Titre: Title:	Similarity solutions of partial differential equations in probability
Auteurs: Authors:	Mario Lefebvre
Date:	2011
Type:	Article de revue / Article
Référence: Citation:	Lefebvre, M. (2011). Similarity solutions of partial differential equations in probability. Journal of Probability and Statistics, 2011, 1-13. https://doi.org/10.1155/2011/689427

Titre de la revue: Journal Title:	Journal of Probability and Statistics (vol. 2011)
Maison d'édition: Publisher:	Hindawi
URL officiel: Official URL:	https://doi.org/10.1155/2011/689427
Mention légale: Legal notice:	

Hindawi Publishing Corporation
Journal of Probability and Statistics
Volume 2011, Article ID 689427, 13 pages
doi:10.1155/2011/689427

Research Article

Similarity Solutions of Partial Differential Equations in Probability

Mario Lefebvre

Département de Mathématiques et de Génie Industriel, École Polytechnique de Montréal, C.P. 6079, Succursale Centre-Ville, Montréal, QC, Canada H3C 3A7

Correspondence should be addressed to Mario Lefebvre, mlefebvre@polymtl.ca

Received 2 May 2011; Accepted 6 June 2011

Academic Editor: Shein-chung Chow

Two-dimensional diffusion processes are considered between concentric circles and in angular sectors. The aim of the paper is to compute the probability that the process will hit a given part of the boundary of the stopping region first. The appropriate partial differential equations are solved explicitly by using the method of similarity solutions and the method of separation of variables. Some solutions are expressed as generalized Fourier series.

1. Introduction

Let $(X_1(t), X_2(t))$ be the two-dimensional diffusion process defined by the stochastic differential equations

$$dX_i(t) = f_i[X_i(t)]dt + \{v_i[X_i(t)]\}^{1/2} dW_i(t), \tag{1.1}$$

for $i = 1, 2$, where $v_i(\cdot)$ is nonnegative and $W_1(t)$ and $W_2(t)$ are independent standard Brownian motions. In this note, the problem of computing the probability that the process $(X_1(t), X_2(t))$, starting between two concentric circles, will hit the larger circle first is solved for the most important particular cases. The process is also considered inside a circle centered at the origin, and, this time, the probability that $(X_1(t), X_2(t))$ will hit the boundary of the circle before either of two radii is treated. Again, the most important particular cases are analyzed.

Suppose that we consider only the process $X_1(t)$ in the interval $[a, b]$. Let

$$\tau(x) := \inf\{t \geq 0 : X_1(t) = a \text{ or } b \mid X_1(0) = x \in [a, b]\}. \tag{1.2}$$

Then, it is well known (see Cox and Miller [1, p. 230], for instance) that the moment generating function (which is a Laplace transform)

$$L(x;s) := E\left[e^{-s\tau(x)}\right] \tag{1.3}$$

of the first passage time $\tau(x)$, where s is a nonnegative parameter, satisfies the Kolmogorov backward equation

$$\frac{v_1(x)}{2}L''(x;s) + f_1(x)L'(x;s) = sL(x;s), \tag{1.4}$$

and it is subject to the boundary conditions

$$L(a;s) = L(b;s) = 1. \tag{1.5}$$

Next, let

$$p(x) := P[X_1(\tau(x)) = a \mid X_1(0) = x]. \tag{1.6}$$

The function $p(x)$ satisfies the ordinary differential equation (see Cox and Miller [1, p. 231])

$$\frac{v_1(x)}{2}p''(x) + f_1(x)p'(x) = 0, \tag{1.7}$$

with

$$p(a) = 1, \qquad p(b) = 0. \tag{1.8}$$

It is therefore a simple matter to compute explicitly the probability $p(x)$ of hitting the point a before b, starting from $x \in [a,b]$. In particular, in the case when $X_1(t)$ is a standard Brownian motion, so that $f_1(x) \equiv 0$ and $v_1(x) \equiv 1$, we find at once that

$$p(x) = \frac{b-x}{b-a} \quad \text{for } a \le x \le b. \tag{1.9}$$

Many papers have been devoted to first passage time problems for diffusion processes, either in one or many dimensions; see, in particular, the classic papers by Doob [2] and Spitzer [3], and also Wendel [4]. However, a rather small number of papers have been written on first hitting *place* problems; see, for instance, the papers by Yin and Wu [5] and by Yin et al. [6]. Guilbault and Lefebvre (see [7, 8]) have considered problems related to the ones treated in the present note; however, in these problems, the processes were considered inside rectangles.

