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**Control of Chaos in Microelectromechanical Systems**

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Département de génie électrique

Thèse présentée en vue de l'obtention du diplôme de *Philosophiæ Doctor*

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**Control of Chaos in Microelectromechanical Systems**

présentée par **Alexander JIMENEZ TRIANA**

en vue de l'obtention du diplôme de *Philosophiæ Doctor*  
a été dûment acceptée par le jury d'examen constitué de :

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**Frédéric NABKI**, membre externe

**DEDICATION**

*To my grandma  
who is always present in my thoughts,  
to my beloved mother and my dear son  
who give me strength and light my way,  
and to Marcela, the love of my life.  
I love you all.*

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## RÉSUMÉ

Les systèmes microélectromécaniques (MEMS) ont révolutionné le marché électronique à travers le monde et constituent l'une des technologies les plus prometteuses pour la prochaine décennie. Les applications encourageantes des résonateurs MEMS, ainsi que la demande de capteurs de tailles plus petits tout en augmentant le rapport signal sur bruit, rendent nécessaire d'augmenter l'amplitude de l'oscillation de ces dispositifs à un effort de contrôle réduit. La recherche de capteurs et d'appareils électroniques ne consommant que de petites quantités d'énergie est constante et la résolution du problème de consommation d'énergie est une tâche prioritaire si nous avons besoin d'économiser de l'énergie et de réduire les changements de batterie.

Le présent travail est consacré au contrôle du chaos dans les MEMS en tant qu'approche pour améliorer la performance des oscillateurs MEMS permettant d'augmenter l'amplitude de leurs oscillations tout en réduisant l'effort de contrôle. L'étude est réalisée dans des conditions opérationnelles réalistes incluant des incertitudes paramétriques, des dynamiques non modélisées, la présence de perturbations, du bruit et des limitations dans la mesure de la vitesse des plaques des résonateurs MEMS.

L'amplitude d'oscillation des résonateurs électrostatiques MEMS est limitée par les phénomènes dits pull-in liés à une bifurcation survenant lors de l'actionnement électrostatique. La plage de déplacement stable est limitée à un tiers de son écart complet en raison de l'instabilité au-delà de ce point. Les phénomènes de pull-in peuvent être utilisés pour accélérer la vitesse de réaction des structures de commutation. Cependant, il limite la plage de fonctionnement stable du dispositif, ce qui est obligatoire pour les applications nécessitant une oscillation soutenue des plaques. Afin de faire fonctionner des dispositifs MEMS électrostatiques au-delà du pull-in, nous exploitons la dynamique non linéaire du système, qui est conçu pour fonctionner en régime chaotique pour générer un nombre infini d'orbites périodiques instables qui sont immergés dans l'attracteur chaotique. L'une de ces orbites est convenablement choisie pour produire une oscillation de forte amplitude, et est stabilisée en utilisant différents schémas de commande. Étant donné que l'orbite instable existe déjà dans le système et qu'il n'est pas nécessaire de dépenser de l'énergie pour la créer, l'effort du contrôleur est réduit.

Plusieurs schémas de contrôle pour stabiliser le système chaotique sont envisagés, et en raison de la difficulté de mesurer la vitesse des plaques pendant le fonctionnement de l'appareil, nous utilisons des techniques de contrôle de rétroaction de sortie comprenant un schéma de retour

en arrière adaptatif de l'observateur et une approche de rejet actif des perturbations (ADRC). Les problèmes liés au bruit et au gain élevé utilisé en ADRC, nous incitent à proposer l'utilisation du mode glissant d'ordre élevé dans un contexte de rejet des perturbations, permettant non seulement de profiter des meilleures caractéristiques des deux approches, mais aussi de réduire l'amplitude du signal de commande et le broutage en rejetant la dynamique rapide qui est généralement négligée dans les modèles idéaux. Nous présentons quelques résultats montrant comment le rejet des perturbations peut être atteint en utilisant une approche à gain élevé, et comment ces résultats peuvent être améliorés en utilisant le mode glissant d'ordre élevé.

Nous abordons également le problème du contrôle de synchronisation d'une classe de systèmes chaotiques. La synchronisation est obtenue en utilisant une loi de commande adaptative, qui garantit une convergence asymptotique globale de l'erreur entre les sorties des systèmes de pilotage et de réponse en présence d'incertitudes paramétriques dans les deux systèmes. Le procédé de synchronisation est appliqué dans un schéma de communication cohérent utilisant une modulation de modulation par changement de chaos.

## ABSTRACT

Microelectromechanical systems (MEMS) have revolutionized the electronic market around the world and is one of the most promising technologies for the next decade. The encouraging applications of MEMS resonators, along with the demand for smaller sensors while increasing signal to noise ratio, make it necessary to increase the amplitude of the oscillation of these devices at a reduced control effort. The search for sensors and electronic devices drawing just small amounts of power is constant, and resolving the power-consumption problem is a priority task if we require to save energy and reduce battery changes.

The present work is devoted to the control of chaos in MEMS as an approach to enhance the performance of MEMS oscillators, improving the amplitude of their oscillations, and at the same time reducing the control effort. The study is carried out under realistic operational conditions including parametric uncertainties, unmodeled dynamics, presence of disturbances, noise, and limitations in the measurement of the speed of the plates of the MEMS resonators.

The oscillation amplitude of electrostatic MEMS resonators is restricted by the so-called pull-in phenomena related to a bifurcation arising in electrostatic actuation. The stable traveling range is limited to one third of its full gap due to the instability beyond that point. The pull-in phenomena can be used to accelerate the reaction speed for switching structures. However, it limits the stable operational range of the device, which is mandatory for applications requiring a sustained oscillation of the plates. In order to operate electrostatic MEMS devices beyond the pull-in, we exploit the nonlinear dynamics of the system, which is designed to operate in chaotic regime to generate an infinite number of unstable periodic orbits that are immersed in the chaotic attractor. One of these orbits is conveniently chosen to produce a high amplitude oscillation, and is stabilized by using different control schemes. Since the unstable orbits already exist in the system and it is not necessary to spend energy to create it, the effort of the controller is reduced.

Several control schemes to stabilize the chaotic system are considered, and due to the difficulty for measuring the velocity of the plates during the operation of the device, we use output feedback control techniques including an adaptive observer backstepping scheme and an active disturbance rejection approach (ADRC). The problems associated with noise and the high gain used in ADRC encourage us to propose the use of high-order sliding-mode in a context of disturbance rejection, allowing not only taking advantage of the best characteristics of both approaches, but also reducing the amplitude of the control signal and the chattering

by rejecting the fast dynamics that are usually neglected in ideal models. We state some results showing how disturbance rejection can be attained using a high gain approach, and how these results can be improved by using high order sliding mode.

We also address the synchronization control problem of a class of chaotic systems. The synchronization is obtained by using an adaptive control law, which guaranties a global asymptotic convergence of the error between the outputs of driving and response systems in the presence of parametric uncertainties in both systems. The method of synchronization is applied in a coherent communication scheme using chaos shift keying modulation.

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## LIST OF SYMBOLS AND ACRONYMS

|         |                                                  |
|---------|--------------------------------------------------|
| AC      | Alternating Current                              |
| ADRC    | Active Disturbance Rejection Control             |
| AI      | Artificial Intelligence                          |
| ANBC    | Adaptive Neural Backstepping Control             |
| CAGR    | Compound Annual Growth Rate                      |
| CMUTs   | Capacitive Micromachined Ultrasonic Transducers  |
| CSK     | Chaos Shift Keying                               |
| C-MEMS  | Carbon Microelectromechanical Systems            |
| DC      | Direct Current                                   |
| DI      | Differential Inclusions                          |
| DOF     | Degree Of Freedom                                |
| EBFCs   | Enzymatic Biofuel Cells                          |
| EMI     | Electromagnetic Interference                     |
| ESC     | Electronic Stability Control                     |
| FSMC    | Fuzzy Sliding Mode Control                       |
| FTS     | Finite-Time Stable                               |
| GAS     | Globally Asymptotically Stable                   |
| GFTS    | Global Uniform Finite Time Stability             |
| HOSM    | High-Order Sliding Mode                          |
| HOSMC   | High-Order Sliding Mode Control                  |
| IC      | Integrated Circuit                               |
| IoT     | Internet-of-Things                               |
| MEMS    | Micro Electromechanical Systems                  |
| NEMS    | Nanoelectromechanical System                     |
| PCB     | Printed Circuit Board                            |
| PD      | Proportional-Derivative                          |
| PDE     | Partial Differential Equation                    |
| PPA     | Parallel Plate Approximation                     |
| RF      | Radio Frequency                                  |
| RF MEMS | Radio Frequency Microelectromechanical Systems   |
| RNTSMC  | Robust Nonsingular Terminal Sliding Mode Control |
| ROM     | Reduced Order Model                              |
| SFTSMC  | Second-Order Fast Terminal Sliding Mode Control  |

|        |                                        |
|--------|----------------------------------------|
| TPMS   | Tire Pressure Monitoring Systems       |
| T2SFNN | Type-2 Sequential Fuzzy Neural Network |
| UPO    | Unstable Periodic Orbit                |
| XTAL   | Crystal                                |

## CHAPTER 1 INTRODUCTION

### 1.1 Context and Motivation

From industry to home, applications of micro electromechanical systems (MEMS) are almost endless, and impulse an important segment of the electronic market. Typical applications range from wearable devices, smartphones, laptops, tablets, navigation devices, media players, automotive electronics equipments and gaming consoles, to mention some.

The global MEMS market was valued at USD 12.74 billion in 2020, and is projected to reach USD 18.46 billion by 2026, registering a compound annual growth rate (CAGR) of 6.34% during the forecast period (2021-2026) [2]. The need for miniaturization, the increasing demand of portable and wearable electronic devices, along with the continued growth of automation in industries and homes, are some of the significant factors that fuel the growth of the MEMS market.

There is an extensive range of markets for MEMS, including automotive, medical, consumer, communication, semi-conductor, defense, aerospace, and more. Although the industry of MEMS was born with pressure sensors as first commercial products, the first market revolution of MEMS, or first wave, began in the early 90s and came from the need for low cost and reliable accelerometers for airbags. The second wave arrived on the market in the late 2000s with the gaming industry and smartphones markets [3], and the third wave of MEMS is on its way with internet-of-things (IoT) and AI-based applications [4].

In the automotive industry, MEMS sensors play an integral role across numerous systems [5–8]. They facilitate vehicle control via electronic-stability control (ESC) and hill hold control, monitor safety factors such as tire pressure through tire pressure monitoring systems (TPMS), and enhance engine performance by assessing intake air and fuel pressures. MEMS sensors also contribute to comfort by adjusting active suspension systems according to road conditions and participate in high-pressure measurements for various systems including brakes and air conditioning.

MEMS sensors are crucial in the consumer electronics sector, with widespread applications in smartphones, wearable devices, and inkjet printing technologies [9–11]. From controlling screen orientations and enhancing gaming experiences in smartphones, to improving functionality in wearables, and enabling precision and cost-effectiveness in microfluidic inkjet printheads, MEMS technology is a key component.

MEMS technology greatly contributes to the medical industry, with applications ranging

from ultrasound imaging to drug delivery and biosensing [12–24]. Capacitive micromachined ultrasonic transducers (CMUTs) and microfluidic devices have brought about miniaturized solutions in imaging and ‘lab-on-a-chip’ scenarios. The usage of carbon MEMS (C-MEMS) in biosensors and enzymatic biofuel cells (EBFCs) further highlights the integration of MEMS in diverse medical and biotechnological contexts.

Radio frequency microelectromechanical systems (RF MEMS) switches are being widely used in numerous applications including transmit/receive modules, like reconfigurable antenna, RF phase shifter, phase-array radar, mixer, and switching networks [25]. Other applications of RF MEMS are impedance tuners, widely used in characterization of transistor noise parameters [26], and tuneable filters [27]. The market prospect for these devices is huge, since they will be used in next generation 5G infrastructure, and will help drive the 4th industrial revolution.

As a motivational example of application of chaotic MEMS resonators, we consider a modulation process of digital communications using chaos, where the digital symbols are mapped into the chaotic signals. Chaotic signals are useful to encode information due to their inherent wideband characteristic. When they are used as the carrier waves in communication systems, produce spread-spectrum signals that enjoy many benefits as security in the transmission and anti-jamming effect, among others [28].

In a chaos shift keying approach (CSK), every symbol is mapped to different chaotic basis signals produced from a dynamical system (e.g. a chaotic resonator) using different values of a parameter. Since the chaotic basis signals are available at the receiver using synchronization, the symbols can be detected by evaluating the synchronization error [29] or using a conventional correlator-type detector [30]. Once the signal is modulated, it is transmitted through an appropriate channel to the destination. In order for the receiver to decode the received signal, a coherent method based on synchronization may be used. However, these channels usually have undesirable effects on the signal being transmitted, e.g., attenuation, band-pass filtering, nonlinear distortion, and noise contamination [28], which make the synchronization a very difficult task [31].

Figure 1.1 shows the coherent (synchronization-based) communication scheme using CSK modulation, used to illustrate the application of the synchronization method from Chapter 4. The driving system is the chaotic MEMS resonator presented in Section 4.3 and the response system is the MEMS resonator given by the equation (4.19). The two chaotic signals of the transmitter side are obtained by changing the frequency of the AC signal applied to the resonator, and the binary symbols 0 and 1 are associated to the frequencies 0.2 and 0.5 respectively. In the receiver side, a MEMS resonator is configured to synchronize with the

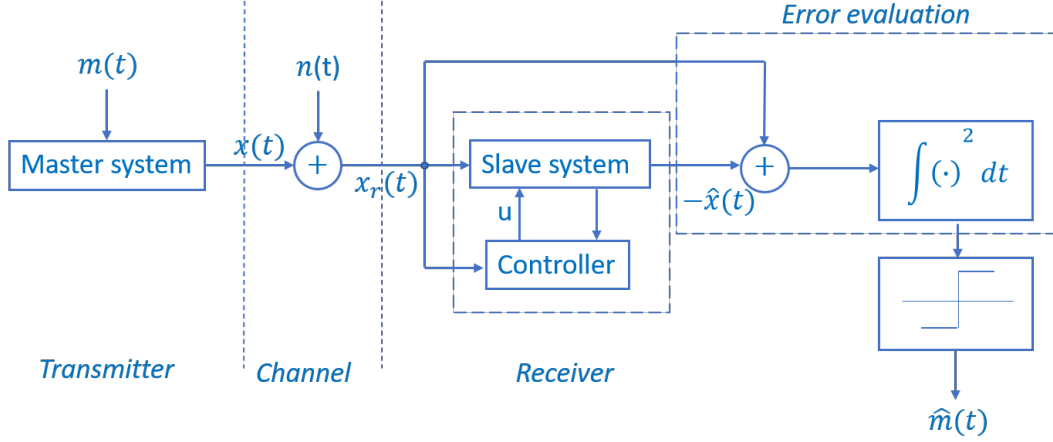


Figure 1.1 Block diagram of the chaotic communication system based on Coherent demodulation

system at the frequency of the AC signal of 0.5. If the frequency differs of this value, an error in the synchronization must be reflected in the receiver. If the error between the transmitted and synchronized signals exceeds a certain threshold, then a bit 1 is transmitted, if the synchronization error is smaller, then a 0 is transmitted instead. The sequence 0011010101 is transmitted, and a decoder based on an error detector is used to recover the data in the receiver. Figure 1.2(a) shows the transmitted signal with its estimation, Fig. 1.2(b) shows the estimation error and the recovered data, and Fig. 1.2(c) shows the estimation errors for variables  $x_1$  and  $x_2$ . Note that the method as used in Chapter 4 assumes the knowledge of the states, which are needed in order to obtain the control signal (4.9) for synchronizing the systems. Under a real application basis, this signal is not available, and an observer must be used to recover the state  $x_2$ . To obtain this state we use the sliding observer described in Chapter 7 given by equation (7.52). In general, some noise  $n(t)$  is added to the transmitted signal  $x(t)$ , and the received signal is equal to  $x_r(t) = x(t) + n(t)$ , so that the observer must be robust against noise and other imperfections in the transmission channel.

Although noise in the communication channel is very restrictive for practical applications using coherent demodulators, which are based on synchronization, it is still a topic that requires attention and remains a very active research area.

## 1.2 Research Problems

Electrostatic actuation is the most popular actuation mechanism used in micro/ nanoelectromechanical systems (MEMS/NEMS) due to its many inherent advantages, as ease of

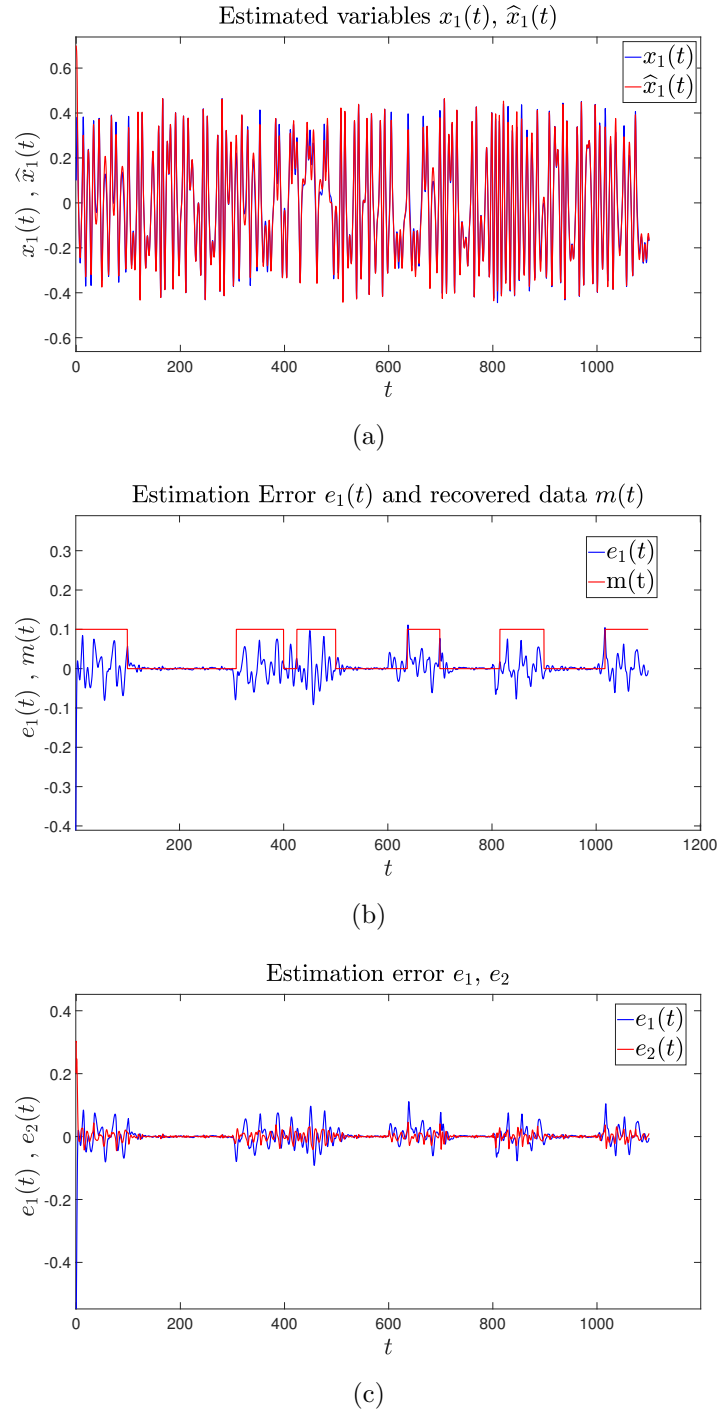


Figure 1.2 Signals of the chaotic communication system: (a) Estimated variables  $x_1$  and  $\hat{x}_1$ ; (b) Synchronization error and recovered data; (c) Estimation error.

manufacture, high efficiency, low power consumption, simple structure, and fast actuation rates [32]. However, this actuation scheme results in highly nonlinear dynamics leading to a saddle-node bifurcation, called pull-in. This phenomenon, which is responsible of the sudden snap between the plates of the structure, has crucial effects on electrostatically actuated MEMS/NEMS and limits their travel ranges, affecting the performance of this kind of actuators.

There are different classes of pull-in, including static, dynamic, and resonant pull-in, which can be classified as one kind of dynamic pull-in due to AC and DC excitations [33]. In resonant pull-in, the system is dynamically actuated at resonance. An accurate determination of the pull-in condition is critical in the design process to determine the dynamic instability and power handling of MEMS/NEMS sensors and actuators, such as gyroscopes, switches, and resonators [34].

Improving the stable range of operation of the electrostatically actuated MEMS under real operational conditions, avoiding pull-in, is of outmost importance. Noise in the measurement of states and environmental disturbances are unavoidable in a real setting, and must be considered in the design of reliable controllers for eliminating the pull-in instability. The control system must also be robust against differences in the manufacturing process, modeling errors and parameter uncertainties [35], which suggest the use of nonlinear approaches to stabilize the system or provide accurate position tracking [36].

How to suppress and control the pull-in instability and extend the travel range of the actuator beyond the pull-in limit, under real operational conditions, taking into account unmodeled dynamic, parametric uncertainties, and external disturbances, is of great significance in electrostatic MEMS/NEMS devices, and is a major issue that we will address in this research. Reducing actuation effort is mandatory in the design of portable applications of MEMS, and also will be addressed by designing a feedback controller with low energy consumption.

### 1.3 Thesis Outline

Chapter 2 presents a literature review on MEMS technology and its applications, modeling nonlinearities in MEMS, and the control of MEMS resonators. Then, the objective of the research conducted in the dissertation, the proposed solutions, and the adopted methodology are stated.

Chapter 3 presents the dynamic models of electrostatic MEMS and mathematical tools used in the dissertation. It begins with an overview of the concept of solutions of differential equations with discontinuous right-hand sides using the Filippov approach, covering some

important aspects of differential inclusions (DI), and their relation with homogeneous vector fields. The concept of homogeneity is reviewed, and in particular, weighted homogeneity of DI. Then, in order to show the equivalence between asymptotic stability of homogeneous DI with negative homogeneous degree and global finite-time stability, some remarkable results that apply to homogeneous DI are presented. In the second part of the chapter, a reduced order model of a MEMS resonator is obtained. Some relevant results related to chaotic behaviour of dynamical systems are described, in particular the Melnikov's method for detection of chaos, and an algorithm to calculate the Lyapunov exponents. It is used as a numerical tool to determine the sensitive dependence on initial conditions of chaotic systems.

The synchronization control problem of a class of chaotic systems, which include chaotic MEMS resonators, is addressed in Chapter 4. In this chapter the importance of synchronization in secure communications is introduced, as well as the model of the MEMS used for synchronization purposes. A theorem that shows how the synchronization is achieved by using an adaptive control law, which guaranties global asymptotic convergence of the error between the outputs of the driving and response systems. At the end of the chapter, two examples of application and some conclusions regarding synchronization of MEMS resonators are presented.

In Chapter 5 chaos is controlled in an electrostatic MEMS oscillator with the aim of enhancing its amplitude of oscillation, using its nonlinear properties and in particular, its chaotic dynamics. The dynamical model of a chaotic MEMS resonator is briefly explained based on a model given in [37], [38]. An unstable orbit immersed in the chaotic attractor is obtained and stabilized using an adaptive backstepping approach. Numerical simulations are carried out and the results are presented along with concluding remarks. Then, in Chapter 6, the procedure to find the unstable orbits is illustrated, and more realistic operational conditions are considered by using an output feedback control scheme to stabilize one of these orbits.

In Chapter 7 we use high-order sliding-mode (HOSM) in a context of disturbance rejection, allowing to reduce the amplitude of the control signal used to stabilize an unstable orbit in the chaotic oscillator, and the effects of chattering on the plant. The chapter is divided in two sections, the first considering high-gain as a mechanism to reject the disturbances, and the last using sliding mode to improve the performance of the controller. In the first part we find a bound for the matrix exponential, which is used to enunciate a theorem which gives conditions to obtain bounded solutions of perturbed linear systems. Then we use these results to design an extended observer to estimate the generalized disturbance which lump together external disturbances and unmodeled dynamics. We show how to cancel the effects of the generalized disturbance using the gathered estimate as part of the feedback control

action. We begin the second part of the chapter with a description of the advances of sliding mode in the last decades, and review HOSM differentiators and observers. We learn about a very important class of high-order sliding-mode controllers (HOSMC), and introduce the extended HOSM observer used in our approach to reduce even more the control gain in the HOSMC. We end this chapter showing some application examples in order to highlight the advantages of the proposed method.

In Chapter 8 we discuss relevant results obtained in this work, and in Chapter 9 we present the main conclusions of the research, as well as the limitations and perspectives for future work.

## CHAPTER 2 MEMS RESONATORS: APPLICATIONS, MODELING, AND CONTROL

This chapter provides a comprehensive literature review on the applications, modeling, and control of micro-electromechanical systems (MEMS) resonators. The chapter begins by introducing the main applications of MEMS resonators in Section 2.1, highlighting their importance as timing references, sensors, and filters in different electronic systems where they are used. The second section of the chapter presents nonlinear phenomena associated with the operation of MEMS, and specifically illustrates the methods used to predict the onset of chaos in these devices. Furthermore, it emphasizes the importance of nonlinearities in the analysis and design of MEMS devices. Section 2.3 explores various methodologies employed for controlling MEMS resonators, with a particular focus on stabilizing the chaotic dynamics exhibited by such resonators. The chapter concludes with Section 2.4, which specifies the research objectives and outlines the adopted methodology.

### 2.1 Present and Emerging Applications of MEMS Resonators

Microelectromechanical systems (MEMS) resonators play a critical role in a wide range of technological applications, including sensors, actuators, and communication devices, among others [7, 8, 10, 19].

In the last 20 years, there has been a series of important technological developments, which made the MEMS resonators useful in many applications and successful in the market. Monolithic integration of CMOS–MEMS oscillator circuits opens new perspectives in the miniaturization of RF transceivers as well as ultrasensitive resonant sensing [39].

#### 2.1.1 MEMS Resonators as Timing References

MEMS resonators offer the potential for very high Q's and thus are ideal for use as the timing element in oscillator implementations [40]. Electronic systems need timing devices, usually crystal (XTAL) resonators which have been used for decades despite their inherent limitations. The use of MEMS resonators and oscillators inside an integrated circuit (IC) as timing devices, offers several benefits compared to XTALs. They simplify system design since oscillators are a completely integrated solution and do not require external capacitors to tune the resonant frequency. Oscillators based on MEMS resonators offer much better quality and reliability, reaching defective parts per million and Mean time between failures 30 times

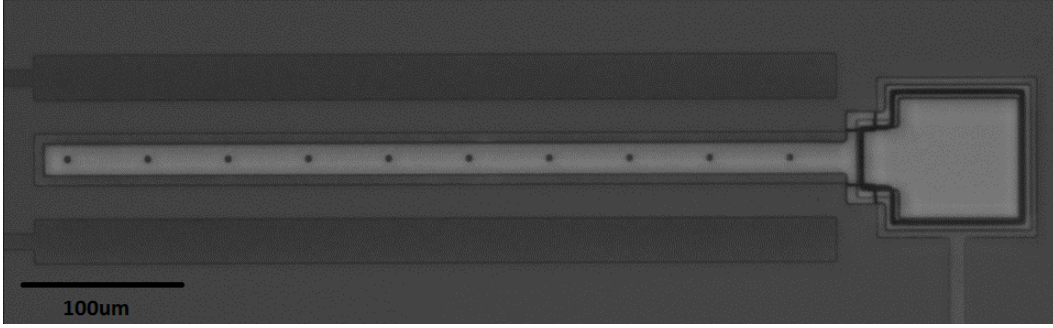


Figure 2.1 PolyMUMPs fabricated resonator with anchor (right), beam (center), and side electrodes (top and bottom). Reproduced from [1].

better than quartz, which requires a very specialized manufacturing process in contrast to MEMS oscillator manufacturers that use standard semiconductor batch mode techniques. In addition, the total printed circuit board (PCB) area of XTAL solutions are usually larger in comparison with the MEMS resonators based oscillators. Other advantage of using oscillators with MEMS resonators is that they are less sensitive to electromagnetic interference (EMI) since the crystal resonator must be attached to a IC containing the oscillator circuit, and the connection makes the circuit sensitive to EMI; MEMS resonators and oscillators are in the same IC and therefore do not have exposed PCB connections. MEMS oscillators are much less sensitive to vibration, since the mass of a MEMS resonator is typically 1,000 to 3,000 times lower than the mass of a quartz resonator and that increases the probability that vibration stresses can induce frequency shift and noise on the crystal resonator. Another advantage over the quartz oscillators is that MEMS oscillators are readily available in any frequency, unlike quartz that suffer from several constraints imposed by availability of suppliers and frequency options.

### 2.1.2 MEMS Resonators for Sensing

MEMS resonators find many applications in sensing, like in the development of small-sized particulate matter sensors for monitoring the pollutant level in the air which is of utmost importance for human health in the prevention of lung diseases [41]. Resonators can also be used to allow the detection of volatile organic chemical compounds in the environment. In this application, a chemo-sensitive polymer layer is usually deposited on the mechanical resonator, modifying its intrinsic mass responsivity and frequency characteristics, that sets the final sensitivity of the sensor [39] and may be used to detect several gas compounds [42]. Other field of application of MEMS sensors is medicine, where the ultrasonic transducers are often used in biomedical sensing and imaging applications due to their biocompatibility and

non-destructive testing abilities [41]. MEMS resonators can also be used as high sensitivity temperature sensors, since their resonant frequency when fabricated with CMOS technology shows a strong dependence on temperature [43].

### 2.1.3 MEMS Resonators as Filters

MEMS resonators operate essentially as a very narrow bandwidth filter with a center frequency corresponding to the resonance frequency [39]. They can also be combined in order to create higher order bandpass filters which can potentially be integrated into transceiver front-ends by replacing costly, off-chip, narrow-bandwidth filters. In order to achieve a bandpass filtering function with sufficient bandwidth, MEMS resonators need to be coupled together as their quality factors are generally too high to provide a wide enough bandwidth standalone [44]. There are two ways to couple resonators in practice. One is the mechanical coupling, which is based on the use of a mechanical link that fulfills the function of transmitting the movement of one resonator to the other, and is usually implemented with springs. The second is the electrical coupling, which does not require a mechanical link and uses the electrostatic force between two adjacent resonators biased with different voltages [39].

## 2.2 Modeling Nonlinearities in MEMS

Accurate models is of the utmost importance in analysis and design of MEMS resonators. There are several aspects that must be considered when modeling a MEMS device, such as nonlinearities and micro-scale phenomena.

Nonlinearities usually are more important in micro-scale devices, and in many cases can not be neglected. Sources of nonlinearities include forcing, damping and stiffness [33]. Forcing introduces nonlinearities through actuation mechanisms, for example when electrostatic forces are used since they are inversely proportional to the square of the distance between voltage electrodes. The squeeze-film damping is inherently nonlinear since it depends on the inverse of the distance cubed between the close surfaces. In microstructures, the movement of plates is usually large enough to be compared with its thickness and size, which leads to changes in its stiffness.

Electrostatic and geometrical non-linearities for beam-like mems have been largely studied [45], and research is still active on the modeling of nonlinear mechanical effects such as stress-stiffening, modal coupling and internal resonances [46]. A simplified numerical technique to quantify the effects of non-linear stiffness in MEMS resonators, along with the generation of a reduced order model (ROM), was proposed in [47]. The nonlinear interaction among

various modes of vibration of nano/micromechanical systems (N/MEMS) has been widely studied in recent years [48]. The two-to-one internal resonance between the first two symmetric vibrational modes of a micromachined arch resonator, electrothermally tuned and electrostatically driven, was investigated experimentally and theoretically in [49]. Varieties of complex behaviors were demonstrated including quasi-periodic and aperiodic motion leading to potential chaotic motion [49, 50]. Internal resonances were also studied in [51], where it was demonstrated that, by coupling two different vibrational modes through an internal resonance, it is possible to stabilize the oscillation frequency of nonlinear self-sustaining micromechanical resonators.

Following different approaches, complex behaviors like chaos have been detected in several MEMS oscillating structures [52]. A Duffing-like model has shown to capture the essential features of nonlinearities in MEMS, like softening and hardening behavior and sub- and superharmonic resonances [53]. The Duffing oscillator is a well-known and useful model for the nonlinear behavior of beams with a nonlinear cubic stiffness term. This model is heuristic in principle, based on experimental observations, in comparison with more elaborated models that analyze the resonator as a distributed-parameter system [54]. Although the results are less accurate using single degree-of-freedom models (1DOF) like Duffing model, they provide the essential nonlinear dynamics of these systems.

In particular, Duffing model has been used in different works to predict the onset of chaos in MEMS resonators with electrostatic actuation [55–57]. Haghighi and Markazi in [56] used the Melnikov function to obtain an analytical criterion for homoclinic chaos in terms of system parameters, and Wang et al. in [55] demonstrated the existence of chaos in the MEMS device using experimental data. Siewe and Hegazy in [57] utilized higher-order terms in the Duffing equation and a Melnikov function to define the regions of parameter space where homoclinic and heteroclinic chaos can occur. Zhang et al. in [58] employed a Duffing model to investigate noise-induced chaos in an electrostatically actuated MEMS resonator subject to a random disturbance. They used a modification of the Melnikov function to establish the threshold of noise for the occurrence of chaos. De and Aluru [59] studied the period doubling bifurcation route to chaos of a MEMS oscillator using a Duffing-like model with higher-order terms obtained from a series expansion of the electrostatic force applied to the MEMS.

With the aim of studying nonlinear phenomena in resonators, other methods have also been used to obtain 1DOF models. In [60], a 1DOF equation of motion, obtained by application of single-mode Galerkin decomposition method to the governing Euler-Bernoulli shallow arch equation with mid-plane stretching effects and distributed electrostatic force, was used for investigating chaotic vibrations of a bistable resonator. The Melnikov's method was used to

derive a necessary condition for the initiation of chaos in this type of resonators. Although Melnikov method has been extensively used to detect chaos, other criteria have also been proposed, see for example [61], where it is concluded that the response of the system may become chaotic when its velocity is near the maximum velocity of a homoclinic orbit.

### 2.3 Control of MEMS Resonators

Haghighi and Markazi [56] predicted the chaos in MEMS resonators and presented a robust adaptive fuzzy control method to control the chaotic motion. In [62], a fuzzy sliding mode control (FSMC) scheme was used to stabilize the chaotic motion of a micromechanical resonator. Siewe and Hegazy in [57] applied active vibration control and time-varying stiffness to suppress the amplitude responses and control the chaotic behavior of the MEMS resonator investigated. Zhankui and Sun [63] proposed a second-order fast terminal sliding mode control (SFTSMC) scheme to suppress the chaotic motion of a micro-mechanical resonator. The SFTSMC technique hold robustness and stability in the presence of uncertainties and external disturbance, and drove the system states to a desired state in finite-time. Liu et al. [64] focused on the control of chaotic behavior of MEMS resonators by a delayed feedback proportional-derivative (PD) controller. The results showed that time-delay feedback is a good method for controlling these systems. Zhang et al. [65] proposes a novel method for controlling a network of four Duffing-type MEMS resonators with mechanical and electrostatic coupling. The proposed method uses an analog circuit to implement the control algorithm, which is based on adaptive neural backstepping control (ANBC). ANBC is a robust control technique that can handle uncertainties and disturbances in the system. The proposed method was evaluated using simulations, and the results showed that it was able to effectively control the network of MEMS resonators and achieve the desired performance. The paper by Azizi [66] et al. proposes a control technique for mitigating disturbance effects in MEMS resonators. The proposed technique uses a robust control technique called robust nonsingular terminal sliding mode control (RNTSMC), which can handle uncertainties and disturbances in the system and works by creating a sliding mode that the system is forced to follow. The results showed that the applied technique is highly capable of stabilizing the microstructure and mitigating the effects of disturbances and uncertainties. The paper by Le Zhao et al. [67] studies the chaos analysis and stability control of a MEMS resonator using a type-2 sequential fuzzy neural network (T2SFNN). The authors first use bifurcation diagrams, exponent diagrams, phase diagrams, and time histories to analyze the chaotic behavior of the MEMS resonator. They then propose a T2SFNN-based controller to suppress the chaotic behavior of the MEMS resonator. The controller is designed using a backstepping

control approach. The results show that the proposed controller is able to effectively suppress the chaotic behavior of the MEMS resonator and maintain the system in a stable state.

## 2.4 Objectives and Proposed Methodology

In our study, we will consider a chaotic MEMS resonator modeled by a nonlinear mass-spring-damper system with external electrostatic actuation whose model is detailed in Section 3.1.3. This MEMS device is operated in chaotic mode under certain operational conditions [38], [68], [69], [70], and its dynamic behavior will be used to solve the pull-in problem.

The primary aim of this research is to address the pull-in instability in MEMS resonators, with the goal of enhancing their functionality, reliability, and overall performance. Specifically, the objectives of this research are to:

1. Develop strategies to suppress the pull-in instability in MEMS resonators effectively, and extend the operational range of the resonator beyond the pull-in limit, thereby enhancing its performance.
2. Formulate control strategies capable of maintaining stable, high-amplitude oscillations under real-world operational conditions, as unmodelled dynamics, parametric uncertainties, noisy signals, and external disturbances

To accomplish the outlined objectives, this research encompasses several articles, each with its specific aims:

Article 1, Chapter 4, seeks to investigate the feasibility of inducing chaos in the MEMS device using synchronization. We aim to incorporate uncertainties into the model to enhance the robustness of our synchronization method.

Articles 2 and 3, Chapters 5 and 6, focus on the control of chaos in an electrostatic MEMS resonator. The aim is to stabilize one of the unstable orbits within the chaotic attractor, thus facilitating high-amplitude oscillation of the microstructure's movable plate. These articles will consider parametric uncertainties in the MEMS model and propose the implementation of state and output feedback control. We address challenges associated with measuring the velocity of the microstructure's plates.

