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Cylindrical Shells

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Nonlinear vibration analysis of thin incompressible hyperelastic cylindrical shells

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Thèse présentée en vue de l'obtention du diplôme de *Philosophiæ Doctor*
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Cette thèse intitulée :

Nonlinear vibration analysis of thin incompressible hyperelastic cylindrical shells

présentée par **Morteza SHAYAN ARANI**

en vue de l'obtention du diplôme de *Philosophiæ Doctor*
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DEDICATION

Dedicated to the memory of my beloved father, whose love and support have been my lifelong inspiration. I deeply wish you could be here to share this moment with me.

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Prof. Lakis, for his guidance and mentorship throughout my thesis work. I would also like to express my gratitude to Dr. Toorani and Dr. Bakhtiari for their assistance and support, (my research proposal, papers, response letters, thesis). Your contributions were essential to the success of this research. My heartfelt appreciation goes to my dear friend, Dr. Moeini, whose insightful comments and unwavering support, often shared during our long working sessions at his house, played a pivotal role in my progress. Dr. Moeini's friendship and collaboration have been invaluable to me on this journey. I must also acknowledge the tremendous support and motivation provided by my colleague and friend, Dr. Azimi. His encouragement during challenging times, as well as his effort in applying for the Rio Tinto scholarship on my behalf (which I was fortunate to receive), were instrumental in my academic journey.

To my family, you are the most cherished treasure in my life, a legacy passed down from my father. Your love and sacrifices have provided the foundation upon which I built this journey. You are, and forever will be, the best thing I have in this world, and I carry my father's legacy in each of you.

To my friends, including Mr. Fateh and Mr. Shariati, your support and motivation kept me going strong. You were there to celebrate not only my academic successes but also the little wins along the journey.

I would also like to extend a special and heartfelt thanks to my beloved wife Zahra, We have shared countless moments of joy and faced challenges together, from my bachelor's to my master's and now my Ph.D. You have been by my side, feeling every step of this academic path, and I am deeply grateful for all that you have done. Thank you, my love.

RÉSUMÉ

Les structures de coque sont largement utilisées dans divers domaines de l'ingénierie en raison de leurs caractéristiques distinctives, notamment une capacité de charge remarquable et un rapport résistance-poids élevé. Cette thèse souligne leur importance en enquêtant sur l'analyse des vibrations non linéaires de coques cylindriques minces composées hyperélastique, une classe de structures couramment rencontrée en ingénierie et en biologie. Les matériaux hyperélastiques sont d'une grande importance dans diverses applications en ingénierie en raison de leur capacité exceptionnelle à subir de grandes déformations tout en préservant leur intégrité mécanique. Ces matériaux présentent un comportement contrainte-déformation non linéaire et se trouvent couramment dans les élastomères, les tissus biologiques et les polymères souples. Pour obtenir les équations du mouvement des coques cylindriques minces hyperélastiques, il est nécessaire de prendre en compte à la fois les non-linéarités géométriques et matérielles.

Dans cette étude, la non-linéarité géométrique est abordée en utilisant une théorie non linéaire basée sur la théorie des coques de Donnell, et la non-linéarité du matériau est incorporée dans le modèle en utilisant le modèle de Mooney-Rivlin. La méthode de condensation statique est utilisée pour réduire l'ordre des équations, obtenant ainsi des solutions pour les vibrations libres en utilisant la méthode de Lindstedt-Poincaré et pour les vibrations forcées à travers la méthode de l'échelle multiple. Dans la première étape de l'évaluation (comme détaillé dans le premier article), le modèle a d'abord été vérifié, et l'impact de la méthode de condensation statique a été évalué. Ensuite, l'étude s'est concentrée sur l'analyse de l'effet des fonctions de forme dans la capture de la réponse non linéaire du système. Il a été observé que les fonctions de forme avec des termes de flexion indépendants entraînent un comportement d'adoucissement plus prononcé, caractérisé par un pic plus élevé, par rapport aux fonctions de forme qui n'incluent pas de termes de flexion. De plus, cette recherche explore les courbes de base associées à deux fonctions de forme distinctes. Les résultats indiquent que la fonction de forme avec des termes de flexion présente un comportement de durcissement relativement plus faible.

Dans la deuxième étape de l'évaluation (décrite dans notre deuxième article), nous avons mis en œuvre la théorie améliorée des coques minces de Donnell pour améliorer la précision de nos résultats, en particulier lorsqu'il s'agit de nombres d'ondes circonférentielles inférieurs à cinq. Dans cette recherche, la fréquence naturelle de la coque cylindrique hyperélastique incompressible a été vérifiée par rapport aux résultats obtenus à partir d'ABAQUS. La différence maximale observée dans les résultats varie entre 2 et 2,7 [Hz], se produisant lorsque

le nombre d'ondes circonférentielles est compris entre 7 et 13. Ensuite, nous avons introduit trois imperfections géométriques initiales distinctes: asymétriques, axisymétriques et combinées. Les résultats ont montré que les imperfections géométriques augmentent la fréquence naturelle et modifient le comportement de la réponse en amplitude. Ces imperfections ont la capacité d'influencer l'ampleur des pics d'adoucissement et de durcissement dans la réponse en amplitude.

ABSTRACT

Shell structures find extensive applications in various engineering fields due to their distinctive characteristics, including a remarkable load-bearing capacity and a high stiffness-to-weight ratio. This thesis emphasizes its significance by investigating the nonlinear vibration analysis of thin cylindrical shells composed of hyperelastic material—a class of structures common in engineering and biology. Hyperelastic materials are of significant importance in various engineering applications due to their exceptional ability to undergo large deformations while maintaining mechanical integrity. These materials exhibit nonlinear stress-strain behavior and are commonly found in elastomers, biological tissues, and soft polymers. To derive the equations of motion for thin hyperelastic cylindrical shells, we need to consider both geometrical and material nonlinearities.

In this study, geometrical nonlinearity is considered by employing nonlinear theory based on Donnell’s shell theory, and the material nonlinearity is incorporated into the model using the Mooney-Rivlin model. The static condensation method is employed to reduce the order of the equations, thereby obtaining solutions for free vibrations using the Lindstedt-Poincaré method and for forced vibrations through the Multiple Scales method.

In the initial stage of evaluation (as detailed in the first paper), the model was first verified, and the impact of the static condensation method was assessed. Subsequently, the study focused on analyzing the effect of shape functions in capturing the nonlinear response of the system. It was observed that shape functions with independent bending terms result in a more pronounced softening behavior, characterized by a higher peak, in contrast to shape functions that do not include bending terms. Additionally, this research explores the backbone curves associated with two distinct shape functions. The findings indicate that the shape function with bending terms demonstrates a relatively weaker hardening behavior.

In the second level of assessment (our second paper), we implemented the improved Donnell’s shallow shell theory to enhance the accuracy of our results, particularly when dealing with circumferential wave numbers less than five. In this research, the natural frequency of the incompressible hyperelastic cylindrical shell was verified against results obtained from ABAQUS. The maximum difference observed in the results ranges between 2 and 2.7 [Hz], occurring when the number of circumferential waves is between 7 and 13. Next, we introduced three distinct initial geometric imperfections: asymmetric, axisymmetric, and combined. The findings demonstrated that geometric imperfections increase the natural frequency and alter the amplitude response behavior. These imperfections have the ability to influence the magnitude of softening and hardening peaks in the amplitude response.

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LIST OF SYMBOLS AND ACRONYMS

Latin symbols

t	Time
$\mathbf{E}$	Green-Lagrange strain tensor
$\mathbf{C}$	Right Cauchy-Green deformation tensor
E	Young's modulus
G	Shear modulus
K	Kinetic energy
P	Potential energy
n	Circumferential wave number
m	Longitudinal half-wave number
D_1	Material incompressibility parameters
c_{01}, c_{10}	Hyperelastic material constant
a_1	Axisymmetric mode
a_2	Asymmetric mode

Greek symbols

ν	Poisson ratio
σ	stress tensor
ε	strain tensor
θ	Angle between horizontal and z coordinate
ρ	Density of the material
μ	Initial shear modulus

Abbreviations

DOF	Degrees-Of-Freedom
SED	Strain Energy Density
LMM	Local Model Method
WB	With Bending
WOB	Without Bending
DST	Donnel's Shallow Theory
$IDST$	Improved Donnel's Shallow Theory

CHAPTER 1 INTRODUCTION

1.1 Overview

Shell structures are widely utilized across various engineering domains, primarily due to their impressive load-carrying capacity and lightweight characteristics. When these structures are composed of hyperelastic materials, they gain an additional advantage: enhanced flexibility. This unique property allows them to stretch significantly without losing their structural integrity. However, this also brings complexity to the analysis and study of hyperelastic cylindrical shells, due to the material properties and the geometry of the problem.

The stress-strain relationship in hyperelastic materials is complicated, making the linear stress-strain relationship and linear elastic models invalid for simulating their mechanical behavior. Figure 1.1 illustrates the distinction between stress-strain behaviors in elastic and hyperelastic materials. In elastic materials, stress-strain relationships adhere to Hooke's Law, resulting in a linear relationship. However, hyperelastic materials deviate from this law, exhibiting nonlinear stress-strain relationships. To model such materials accurately, a strain energy function is employed to capture their unique behavior. Furthermore, the nonlinearity arising from large deformations also should be considered in studying the hyperelastic cylindrical shells as geometrical nonlinearity. The presence of both material and geometrical nonlinearities complicates the modeling and analysis of these structures.

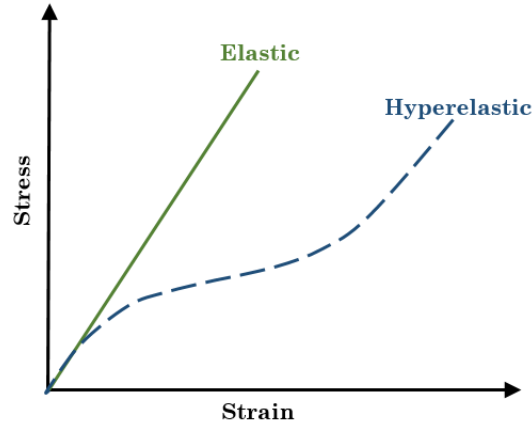


Figure 1.1 Stress-strain relations in elastic and hyperelastic Materials.

1.2 Necessity for this research

An analysis of the available database in Scopus on hyperelasticity and hyperelastic structures indicates that these topics have consistently been a focus of research over the past three decades. With advancements in computational calculations and materials science, interest in these areas has significantly increased. Notably, the number of publications on these topics has doubled from 2010 to 2020, as illustrated in Figure 1.2.

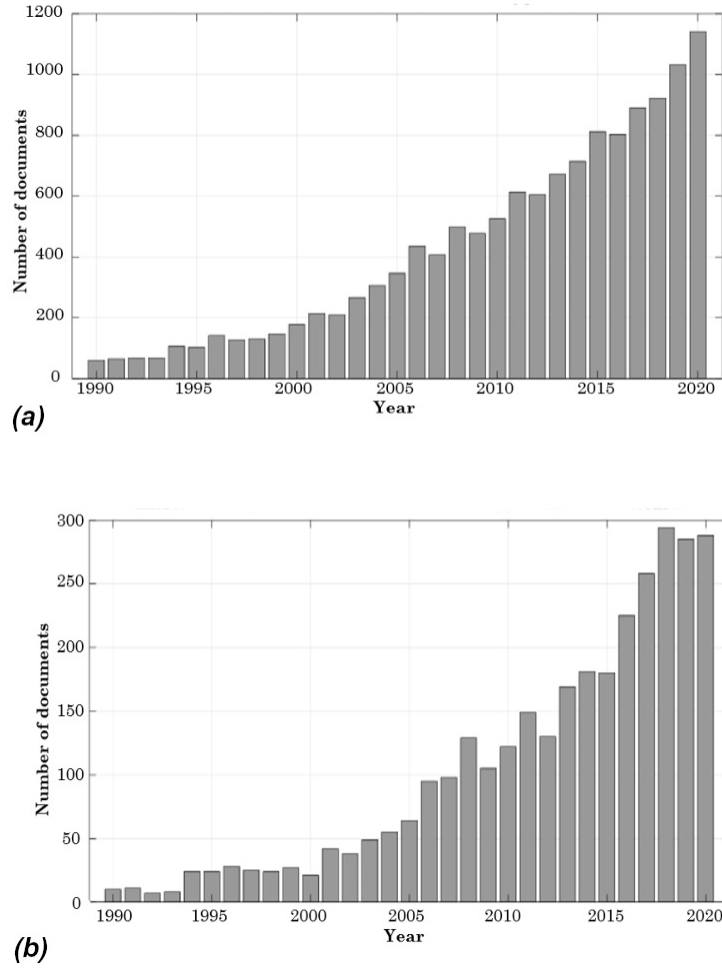


Figure 1.2 Number of documents from 1990 to 2020 on **a)** hyperelasticity and **b)** hyperelastic beams, plates, and shells [1].

This trend underscores the growing relevance and continued exploration in this field.

Another important factor contributing to the surge in research on hyperelastic materials and structures is their wide-ranging applications across various fields. They are increasingly

utilized in soft robotics [2–7]. Their flexibility allows for higher-order degrees-of-freedom, facilitating smoother and more adaptable movements. These advanced capabilities make such robots ideal for real-world applications, where they can assist humans in various tasks. This includes monitoring and exploration, remediation, as well as the detection and handling of biological and chemical entities [8] in harsh or dangerous environments where others would struggle. For instance, as shown in Figure 1.3(a), the electronics-free DraBot is a notable example that utilizes air pressure, specialized microarchitectures, and self-healing hydrogels. This unique design enables DraBot to monitor environmental changes, such as changes in pH, temperature, and the presence of contaminants [8, 9]. In Figure 1.3(b), a soft robot is depicted, which is soft and gentle enough to pick up a jellyfish, an organism composed of about 95 percent water [10].

Hyperelastic structures also are used in 3D printing, as soft actuators. They offer a range of advantages, including the ability to perform complex motions, along with their low weight and reduced energy consumption. These features make them significantly more advantageous compared to traditional rigid actuators [11]. In Figure 1.4, some actuators and a soft glove composed of hyperelastic materials are shown with the ability to grab, lift, and twist. Researchers have utilized hyperelastic strain energy density models to model the mechanical behavior of various biological tissues effectively. For instance, hyperelastic models of white and grey matter in the brain have directly contributed to advancements in brain surgery [12, 13], trauma analysis [14], and injury simulation [15].

Furthermore, in the field of biomechanics, the human aorta, being the largest artery in the body, is subject to serious conditions like aneurysms and acute dissections. These conditions often necessitate open or endovascular surgery for repair [16, 17]. Figure 1.5(a) illustrates how an artificial part, designed as a thin hyperelastic cylindrical shell, has been developed to address such aortic issues. Also, the cardiac support sleeve depicted in Figure 1.5(b), crafted from hyperelastic material, exemplifies the significant applications of such structures. Its design is aimed at increasing cardiac function by augmenting ventricular activity in a failing heart [18].

In summary, the phenomenal growth of studies on hyperelastic materials and their diverse applications in the engineering field aimed to highlight the critical importance of this research. This section has emphasized how these innovative materials are making significant contributions across various domains, underlining their relevance and potential in advancing engineering solutions.

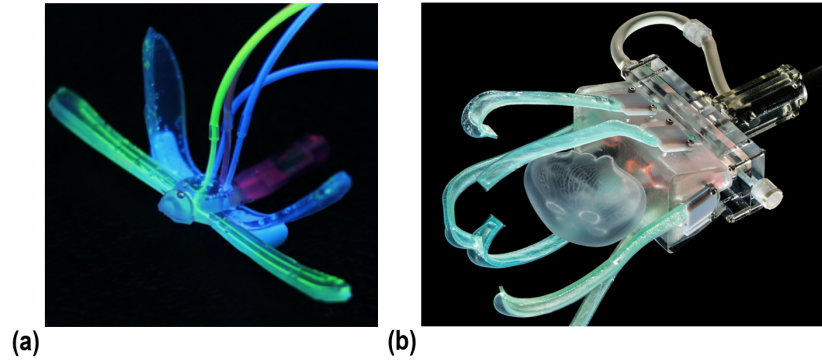


Figure 1.3 Hyperelastic materials in soft robotics **a)** Soft robot (DraBot) [9] and **b)** Ultra-gentle soft gripper [10].

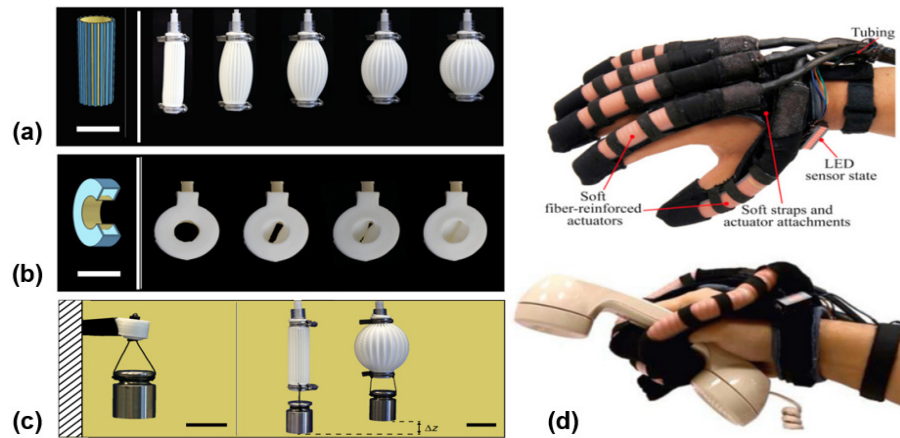


Figure 1.4 Hyperelastic materials in rehabilitation robotics **a)** Soft actuators (contractor) [11]; **b)** Soft actuators (grabber) [11]; **c)** Grabbing and lifting 500g [11] and **d)** The soft and lightweight wearable robotic hand assistance device [19].

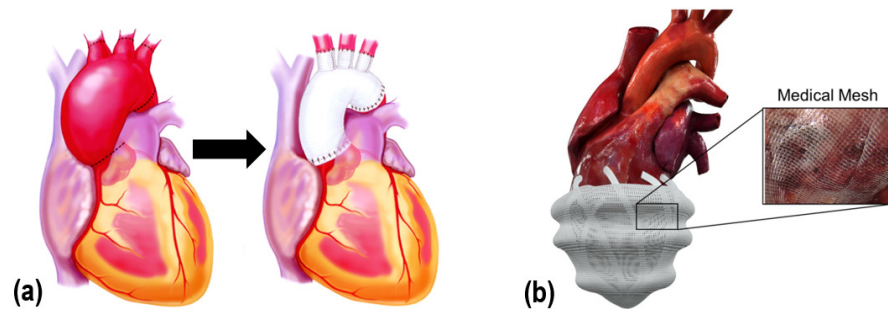


Figure 1.5 Hyperelastic materials in biomedical field **a)** Repair of an aneurysm involving ascending thoracic aorta and arch by using a multilimbed prosthetic graft [16] and **b)** Sleeve support for heart function [18].

1.3 Structure of thesis

In this thesis, each of the challenges has been addressed, aiming to capture the nonlinear behavior response of the structure under various conditions. The objective is to develop a model that closely aligns with real-world dynamics. This dissertation is composed of seven chapters:

- Chapter 2 provides the literature review and summarizes the work that has been done on material nonlinearity (physical nonlinearity), geometrical nonlinearity (large-deformation), and addressing the nonlinear equations of motion.
- Chapter 3 highlights the existing gaps in the literature in line with our application. This chapter also introduces the objectives of this thesis.
- Chapter 4 presents the scientific approach and publication strategy.
- Chapter 5 contains the first article for analyzing softening and hardening behavior in the vibration of a thin incompressible hyperelastic cylindrical shell.
- Chapter 6 contains the second article, focusing on the nonlinear vibration of hyperelastic thin cylindrical shells in the presence of different initial geometric imperfections.
- Chapter 7 provides a summary of the work done in this study and suggestions for future research directions.

CHAPTER 2 LITERATURE REVIEW

2.1 Literature review

To comprehensively study the behavior of thin hyperelastic cylindrical shells, a step-by-step approach is essential. These structures experience geometrical nonlinearity arising from large deformation and material (physical) nonlinearity from their nonlinear stress-strain relation of hyperelastic materials. Our initial step involves formulating models for both the structure and the material. This serves as the basis for deriving the equations of motion, which are subsequently solved to attain meaningful results. In this chapter, we will conduct a comprehensive literature review covering both linear and nonlinear shell theories, hyperelastic material models, and the study of hyperelastic cylindrical shells.

2.1.1 Plate and shell theories

In the Classical Linear theory of plates (CPT), two foundational approaches exist for predicting the behavior of these structures. The first method, introduced by Cauchy in 1828 [20] and subsequently by Poisson in 1829 [21], is based on expanding the displacements and stresses in the plate in power series of the distance z from the middle surface. This method, while innovative, encountered challenges due to the mathematical complexities involved in the series expansion and the precise boundary conditions required, which made this method unpopular. Kirchhoff's theory [22], formulated in 1850, introduced crucial assumptions for understanding thin plate structures based on the Euler-Bernoulli beam theory. He proposed that the thickness of the plate is relatively small compared to its other dimensions, which simplifies the analysis. Additionally, Kirchhoff assumed that plane sections normal to the mid-surface remain plane and normal to the deformed mid-surface. As illustrated in Figure 2.1, in the undeformed state, a material point in the plate is located at coordinates (x, y, z) . When the plate undergoes deformation, this point moves to a new position. The new position of the point is given by $(x + u, y + v, z + w)$, where u , v , and w are the displacements in the x , y , and z directions, respectively. Kirchhoff neglected the effect of transverse shear deformation as well. These assumptions did more than just simplify the mathematical approach; they also introduced physical meaning into the theory of thin plates. The displacement field for

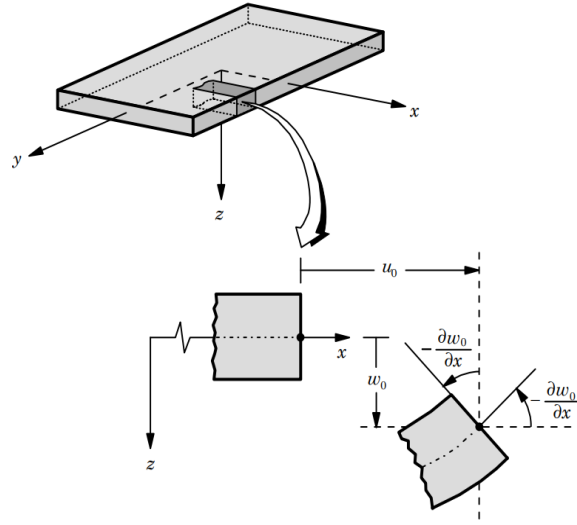


Figure 2.1 Undeformed and deformed geometries of an edge of a plate under the Kirchhoff assumptions [23].

time-dependent deformations, based on this theory, is as follows:

$$\begin{aligned} u(x, y, z, t) &= u_0(x, y, t) - z \frac{\partial w_0}{\partial x}, \\ v(x, y, z, t) &= v_0(x, y, t) - z \frac{\partial w_0}{\partial y}, \\ w(x, y, z, t) &= w_0(x, y, t). \end{aligned} \tag{2.1}$$

Von Kármán expanded upon this method by introducing nonlinear terms, enabling the study of large deformations in plates [24]. His work significantly advanced the understanding of plate behavior beyond the limitations of linear assumptions. To analyze the behavior of thicker plates, as shown in Figure 2.2, the First-order Shear Deformation Theory (FSDT) was proposed. In this theory, the displacement field for time-dependent deformations includes three displacement components and two rotational parameters as follows:

$$\begin{aligned} u(x, y, z, t) &= u_0(x, y, t) + z\phi_x(x, y, t), \\ v(x, y, z, t) &= v_0(x, y, t) + z\phi_y(x, y, t), \\ w(x, y, z, t) &= w_0(x, y, t). \end{aligned} \tag{2.2}$$

As (u_0, v_0, w_0) are the displacement of any point on the middle surface of the plate the ϕ_x and ϕ_y are the rotations of a transverse normal about the x - and y -axes, respectively.

$$\frac{\partial u}{\partial z} = \phi_x, \quad \frac{\partial v}{\partial z} = \phi_y. \quad (2.3)$$

While these elements enhance the representation of shear deformations and rotations, the theory falls short of accurately satisfying the boundary conditions related to transverse shear at the top and bottom surfaces of the structure. This is primarily because the theory assumes a uniform shear angle across the thickness, and plane sections remain plane after deformation. As shown in Figure 2.3, to overcome the limitations of the first-order shear deformation theory

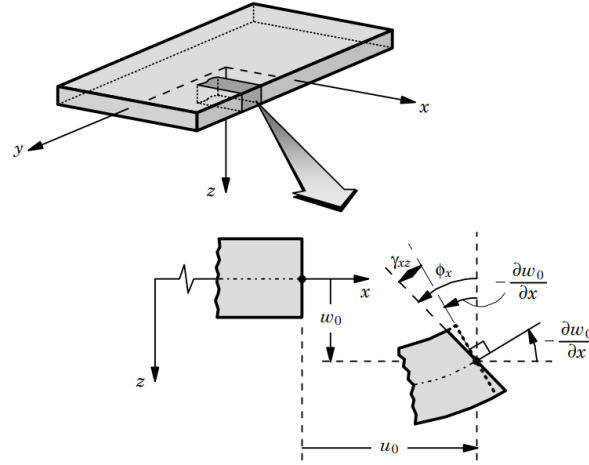


Figure 2.2 Undeformed and deformed geometries of an edge of a plate under the assumptions of the first-order shear deformation plate theory (FSDT) [23].

in handling transverse shear stresses, Reddy developed the third-order shear deformation theory (TSDT) [25]. This advanced theory introduces higher-order terms (cubic terms) for more accurate modeling of shear stress distribution, particularly at the top and bottom of the plates.

Shells are structural components characterized by their thinness in comparison to plates, along with the curvature of their middle surface, and the exploration of these structures traces back to 1910 when Von Kármán [24] introduced nonlinear terms to the classical linear theories of plates. In cylindrical coordinates, by substituting y with $R\theta$ and assuming that $R + z \approx R$ the displacement field equation for a circular cylindrical shell (Figure 2.4), as given in equation 2.1 for plate, can be redefined as follows:

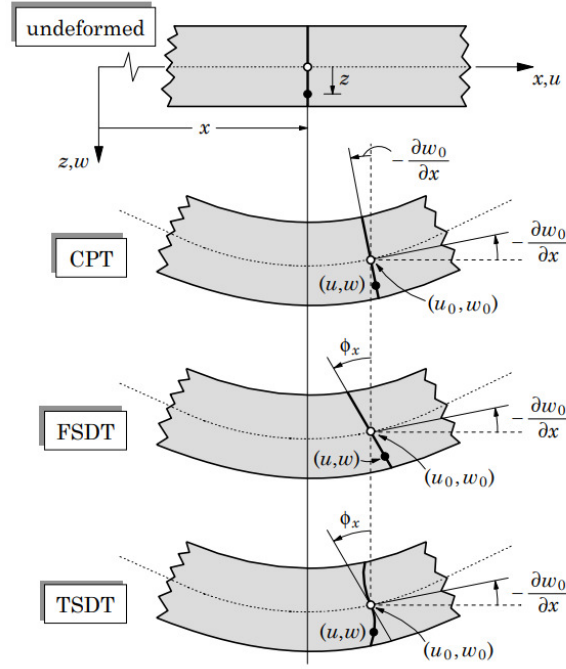


Figure 2.3 Deformation of a transverse normal according to the classical (CPT), first-order (FSDT), and third-order (TSDT) plate theories [23].

$$\begin{aligned}
 u(x, \theta, z, t) &= u_0(x, \theta, t) - z \frac{\partial w_0}{\partial x}, \\
 v(x, \theta, z, t) &= v_0(x, \theta, t) - z \frac{\partial w_0}{R \partial \theta}, \\
 w(x, \theta, z, t) &= w_0(x, \theta, t).
 \end{aligned} \tag{2.4}$$

Following that, various researchers proposed different nonlinear shell theories, including Donnell [26], Sanders [27], Koiter [28], Mushtari and Galimov [29], Novozhilov [30], and Reddy [31]. By neglecting in-plane inertia, a simplified version of Donnell shell theory is obtained, known as Donnell's nonlinear shallow-shell theory or the Donnell-Mushtari-Valsov theory. This theory is widely utilized for its practical accuracy and relative simplicity [32]. In this simplified theory, through the definition and application of a stress function, the three equilibrium equations concerning shell displacement in the radial, circumferential, and axial directions are combined into two equations, which are solely dependent on the radial displacement and stress function. The primary limitation of this theory is its reduced accuracy when dealing with cases where the circumferential wave number is less than five. To address this limitation and enhance the applicability of Donnell's shell theory in scenarios with fewer circumferential wave modes, an improved version has been developed and applied [33, 34].

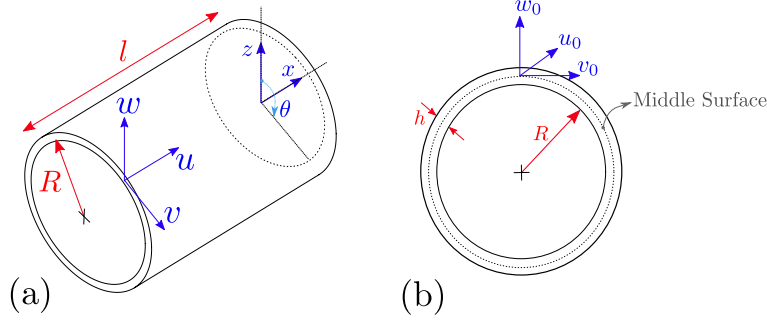


Figure 2.4 (a) Schematic representation of shell geometry, (b) Cross-section of the shell.

This refined version aims to provide increased accuracy and reliability. In Table 2.1, the relationships between strain and displacement for both classical and improved versions are detailed. Another approach to reduce the degrees-of-freedom of the system is through the application of the static condensation method. This technique enables the transformation of axial and circumferential displacements into expressions related to radial displacements [35].

