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Modeling and Optimal Control of Infectious Diseases

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Abstract: We propose a stochastic model of infectious disease transmission that is more realistic than those found in the literature. The model is based on jump-diffusion processes. However, it is defined in such a way that the number of people susceptible to be infected decreases over time, which is the case for a population of fixed size. Next, we consider the problem of finding the optimal control of the proposed model. The dynamic programming equation satisfied by the value function is derived. Estimators of the various model parameters are obtained.

Keywords: SIR model; jump-diffusion processes; parameter estimation; dynamic programming; homing problem

MSC: 60J70; 93E20

1. Introduction

Let $S(t)$ be the number of people susceptible (S) to a certain disease at time t, $I(t)$ the number of infected (I) and $R(t)$ the number of recovered (R) persons. In the classic SIR model proposed by Kermack and McKendrick [1], these variables satisfy the following system of three coupled non-linear ordinary differential equations:

$$S'(t) = -\beta S(t) I(t), \quad (1)$$

$$I'(t) = \beta S(t) I(t) - \gamma I(t), \quad (2)$$

$$R'(t) = \gamma I(t), \quad (3)$$

where β is the infection rate and γ is the recovery rate. Depending on the application, it is sometimes assumed that those who have recovered have developed immunity to the infectious disease. $R(t)$ can also represent the number of people who have died of the disease and thus been removed from the population.

Since the COVID-19 pandemic, numerous papers have been published on mathematical models for the transmission of infectious diseases. One way of making models more realistic is to incorporate stochastic processes into the equations describing the evolution of the epidemic over time. Often, the authors add diffusion processes like geometric Brownian motions in the above equations; see, for example, Gray et al. [2], Tornatore et al. [3], Zhou et al. [4] and Zhang et al. [5]. There are a few papers in which the authors used jump-diffusion processes in their models. An important question is to determine conditions that lead to the extinction or persistence of the disease.

Zhang and Wang [6] studied the asymptotic behavior of a stochastic SIR model with jump-diffusion processes. Zhou and Zhang [7] found a threshold, depending on the accumulated jump size, that can determine the extinction and persistence in mean of the epidemic considered. The same problem was addressed by Dieu et al. [8], Nguyen et al. [9], Tesfay et al. [10] and Privault and Wang [11]. In Zhao and Yuan [12], and also in Albani and Zubelli [13], this problem was considered with the parameter β (the *infection force*) that is assumed to evolve like a jump-diffusion process. In Lefebvre [14], it is the logarithm of new epidemic cases that is modeled as a Wiener process with jumps.

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A common assumption in deterministic or stochastic SIR models is that the size N of the population of interest is fixed, so that

$$S(t) + I(t) + R(t) = N. \tag{4}$$

This assumption can simply mean that we may neglect the births and deaths over a short period of time, when the size of the population is very large, for example, several million people. Notice that $S'(t) + I'(t) + R'(t)$ is indeed equal to zero in Equations (1)–(3). Some authors replace β by β/N in Equations (1) and (2) (or $S(t)$ by $S(t)/N$, which is mathematically equivalent). In the deterministic model, an infected person comes into contact with βN other persons per unit time (the *contact rate*).

Now, in the case of a population of fixed size, it is necessary to ensure that the number of people susceptible to be infected decreases over time. If we simply add a diffusion process to Equation (1), then $S(t)$ can increase with time. Indeed, any diffusion process both increases and decreases in any (however small) interval.

Let us denote $S(t)$, $I(t)$ and $R(t)$ by $X(t)$, $Y(t)$ and $Z(t)$, respectively. We propose the following model:

$$X(t) = X(0) - \beta \int_0^t X(s)\,Y(s)\,\mathrm{d}s - \int_0^t V(s)\,\mathrm{d}s - \sum_{i=1}^{N(t)} R_i, \tag{5}$$

$$V(t) = V(0) + \mu \int_0^t V(s)\,\mathrm{d}s + \sigma \int_0^t V(s)\,\mathrm{d}B(s), \tag{6}$$

$$\begin{aligned} Y(t) &= Y(0) + \beta \int_0^t X(s)\,Y(s)\,\mathrm{d}s - \gamma \int_0^t Y(s)\,\mathrm{d}s \\ &\quad + \int_0^t V(s)\,\mathrm{d}s + \sum_{i=1}^{N(t)} R_i, \end{aligned} \tag{7}$$

$$Z(t) = Z(0) + \gamma \int_0^t Y(s)\,\mathrm{d}s, \tag{8}$$

where μ is a real constant, σ is a positive constant, $\{B(t), t \geq 0\}$ is standard Brownian motion, $\{N(t), t \geq 0\}$ is a Poisson process (independent of $\{B(t), t \geq 0\}$) with rate λ, and $R_1, R_2, \ldots$ are continuous non-negative i.i.d. random variables. Therefore, $\sum_{i=1}^{N(t)} R_i$ is a *compound Poisson process*.

