PolyPublie
Polytechnique Montréal

POLYTECHNIQUE MONTRÉAL
UNIVERSITÉ D'INGÉNIERIE

Titre: Behrens-Fisher problem under the assumption of homogeneous coefficients of variation
Title:

Auteurs: Bimal-Kumar Sinha, & Bernard Clément
Authors:

Date: 1975

Type: Rapport / Report

Référence: Sinha, B.-K., & Clément, B. (1975). Behrens-Fisher problem under the assumption of homogeneous coefficients of variation. (Technical Report n° EP-R-75-40). https://publications.polymtl.ca/6203/
Citation:

Document en libre accès dans PolyPublie
Open Access document in PolyPublie

URL de PolyPublie: https://publications.polymtl.ca/6203/
PolyPublie URL:

Version: Version officielle de l'éditeur / Published version

Conditions d'utilisation: Tous droits réservés / All rights reserved
Terms of Use:

Document publié chez l'éditeur officiel
Document issued by the official publisher

Institution: École Polytechnique de Montréal

Numéro de rapport: EP-R-75-40
Report number:

URL officiel:
Official URL:

Mention légale:
Legal notice:

Ce fichier a été téléchargé à partir de PolyPublie, le dépôt institutionnel de Polytechnique Montréal
This file has been downloaded from PolyPublie, the institutional repository of Polytechnique Montréal

https://publications.polymtl.ca

MATHÉMATIQUES

Rapport Technique EP75-R-40

Classification: Library of Congress no

BEHRENS-FISHER PROBLEM UNDER THE ASSUMPTION OF HOMOGENEOUS COEFFICIENTS OF VARIATION

par

BIMAL KUMAR SINHA
BERNARD CLEMENT

Août 1975

Ecole Polytechnique de Montréal

CA2PQ
UP 4
75R40
(Ang)

Campus de l'Université
de Montréal
Case postale 6079
Succursale 'A'
Montréal, Québec
H3C 3A7

Bibliothèque
Ecole
Polytechnique
MONTRÉAL

CLASSIFICATION
CA2PQ
UP4
75R40

No D'ENTRÉE
77759

(Aug)

17 OCT 1975

A CONSULTER SUR PLACE

BEHRENS-FISHER PROBLEM UNDER THE ASSUMPTION OF HOMOGENEOUS COEFFICIENTS OF VARIATION

par

Bimal Kumar Sinha

et

Bernard Clément

77759

Soumis pour publication

BEHRENS-FISHER PROBLEM UNDER THE ASSUMPTION OF HOMOGENEOUS COEFFICIENTS OF VARIATION

by Bimal Kumar Sinha†
University of Montreal

and

Bernard Clément*
Ecole Polytechnique

SUMMARY

This paper examines and compares by simulation techniques the powers of the likelihood ratio (LR) test and an "asymptotically (in some sense) optimum" invariant test with those of several other relevant tests for the problem of testing the equality of two univariate normal population means under the assumption of heterogeneous variances but homogeneous coefficients of variation. It is seen that the LR test is highly satisfactory for all values of the coefficient of variation and the "asymptotically optimum" invariant test, which is computationally much simpler than the LR test, is a reasonably good competitor for coefficient of variation greater than or equal to 3. Also, the study of robustness of the LR test shows that the test remains as highly satisfactory even for an appreciable deviation of the ratio of two coefficients of variation from unity.

Some keywords: Behrens-Fisher problem; likelihood ratio test; invariant test; simulation techniques; coefficient of variation.

---

* Research supported by the National Research Council of Canada, Grant A-8728.

† Now at the University of Pittsburgh.

1. INTRODUCTION

We consider in this paper what is technically known as the Behrens-Fisher problem of testing the equality of means of two univariate normal populations when the variances are unequal and unknown. There has been a considerable amount of work in the literature on this problem during the last few decades. Starting with Behrens who introduced this problem in 1929 and Fisher (1939) who defended Behrens'work, Welch (1937), Gronow (1951), David and Johnson (1951, a and b), Box (1954, a and b), among others, have contributed significantly towards examining and discussing various aspects of this problem. Lately, Lohrding (1969) derived the likelihood ratio test for this problem with equal sample sizes from both the populations under the assumption of heterogeneous variances but homogeneous coefficients of variation and examined by simulation techniques its power compared to several other relevant tests. As has been noted by Lohrding, the assumption of homogeneous coefficients of variation is a valid assumption in many types of agricultural, biological and psychological experimentation, because quite often the treatment that yields a larger mean also has a larger standard deviation. The motivation underlying the present investigation lies in the empirical findings of Lohrding, which suggest that the use of the likelihood ratio test under the assumption of homogeneous coefficients of variation is extremely helpful as it results in maximum power compared to other tests.

Throughout this paper Lohrding's assumption of homogeneous coefficients of variation is made. In section 2 we generalize the result of Lohrding and derive the likelihood ratio test for unequal sample sizes. The small sample distribution of the test statistic is obtained by simulation techniques. Its power is studied in section 4 and compared with those of several other relevant tests. The conclusion is that this test is highly powerful (among the tests under comparison) for all values of the coefficient of variation despite Lohrding's remark that the test is good only for a certain ~~marge~~ range of the coefficient of variation.