Table 2.1 Strain- Displacement relationships based on Donnell's shallow shell theory and its improved version

$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 - \frac{\partial w_0^2}{\partial x^2} z$ $\varepsilon_{\theta\theta} = \frac{1}{R} \left(\frac{\partial v_0}{\partial \theta} + w_0 \right) + \frac{1}{2R^2} \left(\frac{\partial w_0}{\partial \theta} \right)^2 - \frac{1}{R^2} \frac{\partial w_0^2}{\partial \theta^2} z$ $\gamma_{x\theta} = \frac{1}{R} \frac{\partial u_0}{\partial \theta} + \frac{\partial v_0}{\partial x} + \frac{1}{R} \frac{\partial w_0}{\partial \theta} \frac{\partial w_0}{\partial x} - 2 \frac{1}{R} \frac{\partial w_0^2}{\partial \theta \partial x} z$	<i>Donnell's shallow theory</i>
$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 - \frac{\partial w_0^2}{\partial x^2} z$ $\varepsilon_{\theta\theta} = \frac{1}{R} \left(\frac{\partial v_0}{\partial \theta} + w_0 \right) + \frac{1}{2R^2} \left(\frac{\partial w_0}{\partial \theta} \right)^2 - \frac{1}{R^2} \left(\frac{\partial w_0^2}{\partial \theta^2} - \frac{\partial v_0}{\partial \theta} \right) z$ $\gamma_{x\theta} = \frac{1}{R} \frac{\partial u_0}{\partial \theta} + \frac{\partial v_0}{\partial x} + \frac{1}{R} \left(\frac{\partial w_0}{\partial \theta} \frac{\partial w_0}{\partial x} \right) - 2 \frac{1}{R} \left(\frac{\partial w_0^2}{\partial x \partial \theta} - \frac{\partial v_0}{\partial x} \right) z$	<i>Improved Donnell's shallow theory</i>

2.1.2 Hyperelastic material models

To capture the nonlinear behavior of hyperelastic materials, the deformation gradient (F) is one of the essential concepts from continuum mechanics. It is the second-order tensor that maps line elements in the reference (or undeformed) configuration into line elements (consisting of the same material particles) in the current (or deformed) configuration. In general, the deformation gradient (F) is defined as [36]:

$$F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j}, \quad (2.5)$$

where δ_{ij} represents the Kronecker delta and u_i is the displacement field, which could be rewritten in the principal directions of the structures as follows [37]:

$$F = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \rightarrow F_{ij} = \delta_{ij} \lambda_i, \quad (2.6)$$

λ_i represents the principal stretch corresponding to the principal direction i , and is defined as:

$$\lambda_i = \frac{X_i^c}{X_i^r}. \quad (2.7)$$

Where X_i^c and X_i^r represent the current and reference lengths of the structure through the i direction, respectively. Another important concept in characterizing hyperelastic structures is the right Cauchy–Green strain tensor, denoted as $\mathbf{C}$, which can be derived from the deformation gradient as follows [37]:

$$\mathbf{C} = F^T F. \quad (2.8)$$

To derive the strain energy density function for hyperelastic materials, it is necessary to define the first, second, and third invariants. These invariants are derived from the right Cauchy–Green deformation tensor as follows [38]:

$$\begin{aligned} I_1 &= Tr(\mathbf{C}) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ I_2 &= \frac{1}{2} \left(Tr(\mathbf{C})^2 - Tr(\mathbf{C}^2) \right) = \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2 \\ I_3 &= J^2 = det(\mathbf{C}) = \lambda_1^2 \lambda_2^2 \lambda_3^2, \end{aligned} \quad (2.9)$$

where I_1 is the first invariant and relates to the change in length or extension or compression along the principal axes. I_2 is the second invariant and relates to the change in the shape of a material element in the plane, and I_3 is the third invariant which provides a direct measure

of the volumetric change. For an incompressible material, the third invariant would remain constant and J is equal to one [39]. To analyze the deformation of structures experiencing large strains, the Green-Lagrange strain tensor is required. Including higher-order displacement terms, it calculates the strain gradients for the undeformed (original) configuration of the structure. This strain tensor is obtained based on the right Cauchy-Green strain tensor as follows:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}), \quad (2.10)$$

where $\mathbf{E}$ represent the Green-Lagrange strain tensor and $\mathbf{I}$ is unity matrix. In the context of hyperelasticity, a key concept is the Helmholtz free-energy density function W , which is defined per unit reference volume [38]. In the case when W is only a function of F , another strain tensor, or their respective invariants, then it is known as the strain energy density function and can have different expressions as follows:

$$W = W(\mathbf{F}) = W(\mathbf{C}) = W(\mathbf{E}) = W(I_1, I_2, I_3) = W(\lambda_1, \lambda_2, \lambda_3). \quad (2.11)$$

Neo-Hookean strain energy density function, originally proposed by Ronald Rivlin in 1948 [40], is one of the straightforward models for hyperelastic materials which are defined as follows:

$$W_{NH} = \frac{\mu}{2}(I_1 - 3) + \frac{1}{D_1}(J - 1)^2. \quad (2.12)$$

In this model, μ stands for the initial shear modulus, while D_1 represents the material's incompressibility parameter. This model, characterized by a constant shear modulus, is well-suited for predicting the mechanical behavior of materials like rubber under specific conditions, such as uniaxial tension within the range of 30% to 40% and pure shear in the range of 80% to 90%. However, its accuracy diminishes for large strain deformations, making it less versatile compared to other models, especially in such scenarios.

Later, an extended version of the Neo-Hookean model was proposed by Mooney in 1940 [41] to address the limitations of the original model. In this modified version, the second invariant terms are also considered, resulting in a more comprehensive description of material behavior. This model is defined as follows:

$$W_M = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + \frac{1}{D_1}(J - 1)^2, \quad (2.13)$$

the values of C_{10} , C_{01} , and D_1 are typically determined through experimental observations and measurements. This model was presented by Rivlin in a general form as a polynomial

series [42], expressed as follows:

$$W_{MR} = \sum_{i=0}^n \sum_{j=0}^m C_{ij} (I_1 - 3)^i (I_2 - 3)^j + \frac{1}{D_1} (J - 1)^2. \quad (2.14)$$

This model is commonly referred to as the Mooney-Rivlin model, named after its developers, Mooney and Rivlin. Within the framework of Mooney-Rivlin model, various strain energy functions have been developed, denoted as the Biderman model [43], the Ishihara model [44], and the Haines-Wilson model [45]. Detailed descriptions of these models are presented in Table 2.2. These models find application in the analysis of hyperelastic materials, particularly

Table 2.2 The three models based on the Mooney-Rivlin; NMP stands for the number of material parameters

Model	Year	NMP	Strain energy
Ishihara	1951	three	$W = C_{10}(I_1 - 3) + C_{20}(I_1 - 3)^2 C_{01}(I_2 - 3)$
Biderman	1958	four	$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3$
Haines-Wilson	1975	six	$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + C_{11}(I_1 - 3)(I_2 - 3) + C_{02}(I_2 - 3)^2 + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3$

those subject to deformations of less than 200 percent [46]. All strain energy density models based on the Neo-Hookean and Mooney-Rivlin are also unable to accurately model the rapid increase in the load versus stretch behavior observed in uniaxial extension experiments. These limitations persist across various strain energy density models derived from these foundational formulations. This sudden rise of stiffness occurs even at moderate strains for biological materials [38]. To catch this phenomenon, Ogden [47] defined the strain energy through a series of real powers of $(\lambda_i)_{i=1,3}$ as per:

$$W_{og} = \sum_{i=0}^n \frac{\mu_i}{\alpha_i} (\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3), \quad (2.15)$$

where the material parameters $(\mu_i, \alpha_i)_{i=1,n}$ should fulfilled the following stability condition:

$$\mu_i \alpha_i > 0 \quad \forall i = 1, n, \quad (2.16)$$

and they must be found using experimental testing. This model is well capable of modeling

the typical hardening behavior observed in rubber materials. Such hardening characteristics are not captured by the Neo-Hookean and Mooney-Rivlin models, nor by any models derived from these two. Then, this model has become one of the most widely used for addressing large strain problems. Many other models have been developed by researchers to model the hyperelastic materials while experiencing deformation of more than 400 percent, including those named the Arruda-Boyce model [48], the polynomial model [49], the Blatz and Ko [50], and the Gent model [51]. In his model, Gent introduced J_m as the limiting stretch parameter. This parameter typically ranges from 0.4 to 2.3 for biological tissues, and for plastic structures, it can be on the order of 100. Interestingly, if the maximum permitted value of J_m is increased to infinity, the Gent model transitions into the neo-Hookean incompressible model. Details of these models are presented in Table 2.3.

Table 2.3 Exponential hyperelastic material models

Model	Year	Strain energy
Blatz-Ko	1962	$W = \frac{\mu}{2} \left(\frac{I_1}{I_2} + 2\sqrt{I_3} \right)$
Arruda Boyce	1993	$W = \sum_{i=1}^n C_i (I_1^i - 3^i) + \sum_{i=1}^n D_i (J - 1)^{2i}$
Polynomial model	1996	$W = \sum_{i+j=1}^n C_{ij} (I_1 - 3)^i (I_2 - 3)^j + \sum_{i=1}^n D_i (J - 1)^{2i}$
Gent	1999	$W = -\frac{\mu}{2} J_m \ln \left(1 + \frac{3 - I_1}{J_m} \right)$

2.1.3 Hyperelastic plates and shells

Taking into account both geometrical and material nonlinearities in shell vibration analysis introduces a high level of complexity to the problem. An early investigation on this topic was conducted by Knowles [52, 53]. He studied the free vibration caused by an instantly applied force and subsequently maintained uniform pressure on a cylindrical thin tube composed of incompressible hyperelastic material. In this study, the material was modeled using Mooney-Rivlin model, and the equation of motion was derived for a single degrees-of-freedom. For this single degrees-of-freedom model, an analytical solution was proposed by Shahinpoor and Nowinski [54], and a numerical study was conducted using the fourth order Runge Kutta technique by Shahinpoor and Balakrishnan [55]. The impact of applied forces, geometric characteristics, and material parameters on the dynamic response of long cylindrical tubes

was investigated by Yuan et al. [56]. In this research, the Ogden and Neo-Hookean models were employed to characterize the material's nonlinearity. For specific material parameters, it has been demonstrated that the motion of the structure experiences a class of nonlinear periodic oscillations for any provided pressures, and the oscillation amplitude experiences discontinuities at certain special pressure values. He et al. [57], explored the stability of an incompressible hyperelastic cylindrical shell, employing Neo-Hookean material model. Their findings demonstrated an increase in the shell's thickness correlates with improved system stability. Breslavsky et al. [58] conducted a study on the free and forced large-amplitude vibrations of rectangular hyperelastic plates. In this investigation, the equation of motion, which included quadratic and cubic nonlinear terms to account for both material and geometrical nonlinearities, was solved by applying the harmonic balance method. It was found that under small strains, the material nonlinearity terms have a weak influence, whereas this influence notably increased as strains grew larger. As shown in Figure 2.5, Amabili et al. [59] carried out a comprehensive examination of thin hyperelastic plates, employing analytical, numerical, and experimental approaches.

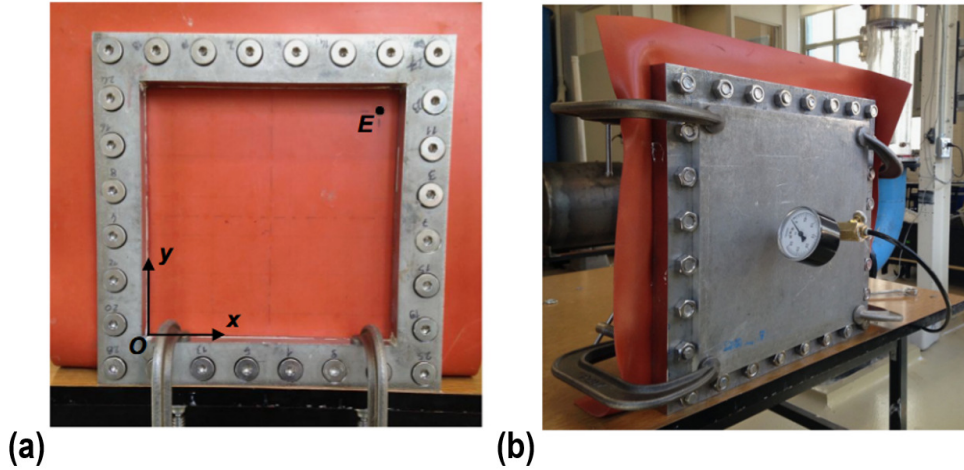


Figure 2.5 Silicone plate inside the steel frame with fixed boundary conditions; **a)** a planar coordinate system o, x, y is fixed on to the square area; **b)** back of the plate complete with pneumatic components used for static pressure loading [60].

This investigation incorporated geometric nonlinearity using Novozhilov nonlinear shell theory, while material nonlinearity was described using Mooney-Rivlin model. It was shown that the experimental results closely matched the dynamic model, especially for the first four natural modes and frequencies. Two years later, the same team conducted further experiments to assess the static hyperelastic behavior, relaxation, dynamic modulus, and loss

factor of woven Dacron as demonstrated in Figure 2.6. This material is notably used in the manufacture of commercial aortic prostheses [60]. The experimental findings regarding storage moduli and loss factors, especially loading frequency, along with the material parameters derived from the generalized Maxwell model. In their study, an 11% difference in the reduced relaxation rates was observed between the axial and circumferential directions as time approached infinity. Additionally, the study noted a significant increase in stiffness when subjected to harmonic loading compared to static loading.

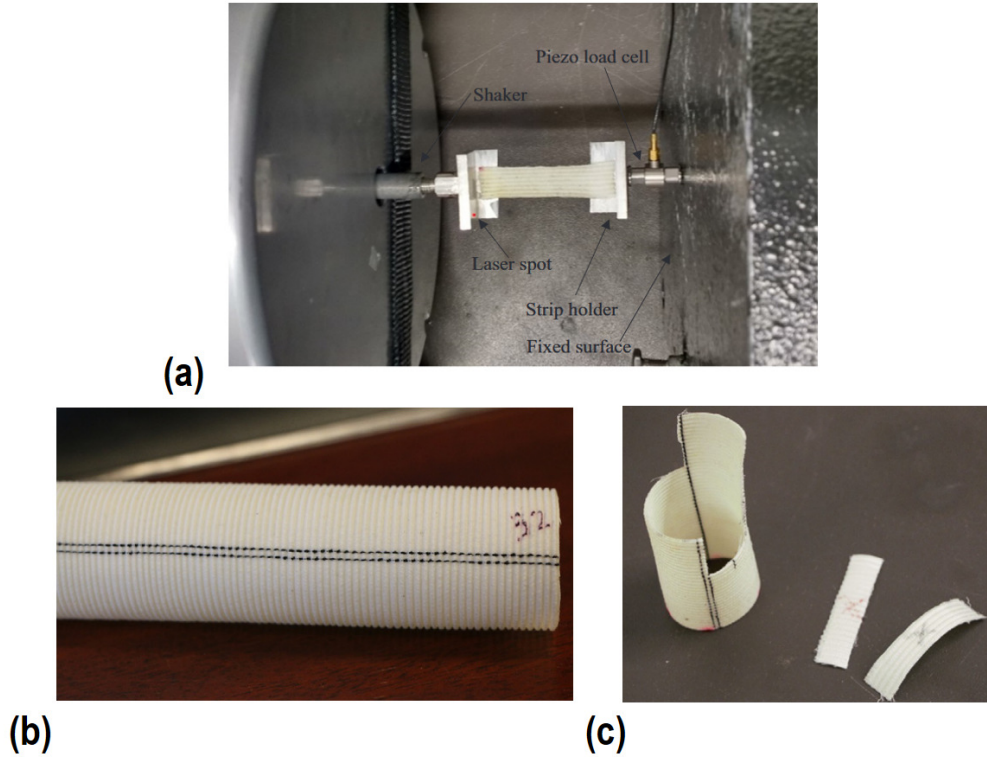


Figure 2.6 Photos of the traction experiments. **a)** Experimental setup for the dynamic stiffness measurement of a circumferential strip. Woven Dacron aortic prosthesis of 32mm diameter with visible axial corrugation; **b)** axial and circumferential strips and one cut prosthesis.

To advance the development of a future generation of aortic grafts that can mimic the natural behavior of the aorta, Amabili et. al [17] conducted an experimental test on 15 human descending thoracic aortas obtained from heart-beating healthy donors. They set up a micro-epsilon triangulation laser sensor at five different locations to measure the thickness of the aortas during quasi-static and dynamic mechanical characterization tests. They discovered limitations in their generalized Maxwell viscoelastic material model and observed a significant

effect of pre-stretch on the dynamic stiffness ratio. Furthermore, they noted a substantial impact of the amplitude of the cyclic load on the viscoelastic parameters. The experimental setup for this test, involving a donated aorta, is illustrated in Figure 2.7.

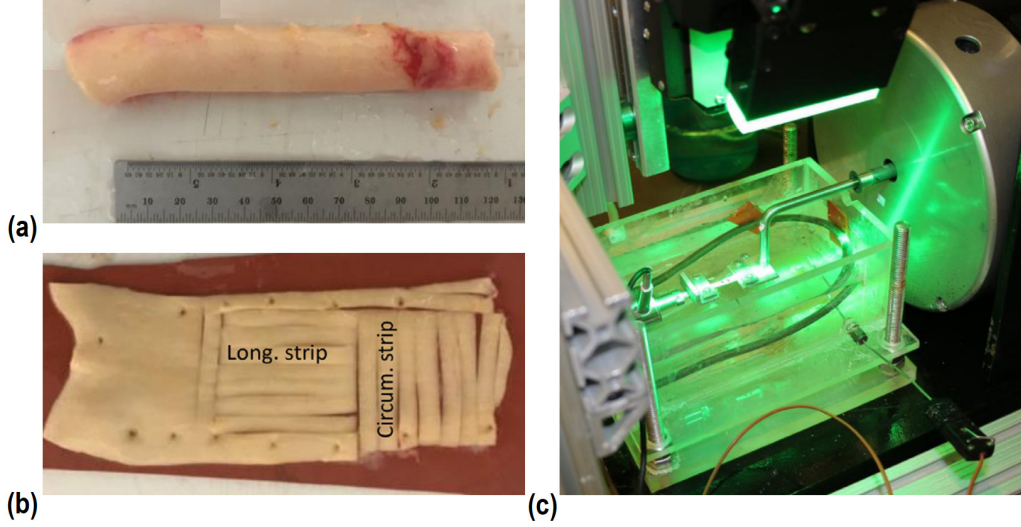


Figure 2.7 The donated aorta; **a)** descending thoracic aorta; **b)** strips cut in longitudinal and circumferential directions; **c)** separation of the intimal layer for the aortic strip [17].

Numerical and analytical methods have been employed in recent studies to investigate the nonlinear dynamics of cylindrical shell structures. Zhang et al. [61] applied Donnell's shallow shell theory to investigate thin hyperelastic cylindrical shells, taking into account three degrees-of-freedom. They used Mooney-Rivlin constitutive model to capture the material's strain energy relationship and employed the static condensation method to reduce the degrees-of-freedom in their analysis. Likewise, Zhao et al. [62] and Shayan et al. [63] adopted a consistent approach, employing the same shell theory, material constitutive model, and reduce order technique in their respective investigations. Zhao's study delved into internal resonance analysis, while Shayan's research explored the effects of various shape functions on the softening and hardening response of the system. In both studies, the coupled nonlinear equations were effectively solved using the Multiple Scales method. Zhang et al. [64] investigated the influence of coupling terms in the equations of motion. They constructed a nine-degree-of-freedom system and employed the fourth-order Runge-Kutta method for numerical solution. The study revealed that the presence of coupling effects led to an enhancement in the stability of the structural response. A general model for analyzing cylindrical, spherical, doubled-curved, and hyperbolic hyperelastic shell panels was proposed by Khaniki et al. [65]. Through the implementation of the Newton-Raphson method and the

two-dimensional Galerkin technique, they successfully solved the coupled nonlinear equations of motion, which encompassed highly nonlinear terms arising from physical nonlinearity, geometrical nonlinearity, and the nonlinearity associated with structural curvature. The researchers conducted a thorough analysis of various aspects, including both linear and nonlinear vibrations, bending behavior, and the phenomenon of internal resonance. Their study highlighted that the fundamental natural frequency is highly sensitive to changes in curvature parameters, while higher modes exhibit comparatively minor variations. It was also demonstrated that all non-dimensional natural frequency parameters exhibit an exponential decrease as the curvature ratio parameter increases. The same research team conducted an exploration of the mass sensitivity of hyperelastic plates, investigating multiple internal resonances and modal interactions through a combination of experiments and theoretical analysis [66]. Utilizing ethylene propylene diene monomer (EPDM) rubber, the experimental setup was shown in Figure 2.8. The theoretical aspect involved employing the Mooney-Rivlin model and addressing coupled nonlinear equations using Galerkin's scheme and dynamic equilibrium techniques. The study delved into the impact of concentrated mass under various time-dependent loads, revealing a rich nonlinear behavior with distinctive responses in axial, lateral, and transverse modes, showcasing complexities in the frequency response plots, and mainly the hardening behavior was observed.

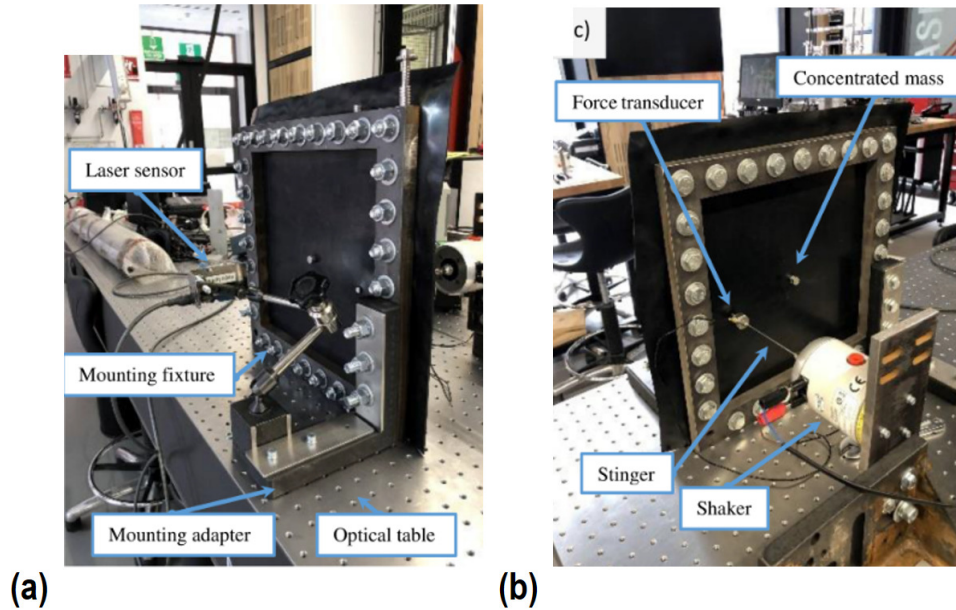


Figure 2.8 The experimental test setup **a)** fixture to hold the plate tight with 32 M12 bolts, and **b)** mounted fixture to the shaking table [66].

Xu et al. [67] investigated the impact of temperature on the mechanical properties of hyperelastic cylinders by incorporating a thermo-mechanical coupling term into Mooney-Rivlin model. The equations of motion were solved using an implicit analytical method. They noted that the impact of temperature on static deformation is not particularly significant. Additionally, in a stable temperature environment, increasing temperature leads to a reduction in the mechanical properties of hyperelastic materials. They extended their research to analyze moderately thick hyperelastic cylindrical shells by incorporating Reddy's third-order shear deformation theory and the Arruda-Boyce constitutive model [68]. They found that while a uniform temperature field had a minimal influence, the presence of a temperature gradient had a significant impact on the nonlinear vibration. Additionally, they observed that the thickness-radius ratio had a notable effect on internal resonance. The influence of geometric and temperature parameters on the natural frequency and dynamic response of thin hyperelastic cylindrical shells was investigated by Xu and colleagues [69]. The nonlinear equations of motion were tackled using both arc-length and harmonic balance methods. The outcomes underscored the substantial impact of these parameters on the response of the system. The influence of initial geometric imperfections on the internal resonance of hyperelastic cylindrical shells was explored by Zhang et al. [70]. They modeled the hyperelastic material using Mooney-Rivlin model and employed Donnell's shallow shell theory to describe the shell structure. The results indicated that both imperfection amplitude and structural parameters had similar effects on the resonance responses. The study also concluded that external excitation and geometric imperfections could significantly influence the dynamic responses of hyperelastic cylindrical shells.

To conclude this section, it is worth mentioning two review papers authored by Khaniki et al. [1, 71]. Reference [71] extensively covers various constitutive material models and their application to model soft structures while reference [1] presented a comprehensive explanation of strain energy function models and also dives deep into the nonlinear dynamic behavior of hyperelastic polymeric structural elements, including beams, plates, and shells.

CHAPTER 3 LITERATURE EVALUATION AND OBJECTIVES

Modeling and analyzing the nonlinear response of the thin hyperelastic cylindrical shell is the main objective of this thesis. The literature review underscores the critical need for research in this field, highlighting several significant gaps that warrant further investigation.

3.1 Critical evaluation of the literature and research gaps

The literature survey highlights that:

- Incorporating various nonlinear shell theories to model the hyperelastic thin cylindrical shell can influence both the accuracy and complexity of the model, leading to increased computational expenses. Thus, the choice of an appropriate shell theory that balances accuracy and computational efficiency poses a noteworthy challenge. To the best of our knowledge, an improved version of Donnell's shallow shell theory has not been previously applied to the study of thin hyperelastic cylindrical shells, nor has there been a comparison of the obtained results with the original Donnell's shallow shell theory in prior research.
- The response of the thin hyperelastic cylindrical shell varies depending on the shape of the estimated deformation of its middle surface. Surprisingly, there has not been conducted a research on how the degree of softening and hardening response is affected by the choice of shape function. In our study, we delve deeply into this topic, exploring both simple shape functions with three degrees-of-freedom and more complex shape functions that incorporate an independent second-order bending term. Therefore, our research comprehensively examines the effect of this independent bending term and its order, alongside the influence of other parameters such as structural and material characteristics, damping, and external forces, on the dynamic response of the system.
- While several studies have examined initial geometric imperfections in hyperelastic cylindrical shells, our research takes a novel approach. To the best of our knowledge, no prior work has explored the effects of different models for initial geometric imperfections and compared their influences on nonlinear system responses. In the present work, we will investigate three distinct types of initial geometric imperfections: asymmetric, axisymmetric, and combined. Furthermore, a thorough investigation into the impact of the parameters associated with each model on the natural frequency and amplitude response of the system should be conducted.

- In the analysis of thin hyperelastic cylindrical shells, the obtained natural frequencies results have been verified versus results available in the open literature. However, comparing these obtained results with powerful commercial finite element software can be very useful in achieving a more precise model. Remarkably, this verification step has not been performed in prior research. In this study, we model the structure using ABAQUS software. After initially verifying the linear model against a hybrid finite element method, the nonlinear model can be verified by considering only geometrical aspects against the existing model in the open literature. Finally, the comprehensive model incorporating both geometrical and material nonlinearities will be verified against ABAQUS model.

3.2 Objectives

To address these research gaps, this thesis outlines the following objectives:

1. Governing the equations of motion of thin hyperelastic cylindrical shell

In this order, we employ nonlinear Donnell's shallow shell theory to address the geometrical nonlinearity. To calculate the potential energy of the material and incorporate material nonlinearity, we utilize Mooney-Rivlin model. Using Lagrangian method, we derive the equations of motion by considering both kinetic and potential energy. Later, we improve the accuracy of the results, especially for cases with a lower number of circumferential waves, by utilizing an improved version of Donnell's shallow shell theory.

2. Reducing the order of the equations

Matrix structural analysis allows us to reduce the size of the problem. In this research, the static condensation method will be employed to define the degrees-of-freedom associated with zero applied forces based on other degrees-of-freedom. This reduction process yields two equations, simplifying the problem significantly.

3. Verification of model

Model verification will involve a three-step process: first, verifying the linear model; second, verifying the model with only geometrical nonlinearity; and finally, verifying the model with both geometrical and material nonlinearities.

- To verify the linear model, we will begin by eliminating the nonlinear terms from the derived equations of motion. Then, we can replace Mooney-Rivlin material nonlinearity with a linear strain energy function to remove material nonlinearity

from the equations. Subsequently, the natural frequency of the linearized equations should be calculated, and be compared with existing literature to assess the model's accuracy.

- By reintegrating the previously excluded nonlinear terms into the linearized equations, we will establish a model that exclusively addresses geometrical nonlinearity while retaining the linear strain energy density function. Employing Lindstedt-Poincaré method, we will solve these nonlinear equations to determine the natural frequency. The accuracy of the results can be verified against corresponding results available in the existing literature.
- To verify the model considering both geometrical and material nonlinearities, We will model the thin hyperelastic cylindrical shell in ABAQUS, and subsequently, the results will be compared with those obtained from this powerful commercial finite element software.

4. Force vibration analysis

To capture the dynamic behavior of the system, we will utilize the multiple-scales method to address the coupled nonlinear equations. This method allows us to create uniformly valid approximations for solutions to perturbation problems, where the solutions depend on widely different scales simultaneously. This will be achieved by introducing fast-scale and slow-scale variables for an independent variable.

5. Adding the geometric imperfection

The model will integrate geometric imperfections through shell theory. In the present work, we will introduce the different models of geometric imperfection, and this will allow us to examine their impact on both natural frequency and amplitude response.

CHAPTER 4 APPROACH

The outcomes of this thesis were presented as two journal papers.

4.1 Article 1: Analysing softening and hardening behavior in vibration of a thin incompressible hyperelastic cylindrical shell

This paper explores the impact of selected shape functions and considers the influence of geometrical and material characteristics on the free and forced vibrations of a thin hyperelastic cylindrical shell. Through the verification process, the linear model is compared against a hybrid finite element method, and the nonlinear model is assessed against existing literature. These evaluations reveal the potential of utilizing an improved version of Donnell's shallow shell theory to enhance model accuracy, especially for cases with fewer than five circumferential waves.

This paper was published in the "Thin-Walled Structures" journal on June 27, 2023. This journal is dedicated to addressing problems that arise due to the thinness of the walls in various structures, with particular attention to shell and plate structures. Furthermore, a significant factor in our selection of this journal was its extensive collection of published papers on hyperelastic structures [62, 69, 72–74].

4.2 Article 2: Investigating the influence of geometric imperfections on nonlinear responses in thin hyperelastic cylindrical shells

We extended our research by introducing three distinct initial geometric imperfections into our model and studied the impact of each imperfection on the dynamic response of cylindrical shells. We replaced Donnell's shallow shell theory with the improved version and compared both results. In this paper, we utilized ABAQUS simulations to perform a comprehensive verification of our model, which includes both geometrical and material nonlinearities.

This paper was submitted to the journal "Thin-Walled Structures" on October 4, 2023.

CHAPTER 5 ARTICLE 1: ANALYSING SOFTENING AND HARDENING BEHAVIOR IN VIBRATION OF A THIN INCOMPRESSIBLE HYPERELASTIC CYLINDRICAL SHELL

Morteza Shayan Arani, Mehrdad Bakhtiari, Aouni lakis

Published in: *Thin-Walled Structures* in June 27th 2023.

5.1 Abstract

This study focuses on nonlinear vibration of a thin incompressible hyperelastic cylindrical shell subjected to radial harmonic excitation. Mooney-Rivlin model is used to find the Strain Energy Density and model the material's nonlinear behavior. The independent bending terms are considered for shape deformation of the middle surface. The nonlinear relevant equations of motion are obtained using Donnell's nonlinear shallow-shell theory and Lagrange's equation based on the small strain hypothesis. In this study, the linear and nonlinear models are verified by comparing the present theory's natural frequency results with corresponding results obtained using a finite element method. Subsequently, the derived equations are solved under undamped and free vibration conditions, leading to an equation of motion in the form of a Duffing equation. This equation is then analyzed using the Lindstedt-Poincaré technique. Multiple Scale Method is used to address the system of coupled nonlinear equations in the presence of harmonic force and damping term. Backbone curves and amplitude response for both asymmetric and axisymmetric modes are presented by considering coupled and, non-coupled governed equations. The effects of the geometrical and material characteristics are studied on the free and forced vibration of shell structure as well as the impact of shape function, which describes the deformation of the middle surface, on both backbone and amplitude-response.

5.2 Introduction

Shell structures are widely used in various engineering fields because of their unique features, such as large bearing capacity of load and high stiffness-to-weight ratio. Regarding the introduction of advanced materials such as composites and hyperelastic materials, more complex theories and analyses should be used to design lightweight and more flexible structures. The linear Love's shell theory is derived based on the assumption that the deformations amplitudes are less than the thickness of the shell [75]. Different types of nonlinearities have been in-

troduced to capture the complexities of large-scale deformations. Geometrical nonlinearities, the important ones, consider the nonlinear relationship between strains and displacements on the mid-surface of shell. Some of those theories include Donnell's [26], Sander's-Koiter [27], Novozhilov [30] and Nemeth [76,77] theories.

In one of the earliest works on geometrical nonlinearity, Evensen [78,79] studied the cylindrical shell structures theoretically and experimentally. He employed Donnell's shell theory to model the nonlinearities considering both single and multi-bending modes. He found the softening type of behavior in the vibration of simply supported cylindrical shells.

Since then considerable studies have covered different aspects of geometrical nonlinearity, in circular shell structures. Sander's-Koiter shell theory, with more complexity and accuracy, was used by Selmane and Lakis to find natural frequency and dynamic behavior of a thin cylindrical shell [80]. Their method deals with complex shell structures due to finite element analysis. Yacine Ben-Youssef and Lakis studied thin cylindrical shell subjected to supersonic flow [81]. They proposed a new method for obtaining nonlinear stiffness matrix in order to catch the effect of shell curvature in the circumferential direction of displacement field. The obtained results showed good agreement those published in the literature. Bakhtiari and Lakis studied supersonic flutter of truncated conical shell [82,83]. They obtained the nonlinear strain energy of shell through nonlinear kinematics of Donnell's, Sanders', and Nemeth's theories. They observed that geometrical nonlinearities have a softening effect on the stability of shell in supersonic flows. Hemmatnezhad et al. [72] theoretically formulated a thin shell structure based on Sanders-Koiter's thin shell theory to study the effect of geometric imperfections in circular cylindrical shells. The results obtained by semi-analytical method showed that geometric imperfections resulted into strong localization on all mode shapes. Rodrigues et al. [73] studied the effect of small geometrical imperfections in the order of shell thickness on the internal resonance. They also investigated, dynamic instability and energy, using perturbation method, diagrams of poincarè map and time histories. Liu et al. [84] has introduced a pioneering approach to study the influence of multiple internal resonances and bifurcations in functionally graded piezoelectric shells under multi-physics fields. This method incorporates the multi-mode Galerkin scheme and Pseudo-arclength continuation techniques to effectively obtain unstable solutions, which are verified to align with previous research results. The same approach was employed to analyze the two types of functionally graded material sandwich cylindrical shells. The nonlinear dynamic response exhibited both hardening and softening behavior. It was found that, owing to the effect of nonlinear internal resonance, energy could be transferred from the driven mode to the first axisymmetric mode to some extent [85]. The effect of various boundary conditions on the behavior of circular cylindrical metal shells was studied by Wang [86]. The Rayleigh-Ritz

method was used to solve for the natural frequency of the system. The study revealed that the natural frequencies of the system under different boundary conditions tend to converge at a large circumferential wave number. Additionally, it was found that the natural frequencies of metal foam circular cylindrical shells increase with the thickness-to-radius ratio but decrease with the length-to-radius ratio. Teng and colleagues discovered that the frequency response of spinning cylindrical shells displays both softening and hardening nonlinearities, and that internal resonances could be avoided by adjusting the spinning speed of the shells [87]. To address this nonlinear behavior, the authors employed the pseudo-arclength continuation technique, which enabled them to examine energy exchanges between different modes that have internal resonance relationships. In another research, the authors utilized a similar technique to conduct a numerical analysis of the internal behavior of metal foam cylindrical shells. The response of the system in the presence of both softening and hardening exhibits regions of unstable solutions as well as regions where two or even three stable solutions occur simultaneously, depending on the frequency [88]. Besides the problem of geometrical or large displacement nonlinearity in hyperelastic structures, the relation between stress and strain of the materials is not linear either. These rubber-like materials which can elongate orders from 100% to 400% [89], are applied in aerospace, bio-mechanical, soft robotics, etc.