The process $\{V(t), t \geq 0\}$ is a geometric Brownian motion, which is always positive (if $V(0) > 0$). It follows that $X(t)$ will strictly decrease with time, as required. This type of process is used to describe the wear of machines; see Rishel [15]. In the case of wear processes, $V(t)$ is a random variable that influences the wear of the machine. For instance, it can be its operating speed. Here, $V(t)$ could be the strength or risk of infection at time t, which changes over time due in particular to vaccination.

Remark 1. *We emphasize that the main difference between the current paper and the various papers in which diffusion or jump-diffusion processes have been added to the basic SIR model is the fact that in the model that we propose, the variable* $X(t)$ $(= S(t))$ *representing the number of susceptible individuals at time* t *is a decreasing function of time. As mentioned above, to obtain this characteristic with diffusion processes, it is compulsory to introduce at least one new variable into the model, in our case,* $V(t)$.

In the next section, we will consider a controlled version of the system defined by Equations (5)–(8) and we will address the problem of finding the optimal control of the proposed model. We will use the mathematical technique called *dynamic programming* to derive the equation satisfied by what is known as the *value function*. We will see how we can estimate the expected value of $Y'(0)$, which is an important quantity in epidemiology. We will also explain how to estimate the various model parameters, and finally an illustrative example will be presented. We will end this paper with some remarks in Section 3.

2. Optimal Control

We now consider a controlled version of the system defined by Equations (5)–(8):

$$X(t) = X(0) - \beta \int_0^t X(s)\,Y(s)\,\mathrm{d}s - \int_0^t V(s)\,\mathrm{d}s - \sum_{i=1}^{N(t)} R_i, \tag{9}$$

$$V(t) = V(0) + \mu \int_0^t V(s)\,\mathrm{d}s + \sigma \int_0^t V(s)\,\mathrm{d}B(s), \tag{10}$$

$$\begin{aligned} Y(t) &= Y(0) + \beta \int_0^t X(s)\,Y(s)\,\mathrm{d}s - \gamma \int_0^t Y(s)\,\mathrm{d}s + \int_0^t V(s)\,\mathrm{d}s + \sum_{i=1}^{N(t)} R_i \\ &\quad - b \int_0^t u[X(s), V(s), Y(s)]\,\mathrm{d}s, \end{aligned} \tag{11}$$

$$Z(t) = Z(0) + \gamma \int_0^t Y(s)\,\mathrm{d}s + b \int_0^t u[X(s), V(s), Y(s)]\,\mathrm{d}s, \tag{12}$$

where b is a positive constant and the function $u(\cdot)$ is the control variable, which is assumed to be a continuous function.

We define the *first-passage time*

$$\tau(x, v, y) = \inf\{t > 0 : Y(t) = y_1 \mid X(0) = x, V(0) = v, Y(0) = y > y_1 \geq 0\}, \tag{13}$$

where x and v are positive.

We are looking for the value $u^*[X(t), V(t), Y(t)]$, for $0 \leq t < \tau(x, v, y)$, of the control variable that minimizes the expected value of the cost function

$$J(x, v, y) := \int_0^{\tau(x,v,y)} \left\{ \frac{1}{2} q\, u^2[X(t), V(t), Y(t)] + \theta \right\} \mathrm{d}t, \tag{14}$$

where $q > 0$ and θ are positive constants. There could be a final cost function $K(\cdot)$.

Because the parameter θ is positive, the optimizer must try to minimize the time it takes $Y(t)$ to decrease to the value y_1, taking the quadratic control costs into account. A stochastic optimal control problem of this type, in which the optimizer controls a stochastic process until a certain random event takes place, is known as a *homing problem*; see Whittle [16] and/or [17].

To solve our homing problem, we will use dynamic programming. We define the *value function*

$$F(x, v, y) = \inf_{\substack{u[X(t),V(t),Y(t)] \\ t \in [0, \tau(x,v,y))}} E[J(x, v, y)]. \tag{15}$$

The function $F(x, v, y)$ gives the expected cost (or reward, if it is negative) obtained if the optimizer chooses the optimal value of $u[X(t), V(t), Y(t)]$ in the interval $[0, \tau(x, v, y))$.

Remark 2. *Notice that the random variable $\tau(x, v, y)$ and the control variable do not depend on the variable $Z(t)$. Therefore, $Z(t)$ can be dropped from the model for the problem we are interested in.*

We will prove the following proposition.

