]$ and $\exp[-j2\pi\eta l]$, respectively, in (42)], while in transforming from a pair of frequency variables to a pair of time variables the opposite signing procedure is to be used [*e.g.*, $\exp[-j2\pi\xi f]$ and $\exp[j2\pi\eta l]$ in (43)].

Examination of Fig. 6 reveals that the pairs of channel correlation functions (R_g, R_G), (R_U, R_T) and their duals (R_h, R_H), (R_V, R_M) are quadruple Fourier transform pairs. The actual fourfold integral relating any of these pairs is readily obtained by performing two successive double Fourier transforms as indicated in Fig. 6.

From (30) it is quickly determined that

$$R_V(\xi, \eta; \nu, \mu) = e^{j2\pi(\nu\xi - \eta\mu)} R_U(\nu, \mu; \xi, \eta) \tag{44}$$

is the relationship between the correlation functions of the Delay-Doppler-Spread and Doppler-Delay-Spread system functions.

Eq. (39) may be used to determine that the relationship between the correlation functions of the Time-Variant Transfer Function and the Frequency-Dependent Modulation Function is as shown below,

$$R_M(t, s; f, l) = \iiiint R_T(f', l'; t', s') \exp[j2\pi(l - l') \cdot (s - s') - j2\pi(f - f')(t - t')]\, df'\, dl'\, dt'\, ds' \tag{45}$$

$$R_T(f, l; t, s) = \iiiint R_M(t', s'; f', l') \exp[j2\pi(f - f') \cdot (t - t') - j2\pi(l - l')(s - s')]\, df'\, dl'\, dt'\, ds'.$$

From (11) and (15) we find the following relationships between the Delay-Spread and Doppler-Spread Functions:

$$\begin{aligned} R_h(t, s; \xi, \eta) &= R_g(t + \xi, s + \eta; \xi, \eta) \\ R_G(f, l; \nu, \mu) &= R_H(f + \nu, l + \mu; \nu, \mu). \end{aligned} \tag{46}$$

Other analytic relationships between channel correlation functions on either side of the dashed line in Fig. 6 are quickly obtained by using (40), (41) and (42) in

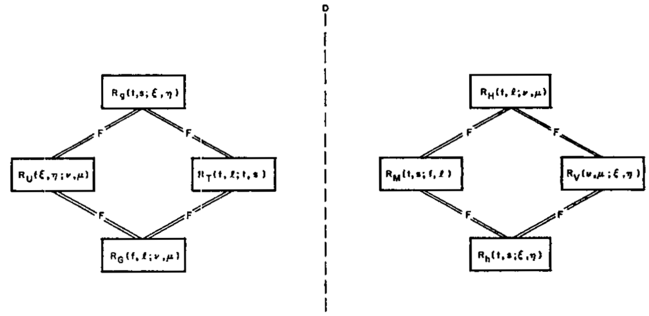

Fig. 6—Relationships between channel correlation functions.

conjunction with the Fourier transform relationships indicated in Fig. 6.

With the aid of the input-output relationship and the correlation function associated with each system function one may readily determine a corresponding double integral relating the autocorrelation function of the output to the autocorrelation function of the system function. Thus consider the system function $g(t, \xi)$. Using (8) to form the product $w^*(t)\, w(s)$ as follows:

$$w^*(t)w(s) = \iint z^*(t - \xi)z(s - \eta)g^*(t, \xi)g(s, \eta)\, d\xi\, d\eta, \tag{47}$$

and then averaging, one finds that

$$R_w(t, s) = \iint z^*(t - \xi)z(s - \eta)R_g(t, s; \xi, \eta)\, d\xi\, d\eta \tag{48}$$

where we have defined

$$R_w(t, s) = \overline{w^*(t)w(s)} \tag{49}$$

as the autocorrelation function of the output time function.

When the input is random rather than deterministic, as in (48), the output autocorrelation function becomes

$$R_w(t, s) = \iint R_z(t - \xi, s - \eta)R_g(t, s; \xi, \eta)\, d\xi\, d\eta \tag{50}$$

where we have defined

$$R_z(t, s) = \overline{z^*(t)z(s)}. \tag{51}$$

The dual system function $H(f, \nu)$ leads to the following expression for the autocorrelation function of the output spectrum

$$\begin{aligned} R_W(f, l) &\equiv \overline{W^*(f)W(l)} \\ &= \iint Z^*(f - \nu)Z(l - \mu)R_H(f, l; \nu, \mu)\, d\nu\, d\mu \end{aligned} \tag{52}$$

when the input is deterministic and

$$R_W(f, l) = \iint R_Z(f - \nu, l - \mu)R_H(f, l; \nu, \mu)\, d\nu\, d\mu \tag{53}$$

when the input is random, where we have defined

$$R_Z(f, l) = \overline{Z^*(f)Z(l)} \tag{54}$$

as the autocorrelation function of the input spectrum.[12] The reader may readily form the input-output correlation function relationships corresponding to the remaining system functions.

B. The Wide-Sense Stationary Channel

In many physical channels the fading statistics may be assumed approximately stationary for time intervals sufficiently long to make it meaningful to define a subclass of channels, called Wide-Sense Stationary (WSS) Channels. A WSS channel has the property that the channel correlation functions $R_g(t, s; \xi, \eta)$, $R_h(t, s; \xi, \eta)$, $R_T(f, l; t, s)$ and $R_M(t, s; f, l)$ are invariant under a translation in time; *i.e.*, these correlation functions depend on the variables t, s only through the difference $\tau = s - t$. Thus for the WSS channel

$$\begin{aligned} R_g(t, t+\tau; \xi, \eta) &= R_g(\tau; \xi, \eta) \\ R_h(t, t+\tau; \xi, \eta) &= R_h(\tau; \xi, \eta) \\ R_T(f, l; t, t+\tau) &= R_T(f, l; \tau) \\ R_M(t, t+\tau; f, l) &= R_M(\tau; f, l). \end{aligned} \tag{55}$$

The restricted nature of the four channel correlation functions in (55) constrains the remaining four channel correlation functions in Fig. 6 to have a singular behavior in the Doppler-shift variables. As an example consider the double Fourier transform relationship between $R_g(t, s; \xi, \eta)$ and $R_U(\xi, \eta, \nu, \mu)$:

$$R_U(\xi, \eta; \nu, \mu) = \iint R_g(t, s; \xi, \eta) e^{i2\pi(\nu t - \mu s)}\, dt\, ds. \tag{56}$$

Upon making the change in variable $\tau = s - t$ in (56) and using the first equation in (55), one finds that

$$R_U(\xi, \eta; \nu, \mu) = \int e^{i2\pi t(\nu-\mu)}\, dt \int R_g(\tau; \xi, \eta) e^{-i2\pi\tau}\, d\tau. \tag{57}$$

The first integral in (57) is recognized as a unit impulse at $\nu = \mu$, *i.e.*, $\delta(\nu - \mu)$. It follows that $R_U(\xi, \eta; \nu, \mu)$ may be expressed in the form

$$R_U(\xi, \eta, \nu, \mu) = P_U(\xi, \eta; \nu)\delta(\nu - \mu) \tag{58}$$

where $P_U(\xi, \eta; \nu)$ is the Fourier transform of $R_g(\tau; \xi, \eta)$ with respect to the variable τ; *i.e.*,

$$P_U(\xi, \eta; \nu) = \int R_g(\tau; \xi, \eta) e^{-i2\pi\nu\tau}\, d\tau. \tag{59}$$

In an analogous fashion it is readily determined that

$$\begin{aligned} R_G(f, l; \nu, \mu) &= P_G(f, l; \nu)\delta(\nu - \mu) \\ R_V(\nu, \mu; \xi, \eta) &= P_V(\nu; \xi, \eta)\delta(\nu - \mu) \\ R_H(f, l; \nu, \mu) &= P_H(f, l; \nu)\delta(\nu - \mu) \end{aligned} \tag{60}$$

where $P_G(f, l; \nu)$, $P_V(\nu; \xi, \eta)$, and $P_H(f, l; \nu)$ are Fourier transforms of $R_T(f, l; \tau)$, $R_h(\tau; \xi, \eta)$, and $R_M(\tau; f, l)$, respectively, with respect to the delay variable τ.

[12] See Bello, *op. cit.*[3], for a discussion of the spectrum of a random process.

The singular behavior of the channel correlation functions in (58) and (60) has interesting implications with regard to the behavior of the associated circuit models. Thus the forms of R_H and R_G as shown in (60) imply that in Figs. 3 and 4 the transfer functions of the random filters associated with different Doppler shifts are uncorrelated. Similarly the forms of R_V and R_U in (60) and (58) imply that in the associated channel models consisting of a number of differential "scatterers" producing delay and Doppler shifts, the complex scattering amplitudes of two different elements are uncorrelated if these elements cause different Doppler shifts.

When the system functions are normally distributed stochastic processes, complete lack of correlation between two processes implies statistical independence. Then wide-sense stationarity implies strict-sense stationarity, and in the circuit models of Figs. 3 and 4 the transfer functions of random filters associated with different Doppler shifts are statistically independent. Similarly in the models consisting of a number of differential "scatterers" producing delay and Doppler shifts, the complex scattering amplitudes of two different elements are statistically independent if these elements cause different Doppler shifts.

The singular behavior of the correlation functions R_H, R_G, R_V and R_U might have been expected *a priori* from the observation that the corresponding system functions are interpretable as (complex) amplitude spectra of random processes and from the fact that the cross-correlation function between the amplitude spectra of two wide-sense stationary noises is an impulse located at zero frequency shift with a complex area equal to the cross-power spectral density between the original processes.[3] Thus, when considered as a function of the Doppler-shift variable ν, the functions $P_U(\xi, \eta; \nu)$, $P_G(t, l; \nu)$ $P_V(\nu; \xi, \eta)$ and $P_H(f, l; \nu)$ may be interpreted as cross-power spectral densities between the pairs of time functions $[g(t, \xi), g(t, \eta)]$, $[T(f, t), T(l, t)]$, $[h(t, \xi), h(t, \eta)]$ and $[M(t, f), M(t, l)]$, respectively. In the particular case that $\xi = \eta$, $P_V(\nu; \xi, \xi)$ and $P_U(\xi, \xi; \nu)$ may be interpreted as power spectral densities of the functions $g(t, \xi)$ and $h(t, \xi)$, respectively; while for $f = l$, $P_G(f, f; \nu)$ and $P_H(f, f; \nu)$ may be interpreted as power spectral densities of the functions $T(f, t)$ and $M(t, f)$, respectively. In view of the above it is clear that the system functions $U(\xi, \nu)$, $G(f, \nu)$, $V(\nu, \xi)$ and $H(f, \nu)$ will behave like nonstationary white noises in the Doppler-shift variable when the channel is WSS.

In Fig. 7 we have summarized the relationships between the channel correlation functions, using only the corresponding density function when the correlation function has an impulsive behavior. Note that the Fourier transform notations of Figs. 5 and 6 have been used.

Let us now consider some analytical relationships between functions on the opposite side of the dashed line in Fig. 7. From (44) and (60) we find that

$$P_V(\xi, \eta; \nu) = e^{-i2\pi\nu(\eta-\xi)} P_U(\mu; \xi, \eta) \tag{61}$$

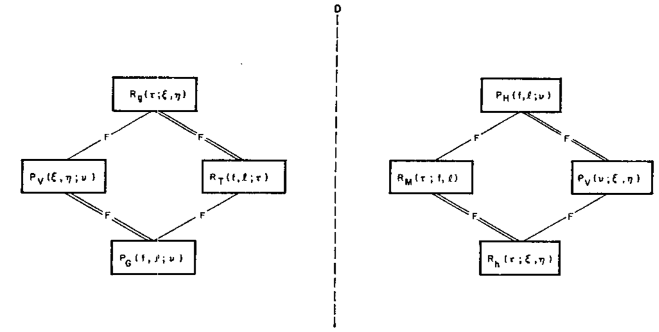

Fig. 7—Relationships between channel correlation functions for WSS channel.

while from (46), (55), (58) and (60) we find that

$$R_h(\tau; \xi, \eta) = R_g(\tau + \eta - \xi; \xi, \eta) \qquad (62)$$
$$P_G(f, l; \nu) = P_H(f + \nu, l + \nu; \nu).$$

The relationship between the correlation functions of the Time-Variant System Function and the Frequency-Dependent Modulation Function is readily determined from (45) and (55) to be

$$R_M(\tau; f, l) = \iint R_T(f', f' + l - f; \tau') \cdot e^{i2\pi(f-f')(\tau-\tau')}\, df'\, d\tau' \qquad (63)$$
$$R_T(f, l; \tau) = \iint R_M(\tau'; f', f' + l - f) \cdot e^{-i2\pi(f-f')(\tau-\tau')}\, df'\, d\tau'.$$

Other analytic relationships between the channel correlation functions on either side of the dashed line in Fig. 7 are quickly obtained by using (61), (62) and (63) in conjunction with the Fourier transform relationships indicated in Fig. 7.

C. The Uncorrelated Scattering Channel

For several physical channels (*e.g.*, troposcatter, chaff, moon reflection) the channel may be modeled approximately as a continuum of uncorrelated scatterers. The mathematical counterpart of this statement is embodied in the following forms for the autocorrelation function of the Input and Output Delay-Spread Functions:

$$R_g(t, s; \xi, \eta) = P_g(t, s; \xi)\delta(\eta - \xi) \qquad (64)$$
$$R_h(t, s; \xi, \eta) = P_h(t, s; \xi)\delta(\eta - \xi).$$

Because of the intimate relationship between the Input and Output Delay-Spread Functions, one of the equations in (64) implies the other. Moreover, these equations imply that the autocorrelation functions of the Doppler-Delay-Spread and Delay-Doppler-Spread Functions must have the form

$$R_U(\xi, \eta; \nu, \mu) = P_U(\xi, \nu, \mu)\delta(\eta - \xi) \qquad (65)$$
$$R_V(\nu, \mu; \xi, \eta) = P_V(\nu, \mu; \xi)\delta(\eta - \xi).$$

The singular behavior of the channel correlation functions in (64) and (65) has implications with regard to the behavior of the associated circuit models. Thus the forms of R_g and R_h as shown in (64) imply that in Figs. 1 and 2 the complex gain functions associated with different path delays are uncorrelated. Similarly the forms of R_V and R_U in (65) imply that in the associated channel models consisting of a number of differential "scatterers" producing delay and Doppler shifts, the complex scattering amplitudes of two different scatterers are uncorrelated if these elements cause different delays. A channel whose system functions have correlation functions of the form shown in (64) and (65) will be called an Uncorrelated Scattering (US) channel.

When the system functions are normally distributed stochastic processes, the uncorrelatedness properties mentioned above for complex gain functions and scattering amplitudes become independence properties.

The forms of the correlation functions for the remaining four system functions which are readily determined from the Fourier transform relationships indicated in Fig. 6 are given by

$$R_t(f, f + \Omega; t, s) = R_T(\Omega; t, s)$$
$$R_M(t, s; f, f + \Omega) = R_M(t, s; \Omega) \qquad (66)$$
$$R_G(f, f + \Omega; \nu, \mu) = R_G(\Omega; \nu, \mu)$$
$$R_H(f, f + \Omega; \nu, \mu) = R_H(\Omega; \nu, \mu).$$

A comparison of the channel correlation functions for the WSS and US channels reveals an interesting fact: the correlation function of a particular system function of the WSS channel and the correlation function of the dual system function of the US channel have identical analytical forms as a function of dual variables. For this reason one may consider the class of WSS channels to be the dual of class of US channels.[13]

As a consequence of this duality we note that the US channel may be regarded as a WSS channel in the frequency variable since, from (66), the channel correlation functions depend upon the frequency variables f, l only through the difference frequency $\Omega = l - f$. Similarly the WSS channel may be regarded as a form of US channel in the Doppler-shift variable.

While the Input and Output Doppler-Spread Function have the character of nonstationary white noise as a function of the Doppler-shift variable in the case of the WSS channel, the dual system functions, *i.e.*, the Input and Output Delay-Spread Functions, the Delay-Doppler-Spread and Doppler-Delay-Spread Functions, respectively, have the character of nonstationary white noise as a function of the dual variable, *i.e.*, the delay variable, in the case of the US channel.

By analogy with the dual functions in the WSS channel, the functions $P_g(t, s; \xi)$, $P_h(t, s; \xi)$, $PU(\xi;), \nu, \mu)$ and

[13] Using the definitions developed in Bello, *op. cit.* [3], one may state that the wide-sense dual of a WSS channel is a US channel and vice versa.

$P_V(\nu, \mu; \xi)$, when considered as a function of ξ, may be regarded as cross-power spectral densities while $P_g(t, t; \xi)$, $P_h(t, t; \xi)$, $P_U(\xi; \nu, \nu)$, $P_V(\nu, \nu; \xi)$ may be regarded as power spectral densities as a function of the delay variable. In Fig. 8 we have summarized the relationships between the channel correlation functions for the US channel using only the corresponding cross-power density function when the correlation function has an impulsive behavior.

We will now obtain some analytical relationships between functions on the opposite side of the dashed line in Fig. 8. First we have the relationship dual to (61) which may be obtained by using (65) in (44),

$$P_U(\nu, \mu; \xi) = e^{j2\pi\nu(\eta-\xi)} P_V(\xi; \nu, \mu). \tag{67}$$

Then, using (64) and the last two equations of (66) in (46) we obtain the dual to (62) as

$$\begin{aligned} R_G(\Omega; \nu, \mu) &= R_H(\Omega + \mu - \nu; \nu, \mu) \\ P_h(t, s; \xi) &= P_g(t + \xi, s + \xi; \xi). \end{aligned} \tag{68}$$

The relationships between the Time-Variant Transfer Function and the Frequency-Dependent Modulation Function dual to (63) are obtained by using the first two equations in (66) in (45) and carrying out two integrations of the appropriate quadruple integrals in (45),

$$\begin{aligned} R_T(\Omega; t, s) &= \iint R_M(t', t' + s - t; \Omega) \\ &\qquad \cdot e^{-j2\pi(t-t')(\Omega-\Omega')}\, dt'\, d\Omega' \\ R_M(t, s; \Omega) &= \iint R_T(\Omega'; t', t' + s - t) \\ &\qquad \cdot e^{j2\pi(t-t')(\Omega-\Omega')}\, dt'\, d\Omega'. \end{aligned} \tag{69}$$

As we have mentioned in a similar vein for the other classes of channels, further analytical relationships may be obtained between system functions on the opposite side of the dotted line in Fig. 8 by using (67), (68) and (69) and the Fourier transform relationships indicated in Fig. 8.

