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Decentralized Control Barrier Functions for Robot Safety under Relative State Estimation

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Mémoire présenté en vue de l'obtention du diplôme de *Maîtrise ès sciences appliquées*
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Ce mémoire intitulé :

Decentralized Control Barrier Functions for Robot Safety under Relative State Estimation

présenté par **Muhammed Bugrahan ARTUÇ**
en vue de l'obtention du diplôme de *Maîtrise ès sciences appliquées*
a été dûment accepté par le jury d'examen constitué de :

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Jérôme LE NY, membre et directeur de recherche
Guchuan ZHU, membre

DEDICATION

Look at the earth, sky, and water . . .
These are signs for people of knowledge.

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RÉSUMÉ

Cette thèse aborde la question cruciale de la sécurité dans les systèmes de contrôle à travers le développement et l'analyse des Fonctions de Barrière de Contrôle (FBC). Elle commence par une revue de la littérature exhaustive, introduisant les FBC pour la sécurité et examinant diverses méthodologies, y compris la rétroaction d'état complet, la rétroaction de sortie avec des incertitudes bornées et non bornées, ainsi que les stratégies de prévention des collisions basées sur les FBC pour les systèmes multi-robots. La problématique se concentre sur un système à deux robots utilisant plusieurs capteurs de distance, détaillant les défis spécifiques et les questions de recherche. L'étude explore l'évitement de collision décentralisé avec une erreur d'observation bornée, en discutant des méthodes sous rétroaction d'état parfaite, rétroaction d'état relative et des approches prenant en compte des erreurs d'estimation bornées. La discussion s'étend à l'évitement de collision décentralisé avec une erreur d'observation stochastique, en mettant l'accent sur les stratégies sous des incertitudes d'estimation stochastiques. Enfin, des études de simulation valident les résultats théoriques, couvrant des scénarios spécifiques qui démontrent l'applicabilité pratique et l'efficacité des méthodes proposées. Dans l'ensemble, cette thèse fait progresser la compréhension et l'application des FBC pour améliorer la sécurité des systèmes dans les applications de navigation relative et de commande multi-robots.

ABSTRACT

This thesis addresses the critical issue of safety in control systems through the development and analysis of Control Barrier Functions (CBFs). It begins with a comprehensive literature review, introducing CBFs for safety and examining various methodologies, including full state feedback, output feedback with both bounded and unbounded uncertainties, and distributed CBF-based collision avoidance strategies for multi-robot systems. The problem statement focuses on a two-robot system using multiple range sensors, detailing specific challenges and research questions. The study explores decentralized collision avoidance with bounded estimation error, discussing methods under perfect state feedback, relative state feedback, and approaches considering bounded estimation errors. The discussion extends to decentralized collision avoidance with stochastic estimation error, emphasizing strategies under stochastic estimation uncertainties. Finally, simulation studies validate the theoretical findings, covering specific scenarios that demonstrate the practical applicability and effectiveness of the proposed methods. Overall, this thesis advances the understanding and application of CBFs for enhancing system safety in multi-robot relative navigation and control applications.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF SYMBOLS AND ABBREVIATIONS

UWB	Ultra-Wideband
GPS	Global Positioning System
CBF	Control Barrier Function
EKF	Extended Kalman Filter
CLF	Control Lyapunov Function
QP	Quadratic Program
ISSf	Input-to-State Safety
GPR	Gaussian Process Regression
ISS	Input to State Stable
EEQ	Estimation Error Quantifier
BE	Bounded Error
TRCBF	Tunable Robust Control Barrier Function
CVaR	Conditional Value at Risk
PDF	Probability Distribution Function
MSE	Mean-Squared Error
PrCBF	Probabilistic Control Barrier Function
DARE	Differential Algebraic Ricatti Equation

CHAPTER 1 INTRODUCTION

In robotic applications, safety, in particular avoiding collisions with obstacles in the environment and other agents, is a primary concern. As robots are increasingly deployed outside of perfectly controlled environments, a fundamental challenge is then to ensure safety in the presence of imperfect information about the state of the robotic system. In this context, two difficulties arise: i) strategies to enforce safety should take the uncertainty of the state estimate into account; and ii) planning and control actions have themselves an indirect effect on safety, as they determine the future information collected along the trajectory and hence ultimately the quality of the system's state estimate; classically, this is referred to as the dual effect of control [1].

This dual effect of control can be observed in relative navigation applications for multi-robot systems. In these systems, each swarm member uses its own sensors to estimate the position information of nearby robots. The obtained position information is then used in collision avoidance algorithms between robots. Because the relative configurations between robots and hence the range sensor positions in space affect estimation errors, the control inputs applied by each robot in the swarm indirectly affect the estimation algorithms and the collision avoidance system. Hence, the effect of sensor errors on the collision avoidance system needs to be considered, especially in such applications where the measurements do not consist of direct information about the estimated state variables. For example, relative positions between robots may have to be estimated from just relative distance measurements obtained from Ultra-Wideband (UWB) sensors.

UWB sensors sensors primarily provide distance measurements but can also be utilized for the transmission of limited data such as Inertial Measurement Unit (IMU) and odometry measurements. They can work in dark environments, where using cameras is challenging. From these measurements, each robot in a swarm system can estimate the relative state of the other agents without Global Positioning System (GPS) information. However, deriving position information from distance measurements requires the design of an additional observers or estimation algorithms. For this reason, the development of relative navigation algorithms has emerged as a significant area of research in swarm drone systems in recent years. Although UWB sensors provide a cost effective solution to localize robots, sensor noise can lead to significant uncertainty in state estimates, which can be amplified in certain geometric configurations of the robotic swarm [2]. Effective collision avoidance needs to take this uncertainty into account, both in a reactive manner and also at the trajectory planning stage.

The development of a real-time robotic system requires specialized modules that assist in completing the desired task. For instance, planning and internal controllers are usually designed independently of each other. Although this layered structure aids in task completion, new structures are needed to enhance safety and prevent collisions among robots in swarm robotics. In this thesis, we focus on distributed swarm robots, where each robot can estimate the relative state of other robots using multiple UWB sensors mounted on their body frames. Using the relative state estimates, we seek to design a high-confidence safety layer to avoid collisions between robots as they reach their targets.

The design of collision avoidance algorithms cannot be considered in isolation from the impacts of the estimation algorithms. The accuracy of the acquired state information directly correlates with the efficacy of preventing collisions among swarm robots. Thus, the accuracy of state estimation is paramount for enhancing the safety and coordination of robotic swarms, making this subject a vital focus for ongoing research and development. Moreover, the development of a collision avoidance system without using GPS information is highly critical for modern swarm robotics, e.g., when robots navigate in indoor environments or around structures that may prevent maintaining line of sight with navigation satellites. In this thesis, we aim at developing new methods to mitigate the effects of the estimation errors on collision avoidance between robots. Consequently, we present theoretical analyses on the interaction between collision avoidance systems, controllers, and estimation algorithms.

Research Objective

Our main objective is to develop a decentralized control strategy for collision avoidance, capable of providing safety guarantees for two robots navigating in proximity of each other and equipped with UWB range sensors. As presented in Chapter 3, each robot is equipped with two UWB sensors, in order to improve the system's observability properties [3]. The collision avoidance system for each robot issues a control input to locally modify the robot's trajectory. We aim by this control input to minimize a quantitative measure of collision risk, but also to minimally perturb the nominal trajectory that each robot is following to perform its tasks. One crucial aspect of this problem is that the uncertainty in the relative position estimates strongly depends on the relative pose (i.e., position and orientation) between the robots, through the geometric configuration of the four range sensors. Because the control inputs of the robots also impact these range sensor configurations, they can affect estimation performance, which in turn affects the ability of the robots to avoid future collisions. Consequently, the collision avoidance system is required to take into account its own impact on estimation errors in addition to acting as a safety filter, and this becomes particularly important when the robots become close to each other.

The remainder of the thesis is organized as follows. Chapter 2 offers a comprehensive literature review, beginning with an introduction to Control Barrier Functions (CBFs) for safety and delving into various related methods, including full state feedback, output feedback with both bounded and unbounded uncertainties, and distributed CBF-based collision avoidance for multi-robot systems. Chapter 3 presents the problem statement, detailing the two-robot system under study and formulating the research questions. Chapter 4 explores decentralized collision avoidance with bounded observer error, discussing methods under perfect state feedback, relative state feedback, and approaches considering bounded estimation errors. Chapter 5 extends the discussion to decentralized collision avoidance with stochastic observer error, emphasizing collision avoidance strategies under stochastic estimation errors. Finally, Chapter 6 concludes the thesis and provides perspectives for future work.

CHAPTER 2 LITERATURE REVIEW

2.1 Introduction

Ensuring safety of a system under control means preventing its state from entering critical conditions. This perspective can be captured by modeling *safe sets* as subsets of the state space that are positively invariant for the dynamic model of the controlled system [4]. A collision avoidance system can be defined within this safety framework. For instance, the trajectory of a robot positioned at any starting point far from a collision region is considered safe if we can ensure that the robot will never encounter a collision in all future trajectories. Safety can also be applied to other problems including robot walking, swarm collision avoidance, automatic cruise control systems, etc [4].

Collision avoidance methods have been the object of significant research. One can distinguish between global and local methods [5]. Global methods focus on replanning on a pre-established map, while local methods are evaluated for collision avoidance at the current position. Local algorithms provide stronger guarantees compared to global methods because global methods do not incorporate sudden changes in the collision region. Local methods can be classified into "geometric", "force field", "optimization", and "sense and avoid" methods [6]. Geometric techniques typically employ analytical strategies but they are susceptible to sensor inaccuracies [7]. Potential or force field methods use repulsive and attractive forces to navigate around obstacles towards a target [8]. Nevertheless, using potential fields for obstacle avoidance requires parameter adjustments based on the specific obstacles encountered. Sense and avoid methods require systematic design and struggle under lack of sensor information and failures [9].

In recent years, the "Control Barrier Function" (CBF) method, which can be seen as a combination of "force field" and "optimization" methods, has emerged as a versatile tool among local algorithms [4]. This method can be combined with the Control Lyapunov Function (CLF) method to simultaneously assess system safety and stability. Control Barrier Functions (CBF) need fewer parameter adjustments, offering a more streamlined approach to collision avoidance [10]. Compared to other collision avoidance methods, the CBF theorem is compatible with the invariant set theorem [4]. Safety conditions can be established in a manner similar to Lyapunov stability. Collision avoidance can be implemented into real-time systems without pre-calculating safe trajectories. However, a critical step to apply the method is the ability to identify an appropriate barrier function [4].

2.2 Background on Control Barrier Functions for Safety

Control Barrier Functions (CBFs) define controllable invariant sets. Early work related to CBFs can be found in [11]. However, the CBFs introduced in [11] diverge at the boundary of the safe set, so that as a result the safety-preserving control inputs can also become unbounded. To address this issue, a CBF-based safety-preserving method that avoids infinite control inputs at the set boundary was proposed in [12]. This method is sensitive to some uncertainties but it is attractive because it guarantees the asymptotic stability of the safe set. Hence, if the system is pushed away from the safe set, the control inputs can return it to this set. In that regard, the CBF method differs from the previously proposed barrier methods [13]. At the same time, it enforces a condition at the boundary of the safe set that should be true to guarantee its invariance [14].

In the rest of this section, we summarize key concepts behind the CBF method, as discussed in [4], [15]. Given (locally Lipschitz) functions $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}$, consider the following control-affine system

$$\dot{x} = f(x) + g(x)u, \tag{2.1}$$

with state $x(t) \in \mathcal{D} \subset \mathbb{R}^n$ and control $u(t) \in \mathcal{U} \subset \mathbb{R}^m$. Consider a set $\mathcal{C} \subset \mathcal{D}$, called a *safe set*, defined by a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, such that

$$\mathcal{C} \coloneqq \{x \in \mathbb{R}^n : h(x) \geq 0\}. \tag{2.2}$$

The boundary $\partial\mathcal{C}$ and interior $\text{Int}(\mathcal{C})$ of $\mathcal{C}$ are defined respectively by

$$\partial\mathcal{C} \coloneqq \{x \in \mathbb{R}^n : h(x) = 0\} \text{ and } \text{Int}(\mathcal{C}) \coloneqq \{x \in \mathbb{R}^n : h(x) > 0\}.$$

Definition 1. *A state feedback controller $x \mapsto u(x)$ renders the system* (2.1) safe *with respect to the set $\mathcal{C} \subset \mathcal{D}$ if $\mathcal{C}$ is forward invariant for the closed-loop system $\dot{x} = f(x) + g(x)u(x)$.*

In Definition 1, recall that a set $\mathcal{S}$ is called forward invariant for a (time-invariant) dynamical system if $x_0 \in \mathcal{S}$ implies that $x(t; x_0) \in \mathcal{S}$ for all $t \geq 0$, where $x(\cdot; x_0)$ denotes a trajectory of the system starting at the initial condition x_0. Next, given a function $h : \mathbb{R}^n \to \mathbb{R}$, we define the shorthand notation

$$\dot{h}(x, u) \coloneqq \frac{\partial h(x)}{\partial x} \left(f(x) + g(x)u \right), \tag{2.3}$$

so that $\dot{h} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$.

Remark 1. *Note that* $\dot{h}(x,u) = L_f h(x) + L_g h(x)u$*, where* $L_f h$ *is the Lie derivative of* h *with respect to the vector field* f *[16], but the shorter notation* (2.3) *is used to simplify some expressions in the following. This notation has the drawback of hiding the dependency on* f *and* g*, which however are fixed for a given system* (2.1)*.*

Definition 2. *An extended class* $\mathcal{K}$ *function is a strictly increasing continuous function* $\alpha : (-b, a) \to \mathbb{R}$*, for some* $a, b \in \mathbb{R}_+ > 0$*, with* $\alpha(0) = 0$*.*

Definition 3. *A continuously differentiable function* $h : \mathbb{R}^n \to \mathbb{R}$ *is a control barrier function (CBF) for the system* (2.1) *and a set* $\mathcal{C}$ *defined by* (2.2) *if there exists an extended class* $\mathcal{K}$ *function* α *such that, for all* $x \in \mathbb{R}^n$*,*

$$\sup_{u \in \mathcal{U}} \left\{ \dot{h}(x,u) \right\} \geq -\alpha(h(x)). \tag{2.4}$$

Motivated by Definition 3, for α an extended class $\mathcal{K}$ function, let

$$K^{\alpha}_{\text{cbf}}(x) \coloneqq \left\{ u \in \mathcal{U} \text{ s.t. } \dot{h}(x,u) + \alpha(h(x)) \geq 0 \right\} \subset \mathcal{U}. \tag{2.5}$$

Then, we introduce following theorem in [4].

Theorem 1. *Let* h *be a control barrier function for the system* (2.1) *and the set* $\mathcal{C}$*, for an extended class* $\mathcal{K}$ *function* α*, and moreover suppose* $\frac{\partial h}{\partial x}(x) \neq 0$ *for all* $x \in \partial\mathcal{C}$*. Then any (Lipschitz continuous) controller* $x \mapsto u(x) \in K^{\alpha}_{cbf}(x)$ *renders the system* (2.1) *safe with respect to the set* $\mathcal{C}$*. Additionally, the set* $\mathcal{C}$ *is (locally) asymptotically stable in* $\mathcal{D}$*.*

A proof of Theorem 1 can be found in [17]. The asymptotic stability of $\mathcal{C}$ is an important property, which allows trajectories to end in the safe set even when the initial state begins in the unsafe set. When it is satisfied, several results from the literature can be used to characterize the robustness of forward invariance of the set $\mathcal{C}$ [17]. A key observation related to the asymptotic stability of $\mathcal{C}$ is that if $\mathcal{D} \supset \mathcal{C}$ is open, h induces a Lyapunov function $V_{\mathcal{C}} : \mathcal{D} \to \mathbb{R}^+_0$, where $\mathbb{R}^+_0$ denotes the non- negative real numbers, defined by:

$$V_{\mathcal{C}}(x) = \begin{cases} 0, & \text{if } x \in \mathcal{C}, \\ -h(x), & \text{if } x \in \mathcal{D} \setminus \mathcal{C}. \end{cases}$$

Indeed, we have the following three properties: 1) $V_{\mathcal{C}}(x) = 0$ for $x \in \mathcal{C}$; 2) $V_{\mathcal{C}}(x) > 0$ for $x \in \mathcal{D} \setminus \mathcal{C}$; and 3) $\dot{V}_{\mathcal{C}}(x)$ satisfies the following inequality for $x \in \mathcal{D} \setminus \mathcal{C}$:

$$\dot{V}_{\mathcal{C}}(x) = -\dot{h}(x, u(x)) \leq \alpha(h(x)) = \alpha(-V_{\mathcal{C}}(x)) < 0.$$

If V_C is continuous on its domain and continuously differentiable at every point $x \in \mathcal{D} \setminus \mathcal{C}$, then the set $\mathcal{C}$ is asymptotically stable whenever (2.1) is forward complete or the set $\mathcal{C}$ is compact [15].

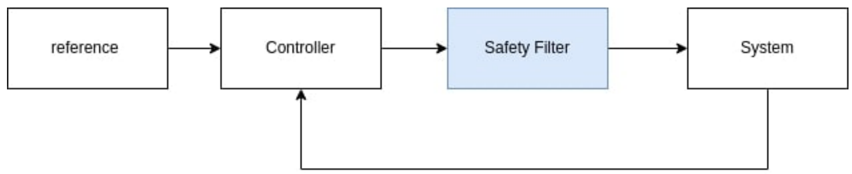

Figure 2.1 Implementation of the CBF QP program.

