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EXACT ALGORITHMS FOR THE QUADRATIC ASSIGNMENT PROBLEM

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EXACT ALGORITHMS FOR THE QUADRATIC ASSIGNMENT PROBLEM

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ABSTRACT:

The Quadratic Assignment Problem is a difficult combinatorial problem, special cases of which are the well-solved Linear Assignment Problem and the Time-Dependent Traveling Salesman Problem, which was recently studied by the authors. These two special cases allow the computation of bounds to be used in branch and bound algorithms for the general Quadratic Assignment Problem. Two distinct cost formulations are given for the general problem. They lead to different versions of the above approach and are illustrated with an example.

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INTRODUCTION

The Quadratic Assignment Problem is one of the most general and difficult problem in Combinatorial Optimization. It differs from the well-known Linear Assignment Problem in the addition of "quadratic" costs C_{ijpq}, which are incurred by the assignments in which activity A_i is assigned to location L_p simultaneously with activity A_j being assigned to location L_q. An important practical application of this problem was first described by Koopmans and Beckmann [1957] , and independently as the "Component Placement" or the "Backboard Wiring" Problem (see, for instance, [Steinberg 1961]). In this formulation the quadratic costs C_{ijpq} are products of a "traffic intensity" (or "interaction") factor Q_{ij} and a "distance factor D_{pq} (see [Los 1976] for a recent survey and a comparison of existing algorithms). When the distances are taken on a single dimension, this defines a generalized version of the "Optimal Linear Ordering" Problem (see [Adolphson and Hu, 1973]).

The general Quadratic Assignment Problem is a possible formulation of sequencing and scheduling problems in which the costs are "sequence-dependent". Some particular cases are the classical Traveling Salesman Problem, and one generalization called the Time Dependent Traveling Salesman Problem defined by K. Fox [1973] and developed by the authors [1976 b].

In the first part of this paper, two distinct cost formulations are given for the Quadratic Assignment Problem, and they lead to distinct versions of the two approaches to be developed. The first such approach consists in approximating the Quadratic Assignment Problem by a related Linear Assignment Problem. It was described by Lawler [1963] for the first cost formulation. In the second part of the paper, this is recalled and the use of the alternative cost formulation is also proposed. The third part deals with the "Time Dependent Traveling Salesman" approach to the Quadratic Assignment Problem, for both cost formulations. An example is used to illustrate the four resulting methods. In the last part, some specific topics for a branch and bound algorithm using this last approach are discussed, in prospect of a possible computer implementation.

1. Definition and cost formulations

The Quadratic Assignment Problem (QAP) is the problem of assigning activities to locations, one activity per location, in order to minimize a total cost function which is a sum of "linear" costs and "quadratic" costs. The problem data are the number n of activities (and locations), the linear costs α_{ip} (i,p= 1,...,n), incurred when activity A_i is assigned to location L_p and the quadratic costs C_{ijpq} (i,j,p,q = 1,...,n) incurred when activity A_i is assigned to location L_p, and A_j is assigned to L_q. A mathematical programming formulation involves the definition of n^2 binary variables x_{ip}, assuming the value one if A_i is assigned to L_p, and zero otherwise . Thus QAP is stated as:

$$(1) \quad \text{minimize } z(x) = \sum_i \sum_p a_{ip} x_{ip} + \sum_i \sum_j \sum_p \sum_q C_{ijpq} x_{ip} x_{jq}$$

$$(2) \quad \text{subject to} \quad \sum_i x_{ip} = 1 \quad (\text{all } p=1,\dots,n)$$

$$(3) \quad \sum_p x_{ip} = 1 \quad (\text{all } i=1,\dots,n)$$

$$(4) \quad x_{ip} = 0 \text{ or } 1 (\text{all } i,p=1,\dots,n)$$

An immediate remark is that the entries C_{iipq} ($p \neq q$) and C_{ijqq} ($i \neq j$), need not be considered since the problem constraints imply that the corresponding quadratic factor is always zero. Similarly the entries C_{iipp} can be introduced into the linear terms a_{ip}