D. The Wide-Sense Stationary Uncorrelated Scattering Channel

The simplest type of randomly time-variant linear channel to describe in terms of channel correlation functions, and one which, fortunately, is of practical interest is the WSSUS channel. As might be suspected from its name, the WSSUS channel is both a WSS and a US channel. Thus, the channel correlation functions of the WSSUS channel must have forms characteristic of both the WSS channel [(55), (58) and (60)] and the US channel [(64), (65) and (66)].

An examination of the correlation functions of the WSS and US channels reveals that for the WSSUS channel, the correlation functions of the Delay-Spread Functions must have the form

$$\begin{aligned} R_g(t, t + \tau; \xi, \eta) &= P_g(\tau, \xi)\delta(\eta - \xi) \\ R_h(t, t + \tau; \xi, \eta) &= P_h(\tau, \xi)\delta(\eta - \xi) \end{aligned} \tag{70}$$

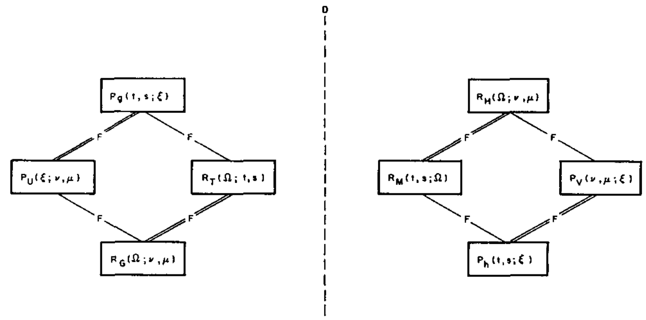

Fig. 8—Relationships between channel correlation functions for US channel.

while the correlation functions of the Doppler-Spread Functions must have the form

$$\begin{aligned} R_G(f, f + \Omega; \nu, \mu) &= P_G(\Omega, \nu)\delta(\mu - \nu) \\ R_H(f, f + \Omega; \nu, \mu) &= P_H(\Omega, \nu)\delta(\mu - \nu). \end{aligned} \tag{71}$$

The equations in (70) show that for the WSSUS channel, the system functions $g(t, \xi)$ and $h(t, \xi)$ have the character of nonstationary white noise in the delay variable and wide-sense stationary noise in the time variable. In terms of the channel models of Figs. 1 and 2, the WSSUS channel has a representation as a continuum of uncorrelated randomly scintillating scatterers with wide-sense stationary statistics.

The equations in (71) show that for the WSSUS channel the system functions $G(f, \nu)$ and $H(f, \nu)$ have the character of nonstationary white noise in the Doppler-shift variable and wide-sense stationary noise in the frequency variable. In terms of the channel models of Figs. 3 and 4, the WSSUS channel has a representation as a continuum of uncorrelated Doppler-shifting filtering (or filtering-Doppler shifting) elements with each filter having a transfer function with wide-sense stationary statistics in the frequency variable.

For the WSSUS channel the correlation functions of the Delay-Doppler-Spread and Doppler-Delay-Spread Functions simplify to

$$\begin{aligned} R_U(\xi, \eta; \nu, \mu) &= P_U(\xi, \nu)\delta(\mu - \nu)\delta(\eta - \xi) \\ R_V(\nu, \mu; \xi, \eta) &= P_V(\nu, \xi)\delta(\mu - \nu)\delta(\eta - \xi). \end{aligned} \tag{72}$$

Eq. (72) shows that for the WSSUS channel the system functions $U(\xi, \nu)$ and $V(\nu, \xi)$ have the character of nonstationary white noise in both the delay and Doppler-shift variables, *i.e.*, a form of two-dimensional nonstationary white noise. It follows that in terms of the corresponding channel models, the WSSUS channel may be represented as a collection of nonscintillating uncorrelated scatterers which cause both delays and Doppler shifts.

Finally, in the case of the WSSUS channel, the correlation functions for the Time-Variant Transfer Function and the Frequency-Dependent Modulation Function take the simple forms

$$\begin{aligned} R_M(t, t + \tau; f, f + \Omega) &= R_M(\tau, \Omega) \\ R_T(f, f + \Omega; t, t + \tau) &= R_T(\Omega, \tau); \end{aligned} \tag{73}$$

i.e., the system functions $T(f, t)$ and $M(t, f)$ are wide-sense stationary processes in both the time and frequency variables.

From previous discussions in Sections IV B and IV C, we know that when considered as a function of the Doppler-shift variable ν the functions $P_G(\Omega, \nu)$ and $P_H(\Omega, \nu)$ may be interpreted as the cross-power spectral densities of the pairs of time functions $T(f, t)$, $T(f + \Omega, t)$ and $M(t, f)$, $M(t, f + \Omega)$, respectively.

Similarly, when considered as a function of the delay variable ξ, the functions $P_g(\tau, \xi)$ and $P_h(\tau, \xi)$ may be interpreted as cross-power spectral densities of the frequency functions $T(f, t)$, $T(f, t + \tau)$ and $M(t, f)$, $M(t + \tau, f)$, respectively.

The functions $P_U(\xi, \nu)$ and $P_V(\nu, \xi)$ may be interpreted as a sort of two-dimensional power density spectrum as a function of delay and Doppler shift corresponding to the combined time and frequency functions $T(f, t)$ and $M(t, f)$, respectively.

In the case of the WSSUS channel the relationships between correlation functions on opposite sides of the dashed line in Fig. 6 become trivial. Thus, use of (73) in (44) shows that

$$P_U(\xi, \nu) = P_V(\nu, \xi) \equiv S(\xi, \nu); \tag{74}$$

i.e., that the two-dimensional power spectral densities in delay and Doppler shift associated with the Doppler-Delay-Spread and Delay-Doppler-Spread functions are identical. We have used the function $S(\xi, \nu)$ to denote this common function which is identical to the Target Scattering Function $\sigma(\xi, \nu)$ defined by Price and Green[5] in their work on radar astronomy. We shall call $S(\xi, \nu)$ the Scattering Function since it has more general applications than to radar problems.

If (72) is used in the last equation of (46) one finds immediately that

$$P_G(\Omega, \nu) = P_H(\Omega, \nu) \equiv P(\Omega, \nu). \tag{75}$$

Thus the Doppler cross-power spectral densities associated with the channel models of Figs. 3 and 4 become equal in the case of the WSSUS channel. We shall call this common function the Doppler Cross-Power Spectral Density and denote it by the function $P(\Omega, \nu)$. In the particular case that $\Omega = 0$, the cross-power spectral densities become simply power spectral densities. Thus we define

$$P(0, \nu) \equiv P(\nu) \tag{76}$$

where $P(\nu)$ is called the Doppler Power Density Spectrum. (This function is called the Echo Power Spectrum by Green.[6])

If (59) is used in the first equation of (42) one finds that

$$P_g(\tau, \xi) = P_h(\tau, \xi) \equiv Q(\tau, \xi). \tag{77}$$

Thus the delay cross-power spectral densities associated with the channel models of Figs. 1 and 2 become equal in the case of the WSSUS channel. We shall call this common function the Delay Cross-Power Spectral Density and denote it by the function $Q(\tau, \xi)$. In the particular case that $\tau = 0$, the cross-power spectral densities become simple power spectral densities. Thus we define

$$Q(0, \xi) \equiv Q(\xi) \tag{78}$$

where $Q(\xi)$ is called the Delay Power Density Spectrum. (This function has been called the Power Impulse Response by Green[6] and the Delay Spectrum by Hagfors.[4])

Finally, if (73) is used in (45) one finds that the quadruple integrals in (45) vanish leaving the interesting result

$$R_M(\tau, \Omega) = R_T(\Omega, \tau) \equiv R(\Omega, \tau). \tag{79}$$

Thus in the case of the WSSUS channel, the correlation functions of the Time-Variant System Function and the Frequency-Dependent Modulation Function become identical, *i.e.*,

$$\overline{T^*(f, t)T(f + \Omega, t + \tau)} = \overline{M^*(t, f)M(t + \tau, f + \Omega)} = R(\Omega, \tau). \tag{80}$$

We shall call this common function the Time-Frequency Correlation Function and denote it by $R(\Omega, \tau)$. (This function has been called the Spaced-Time Spaced-Frequency Correlation Function by Green.[6])

Two correlation functions of practical interest derivable from $R(\Omega, \tau)$ are the Frequency Correlation Function $q(\Omega)$ (called the Spaced-Frequency Correlation Function by Green[6]) given by

$$\overline{T^*(t, f)T(t, t + \Omega)} = \overline{M^*(t, f)M(t, f + \Omega)} = R(\Omega, 0) \equiv q(\Omega) \tag{81}$$

and the Time Correlation Function $p(\tau)$ (called the Echo Correlation Function by Green[6]) given by

$$\overline{T^*(t, f)T(t + \tau, f)} = \overline{M^*(t, f)M(t + \tau, f)} = R(0, \tau) \equiv p(\tau). \tag{82}$$

The relationships between channel correlation functions are shown in Fig. 9. Note that $Q(\tau, \xi)$ and $P(\Omega, \nu)$ are double Fourier transform pairs as are $S(\xi, \nu)$ and $R(\Omega, \tau)$. The double Fourier transform relationship between the Scattering Function and the Time-Frequency Correlation Function appears to have been first pointed out by Hagfors.[4] It is readily determined from the Fourier transform relationships in Fig. 9 that $q(\Omega)$, $Q(\xi)$ and $p(\tau)$, $P(\nu)$ are single Fourier transform pairs.

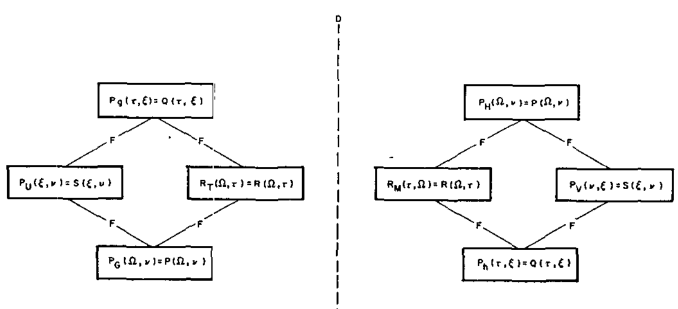

Fig. 9—Relationships between channel correlation functions for WSSUS channel.

V. Radio Channel Characterization

Virtually all radio transmission media may be characterized as linear in regard to their influence upon communication signals. Thus, from a phenomenological point of view, radio channels may be regarded as special cases of random time-variant linear filters. In the case of the transmission of digital signals over radio channels it appears that certain simplifications may be effected in the general characterization of randomly time-variant channels developed in the previous sections of this report. These simplifications arise when the time and frequency selective behavior of the channel may be regarded as wide-sense stationary for time and frequency intervals much greater than the durations and bandwidths, respectively, of the signaling waveforms of interest. Such a situation arises in practice when the channel contains very slow fluctuations superimposed upon more rapid fluctuations, the latter of which exhibit the desired statistical stationarity properties. Most radio channels do, in fact, appear to exhibit such "quasi-stationary" behavior. Moreover, the more rapid fluctuations often appear to be characterizable in terms of appropriately defined Gaussian statistics. Since a Gaussian process can be completely described statistically if its correlation function is known, it follows that a fairly complete statistical description of many quasi-stationary radio channels should be achievable by measuring the channel correlation functions for time and frequency intervals small compared to the fluctuation intervals of the slow channel variations, and then measuring the statistical behavior of these quasi-stationary channel correlation functions as caused by the slow channel variations. In this way one may compute quasi-stationary error probabilities for digital transmission which would accurately reflect the short-time error rate behavior of the channel.[14-16] The long-time error rate behavior of the channel could then be predicted by averaging the short-time error rate behavior over the long-time fading statistics of the channel.

To sum up, our thesis is that a useful way to perform measurements on radio channels is to determine the long-time statistics of short-time channel correlation functions. The resulting data should be sufficient to provide a fairly complete statistical description of some radio channels. To make these ideas more precise we shall now present a mathematical exposition of the above ideas.

The time and frequency selective behavior of a random time-variant linear channel may be described with the aid of several of the system functions described in Section III. For purposes of discussion it is sufficient to start with an examination of $T(f, t)$, the Time-Variant Transfer Function of the channel. It will be recalled that $T(f, t)$ is the complex envelope of the response of the channel to an excitation $\cos 2\pi(f_c + f)t$, where f_c is the "center" or "carrier" frequency at which the channel is being excited. Thus, the magnitude of $T(f, t)$ is the envelope of the channel response and the angle of $T(f, t)$ is the phase of the channel response measured with respect to the phase $2\pi(f_c + f)t$.

For a general input signal with complex envelope $z(t)$, the channel output complex envelope $w(t)$ is given by (19). Although the input time function $z(t)$ may be deterministic (nonrandom), the output time function $w(t)$ will be a random process since $T(f, t)$ is a random process. It is possible that for some radio channels $T(f, t)$ will contain a deterministic component so that $w(t)$ will also contain a deterministic component. However, for the purpose of the present discussion it is sufficient to confine our attention to the purely random part of $T(f, t)$ and $w(t)$. Thus, to avoid introducing unnecessary notation we shall assume in this section that $w(t)$ and $T(f, t)$ are purely random.

The time and frequency selective behavior of the channel is evidenced by the way $T(f, t)$ changes with changes in f and t. As far as conventional usage is concerned the concept of "fading" is associated only with the fact that $T(f, t)$ varies with the time variable t. However, it appears desirable to extend this definition to include variation of $T(f, t)$ with f, and thus talk of "frequency fading," the dual of conventional "time fading."

From a statistical point of view the simplest way to describe the sensitivity of $T(f, t)$ to changes in f and t is to form the correlation function

$$\overline{T^*\left(f - \frac{\Omega}{2}, t - \frac{\tau}{2}\right)T\left(f + \frac{\Omega}{2}, t + \frac{\tau}{2}\right)} \equiv R_{f,t}(\tau, \Omega). \quad (83)$$

In terms of the notation in Section IV,

$$R_{f,t}(\tau, \Omega) \equiv R_T\left(f - \frac{\Omega}{2}, f + \frac{\Omega}{2}; t - \frac{\tau}{2}, t + \frac{\tau}{2}\right). \quad (84)$$

When f and t are fixed, say at $f = f'$, $t = t'$, $R_{f',t'}(\tau, \Omega)$ describes the way in which the Time-Variant Transfer Function becomes decorrelated for a frequency interval Ω and a time interval τ centered on the "local" time-frequency coordinates f', t'. In the case of the WSSUS channel $R_{f,t}(\tau, \Omega)$ becomes independent of f, t; *i.e.*,

$$R_{f,t}(\tau, \Omega) = R(\tau, \Omega). \quad (85)$$

From an analytical point of view the WSSUS channel is the simplest nondegenerate channel that exhibits both time and frequency selective behavior. As discussed in Section IV, such a channel may be modeled as a continuum of uncorrelated scatterers such that each infinitesimal scatterer providing Doppler shifts in the range ν, $\nu + d\nu$ and delays in the range ξ, $\xi + d\xi$ has a scattering cross section of $S(\xi, \nu)d\xi, d\nu$, where

[14] P. A. Bello and B. D. Nelin, "The influence of fading spectrum on the binary error probabilities of incoherent and differentially-coherent matched filter receivers, "IRE Trans. on Communications Systems, vol. CS-10, pp. 160–168; June, 1962.

[15] P. A. Bello and B. D. Nelin, "The effect of frequency selective fading on the binary error probabilities of incoherent and differentially coherent matched filter receivers," IEEE Trans. on Communication Systems, vol. CS-11, pp. 170–186; June, 1963.

[16] P. A. Bello and B. D. Nelin, "The Effect of Combined Time and Frequency Selective Fading on the Binary Error Probabilities of Incoherent Matched Filter Receivers," ADCOM, Inc., Cambridge, Mass., Res. Rept. No. 7, March, 1963.

$$S(\xi, \nu) = \iint e^{i2\pi(\xi\Omega - \nu\tau)} R(\tau, \Omega)\, d\tau\, d\Omega \tag{86}$$

is the Fourier transform of $R(\tau, \Omega)$.

We will now demonstrate that the simplicity of the WSSUS channel can be transferred to a practically interesting class of channels which we shall call Quasi-WSSUS (or QWSSUS). This class contains two subclasses which are (time-frequency) duals. For the purposes of the present discussion we need only introduce that subclass which is based upon the properties of $R_{f,t}(\tau, \Omega)$. The dual subclass is based upon the Frequency Dependent Modulation Function and may be readily constructed by the reader.

To define the QWSSUS channel of interest here one must assume that the typical input signaling waveform has a constraint on bandwidth and that the resulting output waveform has a constraint on time duration. This bandwidth and time constraint can be centered anywhere, but for simplicity of discussion (and with no loss in generality) it is convenient to assume that the input bandwidth constraint is centered at $f = 0$ (*i.e.*, at the carrier frequency) and the output time constraint is centered at $t = 0$. Any other centering can be handled by a redefinition of carrier frequency and time origin.

The QWSSUS channel is defined as that subclass for which certain gross channel parameters have specified inequality relations with respect to the input-bandwidth and output-time constraints defined above. The gross channel parameters of concern here are measures of the maximum rate at which $R_{f,t}(\tau, \Omega)$ varies in the f and t directions. Let the maximum rate of fluctuation $R_{f,t}(\tau, \Omega)$ in the f and t directions be denoted by $\gamma_{\max}$ sec and $\theta_{\max}$ cps, respectively. Let W, T denote the bandwidth, and time duration of the input signal. Let Δ denote the multipath spread of the channel. Then the QWSSUS channel under discussion is a channel for which the following inequalities hold:

$$W \ll \frac{1}{\gamma_{\max}} \tag{87}$$

$$T + \Delta \ll \frac{1}{\theta_{\max}}, \tag{88}$$

or, in other words, one for which $R_{f,t}(\Omega, \tau)$ changes negligibly over "f" intervals equal to the input signal bandwidth (W) and over "t" intervals equal to the output signal time duration ($T + \Delta$).

It will now be demonstrated that if inequalities (87) and and (88) are satisfied, the actual channel may be replaced by a hypothetical WSSUS channel at least as far as the determination of the correlation function of the channel output is concerned. However, when the channel has Gaussian statistics, the actual channel may be replaced by a hypothetical WSSUS channel as far as the determination of any output statistics are concerned, since then the output will be a Gaussian process and thus completely determined statistically from knowledge of its correlation function.