To use Theorem 1 in practice, first, a nominal controller u_{ref} is designed without considering safety, see Figure 2.1. Then, at each instant, with the system in state x, the value u_{ref} of the nominal control input is potentially modified to find the closest control input value u (for the Euclidean distance) that belongs to $K^{\alpha}_{\text{cbf}}(x)$, thus enforcing safety as long as this set is not empty. Hence, at each time instant, the safety filter solves the following linearly constrained quadratic program (QP)

$$\begin{aligned} \min_{u \in \mathcal{U}} \quad & \frac{1}{2}\|u - u_{\text{ref}}\|^2 \\ \text{subject to} \quad & \frac{\partial h(x)}{\partial x} \cdot g(x)u \geq -\frac{\partial h(x)}{\partial x} \cdot f(x) - \alpha(h(x)), \end{aligned} \tag{2.6}$$

and the minimizer u^* of (2.6) constitutes the final control input implemented. Problem (2.6) is convex and can be solved efficiently. The method can also be combined with the CLF method to consider both safety and stability requirements simultaneously, but this approach is more difficult to tune [18].

Using results such as Theorem 1, designing safety-preserving controllers using CBFs under various conditions has been an active area of research in the last decade. Starting from a stabilizing controller, one can add a control input to satisfy the CBF constraints (2.5) and enforce safety, as well as Control Lyapunov Function (CLF) based constraints to maintain stability. Finding the “smallest” control input (i.e., of minimum norm) satisfying these constraints leads to formulating a QP whose solution at each state and time instant produces the desired control input. This idea is further explored in papers such as [15, 17, 19], and applied to multi-agent systems in [20] for instance. Moreover, alternative definitions are presented in several studies, such as input to state safety [21], [22]. Actually, CLF and CBF QPs are inverse optimal filters [23] based on Lyapunov and barrier function types. The CBF method does not require an initial position for optimization and is easy to implement in comparison to Model Predictive Control [4]. It can be applied to collision avoidance by following centralized or decentralized optimization perspectives [24], [25].

2.3 Safety with Control Barrier Functions under Full State Feedback

In this section, variations of the CBF theorem are discussed to ensure the safety of systems under full state feedback. Most of the research literature assumes perfect state feedback, and the primary consideration for safety are then system uncertainties. These theorems are generally constructed using various concepts similar to stability considerations. For instance, in [26], a known disturbance bound is considered for system safety. As an alternative, Input-to-State Safety (ISSf) theorems are presented by initially constructing safe set in [21], [22]. In these theorems, the safe set is constructed on a nominal safe set with matched disturbances. This allows for the establishment of a safe set in real-time without requiring any disturbance estimation even though initial conditions start in the unsafe set [21]. In [27], a general approach to ISSf conditions is presented by applying a tunable function into the ISSf definition in [21]. These methods may be ineffective in systems where the disturbance bounds are unknown, therefore the use of disturbance observers is investigated to incorporate unknown disturbance effects into CBF conditions [28].

Alternative methods are based on the estimation of uncertainties in order to increase safety, such as robust adaptive control [29], indirect adaptive control [30], and L_1 adaptive control [31]. However, the performance of these methods depends on the learning rate of uncertainties. Also, unbounded noise can destabilize the safety conditions. Note that system safety is proved with high guarantees under perfect state feedback even though Guassian disturbances affect the system [32]. However, highly nonlinear systems require learning of disturbances to use theorems in [32]. For this reason, learning methods are combined with CBF methods with reinforcement learning [33], supervisory learning [34], and Gaussian Process Regression (GPR) [35]. Specially, collision avoidance requires memory to calculate unknown uncertainties at each control input calculation time for GPR method [35]. However, GPR does not consider non-Gaussian probability distributions, and a Gaussian assumption must be made at each calculation time for this method. Variation from the true probability distribution between each calculation can disrupt the time uniformity of the user-defined probability in the GPR method. For this reason, new theorems based on Kullback-Leibler

(KL) divergence are considered in [36].

Learning of uncertainty or indirect adaptive control is not common in the control community and is generally not considered for the aerospace industry due to stability problems [37]. The learning rate can be slow to capture unknown uncertainties, leading to unstable motion under switching reference input [38]. For instance, consider the take-off and landing phases of drones or sudden changes in reference trajectories near collision regions. For this reason, persistent excitation conditions must be satisfied for this type of algorithm [38]. However, the nonparametric nature of GPR can make it difficult to analyze the system's stability. Therefore, direct adaptive control methods, such as MRAC, and hybrid adaptive control architectures are used with GPR methods instead of learning uncertain parameters directly [39]. However, detailed theoretical proof of the impact of learning rates on stability and safety remains an open problem.

2.4 Safety with Control Barrier Functions under Output Feedback

In this section, CBF-based safety conditions under output feedback are reviewed. Essentially, when output measurements have a smaller dimension than the state vector, and/or measurement noise is present, observers or estimators are used to provide feedback into controllers. Although many systems of practical interest require observers, there are relatively few studies combining output feedback with CBFs. Generally, measurements with worst-case bounds on the noise have been the focus for safety with output feedback, such as [40]. In cases where (possibly unbounded) stochastic noisy measurement scenarios occur, methods based on Kalman Filtering principles are employed to attempt to obtain the full state estimation [32]. The system states obtained under noisy measurements are not perfect and are referred to as incomplete information [32]. Unfortunately, it is not necessarily possible to completely avoid collisions within a probabilistic framework when feedback is assumed to use incomplete information [41].

The CBF based theorems primarily address bounded system noise and considered with observer or estimated states with measurements for real time applications. For instance, measurement CBF is also examined to handle bounded system uncertainties with output measurements [42]. However, this research is not detailed for observer and safety interconnection. Although safety under full state feedback is discussed in Section 2.3, real-time applications typically require interconnecting an observer with the controller. This connection is studied for several type of observers in the CBF literature [40] and [43]. Bounded system and measurement uncertainties are considered in [40]. Research is trying to find interconnection between safety and two different observers including Input to State Stable (ISS) and Bounded

Error (BE) observers. Connection between safety and Estimation Error Quantified (EEQ) Observer is considered in [43]. Interconnection between observer and safety requires observer gain selection for convergence of estimation error and safety in ISS observer [40]. However, BE observer is relaxing this requirement for collision avoidance application. Unfortunately, the researchers do not explain the convergence of the estimation error under unobservable conditions.

2.4.1 CBF-Based Safety with Input-to-State Stable Observers

Consider the observer-based CBF method discussed in [40]. Given (locally Lipschitz) functions $f : \mathcal{D} \to \mathbb{R}^n$, $g : \mathcal{D} \to \mathbb{R}^{n\times m}$, $c : \mathcal{D} \to \mathbb{R}^{n_y}$, $g_d : \mathcal{D} \to \mathbb{R}^{n\times n_d}$, and $c_d : \mathcal{D} \to \mathbb{R}^{n_y \times n_v}$, consider the following control-affine system

$$\dot{x} = f(x) + g(x)u + g_d(x)d(t), \tag{2.7}$$

$$y = c(x) + c_d(x)v(t), \tag{2.8}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$ is the system state, $u \in \mathcal{U} \subset \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^{n_y}$ is the measured output, $d : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_d}$ is a disturbance on the system dynamics, and $v : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_v}$ is the measurement noise. We assume d and v are piecewise continuous, bounded disturbances, $\sup_t \|d(t)\|_\infty = \bar{d}$ and $\|v(t)\|_\infty \leq \bar{v}$ for some known $\bar{d}, \bar{v} < \infty$. We consider that $g_d(x)d(t)$ accounts for either matched or unmatched disturbances.

Consider an observer providing a state estimate $\hat{x} \in \mathcal{D}$ used by the controller instead of a state feedback. The observer-controller interconnection is formed by

$$\dot{\hat{x}} = p(\hat{x}, y) + q(\hat{x}, y)u, \tag{2.9}$$

$$u = \pi(t, \hat{x}, y), \tag{2.10}$$

where $p : \mathcal{D} \times \mathbb{R}^{n_y} \to \mathbb{R}^n$, $q : \mathcal{D} \times \mathbb{R}^{n_y} \to \mathbb{R}^{n\times m}$ are locally Lipschitz in both arguments. Assume that the feedback controller $\pi : \mathbb{R}_{\geq 0} \times \mathcal{D} \times \mathbb{R}^{n_y} \to \mathcal{U}$ is piecewise-continuous in t and Lipschitz continuous in the other two arguments. Then, the closed-loop system can be described by

$$\dot{x} = f(x) + g(x)\pi(t, \hat{x}, y) + g_d(x)d(t), \tag{2.11}$$

$$\dot{\hat{x}} = p(\hat{x}, y) + q(\hat{x}, y)\pi(t, \hat{x}, y), \tag{2.12}$$

$$x(0) = x_0, \quad \hat{x}(0) = \hat{x}_0, \tag{2.13}$$

where the closed loop solution of this system exist and unique for some interval $I(0, t_{max})$

of time. Now, we give forward invariant definition of safe set (2.2) considering observer-controller interaction and bounded disturbance in the system (2.7) by following reference [40].

Definition 4. *An observer-controller pair* (2.9)-(2.10) *ensures the system* (2.7) *is safe with respect to a set* $\mathcal{C} \subset \mathcal{D}$ *from the initial-condition sets* $\mathcal{D}_0, \hat{\mathcal{D}}_0 \subset \mathcal{C}$ *if the closed-loop system* (2.11)-(2.12) *satisfies the property*

$$x(0) \in \mathcal{D}_0 \text{ and } \hat{x}(0) \in \hat{\mathcal{D}}_0 \implies x(t) \in \mathcal{C} \quad \forall t \in I.$$

Notice that real state $x(t)$ of the system (2.7) must be in the safe set $\mathcal{C}$ for all time t. Here, a natural question arises regarding conditions under which estimated states $\hat{x}(t) \in \hat{\mathcal{C}}$ ensure true states $x(t)$ in the safe set $\mathcal{C}$. For this reason, ISS observer can be used to show this relationship. Let us introduce related definition in [40].

Definition 5. *An observer* (2.9) *is an Input-to-State Stable (ISS) observer for system* (2.7) *if there exist a class* $\mathcal{KL}$ *function* β *continuously differentiable with respect to the second argument, and a class* $\mathcal{K}$ *function* η *such that*

$$\|x(t) - \hat{x}(t)\| \leq \beta(\|x(0) - \hat{x}(0)\|, t) + \eta(\bar{w}), \quad \forall t \in I,$$

where $\bar{w} = \max(\bar{d}, \bar{v})$.

Suppose the initial estimation error is bounded with a known bound $\delta > 0$. Hence, a continuously differentiable, non-increasing function $M_\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ exists for an ISS observer such that

$$\|x(0) - \hat{x}(0)\| \leq \delta \Rightarrow \|x(t) - \hat{x}(t)\| \leq M_\delta(t), \quad \forall t \in I. \tag{2.14}$$

Hence, we can define set $\mathcal{C}(t)$ as follow

$$\mathcal{C}(t) = \{\hat{x} \in \mathcal{D} : h(\hat{x}) - \gamma_h M_\delta(t) \geq 0\}$$

The set $\mathcal{C}$ can be justified by Lipschitz continuity because $|h(x) - h(\hat{x})| \leq \gamma_h \|x - \hat{x}\| \Rightarrow h(\hat{x}) - \gamma_h \|x - \hat{x}\| \leq h(x)$. Therefore, if $\hat{x} \in \hat{\mathcal{C}}(t)$, then $0 \leq h(\hat{x}) - \gamma_h M_\delta(t) \leq h(\hat{x}) - \gamma_h \|x - \hat{x}\| \leq h(x)$, i.e., $x \in \mathcal{C}$. Thus, $\hat{x} \in \hat{\mathcal{C}}(t) \Rightarrow x \in \mathcal{C}$.

Definition 6. *Consider Definition 5 with an ISS observer of known estimation error bound* (2.14)*. A continuously differentiable function* $h : \mathcal{D} \to \mathbb{R}$ *defines an Observer-Robust CBF with ISS observer for system* (2.7) *if there exists an extended class* $\mathcal{K}$ *function* α *s.t.*

$$\sup_{u \in U} \frac{\partial h(\hat{x})}{\partial \hat{x}} p(\hat{x}, y) + \frac{\partial h(\hat{x})}{\partial \hat{x}} q(\hat{x}, y) u \geq -\alpha(h(\hat{x}) - \gamma_h M_\delta(0))$$

for all $\hat{x} \in \mathcal{C}$, *and all* $y \in Y(\hat{x}) = \{y : y = c(x) + c_d(x)v(t), \|x - \hat{x}\| \leq M_\delta(0), \|v\| \leq \bar{v}\}$, *an overapproximation of the set of possible outputs.*

Theorem 2. *Given system* (2.7) *and ISS observer in Definition* (5) *with estimation error bound* (2.14)*. Suppose that safe set* $\mathcal{C}$ *is introduced by an Observer-Robust CBF* $h : \mathcal{D} \to \mathbb{R}$ *including extended class* $\mathcal{K}$ *function* α*. If the initial conditions satisfy*

$$\hat{x}(0) \in \hat{\mathcal{D}}_0 = \{\hat{x} \in \mathcal{C} : h(\hat{x}) \geq \gamma_h M_\delta(0)\},$$

$$x(0) \in \mathcal{D}_0 = \{x \in \mathcal{C} : \|x(0) - \hat{x}(0)\| \leq \delta\},$$

then any Lipschitz continuous estimate-feedback controller $u = \pi(t, \hat{x}, y) \in K_{orcbf}(t, \hat{x}, y)$ *where*

$$K_{orcbf}(t, \hat{x}, y) = \{u \in U : \frac{\partial h(\hat{x})}{\partial \hat{x}} p(\hat{x}, y) + \frac{\partial h(\hat{x})}{\partial \hat{x}} q(\hat{x}, y)u \geq -\alpha(h(\hat{x}) - \gamma_h M_\delta(t)) + \gamma_h \dot{M}_\delta(t)\}$$

renders the system safe from the initial-condition sets $\mathcal{D}_0, \hat{\mathcal{D}}_0$*.*

If a linear class $\mathcal{K}$ function and observer error bound are selected as $\alpha(r) = \gamma_\alpha r$ with $\dot{M}_\delta \leq -\gamma_\alpha M_\delta(t)$, the set $K_{\mathrm{orcbf}}(t, \hat{x}, y)$ reduces into following form

$$K_{\mathrm{orcbf}}(t, \hat{x}, y) = \{u \in U : \frac{\partial h(\hat{x})}{\partial \hat{x}} p(\hat{x}, y) + \frac{\partial h(\hat{x})}{\partial \hat{x}} q(\hat{x}, y)u \geq -\alpha(h(\hat{x}))\}$$

Hence, the condition of Theorem 2 simplifies and does not include the bound $M_\delta(t)$ or Lipschitz constant γ_h. Furthermore, if the observer can be defined as $\dot{M}_\delta \leq -\gamma_\alpha M_\delta$, then a safe control input does not require information of M_δ or γ_h. This satisfies the general principle in control theory that observers should converge faster than controllers [40].

2.4.2 A Tunable CBF-based Method with Bounded Error Observers

In the previous section, an ISS observer is assumed to be used in the closed loop system. However, BE observer can also be used to relax the ISS estimation error bound condition. Then, CBF constraints can be written with bounded error assumptions by the "tunable CBF" method [40], inspired by [27].

Definition 7. *If there exist a continuous, non-increasing function* $\kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *with* $\kappa(0) = 1$ *and a class* $\mathcal{K}$ *function* α *satisfying*

$$\sup_{u \in U} \left\{ \frac{\partial h(x)}{\partial x} f(x) + \frac{\partial h(x)}{\partial x} g(x)u + \alpha(h(x)) \right\} \geq \kappa(h(x)) \| \frac{\partial h(x)}{\partial x} g_d(x) \| \bar{d}, \quad \forall x \in \mathcal{C}$$

Then, a continuously differentiable function $h : X \to \mathbb{R}$ is a Tunable Robust CBF (TRCBF) for system (2.7)*.*

Example of function $\kappa(r)$ in Definition 7 can be $\kappa(r) = \frac{2}{1+\exp(r)}$. Consider a TRCBF h for system (2.7), the set of safe control inputs satisfying

$$K_{\text{trcbf}}(x) = \{u \in U : \frac{\partial h(x)}{\partial x} f(x) + \frac{\partial h(x)}{\partial x} g(x)u - \kappa(h(x))\|\frac{\partial h(x)}{\partial x} g_d(x)\|\bar{d} \geq -\alpha(h(x))\} \tag{2.15}$$

render the system safe. The function $\kappa(r)$ reduces the conservatism because the set of control inputs in $K_{\text{trcbf}}(x)$ is larger if the system (2.7) is away from the boundary of the set $\mathcal{C}$. Hence, barrier function is used to adjust this relaxation in the domain of $\kappa(r)$. This argument is coincides with Nagumo's Theorem [14], which relies on the following sufficient condition for safety: $x \in \partial\mathcal{C} \implies \dot{h}(x) \geq 0$.

Although the TRCBF is useful to relax conservatism in QP program, the original method does not consider observer-controller interconnections. To include the estimated state's impact into the TRCBF, the notion of bounded observer can be defined.

