On a class of linear quadratic Gaussian quantilized mean field games ⋆

Rinel Foguen Tchuendom [a] Roland Malhamé [b] Peter E. Caines [c]

[a] *McGill University, GERAD, and CIM, Montréal (Canada)*

[b] *École Polytechnique de Montréal, and GERAD, Montréal (Canada)*

[c] *McGill University, GERAD, and CIM, Montréal (Canada)*

Abstract

An energy provider faced with energy generation risks and a large homogeneous pool of customers designs its energy price as a time-varying function of a risk-related quantile of the total energy demand, which generalizes pricing through the mean of the total energy demand. In the infinite population limit, we model the pricing problem with a class of linear quadratic Gaussian quantilized mean field games. For these quantilized mean field games, we show existence and uniqueness of an equilibrium which reveals the price trajectory, as well as an approximate Nash property when the quantilized mean field game's feedback control functions are applied to the large but finite game and the rate of convergence of the Nash deviation to zero as a function of the population size and the quantile is provided. Finally, the use of this class of quantilized mean field games is illustrated in the context of equivalent thermal parameter models for households heater and an energy provider using solar generation.

Key words: Mean Field Games; Quantiles ; Pricing.

1 Introduction

In this paper, we study a class of linear quadratic Gaussian quantilized mean field game of controls the framework of which was first introduced in [7]. Here, these quantilized mean field games are used to model a situation in which an energy provider is responsible for servicing a large pool of household heaters making energy demands. Such models have been considered for example in [16], [20], [15]. In particular, because in our model the energy demands are controls we shall utilize the framework of mean field games of controls, e.g. [3] .

Mean Field Games(MFGs) have been used in a number of papers to model price formation in multi-agents systems, [13], [6], [8], [9], [1]. In theses papers, agents are considered exchangeable and interact with all other agents through their aggregate mean state or control. MFGs approaches, introduced in [10], [14], exploits these assumptions to calculate approximate Nash equilibria; this is done by solving a model of the interaction between an individual representative agent and the mass of other agents. The model assumes a framework where the energy provider shapes the individual agents' behaviors via the prior specification of a price function.

Here we go a step beyond the previous MFGs for price formation. Indeed, total demand for a commodity will in general determine the price that it will sell for. However, total demand can fluctuate significantly while its mean tends to have much more predictability. This is why it has been used in the above works as a price formation device. Here however we take account of the fact that power demand fluctuations above the mean involve increased risk for an energy provider as they may induce expensive power imports from private energy producers or neighboring utilities. We reflect this increased risk by allowing power prices to depend on some appropriate quantile of the total electricity demand. Quantiles can capture increased risk while preserving the key property of predictability when the number of agents is large. A quantile corresponding to the mean of the population's distribution, reflects the situation of a risk neutral energy provider.

The price function that we consider is a predetermined

⋆ This paper was not presented at any IFAC meeting. Corresponding author R. Foguen Tchuendom.
Email addresses: fogurine@cim.mcgill.ca (Rinel Foguen Tchuendom), roland.malhame@polymtl.ca (Roland Malhamé), peterc@cim.mcgill.ca (Peter E. Caines)

Preprint submitted to Automatica
Received February 17, 2024 11:26:06 Pacific Time

linear and time-dependent function of the population's distribution quantile evolution over the finite horizon. The price process is determined as an equilibrium through a fixed point of the map defined by the individual representative agent's optimal control problem. Quantiles have been used in the load demand management literature, e.g. [4], where the authors evaluated the performance of their proposed control algorithms using a quantile-based quality of service criterion. Though our study does not involve common noise, we indicate that quantile-based McKean Vlasov equations have been studied in [5] and have been applied to the characterization of auctions. Furthermore ranking based MFGs models, e.g. [2], have some similarities to quantilized MFGs. A theoretical contribution of our paper is the explicit dependence of approximate Nash equilibria errors on the quantiles, which extends the results in classical MFG theory, e.g. [11].

The paper is organized as follows. In Section 2, we introduce the notation, the coefficients and we formulate our class of quantilized MFGs of controls. In Section 3, we establish a sufficient condition for the solvability of the quantilized MFGs of control to be the solvability of an auxilliary system of forward-backward ordinary differential equations(FBODEs). In Section 4, we show that the solvability of the sufficient condition is met under a set of adequate assumptions on the quantilized MFGs of controls coefficients. In Section 5, we consider the finite population game associated to our quantilized MFGs of controls, and obtain an approximate Nash equilibrium. Section 6 presents a numerical example, for a solar energy source at three distinct quantile values. In the concluding Section 7, potential further research avenues are indicated.

2 Notation and Problem Formulation

Notation. Let $n \geq 1$ be an integer, I_n be the $n \times n$ identity matrix. Given a column vector v of dimension n and an $n \times n$ matrix $G^T = G > 0$, we define the norm $\| v \|_G^2 := v^T G v$, where v^T denotes the transpose of v. Let 1_n be the column vector of dimension n filled with ones. For any $\alpha \in (0,1)$, we denote by $q_\alpha(\chi)$ the α-quantile of the distribution of continuous random variable χ, defined as

$$q_\alpha(\chi) := \inf\{r \in \mathbb{R} \mid \mathbb{P}(\chi \leq r) \geq \alpha\}.$$

Problem Formulation. Consider a time horizon $T > 0$ and introduce a random variable ξ with known distribution and an n dimensional Brownian motion $\{W_t\}_{t\in[0,T]}$ with independent components defined on a probability space $(\Omega, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with the filtration being right continuous and $\mathcal{F}_0$ completed with all $\mathbb{P}$-null events, and $\mathbb{F}$ defined as $\mathcal{F}_t = \sigma(\xi, W_s, 0 \leq s \leq t)$, $t \in [0,T]$. We denote by $\mathcal{M}^n(\mathbb{R})$ the set of $n \times n$ matrices with real valued entries equipped with the inner product and norm $< G, H >= \text{Trace}[G^T H]$ and $|G| = \sqrt{< G, G >}$.