Now, define

$$T(x_1,x_2) = \inf\{t \ge 0 : (X_1(t), X_2(t)) \in D \mid X_i(0) = x_i\}, \tag{1.10}$$

where D is a subset of $\mathbb{R}^2$ for which the random variable $T(x_1, x_2)$ is well defined. The moment generating function of $T(x_1, x_2)$, namely,

$$M(x_1, x_2; s) := E\left[e^{-sT(x_1,x_2)}\right] \tag{1.11}$$

satisfies the Kolmogorov backward equation

$$\sum_{i=1}^{2}\left\{\frac{v_i(x_i)}{2}M_{x_i x_i} + f_i(x_i)M_{x_i}\right\} = sM, \tag{1.12}$$

where $M_{x_i} := \partial M/\partial x_i$ and $M_{x_i x_i} := \partial^2 M/\partial x_i^2$. This partial differential equation is valid in the continuation region $C := D^c$ and is subject to the boundary condition

$$M(x_1, x_2; s) = 1 \quad \text{if } (x_1, x_2) \in \partial D. \tag{1.13}$$

In Section 2, the set C will be given by

$$C_1 := \left\{(x_1, x_2) \in \mathbb{R}^2 : d_1^2 < x_1^2 + x_2^2 < d_2^2\right\}, \tag{1.14}$$

and the function

$$\pi(x_1, x_2) := P\left[X_1^2(T_1(x_1, x_2)) + X_2^2(T_1(x_1, x_2)) = d_2^2\right], \tag{1.15}$$

where T_1 is the random variable defined in (1.10) with $D = D_1 = C_1^c$, will be computed in important special cases, such as when $(X_1(t), X_2(t))$ is a two-dimensional Wiener process.

In Section 3, we will choose

$$C_2 := \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 < \left(x_1^2 + x_2^2\right)^{1/2} < d, 0 < \arctan\left(\frac{x_2}{x_1}\right) < \theta_0\right\}. \tag{1.16}$$

We will calculate for important two-dimensional diffusion processes the probability

$$\nu(x_1, x_2) := P\left[X_1^2(T_2(x_1, x_2)) + X_2^2(T_2(x_1, x_2)) = d^2\right], \tag{1.17}$$

where T_2 is the time taken by $(X_1(t), X_2(t))$ to leave the set C_2, starting from $X_i(0) = x_i$ for $i = 1, 2$.

Finally, a few remarks will be made in Section 4 to conclude.

2. First Hitting Place Probabilities when Starting between Two Circles

From the Kolmogorov backward equation (1.12), we deduce that the function $\pi(x_1, x_2)$ defined in (1.15) satisfies the partial differential equation

$$\sum_{i=1}^{2}\left\{\frac{v_i(x_i)}{2}\pi_{x_i x_i} + f_i(x_i)\pi_{x_i}\right\} = 0 \tag{2.1}$$

in the set C_1 defined in (1.14), and is subject to the boundary conditions

$$\pi(x_1, x_2) = \begin{cases} 1 & \text{if } x_1^2 + x_2^2 = d_2^2, \\ 0 & \text{if } x_1^2 + x_2^2 = d_1^2. \end{cases} \tag{2.2}$$

Because the two-dimensional process $(X_1(t), X_2(t))$ is considered between two concentric circles, it seems natural to try to find a solution of the form

$$\pi(x_1, x_2) = q(y), \tag{2.3}$$

where $y := x_1^2 + x_2^2$. Actually, this only works in a few, but very important, special cases, some of which will be presented below. The partial differential equation (2.1) becomes

$$\sum_{i=1}^{2}\left\{2v_i(x_i)x_i^2 q''(y) + \left[v_i(x_i) + 2x_i f_i(x_i)\right]q'(y)\right\} = 0. \tag{2.4}$$

Remark 2.1. Because the region C_1 is bounded, the solution to the problem (2.1), (2.2) is unique. Therefore, if we can find a solution of the form $\pi(x_1, x_2) = q(y)$, then we can state that it is indeed the solution we were looking for.

