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Research Article LQG Homing in a Finite Time Interval

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Let X(t) be a controlled one-dimensional diffusion process having constant infinitesimal variance. We consider the problem of optimally controlling X(t) until time $T(x) = \min\{T_1(x), t_1\}$, where $T_1(x)$ is the first-passage time of the process to a given boundary and t_1 is a fixed constant. The optimal control is obtained explicitly in the particular case when X(t) is a controlled Wiener process.

1. Introduction

Let $\{X(t), t \ge 0\}$ be the one-dimensional controlled diffusion process defined by the stochastic differential equation

$$dX(t) = a[X(t)]dt + b_0u(t)dt + \sigma_0dB(t),$$
(1)

where $a(\cdot)$ is a real function, u(t) is the control variable, $b_0 \neq 0$, and $\sigma_0 > 0$ are constants and $\{B(t), t \ge 0\}$ is a standard Brownian motion. We define the first-passage time

$$T_1(x) = \inf\{t > 0 : X(t) = d \mid X(0) = x\},$$
(2)

where x < d, and the random variable

$$T(x) = \min\{T_1(x), t_1\},$$
(3)

where $t_1 > 0$ is a constant.

Next, we consider the cost criterion

$$J(x) = \int_0^{T(x)} \frac{1}{2} q_0 u^2(t) dt + k \ln[T(x) + 1], \qquad (4)$$

where $q_0 > 0$ and $k \neq 0$ are constants. We want to find the control u^* that minimizes the expected value of J(x). This type of problem is a special case of the ones that Whittle [1, page 289] termed *LQG homing*. Notice that if the constant *k* is negative, then the optimizer tries to maximize the survival time of the process in the interval (0, d), taking the quadratic control costs into account. LQG homing problems have been

treated by various authors; see Kuhn [2], Lefebvre [3], and Makasu [4]. Kuhn and Makasu used a risk-sensitive cost criterion (see also Whittle [5, page 222]).

In the general formulation given by Whittle, $\{X(t), t \ge 0\}$ is an *n*-dimensional process and the random variable T(x) is the moment of first entry of the joint variable (X(t), t) into a stopping set $D \subset \mathbb{R}^n \times (0, \infty)$. However, in practice, it is very difficult to obtain explicit solutions to problems in two or more dimensions (except in special instances). Moreover, in the papers published so far on homing problems, the hitting time T(x) was defined only in terms of X(t). Here, we consider the case when the optimizer stops controlling the diffusion process at most at time t_1 .

Using a theorem in Whittle [1], we can state that the optimal control u^* can be expressed as follows:

$$u^* = \frac{\sigma_0^2}{b_0} \frac{G'(x)}{G(x)},\tag{5}$$

where

$$G(x) = E\left[\exp\left\{-\frac{b_0^2}{q_0\sigma_0^2}k\ln[\tau(x)+1]\right\}\right].$$
 (6)

In the above formula, $\tau(x)$ is a random variable defined by

$$\tau(x) = \min\{\tau_1(x), t_1\},$$
(7)

with

$$\tau_1(x) = \inf\{t > 0 : \xi(t) = d \mid \xi(0) = x\},\tag{8}$$

and $\{\xi(t), t \ge 0\}$ is the uncontrolled process that satisfies the stochastic differential equation

$$d\xi(t) = a[\xi(t)]dt + \sigma_0 dB(t).$$
(9)

That is, $\tau(x)$ is the random variable that corresponds to T(x) for the diffusion process obtained by setting u(t) = 0 in (1).

Hence, the optimal control problem is reduced to the computation of the mathematical expectation G(x). Actually, for this result to hold, we must have $P[\tau(x) < \infty] = 1$. However, in our case this condition is trivially satisfied because $\tau(x) \le t_1$.

In Section 2, we will obtain an explicit solution u^* in the case when $a[X(t)] \equiv \mu > 0$, so that

$$dX(t) = \mu dt + b_0 u(t) dt + \sigma_0 dB(t).$$
⁽¹⁰⁾

Notice that the uncontrolled process $\{\xi(t), t \ge 0\}$ is then a Wiener process with drift μ and diffusion parameter σ_0^2 . Furthermore, we will choose the constant $k = -q_0 \sigma_0^2/b_0^2$. With this choice, the mathematical expectation G(x) simplifies to

$$G(x) = E[\exp\{\ln[\tau(x) + 1]\}] = 1 + E[\tau(x)].$$
(11)

2. Optimal Control of a Wiener Process

Let $m_1(x)$ denote the expected value of the first-passage time $\tau_1(x)$. In the case of the Wiener process defined by

$$d\xi(t) = \mu dt + \sigma_0 dB(t) \tag{12}$$

the function $m_1(x)$ satisfies the ordinary differential equation

$$\frac{1}{2}\sigma_0^2 m_1''(x) + \mu m_1'(x) = -1, \qquad (13)$$

and is such that $m_1(x) = 0$ if x = d. We find (see Lefebvre [6, page 220]) that

$$m_1(x) = \frac{d-x}{\mu}.$$
 (14)