Knowles presented earlier studies on circular cylindrical tubes composed of hyperelastic material [53]. He used second-order model for the strain energy to develop the model of reduction methods for solving nonlinear equations of motion. The results are presented for rubber-like materials, and potential trajectories were also investigated for the periodic motion. Later, Shahinpoor proposed analytical and numerical studies to find the exact solution for simplified single degree of freedom problem [54,55]. The graphs were presented for amplitude variations against time and frequencies regarding different thicknesses and material constants. In recent years, different shell theories have been proposed for thin shell structures composed of hyperelastic materials [90–92]. Akyüz et al. analyzed the behavior of compressible hyperelastic cylindrical shell subjected to compressive dead-load traction and uniform radial tensile was by [93]. They chose the perturbation method to solve nonlinear equations of motion.

Amabili and his colleagues found that the sensitivity of deflection to the material nonlinearity is higher comparing to that of geometrical nonlinearity, especially with moderate strains [58]. He approximated the behavior of structure around the deflection point using Local Model Method (LMM). The simplicity of their method was its most advantage. Zhang addressed the hyperelastic cylindrical shell by considering a simple solution for nonlinear differential equations. He had three degrees of freedom in his study [94]. Later, a more complicated solution was considered for nonlinear equation with six and nine degrees of freedom and investigated the effect of different parameters on the vibration of structure [61,62]. Khaniki et al. [65]

proposed a general model to analyze different types of hyperelastic shell structures under both static and dynamic conditions. The authors employed the New-Raphson method and a two-dimensional Galerkin technique to handle the coupled nonlinear equations of motion. The results illustrate how curvature affected changes in static deformation, natural frequencies, and nonlinear forced vibration. The literature review highlights many studies in the field of linear and nonlinear vibration of cylindrical shell, from modeling the hyperelastic material to the geometrical nonlinear model and the solution of nonlinear governed equations.

A recent review by Khaniki, et al. [1] highlights that the substantial surge in the number of published papers on hyperelasticity and its application to model the nonlinear dynamics of polymeric structures is indicative of the growing interest and importance of this field. The increasing research output not only signifies the advancements in this area but also points towards its potential practical applications. This indicates a promising direction for further research and development in the field of hyperelasticity.

To examine the nonlinear behavior of hyperelastic materials in beams and plates, stress-strain tests were conducted. The data generated from these experiments was then utilized to perform curve fitting using some of the well-known hyperelastic constitutive models [66, 74]. A modified strain energy density function was proposed in reference [95] to capture the imperfections that may occur during the fabrication process or may be intentionally added to a component based on its intended application. The study conducted by the authors dealt with the incorporation of viscosity in hyperelastic material models has addressed as well [96]. Reference [97] explored the influence of various parameters on the nonlinear frequency response of layered thick hyperelastic beams, revealing significant variations in resonance position, maximum amplitude, coupled motion, and natural frequencies depending on the hyperelastic and layer properties used. Xu et al. [68] conducted a study on the effect of temperature on the behavior of a moderately thick hyperelastic cylindrical shell. The equations of motion were derived based on Reddy's third-order shear deformation theory. The study revealed that the thickness-radius ratio had a significant impact on internal resonance. Additionally, while the temperature gradient had a significant effect on internal resonance, a uniform temperature field had little effect on it. Recently, Khaniki, et al. conducted another review study in 2023 that examined the application of various hyperelastic strain energy models and their role in accurately modeling hyperelastic structures in mechanics and biomechanics. The study highlights a significant gap in the understanding of the behavior of such structures [71].

Yet, more detail should be studied about these nonlinear thin structures, particularly, the material analysis and the effect of geometrical properties on the softening and hardening behavior of nonlinear vibration. Assuming different approximations for the middle surface

deformation is another effective parameter in this field that deserves further studies.

In this paper, the effect of geometrical and material parameters is studied on the vibration of a thin hyperelastic cylindrical shell employing a second-power-independent term for bending. The resulted condition showed zero number of circumferential wave. The degree of freedom of nonlinear system is reduced by applying static condensation method rather than stress functions from Donnell's shallow-shell theory. The nonlinear differential equation will be solved by the multiple-scales method and Lindstedt-Poincaré technique [98,99].

This paper is organized as follows: Section 5.3 provides the equations of motion, mathematical model of material, structure, and reducing the degree of freedom of nonlinear system. Section 5.4 defines the different scenarios of vibration analysis for investigating the effect of each mode on hardening and softening behavior and also describes the method of solving nonlinear equations. Section 5.5 presents the results of the system's linear and nonlinear natural frequency and verification. In this section presents the effects of geometrical and material characteristic parameters on the vibration of a thin cylindrical shell composed of a hyperelastic material as well. Finally, section 5.6 gives the obtained conclusion.

5.3 Mathematical model and description of motion

5.3.1 Elasticity model

Unlike linear elastic materials, where the relationship between stress and strain is linear, hyperelastic materials exhibit a nonlinear relationship between stress and strain. This means that as the material is deformed under an applied load, it stores energy in the form of strain energy, which can be released when the load is removed. In order to describe the nonlinear stress-strain relationship of hyperelastic materials, we use strain energy functions. These functions mathematically capture the material's behavior under deformation, and they are typically determined experimentally or theoretically. By incorporating the strain energy function into mathematical models, we can accurately predict the deformation and stress distribution of the material under different loading conditions. Three major elasticity models, Neo-Hookean, Mooney-Rivlin, and Ogden, are presented for defining the Strain Energy Density function (SED) of hyperelastic materials [40, 89, 100] [1]. While Neo-Hookean is a general second-order model, the third-order Mooney-Rivlin model provides more accurate predictions especially in the presence of shear deformation [101]. Reference [74] provides comprehensive information on various hyperelastic strain energy density models and experimental test results for the mechanical properties of hyperelastic backbone samples, making it a valuable resource for researchers interested in this field.

This study, employs Mooney-Rivlin model with two parameters, considering its simplicity and accuracy, formulated as below [102]:

$$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + \frac{1}{D}(J - 1)^2, \quad (5.1)$$

where C_{10} , C_{01} are empirically determined material constants; I_1 , I_2 are the first and second stretch invariant, respectively; D , J are material incompressibility parameters (J is equal to unity for incompressible material). The invariants are defined, based on Green-Lagrange strain tensor $\mathbf{E}$ and right Cauchy-Green deformation tensor $\mathbf{C}$ as follow [103]:

$$\mathbf{E} = \frac{1}{2} \begin{pmatrix} 2\varepsilon_{xx} & \varepsilon_{x\theta} & 0 \\ \varepsilon_{x\theta} & 2\varepsilon_{\theta\theta} & 0 \\ 0 & 0 & 2\varepsilon_{zz} \end{pmatrix}, \quad (5.2)$$

$$\mathbf{C} = 2\mathbf{E} + \mathbf{I}, \quad (5.3)$$

where, $\mathbf{I}$ is unity matrix, then

$$\mathbf{C} = \begin{pmatrix} 2\varepsilon_{xx} + 1 & \varepsilon_{x\theta} & 0 \\ \varepsilon_{x\theta} & 2\varepsilon_{\theta\theta} + 1 & 0 \\ 0 & 0 & 2\varepsilon_{zz} + 1 \end{pmatrix}, \quad (5.4)$$

with following invariants

$$I_1 = Tr(\mathbf{C}) = 2(\varepsilon_{xx} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 3, \quad (5.5)$$

$$\begin{aligned} I_2 = \frac{1}{2}(Tr(\mathbf{C})^2 - Tr(\mathbf{C}^2)) = \\ 4(\varepsilon_{xx} + \varepsilon_{\theta\theta} + \varepsilon_{zz} + \varepsilon_{\theta\theta}\varepsilon_{xx} + \varepsilon_{\theta\theta}\varepsilon_{zz} + \varepsilon_{xx}\varepsilon_{zz}) \\ - 4(\varepsilon_{\theta z}^2 + \varepsilon_{x\theta}^2 + \varepsilon_{xz}^2) + 3, \end{aligned} \quad (5.6)$$

Given the condition of incompressibility, the strain along the z-axis cannot be zero and is instead determined by the incompressibility condition, which requires the third invariant to be equal to one. This particular strain in the z direction has been explored and utilized in past studies [66, 102]. The third stretch invariant is defined as follows:

$$I_3 = J^2 = det\mathbf{C} = 1. \quad (5.7)$$

The z direction of the strain is defined as follows:

$$\varepsilon_{zz} = \frac{1 + (2\varepsilon_{xx} + 1)\varepsilon_{\theta z}^2 + (2\varepsilon_{\theta\theta} + 1)\varepsilon_{xz}^2 - 2\varepsilon_{x\theta}\varepsilon_{xz}\varepsilon_{\theta z}}{2((2\varepsilon_{xx} + 1)(2\varepsilon_{\theta\theta} + 1) - \varepsilon_{x\theta}^2)} - \frac{1}{2}, \quad (5.8)$$

by considering the small strain assumption, Equation (5.8) could be expanded with respect to ε_{xx} , $\varepsilon_{\theta\theta}$ and $\varepsilon_{x\theta}$ [61, 94] as follows:

$$\begin{aligned} \varepsilon_{zz} = & -(\varepsilon_{xx} + \varepsilon_{\theta\theta}) + 2(\varepsilon_{xx}^2 + \varepsilon_{\theta\theta}^2 + \varepsilon_{xx}\varepsilon_{\theta\theta} + \varepsilon_{x\theta}^2) - 4\varepsilon_{xx}(\varepsilon_{xx}^2 + \varepsilon_{\theta\theta}^2 + 2\varepsilon_{x\theta}^2) \\ & - 4\varepsilon_{\theta\theta}(\varepsilon_{xx}^2 + \varepsilon_{\theta\theta}^2 + 2\varepsilon_{x\theta}^2) + 8(\varepsilon_{xx}^3\varepsilon_{\theta\theta} + \varepsilon_{xx}\varepsilon_{\theta\theta}^3 + 4\varepsilon_{xx}\varepsilon_{\theta\theta} + \varepsilon_{x\theta}) \\ & + 8(\varepsilon_{xx}^4 + \varepsilon_{\theta\theta}^4 + \varepsilon_{x\theta}^4 + \varepsilon_{xx}^2\varepsilon_{\theta\theta}^2 + 3\varepsilon_{xx}^2\varepsilon_{x\theta}^2 + 3\varepsilon_{xx}^2\varepsilon_{x\theta}^2). \end{aligned} \quad (5.9)$$

The strain energy density function of the incompressible hyperelastic material will be obtained by substituting Equation (5.9) to (5.5) and (5.6).

5.3.2 Structural modeling

Figure 5.1 shows a thin shell in a cylindrical coordinate system of (x, z, θ) in which x is in the axial direction; z is in the radial direction ; θ is in the circumferential direction ; R , h and l are radius, thickness, and length of the shell, respectively. The displacement vector of a point in the middle surface of the shell is $(u_0, v_0, \text{ and } w_0)$, and the displacement of any other point is (u, v, w) . Based on the Kirchhoff-Love hypothesis, the following relation will define [26].

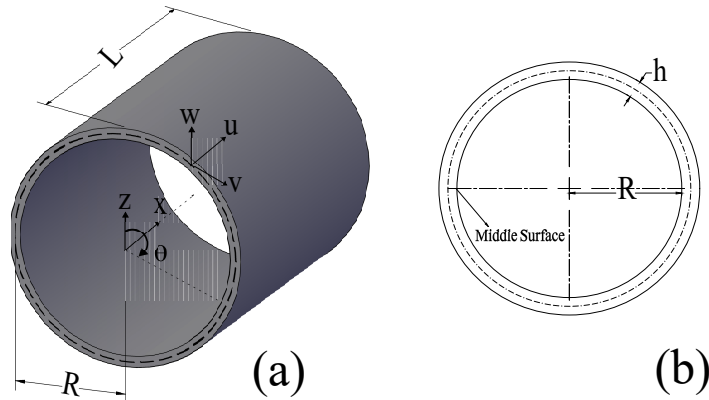


Figure 5.1 Schematic representation of shell geometry. (a) Geometrical dimension symbols (b) Cross-section of shell

$$u = u_0 - \frac{\partial w_0}{\partial x} z, \quad (5.10)$$

$$v = v_0 - \frac{1}{R} \frac{\partial w_0}{\partial \theta} z, \quad (5.11)$$

$$w = w_0. \quad (5.12)$$

To derive the strain-displacement relations, the Donnell's Shallow-Shell theory is a commonly used approach which simplifies the equations of motion for shells by neglecting in-plane inertia [26], it is also a relatively simple and easily programmable theory for numerical analysis. For those who require higher accuracy for lower circumferential wave numbers, the improved Donnell's nonlinear shell theory [33, 34] could be a more suitable alternative. The stress-strain relationship, as defined by Donnell's Shallow-Shell theory, is presented in the following manner [32].

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 - \frac{\partial w_0^2}{\partial x^2} z, \quad (5.13a)$$

$$\varepsilon_{\theta\theta} = \frac{1}{R} \left(\frac{\partial v_0}{\partial \theta} + w_0 \right) + \frac{1}{2R^2} \left(\frac{\partial w_0}{\partial \theta} \right)^2 - \frac{1}{R^2} \frac{\partial w_0^2}{\partial \theta^2} z, \quad (5.13b)$$

$$\varepsilon_{x\theta} = \frac{1}{R} \frac{\partial u_0}{\partial \theta} + \frac{\partial v_0}{\partial x} + \frac{1}{R} \frac{\partial w_0}{\partial \theta} \frac{\partial w_0}{\partial x} - 2 \frac{1}{R} \frac{\partial w_0^2}{\partial \theta \partial x} z. \quad (5.13c)$$

5.3.3 Developing the equations of motion

Nonlinear equation of motion is defined through Lagrange equation for as follows [32]:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q} = Q_i, \quad (5.14)$$

the right side of this equation Q_i is applied forces; i represent the number of the generalized coordinates q_i and $L = K - P$; K is kinetic energy and P is the potential energy presented as follows [32]:

$$K = \frac{1}{2} \rho R \int_0^l \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) dz d\theta dx, \quad (5.15)$$

$$P = R \int_0^l \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} W dz d\theta dx, \quad (5.16)$$

where "." is derivative with respect to time; ρ is the material density. W is the strain energy density which is derived from Equation (5.1). For defining the problem, we consider a thin hyperelastic thin shell, constrained in its v and w motion at both ends. The continuous domain is discretized by developing a displacement function that describes the displacements of middle surface of the shell as a function of degrees of freedom on the edges. It should be noted that discretization has a direct effect on the analysis. Hence, the boundary conditions

should be satisfied by the displacement function itself.

In the current study, the estimation has both companion and driven modes in the first term and pure bending terms in the form of square sine and cosine functions in the second term.

$$u(x, \theta, t) = u_{mn}(t) \cos(n\theta) \cos(\lambda_m x) + u_{m0} \cos^2(\lambda_m x), \quad (5.17a)$$

$$v(x, \theta, t) = v_{mn}(t) \sin(n\theta) \sin(\lambda_m x) + v_{m0} \sin^2(\lambda_m x), \quad (5.17b)$$

$$w(x, \theta, t) = w_{mn}(t) \cos(n\theta) \sin(\lambda_m x) + w_{m0} \sin^2(\lambda_m x), \quad (5.17c)$$

where, u_{mn} , v_{mn} and w_{mn} are generalized coordinates which guide us in time. The associated deformations are shown in Figure 5.2. m and n are longitudinal half-waves and circumferential wave numbers, respectively. u_{m0} , v_{m0} , and w_{m0} are generalized coordinates of bending term when the circumferential wave number is equal to zero; and λ_m is defined as follows:

$$\lambda_m = \frac{m\pi}{l}. \quad (5.18)$$

Nonlinear differential of the equation is defined by substituting Equation (5.17) into Equation (5.13) through Lagrange Equation (5.14) as follows:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}_L\mathbf{q} + \mathbf{K}_{NL1}\mathbf{q}^2 + \mathbf{K}_{NL2}\mathbf{q}^3 = \mathbf{F}, \quad (5.19)$$

where $\mathbf{M}$, $\mathbf{C}$, represent mass matrix and damping matrix respectively. Additionally, the linear, quadratic, and cubic nonlinear stiffness matrices are denoted as $\mathbf{K}_L$, $\mathbf{K}_{NL1}$, and $\mathbf{K}_{NL2}$, respectively. All matrices have dimensions of 6×6 , $\mathbf{q}$ and $\mathbf{F}$ are the column vectors with 6 elements which are the generalized coordinates and forces. The model employed in this study assumes a linear damping term, which may not accurately represent the damping behavior of hyperelastic materials, particularly in situations involving large amplitude vibrations or resonance. Nevertheless, it should be noted that the damping ratio observed in our analysis is relatively small, ranging from 0.02 to 0.06. Therefore, the possible discrepancy associated with the linear damping is not significant for the specific analysis presented in this study. For interested researchers, reference [104] provides a comprehensive study on the damping behavior of hyperelastic materials [104].

5.3.4 Discretization

Static condensation is a widely used technique in matrix structural analysis to reduce the size of the problem and speed up the solution process. It is a method for reducing the number of degrees of freedom (DOFs) in a structural system by eliminating DOFs associated with zero applied forces. Static condensation can significantly reduce the computational

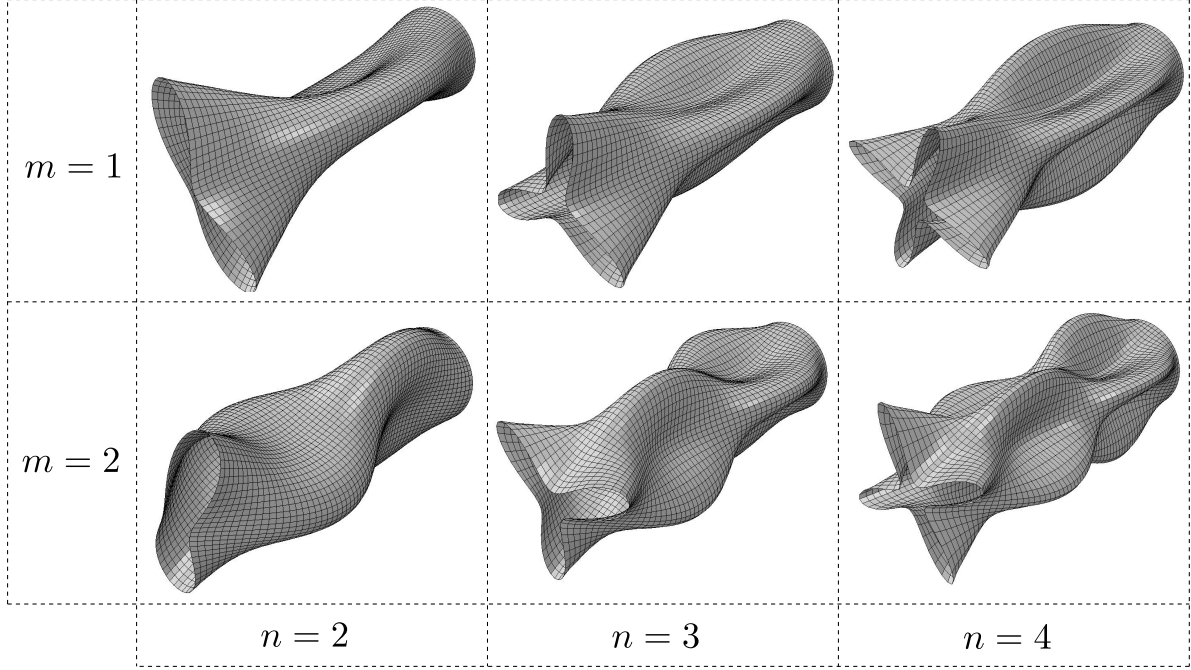


Figure 5.2 Circumferential(n) and axial modes(m) number

effort required for solving large structural systems with many DOFs. For further details on the technique, interested readers are referred to reference [35]. This method is applied to the Equation (5.19) for reducing the degree of freedom. Due to the relatively small membrane displacement compared to the radial displacement in cylindrical structures, it may be reasonable and feasible to neglect the in-plane inertia and damping when the number of circumferential waves is larger than 4 [94]. This approximation can simplify the analysis of the structural dynamic response without significant loss of accuracy. By assuming this particular assumption, the equation of motion for the normal displacement of a cylindrical shell's surface, denoted by w , can be obtained as follows:

$$\begin{bmatrix} M_{11} & \dots & M_{16} \\ \vdots & \ddots & \vdots \\ M_{61} & \dots & M_{66} \end{bmatrix}_{6 \times 6} \begin{bmatrix} \ddot{u}_{mn} \\ \ddot{u}_{m0} \\ \ddot{v}_{mn} \\ \ddot{v}_{m0} \\ \ddot{w}_{mn} \\ \ddot{w}_{m0} \end{bmatrix}_{6 \times 1} + \begin{bmatrix} K_{11} & \dots & K_{16} \\ \vdots & \ddots & \vdots \\ K_{61} & \dots & K_{66} \end{bmatrix}_{6 \times 6} \begin{bmatrix} u_{mn} \\ u_{m0} \\ v_{mn} \\ v_{m0} \\ w_{mn} \\ w_{m0} \end{bmatrix}_{6 \times 1} = \begin{bmatrix} F_{x_1} \\ F_{x_2} \\ F_{\theta_1} \\ F_{\theta_2} \\ F_{z_1} \\ F_{z_2} \end{bmatrix}_{6 \times 1}, \quad (5.20)$$

The above equation is rewritten based on the degree of freedom and reduced as below:

$$\begin{bmatrix} u_{mn} \\ u_{m0} \\ v_{mn} \\ v_{m0} \end{bmatrix}_{4 \times 1} = -J^{-1}N \begin{bmatrix} w_{mn} \\ w_{m0} \\ \ddot{w}_{mn} \\ \ddot{w}_{m0} \end{bmatrix}_{4 \times 1}, \quad (5.21)$$

where, matrices J and N are:

$$J = \begin{bmatrix} K_{11} & \dots & K_{14} \\ \vdots & \ddots & \vdots \\ K_{41} & \dots & K_{44} \end{bmatrix}_{4 \times 4}, N = \begin{bmatrix} K_{15} & K_{16} & M_{15} & M_{16} \\ \vdots & \vdots & \vdots & \vdots \\ K_{45} & K_{46} & M_{45} & M_{46} \end{bmatrix}_{4 \times 4}, \quad (5.22)$$

defining the matrix S as:

$$S = -J^{-1}N, \quad (5.23)$$

four generalised coordinates u_{mn} , u_{m0} , v_{mn} and v_{m0} are defined according to w_{mn} and w_{m0} as follows:

$$\begin{bmatrix} u_{mn} \\ u_{m0} \\ v_{mn} \\ v_{m0} \end{bmatrix}_{4 \times 1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \\ S_{31} & S_{32} \\ S_{41} & S_{42} \end{bmatrix}_{4 \times 2} \begin{bmatrix} w_{mn} \\ w_{m0} \end{bmatrix}_{2 \times 1} + \begin{bmatrix} S_{13} & S_{14} \\ S_{23} & S_{24} \\ S_{33} & S_{34} \\ S_{43} & S_{44} \end{bmatrix}_{4 \times 2} \begin{bmatrix} \ddot{w}_{mn} \\ \ddot{w}_{m0} \end{bmatrix}_{2 \times 1}. \quad (5.24)$$

Assuming w_{mn} and w_{m0} as q_1 and q_2 , the coupled equation axisymmetric and asymmetric modes are defined as below:

$$M_{55}\ddot{q}_1 + c_1\dot{q}_1 + A_1q_1 + A_2q_1q_2 + A_3q_1q_2^2 + A_4q_1^3 = F_{z_1}\cos(\omega t), \quad (5.25a)$$

$$M_{66}\ddot{q}_2 + c_2\dot{q}_2 + B_1q_2 + B_2q_2^2 + B_3q_1^2 + B_4q_1^2q_2 + B_5q_2^3 = F_{z_2}\cos(\omega t). \quad (5.25b)$$

Nondimensionalization process is performed by considering $\tau = \omega_{mn}t$ and $\Omega = \frac{\omega}{\omega_{mn}}$ while ω_{mn} is the natural frequency corresponds to different numbers of circumferential waves (n) and longitudinal half-waves (m). The following terms will be dimensionless as follow:

$$\begin{aligned}
Q_i &= \frac{q_i}{h}, & \bar{B}_1 &= \frac{B_1}{M_{66}\omega_{mn}^2}, & \bar{A}_1 &= \frac{A_1}{M_{55}\omega_{mn}^2}, \\
\bar{c}_1 &= \frac{c_1}{M_{55}\omega_{mn}}, & \bar{B}_2 &= \frac{B_2}{M_{66}\omega_{mn}^2}h, & \bar{A}_2 &= \frac{A_2}{M_{55}\omega_{mn}^2}h, \\
\bar{c}_2 &= \frac{c_2}{M_{66}\omega_{mn}}, & \bar{B}_3 &= \frac{B_3}{M_{66}\omega_{mn}^2}h, & \bar{A}_3 &= \frac{A_3}{M_{55}\omega_{mn}^2}h^2, \\
\bar{F}_1 &= \frac{F_{z_1}}{M_{55}\omega_{mn}^2h}, & \bar{B}_4 &= \frac{B_4}{M_{66}\omega_{mn}^2}h^2, & \bar{A}_4 &= \frac{A_4}{M_{55}\omega_{mn}^2}h^2, \\
\bar{F}_2 &= \frac{F_{z_2}}{M_{66}\omega_{mn}^2h}, & \bar{B}_5 &= \frac{B_5}{M_{66}\omega_{mn}^2}h^2, & &
\end{aligned} \tag{5.26}$$

In this way, Equation (5.25) can be rewritten as a pair of equations as follows:

$$\ddot{Q}_1 + \bar{c}_1\dot{Q}_1 + \bar{A}_1Q_1 + \bar{A}_2Q_1Q_2 + \bar{A}_3Q_1Q_2^2 + \bar{A}_4Q_1^3 = \bar{F}_1\cos(\Omega\tau), \tag{5.27a}$$

$$\ddot{Q}_2 + \bar{c}_2\dot{Q}_2 + \bar{B}_1Q_2 + \bar{B}_2Q_2^2 + \bar{B}_3Q_1^2 + \bar{B}_4Q_1^2Q_2 + \bar{B}_5Q_2^3 = \bar{F}_2\cos(\Omega\tau). \tag{5.27b}$$

5.4 Vibration analysis

5.4.1 Asymmetric and axisymmetric modes

In Equation (5.27), the dynamic behavior of the structure is influenced by both asymmetric and axisymmetric modes. However, simplifying the analysis by neglecting either the axisymmetric or asymmetric modes can help identify the contribution of each mode to the dynamic behavior. This approach can provide valuable insights into the system's behavior. However, by neglecting axisymmetric term in Equation (5.27a), asymmetric in Equation (5.27b) and considering the damping term and force equal to zero, Equation (5.27) yields:

$$\ddot{Q}_1 + \bar{A}_1Q_1 + \bar{A}_4Q_1^3 = 0, \tag{5.28a}$$

$$\ddot{Q}_2 + \bar{B}_1Q_2 + \bar{B}_2Q_2^2 + \bar{B}_5Q_2^3 = 0, \tag{5.28b}$$

Equations (5.28a) and Equation (5.28b) represent asymmetric and axisymmetric modes. Lindstedt-Poincaré technique is applied to solve the two aforementioned equations. The results obtained for asymmetric mode are as follows:

$$\omega_{NL} = \omega_L + \bar{A}_4\omega_1, \tag{5.29a}$$

$$\omega_1 = \frac{3}{8}a^2, \tag{5.29b}$$

and those for axisymmetric mode are as per:

$$\omega_{NL} = \omega_L + \epsilon^2\omega_1, \tag{5.30a}$$

$$\omega_1 = \frac{3}{8}\bar{B}_5 a^2 - \frac{5}{12}(\bar{B}_2)^2 a^2, \quad (5.30b)$$

Where a , and ω_L are amplitude and linear natural frequency, respectively; and ϵ is a small dimensionless parameter amplitude of oscillation.