Definition 8. *An observer is a Bounded-Error (BE) observer if there exist a bounded set $D(\hat{x}_0) \subset X$ and a time-varying bounded set $P(t, \hat{x}) \subset X$ such that*

$$x_0 \in D(\hat{x}_0) \implies x(t) \in P(t, \hat{x}) \quad \forall t \in I. \tag{2.16}$$

A Bounded Error (BE) Observer in Definition 8 is more general than an ISS observer in Definition 5 because the sets $D(\hat{x}_0) = \{x : \|x - \hat{x}_0\| \leq \delta\}$ and $P(t, \hat{x}) = \{x : \|x - \hat{x}(t)\| \leq M_\delta(t)\}$ do not have to be norm bounded and can be time varying, e.g., zonotopes, intervals, or sub-level sets of sum-of-squares polynomials. Now, a modification of the TRCBF condition including estimated states can be found in following theorem.

Theorem 3. *Consider system* (2.7) *with a BE observer and safe set $\mathcal{C}$. Assume that a continuously differentiable function $h : X \to \mathbb{R}$ is a Tunable Robust-CBF for the system. Let $\pi : \mathbb{R}_{\geq 0} \times X \to U$ be an estimate-feedback controller, piecewise-continuous in the first argument and Lipschitz continuous in the second, satisfying*

$$\pi(t, \hat{x}) \in \bigcap_{x \in P(t,\hat{x})} K_{trcbf}(x),$$

where K_{trcbf} is defined in (2.15)*. Then the observer-controller renders the system safe from the sets of initial conditions $x(0) \in X_0 = D(\hat{x}_0)$ and $\hat{x}_0 \in \hat{X}_0 = \{\hat{x}_0 : P(0, \hat{x}_0) \subset \mathcal{C}\}$.*

Proof. The total derivative of h for any $x \in \partial\mathcal{C}$ and $\pi(t,\hat{x}) \in K_{\text{trcbf}}(x)$ satisfies

$$\begin{aligned}&\dot{h} = \frac{\partial h(x)}{\partial x} f(x) + \frac{\partial h(x)}{\partial x} g(x)\pi(t,\hat{x}) + \frac{\partial h(x)}{\partial x} g_d(x) w(t) \geq \\ &\frac{\partial h(x)}{\partial x} f(x) + \frac{\partial h(x)}{\partial x} g(x)\pi(t,\hat{x}) - \kappa(0)\|\frac{\partial h(x)}{\partial x} g_d(x)\|\bar{w} \geq -\alpha(0) = 0\end{aligned} \tag{2.17}$$

since $h(x) = 0$, $\kappa(0) = 1$, and $x(t) \in P(t,\hat{x})$. Hence, for any $x \in \partial\mathcal{C}$, $\dot{h} \geq 0$ leads to the system being safe. □

Generally, Theorem 3 is difficult to apply unless following inequalities are satisfied for all $\hat{x} \in \mathcal{C}$

$$\begin{aligned}&a(t,\hat{x}) \leq \inf_{x \in P(t,\hat{x})} \left\{ \frac{\partial h(x)}{\partial x} f(x) - \kappa(h(x))\|\frac{\partial h(x)}{\partial x} g_d(x)\|\bar{w} + \alpha(h(x)) \right\} \\ &b(t,\hat{x}) \leq \frac{\partial h(x)}{\partial x} g(x) \leq c(t,\hat{x}) \quad \forall x \in P(t,\hat{x})\end{aligned} \tag{2.18}$$

with known functions $a, b, c : \mathbb{R}_{\geq 0} \times X \to \mathbb{R}$, piecewise continuous in the first argument and Lipschitz continuous in the second. The functions a, b, c can be found by Lipshitz continuity arguments.

2.4.3 CBF-Based Safety with Stochastic Estimators

Alternative methods to eliminate unknown estimation error bounds are using probabilistic inequalities such as Conditional Value at Risk (CVaR). Eetimated states are treated as random states to provide constraints under a user defined risk level. Nemirovski and Shapiro initially presented new method leading to CVaR constraint for chance constraint program [44]. In reference [45], CVaR is used to demonstrate the worst-case uncertainty and unbounded noise of system for both CBF and CLF constraints. In addition, Extended Kalman Filter (EKF) and one-sided Chebyshev inequalities were used to demonstrate convex relaxation of primal constraints [46] for swarm drone collision avoidance. Moreover, a general approach to convexifying non-convex Probability Distribution Functions (PDF) was presented in [46]. Unfortunately, these techniques cannot provide absolute guarantees for collision avoidance because concentration inequalities and risk metrics can only ensure safety within small time intervals. Also, convexifying these inequalities can be hard so they may not be practical for real robotic applications.

In stochastic systems, necessary control inputs to avoid collisions can be reduced by applying a user-defined Lipschitz constant to the estimated error state of the Kalman filter [32].

Although the theorems presented in [32] do not depend on the variance of process noise and provide a linear constraint in the control input, process noise affects safety due to the impact of the Kalman filter [41]. Additionally, user defined safety probability for collision avoidance is not considered in [41].

2.5 Decentralized CBF-Based Collision Avoidance for Multi-Robot Systems

Coordination in multi-robot systems has been a subject of research for a long time. A primary objective may be to execute a given task while maintaining formation for the swarm. At a higher planning level, decentralized auction algorithms play a central role for efficient task assignment [47]. However, integrating collision avoidance mechanisms into the high-level planning layer is generally handled separately [24]. Therefore, the completion of the given task is not guaranteed by these strategies. The main reason for this problem is that collision avoidance becomes more critical as the number of robots in the swarm increases [48]. Decentralization of swarm control using the CBF method was first considered in [24]. This research assumes constant velocity of nearby robots to implement the CBF constraint for each individual robot. However, this assumption is removed by distributing the safety constraint among swarm members in [25]. Position synchronization (cohesion), attitude synchronization/velocity matching (alignment), and collision avoidance (separation) are mostly considered separately in many approaches, while the CBF method provides a unifying solution for all of them [25]. A more detailed analysis for deterministic systems can be found in [49].

2.6 Conclusion and Research Questions

This chapter has reviewed control theoretic methods to provide safety guarantees, applicable to dynamic mobile robotic systems, with an emphasis on methods based on the notion of control barrier functions. From this review of existing methodologies, the following research questions can be identified, which will be explored further in the following chapters:

1. Research question 1: How can we design distributed and safe collision avoidance systems under bounded range sensor noise and system uncertainties? What is the impact of relative state estimation on the collision avoidance system?

2. Research question 2: How can we increase safety for collision avoidance between robots using relative state estimation if we have stochastic (possibly unbounded) range sensor noise?

CHAPTER 3 PROBLEM STATEMENT

In this chapter, we present the problem under study, which focuses on designing decentralized collision avoidance strategies for two robots under relative sensing uncertainty. To address this problem, in the rest of the thesis we will adapt and develop methods based on control barrier functions to ensure or at least improve safety under imperfect information.

3.1 Two-Robot System under Study

In this section, we formulate the collision avoidance problem with relative state estimation. In Figure 3.1, two robots are equipped each with two UWB range sensors, such that each pair or sensors on distinct robots can provide noisy range measurements. We aim to investigate the effects of relative state estimation on collision avoidance. Each robot has a safety radius R to define safe relative motions, i.e., the two circles of radious R should not overlap.

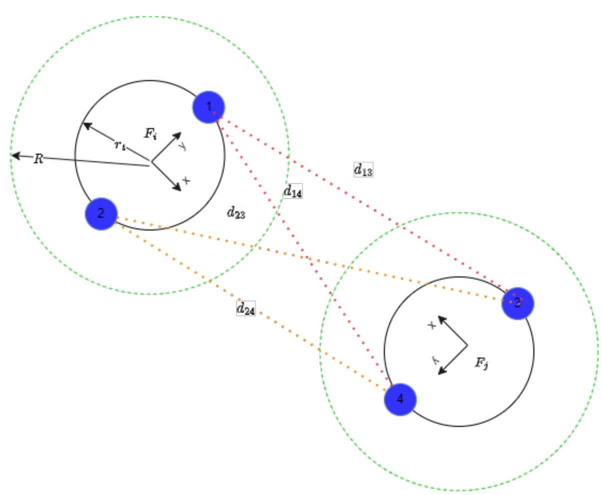

Figure 3.1 Robots using range sensors (in blue), with their unsafe regions (green disks).

Next, we describe the robot kinematics and related measurements. Two omnidirectional robots move in the 2-dimensional Euclidean place $\mathcal{E}$, equipped with a fixed frame $\mathcal{F}_{\mathsf{g}}$. For each robot $i \in \{1, 2\}$, we attach a frame $\mathcal{F}_{\mathsf{i}}$ centered at a point p_i on the robot. The heading of robot i, which is the angle between the x-axes of $\mathcal{F}_{\mathsf{g}}$ and $\mathcal{F}_{\mathsf{i}}$, is denoted ψ_i. Robot i's linear velocity vector with respect to $\mathcal{F}_g$ is denoted $\boldsymbol{v}_{i/\mathsf{g}}$, and its angular velocity is $\omega_{\mathsf{i/g}} \in \mathbb{R}$. The rotation matrix from $\mathcal{F}_\mathsf{g}$ to $\mathcal{F}_\mathsf{i}$ is denoted

$$R_\mathsf{i}^\mathsf{g} = \begin{bmatrix} \cos\psi_i & -\sin\psi_i \\ \sin\psi_i & \cos\psi_i \end{bmatrix}.$$

The kinematics of robot i can be written in coordinates

$$\dot{p}_i^g = R_i^g\, \boldsymbol{v}_{i/\mathsf{g}}^\mathsf{i}, \quad \dot{\psi}_i = \omega_{\mathsf{i}/\mathsf{g}} \tag{3.1}$$

or more compactly

$$\dot{\boldsymbol{x}}_\mathsf{i}^\mathsf{g} = g(\boldsymbol{x}_i^g)\boldsymbol{u_i}, \tag{3.2}$$

where $\boldsymbol{x}_i^g = \begin{bmatrix}(p_i^\mathsf{g})^T & \psi_i\end{bmatrix}^T$ and $\boldsymbol{u}_i = \begin{bmatrix}\left(\boldsymbol{v}_{i/\mathsf{g}}^\mathsf{i}\right)^T & \omega_{\mathsf{i}/g}\end{bmatrix}^T$ are the state and control input vectors, and

$$g(\boldsymbol{x}_i^g) = g(\psi_i) = \begin{bmatrix} R_\mathsf{i}^\mathsf{g} & 0_{2\times 1} \\ 0_{1\times 2} & 1 \end{bmatrix}. \tag{3.3}$$

Define the relative position vector between robot 1 and robot 2 as $\boldsymbol{p}_{12} = p_2 - p_1$. By Coriolis' formula, the relative velocity can be written as

$$^{\mathsf{g}}\dot{\boldsymbol{p}}_{12} = \boldsymbol{v}_{2/g} - \boldsymbol{v}_{1/g} = {}^{1}\dot{\boldsymbol{p}}_{12} + \boldsymbol{\omega}_{1/g} \times \boldsymbol{p}_{12},$$

where the left superscript notation is used to indicate the frame with respect to which a time-derivative is taken [50]. The relative rotation matrix between the two robots is $R_2^1 = (R_1^\mathsf{g})^T R_2^\mathsf{g}$. Define

$$[\omega_{\mathsf{i}/\mathsf{g}}]_\times = \begin{bmatrix} 0 & -\omega_{\mathsf{i}/\mathsf{g}} \\ \omega_{\mathsf{i}/\mathsf{g}} & 0 \end{bmatrix}.$$

Consequently, the relative equations of motion can be written in coordinates in frame $\mathcal{F}_1$

$$^{1}\dot{\boldsymbol{p}}_{12}^1 = \frac{d}{dt}\boldsymbol{p}_{12}^1 = -[\omega_{1/\mathsf{g}}]_\times \boldsymbol{p}_{12}^1 + R_2^1\, \boldsymbol{v}_{2/\mathsf{g}}^2 - \boldsymbol{v}_{1/\mathsf{g}}^1.$$

The relative orientation $\psi_{12} \coloneqq \psi_2 - \psi_1$ between robot frames satisfies

$$\dot{\psi}_{12} = \dot{\psi}_2 - \dot{\psi}_1 = \omega_{2/g} - \omega_{1/g}.$$

Let $\boldsymbol{x}_{12}^1 \coloneqq \begin{bmatrix}(\boldsymbol{p}_{12}^1)^T & \psi_{12}\end{bmatrix}^T$. Overall the relative motion of robot 2 in frame $\mathcal{F}_1$ can be described by

$$\frac{d}{dt}\boldsymbol{x}_{12}^1 = \begin{bmatrix} -[\omega_{1/\mathsf{g}}]_\times \boldsymbol{p}_{12}^1 + R_2^1\, \boldsymbol{v}_{2/\mathsf{g}}^2 - \boldsymbol{v}_{1/\mathsf{g}}^1 \\ \omega_{2/g} - \omega_{1/g} \end{bmatrix} \tag{3.4}$$

or more succinctly as

$$\dot{\boldsymbol{x}}_{12}^{1} = f(\boldsymbol{x}_{12}^{1}, \boldsymbol{u}_1, \boldsymbol{u}_2). \tag{3.5}$$

Robot 1 carries two range sensors indexed by $n \in \{1, 2\}$ and robot 2 carries two other range sensors indexed by $m \in \{3, 4\}$. The locations of each sensor can be selected on the robots' bodies arbitrarily, one possible configuration being illustrated on Figure 3.1. The tag coordinates r_n^1 for $n \in \{1, 2\}$ and r_m^2 for $m \in \{3, 4\}$ are known in the reference frame of the robot to which these tags are attached. Each range sensor obtains two noisy distance measurements with the sensors on the other robot, resulting in a total of four range measurements gathered by each robot. The noisy range measurements at some instant are assumed to be of the form

$$\tilde{d}^{nm} = d^{nm} + \sigma_v^{nm}, \tag{3.6}$$

where σ_v^{nm} is a bounded noise, and the distance d^{nm} satisfies the relation

$$d^{nm}(\boldsymbol{x}_{12}^{1}) = \| \, \boldsymbol{p}_{12}^{1} + R_2^1 \, r_m^2 - r_n^1 \|. \tag{3.7}$$

The real system kinematics are described by (3.2). Each robot estimates its own states in the global frame $\mathcal{F}_{\mathrm{g}}$ and uses these estimates as feedback. In addition, relative state estimation is used to define collision avoidance maneuvers. Hence, robot kinematics can be modeled as

$$\dot{\boldsymbol{x}}_1^{\mathrm{g}} = g(\boldsymbol{x}_1^{\mathrm{g}})\boldsymbol{u_1}, \tag{3.8}$$

$$\dot{\boldsymbol{x}}_2^{\mathrm{g}} = g(\boldsymbol{x}_2^{\mathrm{g}})\boldsymbol{u_2}, \tag{3.9}$$

where the feedback controls use estimated states with $\boldsymbol{u_1} = \pi(\hat{\boldsymbol{x}}_1^{\mathrm{g}}, \hat{\boldsymbol{x}}_{12}^{1})$ and $\boldsymbol{u_2} = \pi(\hat{\boldsymbol{x}}_2^{\mathrm{g}}, \hat{\boldsymbol{x}}_{21}^{2})$. The feedback controls are affected by relative state estimates when the collision avoidance mechanism is active. An example of this type of feedback controller can be found in the passivity-based flocking control literature [51] as follows:

$$\boldsymbol{u_1} = \begin{bmatrix} \hat{\boldsymbol{p}}_{12}^{1} \\ \sin(\hat{\psi}_{12}) \end{bmatrix} + \begin{bmatrix} \hat{R}_{\mathrm{g}}^{1} & 0_{2\times 1} \\ 0_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_d(\hat{\boldsymbol{p}}_1^{\mathrm{g}}) \\ 0 \end{bmatrix}, \tag{3.10}$$

$$\boldsymbol{u_2} = \begin{bmatrix} \hat{\boldsymbol{p}}_{21}^{2} \\ \sin(\hat{\psi}_{21}) \end{bmatrix} + \begin{bmatrix} \hat{R}_{\mathrm{g}}^{2} & 0_{2\times 1} \\ 0_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_d(\hat{\boldsymbol{p}}_2^{\mathrm{g}}) \\ 0 \end{bmatrix}, \tag{3.11}$$

where $\boldsymbol{v}_d$ can be either a proportional controller with estimated positions or a feedforward velocity input in the global frame.

The controllers (4.25)-(3.11) are used for the synchronization of position and orientation purposes. They do not prevent collisions between robots if the control $\boldsymbol{v}_d$ of each robot shares the same values. When the first terms of (4.25) are not active, robots can freely move independently of each other. However, relative estimations can risk the safety of this system because relative states significantly depend on uncertainties and dual effects of control inputs, leading to unobservable sensor configurations in future states. For this reason, the quality of relative state estimates $\hat{\boldsymbol{x}}^1_{12}$, $\hat{\boldsymbol{x}}^2_{21}$ is a main consideration for collision avoidance between robots. Even when the two robots have deterministic dynamics, safe collision avoidance is not straightforward under estimated relative states $\hat{\boldsymbol{x}}^1_{12}$, $\hat{\boldsymbol{x}}^2_{21}$. Hence, we want to develop a CBF method to account for relative state estimation uncertainties for safe collision avoidance. For our problem setup, each robot initially moves independently, following its own trajectory. However, the robots also need to avoid colliding with each other, meaning that the discs of radius R around another robot's center are not allowed to overlap. Therefore, our goal is to design an additional collision avoidance system that activates only when the robots are relatively close, ensuring collision avoidance even under uncertain measurements. To steer the robots without considering synchronization purposes, we use that proportional controller for each initial configuration of a collision scenario. Hence, the CBF QP theorem is an effective solution to realize this application. However, each relative estimation must be considered within the interconnection of the safety framework.