Consider processes with values in n dimensional Euclidean space $(\mathbb{R}^n, ||\cdot||)$ and $\mathcal{P}(\mathbb{R}^n)$, the set of $\mathbb{R}^n$-valued and $\mathbb{F}$-progressively measurable stochastic processes. Let $\mathcal{C}([0,T], \mathbb{R}^n)$ be the space of continuous functions from $[0,T]$ into $\mathbb{R}^n$. We define the following spaces,

$$\mathcal{S}^n = \left\{ M \in \mathcal{M}^n(\mathbb{R}) \;\middle|\; M^T = M \right\},$$
$$\mathbb{L}^2([0,T];\mathbb{R}^n) = \left\{ f : [0,T] \mapsto \mathbb{R}^n \;\middle|\; \int_0^T ||f_t||^2 dt < +\infty \right\},$$
$$L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) = \left\{ f \in \mathcal{P}(\mathbb{R}^n) \middle| \mathbb{E}\left[\int_0^T ||f_t||^2 dt \right] < \infty \right\},$$

We consider a vector valued deterministic process $\{c_t,\ t \in [0,T]\}$, a matrix valued deterministic process $\{Q_t,\ t \in [0,T]\}$, $n \times n$ matrices A, B, R, S, Σ, and an n dimensional deterministic continuous process $\Lambda_t = (\lambda_j(t), j = 1, ..., n) \in \mathbb{R}^n,\ t \in [0,T]$.

Quantilized MFGs of Controls Problem Given any $\alpha \in (0,1)$ and a deterministic function $\{z_t,\ t \in [0,T]\}$ with values in $\mathbb{R}^n$. Find, if it exists, a best response, $u^o := (u^o_t)_{t\in[0,T]} \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ for a generic agent, in the infinite population of agents best response controls, $\mathcal{U}^o$, and a deterministic process $\{P^\alpha_t,\ t \in [0,T]\} \in \mathcal{C}([0,T];\mathbb{R})$ being the α-quantile of $\mathcal{U}^o$, such that:
(i) Best Response holds. That is,

$$J(u^o, P^\alpha) \leq J(u, P^\alpha), \quad \text{for all} \quad u \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n), \tag{1}$$

where for the generic agent,

$$J(u, P^\alpha) :=$$
$$\mathbb{E}\left[\int_0^T (P^\alpha_t) 1_n^T u_t + \frac{1}{2}\left(\| u_t \|_R^2 + \| x_t - z_t \|_{Q_t}^2 \right) dt \right],$$

$$\begin{aligned} &dx_t = \left[A x_t + B u_t + c_t \right] dt + \Sigma dW_t,\ \forall t \in [0,T], \\ &x_0 = \xi \sim \mathcal{N}(m, S) \in \mathbb{R}^n, \quad m \in \mathbb{R}^n, \\ &c \in \mathbb{L}^2([0,T];\mathbb{R}^n), \quad Q \in \mathbb{L}^\infty([0,T];\mathcal{S}^n), \\ &A, B \in \mathcal{M}^n(\mathbb{R}), \quad \Sigma, S, R \in \mathcal{S}^n. \end{aligned} \tag{2}$$

(ii) α-Quantile Equilibrium Condition holds. Namely,

$$P^\alpha_t = q_\alpha(\Lambda_t^T u^o_t), \quad \forall t \in [0,T]. \tag{3}$$

Note that in the context of modeling a pool of homogeneous household heaters, x_t would represent the generic household temperature, u_t would represent its energy demand for heating, P^α_t would represent the price of energy used for heating, Λ_t would represent the cost in

Preprint submitted to Automatica
Received February 17, 2024 11:26:06 Pacific Time

monetary value to service a unit of energy demand, and z_t would represent the heating comfort constraints of the households. We say more about this application in section 6.

3 Solvability of Problem (1-2-3)

Assumption (A1):

(1) $R>0$ and $Q_t \geq 0$, for all $t \in [0,T]$.
(2) $\nu_t := 1+\Lambda_t^T R^{-1} 1_n \neq 0, \quad \forall t \in [0,T]$.

Our first theorem shows that the solvability of some appropriate FBODEs is a sufficient condition for the solvability of the quantilized MFGs of controls problem (1-2-3).

Theorem 1. *Assume (A1) holds and assume there exists a solution $(\Gamma_t, \Pi_t, \bar{x}_t, h_t)_{t\in[0,T]}$ to the FBODEs*

$$\frac{d\Pi_t}{dt} = \Pi_t B R^{-1} B^T \Pi_t - \Pi_t A - A^T \Pi_t - Q_t, \tag{4}$$
$$\frac{d\Gamma_t}{dt} = (A - BR^{-1}B^T\Pi_t)\Gamma_t + \Gamma_t(A - BR^{-1}B^T\Pi_t)^T + \Sigma\Sigma^T, \tag{5}$$
$$\frac{d\bar{x}_t}{dt} = \left[A + \Phi_h(t)\Pi_t\right]\bar{x}_t + \Phi_h(t)h_t + \Phi(\alpha, t, \Pi_t, \Gamma_t), \tag{6}$$
$$\frac{dh_t}{dt} = -\Pi_t O_t \Pi_t \bar{x}_t - \left[A^T + \Phi_h(t)\Pi_t\right]h_t + \left[Q_t z_t - \Pi_t\Phi(\alpha, t, \Pi_t, \Gamma_t)\right], \tag{7}$$
$$\Pi_T = 0, \quad \Gamma_0 = S, \quad \bar{x}_0 = m, \quad h_T = 0.$$

where $\chi \sim \mathcal{N}(0,1)$ and for all $t \in [0.T]$

$$O_t = \nu_t^{-1} B R^{-1} 1_n \Lambda_t^T R^{-1} B^T, \tag{8}$$
$$\Phi_h(t) := -BR^{-1}B^T + O_t, \tag{9}$$
$$G_t = R^{-1}B^T\Pi_t\Gamma_t\Pi_t B R^{-1}, \; V_t = \left(\Lambda_t^T G_t \Lambda_t\right)^{\frac{1}{2}}, \tag{10}$$
$$\Phi(\alpha, t, \Pi_t, \Gamma_t) := c_t - \nu_t^{-1} q_\alpha(\chi) B R^{-1} 1_n V_t. \tag{11}$$

Then there exists a solution to (1-2-3), where the α-quantile equilibrium deterministic process, $\{P_t^\alpha, \; t \in [0,T]\}$, is given, $\forall t \in [0,T]$, by:

$$P_t^\alpha := \nu_t^{-1}\left[-\Lambda_t^T R^{-1} B^T \Pi_t \bar{x}_t - \Lambda_t^T R^{-1} B^T h_t + q_\alpha(\chi) V_t\right],$$

and the best response is given, $\forall t \in [0,T]$, by:

$$u_t^o = -R^{-1}B^T\Pi_t x_t^o - R^{-1}B^T h_t - R^{-1} P_t^\alpha 1_n, \tag{12}$$

with the optimal trajectory, $\forall t \in [0,T]$,

$$x_t^o = \xi + \int_0^t [(A - BR^{-1}B^T\Pi_s)\, x_s^o - BR^{-1}B^T h_s - BR^{-1}P_s^\alpha 1_n + c_s] ds + \Sigma W_t \; .$$