5.4.2 Solving coupled nonlinear equations

Considering both the axisymmetric and asymmetric modes in the analysis of Equation (5.27) is crucial for obtaining a comprehensive understanding of the system's behavior. The coupled nonlinear terms and their interactions with each mode can significantly impact the overall response of the system, making it necessary to solve the equation with all terms included to accurately predict the system's behavior. Multiple-scale method is an analytical way to address this nonlinear system. In this method, ϵ is a parameter to define the nonlinearity of the equations as follows [98]:

$$\begin{aligned} C_1 &= \epsilon \bar{C}_1, & W_1 &= \epsilon Q_1, & F_1 &= \epsilon^2 \bar{F}_1, \\ C_2 &= \epsilon \bar{C}_2, & W_2 &= \epsilon Q_2, & F_2 &= \epsilon^2 \bar{F}_2. \end{aligned} \quad (5.31)$$

Hence, Equation (5.27) becomes:

$$\ddot{W}_1 + \epsilon C_1 \dot{W}_1 + \bar{A}_1 W_1 + \epsilon \bar{A}_2 W_1 W_2 + \epsilon^2 (\bar{A}_3 W_1 W_2^2 + \bar{A}_4 W_1^3) = \epsilon F_1 \cos(\Omega \tau), \quad (5.32a)$$

$$\ddot{W}_2 + \epsilon C_2 \dot{W}_2 + \bar{B}_1 W_2 + \epsilon (\bar{B}_2 W_1^2 + \bar{B}_3 W_2^2) + \epsilon^2 (\bar{B}_4 W_1^2 W_2 + \bar{B}_5 W_2^3) = \epsilon F_2 \cos(\Omega \tau), \quad (5.32b)$$

thus, instead of determining the solution of Equation (5.32) as a function of τ it can be assumed as a function of T_0 and T_1 as per:

$$\begin{aligned} W_1 &= W_{10}(T_0, T_1) + \epsilon W_{11}(T_0, T_1), \\ W_2 &= W_{20}(T_0, T_1) + \epsilon W_{21}(T_0, T_1), \end{aligned} \quad (5.33)$$

where T_i ($i=1, 2$) represent different time scales:

$$T_0 = \tau, T_1 = \epsilon \tau. \quad (5.34)$$

The first and second derivative are obtained, using the chain rule with respect to the new time scales as follows:

$$\frac{d}{d\tau} = D_0 + \epsilon D_1 + \dots, \quad (5.35a)$$

$$\frac{d^2}{d\tau^2} = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2 D_1^2 + \dots, \quad (5.35b)$$

where, D_0 and D_1 are the derivatives with respect to T_0 and T_1 , respectively. The formulas are obtained substituting Equation (5.35) and (5.33) into Equation (5.32), and separating

the zero and first order expression of ϵ , as follows:

$$D_0^2 W_{10} + \bar{A}_1 W_{10} = 0, \quad (5.36a)$$

$$D_0^2 W_{20} + \bar{B}_1 W_{20} = 0, \quad (5.36b)$$

$$D_0^2 W_{11} + \bar{A}_1 W_{11} = F_1 \cos(\Omega T_0) - 2D_0 D_1 W_{10} - D_0 C_1 W_{10} - \bar{A}_2 W_{10} W_{20}, \quad (5.37a)$$

$$D_0^2 W_{21} + \bar{B}_1 W_{21} = F_2 \cos(\Omega T_0) - 2D_0 D_1 W_{20} - D_0 C_2 W_{20} - \bar{B}_2 W_{10}^2 - \bar{B}_3 W_{20}^2. \quad (5.37b)$$

General solution of Equation (5.36) can be expressed as follows:

$$W_{10} = \phi(T_1) e^{i\sqrt{\bar{A}_1} T_0} + \bar{\phi}(T_1) e^{-i\sqrt{\bar{A}_1} T_0}, \quad (5.38a)$$

$$W_{20} = \psi(T_1) e^{i\sqrt{\bar{B}_1} T_0} + \bar{\psi}(T_1) e^{-i\sqrt{\bar{B}_1} T_0}, \quad (5.38b)$$

then Equation (5.37) based on Equation (5.38) is defined as follows:

$$D_0^2 W_{11} + W_{11} = \frac{F_1}{2} e^{i\Omega T_0} - (2D_0 D_1 + D_0 C_1)(\phi e^{iT_0} + \bar{\phi} e^{-iT_0}) - \bar{A}_2(\phi e^{iT_0} + \bar{\phi} e^{-iT_0})(\psi e^{i\Gamma T_0} + \bar{\psi} e^{-i\Gamma T_0}), \quad (5.39)$$

$$D_0^2 W_{21} + \Gamma^2 W_{21} = \frac{F_2}{2} e^{i\Omega T_0} - \bar{B}_2(\phi e^{iT_0} + \bar{\phi} e^{-iT_0})^2 - (2D_0 D_1 + D_0 C_2)(\psi e^{i\Gamma T_0} + \bar{\psi} e^{-i\Gamma T_0}) - \bar{B}_3(\psi e^{i\Gamma T_0} + \bar{\psi} e^{-i\Gamma T_0})^2, \quad (5.40)$$

where:

$$\sqrt{\bar{A}_1} = 1 \text{ and } \sqrt{\bar{B}_1} = \Gamma, \quad (5.41)$$

$$\Omega = 1 + \sigma_1 \epsilon \text{ and } \Gamma = 2 + \sigma_2 \epsilon, \quad (5.42)$$

Equations (5.39) and (5.40) are rewritten as:

$$D_0^2 W_{11} + W_{11} = \frac{F_1}{2} e^{-i\Omega T_0} - (2D_0 D_1 \phi + D_0 C_1 \phi) e^{iT_0} - \bar{A}_2 \phi \psi e^{i(1+\Omega)T_0} - \bar{A}_2 \bar{\phi} \bar{\psi} e^{i(\Omega-1)T_0}, \quad (5.43)$$

$$D_0^2 W_{21} + \Gamma^2 W_{21} = \frac{F_2}{2} e^{-i\Omega T_0} - (2D_0 D_1 \psi + D_0 C_2 \psi) e^{i\Omega T_0} - \bar{B}_2 \phi^2 e^{2iT_0} - \bar{B}_3 \psi^2 e^{2i\Omega T_0} - \bar{B}_2 \phi \bar{\phi} - \bar{B}_3 \psi \bar{\psi}. \quad (5.44)$$

Equation (5.43) is simplified through applying D_0 operator and eliminating the secular terms is presented as follows:

$$D_1 \phi = -\frac{C_1 \phi}{2} - i \frac{F_1}{4} e^{\sigma_1 \epsilon T_0} + i \frac{\psi \bar{\phi}}{2} \bar{A}_2 e^{i\sigma_2 \epsilon T_0}, \quad (5.45)$$

the same operator is applied to Equation (5.44), having:

$$D_1 \psi = -\frac{C_2 \psi}{2} + i \frac{\bar{B}_3 \phi^2}{2\Gamma} e^{-i\sigma_2 \epsilon T_0}, \quad (5.46)$$

and by considering ϕ and ψ as:

$$\phi = \frac{1}{2}a_1e^{i\beta_1} \text{ and } \psi = \frac{1}{2}a_2e^{i\beta_2}, \quad (5.47)$$

where, a_i and β_j ($j = 1, 2$) are amplitude and phase with respect to T_1 , respectively. Equation (5.45) and (5.46) are rewritten based on Equation (5.47) as follows:

$$\dot{a}_1 + (i)a_1\dot{\beta}_1 = -\frac{C_1}{2}a_1 - i\frac{F_1}{2}e^{(\sigma_1\epsilon T_0 - \beta_1)i} + i\frac{a_2a_1\bar{A}_2}{4}e^{(\sigma_2\epsilon T_0 - 2\beta_1 + \beta_2)i}, \quad (5.48)$$

$$\dot{a}_2 + (i)a_2\dot{\beta}_2 = -\frac{C_2}{2}a_2 + i\frac{\bar{B}_3a_1^2}{4\Gamma}e^{2\beta_1 - \sigma_2\epsilon T_0 - \beta_2}i, \quad (5.49)$$

by assuming γ_1 and γ_2 as bellow:

$$\gamma_1 = \sigma_1 T_1 - \beta_1 \text{ and } \gamma_2 = 2\beta_1 - \sigma_2 T_1 - \beta_2, \quad (5.50)$$

the real and imaginary terms of Equations (5.48) and (5.49) are as follows:

$$a_1 = \epsilon\left(-\frac{C_1a_1}{2} + \frac{F_1}{2}\sin\gamma_1 + \frac{1}{4}a_2a_1\bar{A}_2\sin\gamma_2\right), \quad (5.51a)$$

$$\dot{\gamma}_1 = \epsilon\left(\frac{F_1}{2a_1}\cos\gamma_1 - \frac{1}{4}a_2\bar{A}_2\cos\gamma_2 + \sigma_1\right), \quad (5.51b)$$

$$a_2 = \epsilon\left(-\frac{C_2a_2}{2} - \frac{\bar{B}_3a_1^2}{4\Gamma}\sin\gamma_2\right), \quad (5.51c)$$

$$\dot{\gamma}_2 = \epsilon\left(-\frac{\bar{B}_3a_1^2}{4\Gamma a_2}\cos\gamma_2 - \sigma_2 - \frac{F_1}{a_1}\cos\gamma_1 + \frac{1}{2}a_2\bar{A}_2\cos\gamma_2\right). \quad (5.51d)$$

in the steady-state response, a and γ are constant. Then their derivatives will be zero:

$$-\frac{C_1a_1}{2} + \frac{F_1}{2}\sin\gamma_1 + \frac{1}{4}a_2a_1\bar{A}_2\sin\gamma_2 = 0, \quad (5.52a)$$

$$\frac{F_1}{2a_1}\cos\gamma_1 - \frac{1}{4}a_2\bar{A}_2\cos\gamma_2 + \sigma_1 = 0, \quad (5.52b)$$

$$-\frac{C_2a_2}{2} - \frac{\bar{B}_3a_1^2}{4\Gamma}\sin\gamma_2 = 0, \quad (5.52c)$$

$$-\frac{\bar{B}_3a_1^2}{4\Gamma a_2}\cos\gamma_2 - \sigma_2 - \frac{F_1}{a_1}\cos\gamma_1 + \frac{1}{2}a_2\bar{A}_2\cos\gamma_2 = 0. \quad (5.52d)$$

The amplitude of asymmetric and axisymmetric modes are obtained by eliminating γ_1 , γ_2 in Equation (5.52). An in-house computer program in Python programming language is developed to solve these equations.

5.5 Results and discussion

5.5.1 Validation of the hyperelastic elasticity model

To fully understand the material behavior of an isotropic hyperelastic material, it is necessary to obtain stress-strain data under both uniaxial and biaxial loading conditions. This is due to the potential for the material to exhibit different behaviors under different loading conditions. The material's inherent anisotropy and nonlinear deformation behavior can both contribute to its response to various loading conditions [102].

In this research eight percent sulfur rubber is chosen as the hyperelastic material. Figure 5.3 demonstrates the comparison between estimated nominal stress-engineering relationship obtained through proposed Mooney-Rivlin elasticity model with the experimental results provided by Treloar [105]. According to the figure, the values of strain below 30% of thickness, the elasticity model is in excellent accordance with the experimental results for uniaxial and equibiaxial tensions as well as pure shear. It should be noted that to model the material's behavior in the presence of strains more than 30%, employing the Ogden and Polynomial models can provide more accurate results [100, 106].

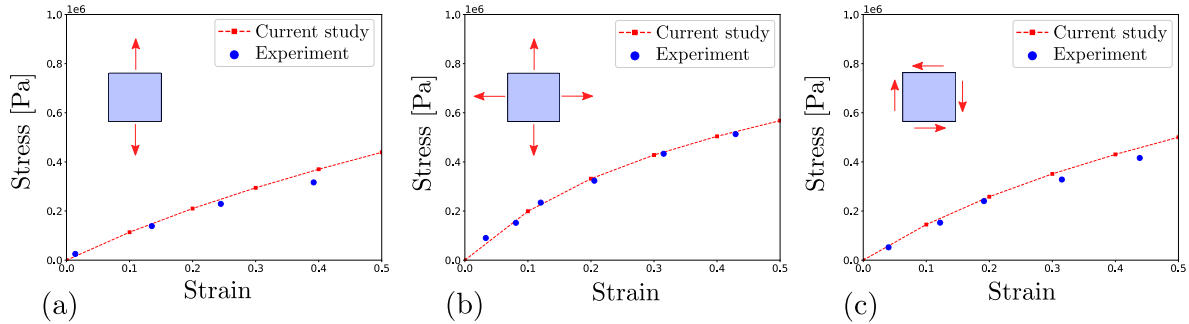


Figure 5.3 Nominal stress-strain curve; the experimental data is from the reference [105] for vulcanised rubber. a) Uniaxial test. b) Equibiaxial test. c) Pure shear test

In order to fit the experimental data with Mooney-Rivlin model, the following expressions for nominal stress are used for uniaxial, equibiaxial and pure shear, respectively [103, 106].

$$S_1 = C_{10}((\varepsilon + 1) - (\varepsilon + 1)^{-2}) + C_{01}(1 - (\varepsilon + 1)^{-3}), \quad (5.53a)$$

$$S_1 = C_{10}((\varepsilon + 1) - (\varepsilon + 1)^{-5}) + C_{01}((\varepsilon + 1)^3 - (\varepsilon + 1)^{-3}), \quad (5.53b)$$

$$S_1 = (C_{10} + C_{01})((\varepsilon + 1) - (\varepsilon + 1)^{-3}), \quad (5.53c)$$

here, C_{10} and C_{01} represent material constants, while ε and S_1 are the nominal strain and stress, respectively.

5.5.2 Verification of linear and nonlinear models

In order to assess the validity of model accuracy and develop the solution, first, linear and nonlinear natural frequencies of selected cases are verified with the results available in the literature.

Static condensation

Linear natural frequency of the system with and without considering the in-plane inertia, is computed for finding the relative difference between the results obtained before and after the static condensation method for a structure with characteristic parameters equal to $\alpha = h/R$ and $\beta = 2R/l$. The material parameters are $\mu_1 = 416185.5Pa$, $\mu_2 = -498.8Pa$, and $\rho = 1100kg/m^3$. According to the results presented in Table (5.1), less than one percent difference is seen between these two methods, considering the circumferential wave numbers larger than five.

Table 5.1 NF and NF-SC: Natural frequency before and after applying static condensation method, respectively.

n	NF(Hz)	NF-SC(Hz)	Error %
4	3.31	3.40	2.71
5	4.72	4.79	1.48
6	6.64	6.70	0.90
7	8.949	9.00	0.67
8	11.62	11.66	0.34
9	14.65	14.69	0.27

Linear natural frequency

Linear natural frequency is the fundamental characteristic of elastic systems. Additionally, nonlinear frequency (as a function of vibration amplitude) is often expressed as a value related to the linear frequency. Hence, this makes it more important to validate the linear natural frequency against those reported in the literature. This is achieved by omitting nonlinear components of the equations of motion and performing an eigenvalue analysis to obtain those frequencies. The incorporation of the strain energy density of an isotropic elastic material into the material model for hyperelastic materials requires the use of the linear strain energy

relation. To accomplish this, Equation (5.54) was utilized to accurately model the mechanical behavior and properties of the isotropic material and to incorporate its strain energy density into the hyperelastic material model [32, 106]:

$$W = \frac{E}{2(1 + \nu)}(4 + \varepsilon_{xx}^2 + \varepsilon_{xx}\varepsilon_{\theta\theta} + \varepsilon_{x\theta}^2), \quad (5.54)$$

E represents the Young's modulus, ν represents the Poisson's ratio, and ε_{xx} , $\varepsilon_{\theta\theta}$, and $\varepsilon_{x\theta}$ denote the normal strains in different directions. Table (5.2) presents the results obtained for two isotropic cylindrical shells (A and B) with the geometry and elastic properties. The obtained results are compared with those of hybrid finite element method used by Lakis [107] as well as ones of Chebyshev method used by Pellicano [108] and presented in Table (5.3). According to the table, in both cases linear natural frequencies are in well agreement with

Table 5.2 The material and structural parameters of the simply supported cylindrical shell.

Cases	R [m]	L [m]	h [m]	ν	E [pa]	ρ [kg/m ³]
<i>CaseA</i>	0.175	0.664	0.001	0.3	2.6×10^{11}	7680
<i>CaseB</i>	0.1	0.2	0.247×10^{-3}	$\times$ 0.31	71.02×10^9	2796

those presented in literature.

Table 5.3 Verification of linear part natural frequency; ω_{LR} is the linear natural frequency of the reference; ω_{LP} is the linear natural frequency of the present model.

Cases	m	n	$\omega_{LR}(Hz)$	$\omega_{LP}(Hz)$	Error%
CaseA Lakis [107]	1	2	1133	1068	5.73
	1	3	629	607	3.49
	1	4	655	671	2.44
	1	5	942	962	2.12
	1	6	1353	1364	0.81
	1	7	1853	1849	0.21
	1	7	1853	1849	0.21
CaseB Pellicano [108]	1	5	722	711	1.52
	1	6	553	538	2.71
	1	7	484	475	1.85
	1	8	489	488	0.20
	1	9	546	548	0.36
	1	10	636	640	0.62
	1	11	750	744	0.26
	1	12	882	881	0.10

Nonlinear natural frequency

For verifying nonlinear model, the ratios of nonlinear to linear frequency of a homogeneous cylindrical shell are computed, employing Lindstedt-Poincaré technique. The obtained results are compared with those of presented by Wang [109]. The properties of isotropic cylindrical shell are: $h = 0.00254m$, $R = 0.254m$, $l = 0.4m$, $\rho = 7800kg/m^3$, $\nu = 0.3$, $E = 200GPa$, and the amplitude=0.00254. The results show in Table(5.4) relatively good agreement. The differences can be attributed to the slightly different methods of discretization.

Table 5.4 Comparing nonlinear and linear frequency ratios for a homogeneous cylindrical shell.

Mode(m,n)		(1,2)	(1,3)	(1,4)
ω_{NL}/ω_L	Present	1.0006	1.0034	1.0144
	Wang [109]	1.0009	1.0033	1.0117
	Difference	0.0003	0.0001	0.0027

5.5.3 Natural frequency of hyperelastic cylindrical shell

Figure 5.4 shows the variation of natural frequencies with the variation of radius to thickness ratio. It is observed that increasing the thickness results in higher natural frequency. This is expected since thicker shells are stiffer, which affects the stiffness matrix and subsequently increases the natural frequency. Figure 5.5 presents the variations of natural frequency with the radius to length ratio variations. As can be seen, taller shells demonstrate lower natural frequencies at a constant thickness to radius. This can be attributed to relatively lower stiffness at a constant thickness. Figure 5.6 shows the variation of natural frequency for different longitudinal mode numbers. This presents a relatively more complex behavior. However, the variation of natural frequency with the circumferential mode numbers evolves into a linear behavior at higher longitudinal mode numbers, in comparison to the second order trend with a clear minimum for the lowest longitudinal mode numbers.

5.5.4 Nonlinear vibration of hyperelastic cylindrical shells

Figures 5.7, and 5.8 show the backbone curves of Duffing equation of the asymmetric mode; Equation (5.28a). Varying the vibration amplitude in Equation (5.29b), the backbone tends to the right side, considered as a hardening behavior. This hardening behavior becomes stronger by increasing circumferential and axial wave numbers. Figure 5.9 shows the backbone curve in the axisymmetric mode. As can be seen the peak of the backbone in the

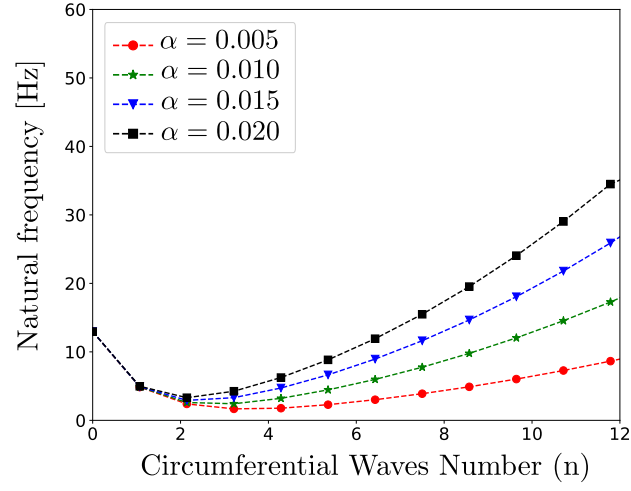


Figure 5.4 Natural frequency of the hyperelastic cylindrical shell with the given material and structural parameters; the axial wave number is equal to one and $\beta = 0.5$.

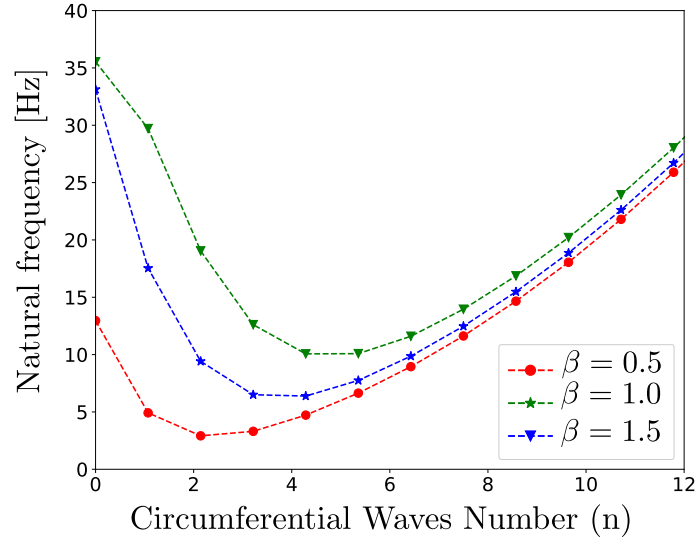


Figure 5.5 Natural frequency of the hyperelastic cylindrical shell with the given material and structural parameters; the axial wave number is equal to one and $\alpha = 0.015$.

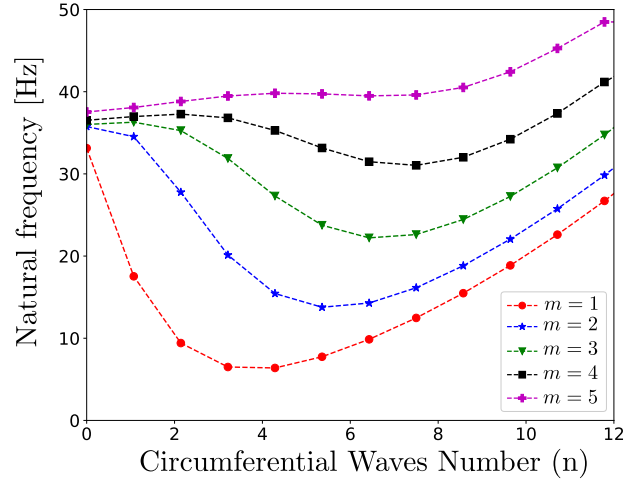


Figure 5.6 Natural frequency of the hyperelastic cylindrical shell; the structural parameters $\alpha = 0.015$ and $\beta = 1.0$.

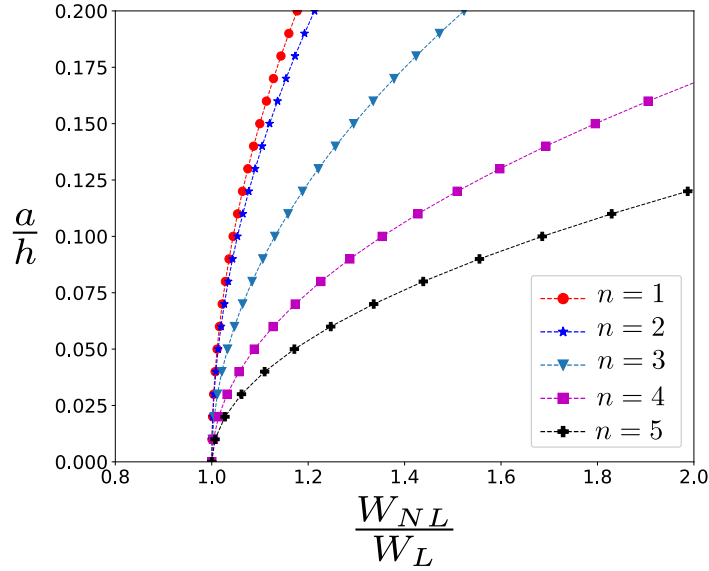


Figure 5.7 Backbone of the asymmetric mode for different circumferential wave numbers with the axial wave number of one.

axisymmetric mode goes to the left side, which indicates a softening behavior. The sign of quadratic non-linearity of the stiffness in Equation (5.28b) shows no effect on the system's behavior. Still, cubic non-linearity with a negative sign causes the softening behavior, while a positive sign causes hardening in the shell. Considering the axial wave numbers more promi-

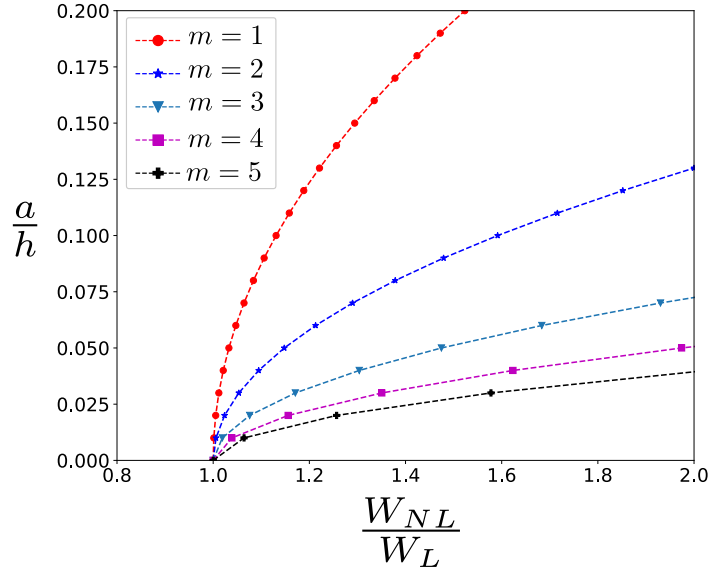


Figure 5.8 Backbone of the asymmetric mode for different axial wave number with the circumferential wave number of three.

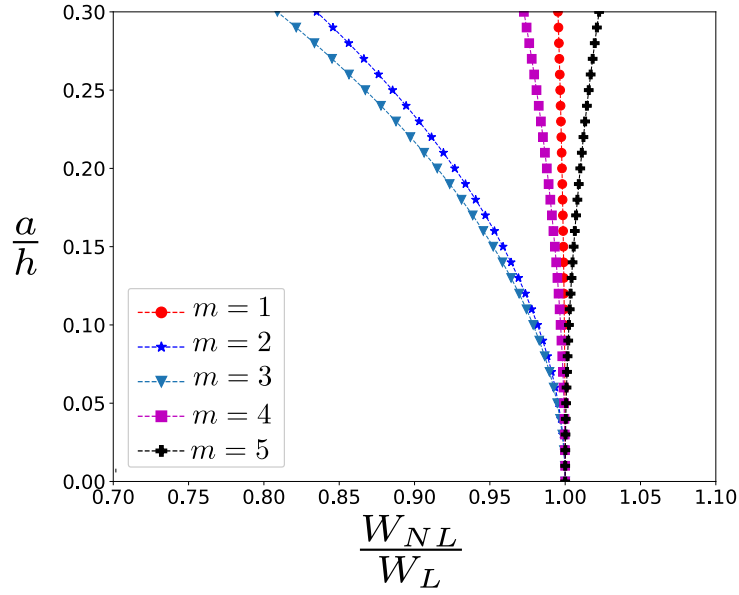


Figure 5.9 Backbone of the axisymmetric mode for different axial wave numbers with the circumferential wave number of zero.

nent than three, the softening behavior is smooth, and the response yields to the right side compared to the lower number of the longitudinal half-waves.

5.5.5 Analysing the effect of independent bending term

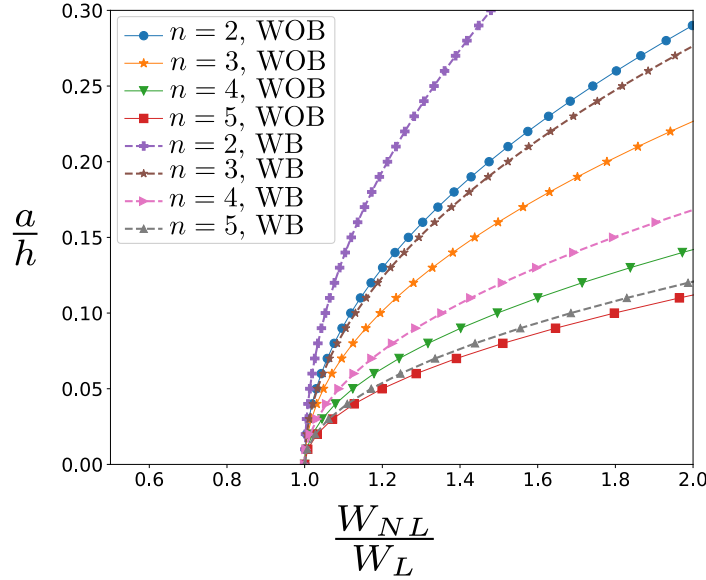


Figure 5.10 Comparison between without bending (WOB) and with bending (WB) with $m = 1$.

In the present study, the middle surface deformation is approximated by including a second-order independent term of either $\sin$ or $\cos$. Neglecting these terms related to bending reduces the number of generalized coordinates from six to three. As a result, the system can be defined using only one Duffing equation. The Lindstedt-Poincaé method is applied to solve this equation. Only including the cubic stiffness non-linearity indicates a hardening-type behavior of the system. We demonstrate the comparison between this situation (without the bending term, WOB) and the situation with a separated term for bending (WB) in Figure 5.10. Eliminating the bending terms in the middle surface deformation approximation causes the system's backbone to display a more pronounced hardening behavior. This is because the bending terms introduce a softening effect that opposes the hardening effect of the cubic stiffness non-linearity.

5.5.6 Forced vibration amplitude response analysis

The amplitude-response curve for axisymmetric and asymmetric modes are obtained using final solution of Equation (5.52), and it is shown in Figure 5.11. There are two peaks in the associated curve of each mode; one bends to the right side and shows the hardening-

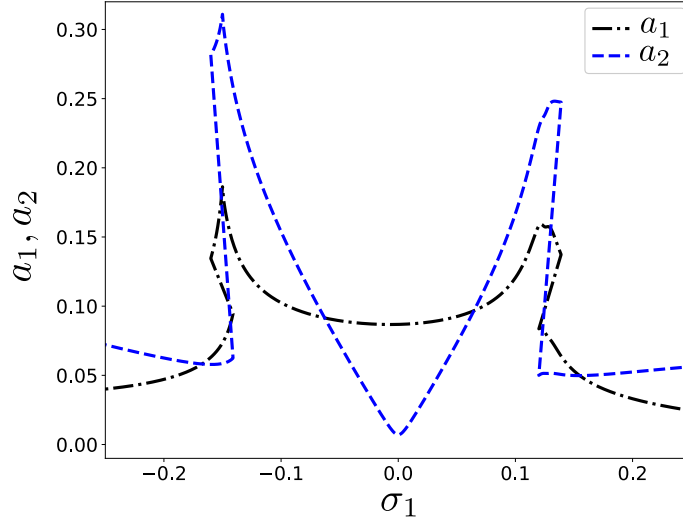


Figure 5.11 Axisymmetric a_1 and asymmetric modes a_2 response.

type behavior; the other bends to the left side in a softening behavior. The presence of these two peaks could be attributed to internal resonance. Internal resonance is a nonlinear phenomenon that can occur when a physical system has multiple modes of vibration that are very close in frequency. In this situation, the energy of one mode can transfer to another mode, leading to a resonant response of the system. This transfer of energy can occur through various nonlinear mechanisms, such as parametric excitation, mode coupling, or modal interaction.

Influence of applied force

By applying different forces $F_1 = [0.13, 0.15, 0.17, 0.19, 0.20]$ Newton, their effects on amplitude-response was shown in Figures 5.12, and 5.13. As can be seen, the excitation amplitude has a direct impact on the response of the system. The system's response exhibited nonlinear softening and hardening behaviors, which cause it to become softer or harder depending on the amount of applied force. By increasing the force amount, the peaks of the curves reach higher.

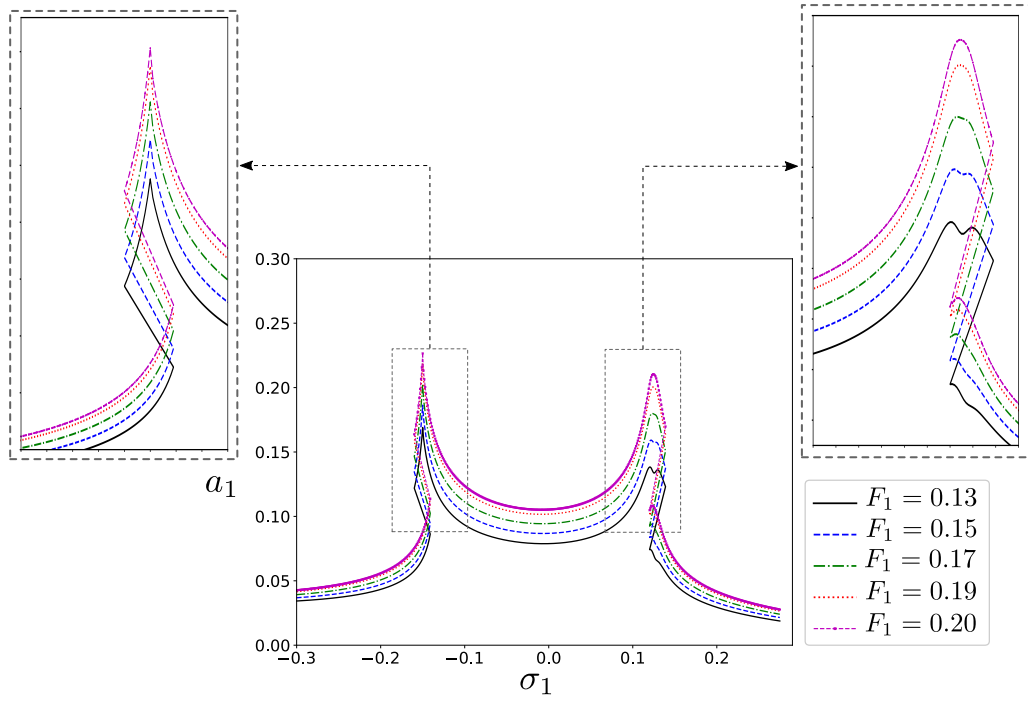


Figure 5.12 Amplitude-frequency response of the axisymmetric mode for different dimensionless forces.