From (19) we determine that the output correlation function is given by

$$\overline{w^*(t)w(s)} = \iint Z^*(f)Z(l)R_T(f, l; t, s)e^{-i2\pi(ft-ls)}\, df\, dl. \tag{89}$$

Upon making the transformations

$$t \to t - \frac{\tau}{2}, \qquad s \to t + \frac{\tau}{2}$$

$$f \to f - \frac{\Omega}{2}, \qquad l \to f + \frac{\Omega}{2}$$

in (89) one finds that

$$\overline{w^*\left(t - \frac{\tau}{2}\right)w\left(t + \frac{\tau}{2}\right)} = \iint Z^*\left(f - \frac{\Omega}{2}\right)Z\left(f + \frac{\Omega}{2}\right) \cdot R_{f,t}(\tau, \Omega)e^{-i2\pi f}e^{i2\pi\Omega t}\, df\, d\Omega. \tag{90}$$

If we consider the integration with respect to f first in (90) we note that the integrand is nonzero over an interval of f values of maximum width W centered on $f = 0$, because by hypothesis $Z(f)$ is zero for values f outside this interval and thus $Z^*(f - \Omega/2)Z(f + \Omega/2)$ must also be zero outside this interval. According to inequality (87), however, $R_{f,t}(\tau, \Omega)$ will vary negligibly for values of f in this interval and thus for values of f for which the integrand in (90) is nonzero. It follows that insignificant error will result in (90) if we use $R_{0,t}(\tau, \Omega)$ in place of $R_{f,t}(\tau, \Omega)$.

Furthermore we note that, since $w(t)$ is constrained by hypothesis to be nonzero only over an interval of t values of width $T + \Delta$ centered on $t = 0$, then the left-hand side of (90) must perforce exhibit the same property as a function of t. Since the double integral in (90) must vanish for values of t outside an integral of width $T + \Delta$ centered on $t = 0$, and since by inequality (88), $R_{f,t}(\tau, \Omega)$ in the integrand can vary negligibly in this interval, it follows that little error can result by replacing $R_{f,t}(\tau, \Omega)$ by $R_{f,0}(\tau, \Omega)$ in the integrand.

It then follows that if both inequalities (87) and (88) are valid, we have the close approximation

$$\overline{w^*(t)w(s)} = \int \chi(\Omega, \tau)R_{00}(\tau, \Omega)e^{i2\pi\Omega t}\, d\Omega \tag{91}$$

where

$$\chi(\Omega, \tau) = \int Z^*\left(f - \frac{\Omega}{2}\right)Z\left(f + \frac{\Omega}{2}\right)e^{-i2\pi\tau f}\, df \tag{92}$$

is the ambiguity function of the transmitted signal. The expression (91) for the output signal correlation function is identical in form to that for a WSSUS channel in which

$$R(\tau, \Omega) = R_{00}(\tau, \Omega). \tag{93}$$

It will be recalled that initially we had assumed the spectrum of the input and the output time function were centered at $f = 0$ and $t = 0$ respectively. If we had assumed instead that they were centered at $f = f'$ and $t = t'$, the satisfaction of the inequalities (87) and (88) would still have lead us to conclude that the output correlation function can be determined by replacing the actual channel by a hypothetical WSSUS channel. However, instead of (93) we must use

$$R(\tau, \Omega) = R_{f',t'}(\tau, \Omega). \tag{94}$$

We are now in a position to consider the application of the preceding analytical results to the characterization of radio channels. As discussed at the beginning of this section, many radio channels seem to exhibit a combination of fast fading of a nearly Gaussian nature[17] and very slow fading of a generally non-Gaussian nature. We shall assume this to be the case for radio channels using the extended concept of fading described previously. Thus we assume that $T(f, t)$ "fades" along the frequency axis with a "fast" frequency fading superimposed upon a "slow" frequency fading.

Let us momentarily consider that the very slow variations are deterministic by selecting a particular member function of the stochastic process defining the slow fading, and let us assume that the fast fading is Gaussian. Then all statistical information concerning the output of the channel may be obtained once the correlation function $R_{f,t}(\tau, \Omega)$ is known, since then (90) may be used to determine the correlation function of the output Gaussian process $w(t)$.

We may ascribe the variations of $R_{f,t}(\tau, \Omega)$ with f, t as due to the (temporarily deterministic) slow (time and frequency) fading of the channel. In practical channels it appears sufficient to consider

$$R_{f,t}(0, 0) = \overline{|T(f, t)|^2} \tag{95}$$

in order to obtain a feeling as to the degree to which $R_{f,t}(\tau, \Omega)$ varies with f and t. From the definition of $T(f, t)$ we see that $R_{f,t}(0, 0)$ is equal to twice the average power received from a unit amplitude sine wave at time t and of frequency f cps away from carrier frequency, as measured by averaging along an ensemble of channels *all with the same deterministic slow variations*. In practice we do not have available an ensemble of channels. However, we may obtain an approximate measurement of (95) with the determination of the time average;

$$\frac{1}{T_1}\int_{t-T_1/2}^{t+T_1/2} |T(f, t_1)|^2 \, dt_1 \equiv \langle |T(f, t)|^2 \rangle_{T_1} \tag{96}$$

where the averaging T_1 is (hopefully) long enough to produce negligible measurement fluctuations due to the fast fading but yet short enough to reflect the long term fading behavior of the channel.

[17] Some confusion may exist in the reader's mind as to precisely what is meant by Gaussian fading, since fading is usually stated to be Rayleigh distributed. By Gaussian fading it is meant that the transmission of a sinusoid results in the reception of a narrow-band Gaussian process with a possible nonfading specular component present. It is the envelope which will be Rayleigh or Rice distributed depending upon the nonexistence or existence of the specular component.

Measurements such as (96) seem to indicate that the inequalities (87) and (88) will be satisfied for a large percentage of radio channels for operating frequencies and signaling element bandwidths and time durations of practical interest. Thus it appears that a useful model for several radio channels is the QWSSUS channel with Gaussian statistics. Measurement of the correlation function $R_{f,t}(\tau, \Omega)$ on a short-time basis will then provide the necessary statistical information to evaluate the short-time performance of a digital system, assuming the statistics of any additive interferences are known. This short-time performance will be a functional of $R_{f,t}(\tau, \Omega)$. Since $R_{f,t}(\tau, \Omega)$ is in effect a random process due to the slow channel fluctuations (we have removed the deterministic assumption), the performance index (say error probability) computed on a short time basis assuming a Gaussian QWSSUS channel must be averaged over the long term statistics of $R_{f,t}(\tau, \Omega)$ to determine a long-time basis performance index.

VI. Canonical Channel Models

All practical channels and signals have an essentially finite number of degrees of freedom due to restrictions on time duration, fading rate, bandwidth, etc. These restrictions allow a simplified representation of linear time-varying channels in terms of canonical elements or building blocks. Such channel representations, called canonical channel models, can simplify the analysis of the performance of communication systems which employ time-variant linear channels.

Two general classes of channel models, called Sampling Models and Power Series Models, will be considered in this paper. The Sampling Models apply when a system function vanishes for values of an independent variable (time t, frequency f, delay ξ, or Doppler shift ν) outside some interval or when the input or output time function is time-limited or band-limited. The conditions for the applicability of the Power Series Models are not so simply stated. Stated briefly it requires the existence of a power series expansion of a system function in an independent variable and, depending upon the channel model, the existence of derivatives of the input function or spectrum.

A. Sampling Models

In this section we will develop the various sampling canonical channel models referred to above. The models developed by Kailath[2] will also be included not only for completeness but because the significance of some of the new sampling models is enhanced since they are dual to those of Kailath.

It will be convenient to divide our discussion into two parts, one involving time and frequency constraints and the other involving delay and Doppler-shift constraints.

1) *Time and Frequency Constraints*: A quick understanding of the time and frequency limitations that are relevant in the case of the sampling channel models may be arrived at by examining the input-output relationships corresponding to the Time-Variant Transfer Function

$T(f, t)$ and the Frequency-Dependent Modulation Function $M(t, f)$. These relationships are repeated below for convenience:

$$w(t) = \int Z(f)T(f, t)e^{i2\pi ft}\, df$$
$$W(f) = \int z(t)M(t, f)e^{-i2\pi ft}\, dt. \tag{97}$$

If a time-variant linear filter is preceded by a time-invariant linear filter with transfer function $T_i(f)$ and is followed with a multiplication by a time function $M_0(t)$, a combination time-variant linear filter which includes the input filter and output multiplier has a Time-Variant Transfer Function $T'(f, t)$ given by

$$T'(f, t) = T_i(f)T(f, t)M_0(t). \tag{98}$$

Eq. (98) is quickly deduced from the first equation in (97) by noting that preceding the original filter by a time-invariant linear filter with transfer function $T_i(f)$ is equivalent to changing the input spectrum from $Z(f)$ to $Z(f)\, T_i(f)$, while following the original filter by a multiplication with $M_0(t)$ is equivalent to multiplying both sides of the first equation in (97) by $M_0(t)$.

Since a constraint on the bandwidth of the input signal to a frequency region of width W_i centered on f_i, *i.e.*, to $f_i - W_i/2 < f < f_i + W_i/2$ cps may be represented by means of a band-limiting filter at the channel input, it is clear from (98) that such a constraint can be handled conveniently by defining a hypothetical channel whose Time-Variant Transfer Function $T'(f, t)$ is given by

$$T'(f, t) = \text{Rect}\left(\frac{f - f_i}{W_i}\right)T(f, t) \tag{99}$$

where $T(f, t)$ is the actual system function and

$$\text{Rect}(x) = \begin{cases} 1 & |x| < \frac{1}{2}. \\ 0 & |x| \geq \frac{1}{2} \end{cases} \tag{100}$$

Since the Input Delay-Spread Function $g(t, \xi)$ is the inverse Fourier transform of the Time-Variant Transfer Function $T(f, t)$ with respect to the frequency variable, it follows that corresponding to the hypothetical system function $T'(f, t)$ in (99) there is a hypothetical Input Delay-Spread Function $g'(t, \xi)$ given by

$$g'(t, \xi) = \int e^{i2\pi f_i(\xi - \eta)}W_i \cdot \text{sinc}\,[W_i(\xi - \eta)]g(t, \eta)\, d\eta \tag{101}$$

where

$$\text{sinc}\, y = \frac{\sin \pi y}{\pi y}. \tag{102}$$

Eq. (101) is obtained from (99) by noting that multiplication corresponds to convolution in the transform domain and that the inverse transform of Rect $([f - f_i]/W_i)$ is $e^{i2\pi f_i \xi}W_i$ sinc $[W_i \xi]$.

Thus the case wherein the channel responds (*i.e.*, has a nonzero output) only to input frequencies in a given range, say $f_i - W_i/2 < f < f_i + W_i/2$, and the case wherein the channel responds to other frequencies but has an input signal limited to frequencies in the range $f_i - W_i/2 < f < f_i W_i/2$ may be handled by the same analytical approach. In deriving the sampling model relevant to an input signal bandwidth limitation or an input frequency response limitation it will be convenient to assume that $T(f, t)$ is nonzero only for values of f within the relevant frequency interval. It should be kept in mind, however, that when the input signal rather than the input frequency response of the channel is band-limited, one must eventually use an equation such as (99) or (101) in order to express the parameters of the canonical channel model in terms of the true channel system function.

If it is desired to observe the channel output for some finite time interval, say $t_0 - T_0/2 < t < t_0 + T_0/2$ or if, due to some gating operation in the receiver, only a finite time segment of the received waveform in the same interval is available, one has a constraint on the time duration of the channel output. It is clear from (98) that such a constraint may be handled analytically by defining a hypothetical channel whose Time-Variant Transfer Function $T'(f, t)$ is given by

$$T'(f, t) = T(f, t)\, \text{Rect}\left(\frac{t - t_0}{T_0}\right). \tag{103}$$

Since the Output Doppler-Spread Function $G(f, \nu)$ is the Fourier transform of the Time-Variant Transfer Function with respect to the time variable, it follows that corresponding to the hypothetical system function $T'(f, t)$ in (103) there is a hypothetical Output Doppler-Spread Function $G'(f, \nu)$ given by

$$G'(f, \nu) = \int e^{-i2\pi t_0(\nu - \mu)}T_0 \cdot \text{sinc}\,[T_0(\nu - \mu)]G(f, \mu)\, d\mu. \tag{104}$$

Eq. (104) follows from (103) for the same reasons that (101) followed from (99).

Thus the case wherein the channel output vanishes outside some time interval, say $t_0 - T_0/2 < t < t_0 + T_0/2$, and the case where the channel has outputs outside this range but the receiver observes the received waveform only in this range may be handled by the same analytical approach. In deriving the sampling model relevant to a limitation in a receiver observation time it will be convenient to assume that $T(f, t)$ is nonzero only for values of t within the relevant time interval. However, it should be kept in mind that when the output observation time is limited rather than the output time response of the channel, one must eventually use an equation such as (103) or (104) in order to express the parameters of the canonical channel model in terms of the true channel system function.

A discussion entirely dual to the one above concerning input frequency and output time constraints and dealing with $T(f, t)$ may be carried through for an input time and output frequency constraint by dealing with $M(t, f)$ the Frequency-Dependent Modulation Function. Thus, if a time-variant filter is preceded by a multiplier which multiplies the input by $M_i(t)$ and is followed with a time-invariant linear filter of transfer function $T_0(f)$, a combination time-variant linear filter which includes the input multiplier and output filter has a Frequency-Dependent Modulation Function given by

$$M'(t, f) = M_i(t)M(t, f)T_0(f). \tag{105}$$

Eq. (105) is quickly deduced from the second equation in (97) by noting that preceding the original filter with a multiplication by $M_i(t)$ is equivalent to changing the input time function from $z(t)$ to $z(t)M_i(t)$, while following the original filter by a filter with transfer function $T_0(f)$ is equivalent to multiplying both sides of the second equation in (97) by $T_0(f)$.

A constraint on the duration of the input waveform to, say, $t_i - T_i/2 < t < t_i + T_i/2$ can be handled analytically by using a hypothetical Frequency-Dependent Modulation Function $M'(t, f)$ given by

$$M'(t, f) = \operatorname{Rect}\left(\frac{t - t_i}{T_i}\right)M(t, f), \tag{106}$$

or equivalently by using a hypothetical Input Doppler-Spread Function $H'(f, \nu)$ given by

$$H'(f, \nu) = \int e^{-i2\pi t_i(\nu-\mu)}T_i \operatorname{sinc}[T_i(\nu - \mu)]H(f, \mu)\, d\mu \tag{107}$$

in place of the actual system function and then assuming an internal input time constraint.

A constraint on the frequency interval over which the channel output is observed, say to $f_0 - W_0/2 < f < f_0 + W_0/2$ cps, may be handled analytically by using

$$M'(t, f) = M(t, f) \operatorname{Rect}\left(\frac{f - f_0}{W_0}\right) \tag{108}$$

in place of the actual Frequency-Dependent Modulation Function $M(t, f)$, or equivalently by using

$$h'(t, \xi) = \int e^{i2\pi f_0(\xi-\eta)}W_0 \operatorname{sinc} W_0(\xi - \eta)h(t, \eta)\, d\eta \tag{109}$$

in place of the actual Output Delay-Spread Function $h(t, \xi)$ and then assuming an internal output bandwidth constraint.

Having completed our preliminary discussion of time and frequency constraints we may now proceed to the determination of the corresponding sampling canonical channel models.

All the sampling models are derived by application of the Sampling Theorem,[18] which states that if a function $h(x)$ is zero for values of x outside an interval $-X/2 < x < X/2$, then its Fourier transform $H(y)$ may be expressed as the following series

$$H(y) = \sum H\left(\frac{k}{X}\right) \operatorname{sinc}\left[X\left(y - \frac{k}{X}\right)\right] \tag{110}$$

where

$$\operatorname{sinc} y = \frac{\sin \pi y}{\pi y} \tag{111}$$

and

$$H(y) = \int h(x)e^{-i2\pi xy}\, dx \quad \text{or} \quad \int h(x)e^{i2\pi xy}\, dx. \tag{112}$$

When $h(x)$ vanishes outside an interval which is centered on x_1, *i.e.*, when $h(x)$ vanishes outside the interval $x_1 - X/2 < x < x_1 + X/2$, the Sampling Theorem becomes

$$H(y) = \sum H\left(\frac{k}{X}\right)e^{\pm i2\pi x_1(y-k/X)} \operatorname{sinc}\left[X\left(y - \frac{k}{X}\right)\right] \tag{113}$$

where the sign of the exponential used in (113) agrees with the sign of the exponential used in the Fourier transform definition of $H(y)$.

a) *Sampling Models for Input Time and Frequency Constraints*: In this subsection we will derive sampling models appropriate for input time and frequency constraints. Consider first the case of an input time constraint. As discussed above, such a case may be described by stating that $M(t, f)$ vanishes for values of t outside some interval, say, $t_i - T_i/2 < t < t_i + T_i/2$. Since the Input Doppler-Spread Function $H(f, \nu)$ is the Fourier transform of $M(t, f)$ with respect to t (see Fig. 5), it follows from (113) that

$$H(f, \nu) = \sum H\left(f, \frac{n}{T_i}\right)e^{-i2\pi t_i(\nu-n/T_i)} \cdot \operatorname{sinc}\left[T_i\left(\nu - \frac{n}{T_i}\right)\right]. \tag{114}$$

If the summation in (114) is used in place of $H(f, \nu)$ in the input-output relationship [(12)]

$$W(f) = \int Z(f - \nu)H(f, \nu)\, d\nu,$$

one finds that in the case of an input time constraint the output spectrum is given by

$$W(f) = \sum H_n(f) \int Z(f - \nu)T_i \cdot \operatorname{sinc}\left[T_i\left(\nu - \frac{n}{T_i}\right)\right]e^{-i2\pi t_i(\nu-n/T_i)}\, d\nu \tag{115}$$

where

$$H_n(f) = \frac{1}{T_i} H\left(f, \frac{n}{T_i}\right). \tag{116}$$

Note that the integral in (115) is just the convolution of the input spectrum $Z(f)$ with $e^{-i2\pi t_i(f-n/T_i)}T_i$ sinc $[T_i(f - n/T_i)]$. Since convolution becomes multiplication in the transform domain, and since the time function corresponding to $e^{-i2\pi t_i(f-n/T_i)}T_i$ sinc $[T_i(f - n/T_i)]$ is

[18] P. M. Woodward, *op. cit.*,[7] pp. 33–34.

