Available online at www.sciencedirect.com

ScienceDirect

Transportation Research Procedia 82 (2025) 2395–2409

Transportation Research Procedia

www.elsevier.com/locate/procedia

World Conference on Transport Research - WCTR 2023 Montreal 17-21 July 2023

A new stochastic model for carsharing suited to free-floating

Christine Fricker[a], Hanene Mohamed[b,*], Alessia Rigonat[a], Martin Trépanier[c]

[a]INRIA de Paris, DI-ENS, PSL Research University, 2 rue Simone Iff, 75012 Paris, France
[b]MODAL'X, UMR CNRS 9023, UPL, Université Paris Nanterre, Nanterre, France
[c]Mathematical and Industrial Engineering, Polytechnique Montreal, Montreal, Canada, H3T 1J4

Abstract

In car-sharing, free-floating is becoming increasingly popular. It means that the shared cars are parked in the public space without dedicated parking spaces. For the operator, this solves the problem of parking space requirements. But the acute imbalance problem shows the need of stochastic modelling and analysis. In this paper, a new stochastic model adapted to free-floating is proposed, taking into account the sharing of public space between private and free-floating cars. As is generally the case, the model consists of dividing the service area into small zones, with free-floating car dynamics adapted to usage, meaning car reservation, one-way trip and no parking space reservation. The originality of our model is that, due to the presence of private cars, the capacity of a zone seen by free-floating cars is random. We show that, unlike in station-based car-sharing systems, it is not limited. In addition, a stochastic averaging principle governs the behavior of free-floating cars. We exhibit a phase transition between a non-saturated regime where free-floating cars can always be parked and a saturated regime where free-floating cars cannot find an available parking space with positive probability. This probability is entirely determined by the environment - parameters of private cars and public space- which means that the operator cannot act on the proportion of zones without available parking spaces. The solution of the dimensioning problem -finding the optimal fleet size to minimize the number of zones without available free-floating cars or parking spaces- is completely different from that of station-based car-sharing which is a trade-off. It consists in claiming that *the more free-floating cars there are in the system, the more satisfied users are*, assuming always that private cars are much more numerous that free-floating cars.

Keywords: free-floating car-sharing, stochastic model, mean-field approach, averaging principle, dimensioning problem

1. Introduction

Over the last few decades, car-sharing has emerged as an alternative mode of transportation that is more ecological, economical and adapted to the architecture of large cities compared to the individual car ownership. See Shaheen et al. (2012).

* Corresponding author. Tel.: +0-000-000-0000.
E-mail address: hanene.mohamed@parisnanterre.fr

10.1016/j.trpro.2024.12.194

1.1. Station-based and free-floating car-sharing

Car-Sharing exists in two forms. The first one is round-trip where the user can book in advance and needs to return the car at the departure point. It involves station-based systems, with stations with a fixed number of parking spaces (station capacities). The second one proposes one-way trips. The user picks up a car at a departure point, makes a trip and returns the car at destination. These systems called *free-floating* are not based on physical stations. Cars are parked all over the public space. One-way trip systems are more convenient than round-trip ones because of flexibility, see Wielinski et al. (2015). Their usage is increasing, especially for free-floating-ones. See for example (Kolleck, 2021, Figure 1) for the German case-study on the development of car sharing.

On the one hand, for station-based systems with round-trips, because the parking space after departure remains reserved during the trip, the only problem for the user is to find an available car. On the other hand, the one-way systems are faced with a greater imbalance problem, with zones with high demand and no car and zones with low demand and many cars. Moreover, in contrast to round-trip systems, the users have to find both an available car and an available parking space. This issue is not only due to heterogeneity on demand. Even in a homogeneous system, the users could face this situation due to their random behavior. Managing such systems is not easy. An issue is the dimensioning problem: how many cars would be needed per station on average to minimize the proportion of empty stations (with no available cars) or full stations (with no available parking spaces)?

To address these issues, much work has been done to study of station-based car-sharing systems in order to understand their behavior and improve their reliability. Few concern stochastic analysis. Stochastic models of station-based sharing systems have been proposed to take into account the randomness of the system. See Gast et al. (2015); Bourdais et al. (2020). They are devoted to bike-sharing and car-sharing. Car-sharing systems have some additional features compared to bike-sharing, mainly the fact that reservation is proposed. In all the following, a bike-sharing model is the version without reservation of a car-sharing model; a station-based car-sharing model offers both car and parking space reservation while in a free-floating car-sharing model, only car reservation can be done, the cars being parked in the public space. Furthermore the parameters vary from one system to another one. For example, from a parameter estimation on data sets, the mean trip duration is greater for car-sharing, see Fricker et al. (2021), than for bike-sharing, see Gast et al. (2015).

The intuitive result for dimensioning the fleet size is that, in the case of a simple homogeneous model for bike-sharing (or car-sharing without reservation) systems, the stations should be roughly half full. Indeed, Fricker and Gast (2016) prove that the optimal fleet size per station in this case is half the station capacity plus an additional term which is the *load* of the system, i.e. the product of the arrival rate of user demand per station by the average trip time. Other refined stochastic models that take into account the heterogeneity of the system in for example Fricker et al. (2012), the impact of the reservation (of the car, the parking space or both of them) in Bourdais et al. (2020) or the presence of bikes and e-bikes in Ancel et al. (2022) have been proposed and analyzed by probabilistic or mean field techniques. Moreover the bike-sharing model motivates other analytical studies. Indeed, in Massey et al. (2022), motivated by bike-sharing, the exact solutions for transient probabilities of the $M/M/1$ queue are derived using group symmetry and complex analysis. Bražėnas and Valakevičius (2023) deals with trip times under a phase-type distribution.

The other, more recent form of car-sharing without physical stations is gaining in popularity. Recall that, in such free-floating systems, cars are parked in public areas. Users pick up the car of their choice, generally the one closest to their geographical location, complete their trip and park the car in an available public space around their destination. Free-floating offers more flexibility to the user and brings an answer to the problem of saturation at the destination. For Le Vine et al. (2014), because of these structural differences with station-based car-sharing systems, free-floating accounts for the bulk of journeys while station-based systems act as a complement.

1.2. Free-foating modelled as a station-based system

Despite the differences, free-floating has always been analyzed in the literature as station-based where the service area is divided into small zones, from $0.25\,km^2$ to $1\,km^2$, see (Lippoldt et al., 2018, section V) for a discussion, acting as stations. For example, to solve the relocation problem for free-floating car-sharing systems, Weikl and Bogenberger (2015) transform a free-floating system to a station-based car-sharing system by theoretically defining artificial stations by dividing the service area.