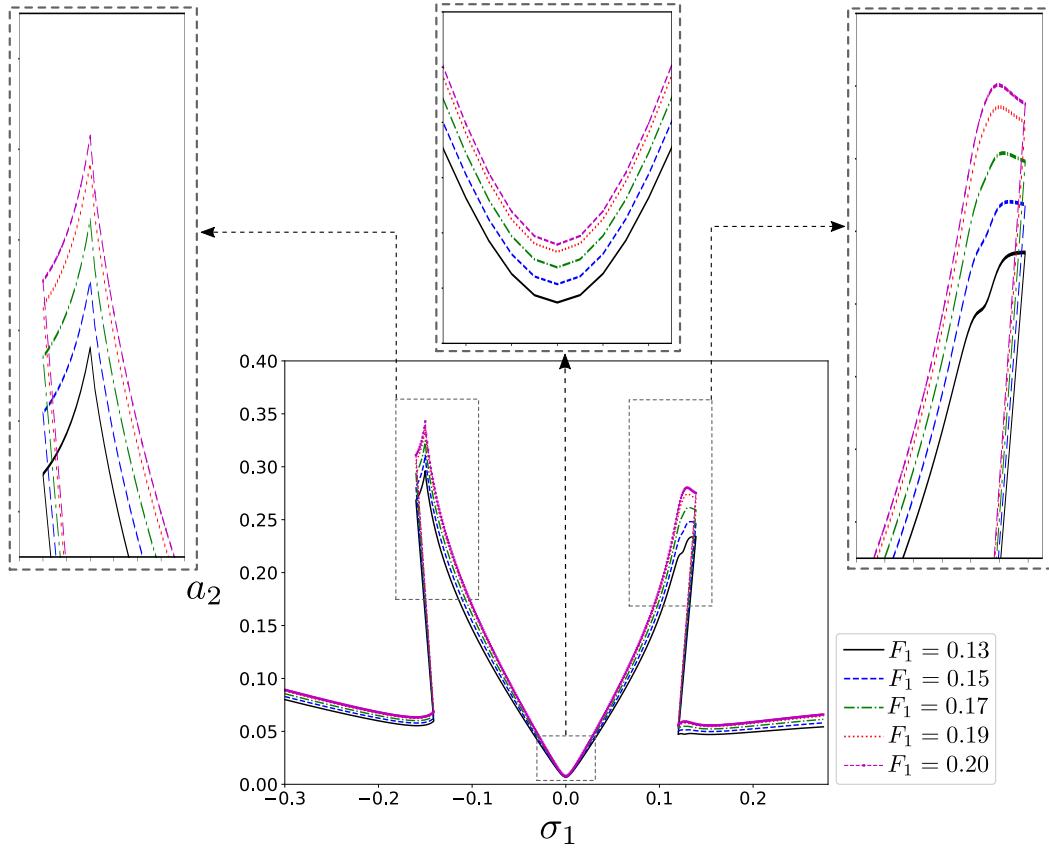


Figure 5.13 Amplitude-frequency response of the asymmetric mode for different dimensionless forces.

Influence of damping term

Figures 5.14, and 5.15 present the effect of the damping parameter (c_1) in both axisymmetric and asymmetric modes. Our analysis showed that (c_1) had a significant impact on the hardening behavior of the system. Specifically, decreasing the value of (c_1) from 0.05 to 0.01 resulted in a more pronounced hardening behavior and an increase in the peak amplitude response. Notably, a slight deviation in the curve pattern was observed in Figure 5.14 for the axisymmetric mode when (c_1) was set to 0.01, while Figure 5.15 showed a more significant unstable behavior for the lowest damping terms before reaching the peaks in the case of the asymmetric mode. This indicates that the system's response for the asymmetric mode is more sensitive to (c_1) changes than the axisymmetric mode. These observations highlight the importance of carefully selecting and tuning the damping parameter (c_1) to optimize the system's response and stability.

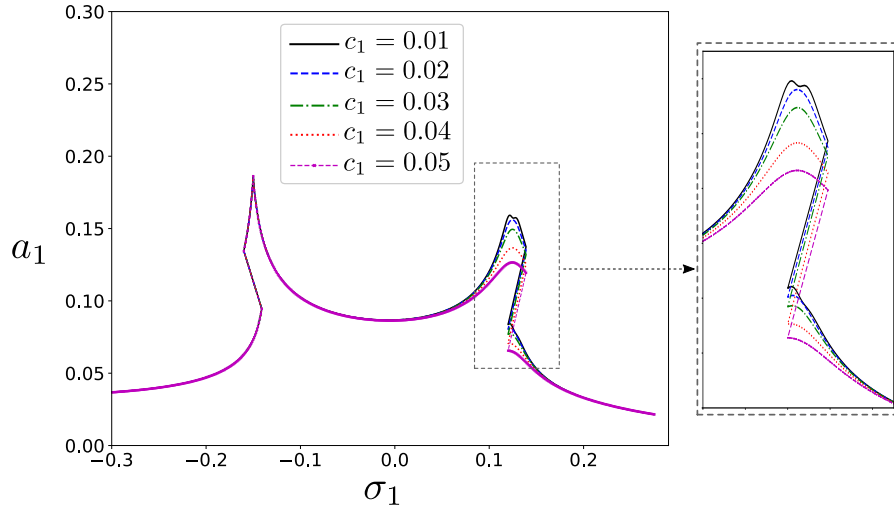


Figure 5.14 Amplitude-frequency response of the axisymmetric mode for different dimensionless damping parameters.

Influence of structural characteristic

Figure 5.16 shows the effect of the structural parameter α , which represents the ratio of thickness to the radius of the cylindrical structure. It was observed that increasing α resulted in higher peaks for both axisymmetric and asymmetric modes. However, at $\alpha = 0.048$, a smooth deviation was observed in the curve for the hardening peak, indicating a change in the system's response behavior. A sharp drop was observed in the hardening peak by having a lower amount for $\alpha = 0.30$. Having a thinner structure would cause it to have a softer

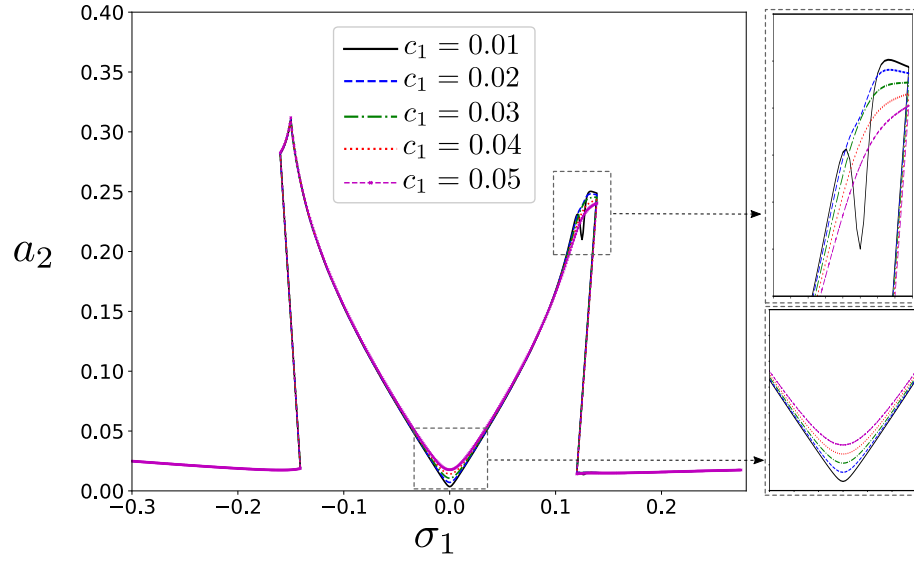


Figure 5.15 Amplitude-frequency response of the asymmetric mode for different dimensionless damping parameters.

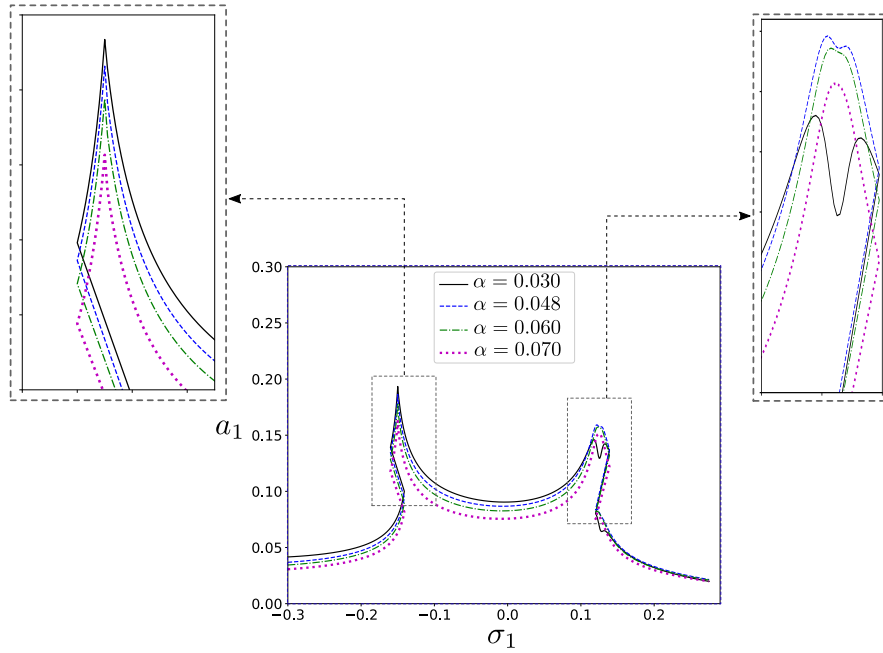


Figure 5.16 Amplitude-frequency response of the axisymmetric mode for different structural parameters α .

structure. These findings highlight the sensitivity of the system's response to changes in the structural parameter (α).

Influence of bending term order

In this study, we have considered independent second-order bending terms for the shape deformation of the middle surface in Equation (5.17). In Section 5.5.5, we analyzed the impact of these terms on the backbone behavior of the system. Figure 5.17 presents the amplitude response for the first-order and second-order bending terms. We observed that while the two models exhibit almost the same hardening behavior, there is a significant difference in the softening peak. The first-order bending terms capture a smaller amount of softening compared to the second-order terms. These findings suggest that accounting for second-order bending terms is important for accurately modeling the softening behavior of the system.

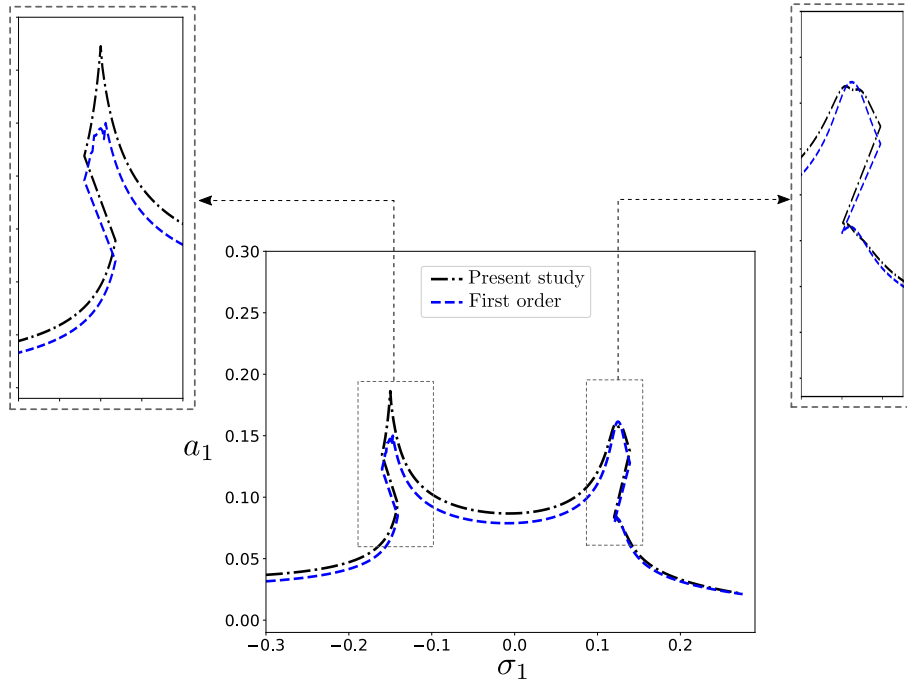


Figure 5.17 Amplitude-frequency response of the axisymmetric mode for first-order independent bending term and second-order.

5.6 Conclusions

In this work, nonlinear vibration is investigated in the incompressible hyperelastic cylindrical shells modeled by Mooney-Rivlin under a radial harmonic load. Lagrangian method is applied to obtain the governing equations of thin cylindrical shells based on Donnell's Shallow Shell theory for geometrical nonlinearities under the small strain assumption. The deformation of the middle surface is approximated using an axisymmetric periodic response augmented by a time-invariant bending term modeled by square sine and cosine trigonometric functions. Lindsedt-Poincare and Multiple Scale Method are applied to Duffing equation. The resulting nonlinear equations have cubic and quadratic nonlinear terms. Static Condensation Method is used to decrease the degree-of-freedom of the problem. The verification of linear and nonlinear models are conducted for a simply supported homogeneous cylindrical shell, showing good agreements with those in the literature. Natural frequencies are calculated for a set of hyperelastic cylindrical shells with varying thickness to radius and radius to length ratios. The effects of geometric parameters were discussed on the nonlinear response. The hardening and the softening behaviors were also investigated in a hyperelastic cylindrical shell. The obtained conclusions are given as follows:

- Asymmetric mode shows the hardening-type behavior; the hardening becomes stronger with increasing circumferential waves numbers.
- Axisymmetric mode with cubic and quadratic nonlinearity terms shows softening-type behavior. However, the sign of cubic nonlinear stiffness can reverse this behavior.
- Coupled nonlinear equation presents two peaks, showing the softening and hardening behavior. It is observed that amplitude of the vibration increases with increasing the force. It is also found that the damping ratio (c_1) exerts a stronger effect on the hardening side of the response. Moreover, softening peak rises considerably with changing the structure's parameters.
- Eliminating the bending terms would lead to a reduction in the amount of hardening in backbone analysis.
- By having a second-order bending term more amount of softening can be captured from the systems. While the bending term doesn't have a significant effect on the hardening peak of amplitude response.

CHAPTER 6 ARTICLE 2: INVESTIGATING THE INFLUENCE OF GEOMETRIC IMPERFECTIONS ON NONLINEAR RESPONSES IN THIN HYPERELASTIC CYLINDRICAL SHELLS

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6.1 Abstract

This study presents a comprehensive analysis of hyperelastic thin cylindrical shells exhibiting initial geometrical imperfections. The nonlinear equations of motion are derived using an improved formulation of Donnell’s nonlinear shallow-shell theory and Lagrange’s equations, incorporating the small strain hypothesis. Mooney-Rivlin constitutive model is employed to capture the hyperelastic behavior of the material. The coupled nonlinear equations of motion are analytically solved using Multiple-Scales method, which effectively accounts for the inherent nonlinearity of the system. To ensure the model’s accuracy, the linear model is verified by comparing the results with those obtained through hybrid finite element method. Subsequently, the model with only geometrical nonlinearity is evaluated against other research works existing in the open literature to ensure its reliability and precision. Finally, the results of the model, considering both geometrical and physical nonlinearity, are verified against the results obtained from Abaqus software. The main objective of this research is to provide a detailed understanding of the response of hyperelastic thin cylindrical shells in the presence of initial geometric imperfections. In this order, the impact of three distinct geometric imperfections—axisymmetric, asymmetric, and a combination of driven and companion modes—on the natural frequency is examined. The behavior of each of these geometric imperfections is investigated by varying their respective coefficients. The numerical results indicate that geometric imperfections enhance the natural frequency, and employing different models for imperfections leads to a variation in this trend. In the amplitude response of hyperelastic cylindrical shells, two peaks coexist, reflecting the softening and hardening responses of the system. Distinct initial geometric imperfections influence these two peaks.

6.2 Introduction

Cylindrical shell structures have found extensive application in diverse engineering fields due to their distinctive characteristics, including a remarkable load-bearing capacity and a high stiffness-to-weight ratio. The study of these structures was approached as a linear problem, assuming that the deformation amplitudes were less than the thickness of the shell [110]. The behavior of cylindrical shell structures under linear assumptions has been extensively studied, providing a comprehensive understanding [108, 111–114]. However, it is important to note that linear assumptions may not fully capture the complex behavior of shell structures, particularly when subjected to larger deformations or nonlinear effects. Therefore, there is increasing recognition of the need to explore beyond linear assumptions and consider the influence of nonlinear behaviors for a more accurate understanding and analysis of shell structures. Geometrical nonlinearities resulting from large deformations, nonlinearity associated with boundary conditions, and physical or material nonlinearity are all factors contributing to the overall nonlinear response of shell structures. Geometrical nonlinearity has been a topic of extensive investigation by numerous researchers throughout the years [32, 58, 79, 81, 83, 86, 115, 116]. The sensitivity of deflection to material nonlinearity has been observed to be greater than that of geometrical nonlinearity, particularly with moderate strains [58]. These findings highlight the significant effect of physical nonlinearity on the response of the structure, underscoring its importance in certain cases. Hyperelastic materials are one example of materials that can undergo significant strains while still maintaining their ability to return to their original shape after the applied load is removed [89]. These materials exhibit a nonlinear stress-strain relationship and do not adhere to Hooke’s law, which characterizes linear elasticity. Understanding and analyzing the response of hyperelastic cylindrical shells requires considering the combined effects of geometric and physical nonlinearity, making it a complex and challenging research problem. In a research paper review by Khaniki et al [1], it was shown that the number of researchers focusing on the hyperelastic structure has noticeably increased over the past three decades, with more than 1100 research studies published in 2020 alone. The significant growth of studies in this subject can be attributed to its wide-ranging applications across various fields, including aerospace [117], biomedical engineering [118], soft structures [119], etc. These studies primarily fall into three research areas: advancing hyperelastic material models, enhancing structural models, and developing methodologies to analyze and solve highly nonlinear equations.

When it comes to material modeling, specialized constitutive models such as Mooney-Rivlin, Neo-Hookean, and Ogden models are employed to characterize the stress–strain behavior of hyperelastic materials [40, 120]. Numerous comprehensive research exist in the literature

regarding hyperelastic constitutive models, including notable references such as [121–123]. In a recent experimental and theoretical study conducted by Khaniki et al. [66], the properties of a hyperelastic material were determined through a series of stress-strain tests. Amabili made a notable contribution by successfully combining the theory of shells with nonlinear materials, marking a significant advancement in the field [38].

Among the nonlinear shell theories, including Sanders-Koiter [27], Novozhilov [30], Flugge-Lur'e-Byrne [124], and Donnell's nonlinear theory [26], Donnell's shallow shell theory (a specific version of Donnell's nonlinear theory with neglected in-plane inertia) has gained significant popularity for practical applications, especially for thin structures. This preference can be attributed to its simplicity and ease of application. Zhang et al. [94] employed Donnell's shallow shell theory to model thin hyperelastic cylindrical shells with three degrees of freedom. They utilized Mooney-Rivlin constitutive model to calculate the material's strain energy and reduced the degrees of freedom using a condensation method. The resulting nonlinear equations of motion were solved using the Lindstedt-Poincaré method. Similarly, Zhao et al. [62] and Shayan et al. [63] also utilized the same shell theory, material model, and reduced method in their research studies. Zhao's work focused on analyzing internal resonance, while Shayan's study examined the impact of different shape functions on system behavior. The system of coupled nonlinear equations in both studies was solved using Multiple Scales method. Zhang and colleagues expanded the modeling of thin hyperelastic cylindrical shells by incorporating nine degrees of freedom to enhance the accuracy of capturing their behavior. They utilized fourth-order Runge-Kutta method to solve the resulting three nonlinear equations and conducted an investigation into the effects of coupled terms in their analysis [61]. It was found that the coupled terms could improve the stability of the system. In their comprehensive investigation, Khaniki and colleagues [65] presented a general model for analyzing different types of hyperelastic shell structures. Through the implementation of Newton-Raphson method and two-dimensional Galerkin technique, they successfully solved the coupled nonlinear equations of motion, which encompassed highly nonlinear terms arising from physical nonlinearity, geometrical nonlinearity, and the nonlinearity associated with the structural curvature. The researchers conducted a thorough analysis of various aspects, including both linear and nonlinear vibrations, bending behavior, and the phenomenon of internal resonance. Their research underscored the significant sensitivity of the fundamental natural frequency to variations in curvature parameters, while the higher modes remained relatively stable and with the least change. Xu et al. [67] conducted a study that revealed how increasing temperature can lead to a reduction in the mechanical properties of a hyperelastic cylindrical structure. In their research, they considered the inclusion of a thermo-mechanical coupling term within Mooney-Rivlin strain energy of the system.

Furthermore, they observed that the impact of temperature on static deformation was not particularly significant. In another study, Xu et al. [68] investigated the effect of temperature on the nonlinear behavior of a moderately thick hyperelastic cylindrical shell. By utilizing ten generalized coordinates as the degrees of freedom of the system, employing Reddy's third-order shear deformation theory for structural modeling, and the Arruda-Boyce constitutive model for material behavior, they found that while the uniform temperature field had minimal influence, the temperature gradient had a significant impact on the observed phenomenon. In their study, Wang et al. [125] investigated various types of nonlinear solitary waves in a thermo-hyperelastic cylindrical shell. They employed the Neo-Hookean model to calculate the strain energy density of the material and addressed the nonlinear equations using bifurcation theory. Through their research, they gained insights into the behavior of nonlinear solitary waves within uniform and non-uniform temperature fields. Xu and colleagues [69], the influence of structural and temperature parameters on the natural frequency and nonlinear vibration of thin hyperelastic cylindrical shells was investigated. The researchers utilized the arc-length and harmonic balance methods to solve the nonlinear equations of motion. Axial forces were applied, and only axisymmetric vibrations were considered in the analysis. The results indicated that both the structural parameter and temperature had a notable impact on the shell's response. Notably, similarities were observed between the effects of temperature and the length-diameter ratio on the shell's behavior. In the study conducted by Khaniki et al. [95] on a hyperelastic beam, the effect of geometrical imperfections on the structure was investigated. They utilized a dynamic equilibrium technique to simultaneously solve the axial and transverse nonlinear coupled equations. The researchers observed that the beam model with geometrical imperfections exhibited a lower maximum amplitude for the first transverse coordinate, but significantly higher amplitudes for the axial coordinates. Zhang et al. [70] conducted a study to explore the influence of geometric imperfections on the resonance responses of hyperelastic cylindrical shells. In their investigation, they employed Donnell's shell theory for structural modeling and Mooney-Rivlin model for material behavior. The researchers utilized the multiple-scales method to solve the coupled highly nonlinear equations of motion. The findings demonstrated that the effect of imperfection amplitude and structural parameters on the resonance responses exhibited similarities. They also concluded that external excitation and geometric imperfections could have a significant impact on the dynamic responses of hyperelastic cylindrical shells.

The literature review emphasizes the significant effects of geometric imperfections on the nonlinear response of thin hyperelastic cylindrical shells. It highlights the importance of conducting studies that consider the effects of different types of geometric imperfections on the system. In this paper, an improved version of Donnell's shallow shell theory will be

employed to attain precise outcomes with a reduced count of circumferential wave modes. Mooney-Rivlin constitutive model will be utilized to capture the nonlinear characteristics of hyperelastic materials. Three distinct geometric imperfections, namely axisymmetric, asymmetric, and combined imperfections, will be introduced and analyzed to assess their influence on both natural frequency and amplitude response. This paper is organized as follows: Section 6.3 provides the derivation of the equations of motion, formulation of the mathematical model for both material and structure, definition of geometric imperfections, and the reduction of degrees of freedom in the nonlinear system. In Section 6.4, Multiple Scales Method is applied as a perturbation technique to the equations of motion. In section 6.5, the linear model and the model incorporating geometrical nonlinearity are verified through comparisons with other models available in the open literature. Additionally, the results of the model that combines both geometrical and material nonlinearity are compared with those obtained using Abaqus. This section also includes a study of Abaqus convergence and an exploration of the impact of various types of geometric imperfections on the system's natural frequency and amplitude response. Finally, Section 6.6 presents the conclusions drawn from this study.

6.3 Mathematical model and description of motion

To characterize a hyperelastic cylindrical shell, it is essential to establish a model for both the material and structure. This involves establishing constitutive equations that capture the nonlinear stress-strain relationship of the hyperelastic material, as well as employing appropriate shell theories to describe the structural response. Once the material and structural models are established, the equations of motion for the system can be derived, considering the specific conditions.

6.3.1 Modeling of material

Hyperelastic materials do not conform to Hooke's law and require a different approach to capture their behavior. To effectively describe their response to deformation, strain energy density function has to be employed. It provides a mathematical representation of the material's behavior, taking into account its nonlinear stress-strain relationship. Ref [74] presents a comprehensive experimental and theoretical study that investigates various strain energy functions and their corresponding curve fitting coefficients. These coefficients are determined through the stress-strain testing. Third-order Mooney-Rivlin model, recognized for its enough accuracy in capturing shear deformation, was chosen over Neo-Hookean model for this research. Moreover, the selection of Mooney-Rivlin model was based on its superior efficiency compared to other models, such as Holzapfel [126], Gent [127], Arruda-Boyce [48] with

the aim of achieving more precise predictions and optimizing computational efficiency [102]. The Mooney-Rivlin strain energy model is as follows:

$$W = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + \frac{1}{D}(J - 1)^2, \quad (6.1)$$

The first and second invariants, I_1 and I_2 , are established utilizing the Green-Lagrange strain tensor $\mathbf{E}$ and the right Cauchy-Green deformation tensor $\mathbf{C}$ in the following manner [103]:

$$\mathbf{E} = \frac{1}{2} \begin{pmatrix} 2\varepsilon_{xx} & \varepsilon_{x\theta} & 0 \\ \varepsilon_{x\theta} & 2\varepsilon_{\theta\theta} & 0 \\ 0 & 0 & 2\varepsilon_{zz} \end{pmatrix}, \quad (6.2)$$

$$\mathbf{C} = 2\mathbf{E} + \mathbf{I}, \quad (6.3)$$

where $\mathbf{I}$ represents the identity matrix, then

$$\mathbf{C} = \begin{pmatrix} 2\varepsilon_{xx} + 1 & \varepsilon_{x\theta} & 0 \\ \varepsilon_{x\theta} & 2\varepsilon_{\theta\theta} + 1 & 0 \\ 0 & 0 & 2\varepsilon_{zz} + 1 \end{pmatrix}, \quad (6.4)$$

with following invariants

$$I_1 = Tr(\mathbf{C}) = 2(\varepsilon_{xx} + \varepsilon_{\theta\theta} + \varepsilon_{zz}) + 3, \quad (6.5)$$

$$\begin{aligned} I_2 &= \frac{1}{2}(Tr(\mathbf{C})^2 - Tr(\mathbf{C}^2)) = \\ &4(\varepsilon_{xx} + \varepsilon_{\theta\theta} + \varepsilon_{zz} + \varepsilon_{\theta\theta}\varepsilon_{xx} + \varepsilon_{\theta\theta}\varepsilon_{zz} + \varepsilon_{xx}\varepsilon_{zz}) \\ &\quad - 4(\varepsilon_{\theta z}^2 + \varepsilon_{x\theta}^2 + \varepsilon_{xz}^2) + 3, \end{aligned} \quad (6.6)$$

Under the assumption of incompressibility, the strain along the z-direction cannot be neglected. This leads to the requirement that the third invariant (I_3) of the strain tensor equals one as follows:

$$I_3 = J^2 = det\mathbf{C} = 1. \quad (6.7)$$

The strain in the z-direction has been discussed in previous studies [66, 102] and is expressed as follows:

$$\varepsilon_{zz} = \frac{1 + (2\varepsilon_{xx} + 1)\varepsilon_{\theta z}^2 + (2\varepsilon_{\theta\theta} + 1)\varepsilon_{xz}^2 - 2\varepsilon_{x\theta}\varepsilon_{xz}\varepsilon_{\theta z}}{2((2\varepsilon_{xx} + 1)(2\varepsilon_{\theta\theta} + 1) - \varepsilon_{x\theta}^2)} - \frac{1}{2}, \quad (6.8)$$

with respect to ε_{xx} , $\varepsilon_{\theta\theta}$ and $\varepsilon_{x\theta}$ and applying the small strain assumption, Equation (6.8) could be expanded around zero up to the fourth-order, yielding:

$$\begin{aligned} \varepsilon_{zz} = & -(\varepsilon_{xx} + \varepsilon_{\theta\theta}) + 2(\varepsilon_{xx}^2 + \varepsilon_{\theta\theta}^2 + \varepsilon_{xx}\varepsilon_{\theta\theta} + \varepsilon_{x\theta}^2) - 4\varepsilon_{xx}(\varepsilon_{xx}^2 + \varepsilon_{\theta\theta}^2 + 2\varepsilon_{x\theta}^2) \\ & - 4\varepsilon_{\theta\theta}(\varepsilon_{xx}^2 + \varepsilon_{\theta\theta}^2 + 2\varepsilon_{x\theta}^2) + 8(\varepsilon_{xx}^3\varepsilon_{\theta\theta} + \varepsilon_{xx}\varepsilon_{\theta\theta}^3 + 4\varepsilon_{xx}\varepsilon_{\theta\theta} + \varepsilon_{x\theta}) \\ & + 8(\varepsilon_{xx}^4 + \varepsilon_{\theta\theta}^4 + \varepsilon_{x\theta}^4 + \varepsilon_{xx}^2\varepsilon_{\theta\theta}^2 + 3\varepsilon_{xx}^2\varepsilon_{x\theta}^2 + 3\varepsilon_{xx}^2\varepsilon_{x\theta}^2). \end{aligned} \quad (6.9)$$

Upon substituting Equation (6.9) into Equations (6.5) and (6.6), the ultimate expression for the strain energy density function of incompressible hyperelastic materials can be derived.

6.3.2 Modeling of structure

Figure 6.1 shows the configuration of a thin shell in a cylindrical coordinate system (x, z, θ) while x corresponds to the axial direction, z represents the radial direction, and θ signifies the circumferential direction. The shell be defined by considering R as the radius, h as the thickness, and l as the length. To analyze the deformation of the structure, the displacement vector (u_0, v_0, w_0) is employed at its middle surface. At any other point on the shell, the displacement vector is denoted as (u, v, w) , effectively capturing the dynamic behavior that occurs throughout the structure. Based on the Kirchhoff-Love hypothesis, the following relation will be defined [26]:

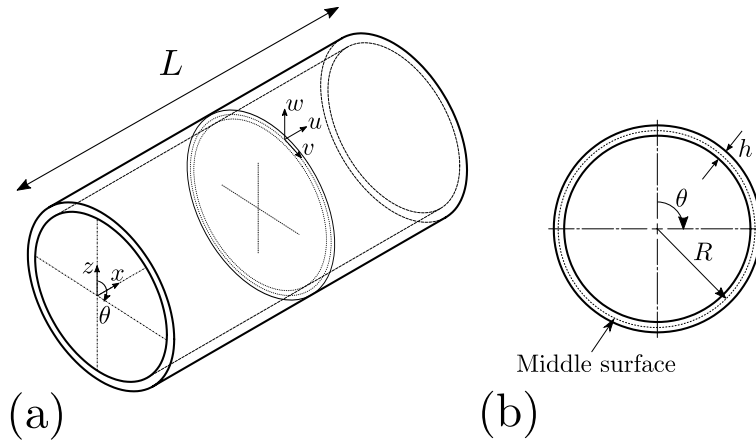


Figure 6.1 Schematic representation of shell geometry. (a) Geometrical dimension symbols
(b) Cross-section of shell

$$u = u_0 - \frac{\partial w_0}{\partial x} z, \quad (6.10)$$

$$v = v_0 - \frac{1}{R} \frac{\partial w_0}{\partial \theta} z, \quad (6.11)$$

$$w = w_0. \quad (6.12)$$

To derive the strain-displacement relations, Donnell's Shallow-Shell theory is a widely used approach that simplifies the equations of motion for shells by neglecting in-plane inertia [26]. However, this theory's accuracy diminishes when the number of circumferential waves is less than five. To overcome this limitation, this study utilizes nonlinear strain-displacement relations derived from the improved Donnell's nonlinear shell theory [33, 34, 128]. This approach is adopted with the intention of achieving improved accuracy, especially for scenarios involving small circumferential wave numbers. Furthermore, this study takes into account the presence of initial geometric imperfection within the analyzed relations, as outlined below:

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 + \frac{\partial w_0}{\partial x} \frac{\partial w_i}{\partial x} - \frac{\partial w_0^2}{\partial x^2} z, \quad (6.13a)$$

$$\varepsilon_{\theta\theta} = \frac{1}{R} \left(\frac{\partial v_0}{\partial \theta} + w_0 \right) + \frac{1}{2R^2} \left(\frac{\partial w_0}{\partial \theta} \right)^2 + \frac{1}{R^2} \left(\frac{\partial w_0}{\partial \theta} \frac{\partial w_i}{\partial \theta} \right) - \frac{1}{R^2} \left(\frac{\partial w_0^2}{\partial \theta^2} - \frac{\partial v}{\partial \theta} \right) z, \quad (6.13b)$$

$$\varepsilon_{x\theta} = \frac{1}{R} \frac{\partial u_0}{\partial \theta} + \frac{\partial v_0}{\partial x} + \frac{1}{R} \left(\frac{\partial w_0}{\partial \theta} \frac{\partial w_0}{\partial x} \right) + \frac{1}{R} \left(\frac{\partial w_0}{\partial x} \frac{\partial w_i}{\partial \theta} \right) + \frac{1}{R} \left(\frac{\partial w_0}{\partial \theta} \frac{\partial w_i}{\partial x} \right) - 2 \frac{1}{R} \left(\frac{\partial w_0^2}{\partial x \partial \theta} - \frac{\partial v}{\partial x} \right) z. \quad (6.13c)$$

6.3.3 Defining geometric imperfection

Geometric imperfections in structures can arise from various sources and contribute significantly to altering their mechanical behavior. These imperfections result from manufacturing processes, assembly errors, or environmental conditions, introducing deviations from the intended geometric configurations. The geometric imperfection effects on the structural response can be profound and complex, affecting parameters such as stability, strength, and natural frequencies. In this study, we investigate three distinct types of geometric imperfections, each with its own unique shape:

$$w_{axi}^i = A_{01} \sin \left(\frac{m_i \pi x}{l} \right), \quad (6.14a)$$

$$w_{asy}^i = A_{02} \sin \left(\frac{m_i \pi x}{l} \right) \cos(n_i \theta), \quad (6.14b)$$

$$w_{combined}^i = A_{03} \sin(n_i \theta) \sin \left(\frac{m_i \pi x}{l} \right) + A_{04} \cos(n_i \theta) \sin \left(\frac{m_i \pi x}{l} \right), \quad (6.14c)$$

Equations 6.14a, 6.14b, and 6.14c correspond to axisymmetric, asymmetric, and combined geometric imperfections, respectively. In these equations, m_i and n_i represent the axial and circumferential mode numbers of the geometric imperfections. Additionally, A_{0i} , where i ranges from 1 to 4, denotes the imperfection amplitude. It's essential to note that the

imperfection amplitude is directly related to the thickness (h) of the structure, and it follows the relationship $A_{0i} = C_{0i}h$, where C_{0i} ($i = 1..4$) are constant coefficients.