Actually, distance and velocity measurements are collected at different update frequencies. Typically, velocity measurements are obtained at higher frequencies than distance measurements. In order to make a more accurate relative state estimation and reduce uncertainties caused by time delays in measurements at different frequencies, each robot computes velocity measurements using preintegration techniques within its own system and shares integrated velocity measurements among robots for every moment when distance measurements are sent. Generally, when this integration is not performed, it creates a time-delayed estimation problem. Our problem, along with the assumption that these effects are minimized, keeps the states in our system model independent of the problems caused by time delays. In other words, our collision avoidance system is considered independent of time delays. This assumption makes it easier to mathematically analyze our collision avoidance system, but we are looking for a collision avoidance system where each robot can operate independently of integrated velocity measurements in order to reduce these effects.

CHAPTER 4 DECENTRALIZED COLLISION AVOIDANCE WITH BOUNDED OBSERVER ERROR

In this chapter, we first review a *decentralized* CBF-based collision avoidance method for the two-robot system presented in Chapter 3. This method provides safety guarantees in the absence of any motion uncertainty and under perfect state information about the robots. Following this, we analyze the observer and safety interconnection for the collision avoidance problem under range measurement uncertainties. Our main focus is the effect of relative state estimation on collision avoidance.

In this thesis, decentralization means the the control inputs of each robot, enforcing safety, are computed independently of each other, i.e., without coordination. However, we still require communication of control inputs between robots for relative state estimation. This become useful when time delays between communicated signals are significant. The decentralization of control inputs reduces the complexity of the computations, especially if the number of robots were to increase.

In the next section, we examine collision avoidance with an observer used solely for estimating the other robot's state. This estimate does not affect the system dynamics during the feedback control of each robot when the robots are far from each other. Hence, the observer's outputs can be considered as feedforward inputs in the CBF QP when the robots are close to each other. Finally, the results in this chapter only ensure safety if certain bounds on the relative estimation errors are maintained, which requires enforcing some observability conditions. These conditions may not hold for some relative configurations of the range sensors, .e.g., if they are almost aligned. How to adjust trajectories to enforce appropriate observability conditions is left for future work.

4.1 Decentralized CBF-Based Collision Avoidance Under Perfect State Feedback

In this section, known deterministic systems are considered for two robot collision avoidance problems. We present decentralized CBF based theorem for collision avoidance between robots to relax implementation problems of quadratic programs due to time delays of communicated UWB sensors. By this way, we can reduce computational requirements in real time applications and time delay effects on the control inputs. Following the CBF methodology outlined in Chapter 2, we define the safe set $\mathcal{C}$ as the set of states where the two robots are not colliding. Let R be a desired safety radius around each robot, define the continuously differentiable function $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$h(\boldsymbol{p}_{12}^g) = \|\boldsymbol{p}_{12}^g\|^2 - 4R^2, \tag{4.1}$$

and let

$$\mathcal{C} \coloneqq \left\{\boldsymbol{p}_{12} \in \vec{\mathcal{E}} \middle| h(\boldsymbol{p}_{12}^g) \geq 0\right\}. \tag{4.2}$$

Note that $\mathcal{C}$ is in fact independent of the (orthonormal) frame in which the relative position $\boldsymbol{p}_{12}$ is expressed in coordinates, by a property of the Euclidean norm.

The following results are considered in [25].

Theorem 4. *Let $\boldsymbol{p}_{12}^1 \mapsto \boldsymbol{v}_{1/\mathsf{g}}^1(\boldsymbol{p}_{12}^1)$, $\boldsymbol{p}_{21}^2 \mapsto \boldsymbol{v}_{2/\mathsf{g}}^2(\boldsymbol{p}_{21}^2)$ be a pair of Lipschitz continuous state feedback controllers satisfying, for all vectors $\boldsymbol{p}_{12} \in \vec{\mathcal{E}}$, the inequality*

$$\left(\boldsymbol{p}_{12}^1\right)^T \boldsymbol{v}_{1/\mathsf{g}}^1 + \left(\boldsymbol{p}_{21}^2\right)^T \boldsymbol{v}_{2/\mathsf{g}}^2 \leq \alpha(h(\boldsymbol{p}_{21}^{\mathsf{g}})), \tag{4.3}$$

where α is an extended class $\mathcal{K}$ function. Then these controls render the system (3.4) *safe with respect to the set* (4.2)*.*

Proof. Let α be an extended class $\mathcal{K}$ function. Then 2α is also an extended class $\mathcal{K}$ function. Given the robot kinematics (3.1), the CBF condition (2.5) can be written

$$2\left(\boldsymbol{p}_{12}^{\mathsf{g}}\right)^T \left(R_2^g\, \boldsymbol{v}_{2/\mathsf{g}}^2 - R_1^g\, \boldsymbol{v}_{1/\mathsf{g}}^1\right) + 2\alpha(h(\boldsymbol{p}_{21}^{\mathsf{g}})) \geq 0, \tag{4.4}$$

which gives immediately the condition (4.3). □

Enforcing the condition (4.3) requires coordination between the two robots when choosing their respective inputs $\boldsymbol{v}_{2/\mathsf{g}}^2$ and $\boldsymbol{v}_{1/\mathsf{g}}^1$. A weaker but sufficient condition to enforce (4.3) is for the inputs to satisfy the condition of the following corollary. An operational advantage of this condition is that it can be implemented in a decentralized manner, with no communication between the robots. A disadvantage however is that the resulting inputs may exercise more action than what is strictly necessary to ensure no collision. Note that control signal can be derived as a formation control [49] or it can be designed as feedback control to reach the target.

Corollary 1. *Let $\boldsymbol{p}_{12}^1 \mapsto \boldsymbol{v}_{1/\mathsf{g}}^1(\boldsymbol{p}_{12}^1)$, $\boldsymbol{p}_{21}^2 \mapsto \boldsymbol{v}_{2/\mathsf{g}}^2(\boldsymbol{p}_{21}^2)$ be a pair of Lipschitz continuous state* *feedback controllers satisfying, for all vectors* $\boldsymbol{p}_{12} \in \vec{\mathcal{E}}$, *the inequalities*

$$\begin{aligned} \left(\boldsymbol{p}_{12}^{1}\right)^{T} \boldsymbol{v}_{1/\mathsf{g}}^{1} &\leq w_1\, \alpha(\|\, \boldsymbol{p}_{12}^{1}\|^2 - 4R^2), \\ \left(\boldsymbol{p}_{21}^{2}\right)^{T} \boldsymbol{v}_{2/\mathsf{g}}^{2} &\leq w_2\, \alpha(\|\, \boldsymbol{p}_{21}^{2}\|^2 - 4R^2), \end{aligned} \tag{4.5}$$

where α *is an extended class* $\mathcal{K}$ *function, and* w_1, w_2 *are positive scalars such that* $w_1 + w_2 = 1$. *Then these controllers render the system* (3.4) *safe with respect to the set* (4.2).

Proof. Summing the inequalities in (4.5) gives (4.3). □

Remark 2. *Note that in Corollary 1, the coordinates of the vector* $\boldsymbol{p}_{12}$ *are written in frame* $i \in \{1, 2\}$ *to implement the controller* $\boldsymbol{u}_i$ *on robot* i.

When the robots pick velocity controls satisfying the constraints (4.5), the safety of the system with respect to the set $\mathcal{C}$ in (4.2) is ensured, but only in the absence of perturbations on the dynamics and with perfect knowledge of the relative state at each robot. In our case, we only have access to a state estimate $\hat{\boldsymbol{x}}_{12}^{1}$ at robot 1 and $\hat{\boldsymbol{x}}_{21}^{2}$ at robot 2.

Example 1. *Suppose* $R = 1/2$. *Consider two robots traveling directly toward each other, i.e., with* $\psi_{12} = \pi$, *and trying to pass each other. Suppose that the estimated relative position is* $\hat{\boldsymbol{p}}_{21}^{2} = \hat{\boldsymbol{p}}_{12}^{1} = [2, 0]^T$, *whereas the true relative position is* $\boldsymbol{p}_{21}^{2} = \boldsymbol{p}_{12}^{1} = [1, 0]^T$. *Then, taking* $\alpha(h) = 2h$, $w_1 = w_2 = 1/2$ *in* (4.5), *the input velocities* $\boldsymbol{v}_{1/\mathsf{g}}^{1} = \boldsymbol{v}_{2/\mathsf{g}}^{2} = [1, 0]^T$ *satisfy these constraints when the estimates* $\hat{\boldsymbol{p}}_{21}^{2}$, $\hat{\boldsymbol{p}}_{12}^{1}$ *are used directly. Nonetheless, since* $h(\boldsymbol{p}_{12}^{\mathsf{g}}) = 0$, *no input* $\boldsymbol{v}_{1/\mathsf{g}}^{1}$, $\boldsymbol{v}_{2/\mathsf{g}}^{2}$ *with positive first component can satisfy these constraints for the true relative position, and so the use of the estimates results in a violation of the decentralized safety constraints, and even of the less restrictive safety constraint* (4.3).

4.2 Decentralized CBF-Based Collision Avoidance with ISS Observers

In this section, we assume that each robot has a access to full state feedback in the fixed global reference frame and to an estimate of the relative state of the other robot for collision avoidance. In this setup, the observers are used for sensing purposes for the robot states in order to avoid collisions. Hence, the closed loop controller is not affected by the safety layer when the collision avoidance system is not active. Then, our purpose is to analyze the interconnection with the observer and safety filter using estimated relative states.

Hence we have the following kinematic model with range measurements for robot 1

$$\begin{aligned} \dot{\boldsymbol{p}}_1^{\mathrm{g}} &= R_1^{\mathrm{g}}\, \boldsymbol{v}_{1/\mathrm{g}}^1(\boldsymbol{p}_1^{\mathrm{g}}), \\ \tilde{d}^{nm} &= \|\, \boldsymbol{p}_{12}^1 + R_2^1\, r_m^2 - r_n^1 \| + \sigma_v^{nm}, \end{aligned} \tag{4.6}$$

where σ_v^{nm} are bounded uncertainties. Robot 2's model is the same as robot 1. We use relative position estimates $\hat{\boldsymbol{p}}_{12}^1$ and $\hat{\boldsymbol{p}}_{21}^2$ for collision avoidance. We assume that the estimated positions $\hat{\boldsymbol{p}}_{12}^1$ and $\hat{\boldsymbol{p}}_{21}^2$ are obtained by an ISS observer as in Definition 5. Then, we can introduce a continuously differentiable, non-increasing function related to relative estimation error bounds $M_{\delta_1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $M_{\delta_2} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\|\boldsymbol{p}_{12}^1(0) - \hat{\boldsymbol{p}}_{12}^1(0)\| \leq \delta_1 \Rightarrow \|\boldsymbol{p}_{12}^1(t) - \hat{\boldsymbol{p}}_{12}^1(t)\| \leq M_{\delta_1}(t) \quad \forall t \in I, \tag{4.7}$$

$$\|\boldsymbol{p}_{21}^2(0) - \hat{\boldsymbol{p}}_{21}^2(0)\| \leq \delta_2 \Rightarrow \|\boldsymbol{p}_{21}^2(t) - \hat{\boldsymbol{p}}_{21}^2(t)\| \leq M_{\delta_2}(t) \quad \forall t \in I. \tag{4.8}$$

We consider the barrier function $h : \mathbb{D} \to \mathbb{R}$, where we introduce a bounded domain $\mathbb{D} \subset \mathbb{R}^2$ to be consistent with Definition 6

$$h(\boldsymbol{p}_{12}^g) = \|\boldsymbol{p}_{12}^g\|^2 - 4R^2. \tag{4.9}$$

Note that this function is using the Euclidean norm hence it is independent of the reference frame, so that $h(\boldsymbol{p}_{12}^g) = h(\boldsymbol{p}_{12}^1) = h(\boldsymbol{p}_{21}^2)$. Then, let us consider the continuously differentiable function $H : \mathbb{R} \times \mathbb{D} \to \mathbb{R}$

$$H(t, \hat{\boldsymbol{p}}_{12}^{\mathrm{g}}) = h(\hat{\boldsymbol{p}}_{12}^{\mathrm{g}}) - \gamma_h M_{\delta_1}^2(t) - \gamma_h M_{\delta_2}^2(t) = \|\hat{\boldsymbol{p}}_{12}^{\mathrm{g}}\|^2 - 4R^2 - \gamma_h M_{\delta_1}^2(t) - \gamma_h M_{\delta_2}^2(t), \tag{4.10}$$

where γ_h is Lipschitz constant of function h on $\mathbb{D}$. We can define the safe set $\hat{\mathcal{C}}(t)$ as

$$\hat{\mathcal{C}}(t) := \left\{ \hat{\boldsymbol{p}}_{12}^{\mathrm{g}} \in \mathbb{R}^2 \middle| h(\hat{\boldsymbol{p}}_{12}^{\mathrm{g}}) - \gamma_h M_{\delta_1}^2(t) - \gamma_h M_{\delta_2}^2(t) \geq 0 \right\}. \tag{4.11}$$

The set $\hat{\mathcal{C}}$ can be explained by Lipschitz continuity because $|h(\boldsymbol{p}_{12}^{\mathrm{g}}) - h(\hat{\boldsymbol{p}}_{12}^{\mathrm{g}})| \leq \gamma_h \|\, \boldsymbol{p}_{12}^{\mathrm{g}} - \hat{\boldsymbol{p}}_{12}^{\mathrm{g}}\|^2 \Rightarrow h(\hat{\boldsymbol{p}}_{12}^{\mathrm{g}}) - \gamma_h \|\, \boldsymbol{p}_{12}^{\mathrm{g}} - \hat{\boldsymbol{p}}_{12}^{\mathrm{g}}\|^2 \leq h(\boldsymbol{p}_{12}^{\mathrm{g}})$. Therefore, if $\hat{\boldsymbol{p}}_{12}^{\mathrm{g}} \in \hat{\mathcal{C}}$, then $0 \leq h(\hat{\boldsymbol{p}}_{12}^{\mathrm{g}}) - \gamma_h M_{\delta_1}^2(t) - \gamma_h M_{\delta_2}^2(t) \leq h(\boldsymbol{p}_{12}^{\mathrm{g}})$, i.e., $\boldsymbol{p}_{12}^{\mathrm{g}} \in \mathcal{C}$. Thus, $\hat{\boldsymbol{p}}_{12}^{\mathrm{g}} \in \hat{\mathcal{C}} \Rightarrow \boldsymbol{p}_{12}^{\mathrm{g}} \in \mathcal{C}$.

Theorem 5. *Consider two ISS observer error bounds $M_{\delta_1}(t), M_{\delta_2}(t)$ and estimates of relative states $\hat{\boldsymbol{p}}_{12}^1$, $\hat{\boldsymbol{p}}_{21}^2$ in each robot body frame. Let α be a positive scalar and assume that the initial* *conditions satisfy*

$$\|\boldsymbol{p}_{12}^1(0)-\hat{\boldsymbol{p}}_{12}^1(0)\| \le \delta_1 \quad \|\boldsymbol{p}_{21}^2(0)-\hat{\boldsymbol{p}}_{21}^2(0)\| \le \delta_2$$
$$h(\hat{\boldsymbol{p}}_{12}^{\mathsf{g}}(0)) - \gamma_h M_{\delta_1}^2(0) - \gamma_h M_{\delta_2}^2(0) \ge 0.$$

Let $\hat{\boldsymbol{p}}_{12}^1 \mapsto \boldsymbol{v}_{1/\mathsf{g}}^1(\hat{\boldsymbol{p}}_{12}^1)$, $\hat{\boldsymbol{p}}_{21}^2 \mapsto \boldsymbol{v}_{2/\mathsf{g}}^2(\hat{\boldsymbol{p}}_{21}^2)$ *be a pair of Lipschitz continuous state feedback controllers satisfying, for all vectors* $\hat{\mathbf{p}}_{12} \in \overrightarrow{\mathcal{E}}$ *and for all* $t \ge 0$*, the inequalities*

$$\left(\hat{\boldsymbol{p}}_{12}^1\right)^T \boldsymbol{v}_{1/\mathsf{g}}^1 \le w_1 \alpha\, h(\hat{\boldsymbol{p}}_{12}^1) - 2\alpha\gamma_h M_{\delta_1}^2(t) - 2\gamma_h M_{\delta_1}(t)\dot{M}_{\delta_1}(t), \tag{4.12}$$
$$\left(\hat{\boldsymbol{p}}_{21}^2\right)^T \boldsymbol{v}_{2/\mathsf{g}}^2 \le w_2 \alpha\, h(\hat{\boldsymbol{p}}_{21}^2) - 2\alpha\gamma_h M_{\delta_2}^2(t) - 2\gamma_h M_{\delta_2}(t)\dot{M}_{\delta_2}(t), \tag{4.13}$$

where w_1*,* w_2 *are positive scalars such that* $w_1 + w_2 = 1$*. Then, these controls render the robots safe with respect to the set* (4.11)*.*