Proof. Step 0: Assume the existence of a solution to FBODEs (4-5-6-7). We show that there exists a best response to the optimal control problem (1-2) and that the α-quantile equilibrium condition (3) holds.
Step 1: A standard application of the Stochastic Maximum Principle shows that the unique best response $(u_t^o)_{t\in[0,T]} \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ to the optimal problem (1-2) is given by

$$u_t^o = -R^{-1}(B^T y_t^o + P_t^\alpha 1_n), \; \forall t \in [0,T], \tag{13}$$

where the stochastic process $(y_t^o)_{t\in[0,T]}$ is a component of the adapted solution $(x_t^o, y_t^o, L_t)_{t\in[0,T]}$ to the Forward Backward Stochastic Differential Equations(FBSDEs);

$$dx_t^o = \left[Ax_t^o - BR^{-1}(B^T y_t^o + P_t^\alpha 1_n) + c_t\right]dt + \Sigma dW_t, \tag{14}$$
$$dy_t^o = -\left[A^T y_t^o + Q_t(x_t^o - z_t)\right]dt + L_t dW_t, \tag{15}$$
$$x_0 = \xi, \quad y_T = 0, \quad \forall t \in [0,T],$$

where $(x_t^o)_{t\in[0,T]}$ is the optimal state evolution and $(y_t^o)_{t\in[0,T]}$ is the adjoint process. An application of the Ito Calculus shows that the following process satisfies FBSDEs (14-15),

$$x_t^o = \xi + \int_0^t \Big[(A - BR^{-1}B^T\Pi_s)\, x_s^o - BR^{-1}B^T h_s - BR^{-1}P_s^\alpha 1_n + c_s \Big] ds + \Sigma W_t,$$
$$y_t^o = \Pi_t x_t^o + h_t, \qquad L_t = \Pi_t \Sigma, \; \forall t \in [0,T],$$

where the offset $(h_t)_{t\in[0,T]}$ in the adjoint process $(y_t^o)_{t\in[0,T]}$ and $(\Pi_t)_{t\in[0,T]}$ are generated by the ODEs (4) and (7). Thus the best response $(u_t^o)_{t\in[0,T]}$ is

$$u_t^o = -R^{-1}B^T\Pi_t x_t^o - R^{-1}B^T h_t - R^{-1}P_t^\alpha 1_n. \tag{16}$$

Step 2: Given any $\alpha \in (0,1)$, we now introduce the candidate α-quantile equilibrium process, $\{P_t^\alpha, \; t \in [0,T]\}$, as:

$$P_t^\alpha := \nu_t^{-1}\left[-\Lambda_t^T R^{-1} B^T \Pi_t \bar{x}_t - \Lambda_t^T R^{-1} B^T h_t + q_\alpha(\chi) V_t\right]. \tag{17}$$

We define on $[0,T]$,

$$\Theta_t := \Lambda_t^T u_t^o = -\Lambda_t^T R^{-1} B^T \Pi_t x_t^o - \Lambda_T^T R^{-1} B^T h_t - \Lambda_t^T R^{-1} P_t^\alpha 1_n, \tag{18}$$

and show that the α-quantile of Θ_t coincides with (17) for all $t \in [0,T]$. The process Θ_t is a Gaussian stochastic process. We denote by, $\mathbb{E}[\Theta_t]$ and $\mathbb{V}[\Theta_t]$, its mean and variance. For any $\alpha \in (0,1)$, its α-quantile is

$$q_\alpha(\Theta_t) = \mathbb{E}[\Theta_t] + q_\alpha(\chi)\left(\mathbb{V}[\Theta_t]\right)^{\frac{1}{2}}, \tag{19}$$

Preprint submitted to Automatica
Received February 17, 2024 11:26:06 Pacific Time

with $\chi \sim \mathcal{N}(0,1)$, and

$$\mathbb{E}[\Theta_t] = -\Lambda_t^T R^{-1} B^T \Pi_t \bar{x}_t - \Lambda_t^T R^{-1} B^T h_t - \Lambda_t^T R^{-1} P_t^\alpha 1_n,$$
$$\mathbb{V}[\Theta_t] = \mathbb{E}\big[\big(\Theta_t - \mathbb{E}[\Theta_t]\big)\big(\Theta_t - \mathbb{E}[\Theta_t]\big)^T\big] = \Lambda_t^T G_t \Lambda_t = V_t^2.$$

where for all $t \in [0,T]$, $\quad \bar{x}_t = \mathbb{E}[x_t^o]$,

$$\text{and} \quad \Gamma_t = \mathbb{E}\big[\big(x_t^o - \mathbb{E}[x_t^o]\big)\big(x_t^o - \mathbb{E}[x_t^o]\big)^T\big].$$

Therefore from (19), the α-quantile of Θ_t is

$$q_\alpha\big(\Theta_t\big) = -\Lambda_t^T R^{-1} B^T \Pi_t \bar{x}_t - \Lambda_t^T R^{-1} B^T h_t - \Lambda_t^T R^{-1} P_t^\alpha 1_n + q_\alpha(\chi) V_t,$$

with $\chi \sim \mathcal{N}(0,1)$. Finally, we show that the consistency condition (3) holds for the candidate given in (17),

$$\begin{aligned} P_t^\alpha &= \nu_t^{-1}\big[-\Lambda_t^T R^{-1} B^T \Pi_t \bar{x}_t - \Lambda_T^T R^{-1} B^T h_t + q_\alpha(\chi) V_t\big] \\ \implies &\quad P_t^\alpha\big(1 + \Lambda_t^T R^{-1} 1_n\big) = -\Lambda_t^T R^{-1} B^T \Pi_t \bar{x}_t - \Lambda_T^T R^{-1} B^T h_t + q_\alpha(\chi) V_t \quad (20) \\ \implies &\quad P_t^\alpha = -\Lambda_t^T R^{-1} B^T \Pi_t \bar{x}_t - \Lambda_T^T R^{-1} B^T h_t - \Lambda_t^T R^{-1} P_t^\alpha 1_n + q_\alpha(\chi) V_t \quad (21) \\ \implies &\quad P_t^\alpha = q_\alpha(\Theta_t). \quad (22) \end{aligned}$$

Hence the equilibrium (3) holds and the proof is complete. ☐

4 Solvability of FBODEs (4-5-6-7)

In this section, we establish sufficient conditions for the existence and uniqueness of solutions $(\Gamma_t, \Pi_t, \bar{x}_t, h_t)_{t\in[0,T]}$ to FBODEs (4-5-6-7). We first show that there exists a unique solution $(\Gamma_t, \Pi_t)_{t\in[0,T]}$ to FBODEs (4-5). We next show under additional assumptions that there exists a unique solution $(\bar{x}_t, h_t)_{t\in[0,T]}$ to FBODEs (6-7).