6.3.4 Developing the equations of motion

Nonlinear equations of motion are derived using Lagrange relations [32], which can be expressed as follows:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q} = Q_i, \quad (6.15)$$

Q_i (i is the number of generalized coordinates q_i) represents applied forces, and in this study, the excitation occurs only in the z -direction as radial harmonic excitation. L is defined as the difference between kinetic energy (K) and potential energy (P) which are defined as follows [32]:

$$K = \frac{1}{2} \rho R \int_0^l \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\dot{u}^2 + \dot{v}^2 + \dot{w}^2) dz d\theta dx, \quad (6.16)$$

$$P = R \int_0^l \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} W dz d\theta dx, \quad (6.17)$$

where "." is derivative with respect to time; ρ is the material density. The potential energy, as expressed in Equation 6.17, is derived through the integration of the strain energy density function (W). In this study, the Mooney-Rivlin model is used as the strain energy density function of hyperelastic material Equation (6.1). Zero displacements in v and w directions are considered at both ends of the geometry as the boundary conditions. The displacement of the middle surface is defined as follows:

$$u(x, \theta, t) = u_{mn}(t) \cos(n\theta) \cos(\lambda_m x) + u_{m0} \cos^2(\lambda_m x), \quad (6.18a)$$

$$v(x, \theta, t) = v_{mn}(t) \sin(n\theta) \sin(\lambda_m x) + v_{m0} \sin^2(\lambda_m x), \quad (6.18b)$$

$$w(x, \theta, t) = w_{mn}(t) \cos(n\theta) \sin(\lambda_m x) + w_{m0} \sin^2(\lambda_m x), \quad (6.18c)$$

u_{mn} , v_{mn} and w_{mn} represent generalized coordinates, while u_{m0} , v_{m0} , and w_{m0} follow the same pattern but for benign terms. Researchers interested in the impact of various middle surface deformation functions on the system's response are referred to Ref. [63] for more details. And λ_m is equal to $\frac{m\pi}{l}$. By substituting Equation (6.18) into Equation (6.13) and applying the Lagrange Equation (6.15), the nonlinear equation of motion for the hyperelastic thin cylindrical shell, which include two nonlinear terms, are derived as follows:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}_L \mathbf{q} + \mathbf{K}_{NL_1} \mathbf{q}^2 + \mathbf{K}_{NL_2} \mathbf{q}^3 = \mathbf{F}, \quad (6.19)$$

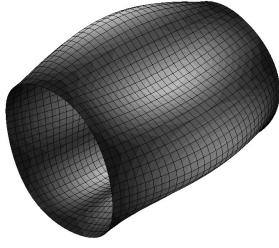
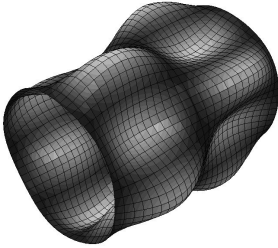
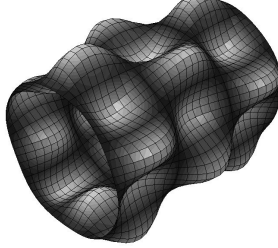
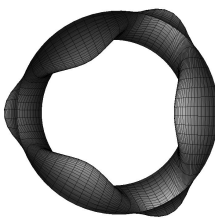
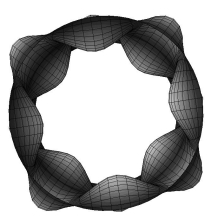
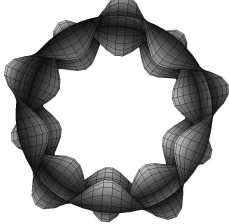
Isometric			
Front			
m	1	2	3
n	3	4	5

Figure 6.2 Circumferential(n) and axial modes(m) number

where $\mathbf{M}$ stands for the mass matrix, $\mathbf{C}$ for the damping matrix, and $\mathbf{K}_L$, $\mathbf{K}_{NL_1}$, and $\mathbf{K}_{NL_2}$ represent the linear, quadratic, and cubic nonlinear stiffness matrices, respectively. Due to the choice of a middle surface deformation model defined by six generalized coordinates, the matrices in this equation have dimensions of 6×6 . Both $\mathbf{q}$ (generalized coordinates) and $\mathbf{F}$ (forces) represent column vectors, each comprising six elements. In this study, the damping ratio falls within the range of 0.02 to 0.06. Therefore, the linear assumption can be applied to the damping term [104]. A higher level of damping must be considered as a nonlinear damping term, which falls outside the scope of this research.

6.3.5 Discretization

In matrix structural analysis, the potential to effectively reduce the problem size exists. In this study, although there are six degrees of freedom (DOFs) for the hyperelastic cylindrical shell, in certain directions, the applied forces can be considered equal to zero. Under these conditions, through the application of the static condensation method [35], DOFs of the hyperelastic cylindrical shell can be reduced from six degrees to two degrees. Since only the forces in the z -direction are non-zero, Equation (6.19) can be redefined based on these two degrees of freedom. This approximation simplifies the structural dynamic response analysis

without a significant loss of accuracy. However, the equation of motion for the normal displacement of a cylindrical shell's surface, denoted by w , can be obtained as follows:

$$\begin{bmatrix} M_{11} & \dots & M_{16} \\ \vdots & \ddots & \vdots \\ M_{61} & \dots & M_{66} \end{bmatrix}_{6 \times 6} \begin{bmatrix} \ddot{u}_{mn} \\ \ddot{u}_{m0} \\ \ddot{v}_{mn} \\ \ddot{v}_{m0} \\ \ddot{w}_{mn} \\ \ddot{w}_{m0} \end{bmatrix}_{6 \times 1} + \begin{bmatrix} K_{11} & \dots & K_{16} \\ \vdots & \ddots & \vdots \\ K_{61} & \dots & K_{66} \end{bmatrix}_{6 \times 6} \begin{bmatrix} u_{mn} \\ u_{m0} \\ v_{mn} \\ v_{m0} \\ w_{mn} \\ w_{m0} \end{bmatrix}_{6 \times 1} = \begin{bmatrix} F_{x_1} \\ F_{x_2} \\ F_{\theta_1} \\ F_{\theta_2} \\ F_{z_1} \\ F_{z_2} \end{bmatrix}_{6 \times 1}, \quad (6.20)$$

the above equation can be rewritten and reduced with respect to the degrees of freedom as follows:

$$\begin{bmatrix} u_{mn} \\ u_{m0} \\ v_{mn} \\ v_{m0} \end{bmatrix}_{4 \times 1} = -J^{-1}N \begin{bmatrix} w_{mn} \\ w_{m0} \\ \ddot{w}_{mn} \\ \ddot{w}_{m0} \end{bmatrix}_{4 \times 1}, \quad (6.21)$$

where, matrices J and N are:

$$J = \begin{bmatrix} K_{11} & \dots & K_{14} \\ \vdots & \ddots & \vdots \\ K_{41} & \dots & K_{44} \end{bmatrix}_{4 \times 4}, N = \begin{bmatrix} K_{15} & K_{16} & M_{15} & M_{16} \\ \vdots & \vdots & \vdots & \vdots \\ K_{45} & K_{46} & M_{45} & M_{46} \end{bmatrix}_{4 \times 4}, \quad (6.22)$$

defining the matrix S as:

$$S = -J^{-1}N, \quad (6.23)$$

four generalized coordinates, namely, u_{mn} , u_{m0} , v_{mn} , and v_{m0} , are defined in terms of w_{mn} and w_{m0} as follows:

$$\begin{bmatrix} u_{mn} \\ u_{m0} \\ v_{mn} \\ v_{m0} \end{bmatrix}_{4 \times 1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \\ S_{31} & S_{32} \\ S_{41} & S_{42} \end{bmatrix}_{4 \times 2} \begin{bmatrix} w_{mn} \\ w_{m0} \end{bmatrix}_{2 \times 1} + \begin{bmatrix} S_{13} & S_{14} \\ S_{23} & S_{24} \\ S_{33} & S_{34} \\ S_{43} & S_{44} \end{bmatrix}_{4 \times 2} \begin{bmatrix} \ddot{w}_{mn} \\ \ddot{w}_{m0} \end{bmatrix}_{2 \times 1}. \quad (6.24)$$

Assuming w_{mn} and w_{m0} as q_1 and q_2 , respectively, the coupled equations for axisymmetric and symmetric modes are defined as follows:

$$M_{55}\ddot{q}_1 + c_1\dot{q}_1 + A_1q_1 + A_2q_1q_2 + A_3q_1q_2^2 + A_4q_1^3 = F_{z_1}\cos(\omega t), \quad (6.25a)$$

$$M_{66}\ddot{q}_2 + c_2\dot{q}_2 + B_1q_2 + B_2q_2^2 + B_3q_1^2 + B_4q_1^2q_2 + B_5q_2^3 = F_{z_2}\cos(\omega t). \quad (6.25b)$$

Nondimensionalization process is performed by considering $\tau = \omega_{mn}t$ and $\Omega = \frac{\omega}{\omega_{mn}}$ while ω_{mn} is the natural frequency corresponds to different numbers of circumferential waves (n) and longitudinal half-waves (m). The following terms will be dimensionless as follow:

$$\begin{aligned} Q_i &= \frac{q_i}{h}, & \bar{B}_1 &= \frac{B_1}{M_{66}\omega_{mn}^2}, & \bar{A}_1 &= \frac{A_1}{M_{55}\omega_{mn}^2}, \\ \bar{c}_1 &= \frac{c_1}{M_{55}\omega_{mn}}, & \bar{B}_2 &= \frac{B_2}{M_{66}\omega_{mn}^2}h, & \bar{A}_2 &= \frac{A_2}{M_{55}\omega_{mn}^2}h, \\ \bar{c}_2 &= \frac{c_2}{M_{66}\omega_{mn}}, & \bar{B}_3 &= \frac{B_3}{M_{66}\omega_{mn}^2}h, & \bar{A}_3 &= \frac{A_3}{M_{55}\omega_{mn}^2}h^2, \\ \bar{F}_1 &= \frac{F_{z_1}}{M_{55}\omega_{mn}^2h}, & \bar{B}_4 &= \frac{B_4}{M_{66}\omega_{mn}^2}h^2, & \bar{A}_4 &= \frac{A_4}{M_{55}\omega_{mn}^2}h^2, \\ \bar{F}_2 &= \frac{F_{z_2}}{M_{66}\omega_{mn}^2h}, & \bar{B}_5 &= \frac{B_5}{M_{66}\omega_{mn}^2}h^2, & & \end{aligned} \quad (6.26)$$

In this way, Equation (6.25) can be rewritten as a pair of equations as follows:

$$\ddot{Q}_1 + \bar{c}_1\dot{Q}_1 + \bar{A}_1Q_1 + \bar{A}_2Q_1Q_2 + \bar{A}_3Q_1Q_2^2 + \bar{A}_4Q_1^3 = \bar{F}_1\cos(\Omega\tau), \quad (6.27a)$$

$$\ddot{Q}_2 + \bar{c}_2\dot{Q}_2 + \bar{B}_1Q_2 + \bar{B}_2Q_2^2 + \bar{B}_3Q_1^2 + \bar{B}_4Q_1^2Q_2 + \bar{B}_5Q_2^3 = \bar{F}_2\cos(\Omega\tau). \quad (6.27b)$$

6.4 Analytical solution

Considering both the axisymmetric and asymmetric modes in the analysis of Equation (6.27) is crucial for obtaining a comprehensive understanding of the system's behavior. The coupled nonlinear terms and their interactions with each mode can significantly impact the overall response of the system. Therefore, it is necessary to solve the equation with all terms included to accurately predict the system's behavior. Multiple-scales method is an analytical way to address this nonlinear system. In this method, ϵ is a parameter to define the nonlinearity of the equations as follows [98]:

$$\begin{aligned} C_1 &= \epsilon\bar{c}_1, & W_1 &= \epsilon Q_1, & F_1 &= \epsilon^2\bar{F}_1, \\ C_2 &= \epsilon\bar{c}_2, & W_2 &= \epsilon Q_2, & F_2 &= \epsilon^2\bar{F}_2. \end{aligned} \quad (6.28)$$

Hence, Equation (6.27) becomes:

$$\ddot{W}_1 + \epsilon C_1\dot{W}_1 + \bar{A}_1W_1 + \epsilon\bar{A}_2W_1W_2 + \epsilon^2(\bar{A}_3W_1W_2^2 + \bar{A}_4W_1^3) = \epsilon F_1\cos(\Omega\tau), \quad (6.29a)$$

$$\ddot{W}_2 + \epsilon C_2\dot{W}_2 + \bar{B}_1W_2 + \epsilon(\bar{B}_2W_1^2 + \bar{B}_3W_2^2) + \epsilon^2(\bar{B}_4W_1^2W_2 + \bar{B}_5W_2^3) = \epsilon F_2\cos(\Omega\tau), \quad (6.29b)$$

thus, instead of determining the solution of Equation (6.29) as a function of τ it can be assumed as a function of T_0 and T_1 as per:

$$\begin{aligned} W_1 &= W_{10}(T_0, T_1) + \epsilon W_{11}(T_0, T_1), \\ W_2 &= W_{20}(T_0, T_1) + \epsilon W_{21}(T_0, T_1), \end{aligned} \quad (6.30)$$

where T_i ($i=1, 2$) represent different time scales:

$$T_0 = \tau, T_1 = \epsilon\tau. \quad (6.31)$$

The first and second derivatives are obtained, using the chain rule with respect to the new time scales as follows:

$$\frac{d}{d\tau} = D_0 + \epsilon D_1 + \dots, \quad (6.32a)$$

$$\frac{d^2}{d\tau^2} = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2 D_1^2 + \dots, \quad (6.32b)$$

where, D_0 and D_1 are the derivatives with respect to T_0 and T_1 , respectively. The formulas can be obtained by substituting Equations (6.32) and (6.30) into Equation (6.29), and separating the zero and first order expression of ϵ , as follow:

$$D_0^2 W_{10} + \bar{A}_1 W_{10} = 0, \quad (6.33a)$$

$$D_0^2 W_{20} + \bar{B}_1 W_{20} = 0, \quad (6.33b)$$

$$D_0^2 W_{11} + \bar{A}_1 W_{11} = F_1 \cos(\Omega T_0) - 2D_0 D_1 W_{10} - D_0 C_1 W_{10} - \bar{A}_2 W_{10} W_{20}, \quad (6.34a)$$

$$D_0^2 W_{21} + \bar{B}_1 W_{21} = F_2 \cos(\Omega T_0) - 2D_0 D_1 W_{20} - D_0 C_2 W_{20} - \bar{B}_2 W_{10}^2 - \bar{B}_3 W_{20}^2. \quad (6.34b)$$

General solution of Equation (6.33) can be expressed as follows:

$$W_{10} = \phi(T_1) e^{i\sqrt{\bar{A}_1} T_0} + \bar{\phi}(T_1) e^{-i\sqrt{\bar{A}_1} T_0}, \quad (6.35a)$$

$$W_{20} = \psi(T_1) e^{i\sqrt{\bar{B}_1} T_0} + \bar{\psi}(T_1) e^{-i\sqrt{\bar{B}_1} T_0}, \quad (6.35b)$$

then Equation (6.34) based on Equation (6.35) is defined as follows:

$$\begin{aligned} D_0^2 W_{11} + W_{11} &= \frac{F_1}{2} e^{i\Omega T_0} - (2D_0 D_1 + D_0 C_1)(\phi e^{iT_0} + \bar{\phi} e^{-iT_0}) - \\ &\quad \bar{A}_2(\phi e^{iT_0} + \bar{\phi} e^{-iT_0})(\psi e^{iT_0} + \bar{\psi} e^{-iT_0}), \end{aligned} \quad (6.36)$$

$$\begin{aligned} D_0^2 W_{21} + \Gamma^2 W_{21} &= \frac{F_2}{2} e^{i\Omega T_0} - \bar{B}_2(\phi e^{iT_0} + \bar{\phi} e^{-iT_0})^2 - \\ &\quad (2D_0 D_1 + D_0 C_2)(\psi e^{iT_0} + \bar{\psi} e^{-iT_0}) - \bar{B}_3(\psi e^{iT_0} + \bar{\psi} e^{-iT_0})^2, \end{aligned} \quad (6.37)$$

where:

$$\sqrt{\bar{A}_1} = 1 \text{ and } \sqrt{\bar{B}_1} = \Gamma, \quad (6.38)$$

$$\Omega = 1 + \sigma_1\epsilon \text{ and } \Gamma = 2 + \sigma_2\epsilon, \quad (6.39)$$

Equations (6.36) and (6.37) are rewritten as:

$$D_0^2 W_{11} + W_{11} = \frac{F_1}{2} e^{-i\Omega T_0} - (2D_0 D_1 \phi + D_0 C_1 \phi) e^{iT_0} - \bar{A}_2 \phi \psi e^{i(1+\Omega)T_0} - \bar{A}_2 \bar{\phi} \psi e^{i(\Omega-1)T_0}, \quad (6.40)$$

$$D_0^2 W_{21} + \Gamma^2 W_{21} = \frac{F_2}{2} e^{-i\Omega T_0} - (2D_0 D_1 \psi + D_0 C_2 \psi) e^{i\Omega T_0} - \bar{B}_2 \phi^2 e^{2iT_0} - \bar{B}_3 \psi^2 e^{2i\Omega T_0} - \bar{B}_2 \phi \bar{\phi} - \bar{B}_3 \psi \bar{\psi}. \quad (6.41)$$

Equation (6.40) is simplified through applying D_0 operator and eliminating the secular terms is presented as follows:

$$D_1 \phi = -\frac{C_1 \phi}{2} - i \frac{F_1}{4} e^{\sigma_1 \epsilon T_0} + i \frac{\psi \bar{\phi}}{2} \bar{A}_2 e^{i\sigma_2 \epsilon T_0}, \quad (6.42)$$

the same operator is applied to Equation (6.41), having:

$$D_1 \psi = -\frac{C_2 \psi}{2} + i \frac{\bar{B}_3 \phi^2}{2\Gamma} e^{-i\sigma_2 \epsilon T_0}, \quad (6.43)$$

and by considering ϕ and ψ as:

$$\phi = \frac{1}{2} a_1 e^{i\beta_1} \text{ and } \psi = \frac{1}{2} a_2 e^{i\beta_2}, \quad (6.44)$$

where, a_i and β_j ($j = 1, 2$) are amplitude and phase with respect to T_1 , respectively. Equation (6.42) and (6.43) are rewritten based on Equation (6.44) as follows:

$$\dot{a}_1 + (i)a_1 \dot{\beta}_1 = -\frac{C_1}{2} a_1 - i \frac{F_1}{2} e^{(\sigma_1 \epsilon T_0 - \beta_1)i} + i \frac{a_2 a_1 \bar{A}_2}{4} e^{(\sigma_2 \epsilon T_0 - 2\beta_1 + \beta_2)i}, \quad (6.45)$$

$$\dot{a}_2 + (i)a_2 \dot{\beta}_2 = -\frac{C_2}{2} a_2 + i \frac{\bar{B}_3 a_1^2}{4\Gamma} e^{2\beta_1 - \sigma_2 \epsilon T_0 - \beta_2)i}, \quad (6.46)$$

by assuming γ_1 and γ_2 as bellow:

$$\gamma_1 = \sigma_1 T_1 - \beta_1 \text{ and } \gamma_2 = 2\beta_1 - \sigma_2 T_1 - \beta_2, \quad (6.47)$$

the real and imaginary terms of Equations (6.45) and (6.46) are as follows:

$$a_1 = \epsilon \left(-\frac{C_1 a_1}{2} + \frac{F_1}{2} \sin \gamma_1 + \frac{1}{4} a_2 a_1 \bar{A}_2 \sin \gamma_2 \right), \quad (6.48a)$$

$$\dot{\gamma}_1 = \epsilon \left(\frac{F_1}{2a_1} \cos \gamma_1 - \frac{1}{4} a_2 \bar{A}_2 \cos \gamma_2 + \sigma_1 \right), \quad (6.48b)$$

$$a_2 = \epsilon \left(-\frac{C_2 a_2}{2} - \frac{\bar{B}_3 a_1^2}{4\Gamma} \sin \gamma_2 \right), \quad (6.48c)$$

$$\dot{\gamma}_2 = \epsilon \left(-\frac{\bar{B}_3 a_1^2}{4\Gamma a_2} \cos \gamma_2 - \sigma_2 - \frac{F_1}{a_1} \cos \gamma_1 + \frac{1}{2} a_2 \bar{A}_2 \cos \gamma_2 \right). \quad (6.48d)$$

in the steady-state response, $a_i (i = 1, 2)$ and $\gamma_i (i = 1, 2)$ are constant. Then their derivatives will be zero:

$$-\frac{C_1 a_1}{2} + \frac{F_1}{2} \sin \gamma_1 + \frac{1}{4} a_2 a_1 \bar{A}_2 \sin \gamma_2 = 0, \quad (6.49a)$$

$$\frac{F_1}{2a_1} \cos \gamma_1 - \frac{1}{4} a_2 \bar{A}_2 \cos \gamma_2 + \sigma_1 = 0, \quad (6.49b)$$

$$-\frac{C_2 a_2}{2} - \frac{\bar{B}_3 a_1^2}{4\Gamma} \sin \gamma_2 = 0, \quad (6.49c)$$

$$-\frac{\bar{B}_3 a_1^2}{4\Gamma a_2} \cos \gamma_2 - \sigma_2 - \frac{F_1}{a_1} \cos \gamma_1 + \frac{1}{2} a_2 \bar{A}_2 \cos \gamma_2 = 0. \quad (6.49d)$$

The amplitude of asymmetric and axisymmetric modes are obtained by eliminating γ_1, γ_2 in Equation (6.49). An in-house computer program in Python programming language is developed to solve these equations.

6.5 Results and discussion

6.5.1 Evaluating model performance

To verify the proposed model presented in this paper, a verification process was undertaken. Initially, the nonlinear terms of the mathematical model were eliminated and the linear terms incorporating linear material properties have been retained for numerical analysis. Subsequently, the nonlinear model incorporating geometrical nonlinearity was verified against existing references. Finally, the validation process has been extended to structures exhibiting both geometrical and physical nonlinearity, with comparison against results obtained from the commercial finite element software Abaqus.

6.5.2 Linear model verification

In this study, the verification of the hyperelastic model involves validating the linear model by neglecting the nonlinear terms. To convert the hyperelastic model into an elastic one and follow the governing equation process, the strain energy density of the hyperelastic material is linearized using the following equation [32, 106]:

$$W = \frac{E}{2(1+\nu)} (4 + \varepsilon_{xx}^2 + \varepsilon_{xx}\varepsilon_{\theta\theta} + \varepsilon_{x\theta}^2), \quad (6.50)$$

E represents the Young's modulus, ν represents the Poisson's ratio, and ε_{xx} , $\varepsilon_{\theta\theta}$, and $\varepsilon_{x\theta}$ denote the normal strains in different directions. The linear natural frequency of a thin shell

Table 6.1 Verification of natural frequency of the linear elastic model using different shell theories. Hybrid Finite Element Method (HFEM) [107], Donnell's Shell Theory (DST), and Improved Donnell's Shell Theory (IDST).

m	n	HFEM(Hz)	DST(Hz)	IDST(Hz)
1	2	1133	1068	1093
1	3	629	607	619
1	4	655	671	651
1	5	942	962	947
1	6	1353	1364	1360
1	7	1853	1849	1858

using the following data: $E = 200\text{GPa}$, $\nu = 0.3$, $h = 0.229\text{m}$, $L = 0.234\text{m}$, $R = 0.0377\text{m}$, and $\rho = 7800\text{kg/m}^3$, is calculated using both Donnell's and Improved Donnell's shell theories in Table 6.1. A comparison is made between these results and those obtained from a hybrid finite element method [107]. In Figure 6.3, a visual representation is presented to compare the relative differences between the hybrid finite element method and both Donnell's and Improved Donnell's shell theories. Notably, the Improved Donnell's method exhibits higher accuracy in predicting natural frequencies, particularly when the number of circumferential waves is low. For cases with fewer than five circumferential waves, the relative difference reduced to nearly half when using Improved Donnell's shell theory, and beyond that point, both methods yield similar accuracy. This indicates the efficacy of the Improved Donnell's approach, particularly in scenarios with a limited number of circumferential waves.

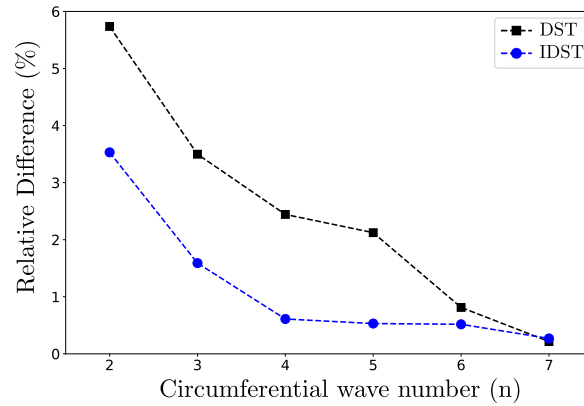


Figure 6.3 Graphical comparison of relative differences: Donnell's Shell Theory (DST) and Improved Donnell's Shell Theory (IDST) in Comparison with Hybrid Finite Element Method (HFEM)

Nonlinear model verification

For the verification of the nonlinear model, the ratios of nonlinear to linear frequencies are computed using the Lindstedt-Poincaré technique for a homogeneous cylindrical shell. The obtained results are then compared with those presented by Wang [109]. The properties of the isotropic cylindrical shell are as follows: $h = 0.00254\text{m}$, $R = 0.254\text{m}$, $l = 0.4\text{m}$, $\rho = 7800\text{kg/m}^3$, $\nu = 0.3$, $E = 200\text{GPa}$, and the amplitude = 0.00254. The comparison results are shown in Table (6.2), demonstrating relatively good agreement. Any differences can be attributed to the slightly different methods of discretization.

Table 6.2 Comparing nonlinear and linear frequency ratios for a homogeneous cylindrical shell.

Mode(m,n)		(1,2)	(1,3)	(1,4)
ω_{NL}/ω_L	Present	1.0006	1.0034	1.0144
	Wang [109]	1.0009	1.0033	1.0117
	Difference	0.0003	0.0001	0.0027

6.5.3 Natural frequency of hyperelastic cylindrical shell

The natural frequency of the hyperelastic cylindrical shell is determined using both DST and IDST shell theories. The obtained results have been verified against those of Abaqus (a powerful commercial finite element software) simulation, and illustrated in figure 6.4. As the number of circumferential wave numbers increases beyond four, the results obtained from both DST and IDST shell theories tend to converge and become nearly identical. However, an observable difference remains between these theoretical results and those obtained from Abaqus. This difference ($|\omega^{IDST} - \omega^{Abaqus}|$) could arise from employing distinct mesh elements, varied numerical integration methods, or other factors inherent in the finite element method calculation process. In our endeavor to explore the natural frequency of a hyperelastic cylindrical shell, we delved into three distinct methodologies: Improved Donnell's Shell Theory (IDST), Donnell's Shell Theory (DST), and Abaqus, a powerful commercial finite element software. While experimental data for thin hyperelastic cylindrical shells was unavailable, Abaqus provided an invaluable estimation, acting as a guiding benchmark for validating the results obtained through IDST and DST.

Abaqus convergence

To determine the natural frequency by Abaqus, a convergence process is necessary. To achieve this, a reasonable mesh size is chosen initially, and natural frequency for a specific mode (in

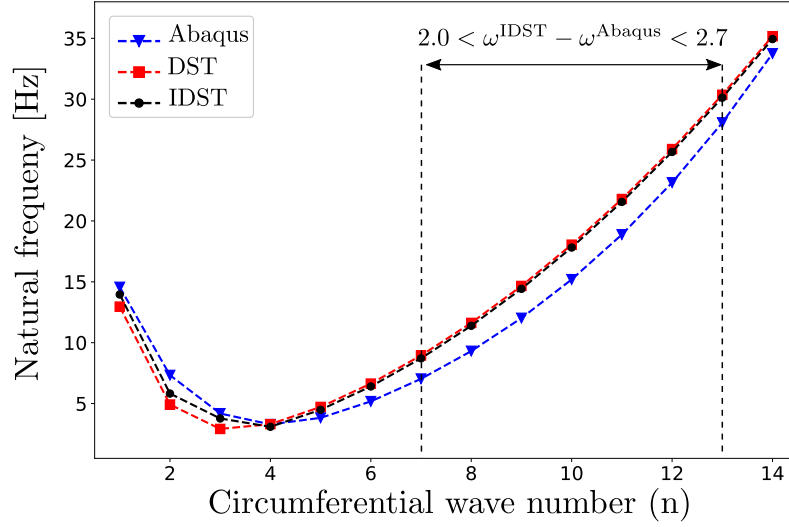


Figure 6.4 Hyperelastic cylindrical shell natural frequency

this study, for mode $m=1, n=3$) is identified. Subsequently, the mesh size is gradually reduced in incremental steps until a point is reached where the relative difference is less than 0.02 Hz, at which point the calculation could be terminated. Figure 6.5 depicts the variation in the obtained natural frequency of the specified mode as the element size decreases. On the other hand, Figure 6.6 presents the degrees-of-freedom in logarithmic order plotted against the relative difference. By cross-referencing the acquired results with Table 6.3, it can be determined that a mesh size of 0.01 led to the occurrence of the convergence point.

$$|\omega^{\text{Fine}} - \omega^{\text{Coarse}}| < 0.02 \quad [Hz], \quad (6.51)$$

in Equation (6.51), coarse mesh refers to the initial mesh size, while finer mesh corresponds to a mesh with improved results.

It's noteworthy that during the convergence process, the number of elements experiences a substantial increase, as evidenced in Table 6.3. This parameter assumes a critical role in optimization efforts and reducing computational costs.