$\exp [j2\pi n(t/T_i)]$ Rect $([t - t_i]/T_i)$, the time function corresponding to the convolution integral in (116) is just the product $z(t) \exp [j2\pi nt/T_i]$ Rect $([t - t_i]/T_i)$. Thus, (115) states that the channel output may be obtained by gating the input with the time function Rect $([t - t_i]/T_i)$, frequency shifting the gated input by harmonics of $1/T_i$ cps, filtering each of these gated frequency-shifted waveforms with an appropriate filter $H_n(f)$ for each harmonic and then summing the result. This series of operations immediately suggests the channel model shown in Fig. 10. Although, in theory, an infinite number of frequency converter-filtering elements would be required, in practice a finite number will suffice since for a physical channel the range of Doppler shifts is finite and thus $H(f, \nu)$ must effectively vanish for ν outside some interval.

When the channel is randomly time-variant the filters $H_n(f)$ become random filters. The correlation properties of these filters are defined by the average

$$\overline{H_n^*(f)H_m(l)} = \left(\frac{1}{T_i}\right)^2 R_H\left(f, l; \frac{n}{T_i}, \frac{m}{T_i}\right). \tag{117}$$

For the case of the US channel (90) simplifies to [see (66)]

$$\overline{H_n^*(f)H_m(f + \Omega)} = \left(\frac{1}{T_i}\right)^2 R_H\left(\Omega; \frac{n}{T_i}, \frac{m}{T_i}\right). \tag{118}$$

It is not difficult to see that a channel may not have an *internal* input (or output) time constraint and be a WSS (or WSSUS) channel since the existence of a time constraint is incompatible with the existence of stationarity. However, an *external* input (or output) time constraint may be associated with any type of channel.

The correlation properties of the random filters as expressed in (117) and (118) are pertinent to the case of an internal input time constraint. In order to obtain from (85) the corresponding cross-correlation function between the random filters for the case of an external input time constraint, *i.e.*, the case of a time-limited input waveform (*e.g.*, a pulse input), we replace the actual Input Doppler-Spread Function by a hypothetical one as indicated in (107). It is quickly seen that instead of (116) we have

$$H_n(f) = \int e^{-j2\pi t_i(\nu - n/T_i)} \operatorname{sinc}\left[T_i\left(\nu - \frac{n}{T_i}\right)\right] H(f, \nu)\, d\nu \tag{119}$$

as the expression for the transfer function of the random filter in the canonic channel model of Fig. 10 when the input time constraint is external to the channel proper. Also, instead of (117) we have

$$\begin{aligned}\overline{H_n^*(f)H_m(l)} &= \iint e^{-j2\pi t_i(\nu - n/T_i - \mu + m/T_i)} \operatorname{sinc}\left[T_i\left(\nu - \frac{n}{T_i}\right)\right] \\ &\cdot \operatorname{sinc}\left[T\left(\mu - \frac{m}{T_i}\right)\right] R_H(f, l; \nu, \mu)\, d\nu\, d\mu\end{aligned} \tag{120}$$

as the cross-correlation function between the random filters for the general channel. By using the appropriate form for the correlation functions, (120) may be specialized

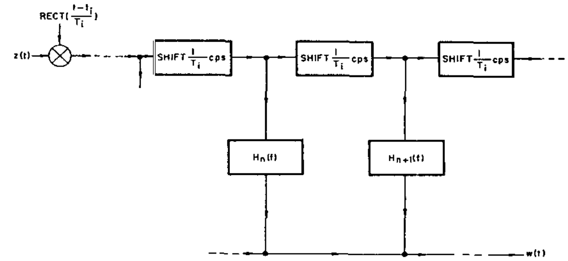

Fig. 10—Canonical channel model for input time constraint, output filter version.

for the US, WSS, and WSSUS channels. We mention only the simplified form it takes in the case of the WSSUS channel,

$$\begin{aligned}\overline{H_n^*(f)H_m(f + \Omega)} &= \int e^{-j2\pi t_i(m-n)/T_i} \operatorname{sinc}\left[T_i\left(\nu - \frac{n}{T_i}\right)\right] \\ &\cdot \operatorname{sinc}\left[T_i\left(\nu - \frac{m}{T_i}\right)\right] P_H(\Omega, \nu)\, d\nu.\end{aligned} \tag{121}$$

It will be recalled that $P_H(\Omega, \nu)$ is equal to the cross-power spectral density between the processes $M(f, t)$ and $M(f + \Omega, t)$. From (121) it is readily seen that if $P_H(\Omega, \nu)$ changes very little for changes in ν of the order of the reciprocal of the duration of the input waveform, $1/T_i$, we have the approximation

$$\overline{H_n^*(f)H_m(f + \Omega)} \approx \begin{cases} 0 & ; \quad m \neq n \\ \dfrac{1}{T_i} P_H\left(\Omega, \dfrac{n}{T_i}\right) & ; \quad m = n \end{cases}. \tag{122}$$

Thus, in the case of the WSSUS channel, when $P_H(\Omega, \nu)$ varies little for changes in ν of the order of the reciprocal of the duration T_i of the input time constraint, the various random filters become uncorrelated and the frequency correlation function of an individual filter transfer function $H_n(f)$ becomes proportional to the value of the density function $P_H(\Omega, \nu)$ at $\nu = n/T_i$.

We will now determine the channel model appropriate to the case of an input bandwidth restriction.[19] This restriction is dual to that of an input time limitation and, as will be seen, leads to a dual channel model.

Let us assume that the channel responds only to frequencies within the interval $f_i - W_i/2 < f < f_i + W_i/2$. From the discussion in Subsection VIA.1), we see that this is equivalent to stating that $T(f, t)$ is zero for values of f outside this interval. Since $g(t, \xi)$ is the inverse Fourier transform of $T(f, t)$, an application of the Sampling Theorem shows that $g(t, \xi)$ may be represented by the series

$$g(t, \xi) = \sum g\left(t, \frac{n}{W_i}\right) e^{j2\pi f_i(\xi - n/W_i)} \cdot \operatorname{sinc}\left[W_i\left(\xi - \frac{n}{W_i}\right)\right]. \tag{123}$$

[19] This channel model has previously been derived by Kailath, *op. cit.*[2]

If the summation in (123) is used in place of $g(t, \xi)$ in the input-output relationship

$$w(t) = \int z(t - \xi) g(t, \xi)\, d\xi,$$

one finds that in the case of an input frequency constraint the output time function may be represented in the form

$$w(t) = \sum g_n(t) \int z(t - \xi) W_i \cdot \operatorname{sinc}\left[W_i\left(\xi - \frac{n}{W_i}\right)\right] e^{j2\pi f_i(\xi - n/W_i)}\, d\xi \tag{124}$$

where

$$g_n(t) = \frac{1}{W_i}\, g\left(t, \frac{n}{W_i}\right). \tag{125}$$

Note that the integral in (124) is just the convolution of the input with a time function $W_i \exp[j2\pi f_i(t - n/W_i)]$ sinc $[W_i(t - n/W_i)]$. Since the spectrum corresponding to this latter time function is $\exp[-j2\pi n(f/W_i)]$ rect $([f - f_i]/W_i)$, it follows that the spectrum corresponding to the convolution in (124) is just the product $Z(f)$ Rect $([f - f_i]/W_i) \exp[-j2\pi n(f/W_i)]$. Thus (124) states that the channel output may be obtained by passing the input through a band-limiting filter with transfer function Rect $([f - f_i]/W_i)$, delaying the resultant by multiples of a basic delay $1/W_i$, multiplying each of these delayed functions by a multiplier $g_n(t)$ appropriate to the delay n/W_i, and then summing the result. This series of operations immediately suggests the channel model shown in Fig. 11.

Although, in theory, an infinite number of taps would be required, in practice a finite number will suffice, since for a physical channel the spread of path delays is finite and thus $g(t, \xi)$ must effectively vanish for ξ outside some interval.

When the channel is randomly time variant the multipliers $g_n(t)$ become random processes. The correlation properties of these multipliers are defined by the average

$$\overline{g_n^*(t) g_m(s)} = \left(\frac{1}{W_i}\right)^2 R_g\left(t, s; \frac{n}{W_i}, \frac{m}{W_i}\right). \tag{126}$$

For the case of the WSS channel,

$$\overline{g_n^*(t) g_m(t, \tau)} = \left(\frac{1}{W_i}\right)^2 R_g\left(\tau; \frac{n}{W_i}, \frac{m}{W_i}\right). \tag{127}$$

It is not difficult to see that a channel may not have an *internal* input (or output) bandwidth constraint and be a US (or WSSUS) channel. To understand this fact recall [see discussion following (66)] that the US channel has a wide sense stationarity property in the frequency variable. Such a property is clearly incompatible with an internal input (or output) frequency constraint. However, an *external* frequency constraint may, of course, be associated with any type of channel.

The correlation properties of the random multipliers as expressed in (126) and (127) are pertinent to the case

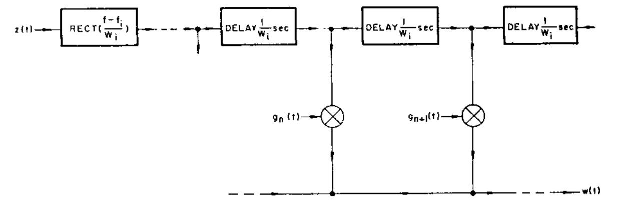

Fig. 11—Canonical channel model for input frequency constraint, output multiplier version.

of an internal input bandwidth constraint. In order to obtain the corresponding cross-correlation function for the case of an external input bandwidth constraint, *i.e.*, a band-limited input signal, we replace the actual Input Delay-Spread Function by a hypothetical one as indicated in (99). It is readily seen that instead of (125), we have

$$g_n(t) = \int e^{-j2\pi f_i(\xi - n/W_i)} \cdot \operatorname{sinc}\left[W_i\left(\xi - \frac{n}{W_i}\right)\right] g(t, \xi)\, d\xi \tag{128}$$

as the expression for the multiplier associated with a delay of n/W_i sec in the canonic model of Fig. 11 when the input frequency constraint is external to the channel proper. Instead of (127), we have

$$\overline{g_n^*(t) g_m(s)} = \iint e^{j2\pi f_i(\xi - \eta - n/W_i + m/W_i)} \operatorname{sinc}\left[W_i\left(\xi - \frac{n}{W_i}\right)\right] \cdot \operatorname{sinc}\left[W_i\left(\eta - \frac{m}{W_i}\right)\right] R_g(t, s; \xi, \eta)\, d\xi\, d\eta. \tag{129}$$

By using the appropriate form for the correlation functions (129) may be specialized for the US, WSS, and WSSUS channels. We mention only the simplified form it takes in the case of the WSSUS channel,

$$\overline{g_n^*(t) g_m(t + \tau)} = \int e^{j2\pi f_i([n-m]/W_i)} \cdot \operatorname{sinc}\left[W_i\left(\xi - \frac{m}{W_i}\right)\right] P_g(\tau, \xi)\, d\xi. \tag{130}$$

Examination of (130) shows that if $P_g(\tau, \xi)$ changes little for changes in ξ of the order of $1/W$, we have the approximation

$$\overline{g_n^*(t) g_m(t + \tau)} \approx \begin{cases} 0 & ; \quad m \neq n \\ \dfrac{1}{W_i} P_g\left(\tau, \dfrac{n}{W_i}\right) & ; \quad m = n \end{cases}. \tag{131}$$

There exist alternate channel models closely related to those of Figs. 10 and 11. Consider first the case of an input time constraint and the channel model of Fig. 10. If we define $Z'(f)$ as the spectrum of the time function, $z(t)$ Rect $([t - t_i]/T)$, which is the input to the frequency conversion chain, it is readily seen that the spectrum of the channel output $W(f)$ may be expressed as

$$W(f) = \sum_n H_n(f) Z'\left(f - \frac{n}{T_i}\right). \tag{132}$$

If we define the filter

$$G_n(f) = H_n\left(f + \frac{n}{T_i}\right) \tag{133}$$

then the output spectrum may be expressed as

$$W(f) = \sum G_n\left(f - \frac{n}{T_i}\right)Z'\left(f - \frac{n}{T_i}\right) \tag{134}$$

which states that the output may be obtained by gating, filtering, and frequency shifting as indicated in Fig. 12. With the aid of (133) and (116) and (119) one may determine expressions for $G_n(f)$ in terms of the system functions for the cases of internal and external time constraints. Also the correlation properties of $G_n(f)$ are readily determined from those of $H_n(f)$ with the aid of (133).

Similarly, if we define $z''(t)$ as the time function resulting from band-limiting the input as shown in Fig. 11, it is seen that the channel output $w(t)$ may be expressed as

$$w(t) = \sum_n g_n(t) z''\left(t - \frac{n}{W_i}\right). \tag{135}$$

If we define the time function

$$h_n(t) = g_n\left(t + \frac{n}{W_i}\right) \tag{136}$$

then the output time function may be expressed as

$$w(t) = \sum_n h_n\left(t - \frac{n}{W_i}\right)z''\left(t - \frac{n}{W_i}\right) \tag{137}$$

which states that the output may be obtained by band-limiting, multiplying, and delaying as indicated in Fig. 13. With the aid of (136) and (125) and (128) one may determine expressions for $h_n(t)$ in terms of the system functions for the cases of internal and external bandwidth constraints. Also the correlation properties of $h_n(t)$ are readily determined from those of $g_n(t)$ with the aid of (136).

b) *Sampling Models for Output Time and Frequency Constraints*: The development of channel models for output time and frequency constraints parallels the previous development of channel models for input time and frequency constraints. When considering an output time constraint one specifies that $T(f, t)$ vanishes for t outside some interval, say, $t_0 - T_0/2 < t < t_0 + T_0/2$ while for an output frequency constraint one specifies that $M(t, f)$ vanishes for f outside an interval $f_0 - W_0/2 < f < f_0 + W_0/2$. Then, for an output time constraint, one makes a sampling expansion of $G(f, \nu)$ [the Fourier transform of $T(f, t)$ with respect to t] in the ν variable while for an output frequency constraint one makes a sampling expansion of $h(t, \xi)$ [the inverse Fourier transform of $M(t, f)$ with respect to f] in the ξ variable. By using these expansions in the appropriate input-output relations and examining the resulting series expressions for the output (as was done for the case of input time and frequency constraints) one may obtain the appropriate canonical channel models for the case of output time and frequency constraints.

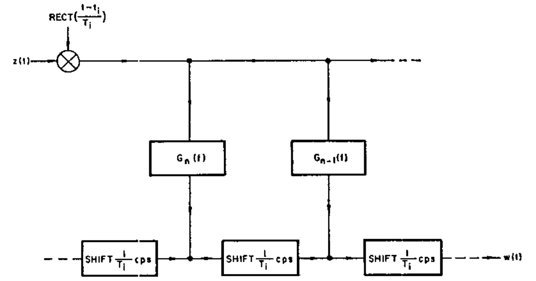

Fig. 12—Canonical channel model for input time constraint, input filter version.

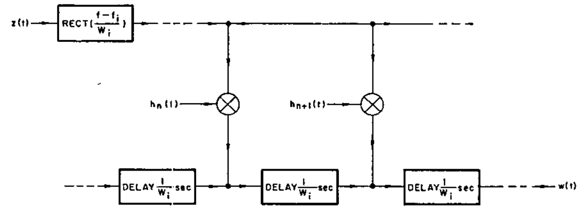

Fig. 13—Canonical channel model for input frequency constraint, input multiplier version.

We shall not give the details of the derivations because of their similarity to the derivations in the previous section. The resulting canonical models are shown in Figs. 14 to 17.[20] Note that the output time constraint models differ in form from the input time constraint models only in having an output time gate instead of an input time gate. Similarly, the output frequency constraint models differ from the input frequency constraint models only in having an output rather than input band-pass filter.

The filter transfer functions in Figs. 14 and 15 are given by

$$\begin{aligned} \hat{H}_n(f) &= \frac{1}{T_0} H\left(f, \frac{n}{T_0}\right) \\ \hat{G}_n(f) &= \frac{1}{T_0} G\left(f, \frac{n}{T_0}\right) = H_n\left(f + \frac{n}{T_0}\right) \end{aligned} \tag{138}$$

in the case of internal output time constraints. When the output time constraint is external, however, one must use

$$\begin{aligned} \hat{G}_n(f) &= \int e^{i2\pi t_0(\nu - n/T_0)} \operatorname{sinc}\left[T_0\left(\nu - \frac{n}{T_0}\right)\right] G(f, \nu)\, d\nu \\ \hat{H}_n(f) &= G_n\left(f - \frac{n}{T_0}\right). \end{aligned} \tag{139}$$

The gain functions in Figs. 16 and 17 are

$$\begin{aligned} \hat{h}_n(t) &= \frac{1}{W_0} h\left(t, \frac{n}{W_0}\right) \\ \hat{g}_n(t) &= \frac{1}{W_0} g\left(t, \frac{n}{W_0}\right) = h\left(t - \frac{n}{W_0}\right) \end{aligned} \tag{140}$$

in the case of internal output bandwidth constraints.

[20] The model in Fig. 16 has been previously derived by Kailath, *op. cit.* [2]

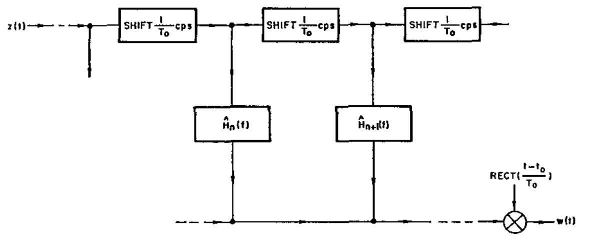

Fig. 14—Canonical channel model for output time constraint, output filter version.

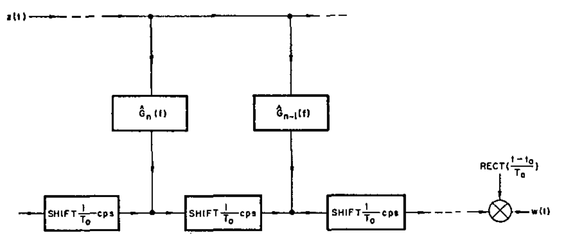

Fig. 15—Canonical model for output time constraint, input filter version.

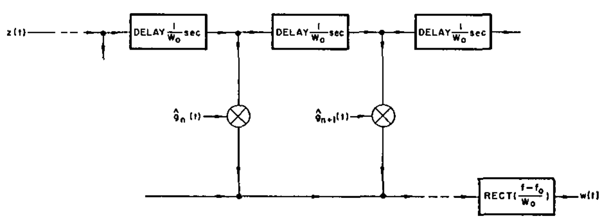

Fig. 16—Canonical channel model for output frequency constraint, output multiplier version.