Proposition 1. *Let $f_R(r)$ be the common density function of the random variables $R_1, R_2, \ldots$ Let $V(0) = v$, $u(t) := u[X(t), V(t), Y(t)]$ and $u_0 := u(x, v, y)$. The value function $F(x, v, y)$ satisfies the dynamic programming equation*

$$\begin{aligned} 0 = \inf_{u_0} \Big\{ & \frac{1}{2} q\, u_0^2 + \theta - (\beta x y + v)\, F_x + \mu v F_v + \frac{1}{2} \sigma^2 v^2 F_{vv} \\ & + (\beta x y - \gamma y + v - b u_0)\, F_y + \lambda \int_0^\infty F(x - r, v, y + r)\, f_R(r)\,\mathrm{d}r \\ & - \lambda F(x, v, y) \Big\}. \end{aligned} \tag{16}$$

Proof. We have

$$J(x,v,y) = \int_0^{\Delta t} \left\{ \frac{1}{2} q\, u^2(t) + \theta \right\} \mathrm{d}t + \int_{\Delta t}^{\tau(x,v,y)} \left\{ \frac{1}{2} q\, u^2(t) + \theta \right\} \mathrm{d}t. \tag{17}$$

We can write that

$$\int_0^{\Delta t} \left\{ \frac{1}{2} q\, u^2(t) + \theta \right\} \mathrm{d}t = \left(\frac{1}{2} q\, u_0^2 + \theta \right) \Delta t + o(\Delta t). \tag{18}$$

Moreover, making use of Bellman's principle of optimality, we can express the infimum with respect to $u(t)$ of the expected value of the second integral in Equation (17) in terms of the value function:

$$\begin{aligned} &\inf_{\substack{u(t) \\ t \in [\Delta t, \tau(x,v,y))}} E\left[\int_{\Delta t}^{\tau(x,v,y)} \left\{ \frac{1}{2} q\, u^2(t) + \theta \right\} \mathrm{d}t \right] \\ &\approx E\Bigg[F\Bigg(x - (\beta x y + v)\,\Delta t - \sum_{i=1}^{N(\Delta t)} R_i, v + \mu v \Delta t + \sigma v B(\Delta t), \\ &\qquad y + (\beta x y - \gamma y + v - b u_0)\,\Delta t + \sum_{i=1}^{N(\Delta t)} R_i \Bigg) \Bigg]. \end{aligned} \tag{19}$$

Next, let

$$\left.\begin{aligned} \xi &:= x - (\beta x y + v)\,\Delta t, \\ \eta &:= v + \mu v \Delta t + \sigma v B(\Delta t), \\ \zeta &:= y + (\beta x y - \gamma y + v - b u_0)\,\Delta t. \end{aligned}\right\} \tag{20}$$

Using the well-known formula $E[X_1] = E[E[X_1 \mid X_2]]$, we can write that

$$\begin{aligned} &E\left[F\left(\xi - \sum_{i=1}^{N(\Delta t)} R_i, \eta, \zeta + \sum_{i=1}^{N(\Delta t)} R_i \right) \right] \\ &= E\left[E\left[F\left(\xi - \sum_{i=1}^{N(\Delta t)} R_i, \eta, \zeta + \sum_{i=1}^{N(\Delta t)} R_i \right) \right] \,\middle|\, N(\Delta t) \right]. \end{aligned} \tag{21}$$

Moreover, $N(\Delta t)$ has a Poisson distribution with parameter $\lambda\,\Delta t$. It follows that

$$P[N(\Delta t) = 0] = e^{-\lambda \Delta t} = 1 - \lambda\,\Delta t + o(\Delta t) \tag{22}$$

and

$$P[N(\Delta t) = 1] = \lambda\,\Delta t\, e^{-\lambda \Delta t} = \lambda\,\Delta t + o(\Delta t), \tag{23}$$

so that

$$P[N(\Delta t) \geq 2] = o(\Delta t). \tag{24}$$

Therefore,

$$\begin{aligned} E\left[F\left(\xi - \sum_{i=1}^{N(\Delta t)} R_i, \eta, \zeta + \sum_{i=1}^{N(\Delta t)} R_i \right) \right] &= E[F(\xi,\eta,\zeta)]\,(1 - \lambda\,\Delta t) \\ &\quad + E[F(\xi - R_1, \eta, \zeta + R_1)]\,\lambda\,\Delta t \\ &\quad + o(\Delta t). \end{aligned} \tag{25}$$