Another feature of the problem under Lohrding's assumption is that the various tests under comparison are all invariant under the action of the multiplicative group of positive real constants that keeps the problem invariant. It is, therefore, natural to enquire if there exists a uniformly most powerful invariant (UMPI) test for this problem. The approach of invariance is adopted in section 3 and certain difficulties in obtaining the UMPI test are noted. However, the situation is extremely simple if the common coefficient of variation is very large ($\to\infty$). The UMP unbiased invariant test derived under this assumption is seen to result in an F-test. It is shown by simulation techniques that this latter test is most powerful compared to the other invariant tests (including obviously the likelihood ratio test of the previous section) for common coefficient of variation equal to or larger than 3.

2. DERIVATION OF THE LIKELIHOOD RATIO TEST OF EQUALITY OF TWO NORMAL POPULATION MEANS UNDER THE ASSUMPTION OF HOMOGENEOUS COEFFICIENTS OF VARIATION

2.1. DERIVATION OF THE LIKELIHOOD RATIO STATISTIC

Let $\{X_{ij}, j = 1,\ldots,n_i\}$ a random sample of size n_i from a univariate normal population $N(\mu_i,\beta^2 \mu_i^2)$, $i = 1,2$. Let the non-restrictive parameter space be $\Omega = \{(\mu_1,\mu_2,\beta):0 < \mu_1 < \infty, 0 < \mu_2 < \infty, 0 < \beta < \infty\}$. In this section we derive the likelihood ratio test of the hypothesis $H_0 : \mu_1 = \mu_2$ against the alternative $H_1 : \mu_1 \neq \mu_2$. Towards this we have first to derive the maximum likelihood estimators of the parameters under both H_0 and H_1.

The likelihood function under H_1 is

$$L(\Omega) = (2\pi)^{-(n_1+n_2)/2} \cdot (\beta\mu_1)^{-n_1} (\beta\mu_2)^{-n_2} \exp\left[-\tfrac{1}{2} \sum_{i=1}^{2} \left\{ \sum_{j=1}^{n_i} (X_{ij}-\mu_i)^2/\beta^2\mu_i^2 \right\}\right] \quad (2.1)$$

Taking derivatives of log $L(\Omega)$ with respect to the parameters β,μ_1 and μ_2 and setting them equal to zero gives the following set of equations:

$$\beta^2 = n_1 \{S_1^2 + (\bar{X}_1-\mu_1)^2\}/\mu_1^2(n_1+n_2) + n_2 \{S_2^2 + (\bar{X}_2-\mu_2)^2\}/\mu_2^2(n_1+n_2) \quad (2.2)$$

and

$$\beta^2\mu_k^2 + \bar{X}_k\mu_k - (\bar{X}_k^2+S_k^2) = 0 \quad , \quad k = 1,2 \quad (2.3)$$

where $\bar{X}_i$ is the average of the i-th sample $(i = 1,2)$ and $S_i^2 = \sum_{j=1}^{n_i} (X_{ij}-\bar{X}_i)^2/n_i$, $i = 1,2$. Writing (2.3) in the form

$$\beta^2 = (\bar{X}_k^2 + S_k^2 - \bar{X}_k\mu_k)/\mu_k^2 \quad , \quad k = 1,2 \quad (2.4)$$

gives

$$\beta^2 = n_1 (\bar{X}_1^2 + S_1^2 - \bar{X}_1 \mu_1)/\mu_1^2 (n_1+n_2) + n_2 (\bar{X}_2^2 + S_2^2 - \bar{X}_2 \mu_2)/\mu_2^2 (n_1+n_2) \tag{2.5}$$

comparing (2.2) and (2.5) gives μ_1 in terms of μ_2 as

$$\mu_1 = n_1 \bar{X}_1 \mu_2 / \{(n_1+n_2)\mu_2 - n_2 \bar{X}_2\} \tag{2.6}$$

Using the above expression for μ_1 in the identity

$$(\bar{X}_1^2 + S_1^2 - \bar{X}_1 \mu_1)/\mu_1^2 = (\bar{X}_2^2 + S_2^2 - \bar{X}_2 \mu_2)/\mu_2^2$$

gives the following equation in μ_2:

$$\mu_2^2 - a\mu_2 + b = 0 \tag{2.7}$$

where

$$a = [(n_1 n_2 + 2n_2^2 - n_1^2)\bar{X}_1^2 \bar{X}_2 + 2n_2 (n_1+n_2)\bar{X}_2 S_1^2]/[(n_1+n_2)\{n_2 \bar{X}_1^2 + (n_1+n_2) S_1^2\}]$$

and

$$b = [\bar{X}_1^2 \bar{X}_2 (n_2^2 - n_1^2) + n_2^2 S_1^2 \bar{X}_2^2 - n_1^2 \bar{X}_1^2 S_2^2]/[(n_1+n_2)\{n_2 \bar{X}_1^2 + (n_1+n_2) S_1^2\}].$$