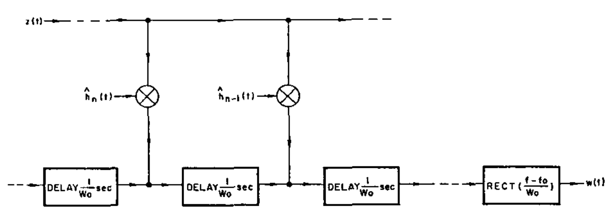

Fig. 17—Canonical channel model for output frequency constraint, input multiplier version.

For external output bandwidth constraints the gain functions become

$$\hat{h}_n(t) = \int e^{-j2\pi f_0(\xi - n/W_0)} \cdot \operatorname{sinc}\left[W_0\left(\xi - \frac{n}{W_0}\right)\right] h(t, \xi)\, d\xi$$
$$\hat{g}_n(t) = \hat{h}_n\left(t - \frac{n}{W_0}\right). \tag{141}$$

For randomly time-varying channels the correlation properties of the filter transfer functions in Figs. 14 and 15 and the gain functions in Figs. 16 and 17 are quickly determined in terms of the appropriate channel correlation functions, as was done for the case of input time and frequency constraints.

c) *Sampling Models for Combined Time and Frequency Constraints*: In Subsections VA. 1) a) and b), we have developed canonic channel models for the case of a single constraint on time or frequency at the input or output of the channel. However, it is possible for certain combinations of these constraints to exist for the same channel. Two interesting combinations, which are dually related, are the cases of a combined input time constraint and output frequency constraint and a combined input frequency constraint and output time constraint. Other combinations are either impossible or do not lead to new models. The impossible combinations are combined internal time and frequency constraints on the same end of the channel, since (as a study of the meaning of an internal constraint readily reveals) such combined constraints imply the existence of functions which are both time- and band-limited.

Consider now the case of an internal input time and output frequency constraint. Mathematically we can represent such a combined constraint by stating that the Frequency-Dependent Modulation Function $M(t, f)$ vanishes for values of t and f outside the rectangle $t_i - T_i/2 < t < t_i + T_i/2$, $f_0 - W_0/2 < f < f_0 + W_0/2$. Thus, $M(t, f)$ satisfies the equation

$$M(t, f) = \operatorname{Rect}\left(\frac{t - t_i}{T_i}\right) M(t, f) \operatorname{Rect}\left(\frac{f - f_0}{W_0}\right). \tag{142}$$

But it is readily seen that we can also write

$$M(t, f) = \operatorname{Rect}\left(\frac{t - t_i}{T_i}\right) \tilde{M}(t, f) \operatorname{Rect}\left(\frac{f - f_0}{W_0}\right) \tag{143}$$

where

$$\tilde{M}(t, f) = \sum_{m,n=-\infty}^{\infty}\sum M(t - mT_i, f - mW_0) \tag{144}$$

since only the $m = 0$, $n = 0$ term in the sum (144) defining $\tilde{M}(t, f)$ contributes nonzero values to the left side of (143).

Eq. (143) states that the channel under discussion may be represented as the cascade of three operations. The first is an input gating operation with the function Rect $([t - t_i]/T_i)$, the second is a filtering operation by means of a time variant filter with Frequency-Dependent Modulation Function $\tilde{M}(t, f)$ and the last is a band-pass filtering operation with transfer function Rect $([f - f_0]/W_0)$. We will now develop a canonical channel model for the second operation and then obtain our desired canonic channel model for the combined input time and output frequency constraint by preceding the canonic channel model of the second operation by the time gate Rect $([t - t_i]/T_i)$ and following it by the band-pass filter Rect $([f - f_0]/W_0)$.

It will be recalled (see Fig. 5 for example) that the Doppler-Delay-Spread Function $V(\nu, \xi)$ is the double Fourier transform of $M(t, f)$,

$$V(\nu, \xi) = \iint M(t, f) e^{-j2\pi(t\nu - f\xi)}\, dt\, df.$$

Thus the Doppler-Delay-Spread Function corresponding to $\tilde{M}(t, f)$ is given by

$$\tilde{V}(\nu, \xi) = \sum_{m,n}\sum \iint M(t - mT_i, f - nW_0) \cdot e^{-j2\pi(t\nu - f\xi)}\, dt\, df$$
$$= V(\nu, \xi)\Big(\sum_m e^{-j2\pi mT_i\nu}\Big)\Big(\sum_n e^{j2\pi nW_0\xi}\Big) \tag{145}$$

where the second equation follows from the first by making the changes of variable $t - mT_i \rightarrow t$, $f - nW_0 \rightarrow f$ in the double integral in the first equation.

The sums in the second equation in (145) may be recognized as Fourier series expansions of periodic impulse trains, *i.e.*,

$$\sum_m e^{-j2\pi mT_i\nu} = \frac{1}{T_i}\sum_m \delta\left(\nu - \frac{m}{T_i}\right)$$
$$\sum_n e^{j2\pi nW_0\xi} = \frac{1}{W_0}\sum_n \delta\left(\xi - \frac{n}{W_0}\right) \tag{146}$$

so that

$$\tilde{V}(\nu, \xi) = \frac{1}{T_iW_0}\sum_{m,n}\sum V\left(\frac{m}{T_i}, \frac{n}{W_0}\right) \cdot \delta\left(\nu - \frac{m}{T_i}\right)\delta\left(\xi - \frac{n}{W_0}\right). \tag{147}$$

If we let $z'(t)$ and $w'(t)$ respectively denote the input and output of the channel with Doppler-Delay-Spread Function $\tilde{V}(\nu, \xi)$, we can express the channel output as

$$w'(t) = \sum_{m,n}\sum V_{mn} z'\left(t - \frac{n}{W_0}\right)e^{j2\pi(m/T_i)(t-n/W_0)} \tag{148}$$

where the complex amplitude

$$V_{mn} = \frac{1}{T_iW_0} V\left(\frac{m}{T_i}, \frac{n}{W_0}\right). \tag{149}$$

Thus, apart from the input gate and the output bandlimiting filter, the channel representation corresponding to an input-time output frequency limitation is a discrete number of point "scatterers" each providing first a fixed Doppler shift which is some multiple of the reciprocal input time duration constraint and then a fixed delay which is some multiple of the reciprocal output bandwidth constraint. The complex amplitude of the reflection from the point scatterer is just $1/T_iW_0$ times the Doppler-Delay-Spread Function sampled at the same value of delay and Doppler shift provided by the scatterer. Fig. 18 demonstrates the realization of such a channel by tapped delay lines and frequency conversion chains.

The canonic model of Fig. 18 can be derived in a somewhat different manner than described above by making use of canonic channel models previously derived for single constraints on input time and output frequency. Thus, consider the canonic channel model for the case of an input time constraint, Fig. 10, and note that when an output frequency constraint exists the filters $H_n(f)$ must

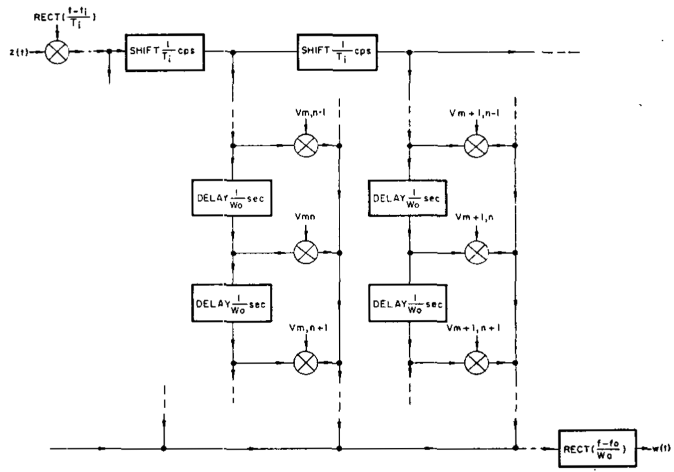

Fig. 18—Canonical channel model for input time-output frequency constraint.

have this same constraint, *i.e.*, $H_n(f)$ must vanish for values of f outside the interval $f_0 - W_0/2 < f < f_0 + W_0/2$. This latter fact becomes quite clear by examination of Fig. 10, since these filters are the only elements in the model which provide frequency selectivity. An analytical proof is readily obtained by noting that $H_n(f) = (1/T_i)H(f, n/T_i)$ and that the f variable in $H(f, \nu)$ and $M(t, f)$ are the same variables since $H(f, \nu)$ and $M(t, f)$ are Fourier transform pairs in the t, ν variables.

Since the filter $H_n(f)$ has an output frequency constraint (and also an input frequency constraint since it is time-invariant), one may represent this filter by means of a canonic channel model consisting of a tapped delay line as shown in Fig. 11, but with time-invariant gains. When this representation of $H_n(f)$ is made one arrives at the canonic channel model shown in Fig. 18.

When the channel is randomly varying the coefficients V_{mn} become random variables. The correlation between these random gains is given by

$$\overline{V_{mn}^* V_{rs}} = \frac{1}{(T_iW_0)^2} R_V\left(\frac{m}{T_i}, \frac{r}{T_i}; \frac{n}{W_0}, \frac{s}{W_0}\right). \tag{150}$$

The expressions for the gain (149) and its correlation properties (150) are applicable for the case of internal constraints on the input time and output frequency. For the case of external constraints the same canonical channel model applies, but in determining V_{mn} the actual Frequency-Dependent Modulation Function $M(t, f)$ should be replaced by a hypothetical one given by

$$M'(t, f) = \mathrm{Rect}\left(\frac{t - t_i}{T_i}\right)M(t, f)\,\mathrm{Rect}\left(\frac{f - f_0}{W_0}\right) \tag{151}$$

or, equivalently, the actual Doppler-Delay-Spread Function $V(\nu, \xi)$ should be replaced by a hypothetical one $V'(\nu, \xi)$ given by

$$V'(\nu, \xi) = \int \mathrm{Rect}\left(\frac{t - t_i}{T_i}\right)e^{-j2\pi t\nu} \cdot \int \mathrm{Rect}\left(\frac{f - f_0}{W_0}\right)e^{j2\pi f\xi}M(t, f)\, dt\, df. \tag{152}$$

With the aid of (108) and (109) the integral with respect to f can be expressed as

$$\int \operatorname{Rect}\left(\frac{f - f_0}{W_0}\right) e^{i2\pi f\xi} M(t, f)\, df$$

$$= \int e^{i2\pi f_0(\xi-\eta)} W_0 \operatorname{sinc}\,[W_0(\xi - \eta)] h(t, \eta)\, d\eta. \qquad (153)$$

Similarly the integration with respect to t may be expressed as

$$\int \operatorname{Rect}\left(\frac{t - t_i}{T_i}\right) e^{-i2\pi t\nu} h(t, \eta)\, dt$$

$$= \int e^{-i2\pi t_i(\nu-\mu)} T_i \operatorname{sinc}\,[T_i(\nu - \mu)] V(\mu, \eta)\, d\mu \qquad (154)$$

which results in the following expression for $V'(\nu, \xi)$:

$$V'(\nu, \xi) = \iint e^{-i2\pi t_i(\nu-\mu)} e^{i2\pi f_0(\xi-\eta)} T_i W_0$$

$$\cdot \operatorname{sinc}\,[T_i(\nu - \mu)] \operatorname{sinc}\,[W_0(\xi - \eta)] V(\mu, \eta)\, d\mu\, d\eta. \qquad (155)$$

The gain V_{mn} in the case of an external input time-output frequency constraint is given by

$$V_{mn} = \frac{1}{T_i W_0} V'\left(\frac{m}{T_i}, \frac{n}{W_0}\right) = \iint e^{i2\pi t_i(\nu - m/T_i)}$$

$$\cdot e^{-i2\pi f_0(\xi - n/W_0)} \operatorname{sinc}\left[T_i\left(\nu - \frac{m}{T_i}\right)\right]$$

$$\cdot \operatorname{sinc}\left[W_0\left(\xi - \frac{n}{W_0}\right)\right] V(\nu, \xi)\, d\nu\, d\xi. \qquad (156)$$

The correlation $\overline{V_{mn}^* V_{rs}}$ is readily determined as a fourfold integral involving $R_V(\nu, \mu; \xi, \eta)$ by using the integral representation, (156), for V_{mn} and V_{rs} and then averaging under the integral sign. It does not appear desirable to take the space to present this fourfold integral. In the case of the WSSUS channel, for which [(72) and (74)]

$$R_V(\nu, \mu; \xi, \eta) = S(\xi, \nu)\delta(\mu - \nu)\delta(\eta - \xi),$$

the fourfold integral becomes the double integral

$$\overline{V_{mn}^* V_{rs}} = e^{-i2\pi (t_i/T_i)(m-r)} e^{i2\pi (f_0/W_0)(n-s)}$$

$$\cdot \iint \operatorname{sinc}\left[T_i\left(\nu - \frac{m}{T_i}\right)\right] \operatorname{sinc}\left[T_i\left(\nu - \frac{r}{T_i}\right)\right]$$

$$\cdot \operatorname{sinc}\left[W_0\left(\xi - \frac{n}{W_0}\right)\right] \operatorname{sinc}\left[W_0\left(\xi - \frac{s}{W_0}\right)\right] S(\xi, \nu)\, d\nu\, d\xi. \qquad (157)$$

When the Scattering Function $S(\xi, \nu)$ varies very little for changes in ξ of the order of $1/W_0$ and changes in ν of the order of $1/T_i$, (157) simplifies to

$$\overline{V_{mn}^* V_{rs}} \approx \begin{cases} 0 & ; \quad m \neq n, \quad r \neq s \\ \dfrac{1}{T_i W_0} S\left(\dfrac{n}{W_0}, \dfrac{m}{T_i}\right) & ; \quad m = n, \quad r = s \end{cases}. \qquad (158)$$

Thus for the WSSUS channel and a sufficiently smooth Scattering Function, the gains of the point "scatterers" in the canonical channel model become uncorrelated and the strength of the reflection from a particular scatterer becomes proportional to the amplitude of the Scattering Function at the same value of delay and Doppler shift provided by the scatterer.

A somewhat different canonical channel model may be derived for the case of an input-time output-frequency constraint by using the relationship [see (34)]

$$U\left(\frac{n}{W_0}, \frac{m}{T_i}\right) = V\left(\frac{m}{T_i}, \frac{n}{W_0}\right) e^{-i2\pi (m/T_i)(n/W_0)} \qquad (159)$$

in (121) to show that

$$w'(t) = \frac{1}{T_i W_0} \sum_{m,n}\sum U\left(\frac{n}{W_0}, \frac{m}{T_i}\right)$$

$$\cdot z'\left(t - \frac{n}{W_0}\right) e^{i2\pi (m/T_i)t}. \qquad (160)$$

Examination of (160) shows that $w'(t)$ is obtained by first delaying $z'(t)$ by multiplies of $1/W_0$ and then Doppler-shifting by multiples of $1/T_i$. Thus, this model will differ from the one shown in Fig. 18 only in that the order of delay and Doppler shift is reversed and the complex amplitude of the reflection from the point scatterer is equal to $1/T_i W_0\ U(n/W_0, m/T_i)$ rather than $1/T_i W_0\ V(m/T_i, n/W_0)$.

To derive the canonical channel model for the case of an input-frequency output-time constraint we may proceed in a manner entirely analogous to that in the case of the dual constraint, *i.e.*, the input time-output frequency constraint. We have omitted this derivation because of its similarity to that for the dual case. The resulting channel model is shown in Fig. 19, in which

$$U_{mn} = \frac{1}{T_0 W_i} U\left(\frac{m}{W_i}, \frac{n}{T_0}\right) \qquad (161)$$

in the case of internal constraints and

$$U_{mn} = \iint e^{-i2\pi f_i(\xi - m/W_i)} e^{i2\pi t_0(\xi - n/T_0)}$$

$$\cdot \operatorname{sinc}\left[W_i\left(\xi - \frac{m}{W_i}\right)\right] \operatorname{sinc}\left[T_0\left(\nu - \frac{n}{T_0}\right)\right] U(\xi, \nu)\, d\nu\, d\xi \quad (162)$$

in the case of external constraints.

The correlation between the gains is given by

$$\overline{U_{mn}^* U_{rs}} = \frac{1}{(T_0 W_i)^2} R_U\left(\frac{m}{W_i}, \frac{r}{W_i}; \frac{n}{T_0}, \frac{s}{T_0}\right) \qquad (163)$$

for the case of internal time and frequency constraints.

As in the dual situation, the correlation between the gains in the case of external constraint may be expressed in terms of a fourfold integral involving the Delay-Doppler-Spread Function. We present only the expression for the case of WSSUS channel,

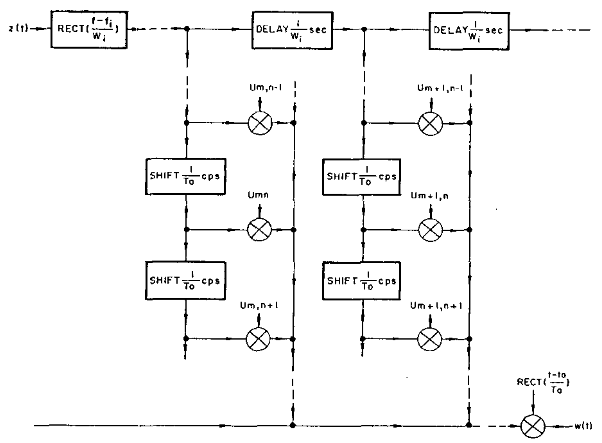

Fig. 19—Canonical model for input frequency-output time constraint.

$$\overline{U_{mn}^* U_{rs}} = e^{j2\pi(f_i/W_i)(m-r)} e^{-j2\pi(t_0/T_0)(n-s)}$$
$$\cdot \iint \operatorname{sinc}\left[W_i\left(\xi - \frac{m}{W_i}\right)\right] \operatorname{sinc}\left[W_i\left(\xi - \frac{m}{W_i}\right)\right]$$
$$\cdot \operatorname{sinc}\left[T_0\left(\nu - \frac{n}{T_0}\right)\right] \operatorname{sinc}\left[T_0\left(\nu - \frac{s}{T_0}\right)\right] S(\xi, \nu)\, d\nu\, d\xi \quad .(164)$$

When the Scattering Function varies very little for changes in ξ of the order of $1/W_i$ and changes in ν of the order of $1/T_0$, (164) simplifies to

$$\overline{U_{mn}^* U_{rs}} \approx \begin{cases} 0 & ; \quad m \neq n, \quad r \neq s \\ \dfrac{1}{W_i T_0} S\left(\dfrac{n}{W_i}, \dfrac{M}{T_0}\right) & ; \quad m = n, \quad r = s \end{cases} \tag{165}$$

yielding a collection of uncorrelated scatterers for the canonic channel model.