Theorem 2. *Assume (A1) holds. There exists a unique solution $(\Gamma_t, \Pi_t)_{t\in[0,T]}$ to FBODEs (4-5), which we recall below,*

$$\frac{d\Pi_t}{dt} = \Pi_t B R^{-1} B^T \Pi_t - \Pi_t A - A^T \Pi_t - Q_t, \quad (23)$$
$$\frac{d\Gamma_t}{dt} = \big(A - BR^{-1}B^T\Pi_t\big)\Gamma_t + \Gamma_t\big(A - BR^{-1}B^T\Pi_t\big) + \Sigma\Sigma^T, \quad (24)$$
$$\Pi_T = 0, \quad \Gamma_0 = S, \quad \forall t \in [0,T].$$

Proof. The existence of a unique solution to Riccati ODE (23) with constant coefficients being well-known [19] and the existence and uniqueness of a solution to the ODE (24) follows from its Lipschitz properties. ☐

Assumption (A2): We assume that $(\nu_t)_{t\in[0,T]}$, $(\Lambda_t)_{t\in[0,T]}$, $(z_t)_{t\in[0,T]}$ introduced above are such that,

(1) $\Lambda_t = \lambda_t 1_n, \ \lambda_t > 0, \ \forall t \in [0,T]$.
(2) $R - \nu_t^{-1}\lambda_t 1_n 1_n^T > 0, \ \forall t \in [0,T]$.
(3) There exists a family of deterministic processes $(e_t^\alpha)_{t\in[0,T]}$, $\alpha \in (0,1)$, such that $\forall t \in [0,T]$,

$$Q_t z_t - \Pi_t O_t \Pi_t e_t^\alpha = \Pi_t \Phi\big(\alpha, t, \Pi_t, \Gamma_t\big).$$

Under the additional assumptions (A2), we show that there is an equivalence between the solvability of a standard Deterministic Linear Quadratic optimal control problem and the solvability of the FBODEs (25-26) below, and use this equivalence to show existence and uniqueness of solutions to FBODEs (25-26). The item (4) of assumption (A2) says that the process $(z_t)_{t\in[0,T]}$ lies in the range space of a linear mapping.

Theorem 3. *Assume (A1) and (A2) hold. There exists a unique solution $(\bar{x}_t, h_t)_{t\in[0,T]}$ to FBODEs (25-26),*

$$\frac{d\bar{x}_t}{dt} = \big[A + \Phi_h\big(t\big)\Pi_t\big]\,\bar{x}_t + \Phi_h\big(t\big)h_t + \Phi\big(\alpha, t, \Pi_t, \Gamma_t\big), \quad (25)$$
$$\frac{dh_t}{dt} = -\Pi_t O_t \Pi_t \bar{x}_t - \big[A^T + \Phi_h\big(t\big)\Pi_t\big]\,h_t + \big[Q_t z_t - \Pi_t \Phi\big(\alpha, t, \Pi_t, \Gamma_t\big)\big], \quad (26)$$
$$\bar{x}_0 = m \quad h_T = 0, \quad \forall t \in [0,T],$$

where the coefficients are given by (8-11).

Proof. Given any $\alpha \in (0,1)$, consider the Deterministic Linear Quadratic optimal control problem

$$J^\alpha(\hat{x}, \hat{u}) = \frac{1}{2}\int_0^T \Big(\| \hat{u}_t \|_{\bar{R}_t}^2 + \| \hat{x}_t - e_t^\alpha \|_{\bar{Q}_t}^2\Big)\, dt, \quad (27)$$
$$\hat{x}_t = m + \int_0^t \big[\bar{A}_s \hat{x}_s + \bar{B}_s \hat{u}_s + \bar{c}_s\big] ds \quad \in \mathbb{R}^n, \quad (28)$$

where the coefficients are: $\forall t \in [0,T]$

$$\bar{B}_t := B, \quad \bar{A}_t := \Phi_x(t, \Pi_t), \quad (29)$$
$$\bar{c}_t := \Phi(\alpha, t, \Pi_t, \Gamma_t), \quad \bar{Q}_t := \Pi_t O_t \Pi_t \quad (30)$$
$$\bar{R}_t := \big[I_n - \nu_t^{-1}\lambda_t 1_n 1_n^T R^{-1}\big]^{-1} R. \quad (31)$$

We first compute, for all $t \in [0,T]$,

$$\bar{Q}_t = -\Pi_t O_t \Pi_t = \nu_t^{-1}\lambda_t(\Pi_t B R^{-1} 1_n)(\Pi_t B R^{-1} 1_n)^T$$

which is positive thanks to items (1) and (2) of assumption (A2). We next compute, for all $t \in [0,T]$,

$$\bar{R}_t^{-1} = R^{-1} - R^{-1}\nu_t^{-1}\lambda_t 1_n 1_n^T R^{-1},$$

Preprint submitted to Automatica
Received February 17, 2024 11:26:06 Pacific Time

which is both positive definite and symmetric by assumptions (A1) and (A2). It follows that $\bar{R}_t$ is also positive definite for all $t \in [0,T]$. We deduce from the assumption that Q_t is positive continuous and R_t is positive definite and continuous that there exists a solution to the Deterministic Linear Quadratic optimal control problem, see [19]. Note that the solvability of the Deterministic Linear Quadratic optimal control problem $(27-28)$ is equivalent to the solvability of the FBODEs below,

$$\frac{d\hat{x}_t^\alpha}{dt} = \bar{A}_t\hat{x}_t^\alpha - \bar{B}_t\bar{R}_t^{-1}\bar{B}_t^T\hat{y}_t^\alpha + \bar{c}_t, \quad \hat{x}_0 = m, \quad (32)$$
$$\frac{d\hat{y}_t^\alpha}{dt} = -\bar{Q}_t\hat{x}_t^\alpha - \bar{A}_t^T\hat{y}_t^\alpha + \bar{Q}_te_t^\alpha, \quad \hat{y}_T = 0. \quad (33)$$

