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Research Article

LQG Homing in a Finite Time Interval

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Let $X(t)$ be a controlled one-dimensional diffusion process having constant infinitesimal variance. We consider the problem of optimally controlling $X(t)$ until time $T(x) = \min\{T_1(x), t_1\}$, where $T_1(x)$ is the first-passage time of the process to a given boundary and t_1 is a fixed constant. The optimal control is obtained explicitly in the particular case when $X(t)$ is a controlled Wiener process.

1. Introduction

Let $\{X(t), t \geq 0\}$ be the one-dimensional controlled diffusion process defined by the stochastic differential equation

$$dX(t) = a[X(t)]dt + b_0 u(t)dt + \sigma_0 dB(t), \tag{1}$$

where $a(\cdot)$ is a real function, $u(t)$ is the control variable, $b_0 \neq 0$, and $\sigma_0 > 0$ are constants and $\{B(t), t \geq 0\}$ is a standard Brownian motion. We define the first-passage time

$$T_1(x) = \inf\{t > 0 : X(t) = d \mid X(0) = x\}, \tag{2}$$

where $x < d$, and the random variable

$$T(x) = \min\{T_1(x), t_1\}, \tag{3}$$

where $t_1 > 0$ is a constant.

Next, we consider the cost criterion

$$J(x) = \int_0^{T(x)} \frac{1}{2} q_0 u^2(t)dt + k \ln[T(x) + 1], \tag{4}$$

where $q_0 > 0$ and $k \neq 0$ are constants. We want to find the control u^* that minimizes the expected value of $J(x)$. This type of problem is a special case of the ones that Whittle [1, page 289] termed *LQG homing*. Notice that if the constant k is negative, then the optimizer tries to maximize the survival time of the process in the interval $(0, d)$, taking the quadratic control costs into account. LQG homing problems have been treated by various authors; see Kuhn [2], Lefebvre [3], and Makasu [4]. Kuhn and Makasu used a risk-sensitive cost criterion (see also Whittle [5, page 222]).

In the general formulation given by Whittle, $\{X(t), t \geq 0\}$ is an n-dimensional process and the random variable $T(x)$ is the moment of first entry of the joint variable $(X(t), t)$ into a stopping set $D \subset \mathbb{R}^n \times (0, \infty)$. However, in practice, it is very difficult to obtain explicit solutions to problems in two or more dimensions (except in special instances). Moreover, in the papers published so far on homing problems, the hitting time $T(x)$ was defined only in terms of $X(t)$. Here, we consider the case when the optimizer stops controlling the diffusion process at most at time t_1.

Using a theorem in Whittle [1], we can state that the optimal control u^* can be expressed as follows:

$$u^* = \frac{\sigma_0^2}{b_0} \frac{G'(x)}{G(x)}, \tag{5}$$

where

$$G(x) = E\left[\exp\left\{-\frac{b_0^2}{q_0 \sigma_0^2} k \ln[\tau(x) + 1]\right\}\right]. \tag{6}$$

In the above formula, $\tau(x)$ is a random variable defined by

$$\tau(x) = \min\{\tau_1(x), t_1\}, \tag{7}$$

with

$$\tau_1(x) = \inf\{t > 0 : \xi(t) = d \mid \xi(0) = x\}, \tag{8}$$

and $\{\xi(t), t \geq 0\}$ is the uncontrolled process that satisfies the stochastic differential equation

$$d\xi(t) = a[\xi(t)]dt + \sigma_0 dB(t). \tag{9}$$

That is, $\tau(x)$ is the random variable that corresponds to $T(x)$ for the diffusion process obtained by setting $u(t) = 0$ in (1).

Hence, the optimal control problem is reduced to the computation of the mathematical expectation $G(x)$. Actually, for this result to hold, we must have $P[\tau(x) < \infty] = 1$. However, in our case this condition is trivially satisfied because $\tau(x) \leq t_1$.

In Section 2, we will obtain an explicit solution u^* in the case when $a[X(t)] \equiv \mu > 0$, so that

$$dX(t) = \mu dt + b_0 u(t)dt + \sigma_0 dB(t). \tag{10}$$

Notice that the uncontrolled process $\{\xi(t), t \geq 0\}$ is then a Wiener process with drift μ and diffusion parameter σ_0^2. Furthermore, we will choose the constant $k = -q_0\sigma_0^2/b_0^2$. With this choice, the mathematical expectation $G(x)$ simplifies to

$$G(x) = E[\exp\{\ln[\tau(x) + 1]\}] = 1 + E[\tau(x)]. \tag{11}$$

2. Optimal Control of a Wiener Process

Let $m_1(x)$ denote the expected value of the first-passage time $\tau_1(x)$. In the case of the Wiener process defined by