Hence, we have $\hat{\mu}_2 = \frac{1}{2}[a \pm \sqrt{a^2 - 4b}]$. It is seen from (2.3) that the equation in μ_k (for $k = 1,2$) has two real roots, one positive and the other negative. Thus $\hat{\mu}_k$ may always be chosen positive and we have in fact the following expression for $\hat{\mu}_2$:

$$\hat{\mu}_2 = [\{(n_1 n_2 + 2n_2^2 - n_1^2)\bar{X}_1^2 + 2n_2 (n_1+n_2) S_1^2\}\bar{X}_2 + n_1 \sqrt{n_1+n_2}\,\bar{X}_1 \{(n_1+n_2)\bar{X}_1^2 \bar{X}_2^2 + 4n_1 \bar{X}_2^2 S_1^2 + 4n_2 \bar{X}_1^2 S_2^2 + 4(n_1+n_2) S_1^2 S_2^2\}^{\frac{1}{2}}]/[2(n_1+n_2)\{n_2 \bar{X}_1^2 + (n_1+n_2) S_1^2\}] \tag{2.8}$$

Using $\hat{\mu}_2$, we get from (2.6)

$$\hat{\mu}_1 = [\{(n_1 n_2 + 2n_1^2 - n_2^2)\bar{X}_2^2 + 2n_1(n_1+n_2)S_2^2\}\bar{X}_1 + n_2\sqrt{n_1+n_2}\,\bar{X}_2\{(n_1+n_2)\bar{X}_1^2\bar{X}_2^2 + 4n_2\bar{X}_1^2 S_2^2 + 4n_1\bar{X}_2^2 S_1^2 + 4(n_1+n_2)S_1^2 S_2^2\}^{\frac{1}{2}}]/[2(n_1+n_2)\{n_1\bar{X}_2^2 + (n_1+n_2)S_2^2\}] \qquad (2.9)$$

which is also otherwise evident due to symmetry of the likelihood equations in the μ's. Finally, $\hat{\beta}^2$ is obtained from (2.2).

This set of estimators solves the set of likelihood equations (2.2) and (2.3). Now a check must be made to see if the joint estimators $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\beta}$ define a maximum. This involves the usual routine procedure of checking for negative definiteness of the matrix

$$\left[\frac{\partial^2 \log L(\Omega)}{\partial\varphi_r\,\partial\varphi_s}\right]_{\hat{\varphi}} = \begin{bmatrix} \frac{n_1}{\hat{\mu}_1^2} - \frac{n_1}{\hat{\beta}^2\hat{\mu}_1^3}(3\hat{\beta}^2\hat{\mu}_1 + \bar{X}_1) & 0 & -\frac{2n_1}{\hat{\beta}\hat{\mu}_1} \\ 0 & \frac{n_2}{\hat{\mu}_2^2} - \frac{n_2}{\hat{\beta}^2\hat{\mu}_2^3}(3\hat{\beta}^2\hat{\mu}_2 + \bar{X}_2) & -\frac{2n_2}{\hat{\beta}\hat{\mu}_2} \\ -\frac{2n_1}{\hat{\beta}\hat{\mu}_1} & -\frac{2n_2}{\hat{\beta}\hat{\mu}_2} & -\frac{2(n_1+n_2)}{\hat{\beta}^2} \end{bmatrix}$$

where $\varphi_1 = \mu_1$, $\varphi_2 = \mu_2$, $\varphi_3 = \beta$.

It has been seen through considerable manipulation of inequalities that the above matrix is negative definite. The details are omitted.

We have thus shown that the estimators $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\beta}$ are joint maximum likelihood estimators of the parameters under H_1. This

gives the maximum of the likelihood under H_1 as

$$L(\hat{\Omega}) = (2\pi)^{-(n_1+n_2)/2} \cdot (\bar{X}_1^2 + S_1^2 - \bar{X}_1\hat{\mu}_1)^{-n_1/2} \cdot (\bar{X}_2^2 + S_2^2 - \bar{X}_2\hat{\mu}_2)^{-n_2/2} \exp[-(n_1+n_2)/2]. \tag{2.10}$$

Under $H_0 : \mu_1 = \mu_2 = \mu$ (say), the restrictive parameter space is $\omega = \{(\mu,\beta) : 0 < \mu < \infty,\ 0 < \beta < \infty\}$ and it is easy to verify that $\hat{\mu} = \bar{X}$, $\hat{\beta} = S/\bar{X}$ where $(n_1+n_2)\,\bar{X} = n_1\bar{X}_1 + n_2\bar{X}_2$ and

$$(n_1+n_2)\,S^2 = n_1S_1^2 + n_2S_2^2 + n_1n_2(\bar{X}_1-\bar{X}_2)^2/(n_1+n_2).$$

Hence, the maximum of the likelihood under H_0 is given by

$$L(\hat{\omega}) = (2\pi)^{-(n_1+n_2)/2} \cdot (S^2)^{-(n_1+n_2)/2} \cdot \exp[-(n_1+n_2)/2] \tag{2.11}$$