We make the following observations on the coefficients of the FBODEs (32 - 33). From (A4-2) and (A4-4), we compute that

$$\begin{aligned} -\bar{Q}_te_t^\alpha &= -\Pi_tO_t\Pi_te_t^\alpha \\ &= \left[-Q_tz_t + \Pi_t\Phi\big(\alpha,t,\Pi_t,\Gamma_t\big)\right], \\ \bar{A}_t^T &= \left[A+\Phi_h(t)\Pi_t\right]^T = \left[A^T+\Phi_h(t)\Pi_t\right], \\ -\bar{B}_t\bar{R}_t^{-1}\bar{B}_t^T &= -\bar{B}_t\left[\left[I_n-\nu_t^{-1}\lambda_t1_n1_n^TR^{-1}\right]^{-1}R\right]^{-1}\bar{B}_t^T \\ &= \Phi_h(t), \end{aligned}$$

such that the FBODEs (32-33) coincides with the FBODEs (25-26).

These computations imply that, if there is a solution to the deterministic control problem (27-28), then there is a solution to the FBODEs (25-26), and vice versa. Therefore, we deduce the existence and uniqueness of a solution $(\bar{x}_t, h_t)_{t\in[0,T]}$ to the FBODEs (25-26) from the existence and uniqueness of a solution to the deterministic control problem (27-28). The proof is complete. □

5 Finite Quantilized Population Games

Consider a finite population of $N \geq 1$ agents. We denote by $\mathcal{A}_i$, the i^{th} agent, $i \in \{1,...,N\}$. All agents have $\mathbb{R}^n$-valued state and control processes over the horizon $[0,T]$. The admissible control space is $L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ and the state processes, $(x_t^i)_{t\in[0,T]}$ $\forall i \in \{1,...,N\}$, evolve according to the SDE below: for all $t \in [0,T]$,

$$dx_t^i = \left[Ax_t^i + Bu_t^i + c_t\right]dt + \Sigma dW_t^i, \quad (34)$$
$$x_0^i = \xi \sim \mathcal{N}(m,S) \quad \forall i \in \{1,...,N\}. \quad (35)$$

All agents aim to minimize their individual cost:

$$J^N(u^i,u^{-i}) := \mathbb{E}\left[\int_0^T P_t^{\alpha,N}1_n^Tu_t^i + \frac{1}{2}\left(\parallel u_t^i\parallel_R^2 + \parallel x_t^i - z_t\parallel_{Q_t}^2\right)dt\right].$$

For all $i \in \{1,...,N\}$, the cost of agent $\mathcal{A}_i$ depends on the control of all other agents denoted, u^{-i}, through the stochastic process $(P_t^{\alpha,N})_{t\in[0,T]}$. This dependence is described as follows; given $\Lambda_t = (\lambda_j(t), j = 1,...,n) \in \mathbb{R}^n, t \in [0,T]$, and agent $\mathcal{A}_i$'s control process $(u_t^i)_{t\in[0,T]} \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$, we define, for all $t \in [0,T]$,

$$\Theta_t^i := \Lambda_t^Tu_t^i \in \mathbb{R}, \ \forall t \in [0,T], \quad (36)$$

and the empirical distribution,

$$F_t^N(\theta) = \frac{1}{N}\sum_{j=1}^N 1_{\{\Theta_t^j\leq\theta\}}, \quad \forall\theta \in \mathbb{R}. \quad (37)$$

Using these empirical distributions, and any given $\alpha \in (0,1)$, we define the stochastic process $(P_t^{\alpha,N})_{t\in[0,T]}$ as follows,

$$P_t^{\alpha,N} := \inf\{\theta \in \mathbb{R} \mid F_t^N(\theta) \geq \alpha\}. \quad (38)$$

Therefore, through the process $(P_t^{\alpha,N})_{t\in[0,T]}$, the cost of each agent depends on the controls of all other agents. We obtain a game by assuming that the agents aim at minimizing their cost without cooperation. The relevant solution notion to this finite quantilized population game is the Nash Equilibrium. As the population size N gets larger, determining a Nash Equilibrium, if it exists, grows in complexity. Instead, we settle for an approximate Nash equilibrium called Epsilon-Nash Equilibrium in MFGs literature, see [10].

Definition 5.1 (Epsilon-Nash Equilibrium). *A collection of controls, $(u_t^{i*})_{t\in[0,T]}$, $i \in \{1,...,N\}$, generates an Epsilon-Nash Equilibrium for the finite quantilized population game with any $\alpha \in (0,1)$, if there exists a sequence of positive constants, denoted $\{\epsilon_N^\alpha,\ N \geq 1\}$, such that for all unilateral deviations of any agent $\mathcal{A}_i$ from the control u^{i*} to another control u^i, such that:*

$$J^N(u^i,u^{-i*}) \leq J^N(u^{i*},u^{-i*}), \quad (39)$$

it holds that

$$J^N(u^{i*},u^{-i*}) - J^N(u^i,u^{-i*}) \leq \epsilon_N^\alpha, \quad (40)$$

where

$$\epsilon_N^\alpha \longrightarrow 0, \ as\ N \longrightarrow +\infty. \quad (41)$$

Theorem 4. *Assume (A1) and (A2) hold. Let any agent $\mathcal{A}_i$ deviate unilaterally to $u^i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$. Then, there exists an Epsilon-Nash Equilibrium for the finite quantilized population game given by the feedback function, $\forall x \in \mathbb{R}^n$ and $t \in [0,T]$,*

$$\Xi(t,x) := -R^{-1}B^T\Pi_tx - R^{-1}B^Th_t - R^{-1}P_t^\alpha 1_n, \quad (42)$$

where $(\Gamma_t, \Pi_t, \bar{x}_t h_t)_{t\in[0,T]}$ *is the solution to the FBODEs (5-4-6-7), and for all* $t \in [0,T]$ *and* $\alpha \in (0,1)$,

$$P_t^\alpha := \nu_t^{-1}\big[-\Lambda_t^T R^{-1} B^T \Pi_t \bar{x}_t - \Lambda_t^T R^{-1} B^T h_t + q_\alpha(\chi) V_t\big],$$