Article 4, Chapter 7, is dedicated to the control of MEMS resonators, with a particular focus on highly uncertain models, with noisy signals, and external disturbances. other objective is to enhance the precision of the resonator by investigating a method for achieving finite-time stabilization of the oscillation.

Having delineated the key objectives of this research, we will now proceed to outline the specific methodology that has been employed to achieve these aims.

We first state the occurrence of chaos in MEMS, and in particular in MEMS resonators, by using mathematical tools like Melnikov's method and the calculation of Lyapunov exponents. In Section 3.2.3 we describe the procedure to detect chaos in a dynamic system by means of Lyapunov exponents, and then we calculate these exponents numerically in Section 4.3. We show the existence of one positive Lyapunov exponent that confirms the chaotic behavior of the system. In Section 3.2.2, the Melnikov's method is explained in details, and we obtain a necessary condition for homoclinic chaos in MEMS electrostatic actuators, and predict analytically (and also numerically) when chaos will be present and under what set of parameters.

After we state the conditions for the onset of chaos in MEMS resonators, we learn that under certain set of parameters chaos is not present in the MEMS device. However, in Chapter 4, we explore the possibility of generate chaos in MEMS using synchronization. We worked on two strategies to generate chaos in MEMS by synchronizing an electrostatic MEMS resonator with unknown damping and stiffness actions with a Duffing chaotic oscillator and a MEMS chaotic oscillator.

Once the occurrence of chaos in the MEMS resonator is fully established, we obtain an unstable periodic orbit (UPO) surrounding the chaotic attractor, which will be used as the reference trajectory in a robust control scheme and will produce a high-amplitude periodic output. This UPO is part of the chaotic dynamic, and is obtained by applying a close return approach as explained in details in Section 5.3.2. In the different control schemes considered, we use a Fourier series approximation of the orbit truncated up to five terms. Different orbits may be found using this procedure, and some are given in Section 5.3.2.

The next step is to evaluate some control strategies to stabilize the UPO under different realistic operational conditions. First, in Chapter 5 we demonstrate asymptotic tracking of the reference trajectory constructing a control law by using backstepping, and then we deal with the uncertainties in the model and stabilize the UPO by using an adaptive control scheme. A drawback of the method is the use of state feedback, since it is very difficult to sense the velocity of the plates during the operation of the device. For more realistic applications, we consider an output feedback control technique in Chapter 6. The output is chosen as the position of the plates, and in order to implement the control, the velocity is estimated by constructing a state observer.

Finally, some considerations related to the errors due to approximations in the nonlinear model of the resonator and other phenomena affecting a real MEMS based oscillator, as

external disturbances, are taken into account by improving the design of the controllers considered. To this end, a disturbance rejection method is introduced in Chapter 7. Some drawbacks to applying the method were identified, related to its sensitivity to noisy signals due to the use of high-gain observers. To avoid the use of high-gain observers, and make the controller robust against noise, a high-order sliding mode controller with an extended high-order finite-time state observer was proposed in Section 7.4.3, which in addition, came with an added advantage, its finite time stability.

## CHAPTER 3    DYNAMIC MODELING AND MATHEMATICAL PRELIMINARIES

This chapter details first in Section 3.1 the dynamic model of MEMS resonators used in our studies, beginning with a distributed-parameter model, and reducing the order of the system to obtain a finite-dimensional nonlinear differential equation. Section 3.2 introduces basic concepts related to chaos and how to detect the chaotic dynamics of a system, with application to MEMS resonators. Then, preliminaries related to the notion of Filippov solution of differential inclusions and homogeneity, which will be used in Chapter 7, are presented in Section 3.3 and Section 3.4, respectively.

### 3.1    Mathematical Model of MEMS Resonators

This section presents first a partial differential equation (PDE)-based model of the motion of a beam in bending and under axial loading. The Galerkin method is then used to convert the distributed-parameter model to a nonlinear ordinary differential equation where mid-plane stretching, axial stress, DC electrostatic and AC harmonic voltages are taken into account. Moreover, a parallel plate approximation using a lumped parameter model is obtained, and the errors of this approximation are considered.

#### 3.1.1    Distributed Parameters Model

We will briefly review the nonlinear equation of motion of a beam with clamped-clamped edges following [33]. We assume that it undergoes moderately large deflection compared to its thickness in response to large forcing. To obtain the partial differential equation model we assume planar motion, linear strain-stress law, and that the beam is initially straight. The beam has a length  $l$ , and its properties like Young's modulus ( $E$ ), second moment of inertia of the cross section ( $I$ ), area of the transverse section ( $A$ ), and density ( $\rho$ ) are assumed uniform and constant. Consider Fig. 3.1, which shows a schematics of the centerline of a small element of the beam before and after deformation, with lengths  $dx$  and  $ds$  respectively. The element experiences an axial displacement  $u(t, x)$  and a vertical displacement  $w(t, x)$ , as indicated in the figure. It is subjected to a tensile axial force  $N(t, x)$ , and a distributed force per unit length  $F(t, x)$ . In Figure 3.1,  $M$  refers to the bending moment,  $V$  is the shear force, and  $\theta$  is the slope angle of the beam deflection curve with the horizontal line. We follow a Newtonian approach, and assume a small change in all the variables on the right side of the

element, which are expanded using Taylor series.

The coordinates of a point  $p$  before and after the deformation are related by

$$\tilde{x} = x + u; \quad \tilde{y} = w. \quad (3.1)$$

Taking the derivative of (3.1) we obtain

$$d\tilde{x} = dx + \frac{\partial u}{\partial x} dx = dx(1 + u'), \quad (3.2a)$$

$$d\tilde{y} = \frac{\partial w}{\partial x} dx = w' dx, \quad (3.2b)$$

where the primes denote the derivative with respect to  $x$ . Hence, the length of the deformed element is

$$ds = \sqrt{d\tilde{x}^2 + d\tilde{y}^2} = \sqrt{(1 + u')^2 + w'^2} dx. \quad (3.3)$$

Defining the axial strain  $\varepsilon$  by

$$\varepsilon = \frac{ds - dx}{dx} = \sqrt{(1 + u')^2 + w'^2} - 1, \quad (3.4)$$

and expanding it in Taylor series, we obtain the approximation for  $\varepsilon$

$$\varepsilon \approx u' + \frac{w'^2}{2} + \dots. \quad (3.5)$$

Using the expressions for  $d\tilde{x}$ ,  $d\tilde{y}$ , and  $ds$  from equations (3.2a) and (3.3), and replacing them in the following relationships for the deformation angle  $\theta$

$$\sin \theta = \frac{d\tilde{y}}{ds}, \quad \cos \theta = \frac{d\tilde{x}}{ds}, \quad (3.6)$$

we obtain

$$\sin \theta = \frac{w'}{\sqrt{(1 + u')^2 + w'^2}}, \quad \cos \theta = \frac{1 + u'}{\sqrt{(1 + u')^2 + w'^2}}, \quad (3.7)$$

whose expansion in Taylor series gives

$$\sin \theta \approx w' - u'w' - \frac{w'^3}{3} + \dots; \quad \cos \theta \approx 1 - \frac{w'^2}{2} + \dots. \quad (3.8)$$

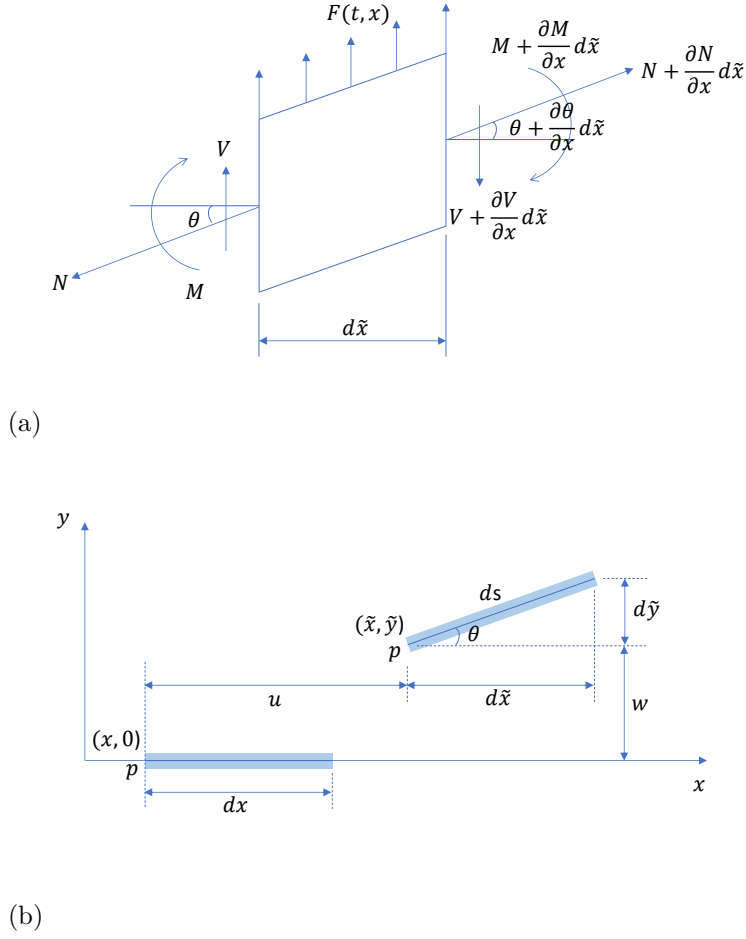


Figure 3.1 Segment on a beam: (a) Before and after deformation. (b) Free-body diagram.

Next, we analyze the axial load on the other end of the deformed element and obtain trigonometric relationships for the deformed angle  $\theta + \theta' d\tilde{x}$ , that is approximated by their Taylor series

$$\begin{aligned}\cos(\theta + \theta' d\tilde{x}) &\approx \cos \theta + (\cos \theta)' d\tilde{x} + \dots, \\ \sin(\theta + \theta' d\tilde{x}) &\approx \sin \theta + (\sin \theta)' d\tilde{x} + \dots.\end{aligned}\tag{3.9}$$

Considering the free-body diagram of Figure 3.1(b), and applying Newton's law in the horizontal direction we obtain

$$(N + N' d\tilde{x}) \cos(\theta + \theta' d\tilde{x}) - N \cos \theta = \rho A d\tilde{x} \ddot{u}.\tag{3.10}$$

Replacing the expression for  $\cos(\theta + \theta' d\tilde{x})$  into (3.10), and dropping the higher order terms

in  $d\tilde{x}$  gives

$$(N \cos \theta)' = \rho A \ddot{u}. \quad (3.11)$$

The application of Newton's second law in the vertical direction gives

$$V - (V + V'd\tilde{x}) + Fd\tilde{x} - N \sin \theta + (N + N'd\tilde{x}) \sin(\theta + \theta'd\tilde{x}) = \rho A d\tilde{x} \ddot{w}. \quad (3.12)$$

Replacing the expression for  $\sin(\theta + \theta'd\tilde{x})$  from equation (3.9) into equation (3.12), and dropping the higher order terms in  $d\tilde{x}$  yield

$$-V' + F + (N \sin \theta)' = \rho A \ddot{w}. \quad (3.13)$$

Using the approximation (3.5) for the axial strain  $\varepsilon$ , the axial force is

$$N = EA\varepsilon = EA \left( u' + \frac{w'^2}{2} + \dots \right), \quad (3.14)$$

and with the approximations for  $N$  (eq. (3.14)) and  $\cos \theta$  (eq. (3.8)), equation (3.11) becomes, after dropping high order terms

$$\rho A \ddot{u} - EA u'' = EA \left( \frac{w'^2}{2} \right)'. \quad (3.15)$$

Equation (3.13) also becomes, after using the approximations for  $N$  (equation (3.14)),  $\sin \theta$  (eq. (3.8)) and  $V$  ( $V = EI w''''$ , see [33]), and dropping high order terms

$$\rho A \ddot{w} + EA w'''' = EA \left( u' w' + \frac{w'^3}{2} \right)' + F. \quad (3.16)$$

Equations (3.15) and (3.16) are coupled nonlinear PDEs governing the dynamics of the beam in axial and transversal directions. In the case of beams of small radius of gyration, the inertial term  $\rho A \ddot{u}$  can be neglected and hence, (3.15) can be reduced to

$$u'' = - \left( \frac{w'^2}{2} \right)'. \quad (3.17)$$

Integrating this equation with respect to  $x$  we obtain

$$u' = - \frac{w'^2}{2} + c_1(t), \quad (3.18)$$

and integrating again

$$u = -\frac{1}{2} \int_0^x w'^2 dx + c_1(t) + c_2(t), \quad (3.19)$$

where  $c_1(t)$  and  $c_2(t)$  are constants in space but variable in time. In order to obtain these constants, the boundary conditions for a clamped-clamped beam are used:

$$u(0, t) = 0, \quad u(l, t) = 0, \quad (3.20)$$

and substituted into (3.18) and (3.19), which gives

$$c_1(t) = \frac{1}{2l} \int_0^l w'^2 dx; \quad c_2(t) = 0. \quad (3.21)$$

Finally, (3.16) becomes, after replacing  $u'$  from (3.18)

$$\rho A \ddot{w} + EI w'''' = w'' \left( \frac{EA}{2l} \int_0^l w'^2 dx \right) + F. \quad (3.22)$$

This equation is used to model beams with midplane stretching.

### 3.1.2 Reduced Order Model

The PDE (3.22) describing the behavior of the beam is nonlinear, and its solution can be computationally demanding even for simple geometries. For this reason, it is advisable to obtain a reduced order model of the system, approximating the beam by 1-degree of freedom (DOF) system. In order to obtain such a model, we use the Galerkin discretization method [33] and follow the approach of [71]. It is assumed that the beam is actuated by an electrostatic force per unit length of the form

$$F = \frac{\varepsilon_0 b}{2} \left[ \frac{V_2(t)^2}{(d-w)^2} - \frac{V_1(t)^2}{(d+w)^2} \right], \quad (3.23)$$

and subjected to a viscous damping of coefficient  $c$ , in such a way that the modified model of the beam becomes

$$\rho A \ddot{w} + c \dot{w} + EI w'''' = w'' \left( \frac{EA}{2l} \int_0^l w'^2 dx \right) + \frac{\varepsilon_0 b}{2} \left[ \frac{V_2(t)^2}{(d-w)^2} - \frac{V_1(t)^2}{(d+w)^2} \right]. \quad (3.24)$$

It is convenient to normalize the deflection of the beam with respect to  $d$ , the width of the gap separating the beam from the substrates, and the position along the length of the beam with respect to  $l$ , the length of the beam. Hence, the following non dimensional variables are introduced

$$\hat{w} = \frac{w}{d}, \hat{x} = \frac{x}{l}, \hat{t} = \frac{t}{T} \quad (3.25)$$

and we obtain the following nondimensional equation

$$\frac{\rho b h d}{T^2} \ddot{\hat{w}} + \frac{c d}{T} \dot{\hat{w}} + \frac{E I d}{l^4} \frac{\partial^4 \hat{w}}{\partial x^4} = \hat{w}'' \left( \frac{E A d^2}{2 l^4} \int_0^l \hat{w}'^2 dx \right) + \frac{\varepsilon_0 b}{2 d^2} \left[ \frac{V_2(t)^2}{(1 - \hat{w})^2} - \frac{V_1(t)^2}{(1 + \hat{w})^2} \right]. \quad (3.26)$$

Dropping the hats from the nondimensional variables for simplicity, the equation becomes

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} + c_n \frac{\partial w}{\partial t} = \alpha_1 \frac{\partial^2 w}{\partial x^2} \int_0^l \left( \frac{\partial w}{\partial x} \right)^2 dx + \alpha_2 \left[ \frac{V_2(t)^2}{(1 - \hat{w})^2} - \frac{V_1(t)^2}{(1 + \hat{w})^2} \right], \quad (3.27)$$

where the following parameters are defined

$$\alpha_1 = 6 \left( \frac{d}{h} \right)^2, \quad \alpha_2 = \frac{6 \varepsilon l^4}{E h^3 d^3}, \quad c_n = \frac{12 c l^4}{E T b h^3}, \quad T^2 = \frac{\rho b h l^4}{E I}. \quad (3.28)$$

In order to obtain a reduced order model, we apply the Galerkin method and use the beam's linear orthonormal modeshapes [33] and approximate  $w$  by  $w(t, x)$

$$w(t, x) = \sum_{i=1}^n u_i(t) \phi_i(x), \quad (3.29)$$

where  $\phi_i(x)$  is the base function which corresponds to the  $i$ -th modeshape. By replacing this solution into (3.27), and then multiplying the resultant equation by the modeshape  $\phi_j(x)$  and integrating along the beam length from 0 to 1

$$\begin{aligned} & \int_0^1 \phi_j \left( \sum_{i=1}^n u_i \phi_i'''' + \sum_{i=1}^n \ddot{u}_i \phi_i \right) dx + c_n \int_0^1 \phi_j \left( \sum_{i=1}^n \dot{u}_i \phi_i \right) dx \\ &= \alpha_1 \int_0^1 \phi_j \left\{ \sum_{i=1}^n u_i \phi_i'' \int_0^1 \left( \sum_{k=1}^n u_k \phi_k' \right)^2 dx \right\} dx + \alpha_2 \int_0^1 \left[ \frac{V_2(t)^2 \phi_j}{(1 - \sum_{i=1}^n u_i \phi_i)^2} - \frac{V_1(t)^2 \phi_j}{(1 + \sum_{i=1}^n u_i \phi_i)^2} \right] dx. \end{aligned} \quad (3.30)$$

Employing the orthogonality property among the normal modes, and the fact that  $\phi_i'''' - k_1^4 \phi_i = 0$ , (3.30) becomes

$$\begin{aligned} \ddot{u}_j + k_1^4 u_j + c_n \dot{u}_j = & \alpha_1 \int_0^1 \phi_j \left\{ \sum_{i=1}^n u_i \phi_i'' \int_0^1 \left( \sum_{k=1}^n u_k \phi_k' \right)^2 \right\} dx \\ & + \alpha_2 \int_0^1 \left[ \frac{V_2(t)^2 \phi_j}{(1 - \sum_{i=1}^n u_i \phi_i)^2} - \frac{V_1(t)^2 \phi_j}{(1 + \sum_{i=1}^n u_i \phi_i)^2} \right] dx, \end{aligned} \quad (3.31)$$

and using the first modeshape  $\phi_1(x)$

$$\ddot{u}_1 + k_1^4 u_1 + c_n \dot{u}_1 + 151.33 \alpha_1 u_1^3 = F_e \quad (3.32)$$

where

$$F_e = \alpha_2 \int_0^1 \left[ \frac{V_2(t)^2 \phi_1}{(1 - u_1 \phi_1)^2} - \frac{V_1(t)^2 \phi_1}{(1 + u_1 \phi_1)^2} \right] dx, \quad (3.33)$$

and

$$\int_0^1 \phi_1 \phi_1'' dx \int_0^1 (\phi_1')^2 dx = 151.33, \quad (3.34)$$

see [33].

### 3.1.3 Parallel Plates Approximation

In our study, we will consider a chaotic MEMS resonator modeled by a nonlinear mass-spring-damper system with external electrostatic actuation expressed as

$$m\ddot{z} + b\dot{z} + c_1 z + c_3 z^3 = F, \quad (3.35)$$

where  $m$  is the mass of the movable structure,  $b$  is the damping constant,  $c_1$  and  $c_3$  are the linear and cubic stiffness coefficients, respectively, and  $F$  is the electrostatic actuation. In the case of small amplitude of oscillations (in comparison with the gap of the resonator), the parallel plates approximation (PPA) considers that the displacement of the midpoint of the beam can be approximated by a rigid movement of the whole beam. That produces the electrostatic force

$$F = \frac{1}{2} \frac{C_0 d}{(d - z)^2} (V_{DC} + u)^2 - \frac{1}{2} \frac{C_0 d}{(d + z)^2} V_{DC}^2. \quad (3.36)$$

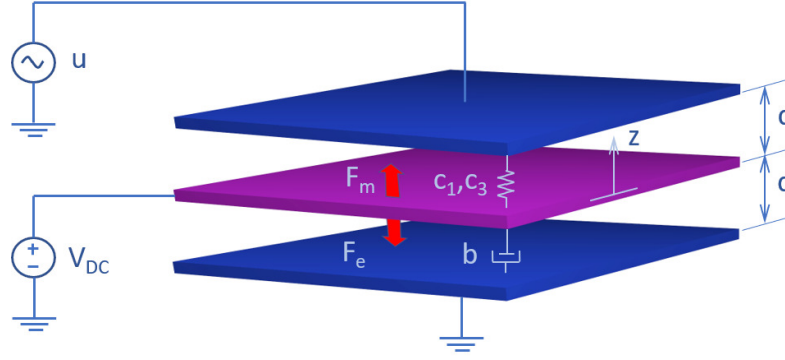


Figure 3.2 Schematic of the electrostatically actuated MEMS.

In (3.36)  $C_0$  is the capacitance of the actuator at rest, which has an initial gap  $d$  between the plates and the resonator. The input voltage  $V_i = (V_{DC} + u)$  is applied as shown in Fig. 3.2, and has two components: a DC voltage  $V_{DC}$  and a forcing voltage  $u$ , which is the actual control signal.

The movement equation (3.35) for the PPA corresponds to the reduced order model of beams actuated by the nonlinear electrostatic force given by equation (3.32), and under certain assumptions they are closely related [71]. However, the right hand side term of equation (3.32) does not correspond to the forcing term modeling the electrostatic force actuating on the beam, and is not exactly the same used in a PPA. That difference may result in significant errors in the analysis when the approximation is used, and makes it necessary to take it into account, e.g., in the design of controllers.

Midplane stretching is by far the most significant source of geometric nonlinearities in MEMS [33]. Using other approximated models, as the parallel plate approximation, must take into account the errors that can be present in the model. Following [71], a series expansion of  $F_e$  in (3.33) shows that the PPA, used in several studies of chaos and control in MEMS resonators, is valid only for a small amplitude of the oscillations in configurations with symmetric electrodes. For this reason, the development of new robust control strategies that also consider modeling errors is mandatory.

## 3.2 Chaos in Dynamic Systems

### 3.2.1 Definition of Chaos

There is no universally accepted definition of chaos, however it is usual to talk about chaos in Devaney's sense [72], and this definition will be adopted here.

Let  $X$  be a metric space and  $f$  a continuous map defined by  $f : X \rightarrow X$ .

**Definition 3.2.1.** *Transitivity of  $f$ .  $f$  is topologically transitive if for all non-empty open subsets  $U$  and  $V$  of  $X$  there exists a natural number  $k$  such that  $f^k(U) \cap V$  is nonempty.*

**Definition 3.2.2.** *Periodic Points and Orbits. The point  $x$  is a periodic point of period  $n$  if  $f^n(x) = x$ . The set of all iterates of a periodic point form a periodic orbit.*

**Definition 3.2.3.** *Density of Periodic Orbits. Periodic orbits are dense if every point in the space is approached arbitrarily closely by periodic orbits.*

**Definition 3.2.4.** *Sensitive Dependence on Initial Conditions.  $f$  has sensitive dependence on initial conditions if there exists a  $\delta > 0$  such that, for any  $x \in V$  and any neighborhood  $N$  of  $x$ , there exists a  $y \in N$  and an  $n \geq 0$  such that  $|f^n(x) - f^n(y)| > \delta$ .*

**Definition 3.2.5.** *Chaotic Systems.  $f$  is said to be chaotic on  $X$  if:*

- $f$  is transitive.
- The periodic points of  $f$  are dense in  $X$ .
- $f$  has sensitive dependence on initial conditions.

Roughly speaking, topological transitivity means that any given neighborhood in the phase space of a chaotic system is eventually visited by the trajectory of some point lying in any other given neighborhood. Density of periodic orbits indicates that every point in the phase space of a chaotic system is approached arbitrarily close by periodic orbits, and finally, sensitivity to initial conditions means that two arbitrarily close points in the phase space will diverge in the future. Although it can be proved that sensitivity to initial conditions is a consequence of the other two properties [73], this remains as the main characteristic of chaotic systems. Sensitivity to initial conditions means that two points arbitrarily close in the state space of the system will diverge in the future, indicating that if we apply an arbitrarily small perturbation to the system, its behavior in the future will be unpredictable. This property is known as “butterfly effect”, and may be used to take advantage of chaos.

### 3.2.2 Detecting Chaos: Melnikov’s Analysis

One of the few theoretical methods to study the onset of chaos in dynamical systems was formulated in 1963 by Melnikov [74]. In Melnikov’s method, the existence of homoclinic chaos is detected by a perturbative first order method.

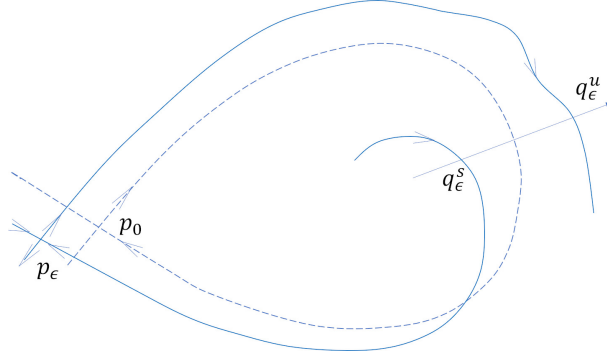


Figure 3.3 Homoclinic orbit (dashed line) and perturbed homoclinic orbit (solid line).

Here we outline the method for chaos suppression. Consider a perturbed system of the form

$$\dot{x} = f(x) + \varepsilon g(x, t) \quad (3.37)$$

where  $x \in \mathbb{R}^2$ ,  $f(x)$  is a Hamiltonian vector field on  $\mathbb{R}^2$ ,  $\varepsilon$  is a small constant and  $g$  is a continuous periodic function of period  $T$ . Assuming that  $q^0(t)$  is a homoclinic orbit of the system 3.37 with  $\varepsilon = 0$ , it is possible to break this orbit by applying a perturbation (i.e.  $\varepsilon > 0$ ), and look for the transversal intersection of the resultants stable and unstable manifolds calculating the Melnikov's function, which represents its distance, and is defined as:

$$M(t_0) = \int_{-\infty}^{+\infty} f(q^0(t)) \wedge g(q^0(t), t + t_0) dt \quad (3.38)$$

where  $(\wedge)$  represents the standard wedge product (See Fig. 3.3).

Based on Melnikov's theorem, a necessary condition for homoclinic chaos is that these orbits intersect, or the sign of  $M(t_0)$  changes for some  $t_0$ . We can conclude that applying a parametric perturbation or a control signal, it is possible to suppress homoclinical chaos guaranteeing that  $M(t_0)$  never changes its sign.

Melnikov's method has been used mainly in smooth systems, although its use in non-smooth systems has been extended recently.

A complete theory of Melnikov method can be found in Holmes and Guckenheimer [75], and Wiggins [76], [77]. In 1964, Arnold [78] generalizes this procedure, and Holmes [79] in 1979 is the first who applies this method to a damped, forced two well Duffing oscillator. With time, Melnikov method becomes a standard procedure to look for sufficient conditions to eliminate chaos, and today it is one of the few analytical techniques to deal with chaos suppression. In 1990 Lima and Pettini [80] are the first in applying this method to suppress chaos in a Holmes-Duffing oscillator theoretically, using parametric perturbation in resonance with

the driving force. Fronzoni et al. [81] apply the same procedure to control the same system experimentally. Although the authors Lima and Pettini made a mistake in the evaluations of the Melnikov function [82], [83] and some conclusions are wrong, the success of this work relies on establishing the procedure to eliminate chaos using periodic parametric perturbations. The procedure established by Pettini was used in the subsequent work by Chacon in 1995 [84], who correct the original sufficient condition to eliminate chaos obtained earlier by Pettini. Finally, this procedure was generalized for a wide class of dynamical systems by Chacon in [85], and the same author in [86] present general results concerning the enhancement of chaos for the same family of oscillators.

There are many papers about applications of Melnikov's method. In [87] Bartucelli et al. obtain an estimate of the threshold curve for the onset of chaos. They apply this method for prediction of chaos in a Josephson-Junction model, considering linear and quadratic damping. In [88] Cicogna and Fronzoni use the method to suppress chaos in a standard Josephson Junction model, and Chacon in [84] corrects and generalizes some of their results. In [89] Chen uses the Melnikov method to obtain a necessary condition for appearance of chaos in a kind of system like Duffing, with both square and cubic nonlinearities.

Experimental demonstrations of the method can be found in Fronzoni and Giocondo [90]. An interesting review of the advances in the applications of the method was carried out by Chacon in [91]. In [86] this author generalizes the application to some class of differential equations, and gives a general view of the subject in [91].

Recently, it is witnessed that Melnikov's method can be extended for certain class of non-smooth planar systems with a discontinuous surface being perturbed periodically. Particularly, Melnikov function is calculated in [92] for a planar system with a homoclinic orbit which intersects the discontinuous surface in two points with no sliding allowed. Under certain assumptions, a more general approach has been used for planar and non smooth systems with or without sliding on the discontinuous surface [93], [94]. For the high dimensional case illustrated in [95], a similar problem can be handled by the assumption of the existence of a homoclinic orbit crossing the discontinuity surface transversally.

For applying the Melnikov's theory to the MEMS resonator under study, it is convenient to write its model, equations (3.35) and (3.36), in normalized coordinates. By assuming that the forcing voltage  $u$  is smaller than the DC voltage, the nondimensional equation of motion becomes

$$\ddot{x} + \mu\dot{x} + \sigma x + \eta x^3 = \gamma \left( \frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} \right) + \frac{u'}{(1-x)^2}. \quad (3.39)$$

A description of the normalization procedure is explained in details in Chapter 4. To evaluate the Melnikov's integral we write the second order system, equation (3.39), as a system of two first order differential equations

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\alpha x - \eta x^3 + \gamma \left( \frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} \right) + \varepsilon \left( -\bar{\mu}y + \frac{\bar{A}}{(1-x)^2} \sin \omega t \right),\end{aligned}\quad (3.40)$$

where  $\mu = \varepsilon \bar{\mu}$ , and  $A = \varepsilon \bar{A}$  (i.e.,  $\mu = O(\varepsilon)$ ,  $A = O(\varepsilon)$ ) and  $\varepsilon$  is a small parameter. The Melnikov's function of (3.40) is

$$M(t_0) = \int_{-\infty}^{+\infty} y^0(t) \left[ -\bar{\mu}y^0(t) + \frac{\bar{A}}{(1-x^0(t))^2} \sin \omega(t+t_0) \right] dt \quad (3.41)$$

where  $q^0(t) = (x^0(t), y^0(t))$  is the homoclinic orbit of the unperturbed system, which, unfortunately cannot be obtained in a closed-form expression. In order to obtain an approximation of this orbit, we expand the terms on the right hand side of (3.40) in a Taylor series up to fourth order,

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\alpha x - \eta x^3 + \gamma (4x + 8x^3 + O(x^5)) + \varepsilon \left( -\bar{\mu}y + \frac{\bar{A}}{(1-x)^2} \sin \omega t \right)\end{aligned}\quad (3.42)$$

and after combining like terms in this expression we obtain

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &\approx (4\gamma - \alpha)x + (8\gamma - \eta)x^3 + \varepsilon \left( -\bar{\mu}y + \frac{\bar{A}}{(1-x)^2} \sin \omega t \right).\end{aligned}\quad (3.43)$$

This system is a perturbed version of Duffing system, for which a homoclinic orbit is well known. For  $\frac{\alpha}{4} < \gamma < \frac{\beta}{8}$  the homoclinic orbit of (3.43) is approximated by

$$(x^0(t), y^0(t)) = (\pm \chi_e \operatorname{sech} \sqrt{\kappa} t, \mp \chi_e) \quad (3.44)$$

where  $\kappa = 4\gamma - \alpha$ , and  $\chi_e$  is a positive solution for  $V(x) = 0$  which is given by

$$\chi_e = \sqrt{\frac{1}{2} - \frac{\alpha}{\eta} - \frac{\sqrt{4\alpha^2 + 4\alpha\eta + \eta^2 - 32\eta\gamma}}{2\eta}}. \quad (3.45)$$

Using the expressions for  $x^0(t)$  and  $y^0(t)$  in the Melnikov's integral (3.41), it becomes

$$M(t_0) = \bar{\mu}I_1 + \bar{A}I_2 \quad (3.46)$$

where  $I_1$  and  $I_2$  are integrals defined by

$$I_1 = - \int_{-\infty}^{+\infty} y_0^2 dt \quad (3.47)$$

and

$$I_2 = \int_{-\infty}^{+\infty} y^0(t) \frac{\sin \omega(t + t_0)}{(1 - x^0(t))^2} dt. \quad (3.48)$$

The first integral is easily evaluated, and for the second the Cauchy's residue theorem can be used [38].

$$I_1 = -\frac{2}{3}\sqrt{\kappa}\chi_e^2 \quad (3.49)$$

$$I_2 = \chi_e \bar{A} \frac{2\pi\bar{\omega}}{\sqrt{1 - \chi_e^2}} \cos(\omega' t_0) \frac{\sinh(\bar{\omega}(-\arccos(\chi_e)))}{\sinh(\bar{\omega}\pi)} \quad (3.50)$$

$$\frac{\bar{A}}{\bar{\mu}} = \left| \frac{\chi_e \sqrt{\kappa} \sqrt{1 - \chi_e^2} \sinh(\bar{\omega}\pi)}{3\pi\bar{\omega} \sinh(\bar{\omega}(-\arccos(\chi_e)))} \right| \quad (3.51)$$

### 3.2.3 Lyapunov's Exponents

The Lyapunov exponents (or characteristic exponents) associated with a trajectory, are essentially a measure of the average rates of expansion and contraction of trajectories surrounding it. They describe the exponential rate at which a perturbation to a trajectory of a system grows or decays with time at a certain location in the state space. Using Lyapunov exponents, we can distinguish among fixed points, periodic motions, quasiperiodic motions, and chaotic motions. Consider a  $n$ -dimensional autonomous system defined by the equation

$$\dot{x}(t) = f(x(t), \varphi), \quad (3.52)$$

where the vector  $x$  is made up of  $n$  state variables and  $\varphi$  is a vector of parameters. Let  $x(t)$  represents a trajectory of the system initiating in  $x_0$ . If a small perturbation  $y(t)$  is applied to that trajectory, and (3.52) is linearized in the disturbance terms, the perturbation

is governed by

$$\dot{y}(t) = Ay(t), \quad (3.53)$$

where  $A = \frac{\partial f}{\partial x}$  is the  $n \times n$  Jacobian matrix. An initial perturbation  $y(0)$  will evolve according to

$$y(t) = \Phi(t)y(0) \quad (3.54)$$

where  $\Phi(t)$  is the fundamental (transition) matrix solution of (3.53) associated with the trajectory  $x(t)$ . For an appropriately chosen  $y(0)$ , the rate of exponential expansion or contraction in the direction of  $y(0)$  on the trajectory passing through  $y(0)$  defines the Lyapunov exponent along that direction

$$\lambda_i = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \left( \frac{\|y(t)\|}{\|y(0)\|} \right), \quad (3.55)$$

where  $\|\cdot\|$  denotes a vector norm.

Since the state space is  $n$ -dimensional, we can use a set of  $n$  linearly independent vectors as a basis:  $y_1, y_2, \dots, y_n$ . By applying a small perturbation along each of these vectors, we can determine  $n$  Lyapunov exponents  $\lambda_i(y_i)$ , which ordered in descending order form the Lyapunov spectrum:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n. \quad (3.56)$$

Following the definitions given by Lyapunov, the fundamental matrix  $\Phi(t)$  is called regular if

$$\lim_{t \rightarrow \infty} \frac{1}{t} \ln |\det \Phi(t)| \quad (3.57)$$

exists and there exists a normal basis of the  $n$ -dimensional state space such that

$$\sum_{i=1}^n \lambda_i = \lim_{t \rightarrow \infty} \frac{1}{t} \ln |\det \Phi(t)|. \quad (3.58)$$

If  $\Phi(t)$  is regular, the Lyapunov exponents exist and are finite for any initial deviation  $y(0)$  belonging to the  $n$ -dimensional space, and are independent of the initial condition  $y(0)$ .

Lyapunov exponents are related to the eigenvalues of the linear matrix in a linear approximation of an autonomous system around a fixed point. In that case the Lyapunov exponents

are defined as

$$\lambda_i = \lim_{t \rightarrow \infty} \frac{1}{t} \text{Re}(\bar{\lambda}_i), \quad (3.59)$$

where  $\text{Re}$  denotes the real part, and  $\bar{\lambda}_i$  is an eigenvalue of the Jacobian matrix. If all the eigenvalues have a negative real part, then the fixed point is stable.

If we consider the Lyapunov exponents of a limit cycle of an autonomous system, one of them is always zero. If it is stable, all other Lyapunov exponents are negative. In the case of dissipative systems, chaotic motions are characterized by one or more positive Lyapunov exponents. These positive exponents correspond to perturbations provided along some directions normal to a trajectory and imply that, locally, trajectories separated from the trajectory in question diverge from it along these directions.

The Lyapunov exponents for the chaotic MEMS considered in this study are evaluated in Section 4.3 and shown in Figure 4.3. In order to evaluate them, it is necessary to numerically integrate (3.52) and (3.53), and use (3.55) to compute the Lyapunov exponents associated to an orbit  $x(t)$  with initial value  $x(0)$ . We choose  $m$  orthonormal initial vectors  $y_i$ , and for each of these initial vectors, we integrate (3.52) and (3.53) for a finite time  $T$  to obtain a set of vectors  $y_1(T), y_2(T), \dots, y_m(T)$ . This set of vectors is orthonormalized using the Gram-Schmidt procedure, and the procedure is repeated using  $x(T)$  as initial condition for (3.52) and the orthonormalized set of vectors as initial condition for (3.53). These equations are integrated for a finite time  $T$ , the obtained vectors orthonormalized, and the procedure repeated again until  $r$  times. We obtain the Lyapunov exponents from

$$\lambda_i = \frac{1}{rT} \sum_{k=1}^r \ln N_i^k, \quad (3.60)$$

where  $N_i^k$  is the norm in the denominator of the  $k$ th time orthonormalization step of the  $i$ th vector.