Therefore, the likelihood ratio statistic for testing $H_0 : \mu_1 = \mu_2$ against $H_1 : \mu_1 \neq \mu_2$ under the assumption of homogeneous coefficients of variation is given by

$$\lambda = L(\hat{\omega})/L(\hat{\Omega}) = [S_1^2+A_1]^{n_1/2} \cdot [S_2^2+A_2]^{n_2/2} \cdot [S^2]^{-(n_1+n_2)/2} \tag{2.12}$$

where

$$A_1 = n_2[(n_1+n_2)\,\bar{X}_1^2(\bar{X}_2^2+2S_2^2) - \sqrt{n_1+n_2}\,\bar{X}_1\bar{X}_2\,B]/2(n_1+n_2)\{n_1\bar{X}_2^2 + (n_1+n_2)\,S_2^2\},$$

$$A_2 = n_1[(n_1+n_2)\,\bar{X}_2^2(\bar{X}_1^2+2S_1^2) - \sqrt{n_1+n_2}\,\bar{X}_1\bar{X}_2\,B]/2(n_1+n_2)\{n_2\bar{X}_1^2 + (n_1+n_2)\,S_1^2\}$$

with

$$B = \{(n_1+n_2)\,\bar{X}_1^2\bar{X}_2^2 + 4n_2\bar{X}_1^2S_2^2 + 4n_1\bar{X}_2^2S_1^2 + 4(n_1+n_2)\,S_1^2S_2^2\}^{\frac{1}{2}}.$$

In particular, when the sample sizes are equal $n_1 = n_2 = n$ (say), λ is reduced to $[\{[(\bar{X}_1^2+2S_1^2)(\bar{X}_2^2+2S_2^2)]^{\frac{1}{2}} - \bar{X}_1\bar{X}_2\}/(2S^2)]^n$, as obtained already by Lohrding (1969).

2.2. DETERMINATION OF THE SAMPLING DISTRIBUTION OF (-2 log λ)

To get a complete description of the LR test what is needed now is the specification of a constant corresponding to a specified level of significance. This in term involves the problem of determination of the sampling distribution of (-2 log λ) under H_0. It is known (Wilks, 1963) that in large samples the distribution of (-2 log λ) under H_0 is that of a central χ^2 variable with 1 d.f. However, for small samples, this distribution is not known and seems to be very complicated in view of the form of (-2 log λ) as evidenced from (2.12).

To investigate the small sample distribution, we have adopted the simulation technique. 10,000 values of the test statistic (-2 log λ) are computed from random samples of various sizes, assuming the null hypothesis $H_0 : \mu_1 = \mu_2$ to be true. These have been obtained by using the Gauss subroutine generating $N(\mu,\sigma^2)$ variates. The empirical distribution function of $X = -2 \log \lambda$ has first been obtained for values of X from 0 to 15 at intervals of 0.1 and then its upper tail has been smoothed by using the exponential model $F(x) = 1-\alpha_1 e^{-\beta x}$. These smoothed values have further been smoothed out by using the formula

$$X - \chi^2_{1,\alpha} = \xi_0 + \xi_1/n_1 + \xi_2/n_2$$

for $n_1 \leqslant n_2$. The two-stage smoothing process gave highly satisfactory results and the smoothed critical values are reproduced in table 1 to table 4.

TABLE 1: SMOOTHED APPROXIMATE CRITICAL VALUES FOR THE TEST STATISTIC $-2 \log \lambda$ ($\alpha = 0.10$)

n_1 \ n_2	2	4	6	10	15	20	30
2	5.60	4.25	3.87	3.57	3.42	3.35	3.27
4		3.93	3.56	3.26	3.11	3.03	2.96
6			3.45	3.15	3.00	2.93	2.85
10				3.07	2.92	2.85	2.77
15					2.87	2.80	2.72
20						2.78	2.71
30							2.70

For intermediate values use $X = 2.49 + \frac{1.27}{n_1} + \frac{4.50}{n_2}$ where $n_1 \leqslant n_2$ and X is the critical value of $-2 \log \lambda$ at $\alpha = 0.10$.

TABLE 2: SMOOTHED APPROXIMATE CRITICAL VALUES FOR THE TEST STATISTIC $-2 \log \lambda$ ($\alpha = 0.05$)

n_1 \ n_2	2	4	6	10	15	20	30
2	8.03	6.00	5.46	5.03	4.82	4.71	4.60
4		5.60	5.06	4.63	4.42	4.31	4.21
6			3.93	4.50	4.29	4.18	4.07
10				4.39	4.18	4.07	3.96
15					4.13	4.02	3.92
20						3.99	3.88
30							3.86

For intermediate values use $X = 3.59 + \frac{1.60}{n_1} + \frac{6.45}{n_2}$ where $n_1 \leqslant n_2$ and X is the critical value of $-2 \log \lambda_1$ at $\alpha = 0.05$.

TABLE 3: SMOOTHED APPROXIMATE CRITICAL VALUES FOR THE TEST STATISTIC $-2 \log \lambda$ ($\alpha = 0.025$)

n_1 \ n_2	2	4	6	10	15	20	30
2	10.57	7.78	7.02	6.42	6.12	5.97	5.82
4		7.37	6.62	6.01	5.71	5.56	5.41
6			6.49	5.88	5.58	5.43	5.28
10				5.78	5.47	5.32	5.17
15					5.42	5.27	5.12
20						5.24	5.09
30							5.06

For intermediate values use $X = 4.71 + \frac{1.61}{n_1} + \frac{9.05}{n_2}$ where $n_1 \leqslant n_2$ and X is the critical value of $-2 \log \lambda_1$ at $\alpha = 0.025$.

