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GRAPHS OBTAINED BY DISJOINT UNIONS AND JOINS OF CLIQUES AND STABLE SETS

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Abstract. We consider the set of graphs that can be constructed from a one-vertex graph by repeatedly adding a clique or a stable set linked to all or none of the vertices added in previous steps. This class of graphs contains various well-studied graph families such as threshold, domishold, co-domishold and complete multipartite graphs, as well as graphs with linear clique-width at most 2. We show that it can be characterized by three forbidden induced subgraphs as well as by properties involving maximal stable sets and minimal dominating sets. We also give a simple recognition algorithm and formulas for the computation of the stability and domination numbers of these graphs.

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1. Introduction

The *disjoint union* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G = (V, E)$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$, while the *join* of G_1 and G_2 is the graph obtained from their disjoint union by linking each vertex of G_1 to each vertex of G_2. The *complement* $\overline{G}$ of a graph G is the graph with the same vertex as G and such that two distinct vertices are adjacent if and only if they are not adjacent in G. A set W of vertices is *stable* if no two vertices in W are linked by an edge, it is a *clique* if every two vertices in W are adjacent, and it is *dominating* if every vertex not in W is adjacent to at least one member of W. A *complete graph* is a graph whose vertex set is a clique, and an *empty graph* is the complement of a complete graph (*i.e.*, its vertex set is stable). The *stability number* $\alpha(G)$ of a graph G is the maximum size of a stable set in G, and the *domination number* $\beta(G)$ of G is the smallest size of a dominating set in G. The *order* of a graph is its number of vertices. A graph is H-free if it does not contain H as induced subgraph. In what follows, P_n, K_n and $K_{n,m}$ denote the path of order n, the complete graph of order n, and the join of an empty gaph of order n with an empty graph of order m, respectively.

In this paper, we define and give various characterization of *junion* graphs (contraction of *join* and *union*) which are built by repeatedly performing joins and disjoint unions with empty and complete graphs.

Definition 1.1. A *junion* graph is a graph that can be built from the one-vertex graph by iteratively performing one of the following four operations:

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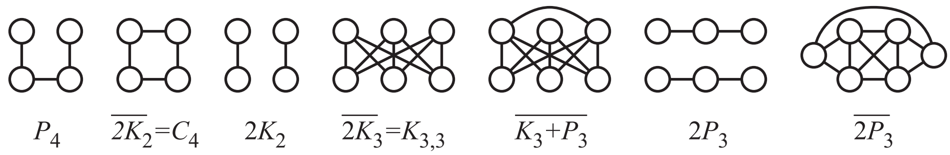

FIGURE 1. Some graphs.

– join with an empty graph;
– join with a complete graph of order at least 2;
– disjoint union with an empty graph;
– disjoint union with a complete graph of order at least 2.

Clearly, junion graphs are cographs (*i.e.*, P_4-free graphs) [6], the class of junion graphs is hereditary (*i.e.*, it is closed under induced subgraph containment), and complements of junion graphs are also junion graphs. Various well-studied graph families belong to the class of junion graphs. For example, *threshold* graphs [3] are constructed from the one-vertex graph by repeatedly applying one of the following two operations:

– join with the empty graph of order 1;
– disjoint union with the empty graph of order 1.

As another example, the *domishold* graphs [1] are constructed from the one-vertex graph by repeatedly applying one of the following two operations:

– join with the empty graph of order 1 or 2;
– disjoint union with the empty graph of order 1.

Additional junion graphs are *co-domishold* (complements of domishold) graphs, [2] *complete multipartite* graphs which are joins of empty graphs, and the *friendship* graphs which are the join of an empty graph of order 1 with the disjoint union of complete graphs of order 2.

As shown in [3] and [1], threshold and domishold graphs can be defined by properties on their stable and dominating sets, respectively. More precisely:

– a graph $G = (V, E)$ is a threshold graph if there are a real number T and for each vertex v a real vertex weight $\omega(v)$ such that a set $S \subseteq V$ is stable if and only if $\sum_{v \in S} \omega(v) \leq T$.
– a graph $G = (V, E)$ is a domishold graph if there are a real number T and for each vertex v a real vertex weight $\omega(v)$ such that a set $D \subseteq V$ is dominating if and only if $\sum_{v \in D} \omega(v) \geq T$.