### 3.3 Filippov Solution of Differential Inclusions

In this section we describe the concept of solution of differential equations with discontinuous right-hand sides in the sense of Filippov [96]. To this end, consider the differential equation

$$\dot{x}(t) = f(t, x), \quad (3.61)$$

where  $x \in \mathbb{R}^n$  is the state and  $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a vector field, discontinuous on a smooth surface  $\mathcal{S}$  defined by

$$\mathcal{S} = \{x : \sigma(x) = 0\}. \quad (3.62)$$

At a given state  $x$ , instead of focusing on the value of the vector field at  $x$ , the idea behind Filippov solutions is to introduce a set of velocities that are determined by the values of the vector field in a neighborhood of  $x$ . This idea plays an important role in such applications as sliding mode control.

Filippov solutions replace the differential equation by a DI of the form

$$\dot{x}(t) \in F(t, x), \quad (3.63)$$

where  $F : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathcal{B}(\mathbb{R}^n)$  with  $\mathcal{B}(\mathbb{R}^n)$  denoting the collection of all subsets of  $\mathbb{R}^n$ . At points of continuity of the function  $f$ , the set  $F(t, x)$  consists of one point  $f(t, x)$  and the solution satisfies (3.61) in the usual sense. If the point  $x_0$  lies on  $\mathcal{S}$ , and defining  $f_-(x_0)$  and  $f_+(x_0)$  as the limits of  $f(x)$  as the point  $x_0$  is approached from opposite sides of the tangent to  $S$  at  $x_0$  (see Fig. 3.4), then the set  $F$  is a linear segment joining the endpoints of the vectors  $f_-$  and  $f_+$

$$F(t, x) = (1 - \alpha)f_-(x_0) + \alpha f_+(x_0) \quad (3.64)$$

where  $0 < \alpha < 1$ . The scalar  $\alpha$  is chosen so that

$$f_a = (1 - \alpha)f_- + \alpha f_+ \quad (3.65)$$

is tangential to  $S$ , as represented in Figure 3.4.

Equation (3.61) can be seen as a differential equation whose right-hand side is defined as the convex set

$$F(t, x) = \{(1 - \alpha)f_- + \alpha f_+ : \text{for all } \alpha \in [0 \ 1]\}. \quad (3.66)$$

**Definition 3.3.1.** *A differential equation  $\dot{x}(t) = F(t, x)$  with locally bounded Lebesgue-measurable right-hand side is understood in the Filippov sense, if it is replaced by a special Filippov differential inclusion  $\dot{x}(t) \in F(t, x)$ , where*

$$F(t, x) = \bigcap_{\delta > 0} \bigcap_{\mu(N)=0} \overline{\text{co}} f(B(x, \delta) \setminus N), \quad x \in \mathbb{R}^n, \quad (3.67)$$

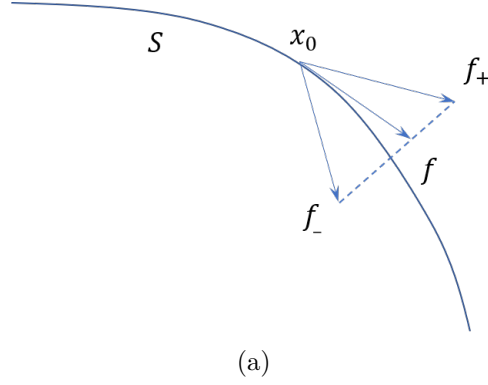


Figure 3.4 Filippov's construction of solution.

and  $\mu$  is the Lebesgue measure,  $B(x, \delta)$  is the  $\delta$ -vicinity of  $x$ , and  $\overline{\text{co}} f$  denotes the convex closure of  $f$ . The intersection is taken over all sets  $N$  of measure zero and over all  $\delta > 0$ .

Note that for  $x \in \mathbb{R}^n$ , the vector field  $f$  is evaluated at the points belonging to  $B(x, \delta)$ , which is the open ball centered at  $x$  with radius  $\delta > 0$ . In the evaluation of  $f$ , any arbitrary set of measure zero in  $B(x, \delta)$  is excluded, so that the value is the same for two vector fields that differ on a set of measure zero [96].

The Filippov's regularization procedure consists in the construction of the set-valued map  $F$  as in (3.67), starting with a vector field  $f$ . By construction, for all  $x \in \mathbb{R}^n$ , the set  $F(t, x)$  is closed and convex. Moreover, the set-valued map  $F(t, x)$  is upper semicontinuous. If the DI is given by the set-valued map coming from the Filippov's procedure, we say that  $F$  verifies the standard assumptions.

### 3.4 Homogeneity and Finite-Time Stability

Homogeneity is a property whereby objects such as functions or vector fields scale with respect to a scaling operation called dilation [97]. This definition involve many qualitative properties for a homogeneous object and is a matter of especial importance in stability. Homogeneity is one of the most used tools to analyze high-order sliding mode observers and controllers, and helps us to obtain one of the more important properties of sliding mode control, which is finite time stability.

#### 3.4.1 Weighted Homogeneity

The weighted homogeneity (introduced by Zubov in the late 1950s [98]) allows to obtain important results related to the global equivalence of local stability, existence of homoge-

neous Lyapunov functions, design of observers and controllers, finite time stability, and the robustness analysis of nonlinear systems.

The weighted dilation  $\Lambda_r(\lambda, x)$  of the vector  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  is defined by

$$\begin{aligned} \Lambda_r : \mathbb{R}_+ \times \mathbb{R}^n &\rightarrow \mathbb{R}^n, \\ (\lambda, x) &\mapsto (\lambda^{r_1}x_1, \lambda^{r_2}x_2, \dots, \lambda^{r_n}x_n), \end{aligned} \quad (3.68)$$

where  $\lambda > 0$  is the scaling factor and  $r = (r_1, \dots, r_n)$  is a  $n$ -uplet of positive real numbers named the generalized weight, which specify dilation rates along different coordinates. For simplicity and shortness we omit the variable  $\lambda$  when we write the dilation function, namely, let us denote  $\Lambda_r z = \Lambda_r(\lambda, z)$ , where  $z \in \mathbb{R}^n$  and  $\lambda > 0$ .

The homogeneity of a scalar valued function with respect to the weighted dilation is defined as follows.

**Definition 3.4.1.** *A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be  $r$ -homogeneous with degree  $m$  ( $m \in \mathbb{R}$ ) if for all  $x \in \mathbb{R}^n$  and all  $\lambda > 0$  we have*

$$\lambda^{-m} f(\Lambda_r x) = f(x). \quad (3.69)$$

Sometimes we denote the degree of a function as  $\deg f = m$ . A significant definition for dynamic systems is the weighted homogeneity of a vector field.

**Definition 3.4.2.** *A vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is said to be  $r$ -homogeneous with degree  $m$  if for all  $x \in \mathbb{R}^n$  and all  $\lambda > 0$  we have*

$$\lambda^{-m} \Lambda_r^{-1} f(\Lambda_r x) = f(x). \quad (3.70)$$

We may denote the degree of a function as  $\deg f = m$ . Generalizing this definition, we may say that a system  $\dot{x} = f(x)$  is homogeneous if  $f$  is so.

**Definition 3.4.3.** *A set-valued map  $F$  is  $r$ -homogeneous with degree  $m \in \mathbb{R}$  if for all  $x \in \mathbb{R}^n$  and all  $\lambda > 0$  we have*

$$\lambda^{-m} \Lambda_r^{-1} F(\Lambda_r x) = F(x). \quad (3.71)$$

We denote the degree of  $F$  as  $\deg F = m$ . The system (3.63) is  $r$ -homogeneous of degree  $m$  if the set-valued map  $F$  is homogeneous of degree  $m$ .

If we have a vector field  $f$ , which is homogeneous as defined in Definition 3.4.2, and we apply a regularization procedure to obtain a DI with a set-valued map  $F$ , its homogeneity property is preserved.

**Proposition 1.** *Let  $f$  be a vector field and  $F$  be the associated set-valued map. Suppose  $f$  is  $r$ -homogeneous of degree  $m$ . Then  $F$  is  $r$ -homogeneous of degree  $m$ .*

### 3.4.2 Stability of Differential Inclusions

In the remainder of this section, we present several definitions of stability that are of fundamental significance in the context of control for systems with discontinuities. We begin with the continuous system

$$\begin{cases} \dot{x}(t) = f(x(t)), & t \geq 0, \\ x(0) = x_0, & x_0 \in \mathbb{R}^n, \end{cases} \quad (3.72)$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $f(0) = 0$  is considered such that (3.72) possesses a unique solution in forward time for all initial conditions  $x_0 \in \mathbb{R}^n$  except the origin, denoted as  $x(t, x_0)$ .

For presenting the stability definitions, it is useful to first define the following classes of functions [99]. A continuous function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}$  if  $\alpha(0) = 0$  and the function is strictly increasing. The function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}_\infty$  if  $\alpha \in \mathcal{K}$  and it is increasing to infinity. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{KL}$  (resp.,  $\mathcal{GKL}$ ) if  $\beta(\cdot, t) \in \mathcal{K}_\infty$  for each fixed  $t \in \mathbb{R}_+$  and  $\lim_{t \rightarrow +\infty} \beta(s, t) = 0$  (resp., if for each fixed  $t \in (0, T)$ ,  $\beta$  is a strictly decreasing function of its second argument  $t \in \mathbb{R}_+$  for any fixed first argument  $s \in \mathbb{R}$  and  $\beta(s, T) = 0$  for some  $0 \leq T < +\infty$ ) for each fixed  $s \in \mathbb{R}_+$ .

**Definition 3.4.4.** *The origin of system (3.72) is Lyapunov stable if there exists a nonempty open neighborhood of the origin  $\mathcal{V} \subset \mathbb{R}^n$  and a function  $\alpha \in \mathcal{K}_\infty$  such that for all  $x_0 \in \mathcal{V}$  we have  $\|x(t, x_0)\| \leq \alpha(\|x_0\|), \forall t \geq 0$ .*

**Definition 3.4.5.** *The origin of system (3.72) is asymptotically stable if it is Lyapunov stable and for all  $x_0 \in \mathcal{V}$ ,  $\lim_{t \rightarrow +\infty} x(t, x_0) = 0$ .*

**Definition 3.4.6.** *The origin of system (3.72) is Finite-time stable (FTS) if it is Lyapunov stable and there exists a function  $T : \mathcal{V} \rightarrow \mathbb{R}_+$  such that  $\forall x_0 \in \mathcal{V}, x(t, x_0) = 0, \forall t \geq T(x_0)$ .  $T$  is called a settling-time function of (3.72).*

All these definitions hold globally, if the set  $\mathcal{V}$  is defined as  $\mathcal{V} = \mathbb{R}^n$ . In particular, we can define a globally asymptotically stable system (GAS) as a system that is asymptotically stable with  $\mathcal{V} = \mathbb{R}^n$ .

The following definitions are related to stability properties for the differential inclusion

$$\begin{cases} \dot{x}(t) \in F(x(t)), & \forall t \geq 0, \\ x(0) = x_0, & x_0 \in \mathbb{R}^n, \end{cases} \quad (3.73)$$

where  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ .

For each  $\Omega \subseteq \mathbb{R}^n$ , we let  $S(\Omega) \subset \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$  denote the set of solutions of (3.73) satisfying  $x(0) \in \Omega$ . If  $\Omega$  is a singleton  $\{x_0\}$ , we will use the shorthand  $S(x_0)$ . We denote  $S = S(\mathbb{R}^n)$  as the set of all solutions. The domain of a solution  $x(\cdot)$  will be denoted by  $\mathbb{R}_+$  if it is forward complete. The next definitions concern GAS and FTS properties of DI, and make reference to system (3.73).

**Definition 3.4.7.** *The origin of the system is strongly GAS (resp., FTS) if all solutions  $x(\cdot) \in S$  are GAS (resp., FTS).*

**Definition 3.4.8.** *The origin of the system is weakly GAS (resp., FTS) if there exists a subset  $\hat{S}$  of the set  $S$  of solutions, such that all  $x(\cdot) \in \hat{S}$  are uniformly GAS (resp., FTS).*

In the study of sliding mode control, it is useful to understand the solutions of the resultant discontinuous systems in the sense of Filippov. That take us to differential inclusions, and in this context, it is natural to extend some useful properties of homogeneous ordinary differential equations to differential inclusions.

The following theorem asserts that a strongly GAS homogeneous DI admits a homogeneous Lyapunov function. This result is a generalization given by Rosier [100] and Zubov [98] for continuous systems [101]. If  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is a function, we denote  $d_x V$  the differential of the function  $V$  at point  $x \in \mathbb{R}^n$ .

**Theorem 3.4.1.** *Let  $F$  be a  $v$ -homogeneous set-valued map of degree  $k$ , satisfying the standard assumptions. Then the following statements are equivalent:*

- *The origin of system (3.63) is (strongly) GAS.*
- *For all  $\mu > \max(-k, 0)$ , there exists a pair  $(V, W)$  of continuous functions, such that:*
  1.  *$V$  is of class  $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ ,  $V$  is positive definite and  $v$  homogeneous of degree  $\mu$ ;*
  2.  *$W$  is of class  $\mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ ,  $W$  is strictly positive outside the origin and  $v$  homogeneous of degree  $\mu + k$ ;*
  3.  *$\max_{v \in F(x)} d_x V v \leq -W(x)$  for all  $x \neq 0$*

The importance of the notions introduced is justified by the following remarkable result: asymptotic stability of a homogeneous DI with negative homogeneous degree is equivalent to its global uniform finite-time stability (GFTS).

**Definition 3.4.9.** *A differential inclusion  $\dot{x}(t) \in F(x)$  is further called globally uniformly finite-time stable at 0, if it is Lyapunov stable, and for any  $R > 0$ , there exists  $T > 0$  such that any trajectory starting within the disk  $\|x\| < R$  stabilizes at zero in the time  $T$ .*

Applying Theorem 3.4.1, it is possible to prove that the existence of a smooth homogeneous Lyapunov function implies the finite time stability of a homogeneous set-valued map of negative degree.

**Corollary 3.4.1.1.** *Let  $F$  be a  $v$ -homogeneous set-valued map of degree  $k < 0$ , satisfying the standard assumptions. Assume also that the origin is GAS for  $F$ . Then the origin is GFTS for  $F$ .*

## CHAPTER 4    ARTICLE 1: CHAOS SYNCHRONIZATION OF AN ELECTROSTATIC MEMS RESONATOR IN THE PRESENCE OF PARAMETRIC UNCERTAINTIES

This Chapter is reproduced from [102]:

A. Jimenez-Triana, G. Zhu, and L. Saydy, “Chaos synchronization of an electrostatic MEMS resonator in the presence of parametric uncertainties,” in *2011 American Control Conference*, San Francisco, California, USA, 18 august 2011, pp. 5115–5120.

### 4.1 Abstract

This paper addresses the synchronization control problem of a class of chaotic systems. The synchronization is obtained by using an adaptive control law, which guarantees a global asymptotical convergence of the error between the outputs of driving and response systems in the presence of parametric uncertainties in both systems. The method of synchronization can be applied to a wide class of systems though the present work focuses only on a particular case, mainly, Micro Electro Mechanical Systems (MEMS). As an example of application, an electrostatic MEMS resonator with unknown damping and stiffness actions is synchronized with a Duffing chaotic oscillator and a MEMS chaotic oscillator. This component can be used in such applications as secure communication. Numerical simulations are carried out to confirm the validity of the control schemes obtained.

### 4.2 Introduction

Chaos synchronization [103] is a phenomenon which may occur when a chaotic system drives another chaotic system by coupling them in a correct way and ensuring that both systems evolve in synchrony. Since the pioneering work of [104], a lot of research has been conducted concerning chaos synchronization, which is nowadays a widely understood phenomenon and an active topic of research [105].

Chaos synchronization is important in many fields, including biology, nonlinear optics, fluid dynamics, and electronics [103]. In particular, it finds applications in secure communications, where different methods have been developed to encode data in a chaotic signal. One of the methods involves using a chaotic driving system to generate a chaotic carrier that is modulated by a message matching the transmitter’s signal. The receiver includes a chaotic response system synchronized with the driver, in such a way that a replica of the chaotic

carrier can be obtained in the receiver, and the message can be decoded by subtracting the “chaotic” part of the signal.

This standard application of chaos synchronization has motivated a great deal of research activity and many methods have been proposed to synchronize chaotic systems, including parametric perturbations [106], adaptive control [107–110], variable structure control [111], impulsive control [106], backstepping [112],  $H_\infty$  control [113, 114], and so on.

It is interesting to note that in order to apply the ideas from chaos synchronization in secure communications, the response system does not have to be the same as the driving system [107–110], or as we note in the results of this paper, it does not have to be even chaotic. To obtain their results, many works using chaos synchronization and claiming applications in secure communications do not use the fact that the response system should be chaotic. However, it is common that assumptions on the dynamics of the system or its structure, such as boundedness of the solutions and Lipschitz conditions, should be easy to fulfil if the response system is chaotic since it evolves over time into a strange attractor [115].

In this paper, an algorithm to synchronize two systems is given, one of which, the driving system, is chaotic. The response system can be chaotic or not, can have the same form as the driving system, or can even be structurally different. In order to apply this idea, affine control systems are considered.

Many MEMS devices exhibit chaotic behavior in certain operational conditions [38, 68–70]. These MEMS can be seen as chaotic capacitors, which can be used, for instance, in the transmitter or the receiver circuitry in security communications schemes using chaos [55]. In addition, as have been observed in recent results concerning the improvement of the output energy in oscillators using MEMS [116], controlling MEMS in chaotic mode can be useful to increase the amplitude of the periodic oscillations of these systems. Therefore, introducing chaos in MEMS by synchronizing them with a chaotic system can also be useful in improving the performance of different systems in which an accurate control of amplitude of oscillation is mandatory.

The organization of the rest of the paper is as follows. In Section 4.3 the dynamic of a chaotic MEMS actuator is briefly explained based on a model given in [37], [38]. Next, in Section 4.4, the synchronization scheme using adaptive control is explained in details. Simulations are then carried out and the results are presented in Section 4.5. Finally, conclusions are given in Section 4.6.

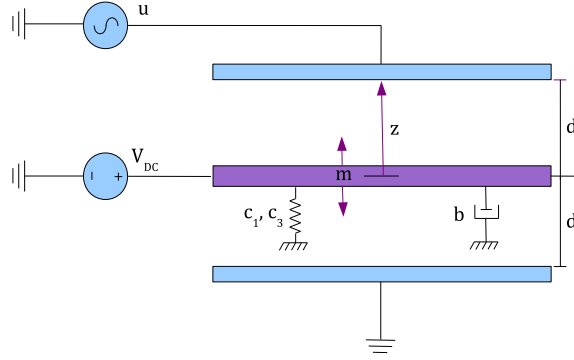


Figure 4.1 Schematic of the electrostatically actuated MEMS.

### 4.3 Dynamics of a Chaotic MEMS Actuator

The chaotic MEMS model used in this study is presented in Fig. 4.1, which is taken from [37], [38]. The MEMS is modeled by a nonlinear mass-spring-damper system with external electrostatic actuation expressed as

$$m\ddot{z} + b\dot{z} + c_1z + c_3z^3 = F, \quad (4.1)$$

where  $m$  is the mass of the movable structure,  $b$  is the damping constant,  $c_1$  and  $c_3$  are the linear and cubic stiffness coefficients, respectively, and  $F$  is the electrostatic actuation given by

$$F = \frac{1}{2} \frac{C_0 d}{(d - z)^2} (V_{DC} + u)^2 - \frac{1}{2} \frac{C_0 d}{(d + z)^2} V_{DC}^2. \quad (4.2)$$

In (4.2)  $C_0$  is the capacitance of the actuator at rest, which has an initial gap  $d$  between the plates and the resonator. The input voltage  $V_i = (V_{DC} + u)$  is applied as shown in Fig. 4.1, and has two components: a DC voltage  $V_{DC}$  and a forcing voltage  $u$ , which is the actual control signal. Note that in the present work, we ignore the dynamics of the driving circuit by assuming that they are much faster than that of the mechanical structure. Note also that for simplicity, we do not take into account contact dynamics. This means that the system should operate in such a way that the moveable and fixed plates will not come into contact.

In order to obtain a simpler model, it is convenient to describe the system dynamics in

normalized coordinates [117]. We introduce then the following dimensionless variables:

$$\begin{aligned}\tau &= \omega_0 t, \quad \omega = \frac{\Omega}{\omega_0}, \quad x = \frac{z}{d}, \quad \mu = \frac{b}{m\omega_0}, \quad \sigma = \frac{c_1}{m\omega_0^2}, \\ \eta &= \frac{c_3 d^2}{m\omega_0^2}, \quad \gamma = \frac{C_0 V_{DC}^2}{2m\omega_0^2 d^3}, \quad A = 2\gamma \frac{1}{V_{DC}}, \quad uA = u',\end{aligned}$$

where  $\omega_0$  is the natural frequency defined by  $\omega_0 = \sqrt{c_1/m}$ .

In the normalized coordinates, by assuming that the forcing voltage  $u$  is smaller than the DC voltage, the nondimensional equation of motion becomes

$$\ddot{x} + \mu\dot{x} + \sigma x + \eta x^3 = \gamma \left( \frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} \right) + \frac{u'}{(1-x)^2}. \quad (4.3)$$

System (4.3) behaves chaotically by selecting  $u = V_{AC} \sin(\omega t)$ , and the set of parameters  $V_{AC}A = 0.04$ ,  $\omega = 0.5$ ,  $\gamma = 0.338$ ,  $\mu = 0.01$ ,  $\sigma = 1$ , and  $\eta = 12$ , as may be observed in the phase diagram of Fig. 4.2 and the Lyapunov exponents diagram of Fig. 4.3. The existence of one positive Lyapunov exponent confirms the chaotic behavior of the system.

#### 4.4 Synchronization of Affine Chaotic Systems

Consider a class of two dimensional uncertain chaotic system

$$\dot{x} = F(x)\alpha + f(t, x) \quad (4.4)$$

which is named the driving or master system, and

$$\dot{y} = G(y)\beta + g(t, y) + h(y)u \quad (4.5)$$

which is named the response or slave system, where  $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^\top \in \mathbb{R}^2$  and  $y = \begin{pmatrix} y_1 & y_2 \end{pmatrix}^\top \in \mathbb{R}^2$  are bounded state vectors,  $\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}^\top \in D \subset \mathbb{R}^n$ , with  $D$  a subset of  $\mathbb{R}^n$  which guarantees that the driving system is chaotic, and  $\beta = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_p \end{pmatrix}^\top \in \mathbb{R}^p$  are uncertain parameters,  $F(x): \mathbb{R}^2 \rightarrow \mathbb{R}^2 \times \mathbb{R}^n$  and  $G(y): \mathbb{R}^2 \rightarrow \mathbb{R}^2 \times \mathbb{R}^p$  are matrices defined by

$$F(x) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ F_{21}(x) & F_{22}(x) & \cdots & F_{2n}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}, \quad (4.6a)$$

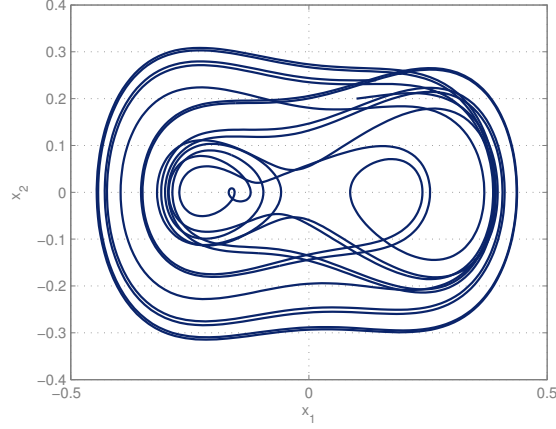


Figure 4.2 State space evolution for the system (4.3) with the set of parameters  $\gamma = 0.338$ ,  $A = 0.04$ ,  $\omega = 0.5$ ,  $\sigma = 1$ ,  $\mu = 0.01$ , and  $\eta = 12$ .

$$G(y) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ G_{21}(y) & G_{22}(y) & \cdots & G_{2p}(y) \end{bmatrix} = \begin{bmatrix} 0 \\ G_2 \end{bmatrix}, \quad (4.6b)$$

$f(t, x) : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $g(t, y) : \mathbb{R}^+ \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $h(y) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  are vectors given by

$$f(t, x) = \begin{bmatrix} x_2 \\ f(t, x) \end{bmatrix}, g(t, y) = \begin{bmatrix} y_2 \\ g(t, y) \end{bmatrix}, \quad (4.7)$$

$$h(y) = \begin{bmatrix} 0 & h_2(y) \end{bmatrix}^\top, \quad (4.8)$$

and  $u(t, x, y) : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is a scalar control input. We suppose that  $F(x)$ ,  $f(t, x)$ ,  $G(y)$ ,  $g(t, y)$ , and  $h(y)$  satisfy some regularity conditions, such as Lipschitz continuity, to ensure, e.g., the existence and the uniqueness of solutions of systems (4.4) and (4.5). The control objective is to design a state feedback control  $u$  such that the states of the response system,  $y$ , asymptotically track that of the driving system,  $x$ , or equivalently, the response system synchronizes the driving system. To this end, we define the error vector by  $e = y - x = \begin{pmatrix} e_1 & e_2 \end{pmatrix}^\top \in \mathbb{R}^2$  and the parameter estimation errors as  $\tilde{\alpha} = \alpha - \hat{\alpha}$ ,  $\tilde{\beta} = \beta - \hat{\beta}$ , where  $\hat{\alpha}$ ,  $\hat{\beta}$  are the estimated values of  $\alpha$  and  $\beta$ , respectively. The control objective can then be thought as to asymptotically stabilize the error dynamics around the origin, implying  $\lim_{t \rightarrow \infty} \|e\| = 0$ .

Based on these definitions, one can assert the following result.

**Theorem 4.4.1.** *For given positive constants  $k_1$ ,  $k_2$ ,  $\gamma_1$ , and  $\gamma_2$ , if the control input  $u$  is*

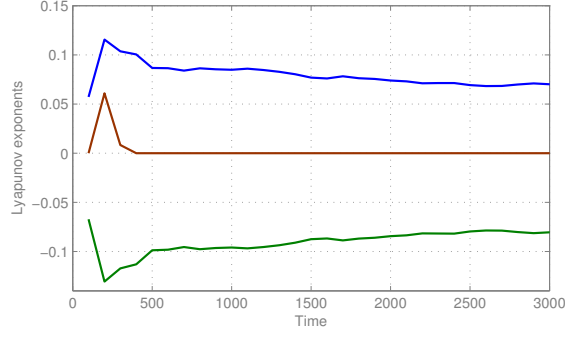


Figure 4.3 Lyapunov exponents for the system (4.3) with the set of parameters:  $\gamma = 0.338$ ,  $A = 0.04$ ,  $\omega = 0.5$ ,  $\sigma = 1$ ,  $\mu = 0.01$ , and  $\eta = 12$ .

given by

$$u = h_2^{-1}(y) \left( -G_2 \hat{\beta} + F_2 \hat{\alpha} - g(t, y) + f(t, x) - k_2^{-1}(k_1 e_1 + k_2 e_2) - k_2^{-1} k_1 e_2 \right), \quad (4.9)$$

and the parameters are updated according to the laws

$$\dot{\hat{\alpha}}^\top = -\gamma_1 (k_1 e_1 + k_2 e_2) k_2 F_2, \quad (4.10a)$$

$$\dot{\hat{\beta}}^\top = \gamma_2 (k_1 e_1 + k_2 e_2) k_2 G_2, \quad (4.10b)$$

then the response system (4.5) synchronizes the driving system (4.4) globally and asymptotically.

*Proof.* It can be derived from (4.4) and (4.5) that the synchronization error dynamics are given by

$$\dot{e} = \dot{y} - \dot{x} = G(y) \beta + g(t, y) + h(y) u(t, x, y) - F(x) \alpha - f(t, x). \quad (4.11)$$

The derivatives of the parameter estimation errors are

$$\dot{\tilde{\alpha}} = -\dot{\hat{\alpha}}, \quad (4.12a)$$

$$\dot{\tilde{\beta}} = -\dot{\hat{\beta}}. \quad (4.12b)$$

By defining the following Lyapunov function candidate:

$$V(e, \tilde{\alpha}, \tilde{\beta}) = \frac{1}{2} \left( (k_1 e_1 + k_2 e_2)^2 + 2k_1 k_2 e_1^2 + \frac{1}{\gamma_1} \tilde{\alpha}^\top \tilde{\alpha} + \frac{1}{\gamma_2} \tilde{\beta}^\top \tilde{\beta} \right), \quad (4.13)$$

and taking its time derivative, one obtains

$$\begin{aligned}
& \dot{V}(e, \tilde{\alpha}, \tilde{\beta}) \\
&= (k_1 e_1 + k_2 e_2) (k_1 \dot{e}_1 + k_2 \dot{e}_2) + 2k_1 k_2 e_1 \dot{e}_1 + \frac{1}{\gamma_1} \tilde{\alpha}^\top \dot{\tilde{\alpha}} + \frac{1}{\gamma_2} \tilde{\beta}^\top \dot{\tilde{\beta}} \\
&= (k_1 e_1 + k_2 e_2) [k_2 G_2 \beta - k_2 F_2 \alpha + k_2 h_2 u(t, x, y) + k_2 g(t, y) - k_2 f(t, x) \\
&\quad + k_1 e_2] + 2k_1 k_2 e_1 \dot{e}_2 - \frac{1}{\gamma_1} \tilde{\alpha}^\top \dot{\tilde{\alpha}} - \frac{1}{\gamma_2} \tilde{\beta}^\top \dot{\tilde{\beta}} \\
&= (k_1 e_1 + k_2 e_2) k_2 G_2 \tilde{\beta} - (k_1 e_1 + k_2 e_2) k_2 F_2 \tilde{\alpha} - (k_1 e_1 + k_2 e_2)^2 + 2k_1 k_2 e_1 \dot{e}_2 \\
&\quad - \frac{1}{\gamma_1} \tilde{\alpha}^\top \dot{\tilde{\alpha}} - \frac{1}{\gamma_2} \tilde{\beta}^\top \dot{\tilde{\beta}}, \tag{4.14}
\end{aligned}$$

which can be written as

$$\begin{aligned}
\dot{V}(e, \tilde{\alpha}, \tilde{\beta}) &= - \left( \frac{1}{\gamma_1} \dot{\tilde{\alpha}}^\top + (k_1 e_1 + k_2 e_2) k_2 F_2 \right) \tilde{\alpha} - \left( \frac{1}{\gamma_2} \dot{\tilde{\beta}}^\top - (k_1 e_1 + k_2 e_2) k_2 G_2 \right) \tilde{\beta} \\
&\quad - k_1^2 e_1^2 - k_2^2 e_2^2. \tag{4.15}
\end{aligned}$$

By replacing (4.10a) into (4.15), one obtains

$$\dot{V}(e, \tilde{\alpha}, \tilde{\beta}) = -k_1^2 e_1^2 - k_2^2 e_2^2 \leq 0, \tag{4.16}$$

which is negative semi-definite. Then we conclude that  $V(e, \tilde{\alpha}, \tilde{\beta}) \leq V(0)$  and the error dynamics are stable. Noting that with the control given in (4.9) the synchronization error dynamics are autonomous, we can apply LaSalle's invariance theorem to show the asymptotic stability of the error dynamics. To this end, we consider the set  $E = \{(e_1, e_2) \in \mathbb{R}^2 \mid \dot{V} = 0\}$ . Since the equilibrium point of the error dynamics ( $e_1 = 0, e_2 = 0$ ) is the unique invariant set in  $E$ , we can conclude that  $\lim_{t \rightarrow \infty} \|e\| = 0$ . Hence (4.5) follows (4.4) globally and asymptotically, which completes the proof.  $\square$

## 4.5 Chaos Synchronization of MEMS Resonator and Simulation Validation

The synchronization method presented above can be applied to a great variety of systems, such as Duffing, Helmholtz, pendulum, MEMS, and many other chaotic oscillators. In order to confirm the generality of the developed method, we apply it firstly to synchronize two structurally different systems: a chaotic Duffing system and a non chaotic MEMS. Then we will apply this method to synchronize two structurally similar systems, a chaotic MEMS with a non chaotic one.

#### 4.5.1 Synchronizing a Chaotic Duffing System with a MEMS

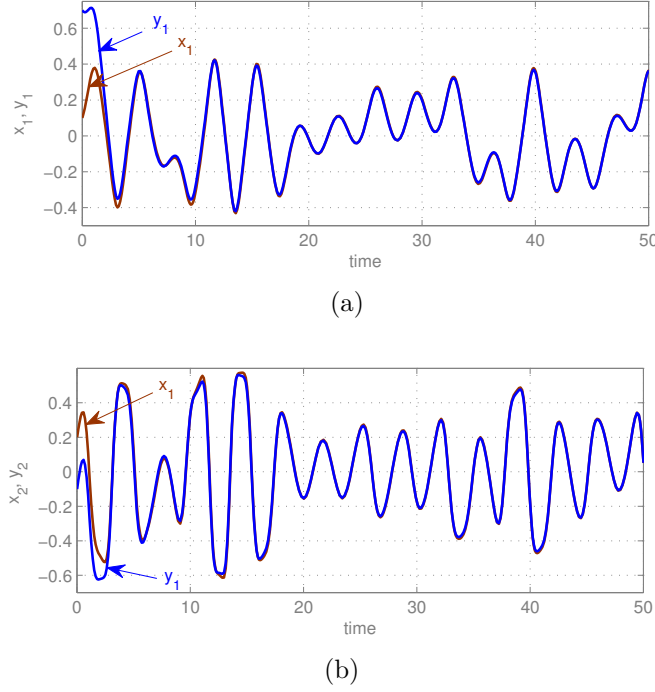


Figure 4.4 Trajectories of the synchronized systems (4.18) with control gains  $k_1 = k_2 = 1$ : (a)  $(x_1, y_1)$ ; (b)  $(x_2, y_2)$ .

In this example the driving system is a Duffing system and the response system is the MEMS described in the Sect. 4.3 with a control input  $u$ .

We consider uncertainties in the parameters of the driving and response systems for which the damping factor as well as the linear and cubic nonlinear stiffness coefficients are unknown constants.

The Duffing system is given by

$$\ddot{x} = -\alpha_1 x - \alpha_2 \dot{x} - \alpha_3 x^3 + A \cos(\omega t), \quad (4.17)$$

which is chaotic for the set of parameters  $\alpha_1 = -1.1$ ,  $\alpha_2 = 0.4$ ,  $\alpha_3 = 25$ ,  $A = 0.42$ , and  $\omega = 1.8$ . This system can be written in the form of (4.4), by defining the state variables  $x_1$  and  $x_2$  as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -x_1 & -x_2 & -x_1^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} x_2 \\ A \cos(\omega t) \end{bmatrix}. \quad (4.18)$$

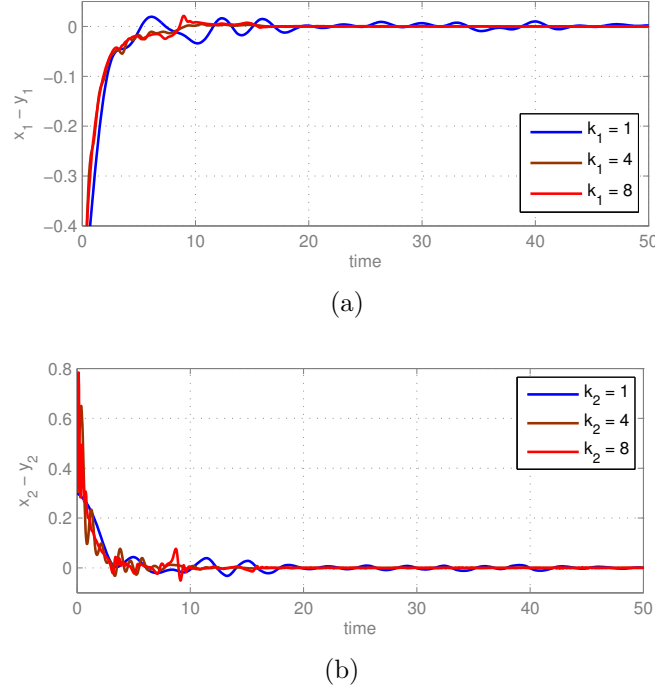


Figure 4.5 Error signals for the synchronized systems (4.18) and (4.20): (a) error between the states  $(x_1, y_1)$ ; (b) error between the states  $(x_2, y_2)$ .