Therefore, in the case when t_1 tends to infinity, the function G(x) is given by

$$G(x) = 1 + \frac{d-x}{\mu}.$$
 (15)

It follows from (5) that

$$u_{t_1=\infty}^* = -\frac{\sigma_0^2}{b_0} \frac{1}{\mu + d - x}.$$
 (16)

Now, to obtain the expected value of the random variable $\tau(x)$, we can condition on $\tau_1(x)$:

$$E[\tau(x)] = E[\tau(x) \mid \tau_1(x) \le t_1] P[\tau_1(x) \le t_1] + E[\tau(x) \mid \tau_1(x) > t_1] P[\tau_1(x) > t_1].$$
(17)

We may write that

$$E[\tau(x) \mid \tau_1(x) > t_1] = t_1.$$
(18)



FIGURE 1: Optimal controls $u_{t_1=\infty}^*$ (dotted line) and u^* (solid line) when $x \in (-10, 5)$, $b_0 = q_0 = \sigma_0 = \mu = 1$, and $d = t_1 = 5$.

Moreover, because the conditional probability density function of $\tau_1(x)$, given that $\tau_1(x) \le t_1$, is given by

$$f_{\tau_1(x)}(t \mid \tau_1(x) \le t_1) = \frac{f_{\tau_1(x)}(t)}{P[\tau_1(x) \le t_1]} \quad \text{for} \quad 0 < t \le t_1,$$
(19)

we have:

$$E[\tau(x)] = \int_0^{t_1} t f_{\tau_1(x)}(t) dt + t_1 P[\tau_1(x) > t_1].$$
(20)

The function $f_{\tau_1(x)}(t)$ is known to be (see Lefebvre [6, page 219])

$$f_{\tau_1(x)}(t) = \frac{d-x}{\sqrt{2\pi\sigma_0^2 t^3}} \exp\left\{-\frac{(d-x-\mu t)^2}{2\sigma_0^2 t}\right\} \quad \text{for} \quad t > 0.$$
(21)

Making use of this formula, we can obtain an explicit expression for the mathematical expectation $E[\tau(x)]$ and hence, for the optimal control u^* .

To illustrate the results, we computed (numerically) the optimal control when $b_0 = q_0 = \sigma_0 = \mu = 1$ and $d = t_1 = 5$. Looking at Figure 1, we see that the optimal control u^* tends to zero much faster than $u_{t_1=\infty}^* = 1/(x-6)$ as x tends to $-\infty$. However, for x close to the boundary at d = 5, the two functions are similar.

Next, to see the effect of the constant t_1 on the optimal control, we computed the value of u^* when x = 0 and t_1 varies from 0 to 20. This value is compared to $u^*_{t_1=\infty} = -1/6$ in Figure 2. When t_1 decreases to 0, so does u^* , as it should be. For $t_1 \ge 15$ (approximately), we have $u^* \simeq u^*_{t_1=\infty}$.

Finally, in Figure 3, we show the optimal control u^* when d = 15 and $t_1 = 5$. Because $E[\tau_1(x)] = 15 - x$ (see above), when $x \le 5$ (approximately), it is very unlikely that the uncontrolled process will hit the boundary at d = 15 before time $t_1 = 5$. Therefore, the optimal control is close to 0. Notice that $\lim_{d\to\infty} u^* = 0$ (for a finite value of x). Indeed, we can write that $\lim_{d\to\infty} P[\tau_1(x) > t_1] = 1$. Hence, we deduce from (17) that $E[\tau(x)] = t_1$, which implies that $G(x) = 1+t_1$, and thus $u^*_{d=\infty} = 0$.



FIGURE 2: Optimal controls $u_{t_1=\infty}^* = -1/6$ (dotted line) and u^* (solid line) when x = 0, $b_0 = q_0 = \sigma_0 = \mu = 1$, d = 5, and $t_1 \in (0, 20)$.



FIGURE 3: Optimal control u^* when $x \in (5, 15)$, $b_0 = q_0 = \sigma_0 = \mu = 1$, d = 15, and $t_1 = 5$.

3. Conclusion

We have considered LQG homing problems for which the optimizer controls the diffusion process in the time interval (0, T(x)], where the random variable T(x) is smaller than or equal to a fixed constant t_1 . Moreover, the termination cost function was chosen so that the optimal control was expressed in terms of the expected value of a first-passage time for the corresponding uncontrolled process.

An application of this type of problem is the following: suppose that X(t) denotes the wear of a machine. The optimizer wants to maximize the lifetime of the machine. However, it is natural to assume that the machine will be replaced after a certain time, even if it is still in working order, because it might become obsolete.

To obtain a more realistic model for the wear of a device, we could use a degenerate two-dimensional diffusion process, as in Lefebvre [7]. The most difficult problem would then be to compute the probability density function of the first-passage time τ_1 .

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