Now, assuming that $F(x,v,y)$ is differentiable with respect to x and y, and twice differentiable with respect to v, we deduce from Taylor's formula for functions of three variables that

$$\begin{aligned} F(\xi,\eta,\zeta) &= F(x,v,y) - (\beta x y + v)\,\Delta t\, F_x + [\mu v \Delta t + \sigma v B(\Delta t)]\, F_v \\ &\quad + (\beta x y - \gamma y + v - b u_0)\,\Delta t\, F_y + \frac{1}{2} [\mu v \Delta t + \sigma v B(\Delta t)]^2 F_{vv} \\ &\quad + o(\Delta t). \end{aligned} \tag{26}$$

For a standard Brownian motion $\{B(t), t \geq 0\}$, we have $E[B(\Delta t)] = 0$ and $E[B^2(\Delta t)] = V[B(\Delta t)] = \Delta t$. Hence,

$$\begin{aligned} E[F(\xi,\eta,\zeta)] &= F(x,v,y) - (\beta xy + v)\,\Delta t\, F_x + \mu v \Delta t\, F_v \\ &\quad + (\beta xy - \gamma y + v - b u_0)\,\Delta t\, F_y + \frac{1}{2}\sigma^2 v^2 \Delta t\, F_{vv} \\ &\quad + o(\Delta t), \end{aligned} \tag{27}$$

which implies that

$$\begin{aligned} E[F(\xi,\eta,\zeta)]\,(1-\lambda\Delta t) &= F(x,v,y) - (\beta xy + v)\,\Delta t\, F_x + \mu v \Delta t\, F_v \\ &\quad + (\beta xy - \gamma y + v - b u_0)\,\Delta t\, F_y + \frac{1}{2}\sigma^2 v^2 \Delta t\, F_{vv} \\ &\quad - F(x,v,y)\,\lambda\Delta t + o(\Delta t). \end{aligned} \tag{28}$$

Next, by independence, we have

$$E[F(\xi - R_1, \eta, \zeta + R_1)] = \int_0^\infty E[F(\xi - r, \eta, \zeta + r)]\, f_R(r)\, \mathrm{d}r \tag{29}$$

and we deduce from Equation (27) that

$$E[F(\xi - r, \eta, \zeta + r)]\,\lambda\Delta t = F(x - r, v, y + r)\,\lambda\Delta t + o(\Delta t). \tag{30}$$

Thus,

$$E[F(\xi - R_1, \eta, \zeta + R_1)]\,\lambda\,\Delta t = \lambda\,\Delta t \int_0^\infty F(x - r, v, y + z)\, f_R(r)\, \mathrm{d}r + o(\Delta t). \tag{31}$$

From what precedes, we can write that

$$\begin{aligned} 0 &= \inf_{u_0}\left\{\left(\frac{1}{2}q u_0^2 + \theta\right)\Delta t - (\beta xy + v)\,\Delta t\, F_x + \mu v \Delta t\, F_v \right. \\ &\quad + (\beta xy - \gamma y + v - b u_0)\,\Delta t\, F_y + \frac{1}{2}\sigma^2 v^2 \Delta t\, F_{vv} - F(x,v,y)\,\lambda\Delta t \\ &\quad \left. + \lambda\,\Delta t \int_0^\infty F(x - r, v, y + r)\, f_R(r)\, \mathrm{d}r + o(\Delta t)\right\}. \end{aligned} \tag{32}$$

Finally, dividing both sides of the above equation by Δt and letting Δt decrease to zero, we obtain Equation (16). □

Now, we deduce from Equation (16) that the optimal control $u_0^* = u^*(x,v,y)$ is given by

$$u^*(x,v,y) = -\frac{b}{q} F_y(x,v,y). \tag{33}$$

Substituting this expression for $u^*(x,v,y)$ into Equation (16), we obtain the following proposition.

Proposition 2. *The value function $F(x,v,y)$ satisfies the second-order, non-linear partial integro-differential equation (PIDE)*

$$\begin{aligned} 0 &= \theta - \frac{1}{2}\frac{b^2}{q} F_y^2 - (\beta xy + v)\,F_x + \mu v F_v + (\beta xy - \gamma y + v)\,F_y \\ &\quad + \frac{1}{2}\sigma^2 v^2 F_{vv} + \lambda \int_0^\infty F(x - r, v, y + r)\, f_R(r)\, \mathrm{d}r - \lambda F(x,v,y), \end{aligned} \tag{34}$$

subject to the boundary condition (which follows at once from the fact that $\tau(x,v,y_1) = 0$)

$$F(x,v,y_1) = 0. \tag{35}$$

Remark 3. *(i) Since $X(t) \geq 0$, the integral in Equation (34) can be written as*

$$\int_0^x F(x-r,v,y+r)\, f_R(r)\, \mathrm{d}r. \tag{36}$$

(ii) We can consider the case when $\mu = \sigma = 0$, so that the jump-diffusion process reduces to a compound Poisson process. Similarly, if we set $\lambda = 0$, then there are no jumps and the noise in the model is a simple diffusion process. Moreover, we could replace the geometric Brownian motion by any non-negative diffusion process. We could also replace $\int_0^t V(s)\,\mathrm{d}s$ in Equations (9) and (11) by $\int_0^t h[V(s)]\,\mathrm{d}s$, where $h(\cdot)$ is a non-negative function. Then, we could use any diffusion process $\{V(t), t \geq 0\}$ in the model.