TABLE 4: SMOOTHED APPROXIMATE CRITICAL VALUES FOR THE TEST STATISTIC $-2 \log \lambda$ ($\alpha = 0.01$)

n_1 \ n_2	2	4	6	10	15	20	30
2	13.82	10.14	9.13	8.36	7.97	7.77	7.57
4		9.64	8.65	7.86	7.47	7.27	7.07
6			8.48	7.69	7.30	7.10	6.90
10				7.56	7.17	6.97	6.77
15					7.10	6.90	6.71
20						6.87	6.70
30							6.64

For intermediate values use $X = 6.18 + \frac{1.99}{n_1} + \frac{11.84}{n_2}$ where $n_1 \leq n_2$ and X is the critical value of $-2 \log \lambda_1$ at $\alpha = 0.01$.

3. "ASYMPTOTICALLY (as $\beta \to \infty$) OPTIMUM" INVARIANT TEST OF EQUALITY OF TWO NORMAL POPULATION MEANS UNDER THE ASSUMPTION OF HOMOGENEOUS COEFFICIENTS OF VARIATION

Assume the same formulation as in section 2. Let $G = \{g : g > 0\}$ be the multiplicative group of positive real constants. It is then easy to see that under the action of the group G acting as

$$X_1 \to gX_1,\ X_2 \to gX_2,\ \mu_1 \to g\mu_1,\ \mu_2 \to g\mu_2,\ \beta \to \beta \tag{3.1}$$

the basic testing problem remains invariant. Since $(\bar{X}_1, \bar{X}_2, S_1, S_2)$ are jointly sufficient for the parameters involved, it follows that we need consider test functions depending only on these quantities. A maximal invariant statistic under the action (3.1) is

$$(\bar{X}_1/S_1,\ \bar{X}_2/S_2,\ S_1/S_2) \equiv (T_1, T_2, T_3) \quad \text{(say)}.$$

It is then well-known that any invariant test must depend only on (T_1, T_2, T_3). To get a reasonably good invariant test what is needed first is the sampling distribution of (T_1, T_2, T_3). It may be noted that a maximal invariant parameter is $(\theta = \mu_1/\mu_2, \beta)$ so that the above distribution will depend only on θ and β, and our problem in this reduced set-up is that of testing $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$ with β as a nuisance parameter. The derivation of this distribution is rather straightforward and comes out as follows:

$$f_{\theta,\beta}(t_1, t_2, t_3) = c \cdot \theta^{n_2} \cdot t_3^{n_1 - 1} \left[(1+n_1 t_1^2)\, t_3^2 + \theta^2 (1+n_2 t_2^2) \right]^{-(n_1+n_2)/2} \cdot \exp\left[-(n_1+n_2)/2\beta^2\right]$$

$$\times \left[\sum_{j=0}^{\infty} \frac{\Gamma(n_1+n_2+j/2)}{\Gamma(j+1)} \cdot \frac{1}{\beta^j} \left\{ \frac{(n_1 t_1 t_3 + \theta n_2 t_2)\sqrt{2}}{\sqrt{(1+n_1 t_1^2)\, t_3^2 + \theta^2 (1+n_2 t_2^2)}} \right\}^j \right],$$

$$-\infty < t_1 < \infty,\ -\infty < t_2 < \infty,\ 0 < t_3 < \infty \tag{3.2}$$

where c is the normalizing absolute constant (independent of θ and β). Any step to obtain an "optimum" invariant test from (3.2) proceeds with the following standard argument.

Observe that under $H_0 : \theta = 1$, β is the only nuisance parameter and $V(t_1,t_2,t_3) = (n_1 t_1 t_3 + n_2 t_2)/\{(1+n_1 t_1^2)\, t_3^2 + (1+n_2 t_2^2)\}^{\frac{1}{2}}$ is a sufficient statistic for β. From (3.2), it follows that the distribution of V under H_0 is of the form

$$g(V) = \left\{ \sum_{j=0}^{\infty} \frac{\Gamma((n_1+n_2+j)/2)}{\Gamma(j+1)} \cdot \frac{1}{\beta^j} \cdot V^j \right\} h(V) \quad , \quad -\infty < V < \infty \tag{3.3}$$

for some function $h(V)$ which is positive almost every where. Since then for any (measurable) function $\ell(V)$ of V, $E_\beta\{\ell(V)\} = 0$ identically in $\beta > 0$ implies $\int_{-\infty}^{\infty} V^j\, \ell(V)\, h(V)\, dV = 0$, for all $j = 0,1,2,\ldots$, it follows that $\int_{-\infty}^{\infty} e^{\theta V}\, \ell(V)\, h(V)\, dV = 0$ for all $\theta \in (-\gamma,\gamma)$, $\gamma > 0$ and hence with the usual argument that the distribution of V is complete. This is no doubt favourable towards obtaining an "optimum" invariant test. However, the usual approach through conditioning argument does not seem to make any progress in this particular problem not even locally. Thus, we have been unable to obtain any "optimum" invariant test for this problem (locally or uniformly).