Proof. Given robot models as in (4.6), and barrier function (4.10) including error bounds, consider the extended class $\mathcal{K}$ function $2\alpha \times r$, where α is a positive scalar. If $H(t, \hat{\boldsymbol{p}}_{12}^{\mathsf{g}}) \ge 0$, then $h(\boldsymbol{p}_{12}^{\mathsf{g}}) \ge 0$. Hence, the real states are in the safe set $\mathcal{C}$. Then, $\dot{H}(t, \hat{\boldsymbol{p}}_{12}^{\mathsf{g}}) \ge -\alpha(H(t, \hat{\boldsymbol{p}}_{12}^{\mathsf{g}}))$ can be obtained explicitly as

$$\begin{aligned}&\left(\hat{\boldsymbol{p}}_{12}^1\right)^T \boldsymbol{v}_{1/\mathsf{g}}^1 + \left(\hat{\boldsymbol{p}}_{21}^2\right)^T \boldsymbol{v}_{2/\mathsf{g}}^2 \le 2w\alpha h(\hat{\boldsymbol{p}}_{12}^{\mathsf{g}}) - 2w\alpha\gamma_h M_{\delta_1}^2(t) - 2w\alpha\gamma_h M_{\delta_2}^2(t)\\ &- 2\gamma_h M_{\delta_1}(t)\dot{M}_{\delta_1}(t) - 2\gamma_h M_{\delta_2}(t)\dot{M}_{\delta_2}(t),\end{aligned}$$

by summing the two inequalities in Theorem 5. By applying the comparison Lemma to $\dot{H}(t, \hat{\boldsymbol{p}}_{12}^{\mathsf{g}}) \ge -\alpha(H(t, \hat{\boldsymbol{p}}_{12}^{\mathsf{g}}))$, it can be shown that $H(t, \hat{\boldsymbol{p}}_{12}^{\mathsf{g}}) \ge 0, \forall t \in I$, for initial conditions satisfying $H(t, \hat{\boldsymbol{p}}_{12}^{\mathsf{g}}(0)) \ge 0$. □

Remark 3. *Note that the derivatives of the error bounds satisfy* $\dot{M}_{\delta_1}(t) \le 0$ *and* $\dot{M}_{\delta_2}(t) \le 0$*, because* $M_\delta(t) = \beta(\delta, t) + \eta(\bar{w})$*, and* β *is a class* $\mathcal{KL}$ *function (so* $\dot{M}_\delta(t) = \frac{\partial \beta}{\partial t} < 0$*). If the observer bounds can be adjusted such that* $\dot{M}_{\delta_1}(t) \le -\alpha M_{\delta_1}(t)$ *and* $\dot{M}_{\delta_2}(t) \le -\alpha M_{\delta_2}(t)$*, Theorem 5 does not require information about the error bounds. For instance, bounds can be selected as* $\dot{M}_{\delta_1}(t) = -(\alpha+1)M_{\delta_1}(t)$ *and* $\dot{M}_{\delta_2}(t) = -(\alpha+1)M_{\delta_2}(t)$*. Since* $\dot{H}(t, \hat{\boldsymbol{p}}_{12}^{\mathsf{g}}) \ge -\alpha H(t, \hat{\boldsymbol{p}}_{12}^{\mathsf{g}}) + M_{\delta_1}(t) + M_{\delta_2}(t) \ge -\alpha H(t, \hat{\boldsymbol{p}}_{12}^{\mathsf{g}})$ *gives same result with* α *convergence rate of errors. Hence, if the observer errors converge faster than convergence of the robots into the safe set boundary, there is no need to know the error bounds and Lipschitz constant* γ_h *for the proposed barrier function* $H(t, \hat{\boldsymbol{p}}_{12}^{\mathsf{g}})$*.*

4.3 Decentralized CBF-Based Collision Avoidance with Bounded Estimation Errors

In this section, we consider bounded estimation errors for our problem. Consequently, we search for bounds similar to (2.18) and obtain solutions analogous to those in Theorem 3. In this manner, the ISS observer error definition can be relaxed to accommodate more general observers. The following propositions propose alternative methods to address the robustness issue of the safety constraint, assuming a known bound on the worst-case error of the position estimators. Let us introduce the full system setup of robot-1 as

$$\begin{aligned} \dot{\boldsymbol{p}}_1^{\mathtt{g}} &= R_1^{\mathtt{g}}\, \boldsymbol{v}_{1/\mathtt{g}}^1(\boldsymbol{p}_{12}^1, \boldsymbol{p}_1^{\mathtt{g}}), \\ \tilde{d}^{nm} &= \|\, \boldsymbol{p}_{12}^1 + R_2^1\, r_m^2 - r_n^1\| + \sigma_v^{nm}, \end{aligned} \tag{4.14}$$

In this section, the state feedback can be expressed as either $\boldsymbol{v}_{1/\mathtt{g}}^1(\hat{\boldsymbol{p}}_{12}^1, \boldsymbol{p}_1^{\mathtt{g}})$ or $\boldsymbol{v}_{1/\mathtt{g}}^1(\hat{\boldsymbol{p}}_{12}^1)$, since we are searching for infimum bounds as in (2.18). Additionally, we define the estimation errors as $\boldsymbol{e}_{12}^1 = \boldsymbol{p}_{12}^1 - \hat{\boldsymbol{p}}_{12}^1$ and $\boldsymbol{e}_{21}^2 = \boldsymbol{p}_{21}^2 - \hat{\boldsymbol{p}}_{21}^2$.

In the previous section, we used the ISS observer definition for error bounds. Now, we consider the more general case of bounded estimation error. Thus, we define $M_{\delta_1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $M_{\delta_2} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\|\boldsymbol{p}_{12}^1(0) - \hat{\boldsymbol{p}}_{12}^1(0)\| \leq \delta_1 \Rightarrow \|\boldsymbol{p}_{12}^1(t) - \hat{\boldsymbol{p}}_{12}^1(t)\| \leq M_{\delta_1}(t) \leq \gamma_1 \quad \forall t \in I, \tag{4.15}$$

$$\|\boldsymbol{p}_{21}^2(0) - \hat{\boldsymbol{p}}_{21}^2(0)\| \leq \delta_2 \Rightarrow \|\boldsymbol{p}_{21}^2(t) - \hat{\boldsymbol{p}}_{21}^2(t)\| \leq M_{\delta_2}(t) \leq \gamma_2 \quad \forall t \in I. \tag{4.16}$$

The difference from the previous estimation error bounds is that the time derivatives of $M_{\delta_1}(t)$ and $M_{\delta_2}(t)$ can be positive during unknown time intervals. Secondly, the error bounds $M_{\delta_1}(t)$ and $M_{\delta_2}(t)$ are bounded by known real numbers γ_1 and γ_2. Since the time derivatives of these error bounds are unknown, the theorems from the previous section may not be practical for certain applications. The next theorems address this issue by considering the overall bounds of CBFs under the following assumption.

Assumption 1. *We assume there are γ_1, γ_2, k_b positive numbers such that $\|\, \boldsymbol{e}_{12}^1\| \leq \gamma_1$, $\|\, \boldsymbol{e}_{21}^2\| \leq \gamma_2$, $\|\, \boldsymbol{v}_{1/\mathtt{g}}^1\| \leq k_b$, $\|\, \boldsymbol{v}_{2/\mathtt{g}}^2\| \leq k_b$ for all time t.*

Let us define the constants $\Gamma_1 = \gamma_1 k_b$, $\Gamma_2 = \gamma_2 k_b$.

Proposition 1. *Consider BE observers and robots with barrier functions $h(\hat{\boldsymbol{p}}_{12}^1)$ and $h(\hat{\boldsymbol{p}}_{21}^2)$. Suppose that Assumption 1 is satisfied. Let $\boldsymbol{v}_{1/\mathtt{g}}^1$, $\boldsymbol{v}_{2/\mathtt{g}}^2$ be a pair of output-feedback controllers* *satisfying at all time the constraints*

$$\left(\hat{\boldsymbol{p}}_{12}^{1}\right)^{T}\boldsymbol{v}_{1/\mathrm{g}}^{1}+\Gamma_{1}\leq w_{1}\alpha(\max\{\|\,\hat{\boldsymbol{p}}_{12}^{1}\|-\gamma_{1},0\}^{2}-4R^{2}),$$
$$\left(\hat{\boldsymbol{p}}_{21}^{2}\right)^{T}\boldsymbol{v}_{2/\mathrm{g}}^{1}+\Gamma_{2}\leq w_{2}\alpha(\max\{\|\,\hat{\boldsymbol{p}}_{21}^{2}\|-\gamma_{1},0\}^{2}-4R^{2}),$$

where α is an extended class $\mathcal{K}$ function and w_1, w_2 positive scalars such that $w_1+w_2=1$. Then, these controllers render the system (3.4) *safe with respect to the set* (4.2).

Proof. For robot 1, the left hand side of the inequality (4.5) can be upper bounded using Assumption 1

$$\begin{aligned}\left(\boldsymbol{p}_{12}^{1}\right)^{T}\boldsymbol{v}_{1/\mathrm{g}}^{1}&=\left(\hat{\boldsymbol{p}}_{12}^{1}\right)^{T}\boldsymbol{v}_{1/\mathrm{g}}^{1}+\left(\boldsymbol{e}_{12}^{1}\right)^{T}\boldsymbol{v}_{1/\mathrm{g}}^{1}\\&\leq\left(\hat{\boldsymbol{p}}_{12}^{1}\right)^{T}\boldsymbol{v}_{1/\mathrm{g}}^{1}+\gamma_{1}k_{b}=\left(\hat{\boldsymbol{p}}_{12}^{1}\right)^{T}\boldsymbol{v}_{1/\mathrm{g}}^{1}+\Gamma_{1}.\end{aligned}\tag{4.17}$$

The right-hand side of (4.5) can be lower bounded as follows. By the triangle inequality, we have $\|\,\boldsymbol{p}_{12}^{1}\|\geq\|\,\hat{\boldsymbol{p}}_{12}^{1}\|-\|\,\boldsymbol{e}_{12}^{1}\|$, so

$$\|\,\boldsymbol{p}_{12}^{1}\|\geq\max\{\|\,\hat{\boldsymbol{p}}_{12}^{1}\|-\gamma_{1},0\}.$$

Hence,

$$\alpha(\|\,\boldsymbol{p}_{12}^{1}\|^{2}-4R^{2})\geq\alpha(\max\{\|\,\hat{\boldsymbol{p}}_{12}^{1}\|-\gamma_{1},0\}^{2}-4R^{2}).\tag{4.18}$$

As a result, if the right-hand side of (4.17) is smaller than w_1 times the right-hand side of (4.18), then the first constraint of (4.5) is satisfied. The result follows by proceeding similarly for the second robot. □

Next, we provide an alternative safety condition, still under bounded estimation error.

Proposition 2. *Consider BE observers and robots with barrier functions $h(\hat{\boldsymbol{p}}_{12}^{1})$ and $h(\hat{\boldsymbol{p}}_{21}^{2})$ under Assumption 1. Let $\boldsymbol{v}_{1/\mathrm{g}}^{1}$, $\boldsymbol{v}_{2/\mathrm{g}}^{2}$ be a pair of output-feedback controllers satisfying at all time the constraints*

$$\left(\hat{\boldsymbol{p}}_{12}^{1}\right)^{T}\boldsymbol{v}_{1/\mathrm{g}}^{1}+\Gamma_{1}\leq w_{1}\alpha(\|\,\hat{\boldsymbol{p}}_{12}^{1}\|^{2}-4R^{2}-4R\gamma_{1}-\gamma_{1}^{2})$$
$$\left(\hat{\boldsymbol{p}}_{21}^{2}\right)^{T}\boldsymbol{v}_{2/\mathrm{g}}^{2}+\Gamma_{2}\leq w_{2}\alpha(\|\,\hat{\boldsymbol{p}}_{21}^{2}\|^{2}-4R^{2}-4R\gamma_{2}-\gamma_{2}^{2})$$

where α is an extended class $\mathcal{K}$ function and w_1, w_2 positive scalars such that $w_1+w_2=1$. Then these controllers render the system (3.4) *safe with respect to the set* (4.2).

Proof. By employing a similar approach to the proof of Proposition 1, we aim to demonstrate overall bounds. Suppose the conditions of Proposition 2 are satisfied at all times. We then show that the conditions of Nagumo's theorem, as presented in [14], are met. Consequently, the set (4.2) is forward invariant.

Let $\boldsymbol{p}_{12} \in \partial\mathcal{C}$, i.e., such that $\|\boldsymbol{p}_{12}\| = 2R$. We want to show that necessarily in such a state, the controllers choose inputs $\boldsymbol{v}^1_{1/\mathtt{g}}$ and $\boldsymbol{v}^2_{2/\mathtt{g}}$ such that

$$\left(\boldsymbol{p}^1_{12}\right)^T \boldsymbol{v}^1_{1/\mathtt{g}} \leq 0 \text{ and } \left(\boldsymbol{p}^2_{21}\right)^T \boldsymbol{v}^2_{2/\mathtt{g}} \leq 0. \tag{4.19}$$

By an argument similar to that of Corollary 1, this gives a sufficient condition for the forward invariance of $\mathcal{C}$. To verify that (4.19) indeed holds, we use Assumption 1 and the constraints of the proposition to get

$$\begin{aligned}&\left(\boldsymbol{p}^1_{12}\right)^T \boldsymbol{v}^1_{1/\mathtt{g}} = \left(\hat{\boldsymbol{p}}^1_{12}\right)^T \boldsymbol{v}^1_{1/\mathtt{g}} + \left(\boldsymbol{e}^1_{12}\right)^T \boldsymbol{v}^1_{1/\mathtt{g}} \leq \left(\hat{\boldsymbol{p}}^1_{12}\right)^T \boldsymbol{v}^1_{1/\mathtt{g}} + \gamma_1 k_b \leq \left(\hat{\boldsymbol{p}}^1_{12}\right)^T \boldsymbol{v}^1_{1/\mathtt{g}} + \Gamma_1 \\ &\leq w_1\alpha(\|\,\hat{\boldsymbol{p}}^1_{12}\|^2 - 4R^2 - 4R\gamma_1 - \gamma_1^2),\end{aligned} \tag{4.20}$$

where we use the Cauchy-Schwarz inequality. Then, since $\boldsymbol{p}_{12} \in \partial\mathcal{C}$,

$$\begin{aligned}\alpha(\|\,\hat{\boldsymbol{p}}^1_{12}\|^2 - 4R^2) &= \alpha(\|\,\boldsymbol{p}^2_{21} - \boldsymbol{e}^1_{12}\|^2 - 4R^2) \\ &= \alpha\left(-2\left(\boldsymbol{p}^2_{21}\right)^T \boldsymbol{e}^1_{12} + \|\,\boldsymbol{e}^1_{12}\|^2\right) \\ &\leq \alpha(4R\gamma_1 + \gamma_1^2)\end{aligned} \tag{4.21}$$

by the Cauchy-Schwarz inequality again and the monotonicity of α. But this shows by (4.20) that $\left(\boldsymbol{p}^1_{12}\right)^T \boldsymbol{v}^1_{1/\mathtt{g}} \leq 0$. The same steps can be followed for the second controller. □

Returning to Example 1, suppose $k_b = \gamma_1 = \gamma_2 = 1$ and use the same function α and weights w_1, w_2. The constraints from Proposition 1 then yield

$$\begin{bmatrix}1 & 0\end{bmatrix} \boldsymbol{v}^1_{1/\mathtt{g}} \leq -\frac{1}{2}, \quad \begin{bmatrix}1 & 0\end{bmatrix} \boldsymbol{v}^2_{2/\mathtt{g}} \leq -\frac{1}{2},$$

indicating that the velocity controls appropriately push the robots away from each other, even though the distance between the position estimates is more than the safe distance of 1.

For the same example, the constraints in Proposition 2 give

$$\begin{bmatrix}2 & 0\end{bmatrix} \boldsymbol{v}^1_{1/\mathtt{g}} \leq -1, \quad \begin{bmatrix}2 & 0\end{bmatrix} \boldsymbol{v}^2_{2/\mathtt{g}} \leq -1$$

Thus, in this case, we obtain the same constraints as those given by Proposition 1. However, Proposition 1 may exhibit greater conservatism compared to Proposition 2 when the actual position estimates lie within the set $\mathcal{C}$ or away from its boundary. This conservatism can prevent motion from converging towards the boundary, thereby creating issues when attempting to reach the target.

The difference between the two results can be further explained as follows. In Proposition 2, we have the following inequality

$$\dot{h}(\boldsymbol{p}_{12}^1) \geq \dot{h}(\hat{\boldsymbol{p}}_{12}^1) \geq -\alpha(h(\hat{\boldsymbol{p}}_{12}^1)) + c \geq -\alpha(h(\boldsymbol{p}_{12}^1)) + c \tag{4.22}$$

where $\alpha(r)$ is chosen as $\alpha \times r$ with a positive scalar α, and $c = \gamma_1 k_b + \alpha 4R\gamma_1 + \alpha\gamma_1^2$. Equation (4.22) implicitly represents the constraint for the first robot in Proposition 2, as we separate the safety of relative dynamics according to [49]. Applying the comparison lemma to Proposition 2 gives

$$h(\boldsymbol{p}_{12}^1(t)) \geq e^{-\alpha(t-t_0))} h(\boldsymbol{p}_{12}^1(0)) + \frac{c(1-e^{-\alpha t})}{\alpha} \tag{4.23}$$

If the robot is far from the set boundary $\partial\mathcal{C}$, the constraints in Proposition 2 allow the robots to ignore the effect of uncertainties limiting control variable. As a robot approaches the set boundary, this effect gradually increases and enables the necessary control choice to remain within the set boundary. This means that extended class $\mathcal{K}$ function is important to relax the constraint on the control input far away from set boundary.

