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Simple Validation of Maximum Closure of a Graph

par

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et
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Novembre 1975.

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ABSTRACT

In [2], a quadratic 0-1 programming formulation of the problem of finding a Maximum Closure in a Graph, is used for defining a minimum cut problem in a related network. Paraphrasing the note of Balinski [1] about a paper by Rhys [3], the present note gives a direct solution of the problem, by identifying closures in the original graph and finite cuts in the related network. It is also shown how the given problem may be reduced to the bipartite case of Rhys.

The definitions and notations used in the following are from [2]. Consider a finite directed graph $G = (V,A)$, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices, A the set of arcs (v_i, v_j), and values m_i are assigned to the vertices $v_i \in V$ (the values are any real number). A closure Y is a set of vertices such that, if $v_i \in Y$ and $(v_i, v_j) \in A$, then $v_j \in Y$. The value $m(Y)$ of a closure Y is the sum of the values of the vertices in Y. The problem considered in [2] is the one of finding a closure in G, with maximum value.

The related network $N = (V', A', C)$ is defined in [2] as follows: the node set is $V' = \{v_o\} \cup V \cup \{v_{n+1}\}$ where v_o (resp v_{n+1}) is an artificial source (resp sink); the arcs in A' are:
arcs (v_o, v_i) with capacity $c_{oi} = m_i$, for all v_i with $m_i > 0$,
arcs (v_j, v_{n+1}) with capacity $c_{j,\, n+1} = -m_j$, for all v_j with $m_j < 0$,
arcs (v_k, v_h) with infinite capacity, for all arcs $(v_k, v_h) \in A$.

A cut $(S; \overline{S})$ in N separating v_o and v_{n+1} is a node partition where $v_o \in S$, $v_{n+1} \in \overline{S}$, $S \cup \overline{S} = V'$ and $S \cap \overline{S} = \emptyset$. The capacity of a cut is the sum of the capacities of the edges directed from a node in S to a node in $\overline{S}$. A finite cut is a cut with finite capacity.

<u>Lemma 1:</u> There is a one-to-one correspondence between closures in G and finite cuts in N.

PROOF: This result is obtained in a straightforward manner by letting $Y = S - \{v_o\}$ (or equivalently $S = Y \cup \{v_o\}$), and by noticing that a cut is finite if and only if there is no edge (v_i, v_j) from A connecting a node $v_i \in Y$ to a node $v_j \in X - Y$.

Lemma 2: A minimum cut corresponds to a maximum closure.

PROOF: The value $m(Y)$ of the closure Y associated with a given cut is:

$$m(Y) = \sum_{i \in I^+ \cap S} m_i + \sum_{j \in I^- \cap S} m_j$$

where $I^+ = \{i / m_i > 0\}$ and $I^- = \{i / m_i < 0\}$

observing that $\sum_{i \in I^+ \cap S} m_i = \sum_{i \in I^+} m_i - \sum_{k \in I^+ \cap \bar{S}} m_k$

and that $C(S; \bar{S}) = \sum_{k \in I^+ \cap \bar{S}} m_k - \sum_{j \in I^- \cap S} m_j$

gives $m(Y) = \sum_{i \in I^+} m_i - C(S; \bar{S})$.

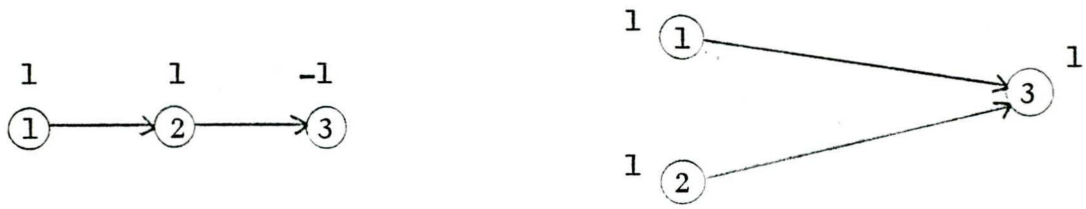
Note that the above arguments do not lead to the construction of the related network, while the quadratic 0-1 programming formulation given in [2] does.