(iii) We can generalize the definition of the first-passage time $\tau(x,v,y)$. Let $X(0) = x$, $V(0) = v$ and $Y(0) = y$. We define

$$\tau(x,v,y) = \inf\{t > 0 : Y(t) = y_1 \text{ or } Y(t) \geq y_2\}, \tag{37}$$

where $0 \leq y_1 < y < y_2$. If $Y(t)$ increases to a value greater than or equal to y_2, we stop the optimal control problem and we set

$$K[Y(\tau(x,v,y))] = \begin{cases} K_1 & \text{if } Y(\tau(x,v,y)) = y_1, \\ K_2 & \text{if } Y(\tau(x,v,y)) \geq y_2, \end{cases} \tag{38}$$

where $0 \leq K_1 < K_2$. We would then have the following boundary conditions: $F(x,v,y_1) = K_1$ and $F(x,v,y) = K_2$ if $y \geq y_2$.

2.1. Particular Cases

Case I. The simplest problem that we can consider is the one for which the jumps are of constant size $r_0 > 0$, so that

$$f_R(r) = \delta(r - r_0), \tag{39}$$

where $\delta(\cdot)$ is the Dirac delta function. Equation (34) then reduces to

$$\begin{aligned} 0 \;=\; \theta - \frac{1}{2}\frac{b^2}{q} F_y^2 - (\beta x y + v) F_x + \mu v F_v + (\beta x y - \gamma y + v) F_y \\ + \frac{1}{2}\sigma^2 v^2 F_{vv} + \lambda F(x - r_0, v, y + r_0) - \lambda F(x,v,y). \end{aligned} \tag{40}$$

Moreover, if $r_0 \geq x$, $F(x - r_0, v, y + r_0) = 0$. This case can be generalized to any positive discrete random variable R.

Case II. If R is uniformly distributed on the interval $(0, r_0)$, where $r_0 \leq x$, then

$$\int_0^\infty F(x-r,v,y+r)\, f_R(r)\, \mathrm{d}r = \frac{1}{r_0}\int_0^{r_0} F(x-r,v,y+r)\, \mathrm{d}r. \tag{41}$$

If $r_0 > x$, we have

$$\int_0^\infty F(x-r,v,y+r)\, f_R(r)\, \mathrm{d}r = \frac{1}{r_0}\int_0^{x} F(x-r,v,y+r)\, \mathrm{d}r. \tag{42}$$

Case III. A common assumption for jump-diffusion processes is that R is exponentially distributed with parameter α. Then, we have

$$\int_0^\infty F(x-r,v,y+r)\, f_R(r)\, \mathrm{d}r = \int_0^{x} F(x-r,v,y+r)\, \alpha e^{-\alpha r}\, \mathrm{d}r. \tag{43}$$

2.2. Estimation of the Expected Value of $Y'(0)$

The quantity $Y'(0)$ is important in determining whether the epidemic will die out or continue to spread. In our case, $Y'(0)$ is a random variable. We will try to estimate its expected value. We have

$$X(t) + Y(t) + Z(t) \equiv X(0) + Y(0) + Z(0) := N. \tag{44}$$

For the uncontrolled process, we have

$$\begin{aligned} \frac{Y(t)}{N} &= \frac{Y(0)}{N} + \beta \int_0^t \frac{X(s)}{N} Y(s)\,\mathrm{d}s - \gamma \int_0^t \frac{Y(s)}{N}\,\mathrm{d}s \\ &\quad + \frac{1}{N}\left[\int_0^t V(s)\,\mathrm{d}s + \sum_{i=1}^{N(t)} R_i\right]. \end{aligned} \tag{45}$$

If t is small, we can write that

$$\frac{X(t)}{N} \approx 1. \tag{46}$$

It follows that

$$\begin{aligned} E[Y(t)] &\approx Y(0) + \beta N \int_0^t E[Y(s)]\,\mathrm{d}s - \gamma \int_0^t E[Y(s)]\,\mathrm{d}s \\ &\quad + \int_0^t E[V(s)]\,\mathrm{d}s + E\left[\sum_{i=1}^{N(t)} R_i\right] \\ &\approx Y(0) + \beta N Y(0)\,t - \gamma Y(0)\,t + \int_0^t E[V(s)]\,\mathrm{d}s + E\left[\sum_{i=1}^{N(t)} R_i\right]. \end{aligned} \tag{47}$$