Now, consider Proposition 1 for robot-1, we have

$$\dot{h}(\boldsymbol{p}_{12}^1) \geq \dot{h}(\hat{\boldsymbol{p}}_{12}^1) \geq -\alpha(\sigma(\| \hat{\boldsymbol{p}}_{12}^1\| - \gamma_1)^2 - 4R^2) \tag{4.24}$$

where we are using $\sigma(x) = \frac{|x|}{1+e^{-x}}$ to approximate the maximum function in Proposition 1. Due to complexity, we can not apply the comparison lemma here to understand the control relaxation. However, the right hand side of (4.24) is invalid for the extended class $\mathcal{K}$ definition when $x := \| \hat{\boldsymbol{p}}_{12}^1\| - \gamma_1 \mapsto \pm 1$, since $0 \leq \sigma(x) \leq 1$ and $\dot{h}(\hat{\boldsymbol{p}}_{12}^1) \geq 0$ far way from the boundary if safe distance $4R^2 \geq 1$. Hence, each level set in the safe set $\mathcal{C}$ tends to push the system states away from its previous set. This means that robots can not approached each other easily even though they are not close. Especially, if we assume that γ_1 also varies with real positions, Proposition 1 can lead to the generation of numerous control inputs in the opposite direction of robot motion. This is the primary issue that the class $\mathcal{K}$ function seeks to address in the CBF literature.

Now, consider Example 1 with an estimated position of $\hat{\boldsymbol{p}}_{21}^{2} = [3, 0]$ and set input bound $k_b = 2$, $\gamma_1 = \gamma_2 = 1$ and $\kappa = 1$ for all time. Under these conditions, Proposition 1 yields,

$$\begin{bmatrix}3 & 0\end{bmatrix} \boldsymbol{v}_{1/\mathsf{g}}^{1} \leq 1, \quad \begin{bmatrix}3 & 0\end{bmatrix} \boldsymbol{v}_{2/\mathsf{g}}^{2} \leq 1,$$

whereas the Proposition 2 result

$$\begin{bmatrix}3 & 0\end{bmatrix} \boldsymbol{v}_{1/\mathsf{g}}^{1} \leq 3, \quad \begin{bmatrix}3 & 0\end{bmatrix} \boldsymbol{v}_{2/\mathsf{g}}^{2} \leq 3,$$

Consequently, Proposition 2 gives a larger admissible set of control inputs, which facilitates reaching the target.

4.4 Simulation Studies

In this section, we perform simulations to illustrate the results of this chapter. In these simulations, the robots have a diameter of 0.4 cm and are controlled using a proportional controller in the global reference frame $\mathcal{F}_g$

$$\boldsymbol{u_1} = \begin{bmatrix} \boldsymbol{v}_{1/\mathsf{g}}^{1} \\ \omega_{\mathsf{i}/g} \end{bmatrix} = \begin{bmatrix} R_{\mathsf{g}}^{1} & 0_{2\times 1} \\ 0_{1\times 2} & 1 \end{bmatrix} K_p \begin{bmatrix} (\boldsymbol{p}_{1,ref}^{\mathsf{g}} - \boldsymbol{p}_{1}^{\mathsf{g}}) \\ (\psi_{1,ref} - \psi_1) \end{bmatrix} \tag{4.25}$$

where $K_p \in \mathbb{R}^{3\times 3}$ contains diagonal elements for the control gains. This control signal is then used as a reference signal for the CBF QP programs, as shown in Figure 2.1. Note that the QP program is only executed when the distance is below 2 meters, based on range measurements.

We selected the Extended Kalman Filter (EKF) for relative state estimation, due to its popularity in navigation systems. We ensure empirically, by manual tuning, that this estimator operates as a BE observer for the scenario with bounded noise described below. To implement the EKF for relative state estimation, we apply Euler discretization

$$\frac{d\,\boldsymbol{x}_{12}^{1}}{dt}(kT_s) \approx \frac{\boldsymbol{x}_{12}^{1}((k+1)T_s) - \boldsymbol{x}_{12}^{1}(kT_s)}{T_s},$$

where T_s is the sampling period, and $\boldsymbol{x}_{12}^{1}$ is defined above (3.4). For simplicity, in the following we use the notation $x(kT_s) := x_k$ for any signal x and integer k. Then, we define an additive noise vector $\boldsymbol{w} = [\boldsymbol{w}_1^T, \boldsymbol{w}_2^T]^T \in \mathbb{R}^6$ for velocity measurements, such that we assume that the relative navigation system measures $\tilde{\boldsymbol{u}}_{\boldsymbol{1}} = \boldsymbol{u_1} + \boldsymbol{w}_1$ and $\tilde{\boldsymbol{u}}_{\boldsymbol{2}} = \boldsymbol{u_2} + \boldsymbol{w}_2$. After discretizing the nonlinear relative motion (3.4), with f defined in (3.5), the Jacobian of the system with respect to state and input noise is obtained as

$$A_k = \frac{\partial f}{\partial \boldsymbol{x}_{12}^1}\Big|_{t=kT_s} \approx I_3 + T_s \begin{bmatrix} -[\omega_{1/g,k}]_\times & R_{2,k}^1[\mathbf{1}]_\times \boldsymbol{v}_{2/g,k}^2 \\ \mathbf{0}_{1\times 2} & 0 \end{bmatrix}, \tag{4.26}$$

$$B_k = \frac{\partial f}{\partial \boldsymbol{w}}\Big|_{t=kT_s} \approx T_s \begin{bmatrix} -I_2 & -[\mathbf{1}]_\times \boldsymbol{p}_{12,k}^1 & R_{2,k}^1 & 0 \\ \mathbf{0}_{1\times 2} & -1 & \mathbf{0}_{1\times 2} & 1 \end{bmatrix}. \tag{4.27}$$

The Jacobian $H(\boldsymbol{x}_{12,k}^1) \in \mathbb{R}^{4\times 3}$ of the measurement model with respect to the state is defined as

$$H(\boldsymbol{x}_{12,k}^1) := \frac{\partial}{\partial \boldsymbol{x}_{12}^1}\Big|_{t=kT_s} \begin{bmatrix} d^{13} \\ d^{23} \\ d^{14} \\ d^{24} \end{bmatrix}$$

and, based on (3.7), we have for $n \in \{1,2\}$, $m \in \{3,4\}$,

$$\frac{\partial d^{nm}}{\partial \boldsymbol{x}_{12}^1}\Big|_{t=kT_s} = \begin{bmatrix} \frac{(\boldsymbol{p}_{12,k}^1 + R_{2,k}^1\, r_m^2 - r_n^1)^T}{\| \boldsymbol{p}_{12,k}^1 + R_{2,k}^1\, r_m^2 - r_n^1 \|} & \frac{(\boldsymbol{p}_{12,k}^1 + R_{2,k}^1\, r_m^2 - r_n^1)^T}{\| \boldsymbol{p}_{12,k}^1 + R_{2,k}^1\, r_m^2 - r_n^1 \|} R_{2,k}^1 [\mathbf{1}]_\times\, r_m^2. \end{bmatrix} \tag{4.28}$$

Then, we apply a standard EKF algorithm by computing the following iterates

$$\begin{aligned}
\hat{\boldsymbol{x}}_{12,k+1/k}^1 &= \hat{\boldsymbol{x}}_{12,k}^1 + T_s\, f(\hat{\boldsymbol{x}}_{12,k}^1, \tilde{\boldsymbol{u}}_{1,k}, \tilde{\boldsymbol{u}}_{2,k}) \\
\hat{P}_{k+1/k} &= A_k \hat{P}_k A_k^T + B_k Q_k B_k^T \\
S_{k+1} &= H(\hat{\boldsymbol{x}}_{12,k+1/k}^1) \hat{P}_{k+1/k} H(\hat{\boldsymbol{x}}_{12,k+1/k}^1)^T + V_k \\
K_{k+1} &= \hat{P}_{k+1/k} H(\hat{\boldsymbol{x}}_{12,k+1/k}^1)^T S_{k+1}^{-1} \\
\hat{\boldsymbol{x}}_{12,k+1}^1 &= \hat{\boldsymbol{x}}_{12,k+1/k}^1 + K_{k+1}\left(\boldsymbol{d}_{k+1} - \boldsymbol{d}\left(\hat{\boldsymbol{x}}_{12,k+1/k}^1\right)\right) \\
\hat{P}_{k+1} &= \left(I_3 - K_{k+1} H(\hat{\boldsymbol{x}}_{12,k+1/k}^1)\right) \hat{P}_{k+1/k}
\end{aligned} \tag{4.29}$$

where the state estimates at the prediction and update steps are denoted $\hat{\boldsymbol{x}}_{12,k+1/k}^1$ and $\hat{\boldsymbol{x}}_{12,k}^1$, $\boldsymbol{d}_k \in \mathbb{R}^4$ denotes the vector of four distance measurements and $\boldsymbol{d}(\boldsymbol{x}_{12}^1) \in \mathbb{R}^4$ is the vector with components given by (3.7). In our implementation, we tune the covariance matrices to $Q_k = 0.0025 \cdot I_6$ and $V_k = 0.01 \cdot I_4$ for the process and measurement noise. The initial prediction covariance is set to $\hat{P}_{0/-1} = 0.1 \cdot I_3$ in the simulations.

To simulate the robot trajectories, we introduce bounded noise for the UWB sensors as $\sigma_v^{nm} = \pm 0.5$ (uniform distribution) and velocity measurements $\tilde{\boldsymbol{u}}_{\mathbf{1}}$, where $\boldsymbol{w}_1$ follows a uniform distribution with support ± 0.05 and ± 0.05 rad/s for translational and rotational additive noises respectively. These distributions are implemented using the “unifrnd” function in MATLAB. Note that we send the velocity measurements $\tilde{\boldsymbol{u}}_1$ into the relative state estimation (EKF) module. We set the diagonal elements of the control matrix K_p to 0.05 to stabilize the system before realizing the scenario. The EKF algorithm operates with a sample time of 0.25 seconds for relative state estimation, which is suitable for real-time estimation and control. The control input has maximum and minimum bounds of $(\pm 0.5, \mathrm{m/s}, \pm 0.5, \mathrm{m/s}, \pm 0.15, \mathrm{rad/s})^T$. The robots' initial positions are near (0, 3) and (0, -3), with slight perturbations to avoid deadlocks. Their task is to reach opposite directions on the y-axis in $\mathcal{F}_g$. After several simulation studies, we determined the upper bound on estimation errors for this scenario. We then introduced this bound into the constraints in Proposition 2. Finally, we chose $\alpha(r) = 0.5 \times r$, $w_1 = w_2 = 0.5$, and a radius $R = 0.5$.

4.4.1 Scenario 1: Perfect Model and Measurements

In this scenario, we are interested in evaluating the effect of estimation error on collision avoidance, as addressed by Theorem 5. To assess this theorem, we utilize the Deterministic Extended Kalman Filter (DEKF), as introduced in [52] and reconsidered in [53]. Consequently, we do not introduce uncertainties into the robot model and range measurements, since the ISS observer, defined in Definition 5, is satisfied with a noiseless system and range measurements. To summarize the DEKF, we provide the observer equations presented in [53].

$$
\begin{aligned}
\frac{d}{dt}\hat{\boldsymbol{x}}_{12}^1 &= f(\hat{\boldsymbol{x}}_{12}^1, \tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2) + P(t)H(\hat{\boldsymbol{x}}_{12}^1, t)^T R^{-1}\left(y(t) - d(\hat{\boldsymbol{x}}_{12}^1, t)\right)\\
\frac{d}{dt}P &= A(\hat{\boldsymbol{x}}_{12}^1, t)P(t) + P(t)A(\hat{\boldsymbol{x}}_{12}^1, t)^T + Q - P(t)H(\hat{\boldsymbol{x}}_{12}^1, t)^T R^{-1} H(\hat{\boldsymbol{x}}_{12}^1, t)P(t) + 2\beta P(t)
\end{aligned}
$$

where $A(\hat{\boldsymbol{x}}_{12}^1, t)$ and $H(\hat{\boldsymbol{x}}_{12}^1, t)$ are the continuous versions of the matrices described in (4.26) and (4.28). Note also that matrices Q and R are user-defined constants. For this observer, β determines the lower rate of convergence for the estimation error. To reduce differences between simulations, we calculate the posterior covariance of the EKF algorithm as follows

$$
\hat{P}_{k+1} = \left(I_3 - K_{k+1}H(\check{\boldsymbol{x}}_{12,k+1}^1)\right)\check{P}_{k+1}\left(I_3 - K_{k+1}H(\check{\boldsymbol{x}}_{12,k+1}^1)\right)^T + K_{k+1}V_k {K_{k+1}}^T + 2\beta\check{P}_{k+1} \tag{4.30}
$$

where we can chose $\beta = 0.5$ to satisfy Remark 3. Then, we adopt covariance matrices similar to those used in the previous section and initialize the EKF with an initial state error defined as $\hat{\boldsymbol{x}}_{12,0}^1 = 0.9\boldsymbol{x}_{12,0}^1$. For the CBF QP program, we refer to Corollary 1 and evaluate the impact of two different choices for β, as illustrated in Figure 4.1.

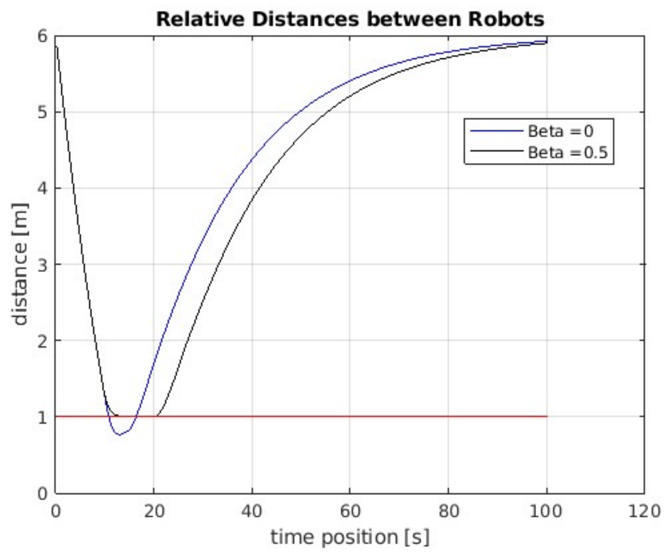

Figure 4.1 Comparison of Corollary 1 with $\beta = 0$ and $\beta = 0.5$.

It can be seen that observer error affects safety in collision avoidance. Additionally, $\beta = 0.5$ provides the desired distances. Note that we did not introduce any noise into the simulation because the DEKF satisfies Definition 5 under noiseless conditions. However, this is impractical in real-life applications. Designing an ISS observer for relative state estimation is considered for future studies and is beyond the scope of this thesis. Nevertheless, the simulation studies in this section show that Theorem 5 is effective under bounded estimation error, satisfying Definition 5, and providing easy implementation as discussed in Remark 3.

4.4.2 Scenario 2: Perfect Model under Relative State Estimation

In the second scenario, the robots are not subject to any disturbances. However, they receive noisy velocity and distance measurements as they estimate the states of other robots for collision avoidance. This scenario is considered because, despite the absence of disturbances in the models, velocity measurements can still be noisy. We compare the CBF constraints outlined in Corollary 1 and Proposition 2, utilizing a safe set with initial distances of 2 meters.

In Figure 4.2, we can see that the robots reach the goal points within both constraints, but it is seen that the safe set cannot be preserved for Corollary 1 in Figure 4.3. This situation proves the effect of relative navigation on the collision system although there is no external disturbances under perfect state estimation.

In Figure 4.4, higher error bounds are used. For this reason, robots do not approach desired safe distance but they are still safe for collision avoidance.

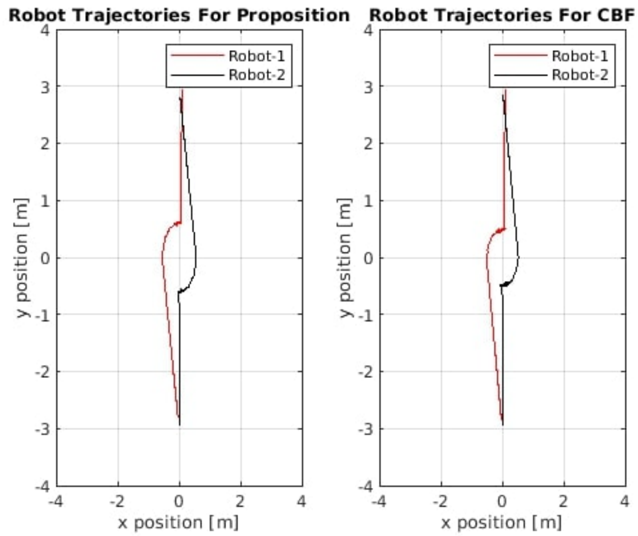

Figure 4.2 2D trajectories for Corollary and Proposition.

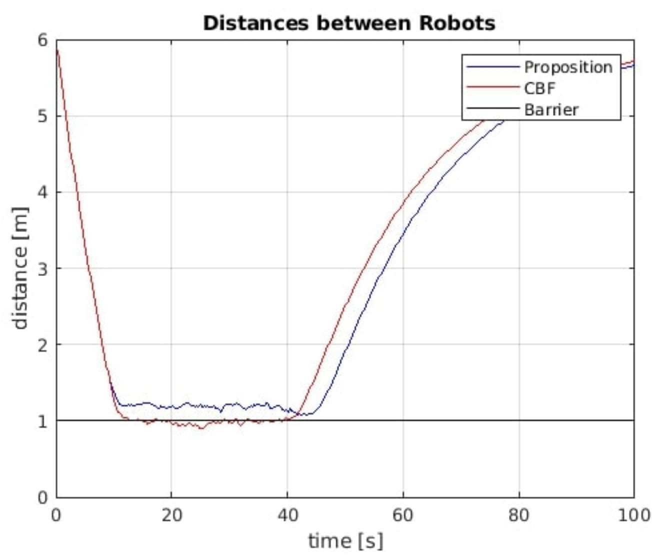

Figure 4.3 Comparison of the results in the Corollary and Proposition.