2.1. The Two-Dimensional Wiener Process

First, we take $f_i(x_i) \equiv 0$ and $v_i(x_i) \equiv v_0 > 0$. Then $(X_1(t), X_2(t))$ is a two-dimensional Wiener process with zero infinitesimal means and infinitesimal variances both equal to v_0. Equation(2.4) can be rewritten as

$$y q''(y) + q'(y) = 0. \tag{2.5}$$

Notice that this is a first-order linear ordinary differential equation for $h(y) := q'(y)$. It is a simple matter to find that

$$q(y) = c_1 \ln(y) + c_0, \tag{2.6}$$

where c_1 and c_0 are constants. Therefore,

$$\pi(x_1, x_2) = c_1 \ln\left(x_1^2 + x_2^2\right) + c_0. \tag{2.7}$$

The boundary condition (2.2) yields that

$$\pi(x_1, x_2) = \frac{\ln\left(\left(x_1^2 + x_2^2\right)/d_1^2\right)}{\ln\left(d_2^2/d_1^2\right)} \quad \text{for } d_1^2 \leq x_1^2 + x_2^2 \leq d_2^2. \tag{2.8}$$

Remark 2.2. If we choose $f_i(x_i) \equiv f_0 \neq 0$ or if $v_i(x_i) \equiv v_{0i} > 0$ for $i = 1, 2$, with $v_{01} \neq v_{02}$, then the particular case of the method of similarity solutions that we have used above fails. Notice also that the solution does not depend on the parameter v_0.

2.2. The Two-Dimensional Ornstein-Uhlenbeck Process

Next, we choose $f_i(x_i) = -\alpha x_i$ and $v_i(x_i) \equiv v_0$ for $i = 1, 2$, where α is a positive parameter, so that $(X_1(t), X_2(t))$ is a two-dimensional Ornstein-Uhlenbeck process with the same infinitesimal parameters. This time, (2.4) becomes

$$v_0 y q''(y) + (v_0 - \alpha y) q'(y) = 0, \tag{2.9}$$

the general solution of which can be expressed as

$$q(y) = c_1 \mathrm{Ei}\left(\frac{\alpha y}{v_0}\right) + c_0, \tag{2.10}$$

where $\mathrm{Ei}(\cdot)$ is the exponential integral function defined by

$$\mathrm{Ei}(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt \quad \text{for } z > 0, \tag{2.11}$$

in which the principal value of the integral is taken. It follows that

$$\pi(x_1, x_2) = \frac{\mathrm{Ei}\left(\alpha\left(x_1^2 + x_2^2\right)/v_0\right) - \mathrm{Ei}\left(\alpha d_1^2/v_0\right)}{\mathrm{Ei}\left(\alpha d_2^2/v_0\right) - \mathrm{Ei}\left(\alpha d_1^2/v_0\right)} \quad \text{for } d_1^2 \leq x_1^2 + x_2^2 \leq d_2^2. \tag{2.12}$$

2.3. The Two-Dimensional Bessel Process

The last particular case that we consider is the one when $f_i(x_i) = (\alpha - 1)/2x_i$ and $v_i(x_i) \equiv 1$ for $i = 1, 2$. Again, α is a positive parameter, so that $(X_1(t), X_2(t))$ is a two-dimensional Bessel process. We assume that $0 < \alpha < 2$ (and $\alpha \neq 1$); then, the origin is a regular boundary for $X_1(t)$ and $X_2(t)$ (see Karlin and Taylor [9, p. 238-239]).

Equation (2.4) takes the form

$$y q''(y) + \alpha q'(y) = 0. \tag{2.13}$$

We find that, for $\alpha \neq 1$, the function $q(y)$ is

$$q(y) = c_1 y^{1-\alpha} + c_0. \tag{2.14}$$

Finally, the solution that satisfies the boundary condition (2.2) is

$$\pi(x_1, x_2) = \frac{\left(x_1^2 + x_2^2\right)^{1-\alpha} - d_1^{2(1-\alpha)}}{d_2^{2(1-\alpha)} - d_1^{2(1-\alpha)}} \quad \text{for } d_1^2 \leq x_1^2 + x_2^2 \leq d_2^2. \tag{2.15}$$

Remarks 2.3. (1) When $\alpha = 1$, the ordinary differential equation (2.14) reduces to the one obtained in Section 2.1 with the two-dimensional Wiener process, if $v_0 = 1$.