The same authors also characterize threshold and domishold graphs by a list of forbidden induced subgraphs which are all depicted in Figure 1:

– a graph is a threshold graph if and only if it is $(P_4, 2K_2, \overline{2K_2})$-free;
– a graph is a domishold graph if and only if it is $(P_4, 2K_2, \overline{2K_3}, \overline{K_3 + P_3}, \overline{2P_3})$-free.

In the next section, we show that junion graphs are characterized by properties on their maximal stable sets and minimal dominating sets and by a list of three forbidden induced subgraphs. More precisely, we show that a graph is a junion graph if and only if it is $(P_4, 2P_3, \overline{2P_3})$-free.

A graph is a *split* graph if its vertex set can be partitioned in an stable set and a clique. Threshold graphs form a subclass of split graphs. A graph is $(P_4, \overline{2K_2}, 2P_3)$-free if and only if it can be obtained from a threshold graph by replacing every vertex in the stable set of the split partition by a clique. These graphs are studied in [5]. They are junion graphs since $\overline{2K_2}$ is an induced subgraph of $\overline{2P_3}$. They complement (*i.e.*, the class of $(P_4, 2K_2, \overline{2P_3})$-free graphs) are also junion graphs and are characterized in [4] as those graphs with linear-clique width at most 2. A picture of the inclusion relations between the mentioned subclasses of junion graphs is given in Figure 2.

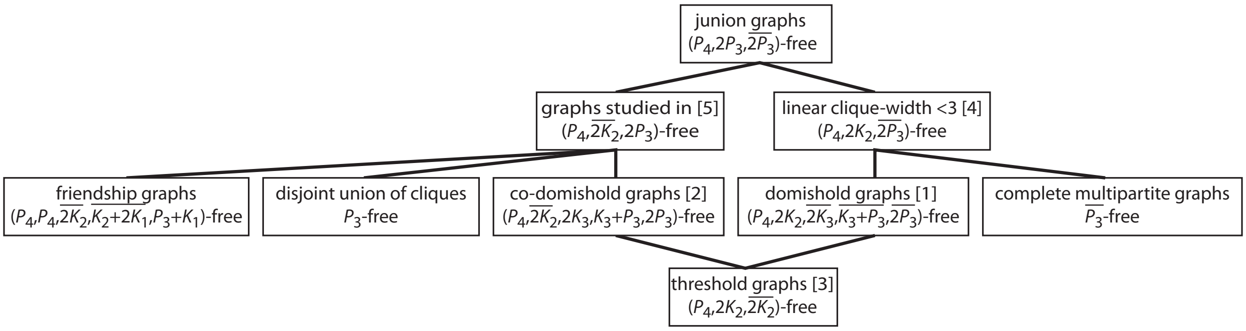

FIGURE 2. Subclasses of junion graphs.

2. Various characterizations of junion graphs

For a set W, an element $u \in W$ and an element $v \notin W$, let $W - u$ be the subset of W obtained by removing u and let $W + v$ be equal to $W \cup \{v\}$. Also, let $N_G(u)$ be the set of vertices adjacent to u in G. Consider the following property, where $\mathcal{S}(G)$ is the set of maximal stable sets in G.

Definition 2.1. A graph $G = (V, E)$ has Property P_S if and only if all pairs S_1, S_2 of disjoint sets in $\mathcal{S}(G)$ satisfy at least one of the following two conditions:

(a) there are $u_1 \in S_1$ and $u_2 \in S_2$ such that both $S_1 - u_1 + u_2$ and $S_2 - u_2 + u_1$ belong to $\mathcal{S}(G)$.
(b) $S_1 \cup N_G(u_1) = V$ for all $u_1 \in S_1$ or (non-exclusive) $S_2 \cup N_G(u_2) = V$ for all $u_2 \in S_2$.

Similarly, let $\mathcal{D}(G)$ be the set of minimal dominating sets in G.

Definition 2.2. A graph G has Property P_D if and only if all pairs D_1, D_2 of disjoint sets in $\mathcal{D}(G)$ satisfy at least one of the following two conditions:

(a) both $D_1 - u_1 + u_2$ and $D_2 - u_2 + u_1$ belong to $\mathcal{D}(G)$ for every two non-adjacent vertices $u_1 \in D_1$ and $u_2 \in D_2$,
(b) there is $u_1 \in D_1$ and $u_2 \in D_2$ such that $N_G(u_1) \cap D_2 = \{u_2\}$ and $N_G(u_2) \cap D_1 = \{u_1\}$.