Example: Consider the QAP with $n = 4$ and costs given by Table I. The linear costs a_{ip} are given as the diagonal entries C_{iipp}. The cost of the assignment defined by

$$x_{31} = x_{12} = x_{43} = x_{24} = 1$$

is the sum of the circled values, so $z(n) = 32$

The entry C_{jiqp} will be called the "symmetric" of C_{ijpq}. Since a quadratic cost C_{ijpq} is incurred if and only if its symmetric is incurred, an equivalent formulation of the cost function (1) is

$$(5) \quad z(x) = \sum_i \sum_p a_{ip} x_{ip} + \sum_i \sum_j \sum_{p=1}^{n} \sum_{q=p+1}^{n} d_{ijpq} x_{ip} x_{jq}$$

$$(6) \quad \text{where } d_{ijpq} = C_{ijpq} + C_{jiqp} \quad (\text{all } i,j,p,q \text{ with } p<q)$$

(i p) \ (j q)	11	21	31	41	12	22	32	42	13	23	33	43	14	24	34	44
11	6					2	0	2		4	0	3		0	3	3
21		1			4		2	0	1		5	0	2		1	4
31			3		3	0		3	2	2		3	1	1		3
41				5	2	0	2		3	3	1		1	1	3	
12		1	1	0	4					2	2	0		4	1	3
22	1		2	3		2			3		3	1	1		3	1
32	1	1		2			1		1	1		0	1	1		4
42	2	2	2					6	1	0	2		4	4	2	
13		3	0	2		1	0	0	3					3	1	1
23	1		1	3	2		1	2		2			0		0	2
33	2	1		0	1	2		1			1		1	2		3
43	3	1	3		2	0	3					0	1	1	0	
14		0	1	0		1	1	1		1	1	1	2			
24	2		2	1	2		0	0	1		2	2		1		
34	0	2		0	4	3		1	0	3		1			2	
44	1	1	1		1	1	1		1	0	2					4

Table I Values of C_{ijpq} for Example 1

So a QAP can be defined with about half the data of the first formulation. The values of d_{ijpq} for Example 1 are given in Table II

(i,p)	12	22	32	42	13	23	33	43	14	24	34	44
11		3	1	4		5	2	6		2	3	4
21	5		3	2	4		6	1	2		3	5
31	4	2		5	2	3		6	2	3		4
41	2	3	4		5	6	1		1	2	3	
12						4	3	2		6	5	4
22					4		5	1	2		6	2
32					1	2		3	2	1		5
42					1	2	3		5	4	3	
13										4	1	2
23									1		3	2
33									2	4		5
43									2	3	1	

<u>Table II</u>: Values of d_{ijpq} for Example 1.

The quadratic terms in the cost of the assignment defined by

(5) are circled and

$z(n) = 8 + 24 = 32$

The choice between the cost formulations (1) or (6) will lead to distinct bounding schemes within each of the enumeration algorithms that will be described.

2. Lower bounds by Linear Assignment

The classical Linear Assignment Problem LAP is a well-solved case of the Quadratic Assignment Problem in which all the quadratic costs are zero. It is tempting to define a LAP for providing bounds for the QAP; these will be used in developing an implicit enumeration algorithm.

The "linear" entries b_{ip} of the LAP relaxation of QAP are defined in order that, for each feasible assignment defined by a vector x satisfying (2),(3),(4) the corresponding LAP cost

$$y(x) = \sum_i \sum_p b_{ip} x_{ip}$$

must be a lower bound for the "real" cost z(x).