(2) If the parameter α is greater than or equal to 2, the origin is an inaccessible boundary for $X_1(t)$ and $X_2(t)$; that is, it cannot be reached in finite time. Therefore, in this case the continuation region could be the region between the two concentric circles, but inside the first quadrant (for instance).

In the next section, the problem of computing explicitly the function $\nu(x_1, x_2)$ defined in (1.17) for important two-dimensional diffusion processes in angular sectors will be treated. This time, we will work in polar coordinates and make use of the method of separation of variables, which can be viewed as a special case of the method of similarity solutions. The solutions will be expressed as generalized Fourier series and will therefore be more involved than the simple solutions obtained in this section.

3. First Hitting Place Probabilities when Starting in Angular Sectors

We consider the two-dimensional processes defined by the stochastic differential equations (1.1) inside the circle of radius d centered at the origin. In polar coordinates, the function $M(x_1, x_2; s) = N(r, \theta; s)$ satisfies the Kolmogorov backward equation (see (1.12))

$$\begin{aligned} sN = {} & \frac{1}{2} v_1(x_1) \left\{ \frac{x_1^2}{r^2} N_{rr} - 2\frac{x_1 x_2}{r^3} N_{r\theta} + \frac{x_2^2}{r^4} N_{\theta\theta} + \frac{x_2^2}{r^3} N_r + 2\frac{x_1 x_2}{r^4} N_\theta \right\} \\ & + \frac{1}{2} v_2(x_2) \left\{ \frac{x_2^2}{r^2} N_{rr} + 2\frac{x_1 x_2}{r^3} N_{r\theta} + \frac{x_1^2}{r^4} N_{\theta\theta} + \frac{x_1^2}{r^3} N_r - 2\frac{x_1 x_2}{r^4} N_\theta \right\} \\ & + f_1(x_1) \left\{ \frac{x_1}{r} N_r - \frac{x_2}{r^2} N_\theta \right\} + f_2(x_2) \left\{ \frac{x_2}{r} N_r + \frac{x_1}{r^2} N_\theta \right\}, \end{aligned} \tag{3.1}$$

where $r := \left(x_1^2 + x_2^2\right)^{1/2}$ and $\theta := \arctan(x_2/x_1)$. Let

$$T(x_1, x_2) := \inf\{t \geq 0 : r = d \text{ or } \theta = 0 \text{ or } \theta_0 \ (>0) \mid X_i(0) = x_i, i = 1, 2\}, \tag{3.2}$$

that is,

$$T(x_1, x_2) = \inf\{t \geq 0 : (x_1, x_2) \notin C_2 \mid X_i(0) = x_i, i = 1, 2\}, \tag{3.3}$$

with C_2 defined in (1.16). The probability $\nu(x_1, x_2)$ defined in (1.17) satisfies the same partial differential equation as $M(x_1, x_2; s)$ in polar coordinates, with $s = 0$. Furthermore, $\nu(x_1, x_2) = \rho(r, \theta)$ is such that

$$\begin{gathered}\rho(d, \theta) = 1 \quad \forall \theta \in [0, \theta_0], \\ \rho(r, 0) = \rho(r, \theta_0) = 0 \quad \text{if } r < d.\end{gathered} \tag{3.4}$$

As in the previous section, we will obtain explicit (and exact) solutions to the first hitting place problem set up above for the most important particular cases.

3.1. The Two-Dimensional Wiener Process

When $(X_1(t), X_2(t))$ is a two-dimensional Wiener process, with independent components and infinitesimal parameters 0 and v_0, the partial differential equation that we must solve reduces to

$$\rho_{rr} + \frac{1}{r}\rho_r + \frac{1}{r^2}\rho_{\theta\theta} = 0. \tag{3.5}$$