We are now ready to state the theorem that gives equivalent characterizations for junion graphs.
Theorem The following statements are equivalent for all graphs G:

(i) G is a junion graph;
(ii) all induced subgraphs G' of G satisfy Property P_S;
(iii) all induced subgraphs G' of G satisfy Property P_D;
(iv) G is $(P_4, 2P_3, \overline{2P_3})$-free.

Proof.
(i)⇒(ii) and (iii). We proceed by induction on the number of vertices in G. Clearly, the graph of order 1 is a junion graph that satisfies Properties P_S and P_D. So assume that all junion graphs with at most n vertices satisfy Properties P_S and P_D and let G be a junion graph of order $n + 1$. Since the class of junion graphs is hereditary, it remains to prove that G satisfies P_S and P_D. Let O be the last operation performed to obtain G.

- If O is the disjoint union of a subgraph G' of G with an empty graph H, then all maximal stable sets and all minimal dominating sets of G contain all vertices of H. Hence, there are no pairs S_1, S_2 of disjoint sets in $\mathcal{S}(G)$ and no pairs D_1, D_2 of disjoint sets in $\mathcal{D}(G)$, which means that G satisfies P_S and P_D.
- If O is the disjoint union of a subgraph G' of G with a complete graph H of order at least 2, then all sets in $\mathcal{S}(G)$ and in $\mathcal{D}(G)$ contain exactly one vertex of H. Hence,

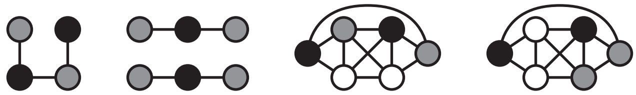

FIGURE 3. Pairs of maximal stable sets and minimal dominating sets in P_4, $2P_3$ and $\overline{2P_3}$.

- if S_1, S_2 are two disjoint maximal stable sets of G, S_1 contains a vertex u_1 of H and S_2 contains a vertex $u_2 \neq u_1$ of H. So (a) of P_S is satisfied since both $S_1 - u_1 + u_2$ and $S_2 - u_2 + u_1$ belong to $\mathcal{S}(G)$
- if D_1, D_2 are two disjoint minimal dominating sets of G, D_1 contains a vertex u_1 of H and D_2 contains a vertex $u_2 \neq u_1$ of H. So (b) of P_D is satisfied since $N_G(u_1) \cap D_2 = \{u_2\}$ and $N_G(u_2) \cap D_1 = \{u_1\}$.

– If O is the join of a subgraph G' of G with an empty graph H, then $\mathcal{S}(G') \subset \mathcal{S}(G)$ and $\mathcal{D}(G') \subset \mathcal{D}(G)$. Let S_1, S_2 be two disjoint sets in $\mathcal{S}(G)$ and let D_1, D_2 be two disjoint sets in $\mathcal{D}(G)$. If S_1 and S_2 do not contain any vertices of H, we know by induction hypothesis that (a) or (b) of P_S is satisfied. Similarly, (a) or (b) of P_D is satisfied if D_1 and D_2 do not contain any vertices of H. So assume, without loss of generality that S_1 and D_1 contain at least one vertex of H. Then
  - S_1 is the vertex set of H and (b) of P_S is satisfied since $S_1 \cup N_G(u) = V$ for all $u \in S_1$.
  - if D_2 contains no vertex of H then (a) of P_D is satisfied (since all $u_1 \in D_1$ are adjacent to all $u_2 \in D_2$); otherwise, $D_1 = \{v_1, w_1\}$ and $D_2 = \{v_2, w_2\}$ with v_1, v_2 in H and w_1, w_2 in G' which means that (a) of P_D is satisfied since both $\{v_1, w_2\}$ and $\{v_2, w_1\}$ belong to $\mathcal{D}(G)$.

– If O is the join of a subgraph G' of G with a complete graph H of order at least 2, then $\mathcal{S}(G)$ and $\mathcal{D}(G)$ are obtained from $\mathcal{S}(G')$ and $\mathcal{D}(G')$ by adding the singletons $\{u\}$ for all vertices u in H. Let S_1, S_2 be two disjoint sets in $\mathcal{S}(G)$ and let D_1, D_2 be two disjoint sets in $\mathcal{D}(G)$. As in the previous case, we can assume that S_1 and D_1 contain a vertex u of H, else (a) or (b) of P_S and P_D is satisfied. Then
  - $S_1 = \{u\}$ and (b) of P_S is satisfied since $S_1 \cup N_G(u) = V$.
  - $D_1 = \{u\}$ and (a) of P_D is satisfied since u is adjacent to all vertices in D_2.