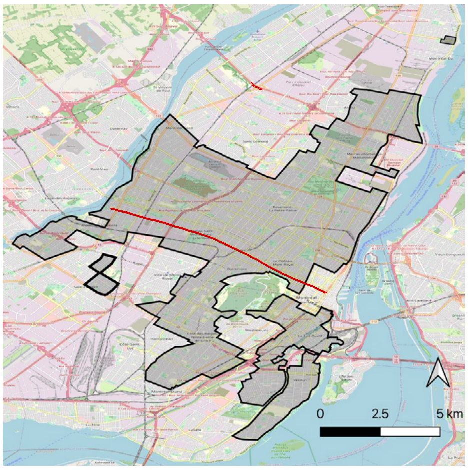

Fig. 1: Communauto service area, Montreal

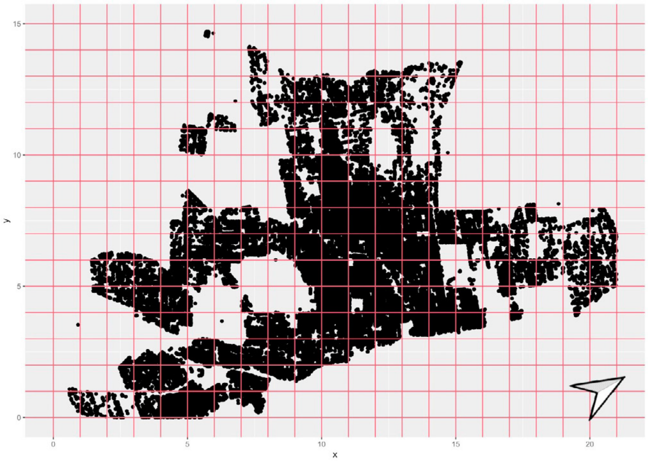

Fig. 2: Reconstruction of Communauto service area with 2020 dataset

In this paper, for our case study on free-floating in Montreal, Figures 1 and 2 show the Communauto free-floating service area in Montreal. Figure 1 shows a capture of this zone with St. Laurent Boulevard in red as a landmark. In Figure 2, a reconstruction of this zone is made through GPS data of the positions of Communauto's free-floating cars. A -62 degree rotation is made in order to make the Saint Laurent Boulevard coincide with the vertical axis and a mesh is applied to the whole service area. Note that in Figure 2 every red square represents a zone. In this case, Communauto's 100 km^2 service area is divided into approximately 100 zones. Every dot is a GPS position of a parked Communauto free-floating car. In this data set for the whole year 2020, each transaction corresponds to one GPS position, for example the destination position.

Such an approach allows to use a framework where results have been obtained. Nevertheless, this approach has some drawbacks already mentioned in the literature. First, the parameters of such a station-based model depend on the mesh which is used, see (Lippoldt et al., 2018, section V). Second Weikl and Bogenberger (2015) raise the issue that *transferring the existing relocation models for station-based systems to FFCS systems is however restricted, as the new systems have other dynamics resulting from spontaneous usage without reservation, without stations and without a priori information on the users' destinations*. Third, a crucial issue is how to define the capacity of each zone. This capacity is limited by the urban space and is arbitrary fixed in the previous literature. See Moreno et al. (2022) where two types of zones are considered, intensive zones and normal zones, according to demand. In Moreno et al. (2022), the capacity of each zone is fixed to the maximum of parked free-floating cars observed on real data in such a zone. In other respects, the problem of fixed capacity does not arise for free-floating bike-sharing systems where the capacity of the physical zones is large enough to be considered infinite. Moreover, it is interesting to note that in the FFBSS relocation problem with *electric fences*, a fixed capacity of artificial stations is relevant, see Zhang et al. (2019) for example.

1.3. Random capacity for free-floating systems

For free-floating systems, the capacity of each zone can be defined as the number of available parking spaces added to the number of the free-floating cars parked in this zone. Thus, the choice of a fixed capacity as in a station-based system does no longer make sense. Indeed, for free-floating, the cars of the system share available places in public areas with the private cars. The number of private cars being random, the residual capacity per zone is therefore perceived as random by free-floating cars. This key feature is specific to free-floating and cannot be captured by the classical modeling of a free-floating system as a station-based system. Note that, for free-floating, the capacity can not be obtained by data analysis without data on public parking spaces in the service area.

The paper deals with free-floating car-sharing. In this paper, our first objective is to propose a new model for car-sharing systems suited to free-floating which take into account that the free-floating cars share public space with much more numerous private cars. Within this framework, our second objective is to measure the difficulty for the user both to find a car at departure and to return his car at destination. The third one is to solve in this framework the dimensioning problem, which is the optimal number of free-floating cars the operator must put to optimize the availibilioty of resources. Then the last objective is to present simulations which validate the results. As far as we know, this is the first model of a free-floating car-sharing system with zones that takes into account the presence of private cars in the dynamics. This point of view is original and seems much more relevant for the system than the classical approaches used up to now to study free-floating car-sharing systems. This model takes into account the interactions between private and free-floating cars and as a result highlights that the residual capacity (the parking spaces not occupied by private cars) *ignore* the free-floating cars. Note that all the results presented in Section 3 are proved in a companion paper Fricker et al. (2023).

1.4. Outline of the paper

Section 1 deals with the introduction. Section 2 presents briefly the model and the probabilistic tools used in the stochastic analysis. In Section 3, the stochastic model is described and the main results of the analysis are presented, then the validation by simulation. Section 4 gives a conclusion and directions for future work.

2. Material and methods

2.1. Modelling

In this paper, we consider an homogeneous framework. Let us clarify the main features of our model suited to free-floating. The service area is divided into N zones. The public space in each zone contains a number of parking spaces of the order of N, called the overall capacity of the zone. Note that this capacity does not depend on pricing or demand. It depends only on the urban characteristics of the zone assumed to be similar for all zones. Moreover the set of N zones is completely symmetric in terms of free-floating car demand (a random user demand with the same rate for each zone), destination choice (uniform among the N zones), and random reservation and trip time. The heterogeneity of the trip times of the users is modelled by a common distribution. Indeed as in previous bike or car sharing models, see Fricker and Gast (2016); Bourdais et al. (2020) for example, the cars moving are *indistinguishable*. Car-Sharing cars coexist with a large number of private cars and share public parking space with them. Private cars arrive in each zone according to a Poisson process of parameter of the same order as N. This choice reflects the numerical imbalance between the private and free-floating cars. The interarrival times and parking times of private cars are assumed independent respectively identically distributed with exponential distribution. In conclusion, the parameters of the model do not depend on the specific zone.

This model aims to describe the randomness of the residual capacity available to the car-sharing system in the public space. The homogeneity is natural for a mean-field approach, to have the simplest framework to deal with for technical reasons. In the following, thanks to our methodology, the model could be modified with various behaviors to model different usages. We will keep in our model that the parked free-floating cars are either available or reserved. This shows how other more refined variants can also be modeled. Moreover, in real systems, there can be higher car-sharing demand where supply of parking is likely lower. Our model is robust enough to be generalized to a spatially heterogeneous setting, making all parameters dependent on the zones considered. Modeling via clusters remains suitable for a mean field approach, such as what was done for bike sharing systems by Fricker et al. (2012). As for the temporal heterogeneity (peak and off-peak hours), it is possible to set the model parameters corresponding to the chosen time window. In this way, we obtain an approximation of the model behavior specific to this time window. But a general framework in which the parameters are time-dependent could be envisaged as a final goal.