Looking for a solution of the form $\rho(r, \theta) = F(r)G(\theta)$, we find that

$$F''(r)G(\theta) + \frac{1}{r}F'(r)G(\theta) + \frac{1}{r^2}F(r)G''(\theta) = 0, \tag{3.6}$$

so that we obtain the ordinary differential equations

$$G''(\theta) = \lambda G(\theta), \tag{3.7}$$

$$r^2 F''(r) + rF'(r) + \lambda F(r) = 0, \tag{3.8}$$

where λ is the separation constant. The ordinary differential equation (3.7) is subject to the boundary conditions

$$G(0) = G(\theta_0) = 0, \tag{3.9}$$

whereas

$$F(0) = 0. \tag{3.10}$$

It is well known that the function $G(\theta)$ must be of the form

$$G_n(\theta) = c_n \sin\left(\frac{n\pi\theta}{\theta_0}\right) \quad \text{for } n = 1, 2, \ldots, \tag{3.11}$$

where c_n is a constant; therefore, the separation constant must be given by

$$\lambda = \lambda_n = -\frac{(n\pi)^2}{\theta_0^2} \quad \text{for } n = 1, 2, \ldots \tag{3.12}$$

Next, the solution of (3.8) (which is an Euler-Cauchy equation), with $\lambda = -(\pi n)^2/\theta_0^2$, that is such that $F(0) = 0$ is

$$F_n(r) = \text{const. } r^{n\pi/\theta_0}. \tag{3.13}$$

We then consider the infinite series

$$\rho(r,\theta) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi\theta}{\theta_0}\right) r^{n\pi/\theta_0}, \tag{3.14}$$

where a_n is a constant. This series, as a function of θ, is a Fourier series. The condition $\rho(d,\theta) = 1$ implies that

$$a_n = \frac{2}{\theta_0}\int_0^{\theta_0} d^{-n\pi/\theta_0} \sin\left(\frac{n\pi\theta}{\theta_0}\right) d\theta = 2d^{-n\pi/\theta_0}\frac{(-1)^{n+1}+1}{n\pi}. \tag{3.15}$$

Hence, the solution is

$$\rho(r,\theta) = 2\sum_{n=1}^{\infty}\left(\frac{r}{d}\right)^{n\pi/\theta_0}\frac{1+(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi\theta}{\theta_0}\right), \tag{3.16}$$

for $0 \le \theta \le \theta_0$ and $0 \le r \le d$.

Remark 3.1. If the infinitesimal mean of $X_i(t)$ is not equal to zero, we cannot separate the variables in the partial differential equation satisfied by the function $\rho(r,\theta)$. So, as in Section 2, the cases for which the technique we have used will work are actually rather few. Fortunately, it *does* work in the most important cases for applications.

3.2. The Two-Dimensional Ornstein-Uhlenbeck Process

When $X_1(t)$ and $X_2(t)$ are independent Ornstein-Uhlenbeck processes with infinitesimal parameters $-\alpha X_i(t)$ and v_0 for $i = 1, 2$, we must solve the partial differential equation

$$\frac{1}{2}v_0\left\{\rho_{rr} + \frac{1}{r}\rho_r + \frac{1}{r^2}\rho_{\theta\theta}\right\} - \alpha r\rho_r = 0. \tag{3.17}$$

Writing $\rho(r,\theta) = F(r)G(\theta)$, we obtain the ordinary differential equations

$$G''(\theta) = \lambda G(\theta), \tag{3.18}$$

$$r^2F''(r) + rF'(r) - 2\frac{\alpha}{v_0}r^3F'(r) + \lambda F(r) = 0. \tag{3.19}$$

The boundary conditions are the same as in Section 3.1. Therefore, we find that we still have $\lambda = \lambda_n = -(\pi n)^2/\theta_0^2$ and

$$G_n(\theta) = c_n \sin\left(\frac{n\pi\theta}{\theta_0}\right) \quad \text{for n} = 1,2,\ldots \tag{3.20}$$

Next, the general solution of (3.19) can be written as

$$F(r) = c_1 r^{-2\sqrt{-\lambda}} M\left(-\frac{1}{2}\sqrt{-\lambda}, 1-\sqrt{-\lambda}, -\frac{1}{2}kr^2\right) + c_2 r^{2\sqrt{-\lambda}} M\left(\frac{1}{2}\sqrt{-\lambda}, 1+\sqrt{-\lambda}, -\frac{1}{2}kr^2\right), \tag{3.21}$$

where $k := -2\alpha/v_0$ and $M(\cdot,\cdot,\cdot)$ is a confluent hypergeometric function (see Abramowitz and Stegun [10, p. 504]). We find at once that we must choose c_1 equal to zero. We then consider the infinite series

$$\rho(r,\theta) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi\theta}{\theta_0}\right) r^{2n\pi/\theta_0} M\left(\frac{n\pi}{2\theta_0}, 1+\frac{n\pi}{\theta_0}, -\frac{1}{2}kr^2\right). \tag{3.22}$$