Using the cost formulation (1), Lawler [1963] derived a value of b_{ip} obtained by solving a (n-1) X (n-1) Linear Assignment Problem: define

$$b_{ip} = a_{ip} + V_{ip}$$

where V_{ip} is the optimum value of the objective function of

$$(LAP_{ip}) \quad \left\{ \begin{array}{lll} \text{minimize} & \sum_{j \neq i} \sum_{q \neq p} C_{ijpq} \, u_{jq} & \\ \text{subject to} & \sum_{j \neq i} u_{jq} = 1 & \text{all } q \neq p \\ & \sum_{q \neq p} u_{jq} = 1 & \text{all } j \neq i \\ & u_{jq} = 0 \text{ or } 1 & \text{all } j \neq i, \text{ all } q \neq p \end{array} \right.$$

Illustration: in Example 1, the computation of b_{12} requires the solution of the 3x3 LAP defined by the cost matrix in Table III. An optimal assignment is defined by the circled entries and

$b_{12} = 4 + 2 = 6$

j \ q	1	3	4
2	1	2	4
3	1	2	1
4	0	0	3

Table III: Values of C_{1j2q}

The computation of all the values b_{ip} leads to the 4x4 matrix given in Table IV. The cost of the assignment defined by (5) is

$b_{31} + b_{12} + b_{43} + b_{24} = 19$

i \ p	1	2	3	4
1	8	6	5	5
2	3	5	5	3
3	7	3	4	5
4	7	10	3	6

Table IV: Values of b_{ip}

For deriving a lower bound from the "triangular" cost formulation (6), it is necessary to define the Incomplete Linear Assignment Problem:(ILAP) this is a LAP in which the number m of activities is greater than the number n of locations, and consequently n-m activities will not be assigned.

(ILAP)

(8) minimize $\sum_{i=1}^{m} \sum_{p=1}^{n} a_{ip}\, x_{ip}$

(9) subject to $\sum_{i=1}^{m} x_{ip} = 1$ all $p=1,\ldots,n$

(10) $\sum_{p=1}^{n} x_{ip} \leqslant 1$ all $i=1,\ldots,n$

(11) $x_{ip} = 0$ or 1 all i,p

Illustration: in the above example, the computation of b'_{12} requires the solution of the 3x2 ILAP defined in Table V, an optimal solution of which is defined by the circled entries: so $b'_{12} = 4 + 7 = 11$

j \ q	3	4
2	4	6
3	3	5
4	2	4

Table V: Values of d_{1j2q}

The computation of all the b'_{ip} gives the matrix pictured in Table VI.

The cost of the assignement defined by (5) is $b'_{31} + b'_{12} + b'_{43} + b'_{24} = 24$, which is a better lower bound than the value 19 given by the other approach.

i \ p	1	2	3	4
1	14	11	4	2
2	7	5	3	1
3	11	3	3	2
4	10	10	1	4

Table VI: Values of b'_{ip}

The optimal assignments in Tables IV and VI are defined by the circled entries, and they are the same. Yet

$b_{21} + b_{32} + b_{43} + b_{14} = 14$ while

$b'_{21} + b'_{32} + b'_{43} + b'_{14} = 13$

and the cost of this assignment in QAP is 17.

The two methods provide lower bounds which are not necessarily the same. The advantage of the second method is that it involves the solution of simpler, sometimes trivial, assignment problems and this could result in a real saving in computer time.

As Gilmore [1962], Lawler [1963] pointed out for the Koopmans-Beckmann problem, the solution of these assignment problems can be obtained through a simple ranking technique. Los [1976] gives an efficient way to perform the involved computation in a recursive and parallel way. In that case, the computation of the b'_{ip} will consume half the time of the computation of the b_{ip}.

3. Lower bounds by the Time-Dependent Traveling Salesman.

The bounds derived in the previous section may be refined by considering not only the assignment of activity A_i to location L_p, but the simultaneous assignment of A_i to L_p and of A_j to the next location L_{p+1}. The model called Time-Dependent Traveling Salesman Problem (TDTSP), defined by, K. Fox [1973] and developed by the authors [1976 b] allows these manipulations.