We have (see Lefebvre [18])

$$E[V(t)] = V(0)\,\exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)t\right\} \tag{48}$$

and

$$E\left[\sum_{i=1}^{N(t)} R_i\right] = \lambda\, t\, E[R_1]. \tag{49}$$

Therefore, assuming that $\mu + \frac{\sigma^2}{2} \neq 0$,

$$\begin{aligned} E[Y(t)] - Y(0) &\approx (\beta N - \gamma)\, Y(0)\, t \\ &\quad + \frac{V(0)}{\mu + \frac{\sigma^2}{2}}\left[\exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)t\right\} - 1\right] + \lambda\, t\, E[R_1]. \end{aligned} \tag{50}$$

We estimate $E[Y'(t)]$ for small t values as follows:

$$\begin{aligned} E[Y'(t)] &\approx \frac{E[Y(t)] - Y(0)}{t} \approx (\beta N - \gamma)\, Y(0) \\ &\quad + \frac{V(0)}{\mu + \frac{\sigma^2}{2}} \frac{\left[\exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)t\right\} - 1\right]}{t} + \lambda\, E[R_1]. \end{aligned} \tag{51}$$

Hence, we can write that

$$E[Y'(0)] \approx (\beta N - \gamma)\, Y(0) + V(0) + \lambda\, E[R_1]. \tag{52}$$

Remark 4. *If $\mu + \frac{\sigma^2}{2} = 0$, then $E[V(t)] \equiv V(0)$ and we obtain the same approximate formula for $E[Y'(0)]$.*

2.3. Model Parameters Estimation

The problem of estimating the various parameters in stochastic SIR models has been considered, in particular, by Rica and Ruz [19]. El Kharrazi and Saoud [20] studied this problem in the case of a SIR model with jump-diffusion.

In the model that we propose, there are five parameters that must be estimated: β, γ, μ, σ^2 and λ. Moreover, we must also estimate the expected value of the random variables R_i, $i = 1, \ldots$ As mentioned above, a common assumption in applications of jump-diffusion processes is that R_i has an exponential distribution with parameter α. In our case, it is rather $R_i - \kappa$ that could be exponentially distributed, where κ is the threshold chosen in the

definition of a jump. We must then estimate the parameter α and perform a goodness-of-fit statistical test to check whether the exponential distribution assumption is plausible.

First, to estimate λ and $E[R_i]$ from a particular random sample, we define what constitutes a *jump* (which is always rather subjective) and we can then easily determine point estimates of λ and $E[R_i]$ from the data collected.

Next, we can estimate the parameter γ by making use of Equation (8). In practice, we need to determine the value of t_0 at which the epidemic began. We write

$$Z(t) - Z(t_0) = \gamma \int_{t_0}^{t} [Y(s) - Y(t_0)]\,\mathrm{d}s. \tag{53}$$

It is not difficult to approximate the above integral from the observed values of $Y(t)$. The point estimate of the parameter γ can be obtained by using the formula for linear regression through the origin.

For the parameters β, μ and σ^2, we can proceed as follows: we replace $V(s)$ in Equation (7) by its expected value, namely

$$E[V(s)] = V(0)\,\exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)s\right\},$$

as well as $\sum_{i=1}^{N(t)} R_i$ by its expected value $\hat{\lambda}\,t\,\widehat{E[R_i]}$, where $\hat{q}$ denotes the point estimate of the quantity q.

Remark 5. *It is generally assumed that a geometric Brownian motion starts at 1, because $V(t) = e^{W(t)}$ and $W(0)$ is usually taken to be 0. However, we could estimate $V(0)$ to be more general.*

It follows (setting $t_0 = 0$ and $V(0) = 1$) that

$$\begin{aligned} Y(t) \;\approx\; & Y(0) + \beta \int_0^t X(s)\,Y(s)\,\mathrm{d}s - \hat{\gamma} \int_0^t Y(s)\,\mathrm{d}s \\ & + \int_0^t \exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)s\right\}\mathrm{d}s + \hat{\lambda}\,t\,\widehat{E[R_i]}, \end{aligned} \tag{54}$$

which implies that

$$Y'(t) \approx \beta\,X(t)\,Y(t) - \hat{\gamma}\,Y(t) + \exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)t\right\} + \hat{\lambda}\,\widehat{E[R_i]}. \tag{55}$$

Let

$$\eta(t) := Y'(t) + \hat{\gamma}\,Y(t) - \hat{\lambda}\,\widehat{E[R_i]} \tag{56}$$

and

$$\nu := \mu + \frac{\sigma^2}{2}. \tag{57}$$

We have the equation

$$\eta(t) = \beta\,X(t)\,Y(t) + \exp\{\nu t\}. \tag{58}$$

We can estimate $\eta(t)$ from the data and perform non-linear regression to find point estimates of β and ν.