Making use of the boundary condition $\rho(d,\theta) = 1$, we find that

$$\rho(r,\theta) = \sum_{n=1}^{\infty} \frac{1+(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi\theta}{\theta_0}\right)\left(\frac{r}{d}\right)^{2n\pi/\theta_0} \frac{M(n\pi/2\theta_0, 1+n\pi/\theta_0, -(1/2)kr^2)}{M(n\pi/2\theta_0, 1+n\pi/\theta_0, -(1/2)kd^2)}, \tag{3.23}$$

for $0 \le \theta \le \theta_0$ and $0 \le r \le d$.

3.3. The Two-Dimensional Bessel Process

Finally, with $f_i(x_i) = (\alpha-1)/2x_i$ $(0 < \alpha < 2, \alpha \neq 1)$ and $v_i(x_i) \equiv 1$ for $i = 1,2$, we obtain the partial differential equation

$$\frac{1}{2}\left\{\rho_{rr} + \frac{1}{r}\rho_r + \frac{1}{r^2}\rho_{\theta\theta}\right\} + \frac{\alpha-1}{2}\left\{\frac{2}{r}\rho_r + \frac{1}{r^2}\left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\right)\rho_\theta\right\} = 0. \tag{3.24}$$

It follows that we must solve the ordinary differential equation

$$G''(\theta) + (\alpha - 1)\left(\frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\right)G'(\theta) + \lambda G(\theta) = 0. \tag{3.25}$$

We assume that θ_0 is in the interval $(0, \pi/2)$. Writing that $G(\theta) = H(z)$, where $z := \sin\theta$, we find that this ordinary differential equation is transformed to

$$\left(1 - z^2\right)H''(z) - zH'(z) + (\alpha - 1)\frac{1 - 2z^2}{z}H'(z) + \lambda H(z) = 0. \tag{3.26}$$

The general solution of (3.26) can be written in the form

$$\begin{aligned} H(z) = z^{2-\alpha} c_1 F&\left(\frac{1}{2} - \frac{\left(\gamma^2 + \lambda\right)^{1/2}}{2}, \frac{1}{2} + \frac{\left(\gamma^2 + \lambda\right)^{1/2}}{2}; \frac{3}{2} - \frac{\gamma}{2}; z^2\right) \\ + c_2 F&\left(\frac{\gamma}{2} - \frac{\left(\gamma^2 + \lambda\right)^{1/2}}{2}, \frac{\gamma}{2} + \frac{\left(\gamma^2 + \lambda\right)^{1/2}}{2}; \frac{1}{2} + \frac{\gamma}{2}; z^2\right), \end{aligned} \tag{3.27}$$

where $\gamma := \alpha - 1$ and $F(a, b; c; z)$ is a hypergeometric function (see Abramowitz and Stegun [10, p. 556]). Hence, we have

$$\begin{aligned} G(\theta) = (\sin\theta)^{2-\alpha} c_1 F&\left(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 2 - \frac{\alpha}{2}; \sin^2\theta\right) \\ + c_2 F&\left(\frac{\alpha - 1}{2} - \delta, \frac{\alpha - 1}{2} + \delta; \frac{\alpha}{2}; \sin^2\theta\right), \end{aligned} \tag{3.28}$$

where $\delta := (1/2)(\gamma^2 + \lambda)^{1/2}$.

The condition $G(0) = 0$ implies that we must set c_2 equal to zero. Next, we must find the value(s) of the separation constant λ for which $G(\theta_0) = 0$; that is,

$$(\sin\theta_0)^{2-\alpha} F\left(\frac{1}{2} - \delta, \frac{1}{2} + \delta; 2 - \frac{\alpha}{2}; \sin^2\theta_0\right) = 0. \tag{3.29}$$

Now, notice that (3.25) can be written in the form

$$\frac{d}{d\theta}\left[P(\theta)\frac{d}{d\theta}G(\theta)\right] - S(\theta)G(\theta) + \lambda R(\theta)G(\theta) = 0, \tag{3.30}$$

with $P(\theta) = (\sin\theta\cos\theta)^{\alpha-1}$, $S(\theta) \equiv 0$ and $R(\theta) \equiv P(\theta)$. If we assume that $0 < \theta_{00} < \theta < \theta_0 < \pi/2$ in C_2, then the problem of solving (3.25) together with the boundary conditions $G(\theta_{00}) = G(\theta_0) = 0$ is a *regular* Sturm-Liouville problem. It follows that we can state (see Edwards Jr.