Moreover, $\{\epsilon_N^\alpha\ N \geq 1\}$ *satisfy,*

$$\epsilon_N^\alpha = \mathcal{O}\Bigg(\sqrt{\frac{\alpha(1-\alpha)}{N}}\bigg(\int_0^T \left|\frac{1}{f_t(P_t^\alpha)}\right|^2 dt\bigg)^{1/2}\Bigg), \quad (43)$$

where, the function $f_t(\cdot)$ *denotes the probability density function of the* $\mathbb{R}$*-valued random variable* $\Lambda_t^T u_t^o$ *with* $u^o \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ *being the best response given in (12).*

Proof. Step 0: Let (A1) and (A2) hold, then there exists a solution $(u_t^o, P_t^\alpha)_{t\in[0,T]}$ to problem (1-2-3) given by

$$u_t^o := -R^{-1}B^T\Pi_t x_t - R^{-1}B^T h_t - R^{-1}P_t^\alpha 1_n,$$
$$P_t^\alpha := \nu_t^{-1}\big[-\Lambda_t^T R^{-1} B^T \Pi_t \bar{x}_t - \Lambda_t^T R^{-1} B^T h_t + q_\alpha(\chi) V_t\big],$$

where $\chi \sim \mathcal{N}(0,1)$ and $(\Gamma_t, \Pi_t, \bar{x}_t h_t)_{t\in[0,T]}$ is the solution to the FBODEs (5-4-6-7), and $(x_t)_{t\in[0,T]}$ is given by

$$dx_t = \Sigma dW_t + \Big[(A - BR^{-1}B^T\Pi_t)x_t \quad (44)$$
$$+ \big[-BR^{-1}B^T h_t - BR^{-1}P_t^\alpha 1_n + c_t\big]\Big]dt,$$
$$x_0 = \xi \sim \mathcal{N}(m,S), \quad \forall t \in [0,T].$$

Let the finite set of agents $\mathcal{A}_i$, $i \in \{1,...,N\}$, use the feedback function $\Xi(t,x)$ of the form specified by the best reponse $u^o \in \mathcal{U}^o$ of the generic agent in the infinite population of agents. That is to say, for all $t \in [0,T]$, $i \in \{1,...,N\}$,

$$u_t^{i*} := \Xi(t, x_t^{i*}) \quad (45)$$
$$= -R^{-1}B^T\Pi_t x_t^{i*} - R^{-1}B^T h_t - R^{-1}P_t^\alpha 1_n,$$

where $(x_t^{i,*})_{t\in[0,T]}$, $i \in \{1,...,N\}$, are given by

$$dx_t^{i*} = \Sigma dW_t^i + \Big[(A - BR^{-1}B^T\Pi_t)x_t^{i*} \quad (46)$$
$$+ (-BR^{-1}B^T h_t - BR^{-1}P_t^\alpha 1_n + c_t)\Big]dt,$$
$$x_0^{i*} = \xi \sim \mathcal{N}(m,S), \quad (47)$$

Hence $\mathcal{L}(x_t) = \mathcal{L}(x_t^{i*})$, for all $t \in [0,T]$.

Step 1: Because u^o is the best response for cost $J(\cdot, P^\alpha)$,

$$J(u^o, P^\alpha) - J(u^i, P^\alpha) \leq 0, \quad (48)$$

and it holds that

$$J(u^o,P^\alpha) - J^N(u^i, u^{-i*}) =$$
$$J(u^o,P^\alpha) - J(u^i,P^\alpha) + J(u^i,P^\alpha) - J^N(u^i,u^{-i*})$$
$$\leq 0 + J(u^i,P^\alpha) - J^N(u^i,u^{-i*}), \quad (49)$$
$$\leq \left|\mathbb{E}\Big[\int_0^T (P_t^\alpha - P_t^{\alpha,N})1_n^T u_t^i dt\Big]\right|. \quad (50)$$

And

$$J^N(u^{i*}, u^{-i*}) - J(u^o, P^\alpha) = \quad (51)$$
$$= \mathbb{E}\Big[\int_0^T (P_t^{\alpha,N} - P_t^\alpha)1_n^T u_t^{i*} dt\Big]$$
$$+ \mathbb{E}\left[\int_0^T U(t, x_t^{i*})dt\right] - \mathbb{E}\left[\int_0^T U(t,x_t)dt\right],$$

where $U(\cdot,\cdot)$ is defined on $(t,y) \in [0,T]\times\mathbb{R}^n$ by

$$U(t,y) = P_t^\alpha 1_n^T \Xi(t,y) \quad (52)$$
$$+ \frac{1}{2}\Big(\parallel \Xi(t,y) \parallel_R^2 + \parallel y - z_t \parallel_{Q_t}^2\Big).$$

Because, $\mathcal{L}(x_t) = \mathcal{L}(x_t^{i*})$, for all $t \in [0,T]$, we obtain,

$$\mathbb{E}\left[\int_0^T U(t,x_t^{i*})dt\right] - \mathbb{E}\left[\int_0^T U(t,x_t)dt\right] = 0, \quad (53)$$

and so

$$J^N(u^{i*},u^{-i*}) - J(u^o,P^\alpha) = \quad (54)$$
$$\left|\mathbb{E}\Big[\int_0^T (P_t^{\alpha,N} - P_t^\alpha)1_n^T u_t^{i*} dt\Big]\right|. \quad (55)$$

But from Holder's inequality, we obtain the estimate

$$\left|\mathbb{E}\Big[\int_0^T (P_t^\alpha - P_t^{\alpha,N})1_n^T u_t^i dt\Big]\right| \leq \bigg(\mathbb{E}\Big[\int_0^T |1_n^T u_t^i|^2 dt\Big]\bigg)^{1/2}$$
$$\times \bigg(\mathbb{E}\Big[\int_0^T |P_t^\alpha - P_t^{\alpha,N}|^2 dt\Big]\bigg)^{1/2}, \quad (56)$$
$$\leq \quad c_1\bigg(\mathbb{E}\Big[\int_0^T |P_t^\alpha - P_t^{\alpha,N}|^2 dt\Big]\bigg)^{1/2}. \quad (57)$$

Indeed, for some $c_0 > 0$ independent of N and α,

$$|1_n^T u_t^i|^2 = \left|\sum_{k=1}^n u_t^i\right|^2 \leq c_0 \sum_{k=1}^n |u_t^i|^2 = c_0 \parallel u_t^i \parallel^2, \quad (58)$$

and since $u^i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$, there exists a constant $c_1^2 > 0$ also independent of N and α such that

$$\mathbb{E}\Big[\int_0^T |1_n^T u_t^i|^2 dt\Big] \leq \mathbb{E}\Big[\int_0^T c_0 \parallel u_t^i \parallel^2\ dt\Big] \leq c_1^2. \quad (59)$$

Preprint submitted to Automatica
Received February 17, 2024 11:26:06 Pacific Time

and also

$$\left|\mathbb{E}\left[\int_0^T (P_t^\alpha - P_t^{\alpha,N}) 1_n^T u_t^{i*} dt\right]\right| \leq \left(\mathbb{E}\left[\int_0^T |1_n^T u_t^{i*}|^2 dt\right]\right)^{1/2} \times \left(\mathbb{E}\left[\int_0^T |P_t^\alpha - P_t^{\alpha,N}|^2 dt\right]\right)^{1/2}, \quad (60)$$

$$\leq \quad c_2 \left(\mathbb{E}\left[\int_0^T |P_t^\alpha - P_t^{\alpha,N}|^2 dt\right]\right)^{1/2}. \quad (61)$$

where $c_2 > 0$ is a positive constant independent of N and α.