On the other hand, the problem of finding a maximum closure of a graph may be reduced to the bipartite selection problem of Rhys [3]. Consider the vertices with positive value as "activities", the vertices with non-positive value as "facilities" and connect an activity v_i to a facility v_j if and only if there is a path in G, from v_i to v_j.

Lemma 3: An optimum selection defines a maximum closure in G.

PROOF: A closure in G obviously defines a selection (with the same value) but the converse is not necessarily true. In Figure 1, the selection $\{1, 3\}$ defines no closure in G.

FIGURE 1

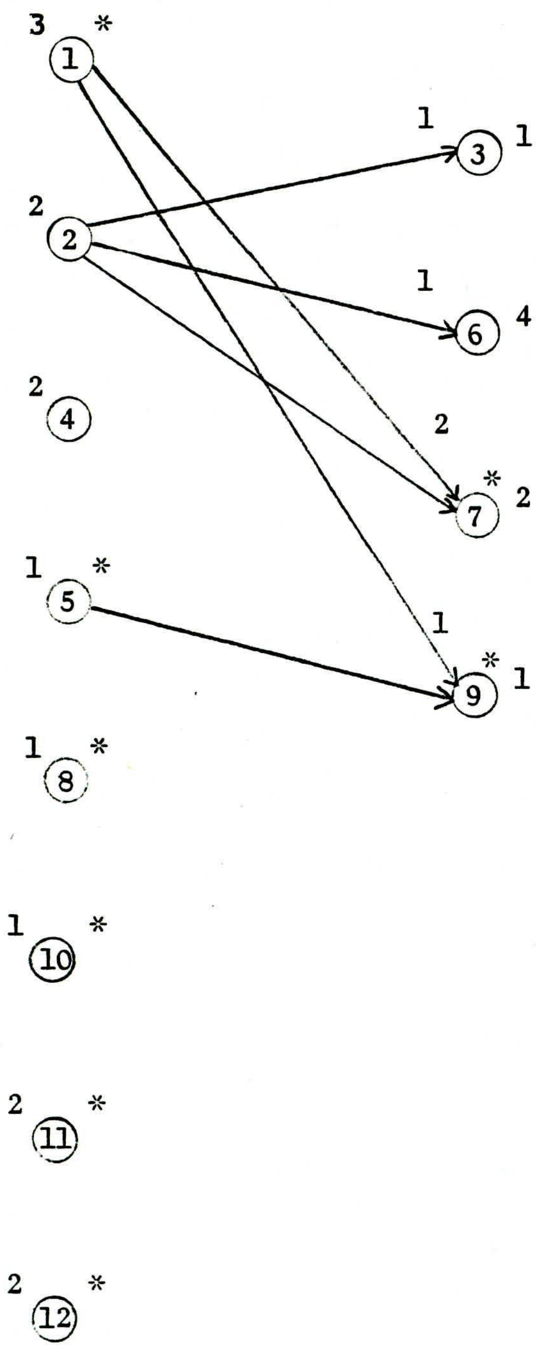


Let S be an optimum selection. If S does not define a closure, let Y be the smallest closure in G containing S (recall that the intersection of closures is a closure). Any nonpositive-valued vertex of Y is in S (otherwise, since there is no path connecting a positive-valued vertex of S to this vertex, Y would not be the smallest closure). If a positive-valued vertex were in Y - S, then S would not be optimum. So Y = S and the lemma holds.

This reduction needs the (partial) determination of the transitive closure of G, but this may be balanced by the simplicity of the bipartite network flow problem. Notice that any cycle can be "shrinked" to a single vertex.

Example: In Figure 2 is pictured the selection problem related to the example from [2]. (See next page)

FIGURE 2



A maximum flow is represented by numbers attached to arcs and the associated minimum cut (maximum closure) by starred vertices.

REFERENCES

[1] Balinski, M.L., "On a Selection Problem", Management Science, Vol. 17, No. 3. (November 1970).

[2] Picard, J.C., "Maximum Closure of a Graph, and Applications to combinatorial Problems", (to appear in Management Science).

[3] Rhys, J.M.W., "Shared fixed Cost and Network flows", Management Science, Vol. 17, No. 3. (November 1970).