Given about n^3 transition costs (lengths) C^t_{ij} (defined for $t=1,\dots,n-1$ and $i,j=1,\dots,n$ and $i\neq j$; for $t=0$, with $i=0$ and $j=1,\dots,n$; for $t=n$, $j=n+1$ and $i=1,\dots,n$), the problem is to find a permutation w of the integers $1,\dots,n$ which minimizes the total length:

$$c(w) = C^o_{o,w(1)} + C^1_{w(1),w(2)} + \dots + C^{n-1}_{w(n-1),w(n)} + C^n_{w(n),n+1}$$

This problem may be pictured by defining a multipartite network with origin (0), n "phases" defined by the values $t = 1,\dots,n$ and each containing n nodes (i,t), and the end node $(n+1)$. The arcs $(0,(i,1),((i,t),(j,t+1))$ and $((i,n), n+1)$ are defined if the corresponding lengths C^o_{oi}, C^t_{ij} and $C^n_{i,n+1}$ are defined. A technique, based on shortest paths and the introduction of penalties, is described in [Picard-Queyranne 1976 b] and provides good bounds on the cost of the optimum permutation. The reader is referred to the above reference for a description of this technique, which is used to define a branch and bound algorithm for the exact solution of TDTSP.

We will now introduce two TDTSP's associated with the two cost formulations of the QAD.

The transition cost C^t_{ij} will be an evaluation of the contribution of the assignment of activity A_i to location L_t, when A_j is assigned to the next location L_{t+1}. When t=n, the cost $C^n_{i,n+1}$ is defined only by considering the assignment of A_i to L_n, and so it is equivalent to the value b_{in} (or b'_{in}) defined in the previous section. The first transition costs $C^o_{o,i}$ will be set to zero, and the diagonal terms C^t_{ii} are not defined.

With the first cost formulation (1), the length C^t_{ij} is derived from the expression of b_{it} by adding a constraint forcing the activity A_j to be assigned to location L_{t+1}:

$$C^t_{ij} = a_{it} + C_{i,j,t,t,+1} + U^t_{ij} \qquad (\text{for } t=1,\ldots,n-1 \text{ and } i \neq j)$$

where U^t_{ij} is the optimal value of the objective function of the (n-2)x(n-2) LAP, denoted by LAP^t_{ij}:

$$\left(LAP^t_{ij}\right) \left\{ \begin{array}{llll} \text{minimize} & \sum\limits_{k\neq i,j} \; \sum\limits_{q\neq t,t+1} C_{iktq} \, u_{kq} & & \\ \text{subject} & \sum\limits_{k\neq i,j} u_{kq} = 1 & & \text{all } q\neq t,t+1 \\ & \sum\limits_{q\neq t,t,+1} u_{kq} = 1 & & \text{all } k\neq i,j \\ & u_{kq} = 0 \text{ or } 1 & & \text{all } q\neq t,t+1, \quad \text{all } k\neq i,j \end{array} \right.$$

<u>Illustration:</u> in the illustrative example, C^2_{14} is computed by solving the LAP defined Table VII,

so $C^2_{14} = 4 + 0 + 2 = 6$

The resulting C^t_{ij} are given in Table VIII.

q		1	4
q	2	1	4
	3	1	1

Table VII: LAP^2_{14}

	$i = 1$	2	3	4
0	0	0	0	0

$t = 0$

i	$t=1$, $j = 1$	2	3	4	$t=2$, $j = 1$	2	3	4	$t=3$, $j = 1$	2	3	4	$t=4$, $n+1 = 5$
1		11	9	8		7	10	6		7	7	5	5
2	6		5	3	8		8	6	5		5	6	3
3	10	7		9	5	5		3	4	4		6	5
4	9	7	11		11	10	14		4	6	3		6

Table VIII: Values of C^t_{ij}

The path length corresponding to the assignment defined by (5), that is, by the permutation 3-1-4-2, is

$$C^o_{o3} + C^1_{31} + C^2_{14} + C^3_{42} + C^4_{25} = 25$$

The shortest path in the above network is defined by 2-3-4-3, with activity 3 appearing twice and activity 1 not at all. It is a simple matter to show, for instance by introducing penalties, that the optimum path which is also a permutation is 2-3-4-1, with length 17, and this is exactly the cost of the corresponding assignment, so QAP is solved.