Finally, the variance of $V(t)$ is given by

$$V[V(t)] = \exp\left\{\left(\mu + \frac{\sigma^2}{2}\right)2t\right\}\left(e^{\sigma^2 t} - 1\right), \tag{59}$$

so that

$$E\left[V^2(t)\right] = \exp\{2\nu t\}\,e^{\sigma^2 t}. \tag{60}$$

Once we have calculated $\hat{\beta}$, we can define

$$\phi(t) = Y'(t) - \hat{\beta}\,X(t)\,Y(t) + \hat{\gamma}\,Y(t) - \hat{\lambda}\,\widehat{E[R_i]}. \tag{61}$$

We have

$$\phi^2(t) \approx \exp\{2\hat{v}\,t\}\, e^{\sigma^2 t}. \tag{62}$$

With the help of the data, it is possible to use non-linear regression to estimate σ^2*, from which we deduce the value of*

$$\hat{\mu} = \hat{v} - \frac{1}{2}\hat{\sigma}^2.$$

2.4. An Illustrative Example

Baldé [21] has used the data for (China and) France from *Wolfram Research* for the COVID-19 pandemic, for the time period 22 January–31 March 2020. The data can be viewed on the https://www.wolframcloud.com website. The number of daily *infected*, *recovered* and *death* cases are shown in Figures 1–3. The random variable $Y(t)$ is the total number of infected persons at time t, while $Z(t)$ is the sum of the persons who have recovered from the disease and the dead ones. Moreover, $X(t) \approx N$, where N is the total population of France, which was 65,241,903 people during the time period considered.

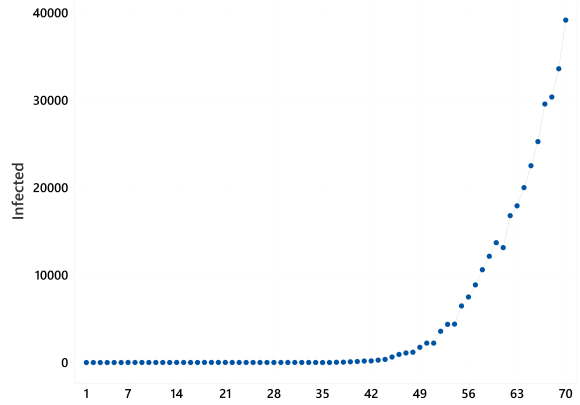

Figure 1. Daily infected COVID-19 cases in France for the time period 22 January–31 March 2020.

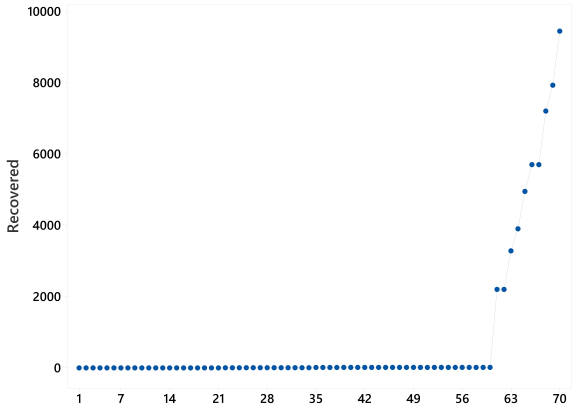

Figure 2. Daily recovered COVID-19 cases in France for the time period 22 January–31 March 2020.

We estimate that the pandemic really began on 7 March (that is, $t_0 = 36$). There are therefore 35 observations of $Y(t)$ and $Z(t)$ in the data set, which is rather small. However, it is sufficient to illustrate the technique that we propose to estimate the various model parameters.

On 7 March, we have $Y(t_0) = 220$ and $Z(t_0) = 68 + 13 = 81$. We define a *jump* to be an increase of at least 500 infected persons in one day. We find that there are 18 jumps from 7 to 31 March, so that our point estimate of the rate λ of the Poisson process is $\hat{\lambda} = 18/35$. The average jump size is 2113. Hence, we set $\widehat{E[R_i]} = 2113$.