As in the dual case, a canonical channel model may be found for the case of an input-frequency output-time constraint which differs from that shown in Fig. 19 only in a reversal of the order of delay and Doppler shift. In this model the gain of a scatterer is set equal to

$$(1/T_0 W_i) V(n/T_0, m/W_i)$$

rather than $(1/T_0 W_i) U(m/W_i, n/T_0)$.

2) *Delay- and Doppler-Shift Constraints*: For physical channels the spread of path delays and the spread of Doppler shifts are effectively limited to finite values. According to our definition of system functions, a limitation in the spread of path delays means that *all* system functions containing the delay variable ξ vanish for values of ξ outside some specified interval. Similarly, a limitation in the spread of Doppler shifts means that *all* system functions containing the Doppler-shift variable ν vanish for values of ν outside some specified interval. This situation is somewhat different from the cases of time and frequency constraints discussed above where different physical interpretations (input as opposed to output constraints) might be associated with a specification that system functions vanish for values of the variables t or f outside specified ranges. The difference in behavior may be traced to the fact that the dual system functions $U(\xi, \nu)$ and $V(\nu, \xi)$ are so simply related both must vanish over the same intervals of ξ and ν, while the dual system functions $M(t, f)$ and $T(f, t)$, with their more complicated relationship, (39), need not vanish over the same intervals of f and t.

In the following subsections we shall develop canonical models for channels which are limited in either path delay spread or Doppler spread or both Doppler- and delay-spread.

a) *Sampling Models for Delay-Spread Constraint*:[21] If we assume that a channel provides path delays only in an interval Δ seconds wide centered at ξ_0 seconds, then $g(t, \xi)$, $U(\xi, \nu)$, $h(t, \xi)$, and $V(\nu, \xi)$ must vanish for values of ξ outside this interval. It follows that the Sampling Theorem, (86), may be applied to the Fourier transforms of these system functions with respect to the delay variable, *i.e.*, to the system functions $T(f, t)$, $G(f, \nu)$, $M(t, f)$, and $H(f, \nu)$, respectively. To derive the canonical channel models appropriate to a delay-spread limitation it is sufficient to deal with $T(f, t)$ and $M(f, t)$. Thus, according to (113), we find that $T(f, t)$ and $M(f, t)$ have the following expansions

$$T(f, t) = \sum_m T\left(\frac{m}{\Delta}, t\right) e^{-j2\pi\xi_0(f - m/\Delta)} \cdot \operatorname{sinc}\left[\Delta\left(f - \frac{m}{\Delta}\right)\right] \tag{166}$$

and

$$M(t, f) = \sum_m M\left(t, \frac{m}{\Delta}\right) e^{-j2\pi\xi_0(f - m/\Delta)} \cdot \operatorname{sinc}\left[\Delta\left(f - \frac{m}{\Delta}\right)\right]. \tag{167}$$

Upon using the expansion (166) to represent $T(f, t)$ in the input-output relationship (19), we find the series representation for the channel output to be

$$w(t) = \sum_m T\left(\frac{m}{\Delta}, t\right) \int Z(f)\, e^{-j2\pi\xi_0(f - m/\Delta)} \cdot \operatorname{sinc}\left[\Delta\left(f - \frac{m}{\Delta}\right)\right] df. \tag{168}$$

Examination of (168) shows that the channel output is represented as the sum of the outputs of a number of elementary parallel channels, each of which filters the input with a transfer function of the form

$$\exp\left[-j2\pi\xi_0\left(f - \frac{m}{\Delta}\right)\right] \operatorname{sinc} \Delta\left(f - \frac{m}{\Delta}\right)$$

for some value of m and then multiplies the resultant by a gain function $T(m/\Delta, t)$. The impulse response of the filter, *i.e.*, the inverse Fourier transform of

$$\exp\left[-j2\pi\xi_0\left(f - \frac{m}{\Delta}\right)\right] \operatorname{sinc} \Delta\left(f - \frac{m}{\Delta}\right)$$

[21] This case has been treated previously by Kailath, *op. cit.*[2]

is readily found to be

$$\exp\left[j2\pi\frac{m}{\Delta}t\right]\frac{1}{\Delta}\operatorname{Rect}\left(\frac{t-\xi_0}{\Delta}\right),$$

which is the complex envelope of a rectangular RF pulse of frequency $f_c + m/\Delta$ (where f_c is the carrier frequency) and of width Δ seconds centered on $t = \xi_0$. Such a filter has frequently been called a band-pass integrator. If we let

$$I_m(f) = \exp\left[-j2\pi\xi_0\left(f-\frac{m}{\Delta}\right)\right]\operatorname{sinc}\Delta\left(f-\frac{m}{\Delta}\right), \tag{169}$$

then we can represent the canonical channel model corresponding to (168) as shown in Fig. 20.

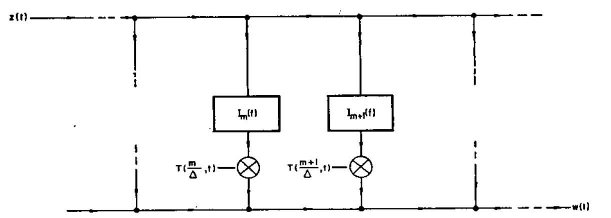

Fig. 20—Canonical channel model for delay-spread limited channel, output multiplier version.

An alternate channel representation which involves multipliers on the input side rather than the output side may be derived by using the series (140) to represent $M(t, f)$ in the input-output relationship (21). The resulting series expression for the output spectrum is given by

$$W(f) = \sum_m I_m(f)\int z(t)M\left(t, \frac{m}{\Delta}\right)e^{j2\pi ft}\,dt. \tag{170}$$

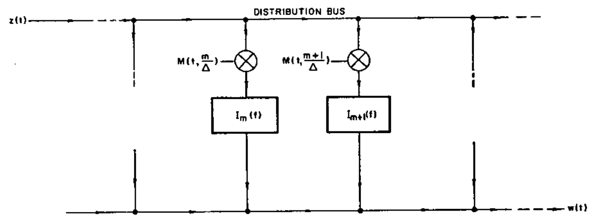

Fig. 21—Canonical channel model for delay-spread limited channel, input multiplier version.

Examination of (170) shows that the mth term in the sum involves a multiplication of the input by a (complex) gain function $M(t, m/\Delta)$ followed by a filtering operation with a filter having transfer function $I_m(f)$. Such a representation is shown in Fig. 21.

As with the previous channel models, although an infinite number of elements is involved, only a finite number is needed in practice. Thus, in Fig. 20, since the approximate bandwidth of each band-pass integrator is $1/\Delta$ cps, and since adjacent integrators are separated by $1/\Delta$ cps, an input signal of bandwidth W would require somewhat more than $W\Delta$, perhaps $10W\Delta$, judiciously selected adjacent multiplier-filter channels to produce a very close approximation to the channel output. More elementary channels may be needed in the model in Fig. 21 where the multipliers precede the filters, because the time varying gains $M(t, m/\Delta)$; $m = 0, \pm 1, \pm 2$, etc. spread the spectra of the inputs to the corresponding filters.

When the channel is randomly time-variant the gain functions $T(m/\Delta, t)$ and $M(t, m/\Delta)$ become random processes. It is clear that the correlation properties of these gain functions are completely determined from the correlation functions of the Time-Variant Transfer Function and Frequency-Dependent Modulation Function. Since these correlation functions have been discussed in detail in Section III, there is no need for further discussion here. However, it is interesting to note that only for the WSSUS channel do the correlation properties of the gain functions in Figs. 20 and 21 become identical, *i.e.*, for the WSSUS channel

$$\begin{aligned}&\overline{M^*(t, m/\Delta)M(t+\tau, n/\Delta)}\\&= \overline{T^*(m/\Delta, t)T(n/\Delta, t+\tau)} = R\left(\frac{n-m}{\Delta}, \tau\right)\end{aligned} \tag{171}$$

where $R(\Omega, \tau)$ is the Time-Frequency Correlation Function defined in Section IV-D.

b) *Sampling Models for Doppler-Spread Constraint*: The Doppler-spread constraint is dual to the delay-spread constraint and the derivations and resulting canonic models follow a dual pattern. Thus, if we assume that a channel provides Doppler shifts only in an interval Γ cps wide centered on ν_0 cps, then $H(f, \nu)$, $V(\nu, \xi)$, $G(f, \nu)$, and $U(\xi, \nu)$ must vanish for values of ν outside this interval. Application of the sampling theorem then produces the expansions

$$T(f, t) = \sum_n T(f, n/\Gamma)e^{j2\pi\nu_0(t-n/\Gamma)}\operatorname{sinc}[\Gamma(t-n/\Gamma)] \tag{172}$$

and

$$M(t, f) = \sum_n M(n/\Gamma, f)e^{j2\pi\nu_0(t-n/\Gamma)}\cdot\operatorname{sinc}[\Gamma(t-n/\Gamma)]. \tag{173}$$

Upon using (172) in (15) and (173) in (21), we obtain the following expansions for the channel output:

$$w(t) = \sum_n e^{j2\pi\nu_0(t-n/\Gamma)}\operatorname{sinc}[\Gamma(t-n/\Gamma)]\cdot\int Z(f)T(f, n/\Gamma)e^{j2\pi ft}\,dt \tag{174}$$

and

$$W(f) = \sum M(n/\Gamma, f)\int z(t)e^{j2\pi\nu_0(t-n/\Gamma)}\cdot\operatorname{sinc}[\Gamma(t-n/\Gamma)]e^{j2\pi ft}\,dt. \tag{175}$$

The nth term of the series (174) may be interpreted as the result of a filtering operation with filter transfer function $T(f, n/\Gamma)$, followed by a multiplication with a gain function $\exp[j2\pi\nu_0(t - n/\Gamma)]\operatorname{sinc}\Gamma(t - n/\Gamma)$. If we let

$$p_n(t) = \exp[j2\pi\nu_0(t-n/\Gamma)]\operatorname{sinc}[\Gamma(t-n/\Gamma)], \tag{176}$$

then the canonical channel model which follows from the above interpretation of (174) is as shown in Fig. 22. In an entirely analogous fashion (175) leads to the model shown in Fig. 23.

Whereas in the dual cases described in the previous section a finite number of multiplier-filter combinations is satisfactory for representing the channel for a band-limited input, in the present cases a finite number of multiplier-filter combinations may be used when the input is time-limited. This fact may be appreciated by noting that the gain function $p_n(t)$ acts as a "gate" of duration $1/\Gamma$ and that the "gates" of adjacent elementary channels are separated by $1/\Gamma$ seconds. Thus, an input signal of duration T will require anywhere from, say $T\Gamma$ to $10T\Gamma$ judiciously selected adjacent elementary channels to characterize the channel.

When the channel is randomly time-variant the filters $T(f, n/\Gamma)$ and $M(n/\Gamma, f)$ become random processes in the frequency variable. The correlation properties of these filters may be determined from the results of Section IV, which deals with the correlation functions of the various system functions. It is interesting to note, as in the dual case, that only for the WSSUS channel do the correlation properties of the random filters in Figs. 22 and 23 become identical.

c) *Sampling Models for Combined Delay-Spread and Doppler-Spread Constraints*: When both a Delay-Spread and Doppler-Spread constraint exist, the sampling theorem may be applied twice to the system functions $T(f, t)$ and $M(t, f)$, *i.e.*, once when they are considered as time functions with the frequency variables fixed and once when they are considered as frequency functions with the time variables fixed. In this manner one may form sampling expansions and determine corresponding canonical channel models. However, this procedure is unnecessary since the desired models may be obtained by inspection of Figs. 20–23 by combining models appropriate to delay-spread and Doppler-spread constraints. To demonstrate this latter approach, examine the model of Fig. 20, which is appropriate to a delay-spread constraint. If we require that a Doppler-spread constraint also exist, the multiplication operation in each parallel channel is in effect a sub-channel with a Doppler-spread constraint and may be represented by the canonical model of Fig. 22 or 23 where the f variable is set equal to m/Δ for the mth branch in Fig. 20. If this procedure is followed using the model of Fig. 22 in Fig. 20, the model of Fig. 24 appears. In an entirely analogous fashion one may generate three additional models, one by using the model of Fig. 23 in Fig. 20 and two more by using the models of Figs. 22 and 23 in Fig. 21.

It is readily demonstrated that for waveforms which are effectively limited in time and frequency duration, *i.e.*, waveforms which have most of their energy located in a finite time-frequency interval, only a finite number of filters and multipliers may be used in the models of Figs. 24 and 25 to provide a close approximation to the actual channel output.

The complex gain constants $T(m/\Delta, n/\Gamma)$, $M(n/\Gamma, m/\Delta)$ become random variables when the channel is randomly time-variant. Their correlation properties are just sampled values of the correlation functions of $T(f, t)$ and $M(t, f)$ respectively.

3) *Combined Time and Delay-Spread or Frequency and Doppler-Spread Constraints*: In Section VIA. 1) we have developed canonical channel models for time and frequency constraints, *i.e.*, for situations in which system functions vanish for values of t and/or f outside specified intervals. In Section VIA. 2) we have developed canonical

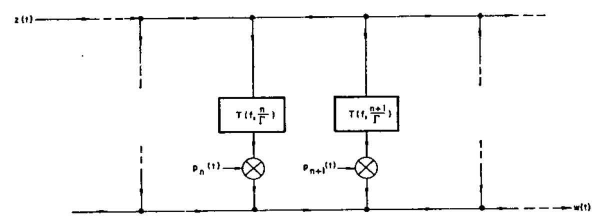

Fig 22—Canonical channel model for Doppler-spread limited channel, output multiplier version.

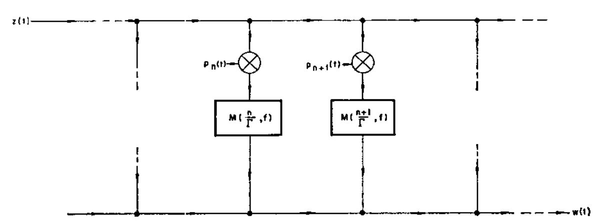

Fig. 23—Canonical channel model for Doppler-spread limited channel, input multiplier version.

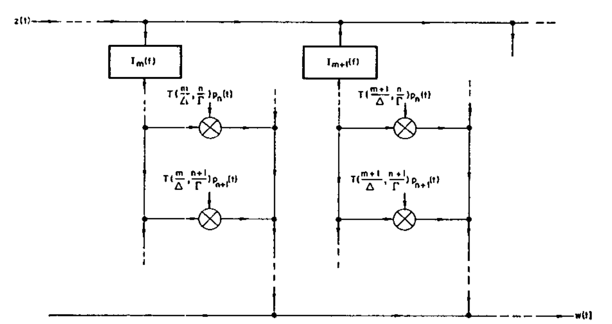

Fig. 24—A canonical channel model for combined delay-spread and Doppler-spread limited channel.

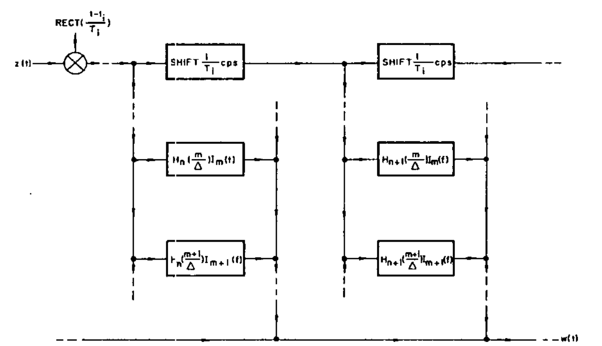

Fig. 25—A canonical channel model for combined input-time and delay-spread constraints.

channel models for delay-spread and Doppler-spread constraints, *i.e.*, for situations in which system functions vanish for values of ξ and/or ν outside specified intervals. Here we consider certain combinations of the constraints in Sections VIA. 1) and 2), namely, combined time and delay-spread constraints and combined frequency and Doppler-spread constraints. Other combinations are not possible since they are equivalent to requiring that a function be limited in both time and frequency.

The combined constraint models may be derived by combining the models appropriate to the individual constraints, as was done in Section VIA. 2) c). One may combine a delay-spread constraint with either an input or output frequency constraint. We shall present here only one model for a combined Doppler-spread and frequency constraint and one model for a combined delay-spread and time constraint. The remaining possible models may be quickly constructed by the reader.

To construct a model appropriate to an input time constraint and a delay-spread constraint we may make use of the models of Figs. 10 and 21. We note first that if the channel is delay-spread limited then the filters $H_n(f)$ in Fig. 10 are also delay-spread limited. Thus, each of these filters may be represented by means of a canonical model of the form of Fig. 21. Note, however, that since these filters are time-invariant, the gain functions in Fig. 21 are also time-invariant. To determine the value of the (time-invariant) gains it should be noted that a time-invariant filter with transfer function $H(f)$ has

$$M(t, f) = H(f). \tag{177}$$

Then, using the model of Fig. 21 to represent each filter in Fig. 10, we arrive at the model shown in Fig. 25.

To construct a model appropriate to an input frequency constraint and a Doppler-spread constraint we may use the Doppler-spread constraint model of Fig. 22 and input frequency constraint model of Fig. 11. In this connection one should note that a multiplier $g(t)$ is a degenerate time-variant linear filter with

$$T(f, t) = g(t). \tag{178}$$

It is readily determined that the model of Fig. 26 results when the model of Fig. 22 is used to represent each multiplier $g_n(t)$ in Fig. 11.

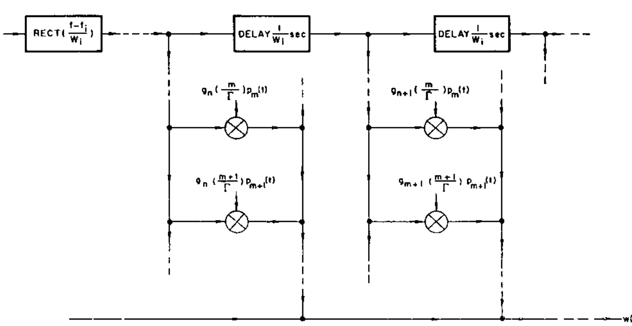

Fig. 26—A canonical channel model for combined input-frequency and Doppler-spread constraints.