2.2. Mean field

For car-sharing, the mean-field approach applied to a set of N zones means that, when N becomes large, the states of two zones become independent and identically distributed. This classical result allows us to reduce the state of the whole system to that of a single zone, when the system is large enough. A nice queueing interpretation of the limiting state process of a zone (when N becomes large) enables us to obtain explicitly the stationary distributions of the number of available parking spaces, the number of available free-floating cars and the number of reserved cars. This simple expression of the steady-state behavior of the free-floating system is used for the dimensioning problem. This means characterizing the optimal number of free-floating cars per zone, that minimizes the number of zones without both available cars and parking spaces.

2.3. Averaging

This new stochastic model takes into account the interaction between private and car-sharing cars sharing public space. The number of car-sharing cars is negligible compared to the total number of private ones that could share the service area (850 for Communauto Vs 800 000 private cars in Montreal). Thus intuitively, the private car behavior ignores the presence of free-floating cars. Told in a different way, ignoring the sparse presence of free-floating cars, the private cars left a residual number of available parking spaces. Moreover, for each zone, since the number of private cars is large, their evolution is fast - with jump rates of order N- and free-floating cars, arriving at a much smaller rate - with jump rates of order 1-, see the number of available parking spaces in average. Averaging has been extensively investigated in stochastic networks since Kurtz (1992), a first famous framework given by loss networks in Hunt and Kurtz (1994). Because of this, loss networks are a key reference or conversely, due to the universality of dynamics of loss networks, averaging is present in a huge number of models. It is the case here. Mathematically, the study deals with the fast process of the number of private cars compared to the slow process of the number of car-sharing vehicles, see also Hunt and Kurtz (1994). It allows to find the behavior of the free-floating cars in a large system although the evolution equation has discontinuities, due to the presence of private cars which condition available parking spaces. To understand the averaging principle, we study the processes at different time scales.

2.4. Phase transition

As in loss networks, Hunt and Kurtz (1994), there are two regimes: One with a large number of available parking spaces (of the order of N) and the free-floating cars always be parked. It is called here *the Montreal regime*. The other is where private cars saturate the public space of the free-floating service area leaving few parking spaces, i.e. a random number with fixed finite mean when N becomes large. There is a positive probability that the free-floating cars can not be parked in the target zone and have to park in another zone. This regime is called *the Paris regime*. Note that in the latter case, the probability that the user has to accept to turn around his destination and park slightly away is an important parameter for the operator. This probability allows him to measure the discomfort suffered by the user in case of saturation.

3. Results

In this section, proofs are omitted and will be presented in a companion paper Fricker et al. (2023).

3.1. The model

The service area is divided into N zones, each of them with a global capacity of order N, say equal to cN where $c > 0$ is a fixed constant depending on the urban structure of the service area. Let M be the total number of free-floating cars in the system. The number M/N of free-floating cars per zone is denoted by s_N in the following. The dynamics of the model are as follows:

- Users of free-floating cars arrive at zone i ($1 \leq i \leq N$) according to a Poisson process of rate λ. The Poisson processes for arrivals in the different zones are independent. Each user reserves a free-floating car, if there is at least one available car. The reservation duration is random with an exponential distribution with parameter η. Otherwise, if no free-floating car is available in zone i when arriving, the user leaves the system, looking for another means of transportation.
- After reservation, they starts a trip whose duration time has an exponential distribution with parameter μ.
- At the end of the trip, the user chooses zone $j \in \{1, ..., N\}$ with probability $1/N$.

- If the user does not find a parking space in zone j, they starts a new trip with exponential distribution with the same parameter μ. Notice that the destination choice is again made uniformly over the N zones.
- Private cars arrive at zone i according to a Poisson process of parameter αN. This choice is consistent with the assumption that private cars outnumber the free-floating ones. The arrival rate of private cars in a zone is thus of order N, larger than that of free-floating cars, of order 1.
- Each private car stays in a zone for a random time with exponential distribution of parameter β then leaves the system.
- If there is no available parking space in the zone, the arriving private car leaves the system.

3.2. Markov process for the state of zones

For a given zone i in $\{1, ..., N\}$, we consider $\left(X_i^N(t), m_i^N(t), V_i^N(t), R_i^N(t)\right)$ a four component stochastic process such that, at time t,

- $X_i^N(t)$ is the number of private cars parked in this zone,
- $m_i^N(t)$ is the number of available spaces in this zone,
- $V_i^N(t)$ is the number of *available* free-floating cars parked in this zone,
- $R_i^N(t)$ is the number of *reserved* free-floating cars parked in this zone.

Note that a distinction is made between available parked free-floating cars and those already reserved in order to obtain a Markovian framework. Note also that, for sake of simplicity, all users make a reservation. Moreover it is obvious that, at any time $t \geq 0$ and for any zone i,

$$X_i^N(t) + m_i^N(t) + V_i^N(t) + R_i^N(t) = cN.$$

Thus the state of a given zone i is reduced to the three component stochastic process $\left(m_i^N(t), V_i^N(t), R_i^N(t)\right)$ since $X_i^N(t) = cN - m_i^N(t) - V_i^N(t) - R_i^N(t)$.

The model presented here is as simple as possible. Variants can be studied. For example, we can consider that, if private cars do not find a parking space in the target zone, they could look for a parking space in another zone. And also we can imagine that, if free-floating users do not find an available car in the target zone, a proportion of them could look for one in another zone. We could also add that only a fraction of users make a reservation.

The state of the N zones is given by $\left(m_i^N(t), V_i^N(t), R_i^N(t),\ 1 \leq i \leq N\right)$ which is a Markov process on state space

$$\left\{(m_i, v_i, r_i)_{1\leq i\leq N} \in \mathbb{N}^{3N}, \forall i \in \{1, \dots, N\},\ m_i + v_i + r_i \leq cN, \sum_{i=1}^{N} v_i + r_i \leq M\right\}.$$