and Penney [11, p. 519], for instance) that there exist an infinite number of eigenvalues λ_n for which the conditions $G(\theta_{00}) = 0$ and $G(\theta_0) = 0$ are satisfied. These eigenvalues constitute an increasing sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ of real numbers with $\lim_{n\to\infty}\lambda_n = \infty$. Moreover, we can also state (see Butkov [12, p. 337–340]) that the eigenfunctions $G_n(\theta)$ corresponding to the eigenvalues λ_n are orthogonal to each other with respect to the weight function $R(\theta)$. However, computing these eigenvalues explicitly is another problem.

Here, we consider the case where $\theta_{00} = 0$, so that we do not have a regular Sturm-Liouville problem. However, one can check graphically, using a computer software, that there exist an infinite number of positive constants λ_n for which

$$F\left(\frac{1}{2} - \delta_n, \frac{1}{2} + \delta_n; 2 - \frac{\alpha}{2}; \sin^2\theta_0\right) = 0, \tag{3.31}$$

where $\delta_n := (1/2)[(\alpha-1)^2 + \lambda_n]^{1/2}$ for $n = 1, 2, \ldots$ We thus have, apart from an arbitrary constant,

$$G_n(\theta) := (\sin\theta)^{2-\alpha} F\left(\frac{1}{2} - \delta_n, \frac{1}{2} + \delta_n; 2 - \frac{\alpha}{2}; \sin^2\theta\right) \tag{3.32}$$

for $0 \le \theta \le \theta_0$.

Finally, we must solve the ordinary differential equation

$$r^2 F''(r) + (2\alpha - 1) r F'(r) = \lambda_n F(r), \tag{3.33}$$

subject to $F(0) = 0$. This is an Euler-Cauchy differential equation; for all positive eigenvalues λ_n, we can write that

$$F_n(r) = \text{const. } r^{k_n}, \tag{3.34}$$

where $k_n := (1-\alpha) + [(\alpha-1)^2 + \lambda_n]^{1/2}$. Since there are an infinite number of such eigenvalues, we can consider the infinite series (a generalized Fourier series)

$$\rho(r,\theta) = \sum_{n=1}^{\infty} a_n r^{k_n} G_n(\theta). \tag{3.35}$$

Making use of the boundary condition $\rho(d,\theta) = 1$, we can write that the constant a_n is given by (see Butkov [12, p. 339])

$$a_n = \frac{d^{-k_n} \int_0^{\theta_0} R(\theta) G_n(\theta) d\theta}{\int_0^{\theta_0} R(\theta) G_n^2(\theta)\, d\theta}. \tag{3.36}$$

Remark 3.2. There is at least another particular case of interest for which we can obtain an explicit expression (when $0 < \theta_{00} < \theta < \theta_0 < \pi/2$). Indeed, if we choose $f_1(x_1) = -1/2x_1$

(i.e., $\alpha = 0$) and $f_2(x_2) = 1/2x_2$ (which corresponds to $\alpha = 2$), with $v_i(x_i) \equiv 2$, we find that the partial differential equation that we must solve is

$$2\left\{\rho_{rr} + \frac{1}{r}\rho_r + \frac{1}{r^2}\rho_{\theta\theta}\right\} + \frac{1}{r^2}\left(\frac{1}{\cos\theta\sin\theta}\right)\rho_\theta = 0. \tag{3.37}$$

This equation is separable; the two ordinary differential equations that result from the separation of variables are

$$G''(\theta) + \left(\frac{1}{2\cos\theta\sin\theta}\right)G'(\theta) + \lambda G(\theta) = 0, \tag{3.38}$$

$$r^2F''(r) + rF(r) = \lambda F(r). \tag{3.39}$$

Writing $G(\theta) = H(z)$ with $z := \sin\theta$, (3.38) becomes

$$\left(1 - z^2\right)H''(z) - zH'(z) + \frac{1}{2z}H'(z) + \lambda H(z) = 0, \tag{3.40}$$

which we can solve to find

$$H(z) = z^{1/2}c_1F\left(\frac{1}{4} - \frac{\sqrt{\lambda}}{2}, \frac{1}{4} + \frac{\sqrt{\lambda}}{2}; \frac{5}{4}; z^2\right) + c_2F\left(-\frac{\sqrt{\lambda}}{2}, \frac{\sqrt{\lambda}}{2}; \frac{3}{4}; z^2\right). \tag{3.41}$$