4.4.3 Scenario 3: Models with Disturbances under Relative State Estimation

In the second scenario, in addition to the assumption of perfect state estimation, we consider the influence of bounded disturbances on the models. As in the previous scenario, relative robot states are determined using the relative EKF module under the same assumptions. Additionally, we consider additive disturbances in the x and y directions of the robots. We use a sinusoidal function, $0.3\sin(2kT_s)$ (m/s), where k and T_s represent the number of

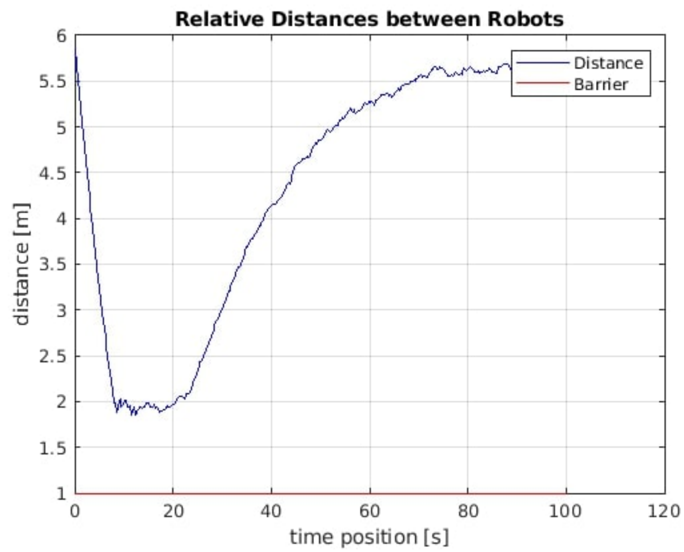

Figure 4.4 Conservatism of the safety conditions.

simulation steps and the sample time, respectively. We are trying to observe how the model and collision avoidance system behaves under an aggressive external disturbance effect. Note that we do not introduce additional disturbances into the relative EKF modules, but we know the upper bound (0.3 m/s) of the added disturbances by the assumption of Proposition 2.

Due to additive disturbances, the robots have much more fluctuations while reaching the goal in Figure 4.5. In Figure 4.6, it can be seen that the constraint in Theorem 6 enforces system safety even though stochastic sensor uncertainties and additional disturbances are present.

As can be expected that, the CBF constraint gives worse results than the conditions in the previous scenario. Propositions 1 and 2 can be used for collision avoidance when worst-case bounds on the estimation error are known. However, in many practical situations such bounds may not be known or could be very conservative. In particular, using a standard estimator such as an extended Kalman filter (EKF) does not provide such bounds. Hence, in the next chapter we consider situations where only bounds on the expectation of stochastic errors are known.

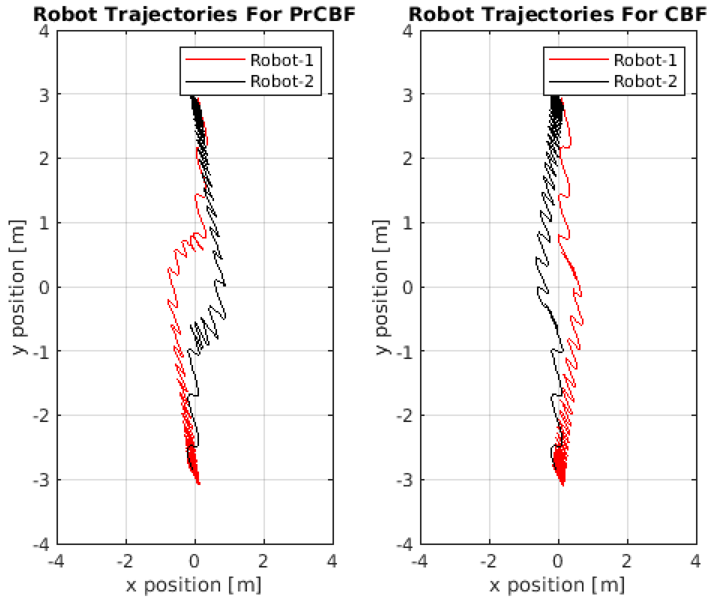

Figure 4.5 Scenario 2: robot trajectories in global frame

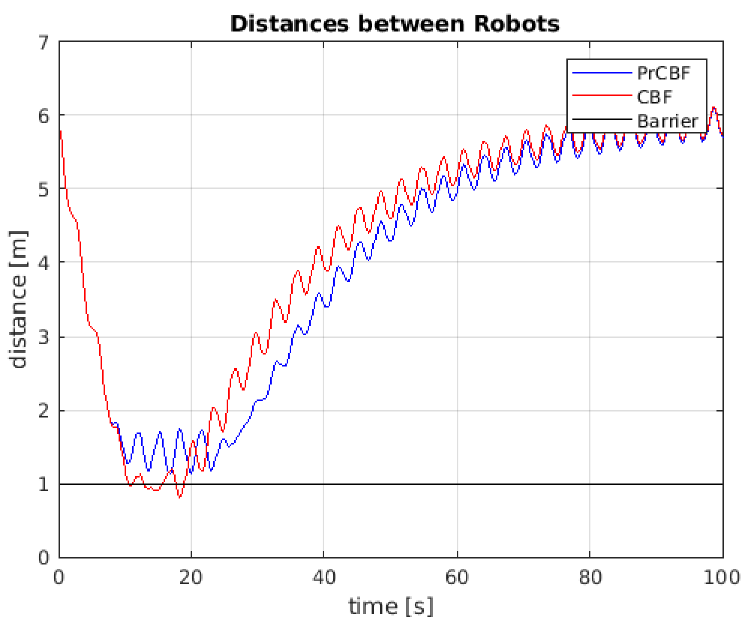

Figure 4.6 Scenario 2: comparison of robot distances

CHAPTER 5 DECENTRALIZED COLLISION AVOIDANCE WITH STOCHASTIC OBSERVER ERROR

In this chapter, we consider the effect of stochastic state estimation errors on CBF-type guarantees. Given a sequence of sensor measurements up to time k, a stochastic state estimator (filter) provides an approximation of the true state distribution at time k. For example, a Gaussian filter approximates the true state distribution by a Gaussian distribution, for which it computes an approximate mean and error covariance matrix, conditioned on the history of sensor measurements. The results discussed in this chapter provide a pointwise (i.e., at each time instant) safety probability guarantee under a condition of boundedness of mean-squared estimation errors.

5.1 Collision-Avoidance with Stochastic Estimation Errors

In the previous chapter, we discussed BE observer-controller architectures for collision avoidance. In some practical applications however, in particular for the vehicle navigation systems of interest in this thesis, errors and noise are generally considered random and some information about their distribution may be known, which we may want to use for control design purposes. Moreover, the support of these distributions may be unbounded (e.g., if we have Gaussian noise), in which case the results of Chapter 4 cannot be used.

Recall the system description and notation of Chapter 3. Hence, the real system kinematics of robot 1 is modeled as

$$\dot{\boldsymbol{p}}_1^{\mathtt{g}} = R_1^{\mathtt{g}}\, \boldsymbol{v}_{1/\mathtt{g}}^1. \tag{5.1}$$

The noisy range measurements at some instant are assumed to be of the form

$$\tilde{d}^{nm} = d^{nm} + \sigma_v^{nm}, \tag{5.2}$$

where σ_v^{nm} is a random noise, and the distance d^{nm} satisfies the relation

$$d^{nm}(\boldsymbol{x}_{12}^1) = \| \, \boldsymbol{p}_{12}^1 + R_2^1\, r_m^2 - r_n^1 \|. \tag{5.3}$$

Denote $\mathcal{I}_t^i$ all the information available until and including time t at robot i (sequence of past inputs and measurements), which can be used by its observer. We then introduce the notation $\mathbb{E}_t^i := \mathbb{E}[\cdot|\mathcal{I}_t^i]$ for the conditional expectation associated to robot i at time t. We define the estimated positions of the robots in global frame as $\hat{\boldsymbol{p}}_1^{\mathsf{g}}(t) = \mathbb{E}_t^1[\boldsymbol{p}_1^{\mathsf{g}}(t)]$, $\hat{\boldsymbol{p}}_2^{\mathsf{g}}(t) = \mathbb{E}_t^1[\boldsymbol{p}_2^{\mathsf{g}}(t)]$, assuming here that these conditional expectations can be computed (in practice, these estimates could be computed approximately, not exactly, by a filter such as the EKF). Then, we define relative state estimates as $\hat{\boldsymbol{p}}_{12}^1(t) = \mathbb{E}_t^1[\boldsymbol{p}_{12}^1(t)]$ and $\hat{\boldsymbol{p}}_{21}^2(t) = \mathbb{E}_t^2[\boldsymbol{p}_{21}^2(t)]$. Hence, robot 1 follows the dynamics

$$\dot{\boldsymbol{p}}_1^{\mathsf{g}} = R_1^{\mathsf{g}}\, \boldsymbol{v}_{1/\mathsf{g}}^1(\hat{\boldsymbol{p}}_1^{\mathsf{g}}, \hat{\boldsymbol{p}}_{12}^1), \tag{5.4}$$

where the feedback control uses estimated states. We define $\gamma_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\gamma_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\begin{aligned} &\mathbb{E}[\|\boldsymbol{p}_{12}^1(0) - \hat{\boldsymbol{p}}_{12}^1(0)\||\mathcal{I}_0^1] \leq \delta_1 \Rightarrow \mathbb{E}[\|\, \boldsymbol{e}_{12}^1\||\mathcal{I}_t^1] = \mathbb{E}[\|\boldsymbol{p}_{12}^1(t) - \hat{\boldsymbol{p}}_{12}^1(t)\||\mathcal{I}_t^1] \leq \gamma_1(t) \quad \forall t \in I \\ &\mathbb{E}[\|\boldsymbol{p}_{21}^2(0) - \hat{\boldsymbol{p}}_{21}^2(0)\||\mathcal{I}_0^2] \leq \delta_2 \Rightarrow \mathbb{E}[\|\, \boldsymbol{e}_{21}^2\||\mathcal{I}_t^2] = \mathbb{E}[\|\boldsymbol{p}_{21}^2(t) - \hat{\boldsymbol{p}}_{21}^2(t)\||\mathcal{I}_t^2] \leq \gamma_2(t) \quad \forall t \in I. \end{aligned} \tag{5.5}$$

Since we defined bounded expected estimation errors, we can consider the probability of bounds passing the thresholds with $\gamma_1(t)$, $\gamma_2(t)$ respectively. We have conditional expected errors with bounded real numbers for all time t.

Definition 9. *Let $\eta \in (0,1)$ and $\mathcal{C}$ be the set defined by (2.2). A control u renders the system (2.1) pointwise $(1-\eta)$-probabilistically safe with respect to the set $\mathcal{C}$ if at any state $x \in \partial\mathcal{C}$ we have*

$$\mathbb{P}\left(\dot{h}(x,u) \geq 0\right) \geq 1 - \eta. \tag{5.6}$$

Definition 9 is inspired by Nagumo's theorem [14], already used in the proof of Proposition 2, providing a probabilistic guarantee that at a state x on the boundary of $\mathcal{C}$, the control will push the system in a safe direction. However, it is important to note that this does not provide rigorous safety guarantees along entire system trajectories, and can only be used as a heuristic for safety. Nonetheless, Definition 9 follows a standard practice in a significant portion of the current literature on CBF-based safety guarantees for stochastic systems. For instance, an similar definition is used in [54].

We now assume in particular that the estimation errors have bounded second moment.

Assumption 2. *We assume there are γ_1, γ_2, k_b positive scalars such that $\mathbb{E}_t^1[\|\, \boldsymbol{e}_{12}^1(t)\|^2] \leq \gamma_1^2$, $\mathbb{E}_t^2[\|\, \boldsymbol{e}_{21}^2(t)\|^2] \leq \gamma_2^2$, and $\|\, \boldsymbol{v}_{1/\mathsf{g}}^1\| \leq k_b$, $\|\, \boldsymbol{v}_{2/\mathsf{g}}^2\| \leq k_b$ for all time t.*

Similarly to the previous chapter, we assume in Assumption 2 that both estimation errors and control input signals are bounded in some sense. This assumption is valid when estimation errors are consistent. Estimators like the EKF can be calibrated using experimental data before being applied in real-time scenarios. For bounded control signals, this assumption holds in cases where the controller is designed in advance and the safety constraints are applied separately. Designing the controller first and using the CBF constraint as a safety filter is referred to as a "minimally invasive safety filter", a common approach in the literature [4]. This approach has also been extensively studied in [18]. Under the previous assumption, we have the following result.

Theorem 6. *Consider 2 robots as in Chapter 3, the CBF $h(\boldsymbol{p}_{12}^g) : \mathbb{R}^2 \mapsto \mathbb{R}$ and initial conditions in the set $\mathcal{C}$. Define the unsafe probability $\eta \in (0,1)$. Let $\alpha > 0$ be constant, and let w_1, w_2 be positive scalars such that $w_1 + w_2 = 1$. Let $\hat{\boldsymbol{p}}_{12}^1$, $\hat{\boldsymbol{p}}_{21}^2$ be minimum mean square estimates of $\boldsymbol{p}_{12}^1$ and $\boldsymbol{p}_{21}^2$ held by agents 1 and 2 respectively, and assume that these estimates satisfy Assumption 2. Moreover, suppose that $\hat{\boldsymbol{p}}_{12}^1 \neq 0_{2\times1}$, $\hat{\boldsymbol{p}}_{21}^2 \neq 0_{2\times1}$. Then, any pair of Lipschitz continuous controllers $\boldsymbol{v}_{1/\mathsf{g}}^1$, $\boldsymbol{v}_{2/\mathsf{g}}^2$ satisfying*

$$\left(\hat{\boldsymbol{p}}_{12}^1\right)^T \boldsymbol{v}_{1/\mathsf{g}}^1 + \frac{2}{\eta}\left(k_b\,\mathbb{E}_t^1[\|\,\boldsymbol{e}_{12}^1\|] + w_1\alpha\,\mathbb{E}_t^1[\|\,\boldsymbol{e}_{12}^1\|^2 + 4R\|\,\boldsymbol{e}_{12}^1\|]\right) \leq w_1\alpha\left(\|\,\hat{\boldsymbol{p}}_{12}^1\|^2 - 4R^2\right), \tag{5.7}$$

$$\left(\hat{\boldsymbol{p}}_{21}^2\right)^T \boldsymbol{v}_{2/\mathsf{g}}^2 + \frac{2}{\eta}\left(k_b\,\mathbb{E}_t^2[\|\,\boldsymbol{e}_{21}^2\|] + w_2\alpha\,\mathbb{E}_t^2[\|\,\boldsymbol{e}_{21}^2\|^2 + 4R\|\,\boldsymbol{e}_{21}^2\|]\right) \leq w_2\alpha\left(\|\,\hat{\boldsymbol{p}}_{21}^2\|^2 - 4R^2\right), \tag{5.8}$$

renders the set $\mathcal{C}$ pointwise $(1-\eta)$-probabilistically safe.

Proof. Assuming $\boldsymbol{p}_{12} \in \partial\mathcal{C}$, we want to prove that if (5.7) and (5.8) are satisfied, then (5.6) is true. As in (4.19), a sufficient condition guaranteeing $\dot{h}(x,u) \geq 0$ for our system is that

$$\left(\boldsymbol{p}_{12}^1\right)^T \boldsymbol{v}_{1/\mathsf{g}}^1 \leq 0 \text{ and } \left(\boldsymbol{p}_{21}^2\right)^T \boldsymbol{v}_{2/\mathsf{g}}^2 \leq 0. \tag{5.9}$$

Moreover, we can repeat the steps of the proof of Proposition 2 to show that a sufficient condition for (5.9) is that the following constraints are both satisfied

$$C_1: \quad \left(\hat{\boldsymbol{p}}_{12}^1\right)^T \boldsymbol{v}_{1/\mathsf{g}}^1 - w_1\alpha(\|\,\hat{\boldsymbol{p}}_{12}^1\|^2 - 4R^2) + \|\,\boldsymbol{e}_{12}^1\|k_b + w_1\alpha\left(\|\,\boldsymbol{e}_{12}^1\|^2 + 4R\|\,\boldsymbol{e}_{12}^1\|\right) \leq 0,$$
$$C_2: \quad \left(\hat{\boldsymbol{p}}_{21}^2\right)^T \boldsymbol{v}_{2/\mathsf{g}}^2 - w_2\alpha(\|\,\hat{\boldsymbol{p}}_{21}^2\|^2 - 4R^2) + \|\,\boldsymbol{e}_{21}^2\|k_b + w_2\alpha\left(\|\,\boldsymbol{e}_{21}^2\|^2 + 4R\|\,\boldsymbol{e}_{21}^2\|\right) \leq 0,$$

where w_1, w_2 are positive scalars summing to 1, and C_1 and C_2 denote the respective event that the corresponding constraint is satisfied.