The transitions from $((m_i, v_i, r_i),\ 1 \leq i \leq N)$ which change the state of zone i move (m_i, v_i, r_i) to

$$\begin{cases} (m_i + 1, v_i, r_i) & \text{at rate} \quad \beta(cN - m_i - v_i - r_i) \\ (m_i - 1, v_i + 1, r_i) & \text{at rate} \quad \dfrac{\mu}{N}\left(M - \sum_{l=1}^{N}(v_l + r_l)\right)\mathbf{1}_{m_i>0} \\ (m_i, v_i - 1, r_i + 1) & \text{at rate} \quad \lambda\mathbf{1}_{v_i>0} \\ (m_i + 1, v_i, r_i - 1) & \text{at rate} \quad \eta r_i \\ (m_i - 1, v_i, r_i) & \text{at rate} \quad \alpha N\mathbf{1}_{m_i>0} \end{cases}$$

and let the remaining $\Big((m_j, v_j, r_j),\ 1 \le j \le N, j \ne i\Big)$ unchanged. The first transition occurs when a private car leaves zone i making a new parking space available. It happens at rate βx_i, i.e. at rate $\beta(cN - m_i - v_i - r_i)$. The second corresponds to the arrival of a free-floating car being available for users and taking a parking place if possible. Note that there are $M - v_1 - r_1 - \ldots - v_N - r_N$ free-floating cars in move for a trip with exponential distribution with parameter μ. For the moving free-floating cars, the probability to park in zone i is $1/N$. Therefore we get the rate associated to this transition. The third one happens when a user reserves an available free-floating car in zone i if there is one. It occurs at rate λ. This car remains parked but reserved at the same parking space. The forth corresponds to the departure of a reserved free-floating car making a new parking space available in the zone. The last one corresponds to the arrival of a private car taking a parking space in zone i if possible.

3.3. Two regimes

Intuitively, due to the homogeneity of the system, all zones have the same limiting behavior. First we focus on the evolution of the number of available parking spaces $(m_i^N(t))_{t\ge 0}$ in a given zone i in $\{1, ..., N\}$. If the state of zone i is (m_i, v_i, r_i) at time t, the number of available parking spaces m_i increases by 1 at rate $\beta(cN - m_i - v_i - r_i) + \eta r_i$ and decreases by 1 if $m_i > 0$ at rate $\alpha N + \mu(M - v_1 - r_1 - \ldots - v_N - r_N)/N$. Roughly speaking v_i and r_i are negligible compared to cN since the sum $v_1 + r_1 + \ldots + v_N + r_N$ is less or equal to M of order N (recall that M/N tends to a constant s, the fleet size per station, called *the fleet size parameter*) . Thus, for N large enough, the process $(m_i(t))$ behaves like a birth and death process with birth rate βcN and death rate αN. It is well known that such a birth-and-death process is ergodic and admits a unique invariant measure which is geometric with parameter $\beta c/\alpha$ if $\beta c/\alpha < 1$, and is transient and tends to $+\infty$ if $\beta c/\alpha > 1$. It should be noted that the limiting number of available parking spaces does not depend on the car-sharing parameters. Intuitively, it is due to the orders of magnitude of random variables, recalling that $m_i^N(t) = cN - X_i^N(t) - V_i^N(t) - R_i^N(t)$.

Thus we highlight two regimes with a phase transition when $\beta c/\alpha$ is equal to 1 clearly identified in term of parameters related to the *environment*, i.e. the private cars and the urban characteristic of the service area (parameters α, β and c). Intuitively,

- *The saturated regime*: When $\beta c/\alpha < 1$, the private cars occupy *all* the parking spaces in the zone which means a number equivalent to cN when N is large. In other words if the arrival rate of the private cars, i.e. αN , exceeds the parking completion rate βcN, the zone will not have time to free a parking space before the arrival of a new private car. Indeed, as previously explained, the number of parking spaces is of order 1, with geometric distribution with parameter $\beta c/\alpha$. With probability $1 - \beta c/\alpha$, a free-floating car arriving in a zone can not be parked in this zone.
- *The unsaturated regime*: When $\beta c/\alpha > 1$, the private cars leave a proportion of order N of the parking spaces in the zone. As the number of free-floating cars are of order 1 in a zone, free-floating cars can be parked without any restriction. They always find an available parking space.

3.3.1. The saturated regime

It holds when $\beta c/\alpha < 1$. We present briefly the behavior.

The mean-field approximation. In this case, the free-floating cars face some positive probability $1 - \beta c/\alpha > 0$ of not finding an available parking space (and thus looking for a space in another zone). This measures the difficulty (and not impossibility) to park at destination for a user of the free-floating car-sharing system. This quantity depends only on the private car parameters (the *environment* for the free-floating system) and could be computed, provided by data on parking spaces in the service area and on the movement of private cars in this area.

Recall that the behavior of the large N-zone system (approximated by the mean-field limit) can be summarized by the behavior of a single zone, two zone behaviors becoming independent. The limiting free-floating state process of a zone is the limiting number of available free-floating cars and the limiting number of reserved cars $(V(t), R(t))$. It behaves like an inhomogeneous Markov process with transitions, from (v, r) at time t,

$$(v, r) \to \begin{cases} (v-1, r+1) & \text{at rate} \quad \lambda \mathbf{1}_{v>0} \\ (v, r-1) & \text{at rate} \quad \eta\, r \\ (v+1, r) & \text{at rate} \quad \dfrac{\beta c}{\alpha} \mu \left(s - \mathbb{E}\left(V(t) + R(t)\right)\right) \end{cases}$$

that is easy to understand intuitively. The couple $(V(t), R(t))$ can be interpreted (see Figure 3) as two queues in tandem with

- the first one, a one-server queue with arrival rate $(\beta c/\alpha)\mu\,(s - \mathbb{E}\,(V(t) + R(t)))$ at time t and service time λ,
- and the second one, an infinite-server queue with service rate η.

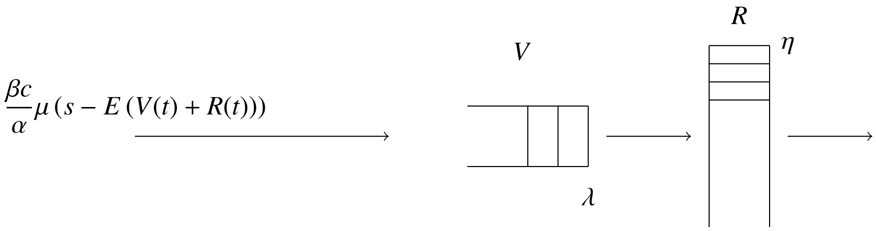

Fig. 3: Dynamics of $(V(t), R(t))$ as a tandem of two queues. The *horizontal* is a $M/M/1$ queue, the *vertical* queue is $M/M/\infty$ queue.

The limiting behavior is given by the following proposition.

Proposition 1 (Behavior of a zone in the saturated case). *When $\beta c/\alpha < 1$, when N gets large, after some time t_o, the number of available parking spaces at time t is stationary with geometric distribution with parameter $\beta c/\alpha$. The free-floating cars which return to a zone find no parking space with probability $\beta c/\alpha$. The joint numbers of available and reserved cars $(V(t), R(t))$ behave after t_0 behave like the inhomogeneous tandem of queues of Figure 3.*

This behavior has to be compared with that of the station-based bike-sharing model with fixed capacity by Fricker et al. (2012) and with car reservation by Bourdais et al. (2020). There are two main differences.