On the other hand, the above testing problem is extremely simple and the solution is almost immediate if we assume that the coefficient of variation β is large. Taking $\beta \to \infty$ in (3.2) gives the distribution of the maximal invariant statistic as

$$f_\theta(t_1,t_2,t_3) = c \cdot \theta^{n_2} \cdot t^{n_1 - 1} \left[(1+n_1 t_1^2)\, t_3^2 + \theta^2 (1+n_2 t_2^2) \right]^{-(n_1+n_2)/2} \tag{3.4}$$

and the solution is obtained by looking at the quantity

$$F = \{n_2 (1+n_1 t_1^2)\ t_3^2\}/\{n_1 (1+n_2 t_2^2)\ \theta^2\}$$

which has a central F-distribution with n_1, n_2 d.f. Since

$$t_3^2 (1+n_1 t_1^2) = \sum_1^{n_1} X_{1j}^2/S_2^2 \quad \text{and} \quad (1+n_2 t_2^2) = \sum_1^{n_2} X_{2j}^2/S_2^2 ,$$

it follows that under the assumption of large common homogeneous coefficient of variation, the "approximate" uniformly most powerful unbiased invariant test at level α has the specification

$$\text{Reject } H_0 \text{ if } \left\{\sum_1^{n_1} X_{1j}^2/n_1\right\} \Big/ \left\{\sum_1^{n_2} X_{2j}^2/n_2\right\} > F_{n_1,n_2,\alpha/2} \text{ or } < F_{n_1,n_2,1-\alpha/2} \qquad (3.5)$$

4. POWER STUDIES OF THE VARIOUS TESTS AND CONCLUSIONS

Extensive study has been made of the powers of the likelihood ratio test, the asymptotically optimum invariant test and the other relevant tests such as the F-test, the t-test and the t'-test (Fryer, 1966) by using simulation techniques and 99 tables have been prepared. But for the sake of space, only a few of the tables are reproduced here illustrating and justifying our conclusions. We refer to the technical reports (Clément, 1975, a and b) for details. The overall findings are:

1) Lohrding remarked that the likelihood ratio test is good for the coefficient of variation only between 0.25 and 1.0. Contrary to Lohrding's conclusion the likelihood ratio test has shown maximum power for all values of the common coefficient of variation. This is evident from table 5 to 12.

2) This is not to say that the other tests such as the t-test and the F-test do not achieve the same power as the likelihood ratio test but they do so only under some restrictive conditions such as when the variances are nearly equal (for the t-test) and widely different (for the F-test). This is again evident from the table 8, 12, 13.

3) Another very important finding is that the likelihood ratio test remains robust in the sense of retaining its power to a maximum even for an appreciable deviation of the ratio of the two coefficients of variation from unity, say from 0.66 to 1.5. This is again evident from table ~~11~~, ~~12~~, ~~14~~. 15, 16, 17, 18

4) However, if the common coefficient of variation is greater than or equal to 3, the optimum invariant test derived in section 3 has nearly the same power as the likelihood ratio test which is evident from the table ~~15~~, ~~16~~, ~~17~~, ~~18~~. 11, 12, 14 But since the former is computationally much simple than the latter, it should be used whenever the assumption of large ($\geqslant 3$) common coefficient of variation is met.

ACKNOWLEDGMENT

The first author is grateful to Professor N. Giri of the University of Montreal for financial support through an NRC grant.

Our thanks are due to H.Q. Lam for his assistance with the computer programming.

TABLE NO 5

TOTAL NUMBER OF REJECTION OUT OF 100 SAMPLES
SIMULATED WITH THE GIVEN POPULATION PARAMETERS
FOR 2 SAMPLES OF SIZE

N1= 10	N2= 10
MU1= 2.670	MU2= 4.670
SIGMA1= 2.000	SIGMA2= 3.500
BETA1=SIGMA1/MU1= 0.750	BETA2=SIGMA2/MU2= 0.750
THETA=MU1/MU2= 0.570	PSI=BETA1/BETA2= 1.000

TEST STATISTIC	ALPHA-LEVEL		
	0.01	0.05	0.10
LAMBDA	24	53	64
U	4	22	40
F	13	30	45
T	13	29	36
TPRIME	4	25	33

TABLE NO 6

TOTAL NUMBER OF REJECTION OUT OF 100 SAMPLES
SIMULATED WITH THE GIVEN POPULATION PARAMETERS
FOR 2 SAMPLES OF SIZE

N1= 10	N2= 10
MU1= 2.000	MU2= 4.000
SIGMA1= 2.000	SIGMA2= 4.000
BETA1=SIGMA1/MU1= 1.000	BETA2=SIGMA2/MU2= 1.000
THETA=MU1/MU2= 0.500	PSI=BETA1/BETA2= 1.000

TEST STATISTIC	ALPHA-LEVEL 0.01	0.05	0.10
LAMBDA	38	69	84
U	25	53	68
F	21	49	67
T	10	25	47
TPRIME	4	19	38

TABLE NO 7

TOTAL NUMBER OF REJECTION OUT OF 100 SAMPLES
SIMULATED WITH THE GIVEN POPULATION PARAMETERS
FOR 2 SAMPLES OF SIZE

N1= 10	N2= 10
MU1= 0.950	MU2= 1.000
SIGMA1= 0.475	SIGMA2= 0.500
BETA1=SIGMA1/MU1= 0.500	BETA2=SIGMA2/MU2= 0.500
THETA=MU1/MU2= 0.950	PSI=BETA1/BETA2= 1.000