Moreover, (3.39) is again an Euler-Cauchy equation; the solution that satisfies the boundary condition $F(0) = 0$ is (for positive eigenvalues λ_n)

$$F(r) = \text{const. } r^{\sqrt{\lambda_n}}. \tag{3.42}$$

Hence, proceeding as above, we can obtain the function $\rho(r, \theta)$, expressed as a generalized Fourier series, in this case too.

4. Concluding Remarks

We have considered, in this note, the problem of computing first hitting place probabilities for important two-dimensional diffusion processes starting between two concentric circles or in an angular sector. We have obtained explicit (and exact) solutions to a number of problems in Sections 2 and 3. Furthermore, we have arbitrarily chosen in Section 2 to compute the probability $\pi(x_1, x_2)$ of hitting the larger circle first. It would be a simple matter to obtain the probability of hitting the smaller circle first instead. Actually, because the continuation region is bounded, the probability of hitting the smaller circle first should simply be $1 - \pi(x_1, x_2)$, at least in the cases treated here. Similarly, in Section 3 we could have computed the probability that the process $(X_1(t), X_2(t))$ will exit the continuation region through the radius $\theta = 0$, or through $\theta = \theta_0$.

Now, there are other important two-dimensional diffusion processes for which the techniques used in this note do not work. In particular, there is the two-dimensional

Wiener process with nonzero infinitesimal means and also the geometric Brownian motion. Moreover, we have always assumed, except in the last remark above, that the two diffusion processes, $X_1(t)$ and $X_2(t)$, had the same infinitesimal parameters; it would be interesting to try to find the solutions to the first hitting place problems in the general cases.

Next, we could also try to find explicit solutions to first hitting place problems, but in three or more dimensions. It should at least be possible to solve some special problems.

Finally, we have computed the probability that the process $(X_1(t), X_2(t))$ will hit a given part of the boundary of the stopping region first. Another problem would be to try to obtain the distribution of $(X_1(T), X_2(T))$.

References

[1] D. R. Cox and H. D. Miller, *The Theory of Stochastic Processes*, Methuen, London, UK, 1965.

[2] J. L. Doob, "A probability approach to the heat equation," *Transactions of the American Mathematical Society*, vol. 80, pp. 216–280, 1955.

[3] F. Spitzer, "Some theorems concerning 2-dimensional Brownian motion," *Transactions of the American Mathematical Society*, vol. 87, pp. 187–197, 1958.

[4] J. G. Wendel, "Hitting spheres with Brownian motion," *The Annals of Probability*, vol. 8, no. 1, pp. 164–169, 1980.

[5] C. Yin and R. Wu, "Hitting time and place to a sphere or spherical shell for Brownian motion," *Chinese Annals of Mathematics*, vol. 20, no. 2, pp. 205–214, 1999.

[6] C. Yin, X. Shao, and H. Cheng, "The joint density of the hitting time and place to a circle for planar Brownian motion," *Journal of Qufu Normal University*, vol. 25, no. 1, pp. 7–9, 1999.

[7] J.-L. Guilbault and M. Lefebvre, "Au sujet de l'endroit de premier passage pour des processus de diffusion bidimensionnels," *Annales des Sciences Mathématiques du Québec*, vol. 25, no. 1, pp. 23–37, 2001.

[8] M. Lefebvre and J.-L. Guilbault, "Hitting place probabilities for two-dimensional diffusion processes," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 49, no. 1, pp. 11–25, 2004.

[9] S. Karlin and H. Taylor, *A Second Course in Stochastic Processes*, Academic Press, New York, NY, USA, 1981.

[10] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, NY, USA, 1965.

[11] C. H. Edwards Jr. and D. E. Penney, *Elementary Differential Equations with Applications*, Prentice Hall, Englewood Cliffs, NJ, USA, 1985.

[12] E. Butkov, *Mathematical Physics*, Addison-Wesley, Reading, Mass, USA, 1968.