- The $\beta c/\alpha$ factor in the arrival rate of the first queue (the available free-floating cars) in Proposition 1 which is interpreted as an acceptance probability since the free-floating cars have a probability $1 - \beta c/\alpha$ of not finding a parking space in a given zone (rejection probability). It is as if private cars is thinning the return of free-floating cars with probability $\beta c/\alpha$. This replaces the role of the finite station capacity of the model usually proposed for station-based car-sharing systems.
- The absence of finite capacity for the tandem of queues in the limiting behavior. See (Bourdais et al., 2020, Figure 1).

Steady-state behavior. Using the product-form of the invariant measure of the underlying two-dimension Markov process (i.e. with queueing interpretation, a tandem of two queues), the long-time behavior (V, R) of the state of a zone is given by two independent random variables with respective distributions: a geometric [1] distribution with parameter

$$\rho = \frac{\beta c}{\alpha} \frac{\mu}{\lambda} \left(s - \mathbb{E}(V + R)\right) \tag{1}$$

[1] a probability measure π on $\mathbb{N}$ such that, $\pi(k) = (1-\rho)\rho^k$, $k \in \mathbb{N}$ with $\rho \in]0, 1[$.

and a Poisson distribution with parameter $\lambda\rho/\eta$ if such a ρ exists. It is important to mention that the residual capacity m available for the free-floating cars, previously considered as a zone-specific constant and arbitrarily fixed, turns out to be unbounded in our modeling, random with a geometric distribution depending only on the environment parameters (private cars and characteristics of the zone).

Equation (1) is a fixed point equation on ρ. Indeed, using that V and R have respectively a geometric distribution with parameter ρ and a Poisson distribution with parameter $\lambda\rho/\eta$, $\mathbb{E}(V) = \rho/(1-\rho)$ and $\mathbb{E}(R) = \lambda\rho/\eta$. Thus, equation (1) can be rewritten as follows

$$\rho = \frac{\beta c}{\alpha}\frac{\mu}{\lambda}\left(s - \frac{\rho}{1-\rho} - \frac{\lambda\rho}{\eta}\right)$$

or equivalently

$$s = \frac{\rho}{1-\rho} + \left(\frac{\lambda}{\eta} + \frac{\alpha}{\beta c}\frac{\lambda}{\mu}\right)\rho. \tag{2}$$

It is clear that the right-hand side of equation (2) is a function of ρ defined on $[0, +\infty[\backslash\{1\}$, strictly increasing on both intervals $[0, 1[$ and $]1, +\infty[$. More precisely, s is strictly increasing from $[0, 1[$ to $[0, +\infty[$, and from $]1, +\infty[$ to $]-\infty, +\infty[$. In conclusion, for a fixed s, there is a unique solution $\rho \in]0, 1[$. Moreover ρ is given explicitly. See Proposition 2. Indeed, equation (2) is a polynomial equation of second order in ρ, with two real solutions, and it can be straightforwardly checked that it has exactly one solution in $]0, 1[$.

This is summarized in the following result.

Proposition 2 (Stationary mean-field limit). *When $\beta c/\alpha < 1$, the limit as N becomes large of the joint numbers of available and reserved free-floating cars $(V(t), R(t))$ in a given zone has the following stationary distribution*

$$geom(\rho) \otimes Poisson\left(\frac{\lambda\rho}{\eta}\right) \tag{3}$$

where ρ is given by

$$\rho = \frac{A + s + 1 - \sqrt{(A + s + 1)^2 - 4sA}}{2A} \quad with \quad A = \lambda\left(\frac{1}{\mu}\frac{\alpha}{\beta c} + \frac{1}{\eta}\right). \tag{4}$$

It means that the limiting number of available and reserved free-floating cars as N is large at equilibrium are independent and with simple distributions with explicit parameters.

These two limits in N and t are not so easy to handle. For example, it is hard, even an explicit limit is obtained, to prove the long-time convergence of the mean-field limit to its *equilibrium point* (V, R). To understand the long-time behavior of the state process of a zone, we use changes of time-scales. It is the aim of the following section.

The different time-scales. What follows is introduced in order to obtain the long-time behavior of the system when it gets large, by results on convergence of the processes. Roughly speaking, to investigate the stationary state as N gets large, it is convenient to accelerate time in Nt. Moreover to understand the behavior, the difference between the order of the rate of transitions concerning the numbers of private cars $X^N(t)$ and free-floating $V^N(t)$ lead us to investigate different timescales (standard in t, accelerated in Nt and slowed down in t/N). The proofs are left to the companion paper Fricker et al. (2023).

The accelerated time gives the simplest point of view. It allows to obtain the steady-state for large N.

Proposition 3 (Accelerated time-scale behavior). *If $\beta c/\alpha < 1$ and $\mu < \lambda$, for the convergence in distribution,*

1. *If $X^N(0)/N \xrightarrow[N\to\infty]{d} x_0$ then $(X^N(Nt)/N) \to_{N\to\infty} (c)$.*
2. *For each $t > 0$, as $N \to \infty$, $m^N(t) \to Y$ where Y is random variable on $\mathbb{N}$ with geometric distribution of parameter $\beta c/\alpha$.*
3. *For each $t > 0$ and each i, the random variables $(V_i^N(Nt), R_i^N(Nt))$ converges to (V, R) as N gets large with distribution geometric $\otimes$ Poisson given by Proposition 2.*

Considering the normal time scale, we capture that saturation by private cars holds after some time t_0. Indeed, in the standard time-scale ($t \mapsto t$), the two processes $(X^N(t))$ and $(m^N(t))$ are both moving fast (at order N) such that $(m^N(t))$, when N tends to $+\infty$, roughly becomes of order 1 after some finite time t_0. They reach instantaneously (at fixed $t > t_0$) an equilibrium for a fixed value (v, r) of the slow process $(V^N(t), R^N(t))$ at t, moving at order 1. Moreover, when $\beta c/\alpha < 1$ and $t > t_0$, the equilibrium distribution of the number of available places is geometric with parameter $\beta c/\alpha$. The following result holds for the saturated case.

Proposition 4 (Standard time-scale behavior). *If $\beta c/\alpha < 1$ and $\mu < \lambda$, for the convergence in distribution,*

1. *If $X^N(0)/N \xrightarrow[N\to\infty]{d} x_0$ then process $(X^N(t)/N)$ converges to a deterministic process given by*

$$\left(\frac{X^N(t)}{N}\right) \to_{N\to\infty} (x(t)) = \left(\frac{\alpha}{\beta} + \left(x_0 - \frac{\alpha}{\beta}\right) e^{-\beta t}\right) \wedge c.$$

2. *There exists $t_0 > 0$ such that, for $t > t_0$ fixed, as $N \to \infty$, $m^N(t) \to Y$ where Y is a random variable on $\mathbb{N}$ with geometric distribution of parameter $\beta c/\alpha$.*
3. *$(V_1^N(t), R_1^N(t))$ converges after t_0 to the inhomogeneous process described by the tandem of queues.*

Considering a slow time-scale ($t \mapsto t/N$), we only capture the dynamics of the fast process (private cars), the slow process (free-floating cars) being static in time.