To satisfy (5.6), it is then sufficient to enforce $\mathbb{P}_t(C_1 \cap C_2) \geq 1-\eta$, with $\mathbb{P}_t$ denoting the probability conditioned on the information available at t. A further sufficient condition is then

$$\mathbb{P}_t(\bar{C}_1) \leq \frac{\eta}{2} \text{ and } \mathbb{P}_t(\bar{C}_2) \leq \frac{\eta}{2}, \tag{5.10}$$

where $\bar{C}_1$ and $\bar{C}_2$ denote the complement of C_1 and C_2, since by the union bound we then have

$$\mathbb{P}_t(\bar{C}_1 \cup \bar{C}_2) \leq \mathbb{P}_t(\bar{C}_1) + \mathbb{P}_t(\bar{C}_2) \leq \eta.$$

Now, defining the nonnegative random variable

$$\Gamma_1(t) := \| \boldsymbol{e}_{12}^1(t)\| k_b + w_1\alpha \left(\| \boldsymbol{e}_{12}^1(t)\|^2 + 4R\| \boldsymbol{e}_{12}^1(t)\|\right),$$

we can write

$$\mathbb{P}_t(\bar{C}_1) = \mathbb{P}_t \left(\Gamma_1(t) > w_1\alpha \left(\| \hat{\boldsymbol{p}}_{12}^1\|^2 - 4R^2\right) - \left(\hat{\boldsymbol{p}}_{12}^1\right)^T \boldsymbol{v}_{1/\mathrm{g}}^1\right). \tag{5.11}$$

For this probability to be less than 1 (and a fortiori less than $\eta/2$), we first need to impose

$$w_1\alpha(\| \hat{\boldsymbol{p}}_{12}^1\|^2 - 4R^2) - \left(\hat{\boldsymbol{p}}_{12}^1\right)^T \boldsymbol{v}_{1/\mathrm{g}}^1 > 0. \tag{5.12}$$

by supposing $\hat{\boldsymbol{p}}_{12}^1 \neq 0_{2\times 1}$ and choosing appropriate control input $\boldsymbol{v}_{1/\mathrm{g}}^1$. Then, since Γ_1 is integrable by Assumption 2, we can use Markov's inequality to write

$$\mathbb{P}_t(\bar{C}_1) \leq \frac{\mathbb{E}_t[\Gamma_1(t)]}{w_1\alpha\left(\| \hat{\boldsymbol{p}}_{12}^1\|^2 - 4R^2\right) - \left(\hat{\boldsymbol{p}}_{12}^1\right)^T \boldsymbol{v}_{1/\mathrm{g}}^1}.$$

To enforce the first constraint in (5.10), it is then sufficient to ensure

$$\mathbb{E}_t[\Gamma_1(t)] \leq \frac{\eta}{2}\left(w_1\alpha \left(\| \hat{\boldsymbol{p}}_{12}^1\|^2 - 4R^2\right) - \left(\hat{\boldsymbol{p}}_{12}^1\right)^T \boldsymbol{v}_{1/\mathrm{g}}^1\right),$$

which corresponds to (5.7). Note also that this constraint (5.7) already implies (5.12), which therefore does not need to be enforced separately.

The same argument leads to the constraint (5.8) to ensure that the probability of $\bar{C}_2$ is small. □

Theorem 6 can also be proven in another way. This relies on the following lemma, proved in [44] (see also the explanation in [55, Section 3.2.3]). Define the convex function $(\cdot)_+ : \mathbb{R} \to \mathbb{R}$ as $(z)_+ = \max\{z, 0\}$.

Lemma 1. *Let X be a random variable, $(x,u) \mapsto \psi(x,u)$ be a real-valued function, $0 < \delta < 1$. For u given and any $z > 0$, the inequality*

$$\frac{\mathbb{E}\left[(\psi(X,u)+z)_+\right]}{\delta} - z \leq 0, \tag{5.13}$$

implies the inequality

$$\mathbb{P}(\psi(X,u) > 0) \leq \delta. \tag{5.14}$$

The result of Lemma 1 allows replacing chance constraints, which are in general non-convex in u, by the more conservative but more tractable constraint (5.13). Moreover, since this constraint implies (5.14) for any $z > 0$, we can replace the latter constraint by the best approximation

$$\inf_{z>0}\left(\frac{\mathbb{E}\left[(\psi(X,u)+z)_+\right]}{\delta} - z\right) \leq 0. \tag{5.15}$$

Now, assume that all condition in Theorem 6 are satisfied. Following the same procedure in the proof of Theorem 6, we define the nonnegative random variable as

$$\hat{\Gamma}_1(t) := \|\,\boldsymbol{e}_{12}^1(t)\|k_b + w_1\alpha\left(\|\,\boldsymbol{e}_{12}^1(t)\|^2 + 4R\|\,\boldsymbol{e}_{12}^1(t)\|\right)$$

Then, we consider unsafe probability as

$$\mathbb{P}_t\Bigg(\left(\hat{\boldsymbol{p}}_{12}^1\right)^T \boldsymbol{v}_{1/\mathrm{g}}^1 - w_1\alpha(\|\,\hat{\boldsymbol{p}}_{12}^1\|^2 - 4R^2) + \hat{\Gamma}_1\Big) \geq 0\Bigg) \leq \frac{\eta}{2} \tag{5.16}$$

Now, we use Lemma 1 to replace (5.16), where ψ is

$$\psi = \left(\hat{\boldsymbol{p}}_{12}^1\right)^T \boldsymbol{v}_{1/\mathrm{g}}^1 - w_1\alpha(\|\,\hat{\boldsymbol{p}}_{12}^1\|^2 - 4R^2) + \hat{\Gamma}_1.$$

Choose z as

$$z = w_1\alpha(\|\,\hat{\boldsymbol{p}}_{12}^1\|^2 - 4R^2) - \left(\hat{\boldsymbol{p}}_{12}^1\right)^T \boldsymbol{v}_{1/\mathrm{g}}^1$$

Notice that we can enforce $z > 0$ by choosing control input $\boldsymbol{v}_{1/\mathrm{g}}^1$ if estimated relative state $(\hat{\boldsymbol{p}}_{12}^1)^T \neq [0,0]^T$. Then, applying Lemma 1 gives the constraints in Theorem 6.

Theorem 6 is useful when an error is bounded for a short period of time. In our case, the constraints are linear hence the EKF information can be easily used. In case of arbitrary selection of barrier function, this constraints require higher order moments of random variables, which can be obtain by Gaussian Process Learning [45] or moment calculation procedures [46]. Additionally, the robot configurations may lead to unobservable states for an extended period of time. Furthermore, Theorem 6 is conservative when the robots are far away from the safe set boundary. The existence of estimation error bounds Γ_1 and Γ_2 is crucial for collision avoidance systems. For a standard EKF, error bounds are ensured by the compactness of the state space or the observability of the system. However, our problem does not necessarily always meet these assumptions because the range sensors are mounted on the robots' bodies and their configurations change according to the robots' movements.

Assume that a controller $\boldsymbol{v}_{ref} \in \mathbb{R}^2$ including state feedback $\boldsymbol{p}_1^g$ is designed to reach given target points. Note that $\boldsymbol{v}_{ref}$ can also be any controller according to control purposes such as trajectory tracking, formation control etc. Then, the PrCBF quadratic program is solved for each time t as follows:

$$\begin{aligned} \min_{\boldsymbol{v}^1_{1/\mathrm{g}} \in U_{ad}} \quad & \|\, \boldsymbol{v}^1_{1/\mathrm{g}} - \boldsymbol{v}_{ref}\|^2 \\ \text{s.t.} \left(\hat{\boldsymbol{p}}^1_{12}\right)^T \boldsymbol{v}^1_{1/\mathrm{g}} &+ \frac{2}{\eta}\left(k_b\, \mathbb{E}^1_t[\|\, \boldsymbol{e}^1_{12}\|] + w_1\alpha\, \mathbb{E}^1_t[\|\, \boldsymbol{e}^1_{12}\|^2 + 4R\|\, \boldsymbol{e}^1_{12}\|]\right) \le w_1\alpha \left(\|\, \hat{\boldsymbol{p}}^1_{12}\|^2 - 4R^2\right) \end{aligned} \tag{5.17}$$

where $U_{ad} \subset \mathbb{R}^2$ is admissible control input set. However, the assumption on MSE bounds may not hold with certain configurations of motion, such as unobservable co-linear configurations between the range sensors. For instance, assume that initial conditions are chosen near the set boundary ∂C and estimated relative positions are $\hat{\boldsymbol{p}}^2_{21} = \hat{\boldsymbol{p}}^1_{12} = [2, 0]^T$, while the true relative positions are $\boldsymbol{p}^2_{21} = \boldsymbol{p}^1_{12} = [0, -1]^T$. This situation can occur due to local minima problems caused by ambiguous poses of range sensors or incorrectly selected initial conditions for the EKF. In this case, Theorem 6 can be conservative according to control input set $U_{ad} := (\boldsymbol{v}_{min}, \boldsymbol{v}_{max})$ where $\boldsymbol{v}_{min}$ and $\boldsymbol{v}_{max}$ are minimum and maximum bounds. However, If initial condition in the set $\mathcal{C}$ are selected or estimated consistently, and bounds $\boldsymbol{v}_{min} \in \mathbb{R}^2_{\le 0}$ and $\boldsymbol{v}_{max} \in \mathbb{R}^2_{\ge 0}$ contain negative and positive elements respectively, the PrCBF quadratic program can be effective for ensuring system safety.

5.2 Simulation Studies

In this section, we first consider the scenario presented in Section 4.4. As a variation, we define all noises as Gaussian noise using the "randn" function in MATLAB. Specifically, we define $\sigma_v^{nm} \in \mathcal{N}(0, 0.5)$ and velocity measurements as $\tilde{\boldsymbol{u}}_{\boldsymbol{1}} = \boldsymbol{u}_{\boldsymbol{1}} + \boldsymbol{w}$, where $\boldsymbol{w} \sim \mathcal{N}(0_3, 0.05 I_3)$ for translational and rotational velocity measurements. We also calculate the control signals $\boldsymbol{u}_{\boldsymbol{1}}$ by adding Gaussian noise $\mathcal{N}(0, 1)$ to the position feedback $\boldsymbol{p}_1^{\mathrm{g}}$ to replicate closed-loop perturbations. In Figure 5.1, the Monte Carlo simulation results can be observed for the

PrCBF QP 5.17 with $\eta = 0.5$.

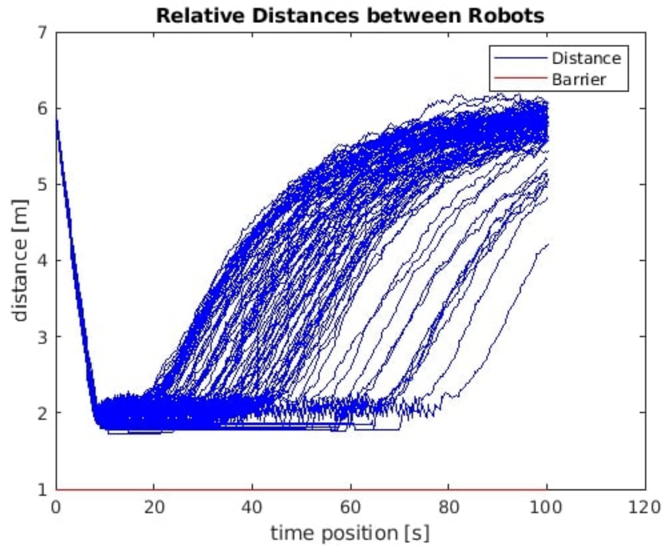

Figure 5.1 Monte Carlo simulation (100 runs) for PrCBF with $\eta = 0.5$.

As expected, the desired distance is not maintained, but collision avoidance is achieved in all simulations. It is also noteworthy that the PrCBF QP sometimes becomes infeasible, but the constraints are always satisfied when $\eta \leq 0.5$ is selected. This is because we define the maximum and minimum bounds of the control set as $(\pm 0.5, \mathrm{m/s}, \pm 0.5, \mathrm{m/s})^T$. Note that this control set selection is applicable to our robot kinematics, which are similar to those of quadrotors.

For the second scenario, we use a circular reference trajectory for the robots. In Figure 5.2, it can be seen that the PrCBF ensures the desired safety requirements for collision avoidance. Finally, we observe that the distance between the robots remains safe, as shown in Figure 5.3.

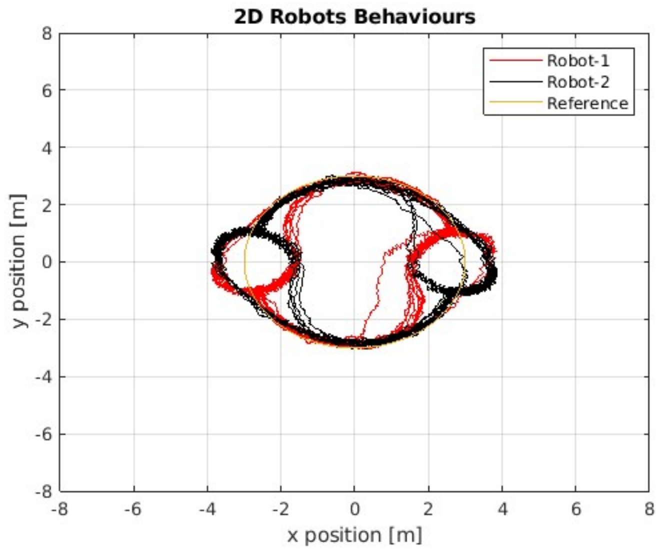

Figure 5.2 Circular trajectory tracking for PrCBF with $\eta = 0.5$.

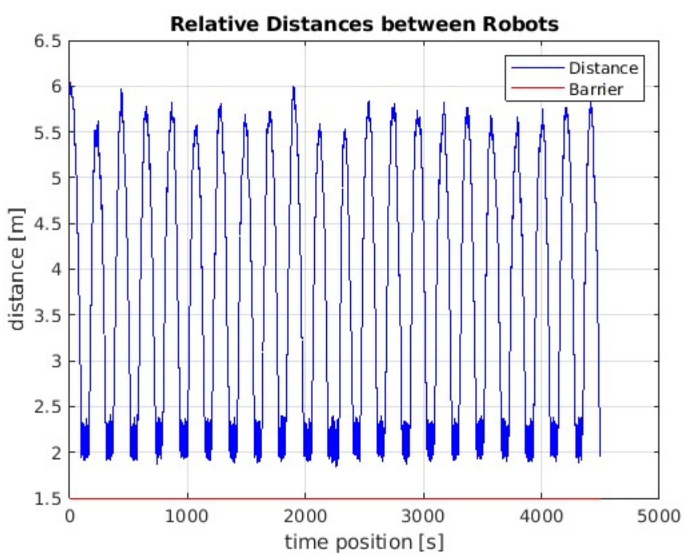

Figure 5.3 Distance between robots for circular trajectory.

CHAPTER 6 CONCLUSION AND FUTURE WORK

In this chapter, we discuss the results presented in the thesis and provide future perspectives on our problem. The main focus of the thesis was to develop methods based on the CBF literature to prevent collisions in decentralized swarm robotic systems. A simplified set-up with only two robots was considered. For the contributions of the thesis, we presented decentralized collision avoidance algorithms under relative state estimation. A detailed literature review was provided in Chapter 2. It can be observed that the collision avoidance problem based on CBFs has not been fully addressed in the presence of relative state estimation uncertainties. We explained our main problem in Chapter 3. In Chapter 4, we proposed our results, which are applicable with ISS and BE observers for relative state estimation. The presented methods can be integrated into real-time systems with known upper bounds on the uncertainties that may affect the systems. We demonstrated that Proposition 2 is useful under bounded estimation errors, while Theorem 5 is more restrictive. Nevertheless, Theorem 5 provides a new perspective for decentralized swarm robotics. In Chapter 5, we proposed a pointwise probabilistic safety condition, assuming stochastic estimation errors on relative states with bounded second moment. We evaluated the PrCBF QP program with different trajectories. Overall, this thesis presents theoretical and simulation results for collision avoidance between robots using multiple range sensors and relative state estimation uncertainties.

The proposed solutions remain limited under some conditions such as high unmodeled antenna delays, variable antenna radiation patterns, non-line-of-sight conditions, and excessive multipath propagation, which can all affect the range measurements. Therefore, this thesis is applicable to environments where these effects are minimized as much as possible. It should be noted that even if an experiment is conducted for sensors, obtaining direct probability distribution information under all these conditions is very difficult. Hence, obtaining practical and tight collision avoidance guarantees still require further work. For this reason, developing new statistical methods is of great importance not only for the two-robot example in the thesis but also for all systems with sensor noises and incomplete information.

In the main result of Chapter 5, Markov's inequality is used, and the proposed method provides only an instantaneous safety probability. On the other hand, in stability theorems, Chebyshev's inequality is applied to a non-negative and continuously decreasing Lyapunov function (supermartingale) and a probability bound can be obtained over entire trajectories [56]. At first glance, it may seem that a similar method can be applied to safety, but some fundamental problems can be foreseen. New barrier functions that can integrate Ville's inequality have been introduced in [57]. However, their effectiveness still remains to be evaluated for practical collision avoidance.

In future, learning algorithms such as Gaussian process regression can be used for unmodeled uncertainties. However, currently the probability guarantee given by these method corresponds the Definition 9. Furthermore, the learning rate poses more significant problems for stability indicated in Chapter 2. Therefore, developing new learning methods that can maintain stability and system safety is crucial.

Collision avoidance systems will continue to hold an important place in the future with the increasing use of robotic systems, automation, and artificial intelligence. Testing the methods in the thesis in real environments, and addressing the open problems discussed in this section, opens the way for future studies.

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