6.5.4 Influence of geometric imperfections on natural frequency.

By setting the coefficients of imperfections, C_{0i} ($i = 1..4$) equal to 0.75, the natural frequency of the structure with initial imperfections with imperfection modes number m_i and n_i equal to

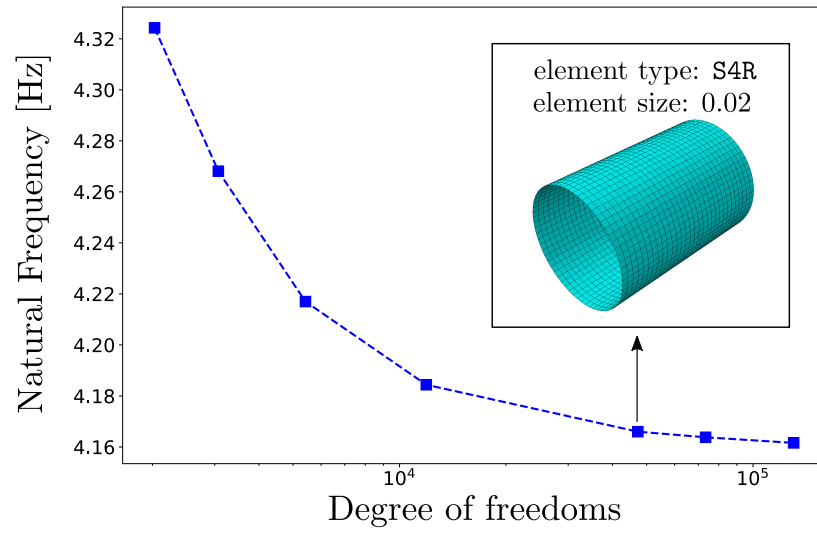


Figure 6.5 Abaqus convergence test

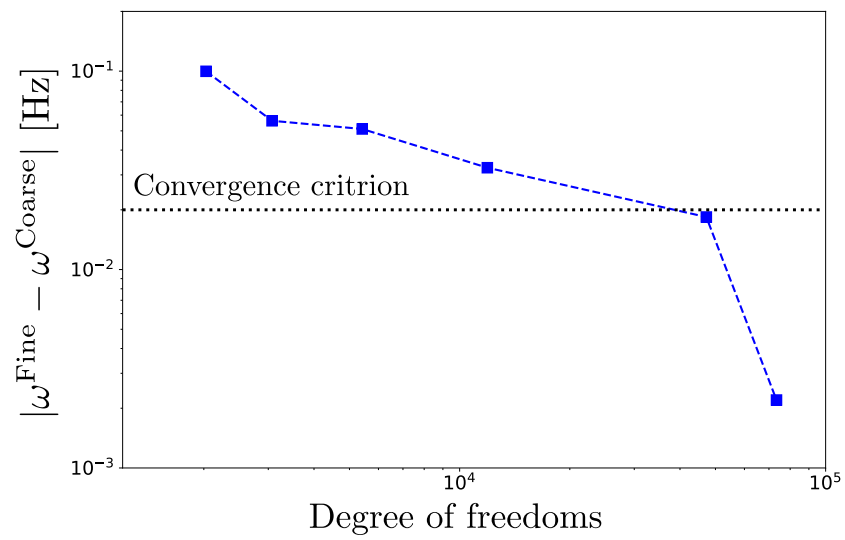


Figure 6.6 Abaqus convergence test

Table 6.3 Convergence analysis of natural frequencies (Abaqus data); S4R is a 4-node, quadrilateral, stress-displacement shell element with reduced integration and a large-strain formulation; the index number is directly linked to the number of modes used in the simulation.

Element type	Element size	Element number	Relative difference
S4R	0.06	210	— — —
	0.05	312	0.0998
	0.04	480	0.0562
	0.03	860	0.511
	0.02	1920	0.0326
	0.01	7740	0.0184
	0.008	12075	0.0022
	0.006	21500	0.0022

unity, has been determined. The comparison between a perfect shell (without imperfections) and shells with various initial geometric imperfections is shown in Figure 6.7. Remarkably when the number of circumferential wave numbers exceeds six, the results converge and exhibit greater consistency. For values below six, the structure with combined asymmetric and axisymmetric imperfections demonstrates a higher natural frequency compared to shells with either asymmetric or axisymmetric geometric imperfections. In general, the presence of geometric imperfections leads to an increase in the natural frequency of the structure. Furthermore, with a rise in the number of circumferential wave numbers, the natural frequency of shells with asymmetric imperfections surpasses that of shells with axisymmetric imperfections.

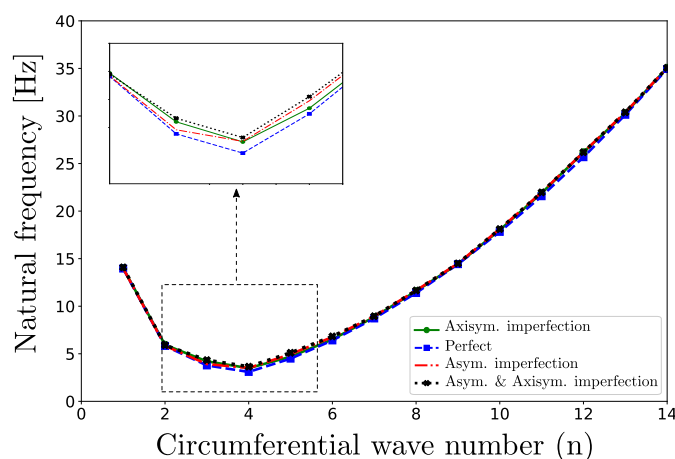


Figure 6.7 Effect of geometrical imperfections on natural frequency.

6.5.5 Exploring the impact of geometric imperfection coefficients

In this section, a closer examination is conducted to understand how alterations in the coefficients associated with three distinct considered geometric imperfections influence the system's natural frequency. The goal is to study the relationship between these coefficients and their effects on natural frequency. When the coefficient C_{0i} ($i = 1..4$) is set to zero, the system attains a state of perfection, without geometric imperfections. By systematically altering the coefficient C_{0i} across a range of positive and negative values, each incremented by half-steps, four distinct response curves were obtained for each geometric imperfection variant. These responses are illustrated in Figures 6.8, 6.9, and 6.10. In each of the three cases (axisymmetric, asymmetric, and combined geometric imperfections), elevating the respective coefficients results in an increase in the natural frequency. Conversely, reducing these coefficients leads to a decrease in the natural frequency. However, these changes in coefficients drive the natural frequency along distinct paths. By adjusting the coefficient C_{01} for axisym-

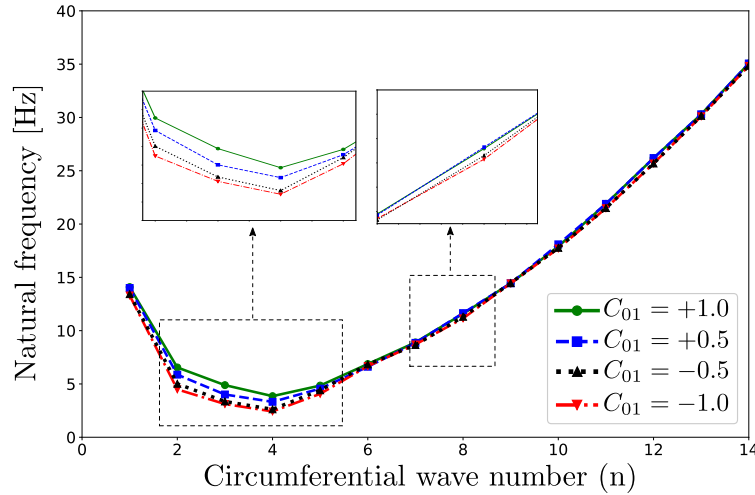


Figure 6.8 Influence of axisymmetric geometric imperfection coefficient C_{01} on system's natural frequency.

metric imperfection, once the number of circumferential wave numbers surpasses six, natural frequencies resulting from different C_{01} values begin to converge. Nevertheless, prior to this point, variations between these frequencies are observable in Figure 6.8.

When considering asymmetric imperfections, modifying the coefficient C_{02} leads to a convergence in the natural frequency at an earlier point ($n = 4$) than observed in the case of axisymmetric imperfection. In the combined geometric imperfection, the manipulation of coefficients C_{03} and C_{04} leads to a significantly more pronounced increase in the natural

frequency, surpassing the effects observed in the other imperfections. Moreover, the convergence of this frequency is delayed. Remarkably, in Figure 6.10 even at higher numbers of circumferential wave numbers, such as seven, noticeable distinctions remain among the results.

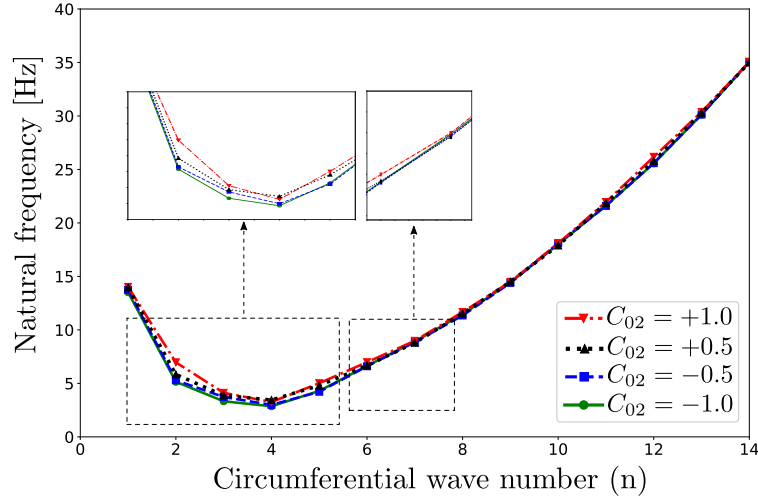


Figure 6.9 Influence of asymmetric geometric imperfection coefficient C_{02} on system's natural frequency.

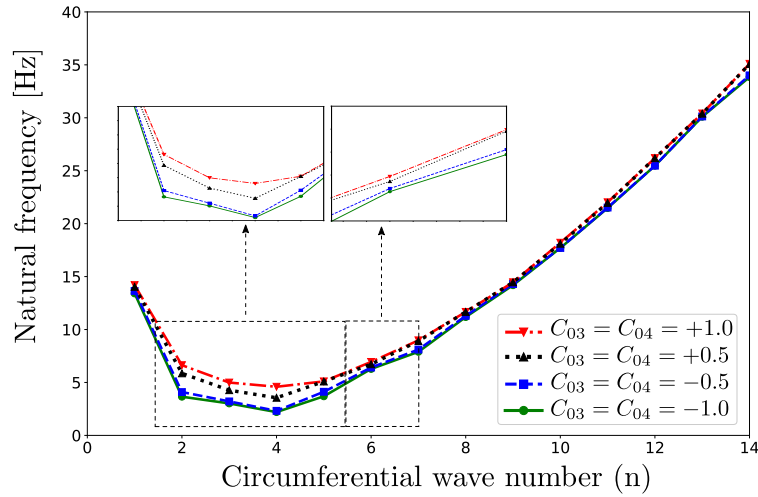


Figure 6.10 Influence of combined geometric imperfection coefficient (C_{03} and C_{04}) on System's natural frequency.

6.5.6 Forced vibration amplitude response analysis

In the vibration analysis of a structure subjected to external forces, amplitude response reveals how far the system moves from its equilibrium position. Within the scope of this study, the amplitude response provides insights into the behavior of each mode when subjected to external excitations. Furthermore, this study investigates how initial geometric imperfections affect the amplitude response of the system. By solving Equation (6.49), the system's amplitude response for both asymmetric and axisymmetric modes can be determined. In Figure 6.11, two peaks are evident for each mode, signifying the simultaneous occurrence of softening and hardening in the hyperelastic cylindrical shell's amplitude response. Notably, asymmetric modes tend to exhibit higher amplitude values than axisymmetric modes.

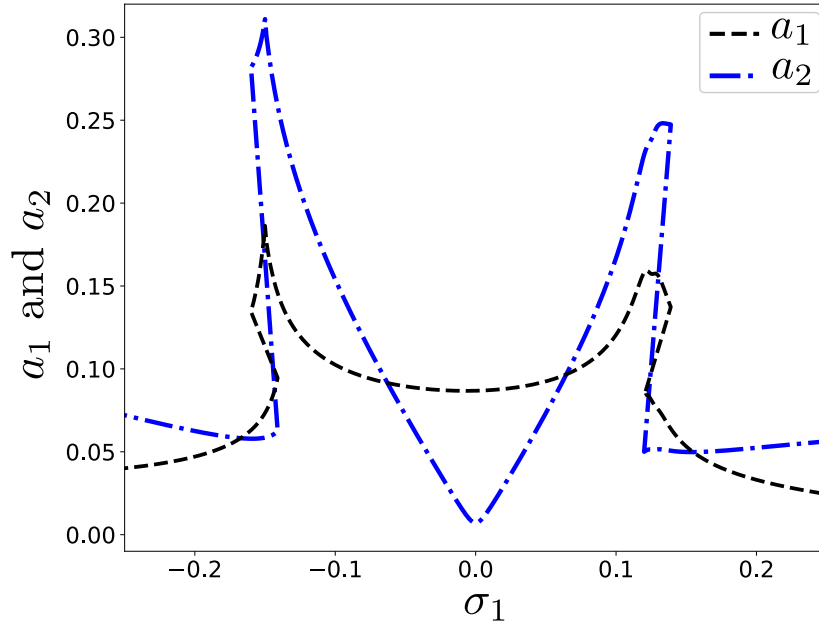


Figure 6.11 Axisymmetric a_1 and asymmetric a_2 modes response.

By incorporating three different geometric imperfections and studying the amplitude response of the axisymmetric mode (a_1), notable patterns emerge in Figure 6.12. It becomes evident that while the axisymmetric geometric imperfection exhibits a higher peak in terms of hardening compared to the other two types, it experiences lower levels of softening. Conversely, the combined imperfection displays more pronounced softening behavior and less hardening than the other two. As for the asymmetric imperfection, it falls somewhere in between the

other two types, showing intermediate levels of both hardening and softening behavior. In contrast to a geometrically perfect shell, all imperfection types lead to a decrease in the degree of softening observed in the structure. When examining the amplitude response of the

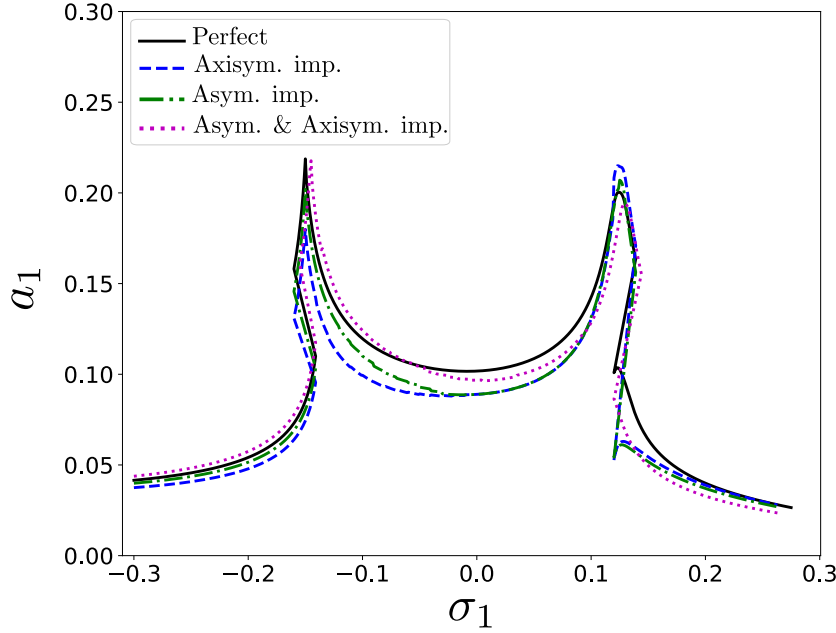


Figure 6.12 Effect of geometric imperfection on axisymmetric a_1 mode.

asymmetric mode, as shown in Figure 6.13, all three types of imperfections exhibit higher softening and hardening behavior in comparison to a perfect shell. Notably, the two peaks observed in the amplitude response of axisymmetric imperfection are higher than those of the other imperfection types, with combined imperfection having the least pronounced effect on the system's response.

Investigating the system under varying degrees of external excitation, the axisymmetric amplitude response has been observed, as depicted in Figure 6.14, across different geometric imperfections. The amplitude increases as the external excitation is raised, while the axisymmetric imperfection displays the highest level of hardening ($a_1^h = 0.24$) and lowest softening ($a_1^s = 0.19$), the combined geometric imperfection showcases the most substantial softening ($a_1^s = 0.24$) and lowest hardening ($a_1^h = 0.21$). The influence of the axisymmetric imperfection coefficient, C_{01} , on the dynamic behavior of the system, is illustrated in Figure 6.15. Increasing (C_{01} from 0.2 to 0.7 results in an observable rise in hardening, while softening weakens. Conversely, setting C_{01} to -0.7 enhances system softening, with a reduced magni-

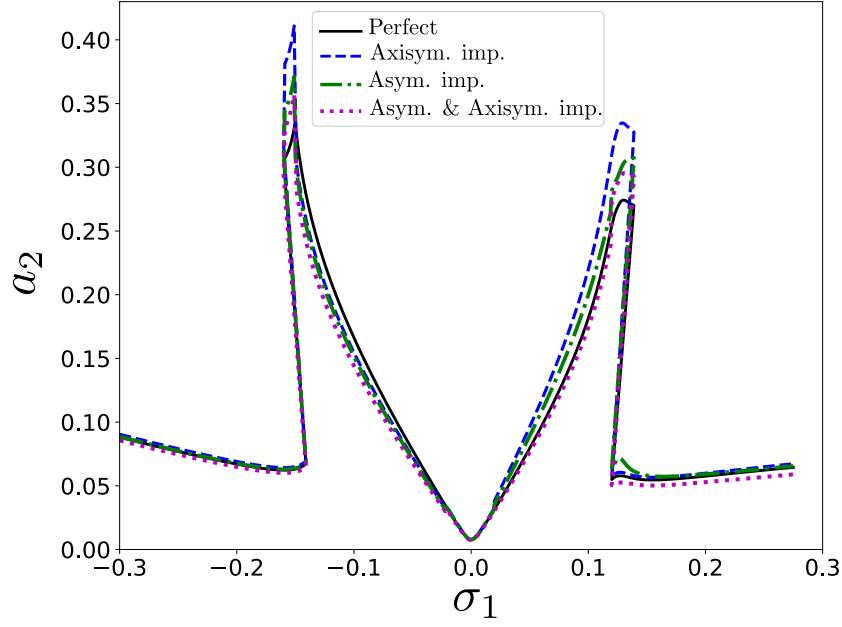


Figure 6.13 Effect of geometric imperfection on asymmetric a_2 mode.

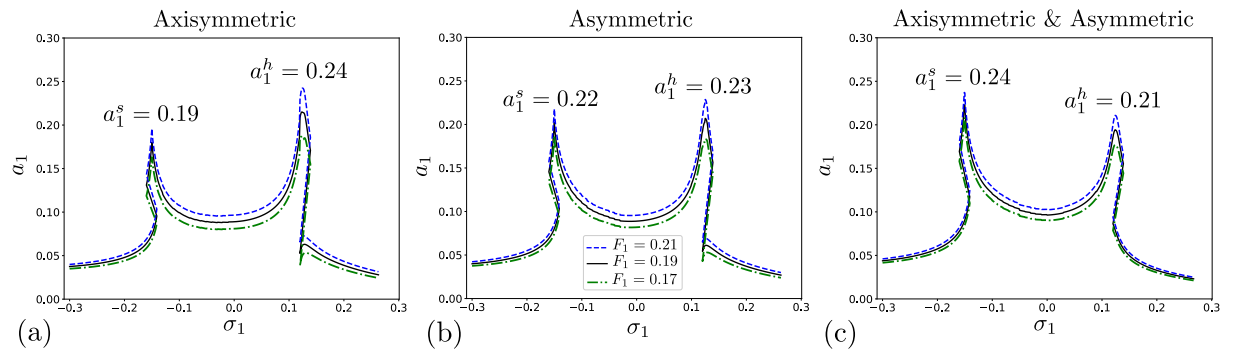


Figure 6.14 Investigation of axisymmetric geometric imperfection effects on axisymmetric mode a_1 under varying external excitation.

tude of the hardening peak. The same pattern is observable in Figures 6.16 and 6.17, as the coefficients of asymmetric and combined imperfections (C_{02}, C_{03} and C_{04}) vary. Similar to the previous parameters (C_{01}), an increase in the imperfection coefficient leads to heightened hardening behavior while softening intensifies as the coefficient approaches negative values with higher absolute magnitudes. The interesting aspect of the coefficients of geometric imperfections is their ability to shift the system's resonance point. As evident in Figures 6.15, 6.16 and 6.17, the resonance in the perfect system occurs when σ_1 is equal to zero. However, as the coefficients of the imperfections change and move towards positive values, this critical point shifts to the right side of the amplitude response figures. Conversely, when the coefficient of geometric imperfections takes on negative values, the resonance point shifts to the left side. Controlling and managing resonance in engineering and structural design is crucial to prevent structural failure or damage caused by excessive vibrations. When a system resonates, it can experience a significant increase in oscillation amplitudes, leading to mechanical stress and potential structural instability.

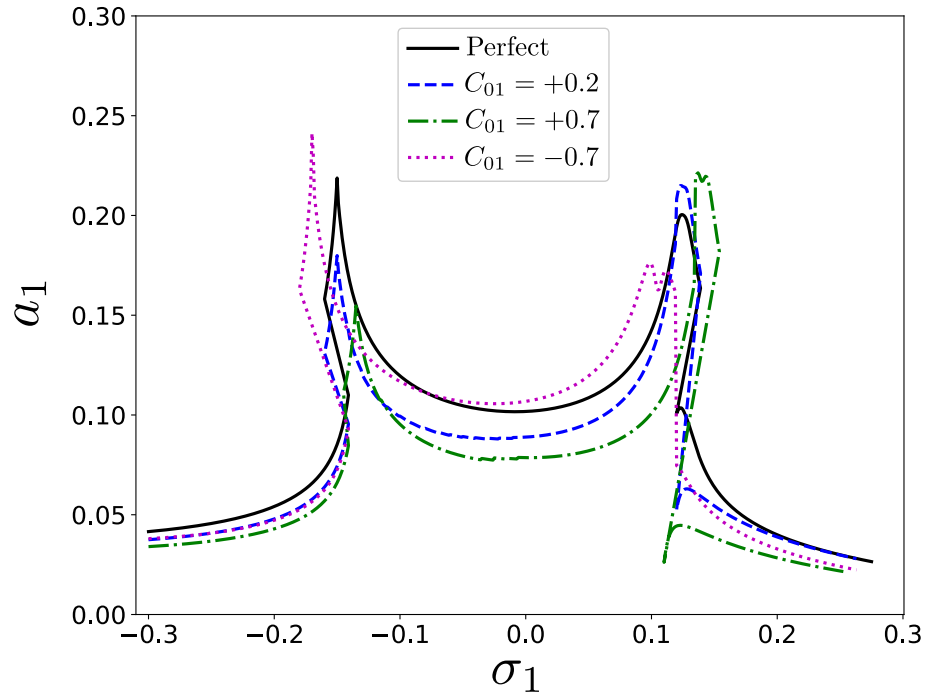


Figure 6.15 Amplitude response of axisymmetric mode with varying coefficient (C_{01}) of axisymmetric imperfection.

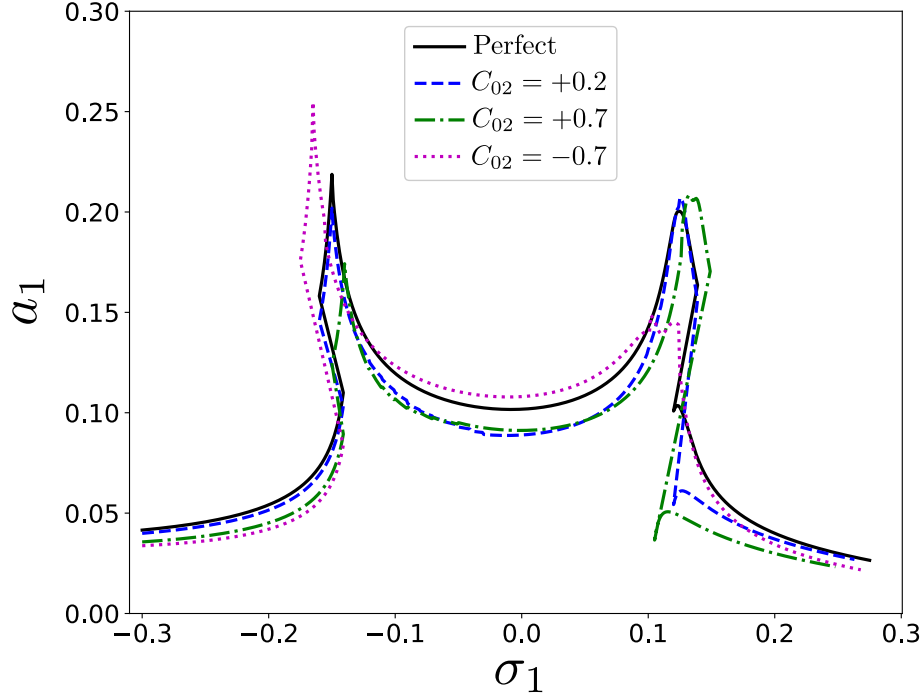


Figure 6.16 Amplitude response of axisymmetric mode with varying coefficient (C_{02}) of asymmetric imperfection.

6.6 Conclusions

In this study, the nonlinear behavior of a thin hyperelastic cylindrical shell in the presence of geometric imperfections is explored. Mooney-Rivlin model is employed as the strain energy function to capture the material's nonlinear behavior. The governing equations of motion are derived using the Lagrangian method and IDST. To verify the linear model, the natural frequency of the system was determined using both DST and IDST shell theories. The relative differences in the results obtained from these two theories were plotted and compared with a hybrid finite element approach. To establish the accuracy of the geometrical nonlinear model, the ratios of nonlinear to linear frequencies were computed using the Lindstedt-Poincaré technique for a homogeneous cylindrical shell and compared with corresponding results available in open references. In the concluding phase of model verification, the natural frequencies of the thin hyperelastic cylindrical shell, considering both physical and geometrical nonlinearities, were compared with results obtained from Abaqus simulation. A convergence test was carried out for the Abaqus model. To investigate the impact of geometric imperfections on the nonlinear system response, three distinct imperfection models were introduced: axisymmetric, asymmetric, and combined models. Static Condensation was employed to re-

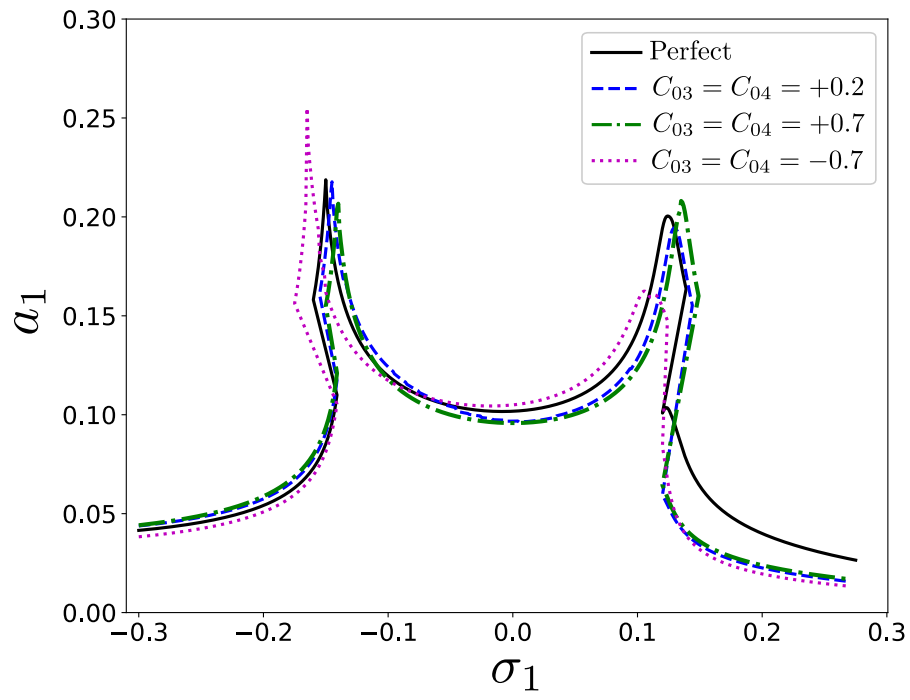


Figure 6.17 Amplitude response of axisymmetric mode with varying coefficients (C_{03} and C_{04}) of combined imperfection.

duce the system's degrees of freedom. Additionally, the Multiple Scales Method was utilized to effectively handle the equations of motion, which included highly nonlinear terms. The conclusions obtained from this study are as follows:

- The natural frequency of the system exhibited an upward trend when the coefficients of geometric imperfections were positive, indicating an increase in natural frequency. Conversely, a decrease in natural frequency was observed when the coefficients of geometric imperfections assumed negative values. This trend was particularly pronounced at lower circumferential wave numbers.
- While both softening and hardening peaks coexist in the amplitude response of the hyperelastic cylindrical shell, their magnitudes increase with higher applied forces.
- In the amplitude response of the axisymmetric mode (a_1), the axisymmetric imperfection exhibits the highest level of hardening and the lowest degree of softening. In contrast, the combined model displays the most pronounced peak in softening behavior.
- The system's behavior is notably impacted by the sign of the imperfection coefficient. A positive coefficient amplifies the hardening effect while diminishing the softening behavior, whereas a negative coefficient yields the opposite effect.
- The occurrence of nonlinear internal resonance in the system shifts as the coefficient of geometric imperfection varies.

CHAPTER 7 GENERAL DISCUSSION

The primary objective of this thesis was to develop a model for a hyperelastic cylindrical shell and investigate its nonlinear behavior. The model was constructed based on essential assumptions, and certain limitations were encountered in the process. This chapter will provide a detailed discussion of these limitations. Additionally, we will explore the advantages of the presented model compared to the hybrid finite element method [111] and ABAQUS. Both of these numerical methods were employed for verification in our study. Finally, we will address the challenges we faced during this work.

7.1 Limitations

- **Analyzing the effect of different boundary conditions on model:** Our model has certain limitations when it comes to selecting boundary conditions. The shape function, which defines how the middle surface deforms, must follow specific rules for the boundaries. This means we face restrictions in choosing and studying different boundary conditions in our model. The requirements of the shape function, along with the need to match prescribed boundaries, make it challenging to analyze and vary boundary conditions freely in our study.
- **Limitations associated with the range of deformation:** Our model represented the hyperelastic material using Mooney-Rivlin strain energy functions. This choice is suitable and accurate when the deformation remains below 400 percent. However, for deformations exceeding this range, a more sophisticated model is recommended. In our study, we chose the Mooney-Rivlin model, taking into account computational cost, complexity, and the objectives of the research. This decision strikes a balance between accuracy and practical considerations within the scope of our study.
- **Validation against experimental test:** During this research, the primary emphasis was on theoretical and computational modeling of hyperelastic cylindrical shell behavior. Consequently, experimental tests were not conducted as they were beyond the scope of this study. It is important to mention that there is a noticeable lack of experimental results for thin hyperelastic cylindrical shells in the existing literature. However, it is acknowledged that the value of this study could be better by validating the theoretical models with experimental results.

7.2 Advantages

- **No need for convergence and mesh sensitive studies:** In finite element methods, creating a mesh is essential, and conducting convergence and mesh sensitivity studies becomes necessary to ensure accurate results. However, in the proposed model, the structure does not undergo meshing, eliminating the need for studying convergence and mesh sensitivity. This represents a significant advantage compared to the finite element approaches, simplifying the modeling process and potentially reducing computational complexities.
- **Adaptability:** This model has the potential to be adapted for investigating more complex scenarios. By incorporating temperature terms, it becomes possible to examine the impact of temperature. Additionally, the model can be adjusted to explore viscosity. The key advantage lies in the flexibility of the model and ease of modification.
- **Potential to study and capture nonlinear behavior of the system more profoundly:** This model can yield more profound results, offering a detailed understanding of the nonlinear behavior of the system. In contrast, widely used commercial finite element software like ABAQUS, to the best of our knowledge, encounters difficulties in obtaining amplitude frequency response results for hyperelastic cylindrical shells. Our model goes beyond, capable of exploring nonlinear behavior and generating valuable outcomes, such as Poincaré maps and other insightful results.

7.3 Challenges

In this research, the initial intention was to investigate the hyperelastic cylindrical shell using the hybrid finite element method. However, upon modeling the system, a significant challenge emerged – the method proved insufficient to fully capture the nonlinear behavior of the system. The primary limitation arose from the shape function of the method, which necessitated a linear form to satisfy mathematical requirements. Faced with this obstacle, a critical decision was made to change the research approach, marking a significant challenge in the progression of this study. We chose to use an alternative approach, utilizing a presented analytical method for a more effective modeling solution and to overcome this challenge.

Another challenge arose when selecting the shell theory and material model. Initially, we opted for Sander’s Koiter nonlinear for the shell theory and Ogden for the material model, aiming to model a broad range of structures and deformations. However, we encountered computational limitations. The model became computationally intensive, making it challenging to calculate potential and kinetic energy efficiently on our PC. Despite applying numerical

integration methods to reduce costs, they proved insufficient, leading us to make the crucial decision to change both the shell theory and material model. To overcome this challenge, we adopted Donnell's shallow shell theory, significantly reducing the size of equations and improving computational efficiency. For the material model, we transitioned to the Mooney-Rivlin model, which not only addressed the computational cost but also provided sufficient accuracy for the range of deformations within the scope of this research.