(ii) or (iii) $\Rightarrow$ (iv). Suppose that G is a P_4, $2P_3$ or $\overline{2P_3}$. It is easy to verify that the black and grey maximal stable sets S_1 and S_2 depicted in Figure 3 (a), (b) and (c) show that G does not satisfy P_S. Also, the black and grey minimal dominating sets D_1 and D_2 depicted in Figure 3 (a), (b) and (d) show that G does not satisfy P_D.

(iv)$\Rightarrow$(i). We proceed by induction on the order of G. The result is clearly valid for the graph of order 1. So assume it is valid for graphs of order at most n and let G be a $(P_4, 2P_3, \overline{2P_3})$-free graph of order $n+1$. Since G has no induced P_4, it is well-known that either G or $\overline{G}$ is disconnected (see for example [6]).

– If G is disconnected then at least one of its connected component is a complete graph H, else G contains a $2P_3$ as induced subgraph. Hence, G is the disjoint union of a subgraph G' with an empty graph of order 1 or a complete graph of order at least 2.

– If $\overline{G}$ is disconnected then at least one of its connected component H is a complete graph, else G contains a $\overline{2P_3}$ as induced subgraph. Hence, G is the join of a graph G' with an empty graph of order 1 or a complete graph of order at least 2.

In both cases, the subgraph G' of G is $(P_4, 2P_3, \overline{2P_3})$-free and we know by induction hypothesis that G' is a junion graph. Hence, G is also a junion graph. □

3. Recognition algorithm and computation of the stability and domination numbers of junion graphs

We say that a connected component of a graph G is complete if it is a complete induced subgraph of G. If follows from Defintion 1.1 that a graph G is a junion graph if and only if exactly one of G and $\overline{G}$ is disconnected and has a complete connected component, and removing that complete connected component results in a junion graph. This leads to the following recognition algorithm for junion graphs G.

Recognition of junion graphs
Input: a graph G.

1. If both G and $\overline{G}$ are connected, then STOP : G is not a junion graph.
2. If G is connected then set $H \leftarrow \overline{G}$, else set $H \leftarrow G$.
3. If H has no complete connected component then STOP : G is not a junion graph.
4. Remove from H the vertices of all of its complete connected components.
5. If H is empty then STOP : G is a junion graph, else set $H \leftarrow \overline{H}$.
6. If H is connected then STOP : G is not a junion graph; else go to 3.

We finally show how to compute the domination and stability numbers of junion graphs. A vertex of a graph G is *universal* if it is adjacent to all other vertices of G. Clearly, $\beta(G) = 1$ if G has a universal vertex and $\beta(G) = 2$ if G is a connected junion graph with no universal vertex. Indeed, if a junion graph is connected than it is a join, and then a set with one vertex of each side of the join forms a dominating set. Note that a junion graph has at most one non-complete connected component. So, let $c(G)$ be the number of complete connected components of a graph G. If G is a junion graph, then

$$\beta(G) = \begin{cases} c(G) & \text{if all connected components of } G \text{ are complete,} \\ c(G)+1 & \text{if } G \text{ has a non-complete connected component with a universal vertex,} \\ c(G)+2 & \text{otherwise.} \end{cases}$$

Now, let $o(G)$ be the largest order of a complete connected component of a graph G and let G^c be the graph obtained from G by removing all the vertices of the complete connected components of G as well as all the vertices of the complete connected components of $\overline{G}$. Note that if G is a junion graphs, then G^c is a proper subgraph of G since exactly one of G or $\overline{G}$ has at least one complete connected component. The stability number $\alpha(G)$ of a junion graph G can easily be determined using the following recursive formula:

$$\alpha(G) = \begin{cases} \max\{o(\overline{G}), \alpha(G^c)\} & \text{if } G \text{ is connected,} \\ c(G) + \alpha(G^c) & \text{otherwise.} \end{cases}$$

Note finally that since the complement of a junion graph is a junion graph, the largest size $\omega(G) = \alpha(\overline{G})$ of a clique in a junion graph G is also easy to compute.

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