The second associated TDTSP is derived from the cost formulation (5) and the expression for b'_{it}. Define

$$d^t_{ij} = a_{it} + d_{i,j,t,t+1} + W^t_{ij} \quad \text{for } t = 1,..,n-2 \quad i \neq j$$

where W^t_{ij} is the optimal value of the $(n-2) \times (n-t-1)$ ILAP:

$$
ILAP_{ij}^{t}
\begin{cases}
\text{minimize} & \displaystyle\sum_{k\neq i,j} \sum_{q=t+2} d_{ijtq}\, u_{kq} & \\
\text{subject} & \displaystyle\sum_{k\neq i,j} u_{kq} = 1 & \text{all } q=t+2,..,n \\
& \displaystyle\sum_{q=t+2} u_{kq} \leqslant 1 & \text{all } k\neq i,j \\
& u_{kq} = 0 \text{ or } 1 & \text{all } q=t+2,...,n \\
& & \text{all } k\neq i,j
\end{cases}
$$

$$d_{ij}^{n-1} = a_{i,n-1} + d_{i,j,n-1,n}$$

and $d_{i,n+1}^{n} = a_{i,n}$

Illustration: in the illustrative example, the computation of d_{14}^{2} requires the solution of a (trivial) 2x1 ILAP defined in Table IX

so $d_{14}^{2} = 4 + 2 + 5 = 11$

The resulting values of d_{ij}^{t} are given in Table X.

	q: 4
k: 2	6
3	5

Table IX: $ILAP_{14}^{2}$

	t = 0: 1	2	3	4	t = 1: 1	2	3	4	t = 2: 1	2	3	4	t = 3: 1	2	3	4	t = 4: 5
0	0	0	0	0													
1						15	15	14		12	11	11		7	4	5	2
2					10		7	10	8		9	5	3		5	4	1
3					14	11		13	3	5		5	3	5		5	2
4					10	10	16		10	11	13		2	3	1		4

Table X: values of d_{ij}^{t}

The length of the illustrative assignment defined by (5) is

$$d^{o}_{o3} + d^{1}_{31} + d^{2}_{14} + d^{3}_{42} + d^{4}_{25} = 29$$

The shortest path is defined by the sequence 2-3-4-3 with length 15, and the optimal permutation is 2-3-4-1 (the same as previously) with length 16, while its cost in QAP is 17. In solving the problem with this bounding approach, some enumeration would be necessary to check that there is no other assignment with cost C* such that $16 \leqslant C^* < 17$

4. Branch and bound structure for the TDTSP approach:

The Branch and Bound structure for handling the QAP via the TDTSP approach is very similar to the one described by the authors [1976 b] for the standard TDTSP. The reader is referred to this paper for its description. Below, only topics specific to the QAP will be discussed.

At each node in the enumeration tree, it is necessary to recompute the multipartite network in order to obtain tigher bounds. When a permutation is found as a shortest path, it is necessary to compute the corresponding cost in QAP. This assignment could possibly become the best known solution. If the cost and the length are equal, then backtracking occurs; otherwise, the length being less than the cost, it is necessary to continue branching in order to reduce this gap. The "dominance test" described for the TDTSP cannot be applied to the QAP, since the cost corresponding to the activities assigned to locations $L_{p+1}, L_{p+2}, \ldots, L_n$ depends not only on the set of activities to be assigned to the p first locations, but also on their mutual assignments. While this dominance test can be applied for certain particular cases of the QAP, such as the Traveling Salesman Problem, the TDTSP or the Linear Ordering Problem, this is not the case for the Koopmans-Beckmann Problem or the general QAP. It is likely that this restriction will make the enumeration more extensive in these cases.

Some remarks pertain to the arbitrary numbering of the locations. The branching being performed on the last location assignments, it appears advisable to select as the last one, a location whose effect is significant for the total cost. Some empirical measures such as the total weight or the variance of the weights associated with each location could lead to more efficient algorithms. However, a similar problem (selection of the origin node) occured in a similar approach to the Traveling Salesman Problem, but no significant result was found (See [Picard and Queyranne 1975 and 1976 a]).