In a similar way, the response system is given by

$$\ddot{y} + \beta_1 \dot{y} + \beta_2 \dot{y} + \beta_3 y^3 = \gamma \left( \frac{1}{(1-y)^2} - \frac{1}{(1+y)^2} \right) + \frac{u}{(1-y)^2} \quad (4.19)$$

and can be written in the form of (4.5), by taking

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -y_1 & -y_2 & -y_1^3 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} + \begin{bmatrix} y_2 \\ g_1(\mathbf{y}, t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \frac{1}{(1-y)^2} \end{bmatrix} u \quad (4.20)$$

with  $g_1(y, t)$  given by

$$g_1(y, t) = \gamma \left( \frac{1}{(1-y_1)^2} - \frac{1}{(1+y_1)^2} \right). \quad (4.21)$$

The control objective is to design a control  $u$  such that the states  $(y_1, y_2)$  track asymptotically the states  $(x_1, x_2)$ . The matrices  $F(x)$ ,  $G(y)$  and the vectors  $f(x, t)$ ,  $g(y, t)$ , and  $h(y)$  are of

the form

$$\begin{aligned}
 F(x) &= \begin{bmatrix} 0 & 0 & 0 \\ -x_1 & -x_2 & -x_1^3 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}, \\
 G(y) &= \begin{bmatrix} 0 & 0 & 0 \\ -y_1 & -y_2 & -y_1^3 \end{bmatrix} = \begin{bmatrix} 0 \\ G_2 \end{bmatrix}, \\
 f(x, t) &= \begin{bmatrix} x_2 \\ A \cos(\omega t) \end{bmatrix}, \\
 g(y, t) &= \begin{bmatrix} y_2 \\ g_1(y, t) \end{bmatrix}, \\
 h(y) &= \begin{bmatrix} 0 & \frac{1}{(1-y)^2} \end{bmatrix}^\top.
 \end{aligned}$$

Based on Theorem 4.4.1, the control signal is given by

$$\begin{aligned}
 u_3 = (1 - y)^2 & \left( y_1 \hat{\beta}_1 + y_2 \hat{\beta}_2 + y_1^3 \hat{\beta}_3 - x_1 \hat{\alpha}_1 - x_2 \hat{\alpha}_2 - x_1^3 \hat{\alpha}_3 - g_1 + A \cos(\omega t) \right. \\
 & \left. - k_2^{-1} (k_1 e_1 + k_2 e_2) - k_2^{-1} k_1 e_2 \right)
 \end{aligned} \tag{4.22}$$

where the parameters are updated by the laws

$$\begin{aligned}
 \dot{\hat{\alpha}}^\top &= -\gamma_1 (k_1 e_1 + k_2 e_2) k_2 F_2 \\
 &= -\gamma_1 (k_1 e_1 + k_2 e_2) k_2 \begin{bmatrix} -x_1 & -x_2 & -x_1^3 \end{bmatrix},
 \end{aligned} \tag{4.23a}$$

$$\begin{aligned}
 \dot{\hat{\beta}}^\top &= \gamma_2 (k_1 e_1 + k_2 e_2) k_2 G_2 \\
 &= \gamma_2 (k_1 e_1 + k_2 e_2) k_2 \begin{bmatrix} -y_1 & -y_2 & -y_1^3 \end{bmatrix}.
 \end{aligned} \tag{4.23b}$$

In this way, the error can be asymptotically diminished and the response system synchronizes the driving system. Figure 4.4 shows the trajectories of  $x_1, y_1$ , and  $x_2, y_2$ , from which we can observe how the systems evolve in synchrony for gains  $k_1 = 1$  and  $k_2 = 1$ . For higher values of the gains, the simulation shows a better agreement between the signals. In Fig. 4.5, the evolution of the errors for different gains can be observed. It is shown how the errors diminish whatever the gain be, although they decrease faster when the gain increases. Figure 4.6 illustrates the evolution of  $\alpha$  and  $\beta$  parameters and shows how they tends to a bounded stationary value.

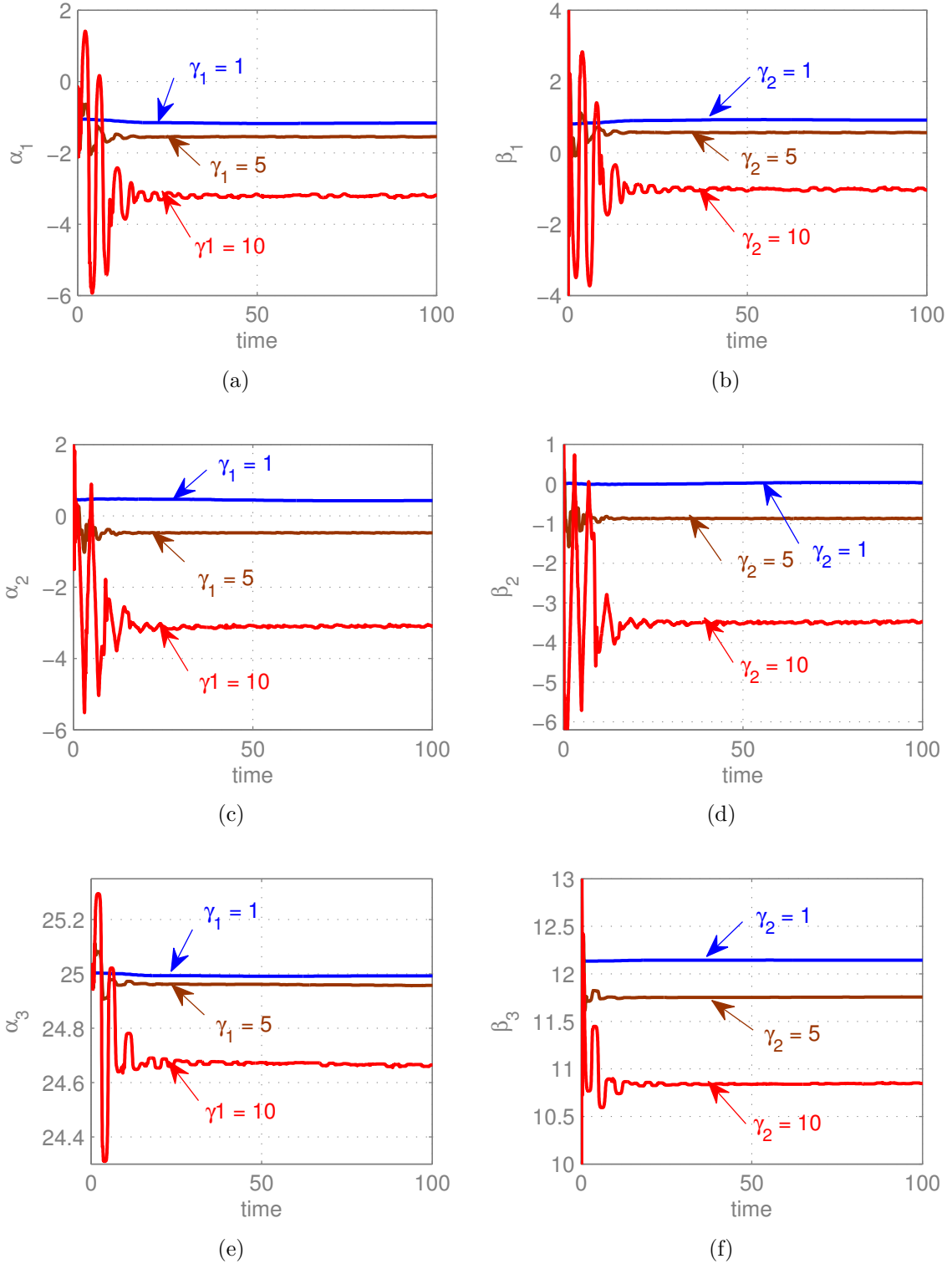


Figure 4.6 Estimation of the parameters: (a)  $\alpha_1$ ; (b)  $\beta_1$ ; (c)  $\alpha_2$ ; (d)  $\beta_2$ ; (e)  $\alpha_3$ ; (f)  $\beta_3$ .

### 4.5.2 Synchronization of two Chaotic and non Chaotic MEMS

In this example, the driving system is the chaotic MEMS presented in Sect. 4.3 and the response system is the MEMS given by the equation (4.19). The driving system can be written in the form of (4.4)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -x_1 & -x_2 & -x_1^3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \begin{bmatrix} x_2 \\ f_1(\mathbf{x}, t) \end{bmatrix} \quad (4.24)$$

with 2

$$f_1(x, t) = \gamma \left( \frac{1}{(1-x_1)^2} - \frac{1}{(1+x_1)^2} \right) + \frac{A}{(1-x_1)^2} \sin(\omega t). \quad (4.25)$$

The response system can be written in the form of (4.5) as in (4.20), in such a way that one can apply Theorem 4.4.1 by identifying the matrices  $F(x)$ ,  $G(y)$  and the vectors  $f(x, t)$ ,  $g(y, t)$ , and  $h(y)$  as

$$\begin{aligned} F(x) &= \begin{bmatrix} 0 & 0 & 0 \\ -x_1 & -x_2 & -x_1^3 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}, \\ G(y) &= \begin{bmatrix} 0 & 0 & 0 \\ -y_1 & -y_2 & -y_1^3 \end{bmatrix} = \begin{bmatrix} 0 \\ G_2 \end{bmatrix}, \\ f(x, t) &= \begin{bmatrix} x_2 \\ f_1(x, t) \end{bmatrix}, g(y, t) = \begin{bmatrix} y_2 \\ g_1(y, t) \end{bmatrix}, \\ h(y) &= \begin{bmatrix} 0 & \frac{1}{(1-y)^2} \end{bmatrix}^T, \end{aligned}$$

and obtaining the following control signal

$$\begin{aligned} u_3 &= (1-y)^2 \left( y_1 \hat{\beta}_1 + y_2 \hat{\beta}_2 + y_1^3 \hat{\beta}_3 - x_1 \hat{\alpha}_1 - x_2 \hat{\alpha}_2 - x_1^3 \hat{\alpha}_3 - g_1 \right. \\ &\quad \left. + f_1(x, t) - k_2^{-1} (k_1 e_1 + k_2 e_2) - k_2^{-1} k_1 e_2 \right) \end{aligned} \quad (4.26)$$

with parameters updated by the laws

$$\begin{aligned} \dot{\hat{\alpha}}^\top &= -\gamma_1 (k_1 e_1 + k_2 e_2) k_2 F_2 \\ &= -\gamma_1 (k_1 e_1 + k_2 e_2) k_2 \begin{bmatrix} -x_1 & -x_2 & -x_1^3 \end{bmatrix}, \end{aligned} \quad (4.27a)$$

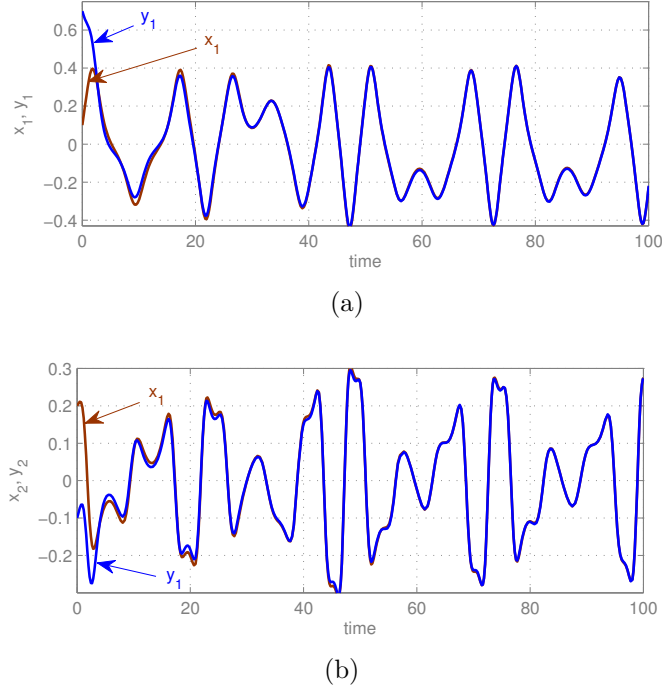


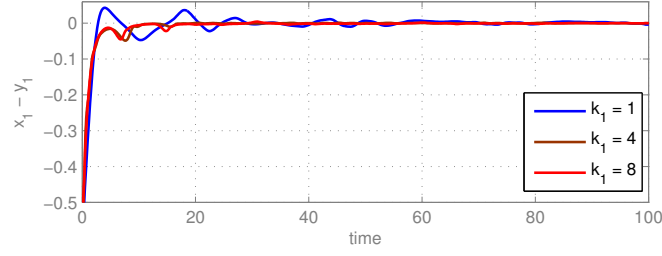
Figure 4.7 Trajectories of the synchronized systems (4.24) and (4.20) with control gains  $k_1 = k_2 = 1$ : (a)  $(x_1, y_1)$ ; (b)  $(x_2, y_2)$ .

$$\begin{aligned}
 \dot{\hat{\beta}}^T &= \gamma_2 (k_1 e_1 + k_2 e_2) k_2 G_2 \\
 &= \gamma_2 (k_1 e_1 + k_2 e_2) k_2 \begin{bmatrix} -y_1 & -y_2 & -y_1^3 \end{bmatrix}.
 \end{aligned} \tag{4.27b}$$

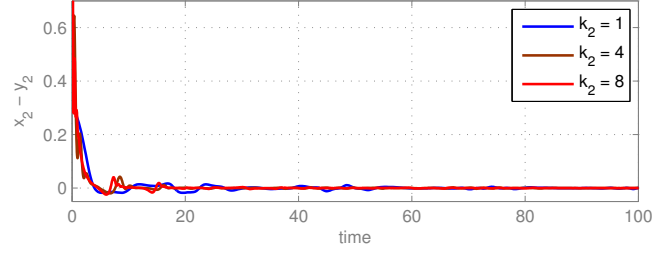
Figure 4.7 shows how the systems are synchronized, with an excellent agreement between signals after a transient time. The error between the synchronized states decreases asymptotically with time, as shown in Fig. 4.8. The estimation of the unknown parameters  $\alpha$  and  $\beta$  is given in Fig. 4.9.

## 4.6 Conclusions

This paper presents an adaptive control scheme for synchronization of a class of chaotic systems with parametric uncertainties. The developed method is applied to the synchronization of two structurally different systems, a Duffing system and a MEMS, and structurally similar systems, a chaotic MEMS resonator and a non-chaotic MEMS. It is confirmed that in chaotic synchronization, the response system does not have to, but can be chaotic. This is a very important property that can be used in such application as secure communication. Other advantage of synchronizing MEMS with a chaotic system is that this would allow an



(a)



(b)

Figure 4.8 Error signals for the synchronized systems (4.24) and (4.20): (a) error between  $(x_1, y_1)$ ; (b) error between  $(x_2, y_2)$ .

enhancement of the amplitude of the oscillation when MEMS are used as resonators and recovered from the chaotic state by controlling it. This is an important aspect for applications which require an accurate control of oscillation amplitude. Finally, it is worth noting that the scheme developed in the present work is based on state-feedback. For more realistic applications, output feedback should be considered. This issue is beyond the scope of this paper and remains an interesting topic of future work.

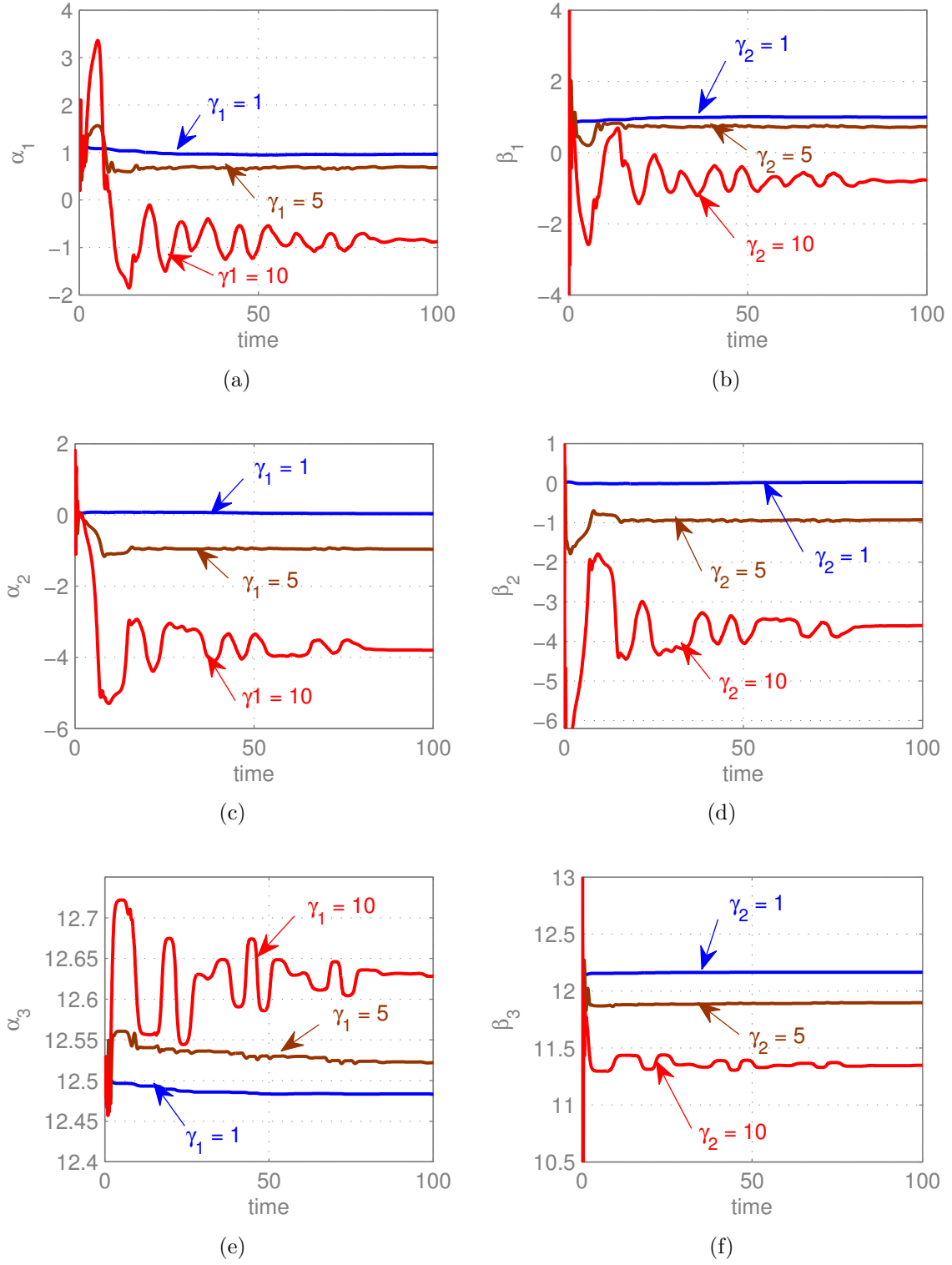


Figure 4.9 Parameter estimates: (a)  $\alpha_1$ ; (b)  $\beta_1$ ; (c)  $\alpha_2$ ; (d)  $\beta_2$ ; (e)  $\alpha_3$ ; (f)  $\beta_3$ .

## CHAPTER 5 ARTICLE 2: OSCILLATION AMPLITUDE ENHANCEMENT OF AN ELECTROSTATIC MEMS RESONATOR VIA CHAOS CONTROL

This Chapter is reproduced from [118]:

A. Jimenez-Triana, G. Zhu, and L. Saydy, “Oscillation amplitude enhancement of an electrostatic MEMS resonator via chaos control,” in *2013 American Control Conference*, Washington, DC, USA, 15 august 2013, pp. 6269–6274.

### 5.1 Abstract

In this work chaos is controlled in an electrostatic MEMS resonator. One of the unstable orbits immersed in the chaotic attractor is stabilized in order to produce a sustained oscillation of the movable plate composing the microstructure. The orbit is chosen carefully so as to produce a high amplitude oscillation. This approach allows us to improve the amplitude of the oscillation at a reduced cost, since the unstable orbit already exists in the system and it is not necessary to spend energy to create it. Parametric uncertainties in the model of the MEMS are considered, and the control law is constructed in detail by using adaptive backstepping. Numerical simulations are carried out to confirm the validity of the developed control schemes, and appreciate the effect of controlling orbits immersed in the chaotic attractor.

### 5.2 Introduction

This paper addresses the problem of chaos control in an electrostatic MEMS resonator with parametric uncertainties. The considered device is a structure with dual drive using differential actuation, as shown in Fig. 5.1.

Many MEMS devices exhibit chaotic behavior in certain operational conditions [56, 68–70]. These MEMS can be seen as chaotic capacitors, which can be used, for instance, in the transmitter or the receiver circuitry in security communications schemes using chaos [55]. In addition, it is observed in recent results concerning the improvement of the output energy in oscillators using MEMS [116] that controlling MEMS in chaotic mode can be useful to increase the amplitude of the periodic oscillations of these systems. Therefore, controlling chaos in MEMS can enhance the performance of different systems in which an accurate control of amplitude of oscillation is mandatory.

Chaos control has been extensively studied in the literature. Different methods have been applied to stabilize equilibrium points and unstable orbits immersed in the chaotic attractor, or to track orbits that can even be chaotic. The considered approaches cover the application of a wide range of control techniques, from the use of Lyapunov theory to stabilize chaotic oscillators designed via local linearization or linear feedback [119], [120], to optimal control [121], robust control [122] and adaptive control [123], just to name a few. Chaos control in MEMS is reported recently [56], [102]. In [56] a fuzzy control algorithm is applied to control the chaotic motion and track a sinusoidal orbit. In [102] an adaptive control is used to control an uncertain MEMS and to synchronize it with a class of uncertain chaotic oscillators. Nevertheless, it does not draw a sufficient attention.

Oscillation amplitude of electrostatic MEMS resonators is restricted by the so-called pull-in phenomena related to a bifurcation arising in electrostatic actuation [124]. The stable traveling range is limited to one third of its full gap due to the instability beyond that point. The pull-in phenomena can be used to accelerate the reaction speed for switching structures. However, it limits the stable operational rang of the device, which is mandatory for applications requiring a sustained oscillation of the plates.

In order to operate electrostatic MEMS devices beyond the pull-in, one can use closed-loop control schemes (see, e.g., [125,126]). However, it may require a considerable control effort to produce desired trajectories. In order to increase the amplitude of periodic oscillations of the MEMS with a low energy consumption, we consider in the present work controlling the resonator in chaotic mode to stabilize one of the infinite unstable orbits immersed in the chaotic attractor. We tackle this problem by stabilizing one of the orbits which produces an oscillation with a high amplitude. We illustrate first control design by backstepping state feedback control and then deal with the uncertainties in the model and stabilize the orbit by

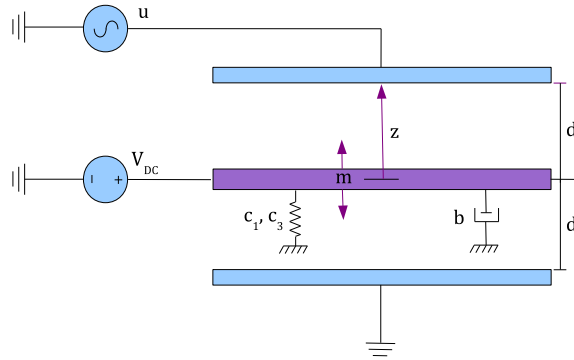


Figure 5.1 Schematic of the electrostatically actuated MEMS resonator.

using an adaptive scheme.

The organization of the rest of the paper is as follows. In Section 5.3, the dynamics of a chaotic MEMS actuator are briefly explained based on a model given in [37], [38]. Next, control design for is presented in Section 5.4. Simulations are then carried out and the results are presented in Section 5.5. Finally, concluding remarks are given in Section 5.6.

### 5.3 Dynamic of a Chaotic MEMS Resonator

#### 5.3.1 Model of the Chaotic MEMS

The considered MEMS device in this study is presented in Fig. 5.1, which is taken from [37,38]. This device is modeled by a nonlinear mass-spring-damper system with external electrostatic actuation, expressed as

$$m\ddot{z} + b\dot{z} + c_1z + c_3z^3 = F, \quad (5.1)$$

where  $m$  is the mass of the movable structure,  $b$  is the damping constant,  $c_1$  and  $c_3$  are the linear and cubic stiffness coefficients, respectively, and  $F$  is the electrostatic actuation given by

$$F = \frac{1}{2} \frac{C_0 d}{(d - z)^2} (V_{DC} + u)^2 - \frac{1}{2} \frac{C_0 d}{(d + z)^2} V_{DC}^2. \quad (5.2)$$

In (5.2),  $C_0$  is the capacitance of the actuator at rest, which has an initial gap  $d$  between the moveable and fixed plates. The input voltage  $V_i = (V_{DC} + u)$  is applied as shown in Fig. 5.1 and contains two components: a DC voltage  $V_{DC}$  and a forcing voltage  $u$ , which may be a control signal that varies according to control objectives.

Note that in the present work, we ignore the dynamics of the driving circuit by assuming that they are much faster than that of the mechanical structure. Note also that for simplicity, we do not take into account contact dynamics. This means that the system should operate in such a way that the moveable and fixed plates will not come into contact.

In order to facilitate control system design and analysis, it is convenient to work with models in normalized coordinates. For this purpose, we introduce the following dimensionless

variables:

$$\begin{aligned}\tau &= \omega_0 t, \quad \omega = \frac{\Omega}{\omega_0}, \quad \xi = \frac{z}{d}, \quad \mu = \frac{b}{m\omega_0}, \\ \sigma &= \frac{c_1}{m\omega_0^2}, \quad \eta = \frac{c_3 d^2}{m\omega_0^2}, \quad \delta = \frac{C_0 V_{DC}^2}{2m\omega_0^2 d^3}, \\ A &= 2\delta \frac{1}{V_{DC}}, \quad uA = u',\end{aligned}\tag{5.3}$$

where  $\omega_0$  is the natural frequency defined by:

$$\omega_0 = \sqrt{\frac{c_1}{m}}.\tag{5.4}$$

In this way, by assuming that the DC voltage is sufficiently higher than the amplitude of forcing voltage  $u$ , the dimensionless equation of motion becomes

$$\ddot{\xi} + \mu\dot{\xi} + \sigma\xi + \eta\xi^3 = \delta \left( \frac{1}{(1-\xi)^2} - \frac{1}{(1+\xi)^2} \right) + \frac{u'}{(1-\xi)^2}.\tag{5.5}$$

It can be shown that system (5.5) behaves chaotically by selecting  $u = V_{AC} \sin(\omega t)$  and the set of parameters  $V_{AC}A = 0.04$ ,  $\omega = 0.5$ ,  $\delta = 0.338$ ,  $\mu = 0.01$ ,  $\sigma = 1$ , and  $\eta = 12$ , as indicated in the Lyapunov exponents diagram of Fig. 5.2. The existence of one positive Lyapunov exponent confirms the chaotic behavior of the system.

System (5.5) can be written in state space form as

$$\begin{aligned}\dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= -\mu\xi_2 - \sigma\xi_1 - \eta\xi_1^3 + f(\xi_1) + \frac{u'}{(1-\xi_1)^2}\end{aligned}\tag{5.6}$$

where

$$f(\xi_1) = \delta \left( \frac{1}{(1-\xi_1)^2} - \frac{1}{(1+\xi_1)^2} \right).$$

### 5.3.2 Unstable Orbit in the Strange Attractor

Unstable periodic orbits in a continuous time chaotic dynamic system can be detected using different methods. By applying the approach of close returns, we obtained a periodic orbit surrounding the chaotic attractor, which produces a high amplitude periodic output. This periodic orbit will be used as the reference trajectory in the control. This orbit is shown in

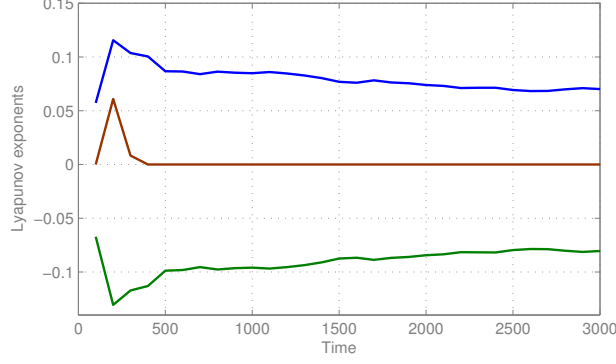


Figure 5.2 Lyapunov exponents for the system (5.5) with the set of parameters:  $\delta = 0.338$ ,  $A = 0.04$ ,  $\omega = 0.5$ ,  $\sigma = 1$ ,  $\mu = 0.01$ , and  $\eta = 12$ .

Fig. 5.3 and can be approximated by its Fourier series truncated up to five terms, given by

$$y_r = a_0 + a_1 \cos(w_1 t) + b_1 \sin(w_1 t) + a_2 \cos(2w_1 t) + b_2 \sin(2w_1 t) + a_3 \cos(3w_1 t) + b_3 \sin(3w_1 t) + a_4 \cos(4w_1 t) + b_4 \sin(4w_1 t) + a_5 \cos(5w_1 t) + b_5 \sin(5w_1 t), \quad (5.7)$$

where

$$\begin{aligned} a_0 &= 0.0175400 & a_1 &= -0.4498000 & b_1 &= -0.0005630 \\ a_2 &= 0.0112600 & b_2 &= 0.00089060 & a_3 &= -0.029140 \\ b_3 &= 0.0000376 & a_4 &= 0.00063300 & b_4 &= 0.0002309 \\ a_5 &= -0.001090 & b_5 &= 0.00008519 \end{aligned}$$

Due to the fact that (5.7) is just an approximation of  $y_r$ , its derivative is not calculated from this equation. Instead, it is approximated by the Fourier series of the data taken from the numerical simulation, which produces

$$\dot{y}_r = a'_0 + a'_1 \cos(w_1 t) + b'_1 \sin(w_1 t) + a'_2 \cos(2w_1 t) + b'_2 \sin(2w_1 t) + a'_3 \cos(3w_1 t) + b'_3 \sin(3w_1 t) + a'_4 \cos(4w_1 t) + b'_4 \sin(4w_1 t) + a'_5 \cos(5w_1 t) + b'_5 \sin(5w_1 t), \quad (5.8)$$

where

$$\begin{aligned} a'_0 &= -0.0001949 & a'_1 &= -0.0009988 & b'_1 &= 0.4164 \\ a'_2 &= 0.001259 & b'_2 &= -0.02081 & a'_3 &= -0.0003363 \\ b'_3 &= 0.08096 & a'_4 &= 0.0004271 & b'_4 &= -0.002241 \\ a'_5 &= -0.00006363 & b'_5 &= 0.005116 \end{aligned}$$

The second derivative of  $y_r$  is obtained by taking the derivative of (5.8).

In the following, we use the following assumption

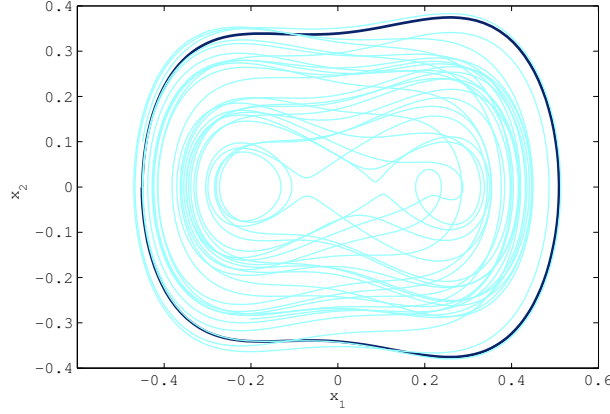


Figure 5.3 Unstable periodic orbit surrounding the chaotic attractor

**Assumption 1.** *The reference signal  $y_r(t)$  and its first two derivatives are known and bounded. In addition,  $y_r^{(p)}(t)$ ,  $\forall p \geq 3$ , is piecewise continuous.*

#### 5.4 Control of Chaotic MEMS

The objective of the control is to stabilize the unstable orbit (5.7) of the chaotic system (5.6). This objective can be achieved by forcing the tracking error to tend to zero:

$$\lim_{t \rightarrow \infty} [y(t) - y_r(t)] = 0. \quad (5.9)$$

We use first state feedback to stabilize the orbit and then consider adaptive control in the presence of parametric uncertainties in the model. In both cases the control law will be obtained by using backstepping.

##### 5.4.1 Backstepping State Feedback Control

In order to apply backstepping design procedure to System (5.6), we transform it into the parametric output-feedback form [127] by using the following change of variables:

$$\begin{aligned} x_1 &= \xi_1 \\ x_2 &= \xi_2 + \mu \xi_1. \end{aligned} \quad (5.10)$$

System (5.6) becomes then

$$\dot{x}_1 = x_2 - \theta_1 x_1 \quad (5.11a)$$

$$\dot{x}_2 = -\theta_2 x_1 - \theta_3 x_1^3 + \theta_4 f(x_1) + \beta(y) u \quad (5.11b)$$

$$y = x_1, \quad (5.11c)$$

where  $\theta = (\theta_1 \ \theta_2 \ \theta_3 \ \theta_4) = (\mu \ \sigma \ \eta \ \delta)$  is a parameter vector and  $f(x_1) = f(\xi_1)$ . For state feedback control, one can assert the following result.

**Proposition 2.** *Given the system (5.11a), the reference signal  $y_r$  satisfying Assumption 2, and two positive constants  $c_1$  and  $c_2$ . Denote by*

$$\begin{aligned} z_1 &= y - y_r \\ z_2 &= x_2 - \alpha_1 - \dot{y}_r, \end{aligned} \quad (5.12)$$

*the tracking errors, where  $\alpha_1$  is a stabilizing function given by*

$$\alpha_1 = -c_1 z_1 + \theta_1 x_1. \quad (5.13)$$

*If the control input  $u$  is chosen as*

$$\beta u = \theta_2 x_1 + \theta_3 x_1^3 - \theta_4 (f(x_1)) + (\theta_1 - c_1)(x_2 - \theta_1 x_1) + c_1 \dot{y}_r + \ddot{y}_r - c_2 z_2 - z_1, \quad (5.14)$$

*then the output  $y(t)$  globally and asymptotically tracks the reference trajectory  $y_r(t)$ .*

The proof of Proposition 2 can be performed by constructing backstepping state feedback control as follows

*Step 1:* The dynamics of the tracking error  $z_1$  defined in (5.12) are given by

$$\dot{z}_1 = \dot{y} - \dot{y}_r = \dot{x}_1 - \dot{y}_r = x_2 - \theta_1 x_1 - \dot{y}_r. \quad (5.15)$$

If we consider  $x_2$  as a virtual control, we obtain from (5.12) and (5.15) that

$$\dot{z}_1 = (z_2 + \alpha_1 + \dot{y}_r) - \theta_1 x_1 - \dot{y}_r. \quad (5.16)$$

By using the stabilizing function  $\alpha_1$  given in (5.13), the dynamics of  $z_1$  become

$$\dot{z}_1 = -c_1 z_1 + z_2. \quad (5.17)$$

Consider the following Lyapunov function candidate:

$$V_1 = \frac{1}{2} z_1^2. \quad (5.18)$$

The time derivative of  $V_1$  along the trajectory of System (5.11a) is given by

$$\dot{V}_1 = z_1 (-c_1 z_1 + z_2) = -c_1 z_1^2 + z_1 z_2. \quad (5.19)$$

*Step 2:* The derivative of  $z_2$  is

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 - \ddot{y}_r \\ &= -\theta_2 x_1 - \theta_3 x_1^3 + \theta_4 (f(x_1)) + \beta(y) u - \dot{\alpha}_1 - \ddot{y}_r \\ &= -\theta_2 x_1 - \theta_3 x_1^3 + \theta_4 (f(x_1)) + \beta(y) u - \frac{\partial \alpha_1}{\partial y} \dot{y} - \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r - \ddot{y}_r \\ &= -\theta_2 x_1 - \theta_3 x_1^3 + \theta_4 (f(x_1)) + \beta(y) u + (\theta_1 - c_1)(x_2 - \theta_1 x_1) - c_1 \dot{y}_r - \ddot{y}_r. \end{aligned} \quad (5.20)$$

If the control signal is chosen as given in (5.14), then the dynamics of the error  $z_2$  are

$$\dot{z}_2 = -c_2 z_2 - z_1. \quad (5.21)$$

Define a Lyapunov function candidate by augmenting  $V_1$  as follows:

$$V_2 = V_1 + \frac{1}{2} z_2^2. \quad (5.22)$$

The time derivative of  $V_2$  is computed as

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + z_2 \dot{z}_2 \\ &= z_1 z_2 - c_1 z_1^2 + z_2 (-c_2 z_2 - z_1) \\ &= -c_1 z_1^2 - c_2 z_2^2 \end{aligned} \quad (5.23)$$

Therefore,  $\dot{V}_2$  is negative definite. As  $V_2$  is positive definite at  $z = 0$  and is radially unbounded,  $z \rightarrow 0$  as  $t \rightarrow \infty$  for any initial condition  $z(0) \in \mathbb{R}^2$ . Since  $z_1 = y - y_r$ , the output  $y(t)$  globally and asymptotically tracks  $y_r(t)$ .

### 5.4.2 Adaptive Control

If the case where  $\mu, \sigma, \eta$ , and  $\delta$  are bound unknown parameters, we can also achieve asymptotic tracking of the same reference trajectory as the one used previously by applying adaptive backstepping. One can state the following result.

**Proposition 3.** *Given the system (5.11a), the reference signal  $y_r$  satisfying Assumption 2, and positive constants  $c_1, c_2, \gamma_1$ , and  $\gamma_2$ . Denote by*

$$\begin{aligned} z_1 &= y - y_r \\ z_2 &= x_2 - \alpha_1 - \dot{y}_r, \end{aligned} \tag{5.24}$$

*the tracking errors, where  $\alpha_1$  is a stabilizing function given by*

$$\alpha_1 = -c_1 z_1 - \vartheta_{11}(-x_1). \tag{5.25}$$

*Suppose that the parameters are updated according to the laws*

$$\dot{\vartheta}_{11} = -\gamma_1 x_1 z_1 \tag{5.26}$$

$$\dot{\vartheta}_2 = \gamma_2 \omega z_2. \tag{5.27}$$

*If the control input  $u$  is chosen as*

$$\beta u = -\vartheta_2 \omega + \frac{\partial \alpha_1}{\partial y} x_2 + \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r + \frac{\partial \alpha_1}{\partial \vartheta_{11}} \dot{\vartheta}_{11} + \ddot{y}_r - c_2 z_2 - z_1, \tag{5.28}$$

*where  $\omega = \left( \frac{\partial \alpha_1}{\partial y} x_1 \quad -x_1 \quad -x_1^3 \quad f(x_1) \right)^\top$  and  $\vartheta_2 = \left( \vartheta_{21} \quad \vartheta_{22} \quad \vartheta_{23} \quad \vartheta_{24} \right)^\top$  is an estimate of  $\theta$ , then the output  $y(t)$  tracks  $y_r(t)$  globally and asymptotically.*

The adaptive control design is carried out as follows.