B. Power Series Models

In this section we will derive certain canonical channel models which arise from power series expansions of $T(f, t)$ and $M(t, f)$ in either the time or frequency variables. The two models arising from expansions in the f variable will be called f-power series models, while those arising from expansions in the t variable will be called t-power series models. As might be expected, the f-power series models are dual to the t-power series models and are useful in dual situations.

1) *f-Power Series Models*: The starting point for our discussion is the input-output relationship corresponding to the Time-Variant Transfer Function $T(f, t)$ (19), which is repeated below:

$$w(t) = \int Z(f)T(f, t)e^{j2\pi ft}\, df.$$

If the input spectrum $Z(f)$ is confined primarily to a specified frequency interval over which $T(f, t)$ varies little in f, with a minimum fluctuation period much greater than the bandwidth of $Z(f)$, then a Taylor series representation of $T(f, t)$ in f will provide a rapidly convergent expansion of $Z(f)T(f, t)$ and $w(t)$.

Since the existence of a mean path delay ξ_o, *i.e.*, a value of ξ about which $g(t, \xi)$ may be considered centered, produces a factor exp $[-j2\pi f\xi_o]$ in $T(f, t)$ which can fluctuate with f quite rapidly, it is desirable to expand only that portion of $T(f, t)$ which does not include this factor. To this end we may define a shifted Input Delay-Spread Function $g_0(t, \xi)$ in which the mean path delay ξ_o has been removed, *i.e.*,

$$g_0(t, \xi) = g(t, \xi + \xi_o) \tag{179}$$

where ξ_o is a mean multipath delay defined according to some convenient criterion. Then

$$T(f, t) = T_0(f, t)e^{-j2\pi f\xi_o} \tag{180}$$

where $T_0(f, t)$ is the Time-Variant Transfer Function of the medium after the mean path delay has been removed, *i.e.*,

$$T_0(f, t) = \int g_0(t, \xi)e^{-j2\pi f\xi}\, d\xi. \tag{181}$$

In the most general situation the input spectrum $Z(f)$ may not be centered at $f = 0$. Thus, assuming that $Z(f)$ is centered at $f = f_i$, the most rapid convergence of $Z(f)T_0(f, t)$ will be obtained by expanding $T_0(f, t)$ about $f = f_i$, *i.e.*,

$$T_0(f, t) = \sum_{n=0}^{\infty} T_n(t)(2\pi j)^n(f - f_i)^n \tag{182}$$

where

$$T_n(t) = \frac{1}{n!\,(2\pi j)^n}\left[\frac{\partial^n T_0(f, t)}{\partial f^n}\right]_{f=f_i}$$

$$= \frac{1}{n!}\int (-\xi)^n g_0(t, \xi)e^{-j2\pi f_i\xi}\, d\xi. \tag{183}$$

A filter with transfer function $(2\pi j)^n f^n$ is an nth order differentiator. We shall define a filter with transfer function $(2\pi j)^n (f - f_i)^n$ as an offset differentiator with an offset of f_i cps. If we let D_{f_i} be an operator denoting such an offset differentiation, it is quickly demonstrated that

$$D^n_{f_i}[f(t)] = e^{j2\pi f_i t} \frac{d^n}{dt^n} \{f(t)e^{-j2\pi f_i t}\}. \tag{184}$$

Then use of (182) and (184) in (19) yields the following series representation of the channel output:

$$\begin{aligned} w(t) &= \sum T_n(t) D^n_{f_i}[z(t - \xi_g)] \\ &= e^{j2\pi f_i t} \sum T_n(t) \frac{d}{dt^n} \{z(t - \xi_g)e^{-j2\pi f_i t}\}. \end{aligned} \tag{185}$$

Examination of the last equation in (185) indicates that the channel output may be represented as the parallel combination of the outputs of an infinite number of elementary channels each consisting of a differentiation of some order followed by a time variant gain with all channels preceded by a delay ξ_g and then a frequency translation of $-f_i$ cps and all channels followed by a frequency translation of $+f_i$ cps. Study of the first equation in (185) shows that the channel output may also be represented as the parallel combination of elementary channels where now the typical channel consists of an offset differentiation of some order followed by the same time-variant gain, with all channels preceded by a delay ξ_g.

This latter channel representation is shown in Fig. 27, where the offset differentiators of different subchannels have been combined into a chain of offset differentiators.

For simplicity, in the following discussion of power series models, we shall present the particular forms that are simplest to diagram. It should be realized, however, that equations such as (185) may have a variety of interpretations in terms of channel models.

An understanding of the conditions leading to rapid convergence of the series (185) may be obtained by first defining a normalized shifted Input Delay Spread Function $\tilde{\tilde{g}}_0(t, \xi)$ whose "width" in the ξ direction is unity and a shifted normalized input time function $\tilde{\tilde{z}}_0(t)$ which is located at $f = 0$ and has unit bandwidth (using any convenient bandwidth criterion). These normalized functions are defined implicitly by the relations

$$\begin{aligned} z(t) &= \tilde{\tilde{z}}_0(G_i t)e^{j2\pi f_i t} \\ g_0(t, \xi) &= \frac{1}{\Delta_g} \tilde{\tilde{g}}_0\left(t, \frac{\xi}{\Delta_g}\right) \end{aligned} \tag{186}$$

where Δ_g is a measure of multipath spread given by the "width" of $g_0(t, \xi)$ in the ξ direction and B_i is the bandwidth of the input.

With the aid of the normalized functions one readily finds that the series (185) may be expressed in the form

$$w(t) = e^{j2\pi f_i(t-\xi_i)} \sum_{n=0}^{\infty} \frac{(j2\pi B_i \Delta_g)^n}{n!} \tilde{\tilde{T}}_n(t) \frac{\tilde{\tilde{z}}_0^{(n)}(B_i t - B_i \xi_g)}{(j2\pi)^n} \tag{187}$$

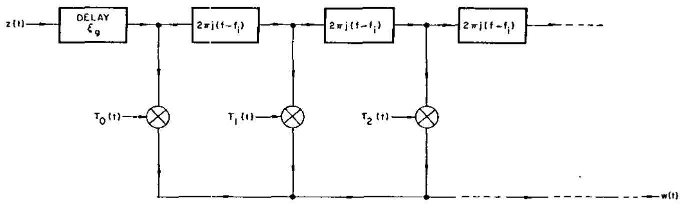

Fig. 27—f-power series channel model, output multiplier version.

where

$$\tilde{\tilde{T}}_n(t) = \int (-\xi)^n \tilde{\tilde{g}}_0(t, \xi)\, d\xi$$

$$\frac{\tilde{\tilde{z}}_0^{(n)}(t)}{(j2\pi)^n} = \frac{1}{(j2\pi)^n} \frac{d^n \tilde{\tilde{z}}_0(t)}{dt^n} = \int f^n \tilde{\tilde{Z}}_0(f) e^{j2\pi f t}\, df \tag{188}$$

in which $\tilde{\tilde{Z}}_0(f)$ is the spectrum of $\tilde{\tilde{z}}_0(t)$.

Examination of (188) reveals that $T_n(t)$ and $[1/(j2\pi)^n] \cdot \tilde{\tilde{z}}_0^{(n)}(B_i t - B\xi_g)$ are both moments of functions having unit "duration." If, indeed, we assume $\tilde{\tilde{g}}_0(t, \xi)$ (as a function of ξ) and $\tilde{\tilde{Z}}_0(f)$ are zero outside the unit interval (centered at $\xi = 0$, and $f = 0$ respectively), then it is readily demonstrated that these moments may not increase and in practice will most likely decrease with increasing n. Examination of (187) then indicates that if

$$2\pi B_i \Delta_g \ll 1 \tag{189}$$

the series will be rapidly convergent.

In the general case $\tilde{\tilde{g}}_0(t, \xi)$ and $\tilde{\tilde{Z}}_0(f)$ will have "tails" extending outside the unit interval. For (189) still to represent a useful convergence criterion, these tails must drop to zero sufficiently rapidly so that the moments do not increase too rapidly with increasing n.[22]

When the channel is randomly time-variant, the multiplier functions become random processes with correlation properties defined by

$$\overline{T^*_n(t)T_m(s)} = \frac{(-1)^{m+n}}{n!\, m!} \cdot \iint \xi^m \eta^n R_g(t, s; \xi + \xi_g, \eta + \xi_g) e^{j2\pi f_i(\xi - \eta)}\, d\xi\, d\eta \tag{190}$$

for the general channel. For the WSSUS channel, the cross-correlation function (190) specializes to (see Fig. 9)

$$\overline{T^*_n(t)T_n(t + \tau)} = \frac{(-1)^{m+n}}{n!\, m!} \int \xi^{m+n} Q(\tau, \xi + \xi_g)\, d\xi. \tag{191}$$

In the case of the WSSUS channel, a desirable choice for ξ_g is given by

$$\xi_g = \frac{\int \xi Q(\xi)\, d\xi}{\int Q(\xi)\, d\xi} \tag{192}$$

[22] The series will diverge if these tails fall too slowly. If the product of the two moments increases exponentially with n as α^n, one may modify (189) into $2\pi\alpha B_i \Delta_g \ll 1$ to obtain a suitable convergence criterion.

where $Q(\xi)$ is the Delay Power Density Spectrum, since such a choice not only minimizes $\overline{|T_1|^2}$ relative to $\overline{|T_0|^2}$ but also leads to $T_1(t)$ and $T_0(t)$ being uncorrelated. The ratio of the strength of T_1 relative to T_0 then takes the simple form

$$\frac{\overline{|T_1|^2}}{\overline{|T_0|^2}} = \frac{\int (\xi - \xi_0)^2 Q(\xi)\, d\xi}{\int Q(\xi)\, d\xi} = \Delta^2 \tag{193}$$

where Δ may be called the rms width of $Q(\xi)$.

When the frequency selective fading in the channel is sufficiently slow, only the first term in the series (185) will be sufficient to characterize the channel output, *i.e.*,

$$w(t) = T_0(t) z(t - \xi_0) \tag{194}$$

which may be recognized as a "flat-fading" or non-frequency-selective channel model. If the first two terms are used,

$$w(t) = T_0(t) z(t - \xi_0) + T_1(t) D_{f_i}[z(t - \xi_0)] \tag{195}$$

which may be called a "linearly frequency-selective fading" channel since it corresponds to approximating $T_0(f, t)$ by a linear term in the frequency variable. One may continue and define a "quadratically frequency-selective fading" channel, etc., depending upon the degree of approximation required.

We shall now investigate the error incurred in using a finite number of terms in the expansion (185) for the case of a WSSUS channel. If we assume the existence of derivatives of $T_0(f, t)$ with respect to f as high as Nth order, then we may expand $T_0(f, t)$ in a finite Taylor series expansion

$$T_0(f, t) = \sum_{n=0}^{N-1} T_n(t)(2\pi j)^n (f - f_i)^n + \frac{1}{N!}(f - f_i)^N \left[\frac{\partial^N T_0(f, t)}{\partial f^N}\right]_{f=f'} \tag{196}$$

where f' lies between f_i and f. Then using (196) in (15) and making use of (183), one obtains the following series expression for the channel output:

$$w(t) = \sum_{n=0}^{N-1} T_n(t) D_{f_i}^n [z(t - \xi_0)] + R_N(t) \tag{197}$$

where $R_N(t)$ is a remainder term given by

$$\begin{aligned} R_N(t) &= \int (2\pi j)^N (f - f_i)^N Z(f) \frac{1}{N!} \frac{1}{(2\pi j)^N} \cdot \left[\frac{\partial^N T_0(f, t)}{\partial f N}\right]_{f=f'} e^{j2\pi f(t-\xi_0)}\, df \\ &= \int (2\pi j)^N (f - f_i)^N Z(f) \frac{1}{N!} \cdot \int (-\xi)^N g_0(t, \xi) e^{-j2\pi f'\xi}\, d\xi\, e^{j2\pi f(t-\xi_0)}\, df. \end{aligned} \tag{198}$$

It follows that

$$\begin{aligned} \overline{|R_N(t)|^2} &= \iint (-2\pi j)^N (f - f_i)^N (2\pi j)^N (l - f_i)^N \frac{1}{(N!)^2} \overline{Z^*(f) Z(l)} \\ &\cdot \iint (-\xi)^N (-\eta)^N \overline{g_0^*(t, \xi) g_0(t, \eta)} \\ &\cdot e^{j2\pi(f'\xi - f''\eta)}\, d\xi\, d\eta\, e^{-j2\pi(f-l)(t-\xi_0)}\, df\, dl \end{aligned} \tag{199}$$

where f'' lies between f_i and l.

For the WSSUS channel

$$\overline{g_0^*(t, \xi) g_0(t, \eta)} = Q(\xi + \xi_0)\delta(\eta - \xi), \tag{200}$$

and for a wide-sense stationary $z(t)$[12]

$$\overline{Z^*(f) Z(l)} = P_z(f)\delta(l - f) \tag{201}$$

where $P_z(f)$ is the power spectrum of $z(t)$.

Using (200) and (201) in (199) we readily find that

$$\begin{aligned} \overline{|R_N(t)|^2} &= \frac{(2\pi)^{2N}}{(N!)^2} \int (f - f_i)^{2N} P_z(f)\, df \int (\xi - \xi_0)^{2N} Q(\xi)\, d\xi \\ &= \frac{(2\pi B_i \Delta_0)^{2N}}{(N!)^2} \int f^{2N} \tilde{\tilde{P}}_z(f)\, df \int \xi^{2N} \tilde{\tilde{Q}}(\xi)\, d\xi \end{aligned} \tag{202}$$

where $\tilde{\tilde{P}}_z(f)$ is the power spectrum of the normalized input signal $\tilde{\tilde{z}}(t)$ (unit bandwidth and centered at zero frequency) and $\tilde{\tilde{Q}}(\xi)$ is the Delay Power Density Spectrum associated with the normalized Delay-Spread Function $\tilde{\tilde{g}}_0(t, \xi)$.

One may show that the right-hand side of (202) is also just equal to the average magnitude squared of the $N + 1$th term in the series (185) for the case of a WSSUS channel. Thus, we have the simple error criterion that the average magnitude squared of the error incurred by using only a finite number of terms in (185) is just equal to the average magnitude squared of the first omitted when the channel is WSSUS and the input is wide-sense stationary.

We shall now derive a channel model which differs from Fig. 27 principally in a reversal of the order of the operations of differentiation and multiplication. This channel model is derived by making use of a Taylor series expansion of $M(t, f)$ in the frequency variable. The input-output relationship corresponding to $M(t, f)$ is given by (25), which is repeated below:

$$W(f) = \int z(t) M(t, f) e^{-j2\pi f t}\, dt.$$

If the output spectrum $W(f)$ is confined primarily to a specified frequency interval over which $M(t, f)$ varies little in f, with a minimum fluctuation period much greater than the bandwidth of $W(f)$, then a Taylor series representation of $M(t, f)$ in f can be used in (25) to obtain a rapidly convergent expansion of $W(f)$.

Since $M(t, f)$ is the Fourier transform of $h(t, \xi)$, the Input Delay-Spread Function, the presence of a nonzero value of ξ, say ξ_h, about which $h(t, \xi)$ is "centered" will result in a factor $\exp[-j2\pi f \xi_h]$ in $M(t, f)$ which can fluctuate with f quite rapidly. Thus, it is desirable to expand only that portion of $M(t, f)$ which does not include this factor. If this portion of $M(t, f)$ is denoted by $M_0(t, f)$, we have

$$M_0(t, f) = \int h_0(t, \xi) e^{-i2\pi f\xi} \, d\xi \tag{203}$$

where

$$h_0(t, \xi) = h(t, \xi + \xi_h) \tag{204}$$

is a shifted Output Delay-Spread Function centered on $\xi = 0$. Using (204) and (203) we note that

$$M(t, f) = M_0(t, f) e^{-i2\pi f\xi} h. \tag{205}$$

The most rapidly convergent expansion of $W(f)$ will be obtained by expanding $M_0(t, f)$ about the "center" frequency of $W(f)$, say, $f = f_0$, as follows:

$$M_0(t, f) = \sum_{n=0}^{\infty} M_n(t)(2\pi j)^n (f - f_0)^n \tag{206}$$

where

$$\begin{aligned} M_n(t) &= \frac{1}{n! \, (2\pi j)^n} \left[\frac{\partial^n M_0(f, t)}{\partial f^n} \right]_{f=f_0} \\ &= \frac{1}{n!} \int (-\xi)^n h_0(t, \xi) e^{-i2\pi f_0 \xi} \, d\xi. \end{aligned} \tag{207}$$

Upon using (205) and (206) in (25) we find that

$$W(f) = \sum_{n=0}^{\infty} (2\pi j)(f - f_0)^n e^{-i2\pi f \xi_h} \int z(t) M_n(t) e^{-i2\pi f t} \, dt. \tag{208}$$

By Fourier transforming (208) we obtain the following series expression for $w(t + \xi_h)$:

$$w(t + \xi_h) = \sum_{n=0}^{\infty} D_{f_0}^n [z(t) M_n(t)]. \tag{209}$$

Examination of (209) indicates that the channel output may be represented as the parallel combination of the outputs of an infinite number of elementary channels each consisting of a time-varying (complex) gain followed by an offset differentiation of some order with all channels followed by a delay of ξ_h seconds. Such a model is shown in Fig. 28, where the differentiators of different subchannels have been combined into a chain of differentiators.

In order to obtain a representation of (209) in terms of normalized functions analogous to (187) it is necessary to assume that the bandwidth of $z(t)M_n(t)$ does not exceed the bandwidth of $w(t)$. Then it may be shown that when $w(t)$ is band-limited to B_0 cps and $h(t, \xi)$ is zero outside a ξ interval of duration Δ_h, the condition for rapid convergence of (209) is given by

$$2\pi B_0 \Delta_h \ll 1. \tag{210}$$

Even when $w(t)$ is not band-limited and $h(t, \xi)$ is not ξ-limited, (210) will still be a useful convergence criterion so long as the "tails" of $W(f)$ and $h(t, \xi)$ (as a function of ξ) drop to zero rapidly enough.

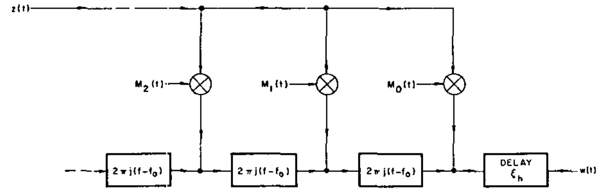

Fig. 28—f-power series channel model, input multiplier version.