CHAPTER 8 CONCLUSION AND RECOMMENDATIONS

This thesis presents a model for analyzing the natural frequency and nonlinear response of a thin hyperelastic cylindrical shell. Nonlinear Donnell's shallow shell theory, including an improved version, was employed to account for geometrical nonlinearity in the problem. The behavior of the incompressible hyperelastic material was modeled using Mooney-Rivlin material model. The equations of motion were derived, and to reduce the degrees-of-freedom, the static condensation method was applied. By using this method, the final equations of motion were redefined to depend solely on the radial displacement. These equations were then addressed using Lindstedt-Poincaré and Multiple Scales methods. This modeling was coded in Python, and the conclusions from this research are summarized as follows:

- Investigating the asymmetric and axisymmetric modes and examining their backbone responses revealed that the axisymmetric mode induces softening due to the quadratic nonlinear stiffness matrix. The nature of this nonlinear matrix, whether positive or negative, contributes to the softening effect in the system. Furthermore, the cubic nonlinear stiffness matrix introduces softening with a negative sign and hardening with a positive sign.
- Exploring the influence of structural parameters and damping coefficients, it was observed that increasing the damping coefficient has a diminishing effect on the hardening peak of the amplitude response. The influence of it on the softening behavior is less pronounced. Regarding structural parameters, thicker structures were found to have higher natural frequencies, leading to lower magnitudes in both the softening and hardening peaks of the amplitude-frequency response.
- By considering different models of shape functions, this research investigated their impact on the response of the system. The results demonstrated that, without incorporating independent second-order bending terms, the model fails to fully capture the softening behavior of the structure. The softening peak of the amplitude-frequency response is approximately 15 percent less when independent bending terms are absent. This underscores the importance of selecting an appropriate shape function for the middle surface.
- The impact of the shape function on the backbone was also studied, revealing that the presence of independent bending terms weakens the hardening observed in the backbone. As the number of circumferential waves increases, this difference in the

ratio of nonlinear natural frequency to linear natural frequency diminishes. However, for lower circumferential wave numbers such as two, it can be observed that for the $\frac{a}{h}$ equal to 0.25, the ratio of nonlinear natural frequency to linear frequency increases from 1.33 to 1.74.

- Applying the improved version of Donnell's shallow shell theory, as opposed to the classical version, enhanced the accuracy of the model for circumferential wave numbers(n) less than six. For $n = 2$, the relative difference with the hybrid finite element method as a reference decreased from 5.74% to 3.52%. Similarly, for $n = 3$, this difference decreased from 3.51% to 1.59%. As the number of circumferential waves increases, the difference between the two models diminishes, and at $n = 7$, they converge closely together.
- By introducing three distinct geometric imperfection models, the results revealed that in the amplitude-frequency response of the axisymmetric mode, the combined geometric imperfection exhibited the highest peak of softening, while this model demonstrated the lowest peak of hardening compared to the other two models. In the amplitude response of this mode, the axisymmetric imperfection exhibited the highest amount of hardening. The axisymmetric imperfection showed both the highest peak of hardening and softening in the amplitude-frequency response of the asymmetric mode.
- Studying the effect of the coefficients of these geometric imperfections revealed that increasing the coefficients towards one with a positive sign leads to an increase in the hardening of the system while softening decreases. Conversely, moving towards a negative one has the opposite effect on the system.

8.1 Recommendations for future study

Concluding our investigation into the nonlinear dynamics of hyperelastic cylindrical shells, the following research areas can be recommended for further investigation.

- Applying more sophisticated strain energy models for modeling biological tissues: Improving model accuracy for human tissues mandates the utilization of sophisticated strain energy models like Ogden, Yeoh, or Holzapfel. However, this enhanced accuracy comes at the cost of increased computational intensity and introduces challenges when performing numerical integration within the strain energy function.
- Incorporating initial geometric imperfections into ABAQUS simulations: The integration of the defined initial geometric imperfections from this research into ABAQUS

will demand the creation of supplementary code and subroutines. Despite the unique challenge this poses, it stands as an essential step for advancing the field.

- Exploring the impact of temperature on hyperelastic material behavior: In future work, the creation of a thermo-hyperelastic model, achieved by introducing temperature parameters into the strain energy density function, is suggested. These logarithmic temperature-dependent terms would significantly enhance the model's complexity.
- Conducting physical experiments on thin hyperelastic cylindrical shells: The absence of experimental testing on thin hyperelastic cylindrical shells underscores the necessity for future work in this domain. Conducting such experiments is vital for achieving a more comprehensive understanding of this structure's behavior.

REFERENCES

- [1] H. B. Khaniki, M. H. Ghayesh, R. Chin, and M. Amabili, “A review on the nonlinear dynamics of hyperelastic structures,” *Nonlinear Dyn*, vol. 110, pp. 963–994, 2022.
- [2] L. Chen, C. Yang, H. Wang, D. T. Branson, J. S. Dai, and R. Kang, “Design and modeling of a soft robotic surface with hyperelastic material,” *Mechanism and Machine Theory*, vol. 130, pp. 109–122, 2018.
- [3] E. Vignali, E. Gasparotti, K. Capellini, B. M. Fanni, L. Landini, V. Positano, and S. Celi, “Modeling biomechanical interaction between soft tissue and soft robotic instruments: importance of constitutive anisotropic hyperelastic formulations,” *The International Journal of Robotics Research*, vol. 40, no. 1, pp. 224–235, 2021.
- [4] C. E. Proulx, M. Beaulac, M. David, C. Deguire, C. Haché, F. Klug, M. Kupnik, J. Higgins, and D. H. Gagnon, “Review of the effects of soft robotic gloves for activity-based rehabilitation in individuals with reduced hand function and manual dexterity following a neurological event,” *Journal of rehabilitation and assistive technologies engineering*, vol. 7, 2020.
- [5] K. Wandke and Y. Z., “Moose-based finite element hyperelastic modeling for soft robot simulations,” *IEEE Access*, vol. 9, pp. 139 627–139 635, 2021.
- [6] M. Roshanfar, S. Taki, A. Sayadi, R. Cecere, J. Dargahi, and A. Hooshier, “Hyperelastic modeling and validation of hybrid-actuated soft robot with pressure-stiffening,” *Micromachines*, vol. 14, no. 5, p. 900, 2023.
- [7] N. G. Kim and J.-H. Ryu, “A soft growing robot using hyperelastic material,” *Advanced Intelligent Systems*, vol. 5, no. 2, p. 2200264, 2023.
- [8] V. Kumar, U. H. Ko, Y. Zhou, J. Hoque, G. Arya, and S. Varghese, “Microengineered materials with self-healing features for soft robotics,” *Advanced Intelligent Systems*, vol. 3, no. 7, p. 2100005, 2021.
- [9] Kumar, U. H. Ko, Y. Zhou, J. Hoque, G. Arya, and S. Varghese, “Dragonfly inspired smart soft robot,” *bioRxiv*, pp. 2020–04, 2020.
- [10] M. Tessler, M. R. Brugler, J. A. Burns, N. R. Sinatra, D. M. Vogt, A. Varma, M. Xiao, R. J. Wood, and D. F. Gruber, “Ultra-gentle soft robotic fingers induce minimal tran-

- scriptomic response in a fragile marine animal,” *Current Biology*, vol. 30, no. 4, pp. R157–R158, 2020.
- [11] M. Schaffner, J. A. Faber, L. Pianegonda, P. A. R  hs, F. Coulter, and A. R. Studart, “3d printing of robotic soft actuators with programmable bioinspired architectures,” *Nature communications*, vol. 9, no. 1, p. 878, 2018.
 - [12] T. Kaster, I. Sack, and A. Samani, “Measurement of the hyperelastic properties of ex vivo brain tissue slices,” *Journal of biomechanics*, vol. 44, no. 6, pp. 1158–1163, 2011.
 - [13] G. Z. Voyiadjis and A. Samadi-Dooki, “Hyperelastic modeling of the human brain tissue: effects of no-slip boundary condition and compressibility on the uniaxial deformation,” *Journal of the mechanical behavior of biomedical materials*, vol. 83, pp. 63–78, 2018.
 - [14] R. Moran, J. H. Smith, and J. J. Garc  a, “Fitted hyperelastic parameters for human brain tissue from reported tension, compression, and shear tests,” *Journal of biomechanics*, vol. 47, no. 15, pp. 3762–3766, 2014.
 - [15] L. Zhang, K. H. Yang, and A. I. King, “A proposed injury threshold for mild traumatic brain injury,” *J. Biomech. Eng.*, vol. 126, no. 2, pp. 226–236, 2004.
 - [16] E. M. Isselbacher, “Thoracic and abdominal aortic aneurysms,” *Circulation*, vol. 111, no. 6, pp. 816–828, 2005.
 - [17] G. Franchini, I. D. Breslavsky, G. A. Holzapfel, and M. Amabili, “Viscoelastic characterization of human descending thoracic aortas under cyclic load,” *Acta Biomaterialia*, vol. 130, pp. 291–307, 2021.
 - [18] M. A. Horvath, C. E. Varela, E. B. Dolan, W. Whyte, D. S. Monahan, C. J. Payne, I. A. Wamala, N. V. Vasilyev, F. A. Pigula, D. J. Mooney *et al.*, “Towards alternative approaches for coupling of a soft robotic sleeve to the heart,” *Annals of biomedical engineering*, vol. 46, pp. 1534–1547, 2018.
 - [19] P. Polygerinos, Z. Wang, K. C. Galloway, R. J. Wood, and C. J. Walsh, “Soft robotic glove for combined assistance and at-home rehabilitation,” *Robotics and Autonomous Systems*, vol. 73, pp. 135–143, 2015.
 - [20] A.-L. Cauchy, “Sur l’equilibre et le mouvement d’une plaque solide,” *Exercices de Mathematique*, vol. 3, no. 1828, pp. 328–355, 1828.

- [21] S. D. Poisson, “Mémoire sur l’équilibre et le mouvement des corp élastique,” *Mémoires de l’Académie Royale des Sciences de l’Institut*, vol. 8, pp. 357–570, 1829.
- [22] G. Kirchhoff, “Über das gleichgewicht und die bewegung einer elastischen scheibe.” *Journal für die reine und angewandte Mathematik (Crelles Journal)*, vol. 1850, no. 40, pp. 51–88, 1850.
- [23] J. Reddy, “Theory and analysis of elastic plates and shells,” 2006. [Online]. Available: <https://api.semanticscholar.org/CorpusID:125214222>
- [24] T. Von Kármán, *Festigkeitsprobleme im maschinenbau*. Teubner, 1910.
- [25] J. Reddy, “A general non-linear third-order theory of plates with moderate thickness,” *International Journal of Non-Linear Mechanics*, vol. 25, no. 6, pp. 677–686, 1990.
- [26] L. H. Donnell, “A new theory for the buckling of thin cylinders under axial compression and bending,” *Transactions of the American society of mechanical engineers*, vol. 473, pp. 795–806, 1934.
- [27] J. L. Sanders Jr, “Nonlinear theories for thin shells,” *Quarterly of Applied Mathematics*, vol. 21, no. 1, pp. 21–36, 1963.
- [28] W. T. Koiter, “On the nonlinear theory of thin elastic shells,” *Proc. Koninkl. Ned. Akad. van Wetenschappen, Series B*, vol. 69, pp. 1–54, 1966.
- [29] K. M. Mushtari and K. Galimov, “Non-linear theory of thin elastic shells(nelineinaya teoriya uprugikh obolochek),” *NASA*, no. 19980231031, 1957.
- [30] V. V. Novozhilov, *Foundations of the nonlinear theory of elasticity*. Courier Corporation, 1999.
- [31] J. N. Reddy, “Exact solutions of moderately thick laminated shells,” *Journal of Engineering Mechanics*, vol. 110, no. 5, pp. 794–809, 1984.
- [32] M. Amabili, *Nonlinear vibrations and stability of shells and plates*. Cambridge University Press, 2008.
- [33] Y. Liu, Z. Qin, and F. Chu, “Nonlinear forced vibrations of rotating cylindrical shells under multi-harmonic excitations in thermal environment,” *Nonlinear Dyn*, vol. 108, pp. 2977–2991, 2022.

- [34] Y. Liu, Z. Qin, and Fulei, “Investigation of magneto-electro-thermo-mechanical loads on nonlinear forced vibrations of composite cylindrical shells,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 107, pp. 106–146, 2022.
- [35] G. Li and K. K.F. Wong, *Theory of nonlinear structural analysis*. John Wiley & Sons, 2014.
- [36] J. Bonet and R. D. Wood, *Nonlinear continuum mechanics for finite element analysis*. Cambridge university press, 1997.
- [37] G. A. Holzapfel, “Nonlinear solid mechanics: a continuum approach for engineering science,” 2002.
- [38] M. Amabili, *Nonlinear Mechanics of Shells and Plates in Composite, Soft and Biological Materials*. Cambridge University Press, 2018.
- [39] K. M. Moerman, B. Fereidoonenezhad, and J. P. McGarry, “Novel hyperelastic models for large volumetric deformations,” *International Journal of Solids and Structures*, vol. 193, pp. 474–491, 2020.
- [40] Z. Yang, *Material Modeling in Finite Element Analysis*. CRC Press, 2019.
- [41] M. Mooney, “A theory of large elastic deformation,” *Journal of applied physics*, vol. 11, no. 9, pp. 582–592, 1940.
- [42] R. S. Rivlin, “Large elastic deformations of isotropic materials iv. further developments of the general theory,” *Philosophical transactions of the royal society of London. Series A, Mathematical and physical sciences*, vol. 241, no. 835, pp. 379–397, 1948.
- [43] V. Biderman, “Calculation of rubber parts,” *Rascheti na prochnost*, vol. 40, 1958.
- [44] A. Isihara, N. Hashitsume, and M. Tatibana, “Statistical theory of rubber-like elasticity. iv.(two-dimensional stretching),” *The Journal of Chemical Physics*, vol. 19, no. 12, pp. 1508–1512, 1951.
- [45] A. James, A. Green, and G. Simpson, “Strain energy functions of rubber. i. characterization of gum vulcanizates,” *Journal of Applied Polymer Science*, vol. 19, no. 7, pp. 2033–2058, 1975.
- [46] G. Marckmann and E. Verron, “Comparison of hyperelastic models for rubber-like materials,” *Rubber chemistry and technology*, vol. 79, no. 5, pp. 835–858, 2006.

- [47] R. W. Ogden, “Large deformation isotropic elasticity—on the correlation of theory and experiment for incompressible rubberlike solids,” *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, vol. 326, no. 1567, pp. 565–584, 1972.
- [48] E. M. Arruda and M. C. Boyce, “A three-dimensional constitutive model for the large stretch behavior of rubber elastic materials,” *Journal of the Mechanics and Physics of Solids*, vol. 41, no. 2, pp. 389–412, 1993.
- [49] A. N. Gent, “A new constitutive relation for rubber,” *Rubber chemistry and technology*, vol. 69, no. 1, pp. 59–61, 1996.
- [50] P. J. Blatz and W. L. Ko, “Application of finite elastic theory to the deformation of rubbery materials,” *Transactions of the Society of Rheology*, vol. 6, no. 1, pp. 223–251, 1962.
- [51] A. Gent, “Elastic instabilities of inflated rubber shells,” *Rubber chemistry and technology*, vol. 72, no. 2, pp. 263–268, 1999.
- [52] J. K. Knowles, “Large amplitude oscillations of a tube of incompressible elastic material,” *Quarterly of Applied Mathematics*, vol. 18, no. 1, pp. 71–77, 1960.
- [53] J. Knowles, “On a class of oscillations in the finite-deformation theory of elasticity,” *Journal of applied mechanics*, vol. 29, pp. 283–286, 1962.
- [54] M. Shahinpoor and J. L. Nowinski, “Exact solution to the problem of forced large amplitude radial oscillations of a thin hyperelastic tube,” *International Journal of Non-Linear Mechanics*, vol. 6, no. 2, pp. 193–207, 1971.
- [55] M. Shahinpoor and R. Balakrishnan, “Large amplitude oscillations of thick hyperelastic cylindrical shells,” *International Journal of Non-Linear Mechanics*, vol. 13, no. 5-6, pp. 295–301, 1978.
- [56] X. Yuan, R. Zhang, and H. Zhang, “Controllability conditions of finite oscillations of hyperelastic cylindrical tubes composed of a class of ogden material models,” *Computers, Materials and Continua*, vol. 7, no. 3, pp. 155–156, 2008.
- [57] X. He, H. Yong, and Y. Zhou, “The dynamics response and stability of a dielectric elastomer cylindrical shell under static and periodic votage,” *Guti Lixue Xuebao/Acta Mechanica Solida Sinica*, vol. 33, pp. 341–348, 08 2012.

- [58] I. D. Breslavsky, M. Amabili, and M. Legrand, “Physically and geometrically non-linear vibrations of thin rectangular plates,” *International Journal of Non-Linear Mechanics*, vol. 58, pp. 30–40, 2014.
- [59] M. Amabili, P. Balasubramanian, I. D. Breslavsky, G. Ferrari, R. Garziera, and K. Ribova, “Experimental and numerical study on vibrations and static deflection of a thin hyperelastic plate,” *Journal of sound and vibration*, vol. 385, pp. 81–92, 2016.
- [60] M. Amabili, P. Balasubramanian, I. D. Breslavsky, G. Ferrari, and E. Tubaldi, “Viscoelastic characterization of woven dacron for aortic grafts by using direction-dependent quasi-linear viscoelasticity,” *Journal of the mechanical behavior of biomedical materials*, vol. 82, pp. 282–290, 2018.
- [61] J. Zhang, J. Xu, X. Yuan, H. Ding, D. Niu, and W. Zhang, “Nonlinear vibration analyses of cylindrical shells composed of hyperelastic materials,” *Acta Mechanica Sinica*, vol. 32, no. 4, pp. 463–482, 2019.
- [62] W. Zhao, J. Zhang, W. Zhang, and X. Yuan, “Internal resonance characteristics of hyperelastic thin-walled cylindrical shells composed of Mooney–Rivlin materials,” *Thin-Walled Structures*, vol. 163, p. 107754, 2021.
- [63] M. Shayan Arani, M. Bakhtiari, and A. A. Lakis, “Analyzing softening and hardening behavior in vibration of a thin incompressible hyperelastic cylindrical shell,” *Thin-Walled Structures*, vol. 189, p. 110943, 2023.
- [64] J. Zhang, J. Xu, X. Yuan, H. Ding, D. Niu, and W. Zhang, “Nonlinear vibration analyses of cylindrical shells composed of hyperelastic materials,” *Acta Mechanica Sinica*, vol. 32, pp. 463–482, 2019.
- [65] H. B. Khaniki and M. H. Ghayesh, “Highly nonlinear hyperelastic shells: Statics and dynamics,” *International Journal of Engineering Science*, vol. 183, p. 103794, 2023.
- [66] H. B. Khaniki, M. H. Ghayesh, and R. Chin, “Theory and experiment for dynamics of hyperelastic plates with modal interactions,” *International Journal of Engineering Science*, vol. 182, p. 103769, 2023.
- [67] J. Xu, X. Yuan, H. Zhang, Z. Zhao, and W. Zhao, “Combined effects of axial load and temperature on finite deformation of incompressible thermo-hyperelastic cylinder,” *Applied Mathematics and Mechanics*, vol. 40(4), pp. 499–514, 2019.

- [68] J. Xu, X. Yuan, H. Zhang, F. Zheng, and L. Chen, “Nonlinear Vibrations of Thermo-Hyperelastic Moderately Thick Cylindrical Shells with 2:1 Internal Resonance,” *International Journal of Structural Stability and Dynamics*, vol. 20, p. 2050067, 2020.
- [69] J. Xu, X. Yuan, H. Zhang, M. Ma, and W. Zhao, “Internal resonance of hyperelastic thin-walled cylindrical shells under harmonic axial excitation and time-varying temperature field,” *Thin-Walled Structures*, vol. 175, p. 109256, 2022.
- [70] J. Zhang, W. Zhang, and Y. Zhang, “Nonlinear resonant responses of hyperelastic cylindrical shells with initial geometric imperfections,” *Chaos, Solitons and Fractals*, vol. 173, p. 113709, 2023.
- [71] H. B. Khaniki, M. H. Ghayesh, R. Chin, and M. Amabili, “Hyperelastic structures: A review on the mechanics and biomechanics,” *International Journal of Non-Linear Mechanics*, vol. 148, p. 104275, 2023.
- [72] M. Hemmatnezhad, G. Iariccio, A. Zippo, and F. Pellicano, “Modal localization in vibrations of circular cylindrical shells with geometric imperfections,” *Thin-Walled Structures*, vol. 181, p. 110079, 2022.
- [73] L. Rodrigues, F. M. Silva, and P. B. Goncalves, “Influence of initial geometric imperfections on the 1: 1: 1 internal resonances and nonlinear vibrations of thin-walled cylindrical shells,” *Thin-Walled Structures*, vol. 151, p. 106730, 2020.
- [74] H. B. Khaniki, M. H. Ghayesh, R. Chin, and L. Chen, “Experimental characteristics and coupled nonlinear forced vibrations of axially travelling hyperelastic beams,” *Thin-Walled Structures*, vol. 170, p. 108526, 2022.
- [75] E. B. Wilson, “AEH Love, A Treatise on the Mathematical Theory of Elasticity,” *Bulletin of the American Mathematical Society*, vol. 34, no. 2, pp. 242–243, 1928.
- [76] M. P. Nemeth, “An exposition on the nonlinear kinematics of shells, including transverse shearing deformations,” Langley Research center, Hampton, Virginia, Tech. Rep., 2013.
- [77] M. Nemeth, “A Leonard-Sanders-Budiansky-Koiter-Type Nonlinear Shell Theory with a Hierarchy of Transverse-Shearing Deformations,” Langley Research center, Hampton, Virginia, Tech. Rep., 2013.
- [78] D. A. Evensen, “Non-Linear Flexural Vibrations of Thin-Walled Circular Rings,” PhD Thesis, Phd Thesis. Pasadena, CA, California Institute of Technology, 1964.

- [79] D. Evensen, *Nonlinear flexural vibrations of thin-walled circular cylinders*. National Aeronautics and Space Administration, 1967.
- [80] A. Selmane and A. A. Lakis, “Influence of geometric non-linearities on the free vibrations of orthotropic open cylindrical shells,” *International Journal for Numerical Methods in Engineering*, vol. 40, no. 6, pp. 1115–1137, 1997.
- [81] Y. Ben-Youssef, Y. Kerboua, and A. A. Lakis, “Analysis of nonlinear vibrations of thin cylindrical shells subjected to supersonic flow,” *Thin-Walled Structures*, vol. 173, p. 108969, 2022.
- [82] M. Bakhtiari, A. A. Lakis, and Y. Kerboua, “Nonlinear supersonic flutter of truncated conical shells,” *Journal of Mechanical Science and Technology*, vol. 34, no. 4, pp. 1375–1388, 2020.
- [83] M. Bakhtiari, A. Lakis, and Y. Kerboua, “Nonlinear vibration of truncated conical shells: Donnell, Sanders and Nemeth theories,” *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 21, no. 1, pp. 83–97, 2020.
- [84] Y. Liu, Z. Qin, and F. Chu, “Nonlinear forced vibrations of functionally graded piezoelectric cylindrical shells under electric-thermo-mechanical loads,” *International Journal of Mechanical Sciences*, vol. 201, p. 106474, 2021.
- [85] Y. Liu, Z. Qin, and C. Fulei, “Nonlinear forced vibrations of FGM sandwich cylindrical shells with porosities on an elastic substrate,” *Nonlinear Dyn*, vol. 104, pp. 1007–1021, 2021.
- [86] Y. Q. Wang, C. Ye, and J. W. Zu, “Vibration analysis of circular cylindrical shells made of metal foams under various boundary conditions,” *Int J Mech Mater Des*, vol. 15, pp. 333–344, 2019.
- [87] M. W. Teng and Y. Q. Wang, “Spin-induced internal resonance in circular cylindrical shells,” *International Journal of Non-Linear Mechanics*, vol. 147, p. 104234, 2022.
- [88] C. Ye and Y. Q. Wang, “Nonlinear forced vibration of functionally graded graphene platelet-reinforced metal foam cylindrical shells: internal resonances,” *Nonlinear Dyn*, vol. 104, pp. 2051–2069, 2021.
- [89] U. Wijaya and R. Soegiarso, “Mechanical properties of Indonesian hyperelastic low-grade rubber for low-cost base isolator,” in *MATEC Web of Conferences*, vol. 276. EDP Sciences, 2019, p. 01017.

- [90] Y. Başar and M. Itskov, “Finite element formulation of the Ogden material model with application to rubber-like shells,” *International Journal for Numerical Methods in Engineering*, vol. 42, no. 7, pp. 1279–1305, 1998.
- [91] E. d. M. B. Campello, P. d. M. Pimenta, and P. Wriggers, “An exact conserving algorithm for nonlinear dynamics with rotational DOFs and general hyperelasticity. Part 2: shells,” *Computational Mechanics*, vol. 48, no. 2, pp. 195–211, 2011.
- [92] K. Luo, C. Liu, Q. Tian, and H. Hu, “Nonlinear static and dynamic analysis of hyperelastic thin shells via the absolute nodal coordinate formulation,” *Nonlinear Dynamics*, vol. 85, no. 2, pp. 949–971, 2016.
- [93] U. Akyüz and A. Ertepinar, “Stability and asymmetric vibrations of pressurized compressible hyperelastic cylindrical shells,” *International Journal of Non-Linear Mechanics*, vol. 34, no. 3, pp. 391–404, 1999.
- [94] J. Zhang, J. Xu, X. Yuan, W. Zhang, and D. Niu, “Strongly nonlinear vibrations of a hyperelastic thin-walled cylindrical shell based on the modified Lindstedt–Poincaré method,” *International Journal of Structural Stability and Dynamics*, vol. 19, no. 12, p. 1950160, 2019.
- [95] H. B. Khaniki, M. H. Ghayesh, R. Chin, and M. Amabili, “Large amplitude vibrations of imperfect porous-hyperelastic beams via a modified strain energy,” *Journal of Sound and Vibration*, vol. 513, p. 116416, 2021.
- [96] H. B. Khaniki, M. H. Ghayesh, R. Chin, and S. Hussain, “Internal resonance and bending analysis of thick visco-hyper-elastic arches,” *Continuum Mech. Thermodyn*, vol. 35, pp. 299–342, 2023.
- [97] H. B. Khaniki, M. H. Ghayesh, R. Chin, and H. Shahid, “Nonlinear continuum mechanics of thick hyperelastic sandwich beams using various shear deformable beam theories,” *Continuum Mech. Thermodyn*, 2022.
- [98] A. H. Nayfeh, *Introduction to perturbation techniques*. John Wiley & Sons, 2011.
- [99] A. H. Nayfeh and D. T. Mook, *Nonlinear oscillations*. John Wiley & Sons, 2008.
- [100] C. P. Okeke, A. N. Thite, J. F. Durodola, and M. T. Greenrod, “Hyperelastic polymer material models for robust fatigue performance of automotive LED lamps,” *Procedia Structural Integrity*, vol. 5, pp. 600–607, 2017.

- [101] L. A. Mihai, L. Chin, P. A. Janmey, and A. Goriely, “A comparison of hyperelastic constitutive models applicable to brain and fat tissues,” *Journal of The Royal Society Interface*, vol. 12, no. 110, p. 20150486, 2015.
- [102] R. W. Ogden, *Non-linear elastic deformations*. Courier Corporation, 1997.
- [103] A. F. Bower, *Applied mechanics of solids*. CRC press, 2009.
- [104] C. Emminger, U. D. Çakmak, R. Preuer, I. Graz, and Z. Major, “Hyperelastic Material Parameter Determination and Numerical Study of TPU and PDMS Dampers,” *materials MDPI*, 2021.
- [105] L. R. G. Treloar, “Stress-strain data for vulcanized rubber under various types of deformation,” *Rubber Chemistry and Technology*, vol. 17, no. 4, pp. 813–825, 1944.
- [106] I. D. Breslavsky, M. Amabili, and M. Legrand, “Nonlinear vibrations of thin hyperelastic plates,” *Journal of Sound and Vibration*, vol. 333, no. 19, pp. 4668–4681, 2014.
- [107] A. A. Lakis, A. Selmane, and A. Toledano, “Non-linear free vibration analysis of laminated orthotropic cylindrical shells,” *International journal of mechanical sciences*, vol. 40, no. 1, pp. 27–49, 1998.
- [108] F. Pellicano, “Vibrations of circular cylindrical shells: theory and experiments,” *Journal of sound and vibration*, vol. 303, no. 1-2, pp. 154–170, 2007.
- [109] Y. Q. Wang, C. Ye, and J. W. Zu, “Nonlinear vibration of metal foam cylindrical shells reinforced with graphene platelets,” *Aerospace Science and Technology*, vol. 85, pp. 359–370, 2019.
- [110] A. E. H. Augustus, “A treatise on the mathematical theory of elasticity,” *Cambridge university*, vol. I., 1892.
- [111] A. Lakis and M. Paidoussis, “Prediction of the response of a cylindrical shell to arbitrary or boundary-layer-induced random pressure field,” *Journal of Sound and Vibration*, vol. 25, pp. 1–27, 1972.
- [112] F. Sabri and A. Lakis, “Efficient Hybrid Finite Element Method for Flutter Prediction of Functionally Graded Cylindrical Shells,” *Journal of Vibration and Acoustics*, vol. 136, p. 011002, 2013.
- [113] Y. Kerboua and A. Lakis, “Numerical model to analyze the aerodynamic behavior of a combined conical–cylindrical shell,” *Aerospace Science and Technology*, vol. 58, pp. 601–617, 2016.

- [114] F. Sabri and A. Lakis, “Finite Element Method Applied to Supersonic Flutter of Circular Cylindrical Shells,” *AIAA Journal*, vol. 48, pp. 72–81, 2010.
- [115] M. Amabili and M. P. Paidoussis, “Review of studies on geometrically nonlinear vibrations and dynamics of circular cylindrical shells and panels, with and without fluid-structure interaction,” *Appl. Mech. Rev.*, vol. 56, no. 4, pp. 349–381, 2003.
- [116] R. Ramzi and A. Lakis, “Effect of geometric nonlinearities on nonlinear vibrations of closed cylindrical shells,” *International Journal of Structural Stability and Dynamics*, vol. 13, no. 04, p. 1250078, 2013.
- [117] J. Antol, P. C. Calhoun, J. J. Flick, G. A. Hajos, R. M. Kolacinski, D. A. Minton, R. N. Owens, and J. J. Parker, *Low Cost Mars Surface Exploration: The Mars Tumbleweed*. NASA Technical Reports Server, 2003.
- [118] L. He, J. Lou, Y. Dong, S. Kitipornchai, and J. Yang, “Variational modeling of plane-strain hyperelastic thin beams with thickness-stretching effect,” *Acta Mechanica*, vol. 229, 12 2018.
- [119] E. Vignali, E. Gasparotti, K. Capellini, B. Fanni, L. Landini, V. Positano, and S. Celi, “Modeling biomechanical interaction between soft tissue and soft robotic instruments: importance of constitutive anisotropic hyperelastic formulations,” *The International Journal of Robotics Research*, vol. 40, pp. 224–235, 07 2020.
- [120] C. P. Okeke, A. N. Thite, and M. T. Greenrod, “Hyperelastic polymer material models for robust fatigue performance of automotive LED lamps,” *Structural Integrity Procedia*, vol. 5, pp. 600–607, 2017.
- [121] M. C. Boyce and E. M. Arruda, “Constitutive Models of Rubber Elasticity: A Review,” *Rubber Chemistry and Technology*, vol. 73, no. 3, pp. 504–523, Jul. 2000.
- [122] A. Elias Zuniga and M. F. Beatty, “Constitutive equations for amended non-Gaussian network models of rubber elasticity,” *International Journal of Engineering Science*, vol. 40, no. 20, pp. 2265–2294, 2002.
- [123] V. Vahapoglu and S. Karadeniz, “Constitutive equations for isotropic rubber-like materials using phenomenological approach: A bibliography (1930–2003),” *Rubber Chemistry and Technology*, vol. 79, 07 2006.
- [124] A. W. Leissa, *Vibration of shells*. Scientific and Technical Information Office, National Aeronautics and Space, 1973, vol. 288.

- [125] R. Wang, H. Ding, X. Yuan, L. Na, and L. Chen, “Different types of solitary waves in a thermo-hyperelastic neo-Hookean cylindrical shell,” *Composite Structures*, vol. 243, p. 112178, 2020.
- [126] G. Holzapfel and T. Gasser, “A New Constitutive Framework for Arterial Wall Mechanics and a Comparative Study of Material Models,” *Journal of Elasticity*, vol. 61, pp. 1–48, 2000.
- [127] A. Gent, “Elastic instabilities in rubber,” *International Journal of Non-Linear Mechanics*, vol. 40, pp. 165–175, 2005.
- [128] Y. Liu, Z. Qin, and F. Chu, “Nonlinear dynamic responses of sandwich functionally graded porous cylindrical shells embedded in elastic media under 1:1 internal resonance,” *Applied Mathematics and Mechanics*, vol. 42, pp. 805–818, 2021.