Theorem 1 (Slow time-scale behavior). *Assume $\beta c/\alpha < 1$. Given $\lim_{N\to\infty}(m^N(0), V^N(0), R^N(0)) = (m_0, v_0, r_0)$, the sequence of processes $(m^N(t/N), V^N(t/N))$ converges in distribution to the process $(L_m(t), v_0, r_0)$, where $L_m(t)$ is the number of customers in a $M/M/1$ queue with arrival rate βc and service rate α.*

3.3.2. The unsaturated regime

When $\beta c/\alpha > 1$, the number of parking spaces is of order N thus the probability for a free-floating car to find an available parking space is equal to 1. Recall that it means that the car is parked in the target zone. The free-floating car process *of returns* is not thinned by the mass of private cars. Thus, for the free-floating car in a zone, the results of the saturated case are valid replacing $\beta c/\alpha$ by one. For example, Proposition 2 holds with only the expression of A changed. It is written as follows.

Proposition 5 (Stationary mean-field limit). *When $\beta c/\alpha > 1$, the limit as N becomes large of the joint numbers of available and reserved free-floating cars $(V(t), R(t))$ in a given zone has the following stationary distribution*

$$geom(\rho) \otimes Poisson\left(\frac{\lambda\rho}{\eta}\right) \tag{5}$$

where ρ is given by

$$\rho = \frac{A + s + 1 - \sqrt{(A + s + 1)^2 - 4sA}}{2A} \quad \textit{with} \quad A = \lambda\left(\frac{1}{\mu} + \frac{1}{\eta}\right). \tag{6}$$

3.4. Application to dimensioning

As in the case of station-based car-sharing systems, users of free-floating car-sharing systems are faced with the problem of finding available cars and parking spaces throughout the service area. To address this major issue, the first question for the operator is the dimensioning problem. The aim is to find the optimal total number of free-floating cars sN to put in circulation which corresponds to a minimum proportion of zones without free-floating cars or parking spaces.

Let us focus on the proportion of zones without parking spaces. In the unsaturated regime, this proportion is 0. In the saturated regime, in contrast with the case of a station-based car-sharing system, the limiting proportion of zones without available parking spaces, which is $1 - \beta c/\alpha$, depends only on the parameters of the environment (private cars and characteristics of the zone). The operator cannot therefore act on this proportion by varying the dimensioning parameter s.

But the operator can reduce the number of zones where there is a lack of cars. Let us denote by P_0 the limiting proportion of zones without free-floating cars, given by $P_0 = \mathbb{P}(V = 0)$. In both regimes, using respectively Proposition 2 or 5 in the saturated or unsaturated regime,

$$P_0 = 1 - \rho$$

where ρ is given by equation (4), respectively equation (6). In both cases, this limiting proportion P_0 is a decreasing function of s since equation (2) shows that s is a strictly increasing function of ρ. Therefore, the more the operator increases the number of free-floating cars per zone, the better the system performs since the probability that a user do not find an available free-floating car in a given zone becomes smaller and smaller. This is nevertheless limited by the assumption that the number of free-floating cars per zone is negligible (of order 1) compared to the number of cars per zone (of order N). Our theoretical study succeeds in evaluating the probability of having no free-floating cars in a zone. In conclusion, our model measures the fact that, for free-floating car-sharing systems, a lack of cars can occur.

3.5. Validation by simulation

We implemented a simulator for our stochastic model. It reproduces the dynamics of a free-floating system of M cars interacting with private cars and N stations each of capacity cN. For the movement of cars, it follows the distributions of our model. In particular, it relies on the generation of random numbers from the exponential distribution to simulate the inter-arrival times of Poisson processes. Two fundamental properties of the exponential distribution are the lack of memory and the fact that, the distribution of $Y = \min\{X_1, ..., X_n\}$ where $X_1, ..., X_n$ are independent random variables exponentially distributed with parameters $\lambda_1, ..., \lambda_n$, is again exponential with parameter $\lambda = \lambda_1 + ... + \lambda_n$. We report page 13 the algorithm used for the simulations.

Plotting in Figure 4 the evolution in time of the number of available parking spaces, available and reserved free-floating cars in a randomly chosen station, we can state what happens when the number of zones grows, i.e. when N doubles from 25 to 50. Indeed, for $N = 25$ we can still detect jumps of the processes related to free-floating cars. But for $N = 50$, the number of jumps for the same processes is much smaller, approximately 1/4 than before. This is due to the fact that as the number of zones doubles, the jumps of the free-floating cars are spread over a larger number of zones.

In Figure 5, we plot the distribution of the number of available parking spaces and available free-floating cars which is a geometric distribution. Indeed plotting the distribution in logarithmic scale gives a straight line. To check

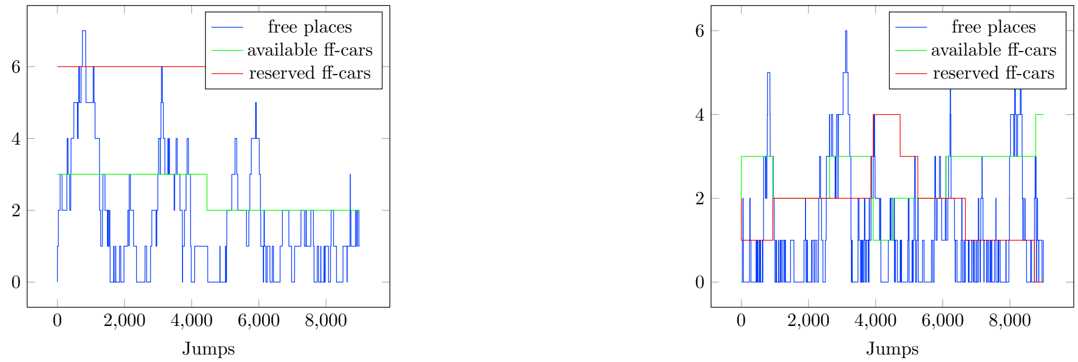

Fig. 4: The evolution in time of the number of available parking spaces, available and reserved FF-cars in a randomly chosen station. On the left, for $N = 50$ and on the right for $N = 25$.

that the number of reserved free-floating cars follows a Poisson distribution, we plot on the same axes the probability mass function of the Poisson with theoretical parameters and the distribution obtained by simulation. The two curves are close.

4. Conclusion and future work

Until now, free-floating car-sharing systems have been modelled as station-based systems with zones considered as stations with fixed capacities. But such a model is not relevant since free-floating cars share the same parking spaces with private cars inducing fluctuations of random *capacities* left to the car-sharing system.