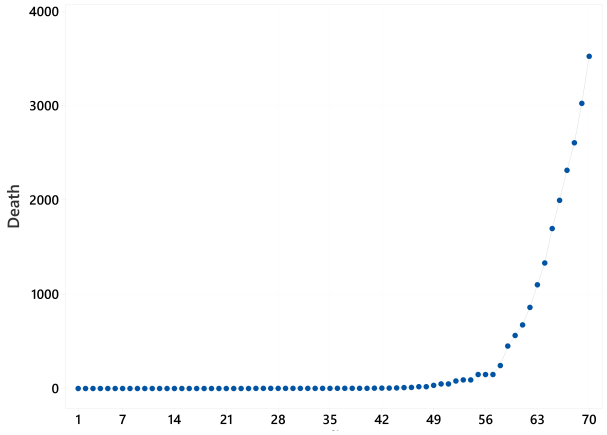

Figure 3. Daily COVID-19 deaths in France for the time period 22 January–31 March 2020.

Moreover, the histogram of the jumps is presented in Figure 4. We see that the form of the histogram corresponds to a shifted exponential distribution. Performing Pearson's goodness-of-fit test with the subintervals $[500, 1500)$, $[1500, 2500)$, $[2500, 3500)$ and $[3500, \infty)$, we find that we can easily accept the hypothesis that $R_i - 500$ follows (approximately) an exponential distribution with parameter $\alpha = 1/2113$. The p-value of the test is equal to 0.51.

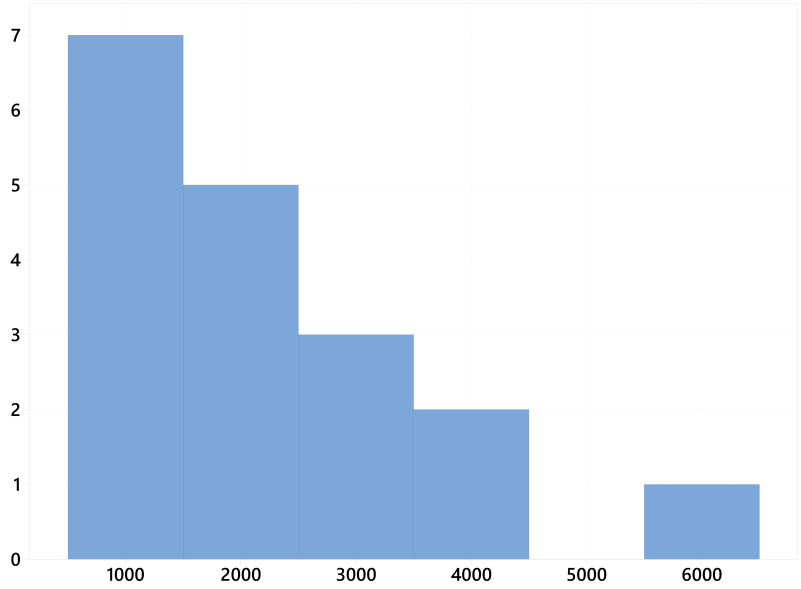

Figure 4. Histogram of the jumps from 7 to 31 March 2020.

Next, proceeding as explained above, we find that $\hat{\gamma} \approx 0.1753$. It follows that $\hat{\beta} \approx 1.78 \times 10^{-8}$ and $\hat{v} \approx 0.2461$. Notice that we have $\hat{\beta} N \approx 1.1615$. Finally, we obtain that $\hat{\sigma}^2 \approx 0.0491$, from which we calculate $\hat{\mu} \approx 0.2216$.

3. Conclusions

In this paper, we proposed a stochastic SIR model that respects the constraint that the number $S(t)$ of people susceptible to becoming infected decreases over time, as it should if the population size is fixed. To do this, when a diffusion process is used in the model, the derivative of $S'(t)$ must depend, for instance, on the integral of a non-negative diffusion process. Therefore, we must add at least one equation to the basic model.

In Section 2, we considered the problem of finding the optimal control of the model in a homing problem. We derived the equation satisfied by the value function, which is a second-order, non-linear partial integro-differential equation. Finding the explicit solution

to this equation, under the appropriate conditions, is of course a very difficult task. We can, however, use numerical methods to obtain the solution in any particular case. We can also simulate the model. All we need is a generator of Gaussian, exponential and Poisson distributions.

At the end of Section 2, an example was presented to illustrate how the various parameters of the model can be estimated. More data would be needed to check whether the model can successfully forecast the evolution of the epidemic. However, even if the proposed model is more realistic than those found in the literature, this does not mean that it will give better results in all cases in practice. Indeed, it is well known that very simple, but clearly wrong, models often give good results, because they are easier to implement. However, if we can implement our model, the results should generally be at least as good as with simpler models, while being closer to reality.

The same kind of work that we have presented in this paper could be carried out for more complex models of infectious disease transmission than the basic SIR model. There could also be more than one diffusion process in the model.

Finally, it would be interesting to obtain the moments of the first-passage time random variable τ, whether in the case of the controlled or uncontrolled process. Again, this could be carried out using simulations.

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