Combining estimates (50), (55), (57), and (61), we get $\{\epsilon_N^\alpha,\ N \geq 1\}$ as

$$J^N(u^{i*}, u^{-i*}) - J^N(u^i, u^{-i*}) = \quad (62)$$

$$J^N(u^{i*}, u^{-i*}) - J(u^o, P^\alpha) + J(u^o, P^\alpha) - J^N(u^i, u^{-i*})$$

$$\leq \mathbb{E}\left[\int_0^T (P_t^\alpha - P_t^{\alpha,N}) 1_n^T u_t^{i*} dt\right]\Bigg| \quad (63)$$

$$+ \left|\mathbb{E}\left[\int_0^T (P_t^{\alpha,N} - P_t^\alpha) 1_n^T u_t^i dt\right]\right|,$$

$$\leq c_3 \left(\int_0^T \mathbb{E}\left[|P_t^\alpha - P_t^{\alpha,N}|^2\right] dt\right)^{1/2} \quad (64)$$

$$=: \epsilon_N^\alpha\ , \quad (65)$$

where $c_3 := 2 \max c_1, c_2 > 0$.

Step 2: It remains to show that, the obtained sequence $\{\epsilon_N^\alpha.\ N \geq 1\}$ converges to 0, as N goes to infinity. Observe that the collection of $\mathbb{R}$-valued random variables, $\{\Lambda_t^T u_t^{j*},\ j \in \{1, ..., N\}\}$ have the same distribution as $\Lambda_t^T u_t^o$, for all $\forall t \in [0,T]$. We deduce as a particular case of a central limit theorem in [17] (p. 77) for quantiles, that for any $\alpha \in (0,1)$, and for all $t \in [0,T]$, the α-quantile associated with the sample $\{\Lambda_t^T u_t^{j*},\ j \in \{1, ..., N\}\}$, denoted $P_t^{\alpha,N}$, and the α-quantile of $\Lambda_t^T u_t^o$, denoted P_t^α, the following central limit convergence hold, as $N \longrightarrow +\infty$,

$$\mathcal{L}(P_t^{\alpha,N}) \longrightarrow \mathcal{N}\left(P_t^\alpha,\ \frac{\alpha(1-\alpha)}{N f_t^2(P_t^\alpha)}\right), \quad (66)$$

where, for all $t \in [0,T]$, the function $f_t(\cdot)$ denotes the probability density function of the $\mathbb{R}$-valued random variable $\Lambda_t^T u_t^o$. Observe from the central limit convergence (66),

$$\epsilon_N^\alpha = \mathcal{O}\left(\sqrt{\frac{\alpha(1-\alpha)}{N}} \left(\int_0^T \left|\frac{1}{f_t(P_t^\alpha)}\right|^2 dt\right)^{1/2}\right).$$

The proof is complete. □

6 Numerical Example

We illustrate the use of quantilized MFGs in the context of equivalent thermal parameter(ETP) models, see [18]. Inspired by [12], we consider a time horizon of $T = 24$ hours (i.e. a day) and define a family of N SDEs representing the thermal dynamics of a population of homogeneous household electric heaters. For all $t \in [0,T]$, and $i \in \{1, ..., N\}$,

$$dx_t^i = [-a(x_t^i - x^{out}) + b u_t^i] dt + \sigma dw_t^i, \quad (67)$$
$$x_0^i = \xi \sim \mathcal{N}(m, s^2),$$

where $s = 1$, $m = 21$, u^i is the energy demand for heating of the i^{th} household and $\{w_t^i,\ i = 1 : N,\ t \in [0,T]\}$ are independent Brownian motions. The heater dynamics are associated with cost functions reflecting the comfort versus financial cost relative priorities of household occupants throughout the day,

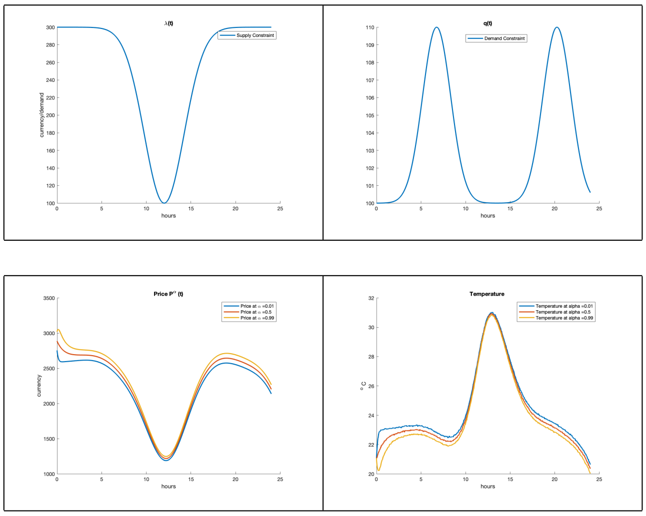
$$J^i(u^i, u^{-i}) = \quad (68)$$
$$\mathbb{E}\left[\int_0^T \left(\frac{q_t}{2}(x_t^i - z_t)^2 + \frac{r}{2}(u_t^i)^2 + P_t^{\alpha,N} u_t^i\right) dt\right],$$

where $(\lambda_t)_{t\in[0,T]}$, $(q_t)_{t\in[0,T]}$, $(z_t)_{t\in[0,T]}$ are deterministic strictly positive continuous processes, $u^i \in L_{\mathcal{F}}^2(0,T;\mathbb{R})$, and $r = 10$. The process $u^i \in L_{\mathcal{F}}^2(0,T;\mathbb{R})$ corresponds to the energy demand of the i^{th} household used for heating. The price of energy is an α-quantile of a linear function of the population of households' demands. We call this the quantilized price of energy and define it below, for all $t \in [0,T]$,

$$P_t^{\alpha,N} = \alpha - \text{quantile of the sample } \{\lambda_t u_t^i,\ i = 1 : N\}.$$