*Step 1:* The dynamics of the tracking error  $z_1$  defined in (5.24) are given by

$$\dot{z}_1 = \dot{y} - \dot{y}_r = \dot{x}_1 - \dot{y}_r = x_2 - \theta_1 x_1 - \dot{y}_r. \tag{5.29}$$

If we consider  $x_2$  as a virtual control, we obtain from (5.24) and (5.29)

$$\dot{z}_1 = (z_2 + \alpha_1 + \dot{y}_r) - \theta_1 x_1 - \dot{y}_r. \tag{5.30}$$

Since  $\theta$  is unknown, we must use its estimate  $\theta_1 = \vartheta_{11}$  in the implementation of stabilizing function. By using  $\alpha_1$  given in (5.25), (5.30) becomes

$$\dot{z}_1 = -c_1 z_1 + z_2 - (\theta_1 - \vartheta_{11}) x_1. \quad (5.31)$$

If we use a Lyapunov function candidate of the form

$$V_1 = \frac{1}{2} z_1^2 + \frac{1}{2\gamma_1} (\theta_1 - \vartheta_{11})^2, \quad (5.32)$$

where  $\gamma_1$  is the adaptation gain, then

$$\begin{aligned} \dot{V}_1 &= z_1 \dot{z}_1 - \frac{1}{\gamma_1} \dot{\vartheta}_{11} (\theta_1 - \vartheta_{11}) \\ &= z_1 (-c_1 z_1 + z_2 + (\theta_1 - \vartheta_{11}) (-x_1)) - \frac{1}{\gamma_1} \dot{\vartheta}_{11} (\theta_1 - \vartheta_{11}) \\ &= -c_1 z_1^2 + z_1 z_2 - (\theta_1 - \vartheta_{11}) \left( x_1 z_1 + \frac{1}{\gamma_1} \dot{\vartheta}_{11} \right). \end{aligned} \quad (5.33)$$

The term  $(\theta_1 - \vartheta_{11})$  can be eliminated by the update law (5.26). Hence, we get

$$\dot{V}_1 = -c_1 z_1^2 + z_1 z_2. \quad (5.34)$$

*Step 2:* The dynamics of  $z_2$  are

$$\begin{aligned} \dot{z}_2 &= \dot{x}_2 - \dot{\alpha}_1 - \ddot{y}_r \\ &= -\theta_2 x_1 - \theta_3 x_1^3 + \theta_4 f(x_1) + \beta(y) u - \dot{\alpha}_1 - \ddot{y}_r \\ &= -\theta_2 x_1 - \theta_3 x_1^3 + \theta_4 f(x_1) + \beta(y) u - \frac{\partial \alpha_1}{\partial y} \dot{y} - \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r - \frac{\partial \alpha_1}{\partial \vartheta_{11}} \dot{\vartheta}_{11} - \ddot{y}_r \\ &= \theta \omega + \beta(y) u - \frac{\partial \alpha_1}{\partial y} x_2 - \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r - \frac{\partial \alpha_1}{\partial \vartheta_{11}} \dot{\vartheta}_{11} - \ddot{y}_r \end{aligned} \quad (5.35)$$

By using the control given in (5.28), the error dynamics of  $z_2$  become

$$\dot{z}_2 = -c_2 z_2 - z_1 + (\theta - \vartheta_2) \omega. \quad (5.36)$$

Consider an augmented Lyapunov function candidate of the following form:

$$V_2 = V_1 + \frac{1}{2} z_2^2 + \frac{1}{2\gamma_2} (\theta - \vartheta_2) (\theta - \vartheta_2)^\top \quad (5.37)$$

Its derivative is computed as

$$\begin{aligned}
 \dot{V}_2 &= \dot{V}_1 + z_2 \dot{z}_2 - \frac{1}{\gamma_2} \dot{\vartheta}_2 (\theta - \vartheta_2)^\top \\
 &= z_1 z_2 - c_1 z_1^2 + z_2 ((\theta - \vartheta_2) \omega - c_2 z_2 - z_1) - \frac{1}{\gamma_2} \dot{\vartheta}_2 (\theta - \vartheta_2)^\top \\
 &= -c_1 z_1^2 - c_2 z_2^2 + (\theta - \vartheta_2) \left( \omega z_2 - \frac{1}{\gamma_2} \dot{\vartheta}_2^\top \right).
 \end{aligned} \tag{5.38}$$

To eliminate the term  $(\theta - \vartheta_2)$ , we use the update law (5.27), which leads to

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2. \tag{5.39}$$

By applying the LaSalle-Yoshizawa theorem [127], we can conclude that  $z_1$ ,  $z_2$ ,  $\vartheta_1$ , and  $\vartheta_2$  are bounded and  $z \rightarrow 0$  as  $t \rightarrow \infty$ . Therefore, the output  $y(t)$  tracks  $y_r(t)$  globally and asymptotically.

## 5.5 Simulation Results

Numerical simulations have been carried out to validate the designs made in Section 5.4. Two orbits are stabilized using the control strategies developed in the previous section with the aim of comparing the corresponding control effort. These orbits are shown in the Fig. 5.4 and their states are compared in Fig. 5.5. Although these two orbits are similar, one is immersed in the chaotic MEMS attractor while the other is not.

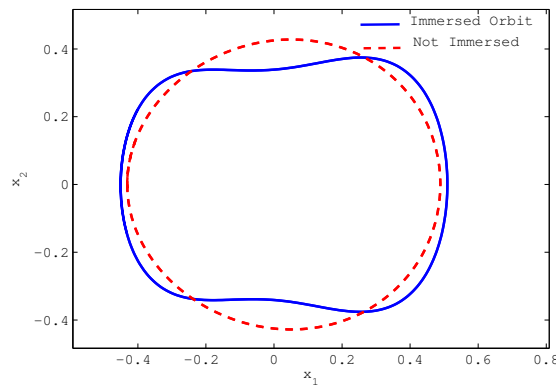


Figure 5.4 Unstable orbit immersed in the chaotic attractor and orbit not immersed

Figure 5.6(a) shows the output of the system controlled by the algorithm developed in Section 5.4. The tracking error is shown in Fig. 5.6(b). It can be seen that the agreement

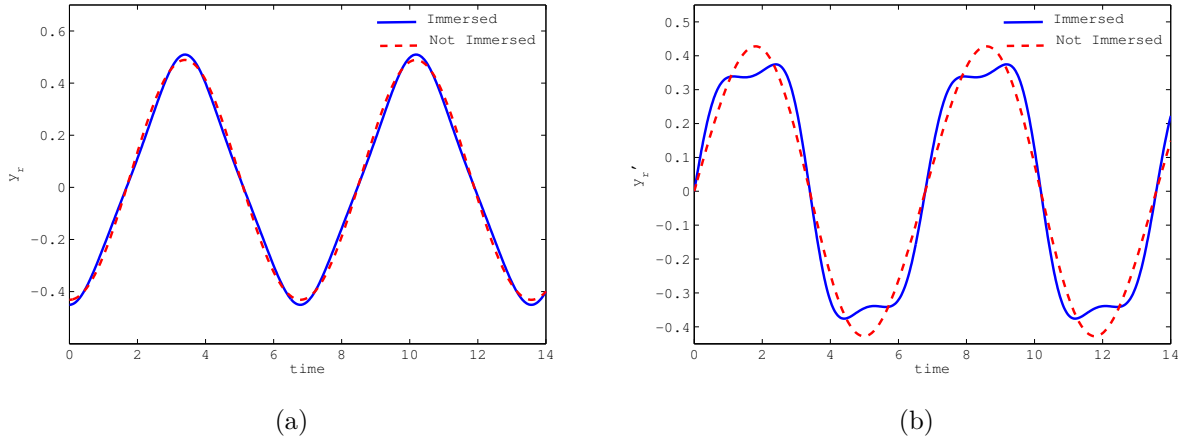


Figure 5.5 Reference signal and its derivative: (a)  $y_r$ ; (b)  $\dot{y}_r$ .

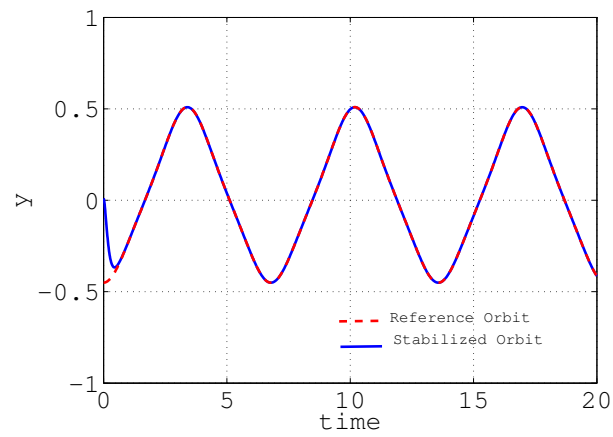
between the curves and the effect of the controller. A comparison between the control signals for stabilizing the two orbits is given in Fig. 5.6(c), which shows clearly that the amplitude of control signal  $u$  for the orbit immersed in the chaotic attractor is significantly smaller than that for non-immersed orbit.

The adaptive control law given in Section 5.4 is also applied to stabilize these two orbits. System output and tracking error are shown in Fig. 5.7(a) and Fig. 5.7(b), respectively. A comparison between the control signals for stabilizing the two orbits is given in Fig. 5.7(c). Again, the required control effort for the orbit immersed in the chaotic attractor is much smaller than that for the orbit that is not inside the attractor. Finally, the evolution of the estimates of the parameter  $\theta$  is shown in Fig. 5.8.

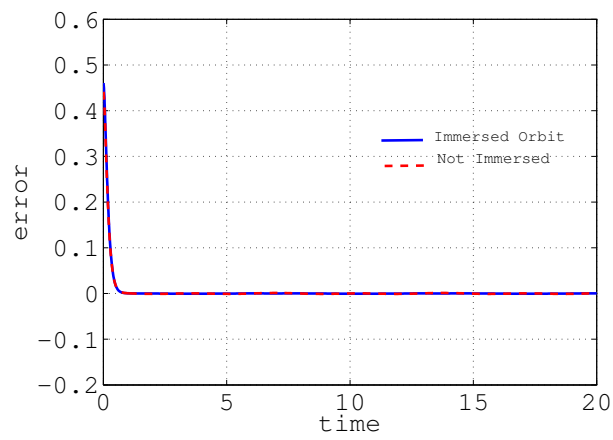
## 5.6 Conclusions

In this work, a scheme is proposed for performance improving of a MEMS resonator based on the properties of a chaotic MEMS. Chaotic systems have immersed an infinite number of unstable periodic orbits, which can be stabilized according to the needs of the system. In order to enhance the amplitude of the oscillation of the MEMS resonator, a high amplitude unstable orbit was chosen to be stabilized. The method of close returns was used to obtain the orbit numerically, and then it was approximated by its Fourier series truncated up to five terms. With the aim of stabilizing this orbit, two control strategies were designed by using state feedback, and parametric uncertainties in the model were considered in one of them. Both were obtained in details by using a backstepping design procedure. Simulations were made and the results showed how the unstable orbits immersed in the attractor are

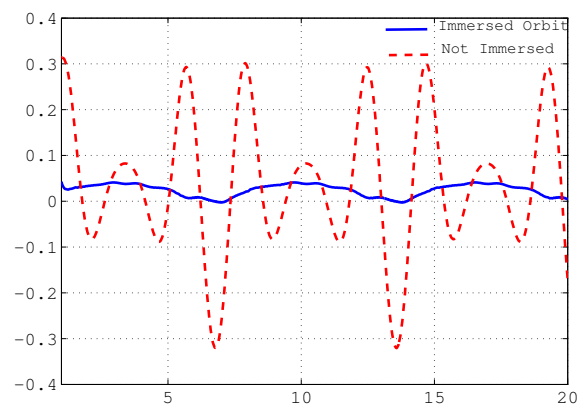
stabilized with a low control signal, in contrast with a high control signal obtained when a not immersed orbit is stabilized. These behavior, high amplitude of oscillations at a small control expenses is very desirable, and in general, it is not an easy task to obtain it. Finally, it is worth noting that the scheme developed in the present work is based on state-feedback. For more realistic applications, output feedback should be considered. This issue is beyond the scope of this paper and remains an interesting topic of future work.



(a)



(b)



(c)

Figure 5.6 Simulation results for state feedback control: (a) output; (b) tracking errors; (c) control signals.

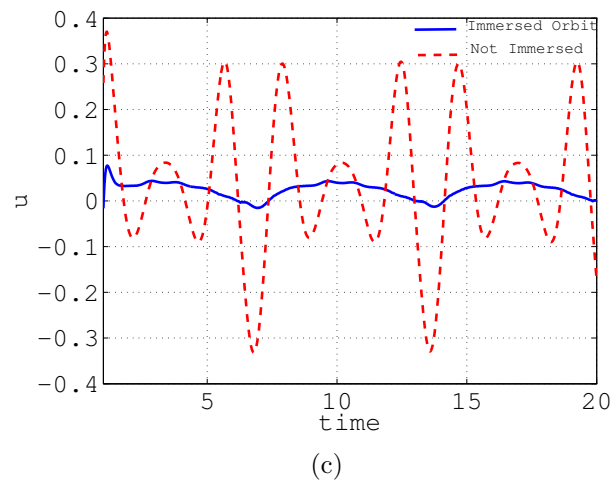
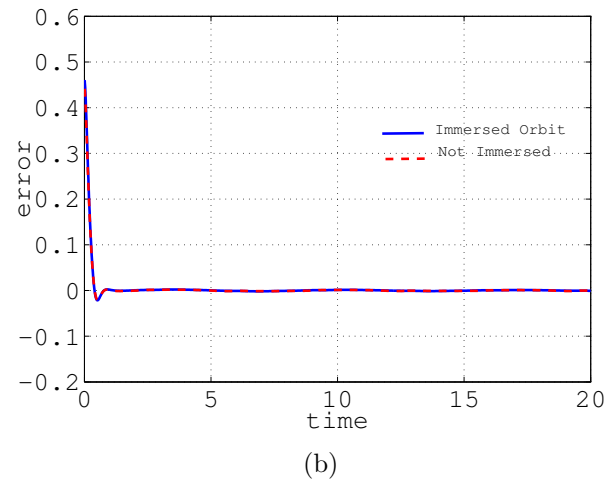
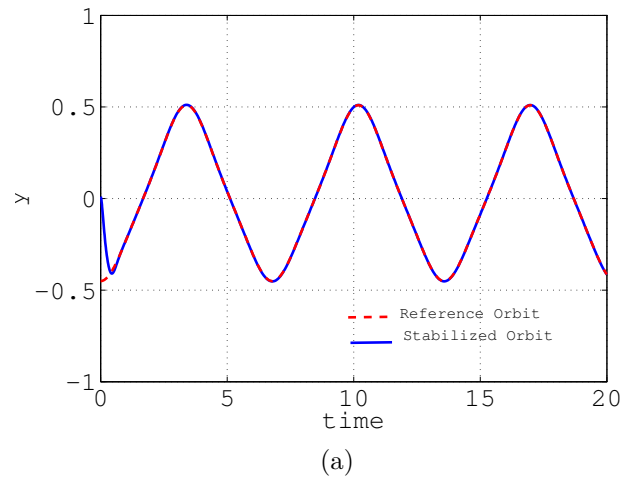


Figure 5.7 Simulation results for adaptive control: (a) output; (b) tracking errors; (c) control signals.

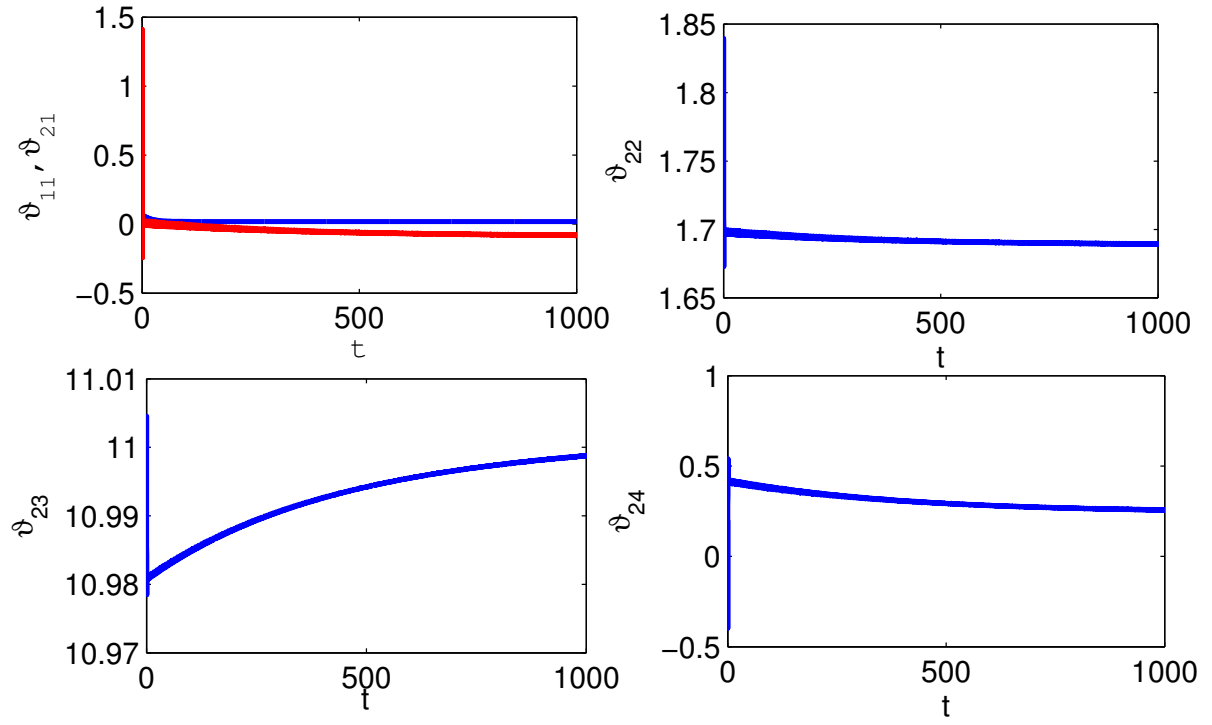


Figure 5.8 Estimated parameter  $\vartheta$ :  $\vartheta_{11}, \vartheta_{21}, \vartheta_{22}, \vartheta_{23}, \vartheta_{24}$ .

## CHAPTER 6    ARTICLE 3: OUTPUT-FEEDBACK CONTROL OF A CHAOTIC MEMS RESONATOR FOR OSCILLATION AMPLITUDE ENHANCEMENT

This Chapter is reproduced from [128]:

A. Jimenez-Triana, G. Zhu, and L. Saydy, “Output-feedback control of a chaotic MEMS resonator for oscillation amplitude enhancement,” *Mathematical Problems in Engineering*, vol. 9 september 2014.

### 6.1 Abstract

The present work addresses the problem of chaos control in an electrostatic MEMS resonator by using an output feedback control scheme. One of the unstable orbits immersed in the chaotic attractor is stabilized in order to produce a sustained oscillation of the movable plate composing the microstructure. The orbit is carefully chosen so as to produce a high amplitude oscillation. This approach allows the enhancement of oscillation amplitude of the resonator at a reduced control effort, since the unstable orbit already exists in the system and it is not necessary to spend energy to create it. Realistic operational conditions of the MEMS are considered including parametric uncertainties in the model and constraints due to the difficulty in measuring the speed of the plates of the microstructure. A control law is constructed recursively by using the technique of backstepping. Finally, Numerical simulations are carried out to confirm the validity of the developed control scheme and to demonstrate the effect of controlling orbits immersed in the chaotic attractor.

### 6.2 Introduction

This paper addresses the problem of chaos control in an electrostatic MEMS resonator with parametric uncertainties and constraints in the measure of its state. The considered device is a structure with dual drive using differential actuation, as shown in Fig. 6.1.

Many MEMS devices exhibit chaotic behaviour in certain operational conditions [56, 68–70]. It is observed in recent results concerning the improvement of the output energy in oscillators using MEMS [116] that controlling MEMS in chaotic mode can be useful to increase the amplitude of the periodic oscillations of these systems. Therefore, controlling chaos in MEMS can enhance the performance of different systems in which an accurate control of amplitude

of oscillation is mandatory.

Chaos control has been extensively studied in the literature. Different methods have been applied to stabilize equilibrium points and unstable orbits immersed in the chaotic attractor, or to track orbits that can be chaotic. The considered approaches cover the application of a wide range of control techniques, from the use of Lyapunov theory to stabilize chaotic oscillators designed via local linearization or linear feedback [119,120], to optimal control [121], robust control [122] and adaptive control [123], to name a few. Chaos control in MEMS is reported recently in [56,102]. In [56] a fuzzy control algorithm is applied to control the chaotic motion and track a sinusoidal orbit. In [102] an adaptive control is used to control an uncertain model of a MEMS device and to synchronize it with a class of uncertain chaotic oscillators. Nevertheless, the enhancement of oscillation amplitude for MEMS resonators by appropriate control techniques remains an interesting and open research subject.

The oscillation amplitude of electrostatic MEMS resonators is restricted by the so-called pull-in phenomena related to a bifurcation arising in electrostatic actuation [124]. The stable traveling range is limited to one third of its full gap due to the instability beyond that point. The pull-in phenomena can be used to accelerate the reaction speed for switching structures. However, it limits the stable operational range of the device, which is mandatory for applications requiring a sustained oscillation of the plates.

In order to operate electrostatic MEMS devices beyond the pull-in, one can use closed-loop control schemes (see, e.g., [125,126]). However, it may require a considerable control effort to produce an arbitrary trajectory. In order to increase the amplitude of periodic oscillations of the device with a low energy consumption, we consider in the present work controlling the resonator in chaotic mode to stabilize an orbit which is part of the chaotic attractor.

There are different ways to select this orbit. One can pick an unstable periodic orbit embed-

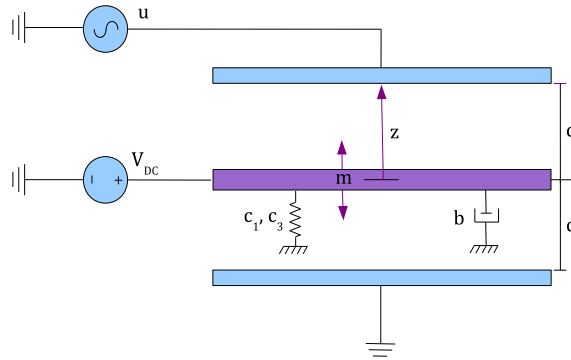


Figure 6.1 Schematic of the electrostatically actuated MEMS resonator.

ded in the chaotic attractor, take an orbit close to an unstable periodic one, or even choose a part of a chaotic trajectory that has an almost periodic behavior.

In the present work, we tackle the amplitude enhancement problem by stabilizing one of these orbits to produce a high amplitude oscillation.

There are some constraints that must be considered in a practical implementation of a chaos control scheme applied to MEMS. In particular, it is very difficult to sense the velocity of the plates during the operation of the device, which motivates the use of output feedback control techniques. In this work, the output is chosen as the position of the plates, and in order to implement the control, the velocity must be estimated by constructing an observer. We deal with the uncertainties in the model and the constraints on sensing the velocity by designing a controller by means of an adaptive observer backstepping scheme.

The organization of the rest of the paper is as follows. In Section 6.3, the dynamics of a chaotic MEMS actuator are briefly explained. Then, the design of an adaptive observer control scheme is presented in Section 6.4. Simulations are carried out and the results are presented in Section 6.5. Finally, concluding remarks are given in Section 6.6.

### 6.3 Dynamics of a Chaotic MEMS Resonator

#### 6.3.1 Model of the Chaotic MEMS

The MEMS device considered in this study is presented in Fig. 6.1, which is based on a model given in [56], and [37]. This device is modeled by a nonlinear mass-spring-damper system

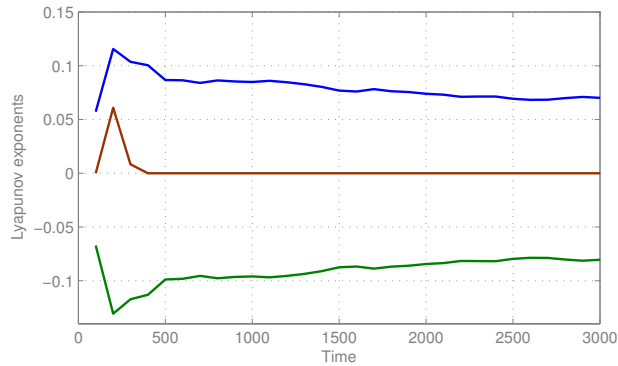


Figure 6.2 Lyapunov exponents for the system (6.5) with the set of parameters:  $\delta = 0.338$ ,  $A = 0.04$ ,  $\omega = 0.5$ ,  $\sigma = 1$ ,  $\mu = 0.01$ , and  $\eta = 12$ .

with external electrostatic actuation, expressed as

$$m\ddot{z} + b\dot{z} + c_1z + c_3z^3 = F, \quad (6.1)$$

where  $m$  is the mass of the movable structure,  $b$  is the damping constant,  $c_1$  and  $c_3$  are the linear and cubic stiffness coefficients, respectively, and  $F$  is the electrostatic actuation force given by

$$F = \frac{1}{2} \frac{C_0 d}{(d - z)^2} (V_{DC} + u)^2 - \frac{1}{2} \frac{C_0 d}{(d + z)^2} V_{DC}^2. \quad (6.2)$$

In (6.2),  $C_0$  is the capacitance of the actuator at rest, which has an initial gap  $d$  between the movable and fixed plates. The input voltage  $V_i = (V_{DC} + u)$  is applied as shown in Fig. 6.1 and contains two components: a DC voltage  $V_{DC}$  and a forcing voltage  $u$ , which may be a control signal that varies according to control objectives.

Note that in the present work, we ignore the dynamics of the driving circuit by assuming that they are much faster than those of the mechanical structure. Moreover, as the MEMS are used as a resonator, we assume that the system operates in such a way that the moveable and fixed plates do not come into contact. Hence, we do not take into account contact dynamics. Nevertheless, for those applications where the contact may occur, a different model should be considered to capture the dynamics of impacts and the control strategy should be modified adequately [129].

In order to facilitate control system design and analysis, it is convenient to work in normalized coordinates and redefine the time variable. For this purpose, we define the new variables  $\psi = \frac{z}{d}$ ,  $\tau = \omega_0 t$ , and  $\tilde{u} = uA$ , and introduce the following dimensionless constants:

$$\sigma = \frac{c_1}{m\omega_0^2}, \quad \eta = \frac{c_3 d^2}{m\omega_0^2}, \quad \delta = \frac{C_0 V_{DC}^2}{2m\omega_0^2 d^2}, \quad A = 2\delta \frac{1}{V_{DC}}, \quad \mu = \frac{b}{m\omega_0}, \quad (6.3)$$

where  $\omega_0$  is the natural frequency defined by:

$$\omega_0 = \sqrt{\frac{c_1}{m}}. \quad (6.4)$$

In this way, by assuming that the DC voltage is sufficiently higher than the amplitude of the forcing voltage  $u$ , the dimensionless equation of motion (6.1) may be fairly approximated by

$$\ddot{\psi} + \mu\dot{\psi} + \sigma\psi + \eta\psi^3 = \delta \left( \frac{1}{(1 - \psi)^2} - \frac{1}{(1 + \psi)^2} \right) + \frac{\tilde{u}}{(1 - \psi)^2}. \quad (6.5)$$

It can be shown that System (6.5) behaves chaotically by selecting  $u = V_{AC} \sin(\tilde{\omega}t)$  and  $V_{AC}A = 0.04$ ,  $\tilde{\omega} = 0.5$ ,  $\delta = 0.338$ ,  $\mu = 0.01$ ,  $\sigma = 1$ , and  $\eta = 12$ , as indicated in the Lyapunov exponents diagram in Fig. 6.2. The Lyapunov exponents are calculated by using the algorithm proposed in [130]. The existence of one positive Lyapunov exponent confirms the sensitive dependence on initial conditions of the system.

System (6.5) can be written in state space form as

$$\dot{\psi}_1 = \psi_2 \quad (6.6a)$$

$$\dot{\psi}_2 = -\mu\dot{\psi}_2 - \sigma\psi_1 - \eta\psi_1^3 + f(\psi_1) + \frac{\tilde{u}}{(1 - \psi_1)^2} \quad (6.6b)$$

$$y = \psi_1 \quad (6.6c)$$

where

$$f(\psi_1) = \delta \left( \frac{1}{(1 - \psi_1)^2} - \frac{1}{(1 + \psi_1)^2} \right),$$

and  $y$  is the output.

### 6.3.2 Unstable Orbits in the Strange Attractor

The main idea to obtain a sustained oscillation of the MEMS with a low energy consumption is not to create a new orbit different from those already present in the attractor. The orbit could be an unstable periodic orbit embedded in the chaotic attractor, an orbit close to an unstable periodic one, or even part of a chaotic trajectory which has an almost periodic behavior.

In order to find one of such orbits, one may measure the phase space dynamics of the system

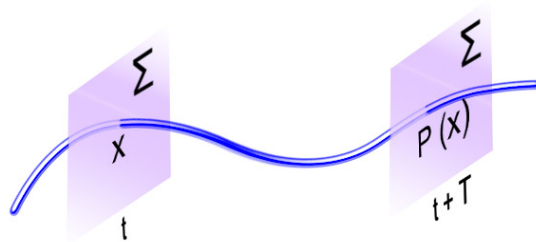


Figure 6.3 Poincare section  $\Sigma$  and Poincare map  $P$  for the periodic vector field defined by (6.7) and (6.8).

and obtain a set of points from which one can extract the orbit by applying the close returns method [131]. Note that other approaches, such as the shooting method [132], can also be used for this purpose. With the aim of simplifying the discussion, we will only consider systems where the state  $x = (x_1, x_2) \in \mathbb{R}^2$  and an external periodic force is applied. Under these conditions, a Poincare map can be easily constructed from stroboscopically logging the value of the variable  $x$  at multiples of the period  $T$  of the forcing. Time can be seen as one of the states, and the problem of finding periodic orbits reduces to find fixed points in  $\mathbb{R}^2$  on the Poincare surface.

The Poincare map may be defined as follows. Given the differential equation

$$\dot{x} = f(x, t), \quad x \in \mathbb{R}^2, \quad (6.7)$$

with periodic phase space of period  $T$  such that

$$f(x, t) = f(x, t + T), \quad (6.8)$$

then the Poincaré map is defined as

$$\begin{aligned} P : \Sigma &\rightarrow \Sigma \\ x &\mapsto \varphi(T, x), \end{aligned} \quad (6.9)$$

where  $\varphi(t, \cdot)$  denotes the flow generated by (6.7), and  $\Sigma$  is a Poincare section as shown in Fig. 6.3.

The procedure for finding orbits with almost periodic behaviour is similar to the one of finding periodic orbits. In both cases the image under the Poincare map of a point close to the orbit, which intersects the Poincare surface, is close to the point. Finding such point is the aim of the close returns method.

For the purpose of applying the method, one records discretely sampled time series  $x(0), x(T_s), \dots, x((N-1)T_s)$ , containing  $N$  points, where  $T_s$  is the sample time. Then, for a  $k$  given, one takes a point  $x(kT_s)$  and its iteration  $m$  on the map  $x(kT_s + mT)$  and evaluates the distance  $d(x(kT_s), x(kT_s + mT))$  between the points, by using for example the Euclidean norm. If the distance is less than a small  $\epsilon_s$ , then this is a  $m$ -period orbit that could be used.

In order to show the application of the close returns method, we use three different time series from which we extract unstable orbits of different periods. Each of these series comprises a set of  $N = 120000$  points with sample time of  $T_s = 0.01$ , which corresponds to a logging time of approximately 100 times the period of the forcing term. The orbits were obtained by using

$\epsilon_s = 0.005$ , and can be seen in Fig. 6.4, 6.5, and 6.6. The figures suggest the bifurcations occurred when the period increases.

In order to enhance the oscillation amplitude of the plates, we chose an unstable orbit surrounding the chaotic attractor, which produces a high amplitude periodic output. This orbit, shown in Fig. 6.7, was used as the reference trajectory in the simulation given subsequently. It can be approximated by its truncated Fourier series up to five terms, given by

$$y_r = a_0 + a_1 \cos(w_1 t) + b_1 \sin(w_1 t) + a_2 \cos(2w_1 t) + b_2 \sin(2w_1 t) + a_3 \cos(3w_1 t) + b_3 \sin(3w_1 t) + a_4 \cos(4w_1 t) + b_4 \sin(4w_1 t) + a_5 \cos(5w_1 t) + b_5 \sin(5w_1 t), \quad (6.10)$$

where

$$\begin{aligned} a_0 &= 0.0175400, & a_1 &= -0.4498000, & b_1 &= -0.000563, \\ a_2 &= 0.0112600, & b_2 &= 0.0008906, & a_3 &= -0.029140, \\ b_3 &= 0.0000376, & a_4 &= 0.0006330, & b_4 &= 0.000231, \\ a_5 &= -0.001090, & b_5 &= 0.0000852. \end{aligned}$$

By taking the time derivative of (6.10), we obtain

$$\dot{y}_r = a'_1 \cos(w_1 t) + b'_1 \sin(w_1 t) + a'_2 \cos(2w_1 t) + b'_2 \sin(2w_1 t) + a'_3 \cos(3w_1 t) + b'_3 \sin(3w_1 t) + a'_4 \cos(4w_1 t) + b'_4 \sin(4w_1 t) + a'_5 \cos(5w_1 t) + b'_5 \sin(5w_1 t), \quad (6.11)$$

where

$$\begin{aligned} a'_1 &= -0.00052, & b'_1 &= 0.4163, & a'_2 &= 0.0016, \\ b'_2 &= -0.0208, & a'_3 &= 0.00010, & b'_3 &= 0.0809, \\ a'_4 &= 0.00085, & b'_4 &= -0.0023, & a'_5 &= 0.00039, \\ b'_5 &= 0.00500. \end{aligned}$$

The second derivative of  $y_r$  is obtained by taking the derivative of (6.11).

In what follows, we use the following general assumption:

**Assumption 2.** *The reference signal  $y_r(t)$  and its first two derivatives are known and bounded. In addition,  $y_r^{(p)}(t)$ ,  $\forall p \geq 3$ , is piecewise continuous.*

## 6.4 Output Feedback Control of Chaotic MEMS

The objective of the control is to stabilize the unstable orbit (6.10) of the chaotic uncertain system (6.6a). This objective can be achieved by forcing the tracking error to tend to zero:

$$\lim_{t \rightarrow \infty} [y(t) - y_r(t)] = 0. \quad (6.12)$$

We use adaptive observer backstepping to deal with the parametric uncertainties in the model and also consider more realistic designs where only the output is available for measurement.

### Adaptive Observer Backstepping Control

In order to apply a backstepping design procedure to System (6.6a), we transform it into the parametric output-feedback form by using the following change of variables [127]:

$$\begin{aligned} x_1 &= \psi_1 \\ x_2 &= \psi_2 + \mu\psi_1. \end{aligned} \quad (6.13)$$

System (6.6a) becomes

$$\dot{x}_1 = x_2 - \theta_1 x_1 \quad (6.14a)$$

$$\dot{x}_2 = -\theta_2 x_1 - \theta_3 x_1^3 + \theta_4 f(x_1) + \beta(y) u \quad (6.14b)$$

$$y = x_1, \quad (6.14c)$$

where  $\theta = (\theta_1 \ \theta_2 \ \theta_3 \ \theta_4) = (\mu \ \sigma \ \eta \ \delta)$  is a vector of uncertain parameters, and

$$\beta(y) = \frac{1}{(1-y)^2}. \quad (6.15)$$

Equation (6.14) can be written in a more compact form

$$\dot{x} = Ax + ky + bu + \varphi \theta \quad (6.16a)$$

$$y = c^\top x, \quad (6.16b)$$

where  $x = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^\top$  is the state vector,  $y$  is the output,  $u$  is the input,  $k = \begin{pmatrix} k_1 & k_2 \end{pmatrix}^\top$

is a vector with  $k_1, k_2 \in \mathbb{Z}^+$ , defined in such a way that the matrix  $A$ , given by

$$A = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}, \quad (6.17)$$

is Hurwitz,  $b = \begin{pmatrix} 0 & \beta \end{pmatrix}^\top$ ,  $\varphi$  is a nonlinear matrix defined by

$$\begin{aligned} \varphi &= \begin{bmatrix} -x_1 & 0 & 0 & 0 \\ 0 & -x_1 & -x_1^3 & f(x_1) \end{bmatrix} \\ &= \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \end{pmatrix}, \end{aligned} \quad (6.18)$$

and  $c = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top$ .

We will reconstruct the state of the system through the use of the filters  $\xi_i, i = 0, 1, \dots, 4$ . The state is reconstructed as

$$x = \xi_0 + \theta_1 \xi_1 + \theta_2 \xi_2 + \theta_3 \xi_3 + \theta_4 \xi_4 + \epsilon, \quad (6.19)$$

where  $\epsilon$  is the error between the state and its estimation. The virtual estimate of  $x$  depends on the uncertain parameter  $\theta$ , but all we need in the design is the vector  $(\xi_0 \ \xi_1 \ \xi_2 \ \xi_3 \ \xi_4)$ . In order to guarantee that the error  $\epsilon$  tends exponentially to zero, we define the filters as

$$\begin{aligned} \dot{\xi}_0 &= \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \begin{bmatrix} \xi_{01} \\ \xi_{02} \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} y + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u \\ \dot{\xi}_1 &= \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \begin{bmatrix} \xi_{11} \\ \xi_{12} \end{bmatrix} + \begin{bmatrix} -y \\ 0 \end{bmatrix} \\ \dot{\xi}_2 &= \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \begin{bmatrix} \xi_{21} \\ \xi_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ -y \end{bmatrix} \\ \dot{\xi}_3 &= \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \begin{bmatrix} \xi_{31} \\ \xi_{32} \end{bmatrix} + \begin{bmatrix} 0 \\ -y^3 \end{bmatrix} \\ \dot{\xi}_4 &= \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \begin{bmatrix} \xi_{41} \\ \xi_{42} \end{bmatrix} + \begin{bmatrix} 0 \\ f(y) \end{bmatrix}, \end{aligned} \quad (6.20)$$

or in a compact way:

$$\begin{aligned}\dot{\xi}_0 &= A\xi_0 + ky + bu \\ \dot{\xi}_j &= A\xi_j + \varphi_j(y), j = 1, 2, 3, 4.\end{aligned}$$

From (6.19) we can obtain the error dynamic equation:

$$\dot{\epsilon} = \dot{x} - \dot{\xi}_0 - \theta_1 \dot{\xi}_1 - \theta_2 \dot{\xi}_2 - \theta_3 \dot{\xi}_3 - \theta_4 \dot{\xi}_4, \quad (6.21)$$

and by using (6.16a) and (6.20) we obtain

$$\begin{aligned}\dot{\epsilon} &= (Ax + ky + bu + \varphi\theta) - (A\xi_0 + ky + bu) - \theta_1 (A\xi_1 + \varphi_1) - \theta_2 (A\xi_2 + \varphi_2) \\ &\quad - \theta_3 (A\xi_3 + \varphi_3) - \theta_4 (A\xi_4 + \varphi_4) \\ &= A(x - \xi_0 - \theta_1 \xi_1 - \theta_2 \xi_2 - \theta_3 \xi_3 - \theta_4 \xi_4) \\ &= A\epsilon.\end{aligned} \quad (6.22)$$

Since  $A$  is Hurwitz,  $\epsilon$  converges to zero exponentially.