When the second bounding scheme is used, the costs appear to usually decrease as t goes from 1 to n. The branching being guided by the length of the paths ending in each of the last phase nodes, it seems advisable to perform the shortest paths iterations in reverse order, that is from the right to the left, and to branch on the first location assignment. This should lead to higher implicit bounds and to earlier fathomings.

A final remark concerns the computational aspect of the definition of the related TDTSP. Each one of the about n^3 entries (C^t_{ij} or d^t_{ij}) requires the solution of an assignment problem. Since exact algorithms are available for solving an nxm LAP in $O(n^3)$ time, it appears that the construction of the multipartite network will require $O(n^6)$ time. This is very time-consuming, but advantage can be taken of the special structure of the LAP involved. Consider the problems LAP^t_{ij} and LAP^t_{ik}: they differ only in the row corresponding to k in LAP^t_{ij}, and to j in LAP^t_{ik}. So, it is possible to derive a solution to the second problem from a solution to the first one. Since only some costs are modified, a primal or a primal-dual algorithm for LAP should easily perform this post-optimization. The Parametric Analysis introduced by Srinivasan and Thompson [1972] for the Transportation Problem could also prove useful in providing faster bounds.

It appears that there are n^2 distinct problems, each one requiring (n-1) post-optimizations. This is also true for the $ILAP_{ij}^{t}$ problems (but note that the computation of d_{ij}^{t} becomes very easy for t close to n, and trivial for $t > n-2$).

For the Koopmans-Beckmann problem, the above reduction takes a neat form: the computation of C_{ij}^{t} (or d_{ij}^{t}) for the successive values of j involves only one exchange per new value of j, if the j's are introduced according to the order of non-decreasing values of their interaction Q_{ij} with i. The recursive and parallel procedure given by Los [1976] will again allow efficient computation.

REFERENCES

D. Adolphson and T.C. Hu (1973) "Optimal Linear Ordering, SIAM J. Appl.Math.25, 403-423 (1973).

K.R. Fox (1973), "Production Scheduling on Parallel Lines with Dependencies", Ph.D. Dissertation, The Johns Hopkins University, Baltimore, Md. (1973).

P.C. Gilmore (1962), "Optimal and Suboptimal algorithms for the Quadratic Assignment Problem" SIAM Journal 10,2, p.305-313.

M. Held, P. Wolfe and H. Crowder (1974), "Validation of Subgradient Optimization", Math. Programming 6, 62-88 (1974).

T.C. Koopmans and M. Beckmann (1957), "Assignment Problems and the Location of Economic Activities", Econometrica, 25,1, P.53-76.

E.L. Lawler (1963),"The Quadratic Assignment Problem", Management Science, 19,4, p.586-599.

M. Los (1976), "Experimental Comparison and Evaluation of Several Exact and Heuristic Algorithms to Solve Quadratic Assignment Problems of the Koopmans-Beckmann Type", Centre de Recherches sur les Transports, Université de Montréal (1976).

J.C. Picard and M. Queyranne(1975), "Le Problème du Voyageur de Commerce: une Formulation par la Programmation Linéaire", Rapport Technique EP75-7, Ecole Polytechnique, Montréal (1975)

J.C. Picard and M. Queyranne(1976 a),"Le Problème du Voyageur de Commerce Plus-Courts Chemins et Optimisation par Sous-Gradients", Rapport Technique EP76-7, Ecole Polytechnique, Montréal (1976).

J.C. Picard and M. Queyranne (1976b), "The Time-Dependent Traveling Salesman Problem and Application to the Tardiness Problem in One-Machine Scheduling", Rapport Technique EP76-17,Ecole Polytechnique, Montréal, (1976).

V. Srinivasan and G.L. Thompson (1972a), "An Operator Theory of Parametric Programming for the Transportation Problem-I", Naval Research and Logistic Quarterly, 19, p.205-225.

V. Srinivasan and G.L. Thompson (1972b), "An Operator Theory of Parametric Programming for the Transportation Problem-II , Naval Research and Logistic Quarterly, 19, p.227-252.

L. Steinberg (1961),"The Backboard Wiring Problem: a Placement Algorithm"
SIAM Review, 3,1, p.37-50.