We choose the process $(z_t)_{t\in[0,T]}$ such that $z_t \equiv 21^o C$, λ_t and q_t to be minimal at mid-day, e.g. when solar generation cheaper and when the household members are absent. We consider the quantilized price of energy P_t^α over the whole horizon for three quantile levels $\alpha \in \{0.01, 0.5, 0.99\}$ associated to three risk sensitivities. When the energy provider prices according level $\alpha = 0.99$, it corresponds to the highest risk sensitivity level, where it is pricing based on the top 1 percent of household heaters. We numerically observe that the quantilized price of energy increases monotonically with quantile levels, while following the general structure of the solar energy generation costs, $(\lambda_t)_{t\in[0,T]}$. The demand response to the quantilized prices of energy and the typical dwelling temperatures are also numerically illustrated. At mid-day, when the prices are lowest the households temperatures increase, thereby preheating as a demand response mechanism.

Preprint submitted to Automatica
Received February 17, 2024 11:26:06 Pacific Time



7 Conclusion

In this paper, we have introduced a novel quantilized pricing scheme for energy providers. It is aimed at reflecting their real energy production costs in highly fluctuating intermittent renewable energy production environments. A quantilized mean field game of controls setting is utilized to identify, under appropriate technical conditions, a pricing trajectory resulting in an Epsilon-Nash equilibrium for a large population of users. Compared to the mean demand based pricing scheme, we can observe that second order demand statistics enter into the characterization of the equilibrium. In future works, one expects higher order demand moments would impact the equilibrium equations for more general non quadratic cost functions or non linear agent dynamics.

Acknowledgements

We gratefully acknowledge for the first and second author the support of grant R2271 (IVADO VORTEX) at the start of the study of this problem and the subsequent support for the first and last author in part by NSERC RGPIN-2019-05336, the U.S. ARL and ARO Grant W911NF1910110, and the U.S. AFOSR Grant FA9550-19-1-0138.

References

[1] Foguen R. T. Haurie A. Babonneau, F. and R. Malhame. Coupling a power dispatch model with a wardrop or mean-field-game equilibrium model. *Dynamic Games and Applications*, 11(2):217–241, 2021.

[2] Cvitanić J. Bayraktar, E. and Y. Zhang. Large tournament games. *The Annals of Applied Probability*, 29(6):3695–3744, 2019.

[3] P. Cardaliaguet and C. A. Lehalle. Mean field game of controls and an application to trade crowding. *Mathematics and Financial Economics*, 12(3):335–363, 2018.

[4] Yue Chen, Ana Bušić, and Sean Meyn. Individual risk in mean field control with application to automated demand response. In *53rd IEEE Conference on Decision and Control*, pages 6425–6432. IEEE, 2014.

[5] Dan Crisan, Thomas G Kurtz, and Yoonjung Lee. Conditional distributions, exchangeable particle systems, and stochastic partial differential equations. In *Annales de l'IHP Probabilités et statistiques*, volume 50, pages 946–974, 2014.

[6] Dena Firoozi and Peter E Caines. An optimal execution problem in finance with acquisition and liquidation objectives: an mfg formulation. *IFAC-PapersOnLine*, 50(1):4960–4967, 2017.

[7] Malhamé R. Foguen Tchuendom, R. and P. Caines. A quantilized mean field game approach to energy pricing with application to fleets of plug-in electric vehicles. In *58th Conference on Decision and Control (CDC)*, pages 299–304. IEEE, 2019.

[8] M. Fujii and A. Takahashi. A finite agent equilibrium in an incomplete market and its strong convergence to the mean-field limit. *SSRN 3714430*, 2020.

[9] Diogo A Gomes and J. Saude. A mean-field game approach to price formation. *Dynamic Games and Applications*, 11(1):29–53, 2021.

[10] Minyi Huang, Roland P Malhamé, and Peter E Caines. Large population stochastic dynamic games: closed-loop mckean-vlasov systems and the nash certainty equivalence principle. *Communications in Information & Systems*, 6(3):221–252, 2006.

[11] Minyi Huang and Xuwei Yang. Linear quadratic mean field games: Decentralized o (1/n)-nash equilibria. *Journal of Systems Science and Complexity*, 34(5):2003–2035, 2021.

[12] Salhab R. Kizilkale, A. C. and R. P. Malhamé. An integral control formulation of mean field game based large scale coordination of loads in smart grids. *Automatica*, 100:312–322, 2019.

[13] Aimé Lachapelle, Jean-Michel Lasry, Charles-Albert Lehalle, and Pierre-Louis Lions. Efficiency of the price formation process in presence of high frequency participants: a mean field game analysis. *Mathematics and Financial Economics*, 10(3):223–262, 2016.

[14] Jean-Michel Lasry and Pierre-Louis Lions. Mean field games. *Japanese journal of mathematics*, 2(1):229–260, 2007.

[15] Quentin Lénet, Md Salman Nazir, and Roland P Malhamé. An inverse nash mean field game-based strategy for the decentralized control of thermostatic loads. In *2021 60th IEEE Conference on Decision and Control (CDC)*, pages 4929–4935. IEEE, 2021.

[16] Johanna L Mathieu, Duncan S Callaway, and Sila Kiliccote. Variability in automated responses of commercial buildings and industrial facilities to dynamic electricity prices. *Energy and Buildings*, 43(12):3322–3330, 2011.

[17] R. J. Serfling. *Approximation theorems of mathematical statistics.* John Wiley Sons, 1980.

[18] R. C. Sonderegger. *Dynamic models of house heating based on equivalent thermal parameters.* Princeton University., 1978.

[19] Eduardo D Sontag. *Mathematical control theory: deterministic finite dimensional systems*, volume 6. Springer Science & Business Media, 2013.

[20] Fatemeh Tahersima, Jakob Stoustrup, Soroush Afkhami Meybodi, and Henrik Rasmussen. Contribution of domestic heating systems to smart grid control. In *2011 50th IEEE Conference on Decision and Control and European Control Conference*, pages 3677–3681. IEEE, 2011.

Preprint submitted to Automatica
Received February 17, 2024 11:26:06 Pacific Time