The adaptive output feedback controller is designed recursively applying Backstepping as follows.

$$z_1 = y - y_r, \quad (6.23)$$

where  $y_r$  is the reference signal which satisfies Assumption 2. Its dynamic is given by

$$\dot{z}_1 = \dot{y} - \dot{y}_r = \dot{x}_1 - \dot{y}_r = x_2 + \theta_1(-x_1) - \dot{y}_r \quad (6.24)$$

, since we cannot measure  $x_2$ , it cannot be used as a virtual control. We replace it with its virtual estimated and the corresponding error:

$$x_2 = \xi_{02} + \theta_1 \xi_{12} + \theta_2 \xi_{22} + \theta_3 \xi_{32} + \theta_4 \xi_{42} + \epsilon_2, \quad (6.25)$$

hence (6.24) becomes

$$\dot{z}_1 = \xi_{02} + \theta_1 \xi_{12} + \theta_2 \xi_{22} + \theta_3 \xi_{32} + \theta_4 \xi_{42} + \theta_1(-x_1) - \dot{y}_r + \epsilon_2. \quad (6.26)$$

By choosing  $\xi_{02}$  as the virtual control, and defining the second error variable as

$$z_2 = \xi_{02} - \alpha_1 - \dot{y}_r, \quad (6.27)$$

(6.26) becomes

$$\dot{z}_1 = (z_2 + \alpha_1 + \dot{y}_r) + \theta_1 \xi_{12} + \theta_2 \xi_{22} + \theta_3 \xi_{32} + \theta_4 \xi_{42} + \theta_1 (-x_1) - \dot{y}_r + \epsilon_2. \quad (6.28)$$

Since  $\theta$  is an uncertain vector, we must use its first estimated value  $\vartheta_1 = \begin{pmatrix} \vartheta_{11} & \vartheta_{12} & \vartheta_{13} & \vartheta_{14} \end{pmatrix}^\top$ . By defining the first stabilizing function  $\alpha_1$  as

$$\alpha_1 = -c_1 z_1 - d_1 z_1 - \vartheta_1 \omega, \quad (6.29)$$

where  $\omega = \begin{pmatrix} \xi_{12} - x_1 & \xi_{22} & \xi_{32} & \xi_{42} \end{pmatrix}^\top$ , then (6.28) becomes

$$\dot{z}_1 = -c_1 z_1 + z_2 - d_1 z_1 + (\theta - \vartheta_1) \omega + \epsilon_2. \quad (6.30)$$

If we use the Lyapunov function candidate

$$V_1 = \frac{1}{2} z_1^2 + \frac{1}{2\gamma_1} (\theta - \vartheta_1) (\theta - \vartheta_1)^\top + \frac{1}{d_1} \epsilon^\top P_0 \epsilon, \quad (6.31)$$

where  $\gamma_1 > 0$  is the adaptation gain and  $P_0 = P_0^\top > 0$  satisfies  $P_0 A + A^\top P_0 = -I$ , then

$$\begin{aligned} \dot{V}_1 &= z_1 \dot{z}_1 - \frac{1}{\gamma} \dot{\vartheta}_1 (\theta - \vartheta_1)^\top - \frac{1}{d_1} \epsilon^\top \dot{\epsilon} \\ &= z_1 (-c_1 z_1 + z_2 - d_1 z_1 + (\theta - \vartheta_1) \omega + \epsilon_2) - \frac{1}{\gamma} (\theta - \vartheta_1) \dot{\vartheta}_1^\top - \frac{1}{d_1} \epsilon^\top \dot{\epsilon} \\ &= -c_1 z_1^2 + z_1 z_2 - d_1 z_1^2 + (\theta - \vartheta_1) \left( z_1 \omega - \frac{1}{\gamma} \dot{\vartheta}_1^\top \right) + z_1 \epsilon_2 - \frac{1}{d_1} \epsilon^\top \dot{\epsilon} \\ &= -c_1 z_1^2 + z_1 z_2 - d_1 \left( z_1 - \frac{1}{2d_1} \epsilon_2 \right)^2 + \frac{1}{4d_1} \epsilon_2^2 - \frac{1}{d_1} \epsilon^\top \dot{\epsilon} + (\theta - \vartheta_1) \left( z_1 \omega - \frac{1}{\gamma} \dot{\vartheta}_1^\top \right) \\ &\leq z_1 z_2 - c_1 z_1^2 + (\theta - \vartheta_1) \left( z_1 \omega - \frac{1}{\gamma} \dot{\vartheta}_1^\top \right) - \frac{3}{4d_1} \epsilon^\top \dot{\epsilon}. \end{aligned} \quad (6.32)$$

It is possible to eliminate the term  $(\theta - \vartheta_1)$  by the update law

$$\dot{\vartheta}_1^\top = \gamma z_1 \omega. \quad (6.33)$$

$$\begin{aligned}
\dot{z}_2 &= \dot{\xi}_{02} - \dot{\alpha}_1 - \ddot{y}_r \\
&= \beta u + k_2 (x_1 - \xi_{01}) - \ddot{y}_r - \dot{\alpha}_1 \\
&= \beta u + k_2 (x_1 - \xi_{01}) - \frac{\partial \alpha_1}{\partial y} (\xi_{02} + \theta_1 \xi_{12} + \theta_2 \xi_{22} + \theta_3 \xi_{32} + \theta_4 \xi_{42} + \epsilon_2 + \theta_1 (-x_1)) \\
&\quad - \frac{\partial \alpha_1}{\partial \xi_{12}} (-k_2 \xi_{11}) - \frac{\partial \alpha_1}{\partial \xi_{22}} (-k_2 \xi_{21} - x_1) - \frac{\partial \alpha_1}{\partial \xi_{32}} (-k_2 \xi_{31} - x_1^3) - \frac{\partial \alpha_1}{\partial \xi_{42}} (-k_2 \xi_{41} + f_1) \\
&\quad - \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r - \frac{\partial \alpha_1}{\partial \vartheta_1} (\gamma z_1 \omega) - \ddot{y}_r.
\end{aligned} \tag{6.34}$$

The control signal is chosen as

$$\begin{aligned}
\beta u &= -k_2 (x_1 - \xi_{01}) + \frac{\partial \alpha_1}{\partial \xi_{12}} (-k_2 \xi_{11}) + \frac{\partial \alpha_1}{\partial \xi_{22}} (-k_2 \xi_{21} - x_1) + \frac{\partial \alpha_1}{\partial \xi_{32}} (-k_2 \xi_{31} - x_1^3) \\
&\quad + \frac{\partial \alpha_1}{\partial \xi_{42}} (-k_2 \xi_{41} + f_1) + \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r + \frac{\partial \alpha_1}{\partial \vartheta_1} (\gamma z_1 \omega) + \ddot{y}_r + \frac{\partial \alpha_1}{\partial y} (\xi_{02} + \vartheta_2 \omega) \\
&\quad - d_2 \left( \frac{\partial \alpha_1}{\partial y} \right)^2 z_2 - c_2 z_2 - z_1
\end{aligned} \tag{6.35}$$

, where  $\vartheta_2 = \begin{pmatrix} \vartheta_{21} & \vartheta_{22} & \vartheta_{23} & \vartheta_{24} \end{pmatrix}^\top$  is a second estimate of  $\vartheta$ . The dynamic of the error  $z_2$  is

$$\dot{z}_2 = -c_2 z_2 - z_1 - d_2 \left( \frac{\partial \alpha_1}{\partial y} \right)^2 z_2 - \frac{\partial \alpha_1}{\partial y} \epsilon_2 - \frac{\partial \alpha_1}{\partial y} (\theta - \vartheta_2) \omega. \tag{6.36}$$

Defining the Lyapunov function

$$V_2 = V_1 + \frac{1}{2} z_2^2 + \frac{1}{2\gamma} (\theta - \vartheta_2) (\theta - \vartheta_2)^\top + \frac{1}{d_2} \epsilon^\top P_0 \epsilon, \tag{6.37}$$

its derivative is computed as

$$\begin{aligned}
\dot{V}_2 &= \dot{V}_1 + z_2 \dot{z}_2 - \frac{1}{\gamma} \dot{\vartheta}_2 (\theta - \vartheta_2)^\top - \frac{1}{d_2} \epsilon^\top \dot{\epsilon} \\
&\leq z_1 z_2 - c_1 z_1^2 - \frac{3}{4d_1} \epsilon^\top \epsilon - c_2 z_2^2 - z_1 z_2 - d_2 \left( \frac{\partial \alpha_1}{\partial y} \right)^2 z_2^2 - \frac{\partial \alpha_1}{\partial y} \epsilon_2 z_2 \\
&\quad - \left( \frac{\partial \alpha_1}{\partial y} \omega z_2 + \frac{1}{\gamma} \dot{\vartheta}_2 \right) (\theta - \vartheta_2) - \frac{1}{d_2} \epsilon^\top \dot{\epsilon} \\
&\leq -c_1 z_1^2 - c_2 z_2^2 - \left( \frac{3}{4d_1} + \frac{1}{d_2} \right) \epsilon^\top \epsilon - \left( \frac{\partial \alpha_1}{\partial y} \omega z_2 + \frac{1}{\gamma} \dot{\vartheta}_2 \right) (\theta - \vartheta_2).
\end{aligned} \tag{6.38}$$

To eliminate the term  $(\theta - \vartheta_2)$  we use the update law

$$\dot{\vartheta}_2 = -\gamma \frac{\partial \alpha_1}{\partial y} \omega z_2, \quad (6.39)$$

and we finally obtain

$$\dot{V}_2 \leq -c_1 z_1^2 - c_2 z_2^2 - \left( \frac{3}{4d_1} + \frac{1}{d_2} \right) \epsilon^\top \epsilon. \quad (6.40)$$

By applying the LaSalle-Yoshizawa theorem [127], we can conclude that  $z_1$ ,  $z_2$ ,  $\vartheta_1$ , and  $\vartheta_2$  are bounded and  $z \rightarrow 0$  as  $t \rightarrow \infty$ . Therefore, the output  $y(t)$  tracks  $y_r(t)$  globally and asymptotically.

## 6.5 Simulation Results

Numerical simulations have been carried out to validate the design made in Section 6.4. Two orbits are stabilized using the control strategy developed in the previous section with the aim of comparing the corresponding control effort.

First we will stabilize the unstable orbit (6.10) of the chaotic system (6.6a), and then we stabilize the orbit:

$$\begin{aligned} y_{r1} &= a_0 + 1.15a_1 \cos(w_1 t) + a_2 \cos(2w_1 t) \\ \dot{y}_{r1} &= -1.15a_1 w_1 \sin(w_1 t) - 2w_1 a_2 \sin(2w_1 t), \end{aligned} \quad (6.41)$$

where

$$a_0 = 0.01754 \quad a_1 = -0.4 \quad a_2 = 0.01126.$$

These orbits have the same fundamental frequency  $w_1$ , and (6.41) is a perturbation of (6.10). The orbits are shown in the Fig. 6.8 and their states are compared in Fig. 6.9. Although these two orbits are similar, one is immersed in the chaotic MEMS attractor while the other is not.

Fig. 6.10(a) shows the output of the system when the orbit immersed in the chaotic attractor is stabilized by the algorithm developed in Section 6.4. In Fig. 6.10(b) the output of the system is shown when the non-immersed orbit is stabilized by using the same algorithm. It can be seen the agreement between the curves and the effect of the controller.

The tracking errors are compared in Fig. 6.10(c), and we can observe how decay as expected. A comparison between the control signals for stabilizing the two orbits is given in Fig. 6.10(d). It clearly shows that in steady-state, an affordable control effort is required for tracking the

orbit immersed in the chaotic attractor, which is significantly smaller than that for the non-immersed orbit.

Finally, the evolution of the estimates of the parameters given by the vector  $\theta$  is shown in Fig. 6.11 for the immersed orbit, and in Fig. 6.12 for the non-immersed orbit. We can verify that the estimated values of the parameters are bounded as expected.

## 6.6 Conclusions

This work presented a solution for the enhancement of the oscillation amplitude of a MEMS resonator based on the chaotic properties of the system. In order to enhance the amplitude of the oscillation at a reduced control effort, a high amplitude unstable orbit immersed in the chaotic attractor was chosen as the reference trajectory. The method of close returns was used to compute the orbit, which was then approximated by a five-terms truncated Fourier series. With the aim of stabilizing this orbit, a control law was designed by using output feedback and considering parametric uncertainties in the model. It has been shown through adaptive observer backstepping that this control law allows the system output to globally and asymptotically track the desired orbits. Numerical simulations were carried out and the results show that the unstable orbits immersed in the chaotic attractor are stabilized with an affordable control effort compared to that required for stabilizing a non-immersed orbit. This demonstrated the advantage of the proposed approach.

Finally, it is worth noting that the scheme developed in the present work is realistic in the sense that it is based on output-feedback and consider parametric uncertainties in the model. Variables as the speed of some components of the MEMS are very difficult to be measured, and a precise knowledge of the parameters of the system is hard due to the fabrication technology. In the approach proposed, the speed of the plates of the device was estimated by means of a virtual observer at the price of increasing the differential equations needed to construct the controller. Reducing the dynamic equations is beyond the scope of this paper and will be considered in a future work.

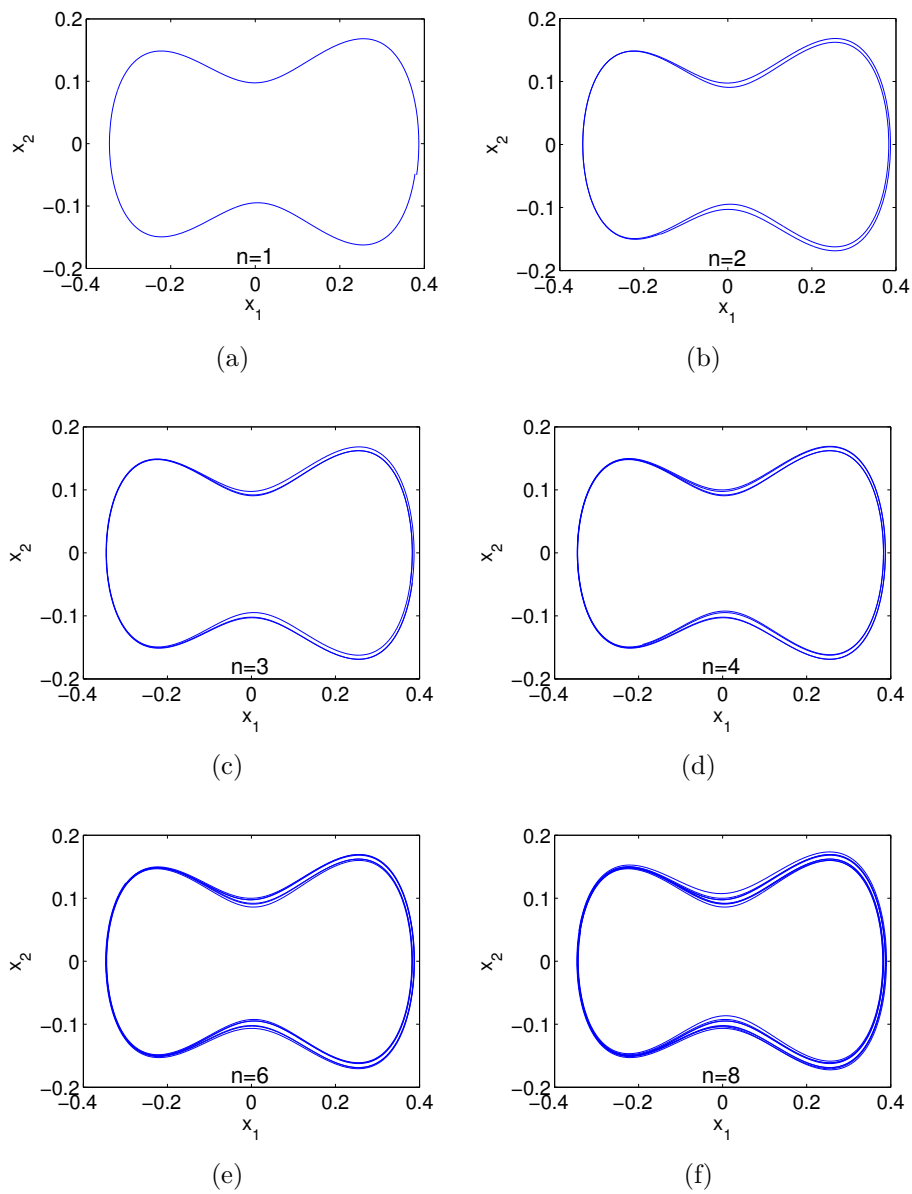


Figure 6.4 Unstable orbits found using close return method. The period of the orbit is identified with  $n$ .

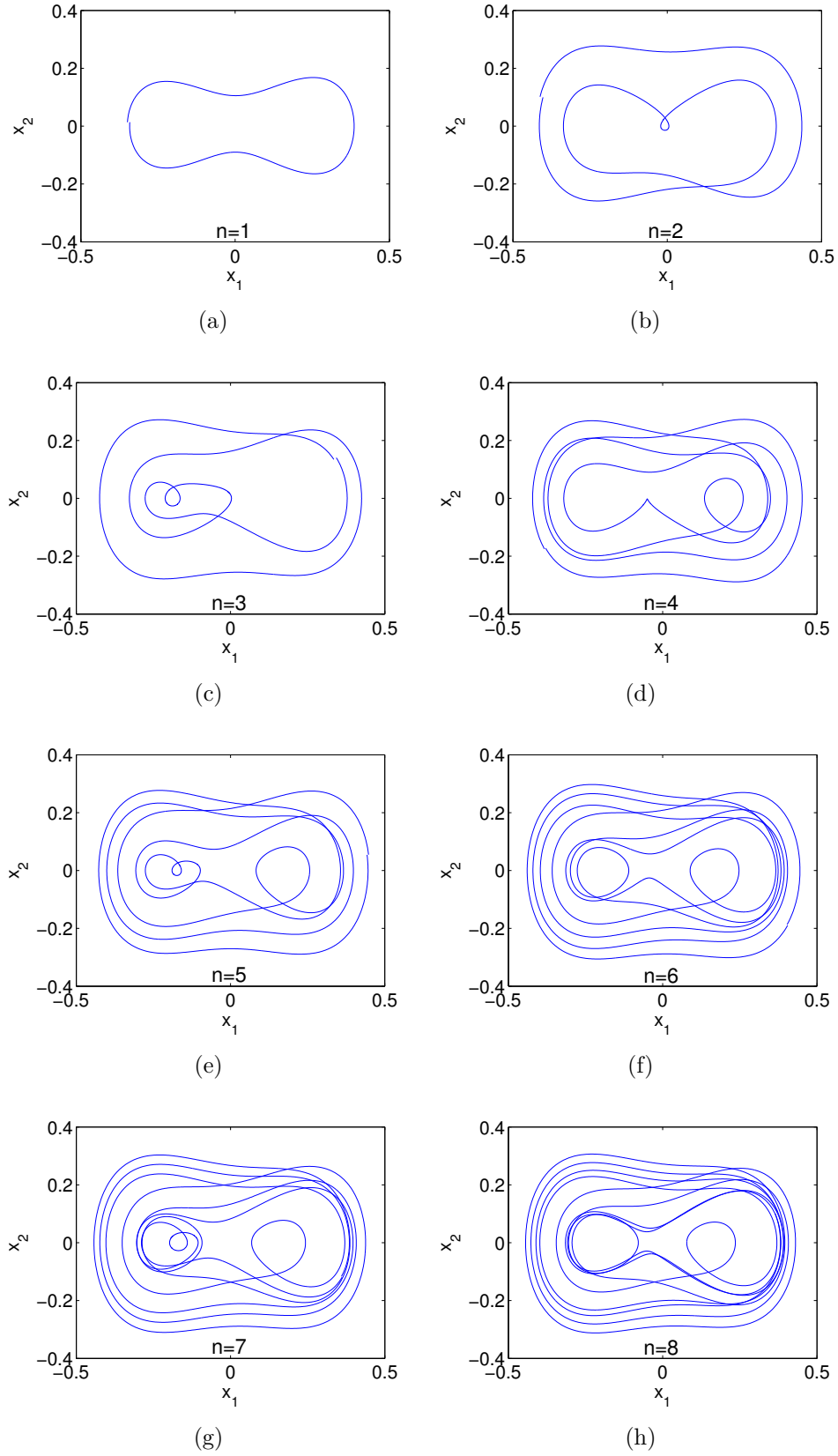


Figure 6.5 Unstable orbits found using close return method for a second set of data. The period of the orbit is  $n$ .

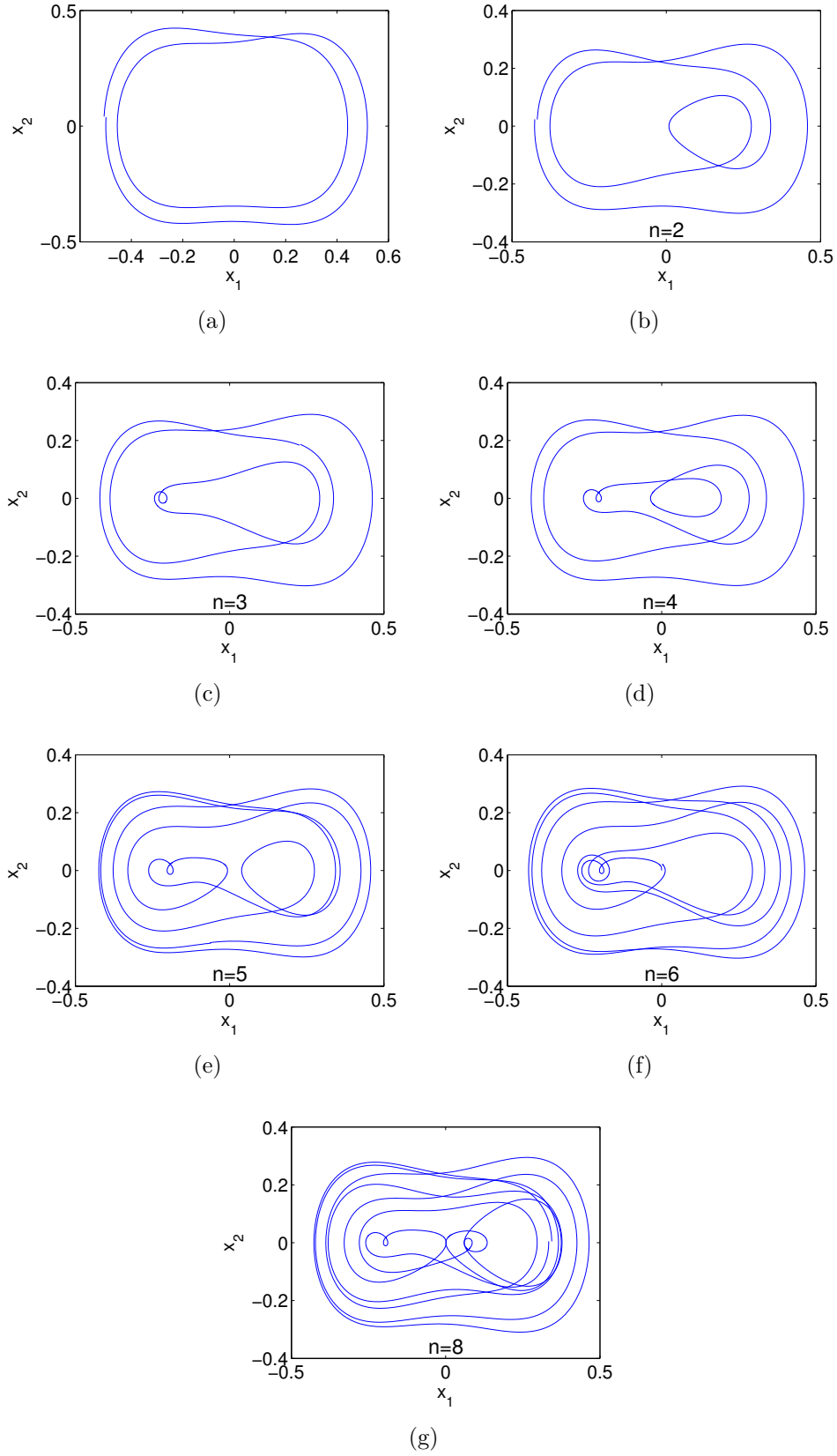


Figure 6.6 Unstable orbits found using close return method for a third set of data.  $n$  identifies the period of the orbit.

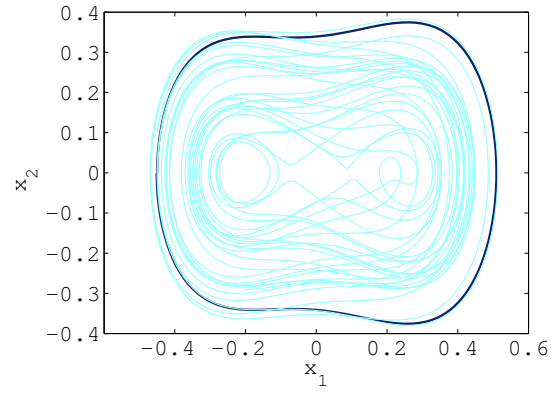


Figure 6.7 Unstable orbit surrounding the chaotic attractor

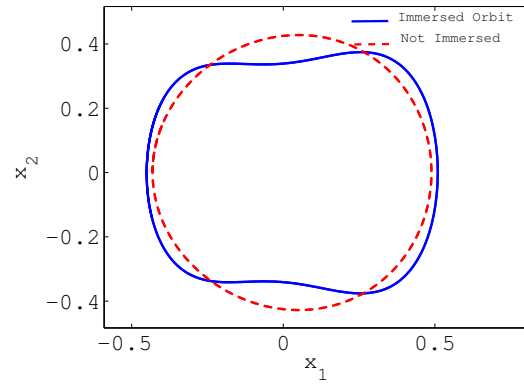


Figure 6.8 Unstable orbit immersed in the chaotic attractor and orbit not immersed

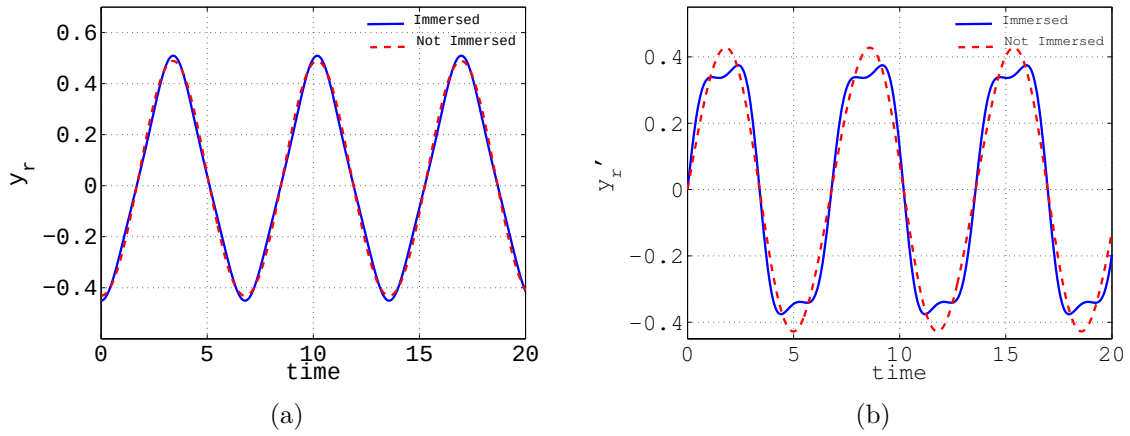


Figure 6.9 Reference signal and its derivative: (a)  $y_r$ ; (b)  $\dot{y}_r$ .

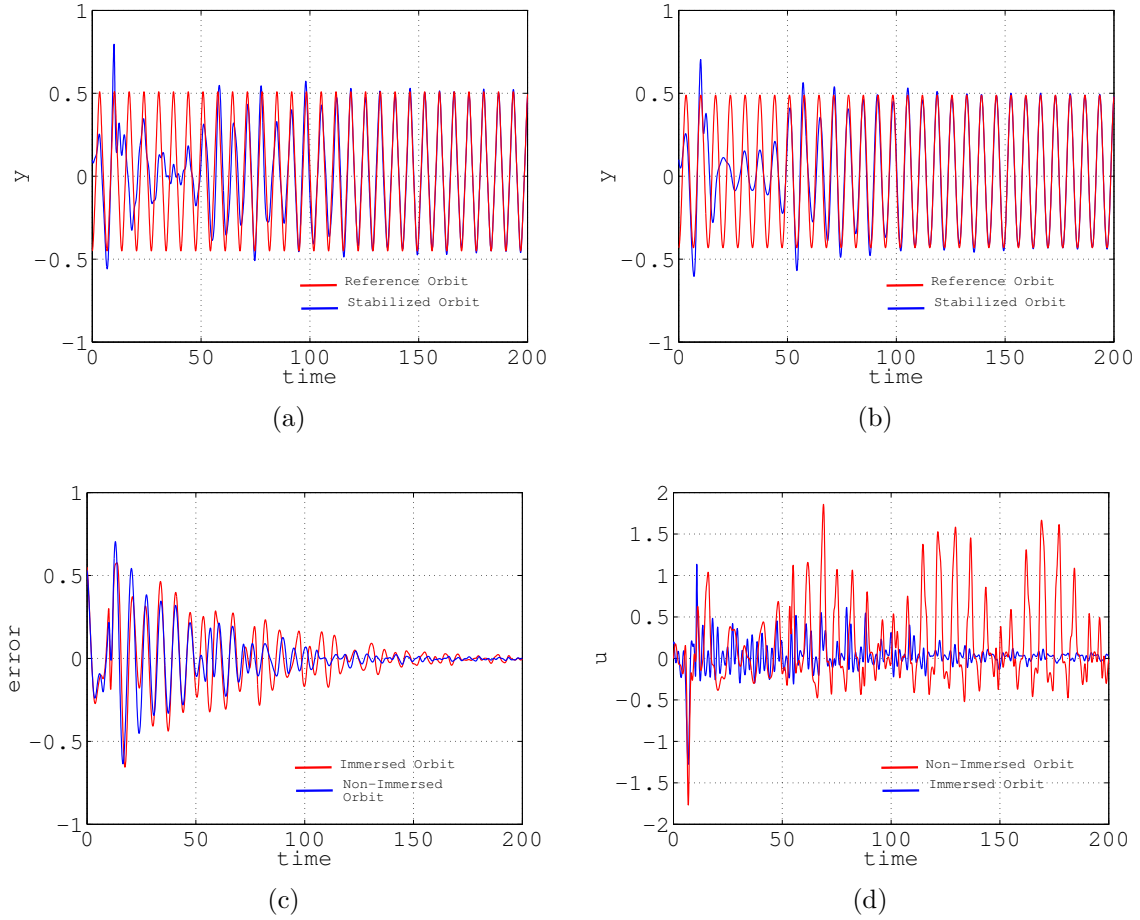


Figure 6.10 Simulation results for the controller: (a) output for the Immersed orbit; (b) output for the Non-Immersed orbit; (c) tracking errors; (d) control signals.

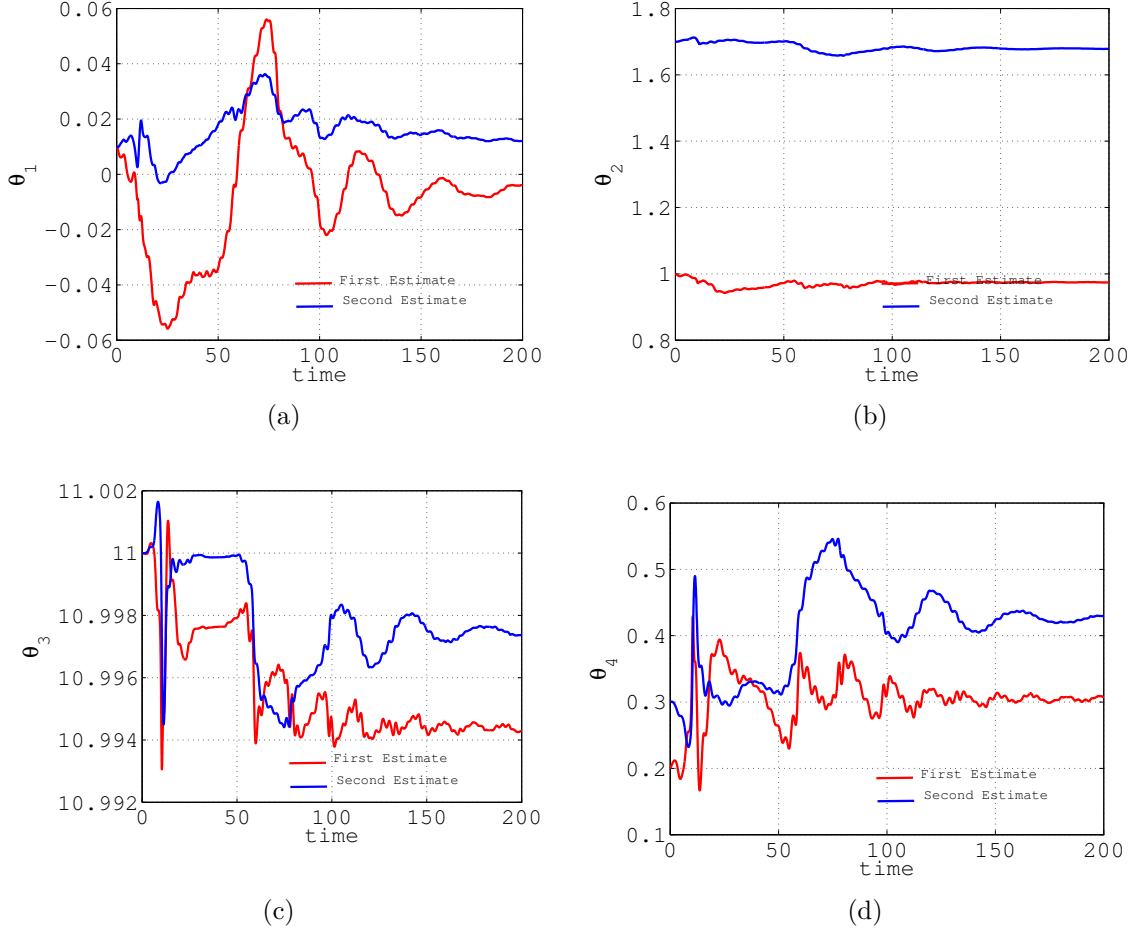


Figure 6.11 Estimated parameter  $\vartheta$  for the immersed orbit: (a)  $\vartheta_{11}$ ; (b)  $\vartheta_{12}$ ; (c)  $\vartheta_{13}$ ; (d)  $\vartheta_{14}$ .

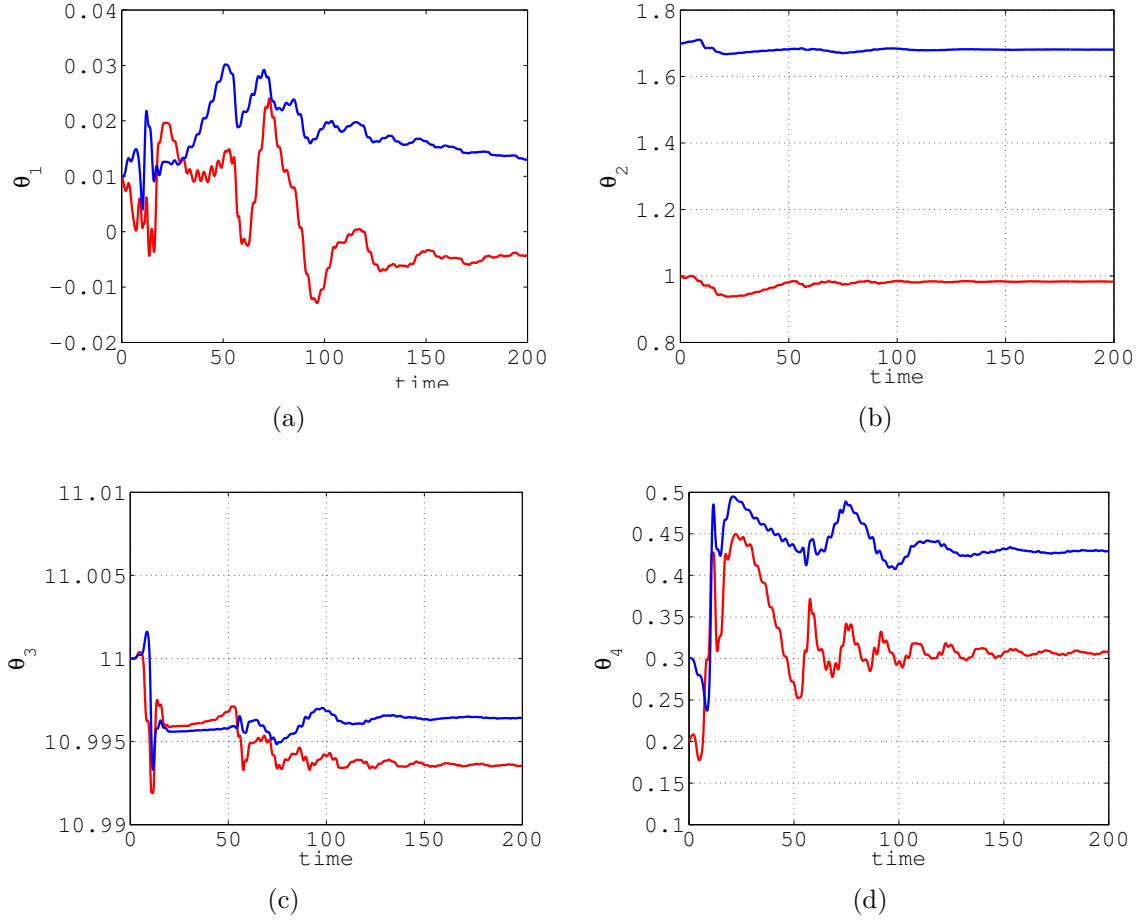


Figure 6.12 Estimated parameter  $\vartheta$  for the Non-immersed orbit: (a)  $\vartheta_{21}$ ; (b)  $\vartheta_{22}$ ; (c)  $\vartheta_{23}$ ; (d)  $\vartheta_{24}$ .