The intuition is that private cars are numerous compared to free-floating cars, ignoring their presence and behaving roughly independently in each zone as an $M/M/cN/cN$ queue. This is the simplest loss system studied. See (Robert, 2013, Chapter 6). Then, because of this difference in order of magnitude, a stochastic averaging principle should naturally govern the free-floating car behavior.

We propose in this paper a new model for car-sharing specific to free-floating that we are able to analyze. The study of this model leads to the mobilisation of several probabilistic techniques, where mean-field and stochastic averaging are combined. The study of stochastic averaging principle in large systems is original, as far as we know. All technical details and theoretical arguments of the proofs are presented in Fricker et al. (2023) in preparation.

A phase transition occurs. Under a critical value for a quantity related to the *environment*, i.e. the private cars and the public space size, the private cars saturate the public space. And because they are more numerous, they prevent the free-floating cars to find a parking space with some probability. This regime is called *saturated regime*. Over this critical value, in the so-called *non-saturated regime*, the free-floating cars find an available parking space with probability one. In both cases, the steady-state behavior when N is large of the non-moving free-floating cars, available or reserved, can be derived in a quite simple explicit form (cf Propositions 2 and 5). The technique for obtaining the long-time behavior is to accelerate the time t to Nt and let N tend to infinity. See the paper Fricker et al. (2023) for details.

As a byproduct, the paper proves the intuitive answer to the dimensioning problem for free-floating. The lack of parking spaces is not governed by free-floating car-sharing but just the *environment*. The operator acts only on providing cars to minimize the lack of available free-floating cars. Thus the more cars the more satisfaction for the user, under the assumption that private cars are much more numerous than free-floating cars.

To investigate further more realistic distributions, especially for trip times of free-floating cars and parking times of private cars, a simulator is currently written. These distributions could be provided by data analysis of a city case-study. The algorithm and simulations presented in in Section 3.5 validate the results for exponential distributions. They should be adapted to general distributions.

Algorithm 1 Simulate the dynamics of the model

$t \leftarrow 0$
$L \leftarrow M - \sum_l (R_l + V_l)$
$t_\eta \leftarrow \exp(\eta R_i, N)$
$t_\alpha \leftarrow \exp(\alpha N, N)$
$t_\beta \leftarrow \exp(\beta X_i, N)$
$t_\lambda \leftarrow \exp(\lambda, N)$
$t_\mu \leftarrow \exp(\mu, L)$
while number of jumps $\leq N_j$ **do**
  $t \leftarrow \min(t_\eta, t_\lambda, t_\mu, t_\alpha, t_\beta)$
  if t is in t_η **then**
    $i \leftarrow$ station where $t_\eta = t$
    $(m_i, V_i, R_i) \leftarrow (m_i + 1, R_i - 1, V_i)$
    $\omega_\mu \leftarrow t + \exp(\mu)$
    append ω_μ to t_μ
    $t_{\eta,i} \leftarrow t + \exp(\eta R_i)$ (set $t_{\eta,i} \leftarrow \infty$ if $R_i = 0$)
  end if
  if t is in t_μ **then**
    $i \leftarrow \mathcal{U}(\{1, ..., N\})$
    if $m_i > 0$ **then**
      $(m_i, V_i, R_i) \leftarrow (m_i - 1, R_i, V_i + 1)$
      remove t from t_η
    else $\{m_i = 0\}$
      $\omega_\mu \leftarrow t + \exp(\mu)$
      substitute t with ω_μ in t_μ
    end if
  end if
  if t is in t_λ **then**
    $i \leftarrow$ station where $t_\lambda = t$
    $t_{\lambda,i} \leftarrow t + \exp(\lambda)$
    if $V_i > 0$ **then**
      $(m_i, V_i, R_i) \leftarrow (m_i, R_i + 1, V_i - 1)$
      $t_{\eta,i} \leftarrow t + \exp(\eta R_i)$
    end if
  end if
  if t is in t_α **then**
    $i \leftarrow$ station where $t_\alpha = t$
    $t_{\alpha,i} \leftarrow t + \exp(\alpha N)$
    if $m_i > 0$ **then**
      $(m_i, V_i, R_i) \leftarrow (m_i - 1, R_i, V_i)$
      $X_i \leftarrow cN - m_i - V_i - R_i$
      $t_{\beta,i} \leftarrow t + \exp(\beta X_i)$
    end if
  end if
  if t is in t_β **then**
    $i \leftarrow$ station where $t_\beta = t$
    $(m_i, V_i, R_i) \leftarrow (m_i + 1, R_i, V_i)$
    $X_i \leftarrow cN - m_i - V_i - R_i$
    $t_{\beta,i} \leftarrow t + \exp(\beta X_i)$ (set $t_{\beta,i} \leftarrow \infty$ if $X_i = 0$)
  end if
end while