It may also be shown in the same manner as was shown for the series (185) that the average magnitude squared error incurred by using a finite number of terms in the series (209) is just equal to the average magnitude squared of the first term omitted when the input is wide-sense stationary and the channel is WSSUS. As a final point we note that relationships analogous to (190) to (196) are readily constructed for the channel model of Fig. 28. Thus, it is readily shown that for the WSSUS channel the correlation properties of the multipliers in Fig. 28 are identical to those in Fig. 27 so that a choice of $\xi_h = \xi_g$ of (192) causes $M_0(t)$ and $M_1(t)$ to be uncorrelated and $\overline{|M_1|^2}$ to be minimized relative to $\overline{|M_0|^2}$.

2) *t-Power Series Models*: The t-Power Series Models are dual to the f-Power Series Models described in the previous section and thus involve power series expansions of $T(f, t)$ and $M(t, f)$ in the t variable. Whereas the f-Power series models are particularly useful when the frequency spread of the input (or output) spectrum and the delay spread of the channel are fairly sharply delimited and their product is much less than unity, the t-Power series models are particularly useful in the dual situation, *i.e.*, when the width of the input (or output) time function and the Doppler spread of the channel are fairly sharply delimited and their product is much less than unity.

Since the models presented in this section are dual to those in the previous section, the analytical part of their derivation is essentially identical to that in the previous section and thus it will be unnecessary to present as detailed derivations.

We will start our discussion by presenting the following expansion of $M(t, f)$:

$$M(t, f) = e^{i2\pi\nu_H t} \sum_{n=0}^{\infty} \hat{M}_n(f)(2\pi j)^n (t - t_i)^n \tag{211}$$

where t_i is a time instant about which the input may be assumed "centered" and ν_H is a value of ν about which $H(f, \nu)$ may be assumed "centered." The frequency function $\hat{M}_n(f)$ is given by

$$\begin{aligned} \hat{M}_n(f) &= \frac{1}{n! \, (2\pi j)^n} \left[\frac{\partial^n}{\partial t^n} \{ M(t, f) e^{-i2\pi\nu_H t} \} \right]_{t=t_i} \\ &= \frac{1}{n!} \int \nu^n H(f, \nu + \nu_H) e^{i2\pi\nu t_i} \, d\nu. \end{aligned} \tag{212}$$

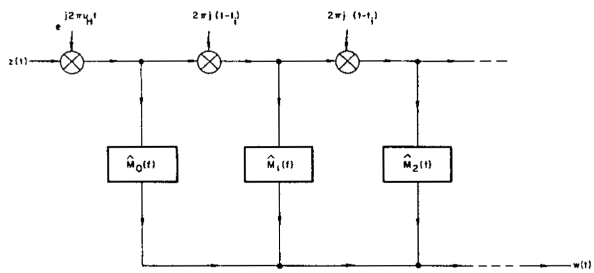

Fig. 29—t-power series model, output filter version.

Use of (211) in (25) leads to the following expansion for the output spectrum:

$$W(f) = \sum_{n=0}^{\infty} \hat{M}_n(f) \int z(t)e^{-j2\pi(f-\nu_H)t}(2\pi j)^n(t - t_i)^n \, dt$$

$$= \sum_{n=0}^{\infty} \hat{M}_n(f)D_{t_i}[Z(f - \nu_H)]$$

$$= e^{-j2\pi t_i f} \sum_{n=0}^{\infty} \hat{M}_n(f) \frac{d^n}{df^n} \{Z(f - \nu_H)e^{j2\pi t_i f}\}. \quad (213)$$

Examination of the channel model in Fig. 29 readily reveals that the summed outputs of its elementary channels is identical to the series (186).

When the input time function is limited to a finite time interval T_i seconds and $H(f, \nu)$ is zero for values of ν outside an interval of duration β_H cps, one may show that the series (186) will be rapidly convergent if

$$2\pi T_i\beta_H \ll 1 \quad (2.14)$$

[c.f. (189)]. Even if $z(t)$ is not time-limited and $H(f, \nu)$ is not ν-limited, (214) will still be satisfactory convergence criterion if the "tails" of $z(t)$ and $H(f, \nu)$ (as a function of ν) drop to zero sufficiently rapidly.

When the channel is randomly time-variant the filter transfer functions $\hat{M}_n(f)$ become random processes in the frequency variable with correlation properties defined by

$$\overline{\hat{M}_n^*(f)\hat{M}_m(l)} = \iint \frac{\nu^n \mu^m}{n!\, m!} R_H(f, l; \nu + \nu_H, \mu + \mu_H) \cdot e^{-j2\pi t_i(\nu-\mu)} \, d\nu \, d\mu \quad (215)$$

for the general channel. In the case of the WSSUS channel, (215) simplifies to

$$\overline{\hat{M}_n^*(f)\hat{M}_m(f + \Omega)} = \frac{1}{n!\, m!} \int \nu^{m+n} P(\Omega, \nu) \, d\nu \quad (216)$$

(see Fig. 9).

A desirable choice for ν_H in the case of the WSSUS channel is given by

$$\nu_H = \frac{\int \nu P(\nu) \, d\nu}{\int P(\nu) \, d\nu} \quad (217)$$

where $P(\nu)$ is the Doppler Power Density Spectrum [see (76)], since in this case $\overline{|\hat{M}_1(f)|^2}$ is minimized relative to $\overline{|\hat{M}_0(f)|^2}$ and

$$\overline{\hat{M}_1^*(f)\hat{M}_0(f)} = 0. \quad (218)$$

The ratio of the strength of $\hat{M}_1(f)$ relative to $\hat{M}_0(f)$ then takes the simple form

$$\frac{\overline{|\hat{M}_1(f)|^2}}{\overline{|\hat{M}_0(f)|^2}} = \frac{\int (\nu - \nu_H)^2 P(\nu) \, d\nu}{\int P(\nu) \, d\nu} \equiv \beta^2 \quad (219)$$

where here β is the rms width of $P(\nu)$.

When the fading is sufficiently slow only the first term in (213) will be sufficient to characterize the channel output, *i.e.*, one may use

$$W(f) = \hat{M}_0(f)Z(f - \nu_H) \quad (220)$$

which, apart from the frequency shift of ν_H cps, represents the channel as a time-invariant linear filter with transfer function $\hat{M}_0(f)$.

If the first two terms are used,

$$W(f) = \hat{M}_0(f)Z(f - \nu_H) + \hat{M}_1(f)D_{t_i}[Z(f - \nu_H)] \quad (221)$$

which may be called a "linearly time-selective fading" channel since it corresponds to approximating $M(t, f)e^{-j2\pi\nu_H t}$ by a linear term in the time variable. One may continue and define a "quadratically time-selective fading channel," etc., depending upon the degree of approximation required.

An exact expression dual to (203) is readily formulated for the average magnitude squared error incurred by using only $N - 1$ terms in (213). However, this expression would be applicable in the dual situation, namely, when the input spectrum (rather than time function) is a wide-sense stationary process and the channel is WSSUS. Since such an input is not very common, the corresponding error expression may not be as useful as in the dual case. Thus, we present a different derivation which yields an upper bound on the average magnitude squared error for the case of arbitrarily specified $z(t)$ and a WSSUS channel.

We first express $M(f, t)e^{-j2\pi\nu_H t}$ in a finite Taylor series,

$$M(f, t)e^{-j2\pi\nu_H t} = \sum_{n=0}^{N-1} \hat{M}_n(f)(2\pi j)^n(t - t_i)^n + \frac{1}{N!}(t - t_i)^N \left[\frac{\partial^N}{\partial t^N}\{M(f, t)e^{-j2\pi\nu_H t}\}\right]_{t=t'} \quad (222)$$

where t' lies between t_i and t. Then using (212) and (222) in (213) we obtain the finite series representation of the output spectrum

$$W(f) = \sum_{n=0}^{N-1} \hat{M}_n(f)D_{t_i}^n[Z(f - \nu_H)] + E_N(f): \quad (223)$$

where the remainder term $E_N(f)$ is given by

$$E_N(f) = (2\pi j)^N \int (t - t_i)^N z(t)e^{j2\pi\nu_H t} \cdot \frac{1}{N!} \int \nu^N H(f, \nu + \nu_H)e^{j2\pi\nu t'} \, d\nu \, e^{-j2\pi f t} \, dt. \quad (224)$$

It follows that

$$\overline{|E_N(f)|^2} = \frac{(2\pi)^{2N}}{(N!)^2} \iint (t - t_i)^N (s - t_i)^N z^*(t) z(s) e^{-i2\pi\nu_H(t-s)}$$
$$\cdot \iint \nu^N \mu^N \overline{H^*(f, \nu + \nu_H) H(f, \mu + \nu_H)}$$
$$\cdot e^{-i2\pi(\nu t' - \mu t'')} \, d\nu \, d\mu \, e^{i2\pi f(t-s)} \, dt \, ds \tag{225}$$

where t'' lies between t_i and s.

For the WSSUS channel,

$$\overline{H^*(f, \nu + \nu_H) H(f, \mu + \nu_H)} = P(\nu + \nu_H)\delta(\mu - \nu). \tag{226}$$

Using (226) in (225), we find

$$\overline{|E_N(f)|^2} = \frac{(2\pi)^{2N}}{(N!)^2} \iint (t - t_i)^N (s - t_i)^N z^*(t) z(s) e^{-i2\pi\nu_H(t-s)}$$
$$\cdot \int \nu^{2N} P(\nu + \nu_H) e^{-i2\pi\nu(t' - t'')} \, d\nu \, e^{i2\pi f(t-s)} \, dt \, ds. \tag{227}$$

Noting that the magnitude of an integral is less than the integral of the magnitude,

$$\overline{|E_N(f)|^2} \leq \frac{(2\pi)^{2N}}{(N!)^2} \iiint |t - t_i|^N \, |s - t_i|^N$$
$$\cdot |z(t)| \, |z(s)| \, P(\nu + \nu_H)\nu^{2N} \, d\nu \, dt \, ds$$
$$= \frac{(2\pi)^{2N}}{(N!)^2} \int (\nu - \nu_H)^{2N} P(\nu) \, d\nu \left| \int |t - t_i|^N \, |z(t)| \, dt \right|^2$$
$$= \frac{(2\pi\beta_H T_i)^{2N}}{(N!)^2} \int \nu^{2N} \tilde{\tilde{P}}(\nu) \, d\nu \left| \int |t|^N \, |\hat{z}(t)| \, dt \right|^2 \tag{228}$$

where $\tilde{\tilde{P}}(\nu)$ is the Doppler Power Density Spectrum corresponding to a normalized Doppler-Spread Function $\tilde{\tilde{H}}(f, \nu)$ which differs from $H(f, \nu)$ in being translated and scaled along the ν axis so that it has $\beta_H = 1$ and $\nu_H = 0$. Similarly, $\hat{z}(t)$ is a shifted scaled version of the input which has unit duration and is located at $t = 0$.

The channel model dual to that in Fig. 28 may be arrived at with the aid of the following expansion:

$$T(f, t) = e^{i2\pi\nu_G t} \sum_{n=0}^{\infty} \hat{T}_n(f)(2\pi j)(t - t_0)^n \tag{229}$$

where t_0 is a time instant about which the output may be assumed "centered" and ν_G is the value of ν about which $G(f, \nu)$ may be assumed "centered." The frequency function $T_n(f)$ is given by

$$\hat{T}_n(f) = \frac{1}{n! \, (2\pi j)^n} \left[\frac{\partial^n}{\partial t^n} \{T(f, t) e^{-i2\pi\nu_G t}\} \right]_{t=t_0}$$
$$= \frac{1}{n!} \int \nu^n G(f, \nu + \nu_G) e^{i2\pi\nu t_0} \, d\nu. \tag{230}$$

Use of (229) in (19) leads to the following expansion for the output time function:

$$w(t) = e^{i2\pi\nu_G t} \sum_{n=0}^{\infty} (2\pi j)^n (t - t_0)^n \int \hat{T}_n(f) Z(f) e^{i2\pi f t} \, df \tag{231}$$

from which we readily infer the channel model shown in Fig. 30.

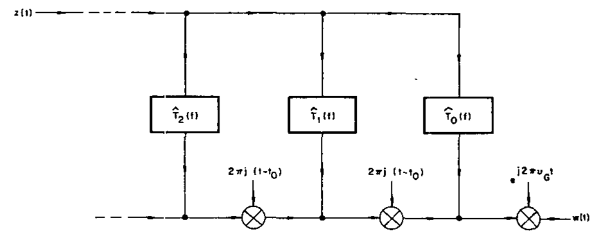

Fig. 30—t-power series model, input filter version.

When the output time function is limited to a finite interval of duration T_0 sec and $G(f, \nu)$ is zero for values of ν outside an interval of duration β_G cps, one may show that the series (231) will be rapidly convergent if

$$2\pi T_0 \beta_G \ll 1. \tag{232}$$

Even if $w(t)$ is not time-limited and $G(f, \nu)$ is not ν-limited, (232) will still be a satisfactory convergence criterion if the "tails" of $w(t)$ and $G(f, \nu)$ (as a function of ν) drop to zero sufficiently rapidly.

When the channel is randomly time-variant the filter transfer functions in Fig. 30, like those in Fig. 29, become random processes in the frequency variable. Relationships analogous to those in (215) to (228) are readily constructed for the model in Fig. 30. In particular, for the WSSUS channel the correlation properties of the random filters in Fig. 30 become identical to those of the random filters in Fig. 29.

3) *ft- and tf-Power Series Models*: In this section we present two channel models, one arising from an expansion of $T(f, t)$ and the other from an expansion of $M(t, f)$ in the t and f variables. As in the previous power series models, it is desirable to remove mean path delays and Doppler shifts before expanding these functions. Thus, we define

$$T_{00}(f, t) = T(f, t) e^{i2\pi(f\xi_0 - t\nu_0)} \tag{233}$$

where ξ_0, ν_0 are a mean delay and Doppler shift defined as the value of ξ and ν, about which the Delay-Doppler-Spread Function $U(\xi, \nu)$ may be assumed "centered", *i.e.*, $U(\xi + \xi_0, \nu + \nu_0)$ is "centered" at $\xi = \nu = 0$.

Since in $T(f, t)$ the variable f is associated directly with the spectrum of the input signal and the variable t is associated with the output time function, we expand $T_{00}(f, t)$ in the following double power series:

$$T_{00}(f, t) = \sum_0^{\infty} \sum_0^{\infty} T_{mn}(2\pi j)^{m+n}(f - f_i)^m (t - t_0)^n \tag{234}$$

where

$$T_{mn} = \frac{1}{m! \, n! \, (2\pi j)^{m+n}} \left[\frac{\partial^{m+n} T_{00}(f, t)}{\partial f^m \, \partial t^n} \right]_{\substack{f=f_i \\ t=t_0}}$$
$$= \frac{1}{m! \, n!} \iint (-\xi)^m (\nu)^n$$
$$\cdot U(\xi + \xi_0, \nu + \nu_0) e^{-i2\pi(f_i\xi - t_0\nu)} \, d\xi \, d\nu. \tag{235}$$

Using (234) in (19), we find that the output time function is represented by the series

$$w(t) = e^{j2\pi\nu_0 t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} T_{mn}(2\pi j)^n (t - t_0)^n D_{f_i}^m [z(t - \xi_0)]. \quad (236)$$

It is readily seen by examination of (236) that an appropriate channel model whose output is given by (236) may be obtained by using the f-Power Series Model of Fig. 27 with each gain function represented by the t-Power Series Model of Fig. 30. We leave it to the reader to sketch this channel model, which we shall call the ft-Power Series Channel Model.

We may make statements concerning the convergence properties which are similar to those pertinent to the output series expansions of the t- and f-Power Series Models. However, since the tf-Power Series Model is, in essence, both a t- and an f-Power Series Model, the convergence requirements of both models need to be imposed. Thus, it is readily shown that when $U(\xi, \nu)$ is zero outside a rectangle whose sides are β_0 cps long in the ν direction and Δ_0 sec long in the ξ direction and when $z(t)$ is band-limited to a bandwidth B_i cps, a sufficient condition for convergence of (236) is the satisfaction of the inequalities

$$2\pi \, |t - t_0| \, \beta_0 \ll 1 \quad (237)$$

$$2\pi B_i \Delta_0 \ll 1. \quad (238)$$

It is clear from (237) that the series (237) may not converge for all values of t. However, if the significant values of the output are confined to an interval of duration T_0 one may change (237) to the inequality

$$2\pi T_0 \beta_0 \ll 1. \quad (239)$$

If the finite Taylor series

$$T_{00}(f, t) = \sum_{m=0}^{M} \sum_{n=0}^{N} T_{mn}(2\pi j)^{m+n}(f - f_i)^m (t - t_0)^n + \frac{(f - f_i)^M (t - t_0)^n}{M! \, N!} \left[\frac{\partial^{M+N} T_{00}(f, t)}{\partial f^M \, \partial t^N} \right]_{\substack{f=f' \\ t=t'}} \quad (240)$$

is used in (19), one readily finds that for the WSSUS channel the average magnitude squared error incurred by using terms up to $m = M - 1$ and $n = N - 1$ is bounded by

$$|E_{MN}|^2 \leq \frac{(2\pi T_0 \beta_0)^{2N} (2\pi B_i \Delta_0)^{2M}}{(N!)^2 \, (M!)^2} \left| \int |f|^M \, |\tilde{Z}(f)| \, df \right|^2 \cdot \iint \xi^{2M} \nu^{2N} \tilde{\tilde{S}}(\xi, \nu) \, d\xi \, d\nu \quad (241)$$

where $\tilde{Z}(f)$ is a shifted scaled version of the input spectrum with unit bandwidth and located at zero frequency, and $\tilde{\tilde{S}}(\xi, \nu)$ is the Scattering Function [see (74)] associated with a shifted scaled version of $U(\xi, \nu)$ which is zero outside a unit square centered at $\xi = \nu = 0$. It is assumed in (241) that only those values of t are of interest for which $|t - t_0| \leq T_0$.

A discussion entirely dual to the one above may be formulated by expanding $M(t, f)$ rather than $T(f, t)$ in a Taylor series and using the resultant series to derive a series expansion for the output spectrum. Because the analytical procedure is identical to that above, except for a replacement of functions and variables by their duals, we shall not present these derivations. We note, however, that the resulting channel model, which we call the tf-Power Series Model may be obtained by using the t-Power Series Model of Fig. 29, with each filter represented by the f-Power Series Model of Fig. 28.