## CHAPTER 7    ARTICLE 4: HIGH-ORDER SLIDING MODE CONTROL AND DISTURBANCE REJECTION OF CHAOTIC MEMS RESONATORS

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### 7.1 Abstract

This article addresses the control of chaotic microelectromechanical (MEMS) resonators, considering the presence of unmodeled dynamics and external disturbances. The paper introduces a theorem that establishes necessary conditions for obtaining bounded solutions of perturbed linear systems. Using this result and in line with the principles of active disturbance rejection control (ADRC), the feasibility of designing a high-gain observer capable of estimating external disturbances and unmodeled dynamics is demonstrated. Furthermore, it is demonstrated how an appropriately designed control signal can effectively steer the regulation error dynamics towards a vicinity of zero. In order to reduce the problems due to the noise in high-gain control schemes and the lack of asymptotic stability inherent in the ADRC approach, an extended sliding mode observer is introduced. Additionally, high-order sliding mode control (HOSMC) is used to reduce the chattering effect and enhance the finite-time convergence of the overall system. The efficacy of the proposed method is demonstrated through its successful application in controlling unstable orbits within the attractor of a chaotic MEMS resonator. Specifically, an orbit with a high amplitude is carefully selected from the chaotic attractor to maximize the oscillation amplitude of the MEMS resonator, thereby highlighting the method’s utility and superior performance.

### 7.2 Introduction

A resonant sensor is a device equipped with a component that vibrates at its resonance frequency and undergoes frequency variations, as a function of a physical or chemical parameter [133]. Resonating MEMS sensors have been used for some time, as mechanical sensors of force [134], flow [135], pressure [136], and acceleration or as chemical sensors [133, 137]. A comprehensive understanding, accurate modeling, and effective control of MEMS resonators are of utmost importance in attaining the desired performance characteristics and optimizing the operation of oscillation to meet specific application requirements. However, the perfor-

mance of MEMS resonators, as well as their associated control strategies, can be significantly affected by the presence of unmodeled dynamics and external disturbances, leading to a degradation in system behavior.

Nonlinear characteristics, such as chaos, can be advantageous in a MEMS oscillator if we select an unstable orbit with high amplitude oscillations from within the chaotic attractor, and then stabilize it, a process usually achieved with low energy consumption. Unmodeled dynamics typically exhibit nonlinear characteristics and have a significant impact on the amplitude of oscillation. These nonlinear effects emerge from diverse factors, including material properties, structural design, and electrostatic or mechanical forces inherent in the resonator's operation [33, 45, 46]. Furthermore, external disturbances arising from environmental factors, mechanical vibrations, electromagnetic interference, or unwanted forces can likewise impact the behavior of MEMS resonators. By employing the correct control strategy, we can exploit the nonlinear chaotic behavior while rejecting undesired nonlinear phenomena.

Employing robust control techniques that can handle uncertainties, unmodeled dynamics, and disturbances effectively is mandatory [56, 57, 62]. To address this challenge, control strategies based on disturbance rejection have emerged as effective approaches [138]. Disturbance rejection is based on the estimation of external disturbances and unknown dynamics affecting the plant by means of high-gain observers, and then cancel them using a high-gain control law that in addition induces in the system an expected behaviour [139]. This approach presents several advantages over other control techniques due mainly to its simplicity and good performance, since it eliminates the negative impact on the plant of external disturbances and unmodeled dynamics. However, there are some significant drawbacks in applying this method due to its sensitivity to noisy signals used as inputs to the high-gain observers and controllers and to its lack of asymptotic stability in the sense of Lyapunov since it only guarantees convergence to a vicinity of an equilibrium.

It is worth mentioning that both the observer and controller can be designed using different strategies, such as sliding mode, that do not involve high gain, provided that an extended observer is designed to estimate disturbances and unmodeled dynamics. Sliding mode control is one of the main strategies for the systematic design of robust controllers for uncertain dynamic systems. However, using this approach to control a system still suffers from disadvantages as the generation of high frequency vibrations or chattering, noise sensitivity, impossibility of rejection of unmatched disturbances, constraints on sliding surface design limited to relative degree one with respect to the control, and while the sliding variables achieve finite-time convergence, the state variables achieve only asymptotic convergence [140, 141].

In [142], Levant introduced the concept of high-order sliding mode (HOSM) to address issues

related to chattering and limitations imposed by the relative degree of the sliding surface and the control signal. He reported the first kind of HOSM controllers called nested due to its structure [142]. Then, in [143] he introduced the arbitrary order differentiator, which is useful to obtain the derivatives of the states needed in HOSM controllers. The combination of the arbitrary-order HOSM differentiator and an arbitrary-order HOSM controller enabled the creation of a versatile HOSM output feedback controller, in spite of disturbances or uncertainties in the model of the system ensuring finite time output stabilization. In this article we define HOSM in a context of disturbance rejection, allowing not only taking advantage of the best characteristics of both approaches, but also reducing the control signal and the chattering by rejecting the fast dynamics which is typically disregarded in idealized models. The article is divided into two main parts, the first is Section 7.3, showing how disturbance rejection can be attained using a high gain approach, and the second in Section 7.4 relating disturbance rejection with HOSM to improve its performance.

The rest of the paper is organized as follows. In Section 7.3.1 we find a bound for the matrix exponential, which will be used in Section 7.3.2 to enunciate a theorem which gives conditions to obtain bounded solutions of perturbed linear systems. Then we use these results in Section 7.3.3 to design an extended observer and estimate the generalized disturbance which lump together external disturbances and unmodeled dynamics. In Section 7.3.4 we show how to cancel the effects of the generalized disturbance using the gathered estimate as part of the feedback control action. In Section 7.4.1 we review HOSM differentiators and observers. Then, in Section 7.4.2 we learn about a very important class of HOSM controllers. Section 7.4.3 introduces the extended HOSM observer used in our approach, and a theorem that state the reduction of the control gain in the HOSM controllers by using disturbance rejection. Finally, in Section 7.5 we show some application examples in order to highlight the advantages of the proposed method in controlling chaotic MEMS resonators.

### 7.3 Disturbance Rejection Using High-Gain Control

The main results of this section concern the  $n$ -dimensional differential equation:

$$\dot{y} = Ay + g(t, y), \quad \text{with} \quad y(t_0) = y_0, \quad (7.1)$$

where  $A$  represents a constant matrix of size  $n \times n$ , and  $g : \Omega \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , with  $\Omega$  an open set of  $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$  including the origin. The point  $(t_0, y_0) \in \Omega$  is the initial condition of the initial value problem given by (7.1). The functions  $g(t, y)$  and its derivative  $\frac{d}{dy}g(t, y)$  are assumed continuous in  $\Omega$ , which guarantees the existence and unicity of the solution of

(7.1).

We will show how the system (7.1) may be stabilized in a neighborhood of its origin, by properly selecting the position of the eigenvalues of the system matrix  $A$ , in spite of the function  $g(t, y)$ , which is considered as a bounded disturbance. We will first introduce some preliminary results regarding bounds for the matrix exponential and bounded solutions of perturbed linear systems, before we apply them in a high-gain disturbance rejection control context in Sections 7.3.3 and 7.3.4.

### 7.3.1 Bound of the Matrix Exponential

In order to find a bound of the matrix exponential, we will use the following result [144]. Suppose we have a constant matrix  $A$  of size  $n \times n$ . Let  $\lambda_1, \lambda_2, \dots, \lambda_k$  represent the unique eigenvalues of  $A$ , each with their respective multiplicity values of  $n_1, n_2, \dots, n_k$ . It is important to note that the sum of these multiplicities,  $n_1 + n_2 + \dots + n_k$ , always equals the matrix size, which is  $n$ .

For each eigenvalue  $\lambda_j$  with a multiplicity of  $n_j$ , we can establish a system of linear equations

$$(A - \lambda_j I)^{n_j} x = 0, \quad (j = 1, 2, \dots, k), \quad (7.2)$$

which solutions, for each such linear systems, span a subspace of  $\mathbb{R}^n$  called  $V_j$  ( $j = 1, 2, \dots, k$ ). Then for each  $x \in \mathbb{R}^n$ , there exists unique vectors  $x_1, x_2, \dots, x_k$ , where  $x_j \in V_j$ , such that  $x = x_1 + x_2 + \dots + x_k$ . Then we have the following result.

**Theorem 7.3.1.** *Given a linear system*

$$\dot{y} = Ay \quad (7.3)$$

*satisfying the initial condition  $y(0) = v_1 + v_2 + \dots + v_k$ , where  $v_j \in V_j$ , ( $j = 1, 2, \dots, k$ ). Then the solution  $y(t)$  of (7.3) that satisfies the initial condition can be expressed as*

$$y(t) = \sum_{j=1}^k e^{\lambda_j t} \left[ \sum_{i=0}^{n_j-1} \frac{t^i}{i!} (A - \lambda_j I)^i \right] v_j, \quad -\infty < t < \infty. \quad (7.4)$$

*Proof.* For a proof of this theorem, see for example [144]. □

We will evaluate the bound for a companion matrix  $A$ .

**Theorem 7.3.2.** Define a  $n \times n$  companion matrix  $A$  of the polynomial

$$f(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n, \quad (7.5)$$

which has a zero of multiplicity  $n$  at  $-\lambda$ ,  $\lambda > 0$ . Then, given  $T > 0$ , there exists  $\lambda$ , and constants  $\sigma = \frac{\lambda}{2}$  and  $K > 0$ , such that

$$|e^{At}| \leq K e^{-\sigma t}, \quad t \geq T. \quad (7.6)$$

*Proof.* Using Theorem 7.3.1, the solution of (7.3) becomes

$$y(t) = e^{-\lambda t} \left[ \sum_{i=0}^{n-1} \frac{t^i}{i!} (A + \lambda I)^i \right] \eta, \quad -\infty < t < \infty. \quad (7.7)$$

Since  $A$  is a companion matrix, all its elements are constants or powers of  $\lambda$ . Then every element of the matrix exponential is in general a term of the form  $p(t, \lambda) e^{(-\lambda t)}$ , where  $p(t, \lambda)$  is a polynomial of  $t$  and  $\lambda$ .

Now, it is possible to show that there exists  $\lambda$  such that

$$|p(t, \lambda)| e^{-\lambda t} < e^{-\sigma t}, \quad \text{for } t \geq T. \quad (7.8)$$

In order to do so, it is enough to prove that

$$\lim_{(\lambda, t) \rightarrow (\infty, \infty)} |p(t, \lambda)| e^{-(\lambda - \sigma)t} = 0.$$

We will use an increasing sequence that converges to  $e^{\lambda t}$ :

$$a_n = \left( 1 + \frac{\lambda t}{n} \right)^n.$$

Define

$$p(t, \lambda) = \sum_{i=0}^r \sum_{j=0}^s a_{ij} (\lambda^i t^j),$$

and take

$$N = \sum_{i=0}^r \sum_{j=0}^s |a_{ij}|,$$

and  $n = \max(r, s) + 1$ . Assume that  $(\lambda t > 1)$ , then one has:

$$\begin{aligned}
\frac{|p(t, \lambda)|}{e^{(\lambda-\sigma)t}} &\leq \frac{N(\lambda t)^{n-1}}{\left(1 + \frac{(\lambda-\sigma)t}{n}\right)^n} = \frac{N\lambda^{n-1}t^{n-1}}{\left(1 + \frac{\lambda t}{2n}\right)^n} \\
&= \frac{N}{\lambda t} \left( \frac{(\lambda t)^n}{(2n + \lambda t)^n} \right) (2n)^n \leq \frac{N}{\lambda t} (2n)^n,
\end{aligned} \tag{7.9}$$

and the limit is zero by squeezing ( $(2n)^n$  can be huge but is constant).

Finally, pick the larger lambda that guarantees that (7.8) is satisfied for every element of the matrix exponential. Then (7.6) holds with  $K$  equal to  $n$ .

□

**Remark 7.1.** *The matrix  $A$  in Theorem 7.3.2 is chosen to be a companion matrix for the sake of simplicity in the presentation of the results, and also because the matrices in the following sections are presented in this form, although the result is valid for a more general structure.*

### 7.3.2 Bounded Solutions of Perturbed Linear Systems

Under appropriate conditions it is possible to obtain bounded solutions of system (7.1). In this section we will show how, for an  $\epsilon$ -vicinity and a time larger than a given  $T$ , the solution is bounded by  $|y(t)| \leq \epsilon$  ( $t \geq T$ ). For this purpose we establish the following theorem.

**Theorem 7.3.3.** *Let system (7.1), where  $A$  is a  $n \times n$  companion matrix with an eigenvalue of multiplicity  $n$  at  $-\lambda$ ,  $\lambda > 0$ , that can be changed by changing the variables entries of  $A$ , and  $|g(t, y)| \leq \eta$ . Given  $\epsilon > 0$  and  $T > t_0$ . Then, for  $y_0$  satisfying  $|y_0| < R$ ,  $R > 0$ , there exist  $\lambda$  such that*

$$|y(t)| \leq \epsilon, \quad (t \geq T). \tag{7.10}$$

*Proof.* The continuity of  $g(t, y)$  and its derivative guarantee the existence and unicity of solutions of (7.1). We can express the solution, by using the variation of constants formula, by means of the equation:

$$y(t) = e^{A(t-t_0)}y_0 + \int_{t_0}^t e^{A(t-s)}g(s, y)ds. \tag{7.11}$$

By using the hypothesis on  $A$  and Theorem 7.3.2, there exist constants  $\sigma > 0$  and  $K > 0$

such that  $e^{A(t-t_0)}$  satisfies

$$\left| e^{A(t-t_0)} \right| \leq K e^{-\sigma(t-t_0)}, \quad (t_0 \leq t < \infty). \quad (7.12)$$

Then, by taking norms to both sides of equation (7.11) and by using (7.12) and the triangle inequality:

$$|y(t)| \leq K e^{-\sigma(t-t_0)} |y_0| + K \int_{t_0}^t e^{-\sigma(t-s)} |g(s, y)| ds. \quad (7.13)$$

Besides,  $|g(t, y)| < \eta$  for  $t \geq t_0$ , so (7.13) reduces to

$$|y(t)| \leq K e^{-\sigma(t-t_0)} |y_0| + K \eta e^{-\sigma t} \left( \frac{e^{\sigma t} - e^{\sigma t_0}}{\sigma} \right), \quad (7.14)$$

and finally

$$|y(t)| \leq K e^{-\sigma(t-t_0)} |y_0| + \frac{K}{\sigma} \eta \left( 1 - e^{-\sigma(t-t_0)} \right). \quad (7.15)$$

Then, the solution of the system is bounded by

$$|y(t)| \leq f(t, \sigma), \quad (7.16)$$

where

$$f(t, \sigma) = K e^{-\sigma(t-t_0)} |y_0| + \frac{K}{\sigma} \eta \left( 1 - e^{-\sigma(t-t_0)} \right). \quad (7.17)$$

This function is well defined since  $\sigma > 0$  by hypothesis.

Now, choose  $\delta_1, \delta_2 > 0$  such that

$$\epsilon = \delta_1 + \delta_2,$$

and define

$$f(t, \sigma) = f_1(t, \sigma) + f_2(t, \sigma),$$

where

$$f_1(t, \sigma) = K e^{-\sigma(t-t_0)} |y_0|,$$

and

$$f_2(t, \sigma) = \frac{K}{\sigma} \eta \left( 1 - e^{-\sigma(t-t_0)} \right).$$

Take  $\delta_1$  and  $T$ . Since  $f_1(t, \sigma)$  decreases exponentially as  $\sigma$  increases, there exists  $\sigma_1$  such that

$f_1(t, \sigma_1) < \delta_1$  for  $t > T$ .

Take  $\delta_2$ . The expression between parentheses in  $f_2(t, \sigma)$  has a maximum value of 1, and  $\frac{K}{\sigma}\eta$  decreases as  $\sigma$  increases. Therefore, there exists  $\sigma_2$  such that  $f_2(t, \sigma_2) < \delta_2$  for  $t > T$ .

Define  $\sigma_e = \max\{\sigma_1, \sigma_2\}$ , such that it satisfies simultaneously  $f_1(t, \sigma_e) < \delta_1$  and  $f_2(t, \sigma_e) < \delta_2$  for  $t > T$ .

Then for  $\sigma_e$  we have:

$$f_1(t, \sigma_e) + f_2(t, \sigma_e) < \delta_1 + \delta_2, \quad t > T,$$

or

$$f(t, \sigma_e) < \epsilon, \quad t > T.$$

□

Now we are prepared to present the first result regarding the design of high gain extended observers. In what follows, and without loss of generality, we will assume that  $t_0 = 0$ .

### 7.3.3 High-Gain Extended Observers Design

**Theorem 7.3.4.** *Given a  $n$ -order nonlinear differential equation*

$$y^{(n)} = \psi(y(t), \dot{y}(t), \dots, y^{(n-1)}(t)) + w(t), \quad (7.18)$$

where  $\psi(\cdot)$  and  $w(t)$  are bounded functions along with its first derivatives. Define the variables:  $y(t) = y_1, \dot{y}(t) = y_2, \dots, y^{(n-1)}(t) = y_n$ , such that system (7.18) may be written as a system of  $n$  first-order differential equations

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ &\vdots \\ \dot{y}_n &= \xi, \end{aligned} \quad (7.19)$$

where  $\xi(t) = \psi(\cdot) + w(t)$ . Define also the following extended observer

$$\begin{aligned}
 \dot{\hat{y}}_1 &= \hat{y}_2 + \alpha_n (y_1 - \hat{y}_1) \\
 \dot{\hat{y}}_2 &= \hat{y}_3 + \alpha_{n-1} (y_1 - \hat{y}_1) \\
 &\vdots \\
 \dot{\hat{y}}_n &= \hat{\xi} + \alpha_1 (y_1 - \hat{y}_1) \\
 \dot{\hat{\xi}} &= \alpha_0 (y_1 - \hat{y}_1),
 \end{aligned} \tag{7.20}$$

where  $\alpha_0, \alpha_1, \dots, \alpha_{n-1}, \alpha_n$  are constants, and the extended estimation error  $\tilde{e}(t)$  as the vector  $(y_1 - \hat{y}_1, y_2 - \hat{y}_2, \dots, y_n - \hat{y}_n, (\xi - \hat{\xi}))^\top$ .

Then, given  $\epsilon > 0$  and  $T_1 > 0$ , there exist constants  $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ , and  $\alpha_n$ , such that

$$|\tilde{e}_1| \leq \epsilon \quad \text{for } t \geq T_1. \tag{7.21}$$

*Proof.* The dynamic of the estimation error is:

$$\begin{aligned}
 \dot{\tilde{e}}_1 &= \tilde{e}_2 - \alpha_n \tilde{e}_1 \\
 \dot{\tilde{e}}_2 &= \tilde{e}_3 - \alpha_{n-1} \tilde{e}_1 \\
 &\vdots \\
 \dot{\tilde{e}}_n &= (\xi - \hat{\xi}) - \alpha_1 \tilde{e}_1 \\
 \dot{\xi} - \dot{\hat{\xi}} &= \xi' - \alpha_0 \tilde{e}_1.
 \end{aligned} \tag{7.22}$$

Defining the following vector of dimension  $(n+1)$ :

$$g(t) = (0, 0, \dots, 0, \dot{\xi}(t))^\top, \tag{7.23}$$

and the companion form matrix

$$A = \begin{bmatrix} -\alpha_n & 1 & 0 & 0 & \cdots & 0 \\ -\alpha_{n-1} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & \\ -\alpha_1 & 0 & 0 & 0 & \cdots & 1 \\ -\alpha_0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{7.24}$$

then system (7.22) becomes

$$\dot{\tilde{e}} = A\tilde{e} + g(t). \quad (7.25)$$

This equation has the same structure of (7.1) with  $g(t)$  bounded since  $\dot{\xi}$  is also bounded by hypothesis. Additionally, from (7.24),  $A$  has the following characteristic polynomial:

$$s^{(n+1)} + \alpha_n s^{(n)} + \alpha_{n-1} s^{(n-1)} + \alpha_1 s^{(1)} + \alpha_0 s = 0, \quad (7.26)$$

whose coefficients may be selected in such a way that its roots have negative real parts and are arbitrarily far from the imaginary axis. Then  $A$  and  $g(t)$  satisfy the hypothesis of Theorem 7.3.3. Therefore, given  $\epsilon > 0$  and  $T_1 > 0$ , there exist  $s = \lambda$ , and hence constants  $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ , and  $\alpha_n$ , such that

$$|\tilde{e}_1| \leq \epsilon, \quad (t \geq T_1). \quad (7.27)$$

□

### 7.3.4 Active Disturbance Rejection Control

Active disturbance rejection control concerns cancelling the external disturbance by injecting the estimated disturbances obtained by an observer as designed above into the control signal.

The control aims to track a reference trajectory  $y_r(t)$ , which can be accomplished by ensuring that the tracking error approaches asymptotically to a neighborhood of zero:

$$|y(t) - y_r(t)| \leq \epsilon, \quad t \geq T, \quad (7.28)$$

where  $\epsilon$  is an arbitrary small constant and  $T > 0$ .

**Theorem 7.3.5.** *Let the dynamic system*

$$y^{(n)} = \xi(w(t), y(t), \dot{y}(t), \dots, y^{(n-1)}(t)) + \phi(t, y) u, \quad (7.29)$$

*and the constants  $0 < T_1 < T$ , where  $\xi$  is a generalized disturbance depending on time, the output, and some of its derivatives, and  $u(t)$  is an input signal depending on the output. If  $\xi$  is bounded along with its time derivative, and  $u(t)$  is given by*

$$u(t) = \begin{cases} 0, & \text{for } t < T_1 \\ \frac{1}{\phi} \left[ y_r^{(n)} - \sum_{j=1}^n \beta_j (\hat{y}_j - y_r^{(j-1)}) - \hat{\xi} \right], & \text{for } t \geq T_1 \end{cases}, \quad (7.30)$$

where  $y_r$  is the reference signal and  $\hat{y}_j$  the estimated values of  $y^{(j-1)}$ , given by the extended observer

$$\begin{aligned} \dot{\hat{y}}_1 &= \hat{y}_2 + \alpha_n (y_1 - \hat{y}_1) \\ \dot{\hat{y}}_2 &= \hat{y}_3 + \alpha_{n-1} (y_1 - \hat{y}_1) \\ &\vdots \\ \dot{\hat{y}}_n &= \hat{\xi} + \alpha_1 (y_1 - \hat{y}_1) + \phi(t, y) u \\ \dot{\hat{\xi}} &= \alpha_0 (y_1 - \hat{y}_1), \end{aligned} \quad (7.31)$$

Then, for every  $\epsilon > 0$  and  $T > T_1$ , there exist constants  $\beta_i$  and  $\alpha_j$ ,  $i = 1, 2, \dots, n$  and  $j = 0, 2, \dots, n$ , such that

$$|x - y_r| < \epsilon, \quad \text{for } t > T. \quad (7.32)$$

*Proof.* First we will replace the value of  $u$  in equation (7.29), so we obtain

$$\begin{aligned} y^{(n)} &= y_r^{(n)} - \beta_1 (\hat{y}_1 - y_r) - \beta_2 (\hat{y}_2 - y_r^{(1)}) - \dots \\ &\quad - \beta_n (\hat{y}_n - y_r^{(n-1)}) - \hat{\xi} + \xi \end{aligned} \quad (7.33)$$

and defining the following tracking error

$$\begin{aligned} e_r(t) &= (e_1, e_2, \dots, e_n, )^\top \\ &= (y_r - y_1, y_r^{(1)} - y_2, \dots, y_r^{(n-1)} - y_n)^\top, \end{aligned} \quad (7.34)$$

then the error dynamics are given as

$$\begin{aligned} &e_1^{(n)} + \beta_n e_1^{(n-1)} + \beta_{n-1} e_1^{(n-2)} + \dots + \beta_2 e_1^{(1)} + \beta_1 e_1 \\ &= \beta_n \tilde{e}_1^{(n-1)} + \beta_{n-1} \tilde{e}_1^{(n-2)} + \dots + \beta_2 \tilde{e}_1^{(1)} + \beta_1 \tilde{e}_1 + \xi - \hat{\xi}. \end{aligned} \quad (7.35)$$

This  $n$ -order differential equation can be expressed as a system of  $n$ -differential equations

$$\begin{aligned}
 \dot{e}_1 &= e_2 \\
 \dot{e}_2 &= e_3 \\
 &\vdots \\
 \dot{e}_{(n-1)} &= e_n \\
 \dot{e}_n &= -\beta_n e_1^{(n-1)} - \beta_{n-1} e_1^{(n-2)} - \cdots - \beta_2 e_1^{(1)} - \beta_1 e_1 \\
 &\quad + \beta_n \tilde{e}_1^{(n-1)} + \beta_{n-1} \tilde{e}_1^{(n-2)} + \cdots + \beta_2 \tilde{e}_1^{(1)} + \beta_1 \tilde{e}_1 + \xi - \hat{\xi}
 \end{aligned} \tag{7.36}$$

and defining the following vector of dimension  $n$ :

$$g(t) = (0, 0, \dots, 0, g_1(t))^T, \tag{7.37}$$

with

$$g_1(t) = \beta_n \tilde{e}_1^{(n-1)} + \beta_{n-1} \tilde{e}_1^{(n-2)} + \cdots + \beta_2 \tilde{e}_1^{(1)} + \beta_1 \tilde{e}_1 + \xi - \hat{\xi}, \tag{7.38}$$

and the companion-form matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -\beta_1 & -\beta_2 & 0 & 0 & \cdots & -\beta_n \end{bmatrix}, \tag{7.39}$$

then system (7.36) becomes

$$\dot{e}_r = A e_r + g(t). \tag{7.40}$$

By using Theorem 7.3.3, and given  $\epsilon_1 > 0$ , there exist constants  $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ , and  $\alpha_n$ , such that for  $t > T_1$ , the estimation error  $|\tilde{e}| < \epsilon_1$ . By using a  $L^1$ -norm:

$$|\tilde{e}| = |y_1 - \hat{y}_1| + |y_2 - \hat{y}_2| + \cdots + |y_n - \hat{y}_n| + |\xi - \hat{\xi}| < \epsilon_1, \tag{7.41}$$

we have

$$|g(t)| < \beta_1 |y_1 - \hat{y}_1| + \beta_2 |y_2 - \hat{y}_2| + \cdots + \beta_n |y_n - \hat{y}_n| + |\xi - \hat{\xi}|$$

$$\begin{aligned}
&<\beta \left( |y_1 - \hat{y}_1| + |y_2 - \hat{y}_2| + \cdots + |y_n - \hat{y}_n| + |\xi - \hat{\xi}| \right) \\
&<\beta\epsilon_1,
\end{aligned} \tag{7.42}$$

which implies that  $g(t)$  should be bounded.

Additionally, from (7.39),  $A$  has the following characteristic polynomial:

$$s^{(n+1)} + \beta_n s^{(n)} + \beta_{n-1} s^{(n-1)} + \beta_1 s^{(1)} + \beta_0 s = 0, \tag{7.43}$$

whose coefficients may be selected in such a way that its roots are negative and arbitrarily far from the imaginary axis. Then,  $A$  and  $g(t)$  satisfy the hypothesis of Theorem 7.3.3. Therefore, given  $\epsilon > 0$ , there exist  $s = \lambda$ , and hence constants  $\beta_0, \beta_1, \dots, \beta_{n-1}$ , and  $\beta_n$ , such that

$$|e_r| \leq \epsilon, \quad (t \geq T). \tag{7.44}$$

□

## 7.4 High-Order Sliding Mode Control

Introducing the notation

$$\sigma_k = (\sigma, \dot{\sigma}, \dots, \sigma^{(k)})^\top, \quad k \in \mathbb{N}, \tag{7.45}$$

the relative degree  $r$  associated with the sliding variable  $\sigma$  is determined by the presence of the control term within  $\sigma^{(r)}$ . The objective of the high-order sliding mode control (HOSMC) is reaching  $\sigma_{r-1} = 0$  in finite time using discontinuous control. By proper design, it improves precision and maintains  $\sigma = 0$ , and also diminishes chattering by shifting the most significant chattering component to  $\sigma^{r-1}$  [145].

Now we describe the objective to be achieved by using HOSMC. Consider an uncertain dynamic system, that is smooth and affine in the input

$$\begin{aligned}
\dot{x} &= a(t, x) + b(t, x)u \\
y &= \sigma(t, x),
\end{aligned} \tag{7.46}$$

where  $x \in \mathbb{R}^n$  is the state vector,  $u \in \mathbb{R}$  is the control input, and  $a(t, x), b(t, x), \sigma(t, x) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  are smooth functions. The vector fields  $a(t, z)$  and  $b(t, z)$  are subject to un-

certainty, and the dimension  $n$  may also be unspecified. Suppose that only the output  $\sigma$  is available in real time. Our objective is to achieve the constraint  $\sigma$  reaching zero within a finite time period, i.e.,  $\sigma(t) = 0, \forall t \geq T$ , using a suitable control input  $u(t)$ , in spite of uncertainties and disturbances.

When we know the system's relative degree  $r$ , such problem becomes equivalent to designing a controller for the system

$$\sigma^{(r)} = h(t, x) + g(t, x)u, \quad (7.47)$$

where a set  $\mathcal{S}$  of constraints  $\sigma = \dot{\sigma} = \dots = \sigma^{(r-1)} = 0$  must be guaranteed. The motion within this set is referred to as operating in an  $r$ -sliding ( $r$ -th order sliding) mode. While the scalar functions  $h(t, z)$  and  $g(t, z)$  can have uncertainty, it is assumed that they remain uniformly bounded for all  $x \in \mathbb{R}^n$  and every  $t \geq 0$

$$0 < K_m \leq g(t, x) \leq K_M, \quad |h(t, x)| \leq C, \quad (7.48)$$

for certain specified positive constants  $C$ ,  $K_M$ , and  $K_m$ .

Under the uncertainties considered,  $C \neq 0$ , there is no continuous output feedback control that can stabilize system (7.47). The control  $u$  must exhibit discontinuity at least on the set  $\mathcal{S}$ , that is to say, we need to establish a motion along the sliding surfaces where conditions  $\sigma = \dot{\sigma} = \dots = \sigma^{(n-1)} = 0$  are met, and this motion should be composed of Filippov trajectories.

Considering the uncertainties of system (7.47), its solutions satisfy the inclusion

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M] u. \quad (7.49)$$

The stated problem can be addressed by designing a control law, denoted as

$$u = \varphi(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}), \quad (7.50)$$

in a way that ensures all trajectories of (7.49) converge to the origin of the  $r$ -sliding phase space  $\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}$  in finite time.

### 7.4.1 HOSM Differentiators and Observers

Given a signal  $f(t)$ , the differentiation problem consists of obtaining an estimate of its successive derivatives  $f^{(i)}(t)$ , for  $i = 1, \dots, n-1$ , assuming a Lipschitz constant  $L > 0$  for the  $n$ -th derivative of the function  $f(t)$ , i.e.,  $|f^{(n)}(t)| \leq L$  for  $t \geq 0$ .

By introducing state variables  $x_1 = f(t), x_2 = \dot{f}(t), \dots, x_n = f^{(n-1)}(t)$ , we obtain the state representation

$$\begin{aligned} \dot{x}_i &= x_{i+1}, & i &= 1, \dots, n-1 \\ \dot{x}_n &= f^{(n)}(t). \end{aligned} \quad (7.51)$$

To compute the signal's derivatives, consider the  $(n-1)$ -th order homogeneous discontinuous exact differentiator defined by Levant [143], which takes the following non recursive form

$$\begin{aligned} \dot{\hat{x}}_i &= -\lambda_i L^{\frac{i}{n}} [\hat{x}_1 - x_1]^{\frac{n-i}{n}} + \hat{x}_{i+1}, & i &= 1, \dots, n-1 \\ \dot{\hat{x}}_n &= -\lambda_n L [\hat{x}_1 - x_1]^0, \end{aligned} \quad (7.52)$$

where  $\hat{x}_i$  is the estimation of the  $(i-1)$ -th derivative of  $f(t)$ ,  $\lambda_i$  are a set of constants chosen appropriately [143], and the notation  $[\cdot]^p$  stands for  $[z]^p = |z| \text{sign } z$ , where it represents the sign-preserving power  $p$  of a real variable  $z \in \mathbb{R}$ , and  $p$  is a real number belonging to  $\mathbb{R}$ .

A major advantage of sliding mode is that a sliding mode observer can ensure that the output estimation error converges to zero in finite time.

Given a  $n$ -th order system

$$x^{(n)} = h(t, x), \quad (7.53)$$

and defining the state variables  $x_1 = x, x_2 = \dot{x}(t), \dots, x_n = x^{(n-1)}(t)$ , we obtain its state representation

$$\begin{aligned} \dot{x}_i &= x_{i+1}, & i &= 1, 2, \dots, n-1 \\ \dot{x}_n &= h(t, x). \end{aligned} \quad (7.54)$$

If the function  $h(t, x)$  has a Lipschitz constant  $L > 0$ , i.e.,  $|h(t, x)| \leq L$  for  $t \geq 0$ , and  $y = x_1$  represents the output of the system (7.53), the problem of estimating the states for system (7.54) can be seen as the problem of differentiating the signal  $f(t)$  in (7.51). In this context,  $\hat{x}_i$  is the estimate of the  $(i-1)$ -th state of  $x(t)$ .

**Theorem 7.4.1.** *Given the differentiation system (observer) defined by (7.51) (by (7.54)).*

For the class of Lipschitz signal (system), satisfying  $|f^{(n)}(t)| \leq L$  ( $|h(t, x)| \leq L$ ), there exist appropriate gains  $\lambda_i > 0$ ,  $i = 1, \dots, n$  such that the differentiation (state) estimation error dynamics at the origin exhibits finite-time stability.

There are two primary approaches for proving the convergence of the observer, both relying on the homogeneity properties of the DI. One approach adopts a geometric perspective, while the other is more engineering-oriented and employs Lyapunov functions.

### 7.4.2 HOSM Controllers

In [146] it is shown how different families of controllers can be derived for (7.49) using Lyapunov functions. In particular, it is obtained the family of discontinuous controllers

$$u_{\rho D} = -k_{\rho} \varphi_D(x) = -k_{\rho} [\sigma_{\rho}(x)]^0. \quad (7.55)$$

One kind of controllers is known as Relay Polynomial Controllers. They take the following forms when the relative degree  $\rho$  is equal to 2, 3 and 4 respectively:

$$\begin{aligned} u_{2R} &= -k_2 \operatorname{sign} \left[ [x_2]^{2+\alpha} + \bar{k}_1 [x_1]^{\frac{2+\alpha}{2}} \right] \\ u_{3R} &= -k_3 \operatorname{sign} \left[ [x_3]^{3+\alpha} + \bar{k}_2 [x_2]^{\frac{3+\alpha}{2}} + \bar{k}_1 [x_1]^{\frac{3+\alpha}{3}} \right] \\ u_{4R} &= -k_4 \operatorname{sign} \left[ [x_4]^{4+\alpha} + \bar{k}_3 [x_3]^{\frac{4+\alpha}{2}} + \bar{k}_2 [x_2]^{\frac{4+\alpha}{3}} + \bar{k}_1 [x_1]^{\frac{4+\alpha}{4}} \right]. \end{aligned} \quad (7.56)$$

In these equations  $\alpha \geq 0$ , and for a general  $\rho$ ,  $\bar{k}_i = \prod_{j=i}^{\rho-1} k_j^{\frac{\rho+\alpha}{\rho-j}}$  for  $i = 1, \dots, \rho - 1$ . The general result is enunciated in the following theorem [146].

**Theorem 7.4.2.** *Every controller from the set of discontinuous controllers (7.55), for any  $\rho \geq 2$ , exhibits  $\rho$ -sliding homogeneity. Additionally, for sufficiently large values of  $k_{\rho}$ , the  $\rho$ -th order sliding mode can be achieved in finite time for the uncertain system (7.47), given appropriately selected gains  $k_1, \dots, k_{\rho-1}$ .*

### 7.4.3 Enhanced High-Order Sliding Mode Control

Enhanced sliding mode concerns canceling the uncertain function  $h(t, x)$  of (7.47) by estimating its value by means of an extended observer (see Section 7.4.1), and then inducing a behavior in the system by using a sliding mode controller, as the one in (7.56). The gains of the obtained controller may be smaller than the gains in (7.56), diminishing the impact of noise on the controlled system. Another effect of canceling  $h(t, x)$  is the decrease of the

amplitude of uncertainties in the controlled system and its finite time convergence. These results are enunciated in the following theorem.