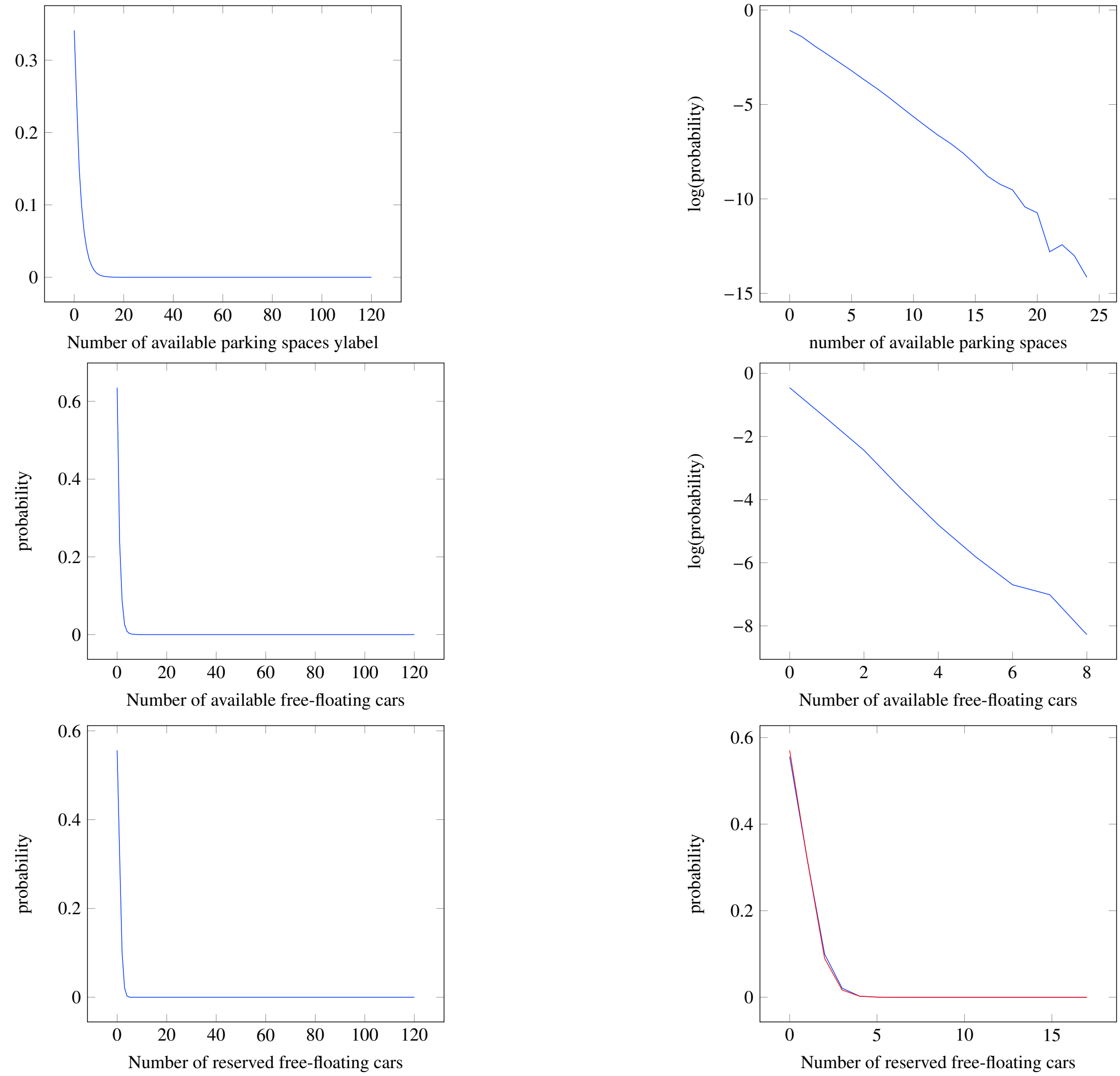

Fig. 5: Estimation by simulation of the distribution of the number of available parking spaces, available and reserved free-floating cars in a given zone, obtained by simulation for $N = 100$ with parameters $\lambda = 1.5$, $\alpha = 1.8$, $\beta = \mu = 1 = \eta = 1$, $c = 1.2$, $s = 2$ (saturated regime). The respective distributions are quite close to two geometric and a Poisson distributions.

Extensions could be made for more complicated dynamics of the private cars where they look for a parking space in another zone when there is no parking space in the target zone. This induces another type of interactions between zones. It is a work in progress. This work also open many avenues for future research.

Finally, it should also be pointed out that, although free-floating car-sharing networks are booming worldwide and address crucial sustainable mobility issues in the urbanized world of the 21st century, the techniques as mean-field approach and stochastic averaging principle employed here could also be used for other applications in the field of interacting systems.

Acknowledgements

The authors would like to thank Communauto for funding, providing data and allowing to do this study. They also thank the Natural Science and Engineering Research Council of Canada (NSERC) for funding.

References

Ancel, J., Fricker, C., Mohamed, H., 2022. Mean field analysis for bike and e-bike sharing systems. ACM SIGMETRICS Performance Evaluation Review 49, 12–14.

Bourdais, C., Fricker, C., Mohamed, H., 2020. A mean field analysis of a stochastic model for reservation in car-sharing systems. ACM SIGMETRICS Performance Evaluation Review 48, 18–20.

Bražėnas, M., Valakevičius, E., 2023. Approximation of non-markovian car sharing systems models by markovian one, in: Proceedings of SAI Intelligent Systems Conference, Springer. pp. 458–474.

Fricker, C., Gast, N., 2016. Incentives and redistribution in homogeneous bike-sharing systems with stations of finite capacity. Euro journal on transportation and logistics 5, 261–291.

Fricker, C., Gast, N., Mohamed, H., 2012. Mean field analysis for inhomogeneous bike sharing systems. Discrete Mathematics & Theoretical Computer Science DMTCS Proceedings vol. AQ, 23rd International Conference on Probabilistic, Combinatorial, and Asymptotic Methods for the Analysis of Algorithms (AofA2012).

Fricker, C., Mohamed, H., Popescu, T., Trépanier, M., 2021. Stochastic modelling of free-floating car-sharing systems .

Fricker, C., Mohamed, H., Rigonat, A., 2023. Stochastic averaging in mean-field analysis for free-floating car-sharing systems. In preparation.

Gast, N., Massonnet, G., Reijsbergen, D., Tribastone, M., 2015. Probabilistic forecasts of bike-sharing systems for journey planning, in: Proceedings of the 24th ACM international on conference on information and knowledge management, pp. 703–712.

Hunt, P., Kurtz, T., 1994. Large loss networks. Stochastic Processes and their Applications 53, 363–378.

Kolleck, A., 2021. Does car-sharing reduce car ownership? empirical evidence from germany. Sustainability 13, 7384.

Kurtz, T.G., 1992. Averaging for martingale problems and stochastic approximation, in: Applied Stochastic Analysis. Springer, pp. 186–209.

Le Vine, S., Lee-Gosselin, M., Sivakumar, A., Polak, J., 2014. A new approach to predict the market and impacts of round-trip and point-to-point carsharing systems: case study of london. Transportation Research Part D: Transport and Environment 32, 218–229.

Lippoldt, K., Niels, T., Bogenberger, K., 2018. Effectiveness of different incentive models in free-floating carsharing systems: A case study in milan, in: 2018 21st International Conference on Intelligent Transportation Systems (ITSC), IEEE. pp. 1179–1185.

Massey, W.A., Ekwedike, E., Hampshire, R.C., Pender, J.J., 2022. A transient symmetry analysis for the m/m/1/k queue. Queueing Systems , 1–43.

Moreno, B.M., Fricker, C., Mohamed, H., Philippe, A., Trépanier, M., 2022. Mean field analysis of an incentive algorithm for a closed stochastic network, in: 33rd International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA2022), Schloss Dagstuhl-Leibniz-Zentrum für Informatik.

Robert, P., 2013. Stochastic networks and queues. volume 52. Springer Science & Business Media.

Shaheen, S.A., Mallery, M.A., Kingsley, K.J., 2012. Personal vehicle sharing services in north america. Research in Transportation Business & Management 3, 71–81. doi:https://doi.org/10.1016/j.rtbm.2012.04.005. flexible Transport Services.

Weikl, S., Bogenberger, K., 2015. A practice-ready relocation model for free-floating carsharing systems with electric vehicles–mesoscopic approach and field trial results. Transportation Research Part C: Emerging Technologies 57, 206–223.

Wielinski, G., Trépanier, M., Morency, C., 2015. What about free-floating carsharing? a look at the montreal, canada, case. Transportation Research Record 2563, 28–36.

Zhang, Y., Lin, D., Mi, Z., 2019. Electric fence planning for dockless bike-sharing services. Journal of cleaner production 206, 383–393.