TEST STATISTIC	ALPHA-LEVEL 0.01	0.05	0.10
LAMBDA	1	7	13
U	0	0	0
F	0	1	7
T	2	7	13
TPRIME	0	4	10

TABLE NO 9

TOTAL NUMBER OF REJECTION OUT OF 100 SAMPLES
SIMULATED WITH THE GIVEN POPULATION PARAMETERS
FOR 2 SAMPLES OF SIZE

N1= 10	N2= 10
MU1= 1.000	MU2= 1.500
SIGMA1= 0.200	SIGMA2= 0.300
BETA1=SIGMA1/MU1= 0.200	BETA2=SIGMA2/MU2= 0.200
THETA=MU1/MU2= 0.333	PSI=BETA1/BETA2= 1.000

TEST STATISTIC	ALPHA-LEVEL 0.01	0.05	0.10
LAMBDA	98	100	100
U	0	0	6
F	7	16	29
T	93	98	99
TPRIME	85	98	99

TABLE NO 10

TOTAL NUMBER OF REJECTION OUT OF 100 SAMPLES
SIMULATED WITH THE GIVEN POPULATION PARAMETERS
FOR 2 SAMPLES OF SIZE

N1= 10	N2= 20
MU1= 8.000	MU2=10.000
SIGMA1= 2.000	SIGMA2= 2.500
BETA1=SIGMA1/MU1= 0.250	BETA2=SIGMA2/MU2= 0.250
THETA=MU1/MU2= 0.800	PSI=BETA1/BETA2= 1.000

TEST STATISTIC	ALPHA-LEVEL 0.01	0.05	0.10
LAMBDA	46	72	84
U	0	0	0
F	2	7	20
T	34	65	76
TPRIME	36	63	78

TABLE NO 11

TOTAL NUMBER OF REJECTION OUT OF 100 SAMPLES
SIMULATED WITH THE GIVEN POPULATION PARAMETERS
FOR 2 SAMPLES OF SIZE

N1= 10	N2= 20
MU1= 1.000	MU2= 2.000
SIGMA1= 3.000	SIGMA2= 6.000
BETA1=SIGMA1/MU1= 3.000	BETA2=SIGMA2/MU2= 3.000
THETA=MU1/MU2= 2.000	PSI=BETA1/BETA2= 1.000

TEST STATISTIC	ALPHA-LEVEL		
	0.01	0.05	0.10
LAMBDA	40	63	82
U	34	63	76
F	29	56	75
T	2	8	10
TPRIME	5	9	10

TABLE NO 12

TOTAL NUMBER OF REJECTION OUT OF 100 SAMPLES
SIMULATED WITH THE GIVEN POPULATION PARAMETERS
FOR 2 SAMPLES OF SIZE

N1= 10	N2= 10
MU1= 1.000	MU2= 2.000
SIGMA1= 4.000	SIGMA2= 8.000
BETA1=SIGMA1/MU1= 4.000	BETA2=SIGMA2/MU2= 4.000
THETA=MU1/MU2= 0.500	PSI=BETA1/BETA2= 1.000

TEST STATISTIC	ALPHA-LEVEL 0.01	0.05	0.10
LAMBDA	26	60	70
U	33	58	75
F	27	54	70
T	1	5	14
TPRIME	0	3	9

TABLE NO 13

TOTAL NUMBER OF REJECTION OUT OF 100 SAMPLES
SIMULATED WITH THE GIVEN POPULATION PARAMETERS
FOR 2 SAMPLES OF SIZE

N1= 10	N2= 10
MU1= 1.000	MU2= 1.500
SIGMA1= 2.000	SIGMA2= 3.000
BETA1=SIGMA1/MU1= 2.000	BETA2=SIGMA2/MU2= 2.000
THETA=MU1/MU2= 0.670	PSI=BETA1/BETA2= 1.000

TEST STATISTIC	ALPHA-LEVEL 0.01	0.05	0.10
LAMBDA	4	17	28
U	4	15	24
F	5	19	27
T	0	3	10
TPRIME	0	1	7

TABLE NO 14

TOTAL NUMBER OF REJECTION OUT OF 100 SAMPLES
SIMULATED WITH THE GIVEN POPULATION PARAMETERS
FOR 2 SAMPLES OF SIZE

N1= 10	N2= 10
MU1= 1.000	MU2= 0.500
SIGMA1= 4.000	SIGMA2= 2.000
BETA1=SIGMA1/MU1= 4.000	BETA2=SIGMA2/MU2= 4.000
THETA=MU1/MU2= 2.000	PSI=BETA1/BETA2= 1.000