**Theorem 7.4.3.** *Let the dynamic system (7.47), and scalar functions  $\hat{h}(t, x)$  and  $\hat{g}(t, x)$  converging in finite time to  $h(t, x)$  and  $g(t, x)$  respectively, with the homogeneous and degree 0 control input*

$$u = -\frac{1}{\hat{g}(t, x)}\hat{h}(t, x) + \varphi(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}), \quad (7.57)$$

where  $\hat{g}(t, x)$  is assumed to be positive, and  $\varphi(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)})$  is an homogeneous function of degree 0 given by (7.55). Then the minimum value of the control gain  $k_\rho$  of the HOSM controller  $\varphi$  that stabilizes (7.47) may be reduced.

*Proof.* With  $u$  given in (7.57), system (7.47) may be written as

$$\sigma^{(r)} = \left[ h(t, x) - \frac{g(t, x)}{\hat{g}(t, x)}\hat{h}(t, x) \right] + g(t, x) \varphi(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}), \quad (7.58)$$

and given the boundedness of functions  $h(t, x)$ ,  $g(t, x)$ ,  $\hat{h}(t, x)$  and  $\hat{g}(t, x)$ , its solution satisfy the inclusion

$$\sigma^{(r)} \in [-C_1, C_1] + [K_m, K_M] \varphi(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}) \quad (7.59)$$

for  $C_1 > 0$  and  $K_m, K_M > 0$  defined in (7.48). According to Theorem 7.4.2, there exists constants  $k_1, \dots, k_\rho$  such that the  $r$ -order sliding mode is reached in finite time ( $\sigma = \dot{\sigma} = \dots = \sigma^{(r-1)} = 0$ ).

Once  $\hat{h}(t, x)$  and  $\hat{g}(t, x)$  approaches  $h(t, x)$  and  $g(t, x)$  respectively,  $C_1 = 0$ , and the gain  $k_\rho$ , given by [146]

$$k_\rho > \frac{1}{K_m}(\mu_\rho k_1^\rho + C_1) \quad (7.60)$$

for some positive constants  $\mu_\rho$  and  $k_1 > 0$ , becomes

$$k_\rho > \frac{1}{K_m}(\mu_\rho k_1^\rho), \quad (7.61)$$

that is,  $k_\rho$  decreases. □

The extended HOSM observer described below, is used to estimate the uncertain function  $h(t, x)$  in system (7.47). Defining the state variables  $x_1 = \sigma, x_2 = \dot{\sigma}, \dots, x_r = \sigma^{(r-1)}(t)$ , the

following representation of the system (7.47) is obtained

$$\begin{aligned}\dot{x}_i &= x_{i+1}, & i &= 1, \dots, r-1 \\ \dot{x}_r &= h(t, x) + g(t, x)u.\end{aligned}\tag{7.62}$$

If we additionally define the state  $x_{r+1}$  as  $x_{r+1} = h(t, x)$ , then we get the following extended system

$$\begin{aligned}\dot{x}_i &= x_{i+1}, & i &= 1, \dots, r-1 \\ \dot{x}_r &= x_{r+1} + g(t, x)u \\ \dot{x}_{r+1} &= \dot{h}(t, x).\end{aligned}\tag{7.63}$$

Assuming that the function  $\dot{h}(t, x)$  has a Lipschitz constant  $L > 0$ , that is,  $|\dot{h}(t, x)| \leq L$  for  $t \geq 0$ , then the following extended observer

$$\begin{aligned}\dot{\hat{x}}_i &= -\lambda_i L^{\frac{i}{r+1}} [\hat{x}_1 - x_1]^{\frac{(r+1)-i}{(r+1)}} + \hat{x}_{i+1}, & i &= 1, \dots, r-1 \\ \dot{\hat{x}}_r &= -\lambda_r L^{\frac{r}{r+1}} [\hat{x}_1 - x_1]^{\frac{1}{(r+1)}} + \hat{x}_{r+1} + g(t, x)u, \\ \dot{\hat{x}}_{r+1} &= -\lambda_{r+1} L [\hat{x}_1 - x_1]^0\end{aligned}\tag{7.64}$$

is equivalent to the observer (7.52) if we define  $n = r + 1$ , for the extended estimation error

$$e_i \triangleq \hat{x}_i - x_i, \quad i = 1, \dots, r+1.\tag{7.65}$$

We obtain the following result.

**Theorem 7.4.4.** *Given the  $r$ -order system defined by (7.62). Assuming that the function  $\dot{h}(t, x)$  has a Lipschitz constant  $L > 0$ , that is,  $|\dot{h}(t, x)| \leq L$  for  $t \geq 0$ . Then, for the observer defined in (7.64), there exist gains  $\lambda_i > 0$ ,  $i = 1, \dots, r+1$  such that the origin of the extended estimation error dynamics is finite time stable.*

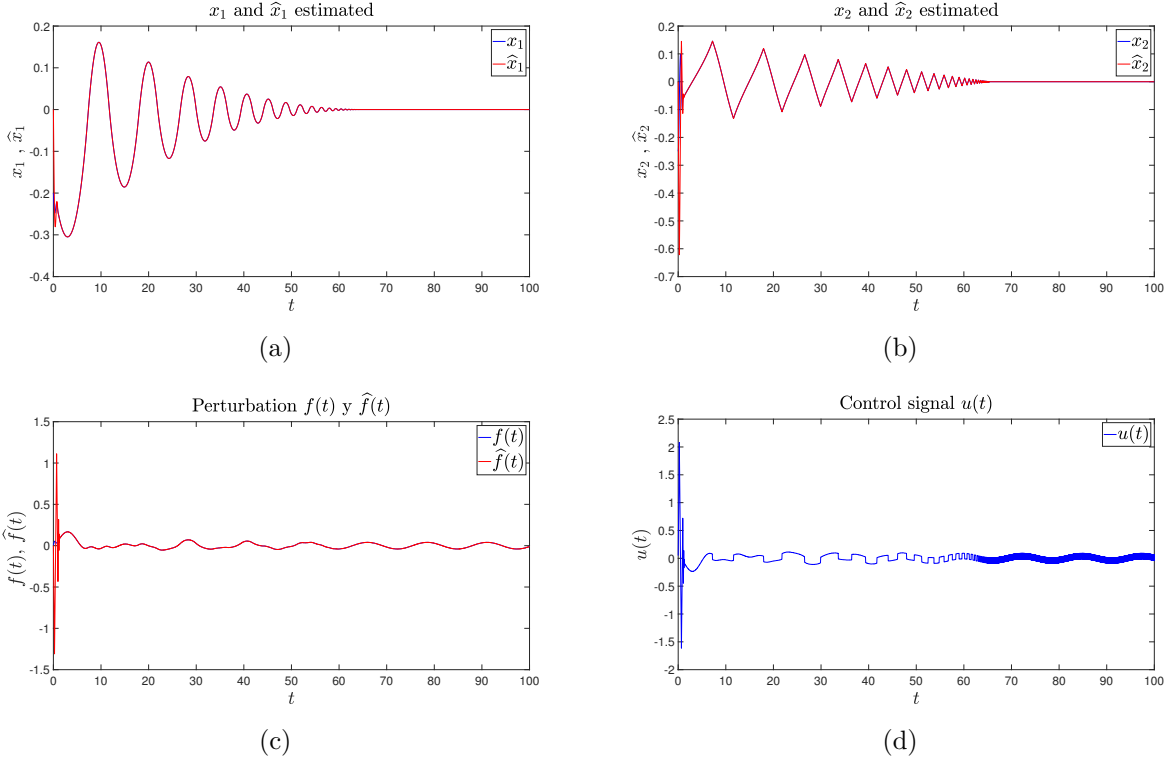


Figure 7.1 HOSM controller and extended observer applied to a MEMS.

(a) and (b), states  $x_1$ ,  $x_2$  and their estimated values  $\hat{x}_1$ ,  $\hat{x}_2$ . (c) Disturbance  $f(t)$  and its estimated value  $\hat{f}(t)$ . (d) Control signal  $u(t)$ .

## 7.5 Application to the Control of Chaotic MEMS Resonators

### 7.5.1 Regulation Control of Chaotic MEMS Resonators

With the aim of illustrating the application of the method developed above, consider the chaotic MEMS [147], written in state space form as

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\mu\dot{x}_2 - \sigma x_1 - \eta x_1^3 + f(x_1) + \frac{\tilde{u}}{(1-x_1)^2},\end{aligned}\tag{7.66}$$

where

$$f(x_1) = \delta \left( \frac{1}{(1-x_1)^2} - \frac{1}{(1+x_1)^2} \right),\tag{7.67a}$$

$$\tilde{u} = A \sin(\omega t) + u,\tag{7.67b}$$

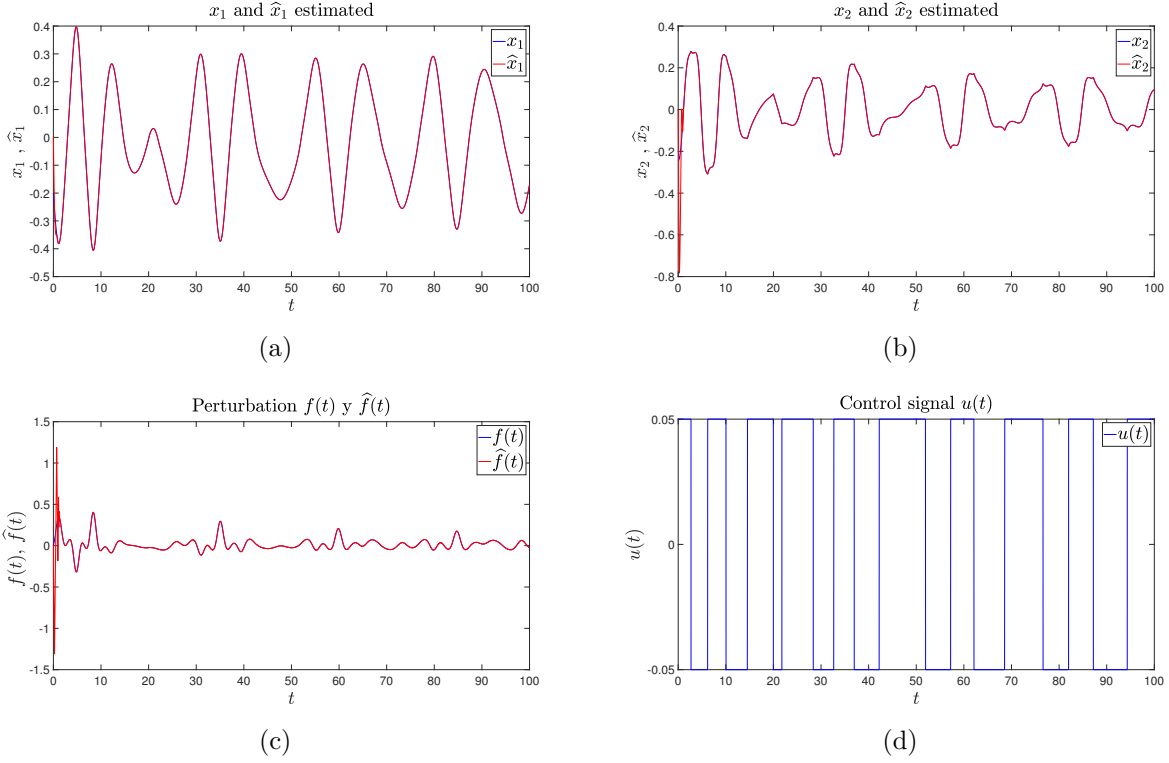


Figure 7.2 HOSM controller with output feedback applied to a MEMS.

(a) and (b), states  $x_1$ ,  $x_2$  and their estimated values  $\hat{x}_1$ ,  $\hat{x}_2$ . (c) Disturbance  $f(t)$  (d) Control signal  $u(t)$ .

and the parameters take the values  $\delta = 0.338$ ,  $\mu = 0.01$ ,  $\sigma = 1$ ,  $\eta = 12$ ,  $A = 0.04$ , and  $\omega = 0.5$ . The objective of the control strategy is to achieve finite-time stabilization of the system at the origin, even in the presence of model uncertainties and external disturbances. The output of the system ( $x_1 = \sigma$ ) has relative degree  $\rho = 2$ , so we can represent the system as

$$\ddot{x}_1 = -\mu\dot{x}_2 - \sigma x_1 - \eta x_1^3 + f(x_1) + \frac{\tilde{u}}{(1 - x_1)^2} \quad (7.68)$$

or in terms of the sliding variable  $\sigma$

$$\sigma^{(2)} = h(t, x) + g(x) u, \quad (7.69)$$

where

$$h(t, x) = -\mu\dot{x}_2 - \sigma x_1 - \eta x_1^3 + \delta \left( \frac{1}{(1 - x_1)^2} - \frac{1}{(1 + x_1)^2} \right) + \frac{A \sin(\omega t)}{(1 - x_1)^2}, \quad (7.70a)$$

$$g(x) = \frac{1}{(1 - x_1)^2}. \quad (7.70b)$$

The HOSM extended observer was designed following equation (7.64)

$$\begin{aligned} \dot{\hat{x}}_1 &= -k_1 [\hat{x}_1 - x_1]^{\frac{2}{3}} + \hat{x}_2, \\ \dot{\hat{x}}_2 &= -k_2 [\hat{x}_1 - x_1]^{\frac{1}{3}} + \hat{x}_3 + g(t, x)u, \\ \dot{\hat{x}}_3 &= -k_3 [\hat{x}_1 - x_1]^0, \end{aligned} \quad (7.71)$$

with the following parameters:  $k_1 = 2 \times 5^{(1/3)}$ ,  $k_2 = 2.12 \times 5^{(2/3)}$ , and  $k_3 = 1.1 \times 5^{(3/3)}$ .

According to Theorem 7.4.3, we use the control signal

$$u = -\frac{1}{\hat{g}(t, x)} \hat{x}_3 + \varphi(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}), \quad (7.72)$$

where  $\varphi$  is the discontinuous relay polynomial of order  $\rho = 2$ ,

$$\varphi = -k \operatorname{sign} \left[ [x_2]^2 + [x_1] \right], \quad (7.73)$$

with gain  $k = 0.05$ . Figure 7.1 shows the resulting estimated variables, along with the estimated uncertain function  $\hat{h}(t, x) = \hat{x}_3$ . It is shown how the HOSM control given by Theorem 7.4.3 stabilizes in finite time the system, even though the gain of the controller is very small. In contrast, Figure 7.2 illustrates that the application of a relay polynomial control with a low gain is insufficient to stabilize the system. This result shows that the improvements obtained with the proposed method are significant.

### 7.5.2 Tracking Orbits in Chaotic MEMS Resonators

In this example we will track an orbit immersed in the chaotic MEMS described in Section 7.5.1. This orbit is unstable and is chosen surrounding the chaotic attractor with the aim of producing a high amplitude periodic output of the MEMS plates once stabilized, at the lowest energy consumption [147].

The orbit selected, shown in Fig. 7.3, was used as the reference path in the simulations presented below. It is represented through an approximation using its truncated Fourier series comprising up to five terms, and is expressed as follows

$$\begin{aligned} y_r = & c_0 + c_1 \cos(\omega_1 t) + d_1 \sin(\omega_1 t) + c_2 \cos(2\omega_1 t) + d_2 \sin(2\omega_1 t) \\ & + c_3 \cos(3\omega_1 t) + d_3 \sin(3\omega_1 t) + c_4 \cos(4\omega_1 t) + d_4 \sin(4\omega_1 t) \end{aligned}$$

$$+c_5 \cos(5\omega_1 t) + d_5 \sin(5\omega_1 t) \quad (7.74)$$

where

$$\begin{aligned} c_0 &= 0.0175400, \quad c_1 = -0.44980, \quad d_1 = -0.0005630, \quad c_2 = 0.011260, \\ d_2 &= 0.0008906, \quad c_3 = -0.02914, \quad d_3 = 3.76e - 0050, \quad c_4 = 0.000633, \\ d_4 &= 0.0002309, \quad c_5 = -0.00109, \quad d_5 = 8.519e - 005, \quad w_1 = 0.925600. \end{aligned}$$

By defining  $\sigma = x_1(t) - y_r$  as the sliding variable, the control objective is to render  $\sigma = 0$  in finite time in spite of perturbations and unknown dynamics, and reduce the controller gain in (7.55) to its lowest value.

The HOSM extended observer was designed following equation (7.64)

$$\begin{aligned} \dot{z}_1 &= -k_1 [z_1 - \sigma]^{\frac{2}{3}} + z_2, \\ \dot{z}_2 &= -k_2 [z_1 - \sigma]^{\frac{1}{3}} + z_3 + g(t, x)u, \\ \dot{z}_3 &= -k_3 [z_1 - \sigma]^0, \end{aligned} \quad (7.75)$$

with the parameters:  $k_1 = 2 \times 15^{(1/3)}$ ,  $k_2 = 2.12 \times 15^{(2/3)}$ , and  $k_3 = 1.1 \times 15^{(3/3)}$ . Theorem 7.4.3 is applied to stabilize in finite time the tracking error dynamics, and give us the control signal

$$u = -\frac{z_3}{\hat{g}(t, x)} + \varphi(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}), \quad (7.76)$$

where  $\varphi$  is the discontinuous relay polynomial of order  $\rho = 2$ ,

$$\varphi = -k \operatorname{sign} \left[ [z_2]^2 + [z_1] \right], \quad (7.77)$$

and  $k = 0.05$ . In the tracking error equation the perturbation is equal to  $h(t, x) - \ddot{y}_r$ , and is estimated by means of  $z_3$ . Figure 7.4 shows the states of the system  $x_1$ ,  $x_2$  and the reference signals  $y_r, \dot{y}_r$  for comparison purposes, along with the estimation of the uncertain function  $h(t, x) - \ddot{y}_r$ . It is shown how the HOSM control proposed in Theorem 7.4.3 stabilizes in finite time the system, despite the disturbances, uncertain dynamics and a very small gain of the controller, see Figures 7.4(d) and 7.4(e). In contrast, Figure 7.5 shows that using the relay polynomial control with a small gain is insufficient to stabilize the orbit. Once again the results confirm the relevance of the proposed method.

We could take advantage of the known dynamics, and eliminate it exactly without estimating it, but we want to illustrate in the examples the extreme cases when the model is almost

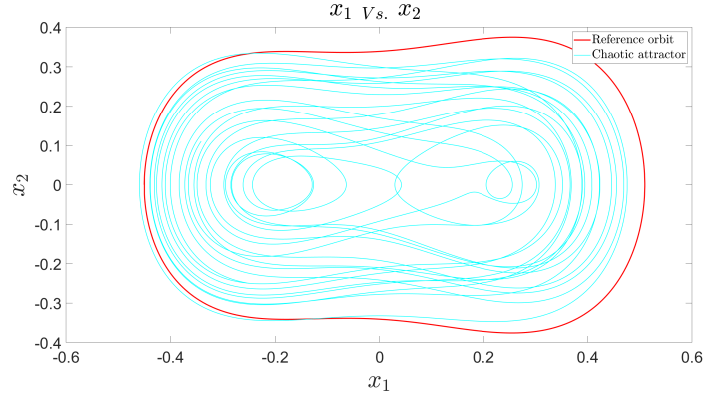


Figure 7.3 Chaotic MEMS attractor and one unstable immersed orbit.

unknown (we only know the input function  $g(x)$  and the relative degree of the system).

Finally, it is worth mentioning that although Theorem 7.4.3 was enunciated for a class of controllers given in (7.55), its application is general and could be extended to other families of HOSM controllers as well.

## 7.6 Conclusions

In the beginning of the paper, a theorem is introduced presenting the necessary conditions for obtaining bounded solutions of perturbed linear systems. By utilizing this theorem, we provide a proof of the feasibility in designing an extended observer capable of estimating the generalized disturbance, which encompasses both external disturbances and unmodeled dynamics. Through the use of this extended observer, we demonstrate how a properly designed control signal effectively stabilizes the system with an error bounded by an  $\epsilon$ -neighborhood of the origin. While the concept of the extended observer and the considered control signal are not novel, the presented proof contributes to the field of ADRC, providing valuable insights into the stability analysis of the system.

Several limitations were identified when applying the high-gain method, including its sensitivity to noisy signals due to the utilization of high-gain observers, as well as the lack of asymptotic stability, restricting convergence to a vicinity of equilibrium. To address these limitations and enhance robustness against noise, an extended sliding mode observer was introduced. Notably, this approach offered the advantage of finite-time stability. Additionally, HOSMC was employed to reduce the chattering effect while enabling the estimation of disturbances, facilitated by the extended observer's relative degree exceeding one.

By combining a sliding mode controller with an extended high-order finite-time state ob-

server for state and disturbance estimation, an enhanced HOSMC was proposed. Noteworthy properties of this method include reduced controller gains, which diminish the influence of noise on the controlled system, and the attenuation of uncertainties in the controlled system, thereby improving finite-time convergence. These notable characteristics position the proposed method as particularly suitable for deployment in highly uncertain and noise-sensitive MEMS oscillators subjected to external disturbances.

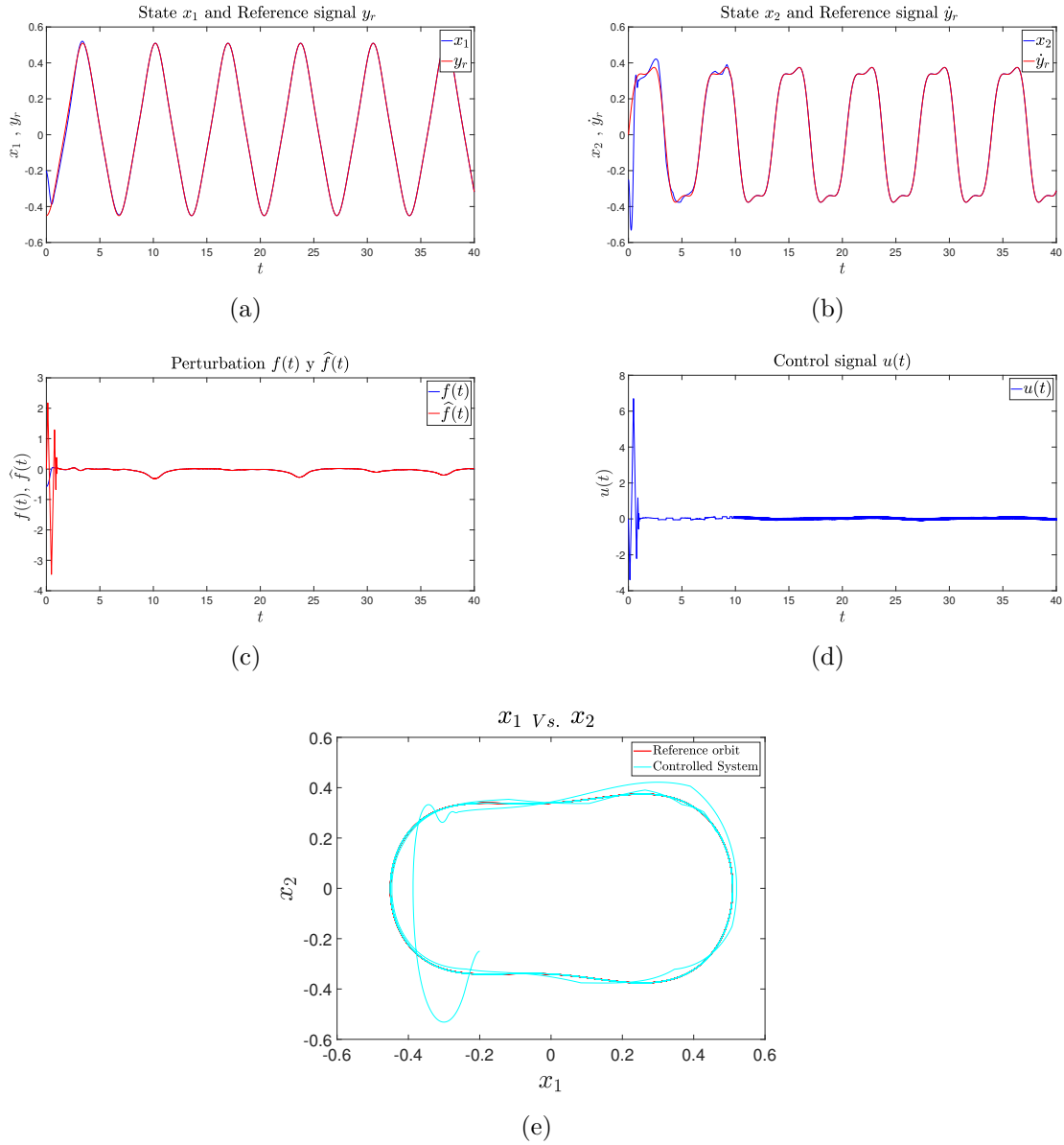


Figure 7.4 Stabilization of an immersed orbit inside a chaotic MEMS attractor using the proposed method, with a small gain in the HOSM controller. (a) and (b), states  $x_1$ ,  $x_2$  and the reference values  $y_r$ ,  $\dot{y}_r$ . (c) Disturbance  $f(t)$  and its estimation  $\hat{f}(t)$ . (d) Control signal  $u(t)$ . (e) Orbit stabilized.

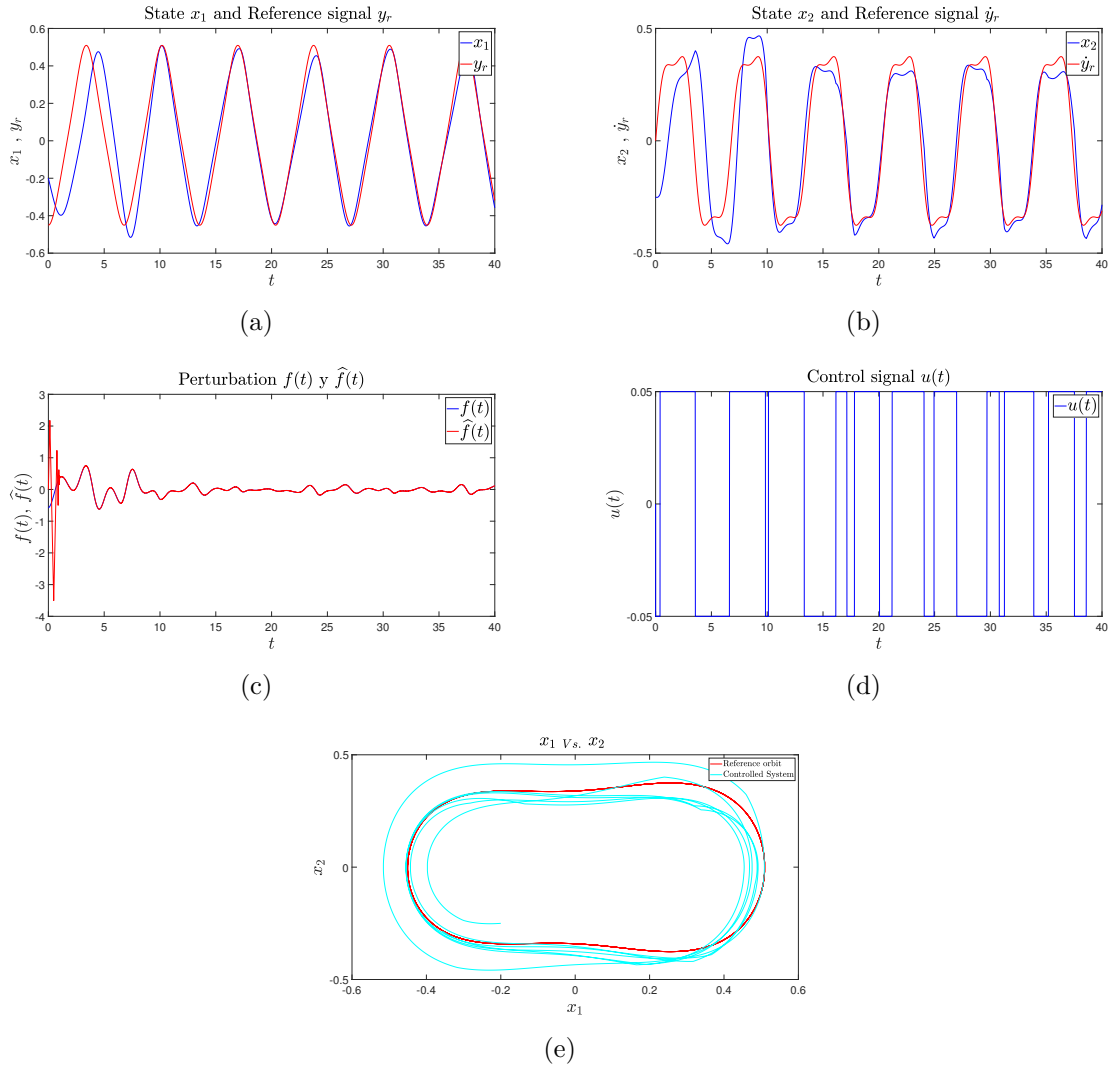


Figure 7.5 Effect of a small gain in the control of an immersed orbit inside a chaotic MEMS attractor using HOSM. (a) and (b), states  $x_1$ ,  $x_2$  and the reference values  $y_r$ ,  $\dot{y}_r$ . (c) Disturbance  $f(t)$  and its estimation  $\hat{f}(t)$ . (d) Control signal  $u(t)$ . (e) Orbit stabilized.

## CHAPTER 8 GENERAL DISCUSSION

Micro-electromechanical systems (MEMS) resonators are inherently nonlinear devices that have found diverse applications in fields such as sensing, signal processing, filtering, and timing, as detailed in Section 2.1. The reduced scale of MEMS resonators lends them the advantage of high oscillation frequencies, thereby enabling the design of ultra-fast mechanical devices [148]. However, this miniaturization introduces challenges, as smaller signals are more susceptible to noise, potentially degrading system performance. One potential solution to overcome this problem is the use of a strong excitation force to induce higher amplitude oscillations. However, the resulting nonlinear effects of the electrostatic force can introduce undesirable consequences, including various types of bifurcations, such as the pull-in instability [149].

In pursuit of optimized system operation, we exploit the inherent nonlinear dynamics of the system. Specifically, we employ differential actuation with high  $V_{DC}$  voltage, which offers several advantages. One of these advantages is the avoidance of singularities due to the quadratic electrostatic actuation force applied to the mechanic dynamics. The singularity can occur when the plates are near the position of rest at which the system is not linearly controllable. By applying a sufficiently high bias voltage  $V_{DC}$ , we can ensure that the system will operate in a safe region without singularities. Additionally, this approach provides the benefit of obtaining a quasi-linear actuation force, implying that the actuator's response is approximately linear within a certain range of operation. This linearity simplifies control and facilitates the modeling and prediction of the actuator's behavior. Furthermore, it allows for the reduction of the amplitude of the driving signal  $u$ , due to the quadratic input that produces high responses in the system.

Besides using differential actuation, we control the system in a chaotic mode. In Chapter 4, we explored the process of generating a chaotic behavior in the oscillator. It is noteworthy that the methodologies presented in the chapter have broader applications, such as in secure communication systems. From an optimization standpoint, initiating chaos only to subsequently suppress it by stabilizing an unstable orbit might not be the most efficient strategy. Instead, designing a MEMS resonator that inherently exhibits chaotic behavior and then subsequently stabilizing an unstable orbit, could be a less energy-expensive and more effective approach.

We strategically select one of the infinite unstable orbits within the chaotic attractor, and stabilize it using an efficient control action. This enhances the oscillator's output energy.

Our method targets an unstable periodic orbit that encompasses the chaotic attractor and exhibits the highest oscillation amplitude, as depicted in Figure 5.3. This strategy not only increases the oscillation amplitude but also reduces control effort by utilizing the system's inherent unstable orbits, thereby eliminating the need for additional energy to form new ones. Evidence of this can be seen in Figure 5.6, which contrasts the control efforts necessary to stabilize inherent unstable orbits versus those artificially created.

Our focus was redirected then toward developing a control method with the aim of handling uncertainties, unmodeled dynamics, noise, and external disturbances more effectively. To this end we designed several controls that potentially result in improved operational efficiency and a reduction in control effort. We first used adaptive backstepping for controlling the MEMS oscillator, and tracking the high-amplitude unstable orbit, which demonstrated effective results, as outlined in Chapter 5. Accounting for parametric uncertainties in the resonator's model is a crucial consideration, given that the parameters of these devices inherently exhibit embedded parasitic components [150], [151] and undergo variations subject to the fabrication process [152]. However, the current utilization of state feedback, detailed in Chapter 5, requires further refinement. This is especially true in real-world operation of resonators using MEMS, where accurately measuring the velocity of the plates in the microstructure presents significant challenges.

In light of the practical challenges associated with direct velocity measurements in MEMS resonators, and the potential for noise amplification when velocity is inferred from position, the use of output feedback employing position proves to be a more feasible, robust, and reliable control approach than state feedback. This concept is elaborated upon in Chapter 6, which is an extension of Chapter 5, wherein a controller is designed by means of an adaptive observer-backstepping scheme that utilizes position for output feedback control. Furthermore, the design accounts for parametric uncertainties inherent in the model. By adopting these methodologies, the system output can track the desired orbit on a global scale, achieving asymptotic convergence.

The disturbance rejection method introduced in Section 7.3 allowed us to consider external disturbances and unknown dynamics, both significant factors affecting a real MEMS-based oscillator. However, the application of the method presented certain drawbacks. These included its sensitivity to noisy signals, a result of the use of high-gain observers, as well as the lack of asymptotic stability, with the method only guaranteeing convergence to a vicinity of an equilibrium. To address these issues, an extended sliding mode observer was proposed in Section 7.4, which made the controller robust against noise and offered the added advantage of finite time stability.

It is widely recognized that the high frequency of the discontinuous signals required to establish sliding mode produces chattering. That is why we use high-order sliding mode, which not only reduces the chattering effect but also facilitates disturbance estimation, given that the relative degree of the extended observer is greater than one. The combination of a sliding mode controller and an extended high-order finite-time state observer for state and disturbance estimations, led to the development of the Enhanced high-order sliding mode control, proposed in Section 7.4.3. The most remarkable properties of the method include reduced controller gains, which minimize the impact of noise on the controlled system, and the reduction of the amplitude of uncertainties in the controlled system, thereby improving its finite time convergence as demonstrated in Section 7.4.3. In addition, sliding mode control algorithms are known for their simplicity in hardware implementation compared to some other advanced control strategies. They demand fewer computational resources and involve less complex circuitry. This inherent simplicity not only reduces the overall complexity and cost of the MEMS control system but also highlights the practical suitability of the proposed control method, especially for highly uncertain and noise-sensitive MEMS oscillators.

## CHAPTER 9 CONCLUSIONS

### 9.1 Summary of Research Results

In Chapter 4 an adaptive control scheme for synchronization of a class of chaotic systems with parametric uncertainties was introduced. The developed method may be applied to the synchronization of two structurally similar or different systems, as Duffing and MEMS, and was successfully implemented. Although this approach has the advantage of being suitable for using with uncertain systems, it is based on state-feedback which is not convenient when applied to communication system based on MEMS. In Chapter 8 a sliding mode observer, which converges in finite-time, was used to overcome this problem.

In Chapters 5 and 6, a scheme was proposed for performance improving of a MEMS resonator based on the properties of a chaotic MEMS. Chaotic systems have immersed an infinite number of unstable periodic orbits, which can be stabilized according to the needs of the system. In order to enhance the amplitude of the oscillation of the MEMS resonator, a high amplitude unstable orbit was chosen to be stabilized. The method of close returns was used to obtain the orbit numerically, and then it was approximated by its Fourier series truncated upto five terms. With the aim of stabilizing this orbit, in Chapter 5 two control strategies were designed by using state feedback, and parametric uncertainties in the model were considered in one of them. Both were obtained in details by using a backstepping design procedure. In Chapter 6 the orbit was stabilized by using output feedback, and it was shown through adaptive observer backstepping that this control law allows the system output to globally and asymptotically track the desired orbits. Numerical simulations were carried out and the results show that the unstable orbits immersed in the chaotic attractor are stabilized with an affordable control effort compared to that required for stabilizing a non-immersed orbit. This demonstrated the advantage of the proposed approach. The scheme developed in these chapters is realistic in the sense that it is based on output-feedback and consider parametric uncertainties in the model. Variables as the speed of some components of the MEMS are very difficult to be measured, and a precise knowledge of the parameters of the system is hard due to the fabrication technology. In the approach proposed, the speed of the plates of the device was estimated by means of a virtual observer at the price of increasing the differential equations needed to construct the controller.

In Chapter 7 an Enhanced sliding mode controller was proposed, which concerns cancelling external disturbances and unmodeled dynamics by estimating their values by means of an extended and finite-time stable sliding-mode observer (see Section 7.4.1), and then inducing

a behavior in the system by using a sliding mode controller, as the one in (7.56). The gains of the obtained controller are smaller than the gains using a conventional sliding mode approach, diminishing the impact of noise on the controlled system. Another effect of canceling  $h(t, x)$  is decreasing the amplitude of uncertainties in the controlled system and therefore its finite time convergence. These results are enunciated in Theorems 7.4.3 and 7.4.4. These theorems were applied successfully in the control of the unstable orbit immersed in the chaotic MEMS resonator, showing the superiority of this control strategy in comparison with the others applied in this work.

## 9.2 Limitations and Perspectives for Future Work

The disadvantages of the proposed methods in the previous works are as follows:

Chapter 4 and 5: The adaptive control schemes in these chapters are based on state-feedback, which is not ideal for controlling MEMS systems. This is because it is difficult to measure the speed of some MEMS components, which are needed for state-feedback.

Chapter 5: The parameter estimation method in this chapter is prone to drift, which can lead to inaccurate parameter estimates. Additionally, the method is sensitive to noise, which can lead to the stabilization of orbits that were not intended. Finally, it will be of great interest to extend the developed control strategies to MEMS resonators while considering the dynamics of driving circuits to further enhance the system performance for practical applications in the real world.

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