TEST STATISTIC	ALPHA-LEVEL 0.01	0.05	0.10
LAMBDA	24	55	71
U	28	59	76
F	26	53	65
T	0	5	11
TPRIME	0	2	6

TABLE NO 15

TOTAL NUMBER OF REJECTION OUT OF 100 SAMPLES
SIMULATED WITH THE GIVEN POPULATION PARAMETERS
FOR 2 SAMPLES OF SIZE

N1= 10	N2= 10
MU1= 1.000	MU2= 2.000
SIGMA1= 0.500	SIGMA2= 1.200
BETA1=SIGMA1/MU1= 0.500	BETA2=SIGMA2/MU2= 0.600
THETA=MU1/MU2= 0.500	PSI=BETA1/BETA2= 0.833

TEST STATISTIC	ALPHA-LEVEL		
	0.01	0.05	0.10
LAMBDA	79	98	100
U	20	66	80
F	40	67	75
T	34	62	79
TPRIME	20	53	78

TABLE NO 16

TOTAL NUMBER OF REJECTION OUT OF 100 SAMPLES
SIMULATED WITH THE GIVEN POPULATION PARAMETERS
FOR 2 SAMPLES OF SIZE

N1= 10	N2= 10
MU1= 1.000	MU2= 2.000
SIGMA1= 1.000	SIGMA2= 3.000
BETA1=SIGMA1/MU1= 1.000	BETA2=SIGMA2/MU2= 1.500
THETA=MU1/MU2= 0.500	PSI=BETA1/BETA2= 0.667

TEST STATISTIC	ALPHA-LEVEL		
	0.01	0.05	0.10
LAMBDA	77	94	97
U	58	82	94
F	70	90	96
T	8	21	26
TPRIME	1	16	23

TABLE NO 17

TOTAL NUMBER OF REJECTION OUT OF 100 SAMPLES
SIMULATED WITH THE GIVEN POPULATION PARAMETERS
FOR 2 SAMPLES OF SIZE

N1= 10 N2= 20

MU1= 4.000 MU2= 6.000

SIGMA1= 2.000 SIGMA2= 2.500

BETA1=SIGMA1/MU1= 0.500 BETA2=SIGMA2/MU2= 0.416

THETA=MU1/MU2= 0.670 PSI=BETA1/BETA2= 1.200

TEST STATISTIC	ALPHA-LEVEL 0.01	0.05	0.10
LAMBDA	43	67	80
U	0	12	24
F	4	14	20
T	27	63	74
TPRIME	27	64	74

TABLE NO 18

TOTAL NUMBER OF REJECTION OUT OF 100 SAMPLES
SIMULATED WITH THE GIVEN POPULATION PARAMETERS
FOR 2 SAMPLES OF SIZE

N1= 10	N2= 20
MU1= 1.000	MU2= 2.000
SIGMA1= 1.500	SIGMA2= 2.000
BETA1=SIGMA1/MU1= 1.500	BETA2=SIGMA2/MU2= 1.000
THETA=MU1/MU2= 0.500	PSI=BETA1/BETA2= 1.500

TEST STATISTIC	ALPHA-LEVEL 0.01	0.05	0.10
LAMBDA	6	34	50
U	7	20	34
F	3	10	22
T	7	20	29
TPRIME	8	21	29

REFERENCES

BEHRENS, W.V. (1929). Ein Beitrag zur Fehlerberechnung bei weinige Beobachtungen. Landiv. Jb. 68, 807-837.

BOX, G.E.P. (1954a). Some theorems on quadratic forms applied in the study of analysis of variance problems, I. Effects of inequality of variance in one-way classifications. Ann. Math. Statist. 25, 290-302.

BOX, G.E.P. (1954b). Some theorems on quadratic forms applied in the study of analysis of variance problems, II. Effects of inequality of variances and of correlation between errors in the two-way classification. Ann. Math. Statist. 25, 484-498.

CLEMENT, B. (1975a). The approximate distribution of the likelihood ratio test for testing the equality of two means with equal coefficient of variation. Technical report EP-75-R-41, Ecole Polytechnique, 63 pages.

CLEMENT, B. (1975b). Power studies of the likelihood ratio test and other allied tests for a restrictive form of the Behrens-Fisher problem. Technical report EP-75-R-42, Ecole Polytechnique, 111 pages.

DAVID, F.N. and JOHNSON, N.L. (1951a). A method of investigating the effect of non-normality and heterogeneity of variance on tests of the general linear hypothesis. Ann. Math. Statist. 22, 382-392.

DAVID, F.N. and JOHNSON, N.L. (1951b). The effect of non-normality on the power function of the F-test in the analysis of variance. Biometrika 38, 43-57.

FISHER, R.A. (1939). The comparison of samples with possibly unequal variances. Ann. Eugenics 9, 174-180.

FRYER, H.C. (1966). Concepts and Methods of Experimental Statistics. Allyn and Bacon, Boston.

GRONOW, D.G.C. (1951). Test for the significance of the difference between means in two normal populations having unequal variances. Biometrika 38, 252-256.

LOHRDING, R.K. (1969). A test of equality of two normal population means assuming homogeneous coefficients of variation. Ann. Math. Statist. 40, 1374-1385.

WELCH, B.L. (1937). The significance of the difference between two means when the two population variances are unequal. Biometrika 29, 350-362.

WILKS, S.S. (1963). Mathematical Statistics. John Wiley and Sons, Inc. New York-London.

ÉCOLE POLYTECHNIQUE DE MONTRÉAL
3 9334 00288787 3