$$d\xi(t) = \mu dt + \sigma_0 dB(t) \tag{12}$$

the function $m_1(x)$ satisfies the ordinary differential equation

$$\frac{1}{2}\sigma_0^2 m_1''(x) + \mu m_1'(x) = -1, \tag{13}$$

and is such that $m_1(x) = 0$ if $x = d$. We find (see Lefebvre [6, page 220]) that

$$m_1(x) = \frac{d - x}{\mu}. \tag{14}$$

Therefore, in the case when t_1 tends to infinity, the function $G(x)$ is given by

$$G(x) = 1 + \frac{d - x}{\mu}. \tag{15}$$

It follows from (5) that

$$u^*_{t_1=\infty} = -\frac{\sigma_0^2}{b_0}\frac{1}{\mu + d - x}. \tag{16}$$

Now, to obtain the expected value of the random variable $\tau(x)$, we can condition on $\tau_1(x)$:

$$\begin{aligned} E[\tau(x)] &= E[\tau(x) \mid \tau_1(x) \leq t_1]P[\tau_1(x) \leq t_1] \\ &\quad + E[\tau(x) \mid \tau_1(x) > t_1]P[\tau_1(x) > t_1]. \end{aligned} \tag{17}$$

We may write that

$$E[\tau(x) \mid \tau_1(x) > t_1] = t_1. \tag{18}$$

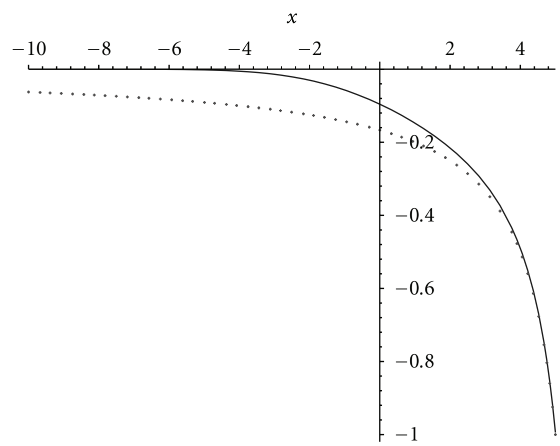

Figure 1: Optimal controls $u^*_{t_1=\infty}$ (dotted line) and u^* (solid line) when $x \in (-10, 5)$, $b_0 = q_0 = \sigma_0 = \mu = 1$, and $d = t_1 = 5$.

Moreover, because the conditional probability density function of $\tau_1(x)$, given that $\tau_1(x) \leq t_1$, is given by

$$f_{\tau_1(x)}(t \mid \tau_1(x) \leq t_1) = \frac{f_{\tau_1(x)}(t)}{P[\tau_1(x) \leq t_1]} \quad \text{for} \quad 0 < t \leq t_1, \tag{19}$$

we have:

$$E[\tau(x)] = \int_0^{t_1} t f_{\tau_1(x)}(t)dt + t_1 P[\tau_1(x) > t_1]. \tag{20}$$

The function $f_{\tau_1(x)}(t)$ is known to be (see Lefebvre [6, page 219])

$$f_{\tau_1(x)}(t) = \frac{d - x}{\sqrt{2\pi\sigma_0^2 t^3}} \exp\left\{-\frac{(d - x - \mu t)^2}{2\sigma_0^2 t}\right\} \quad \text{for} \quad t > 0. \tag{21}$$

Making use of this formula, we can obtain an explicit expression for the mathematical expectation $E[\tau(x)]$ and hence, for the optimal control u^*.

To illustrate the results, we computed (numerically) the optimal control when $b_0 = q_0 = \sigma_0 = \mu = 1$ and $d = t_1 = 5$. Looking at Figure 1, we see that the optimal control u^* tends to zero much faster than $u^*_{t_1=\infty} = 1/(x - 6)$ as x tends to $-\infty$. However, for x close to the boundary at $d = 5$, the two functions are similar.

Next, to see the effect of the constant t_1 on the optimal control, we computed the value of u^* when $x = 0$ and t_1 varies from 0 to 20. This value is compared to $u^*_{t_1=\infty} = -1/6$ in Figure 2. When t_1 decreases to 0, so does u^*, as it should be. For $t_1 \geq 15$ (approximately), we have $u^* \simeq u^*_{t_1=\infty}$.

Finally, in Figure 3, we show the optimal control u^* when $d = 15$ and $t_1 = 5$. Because $E[\tau_1(x)] = 15 - x$ (see above), when $x \leq 5$ (approximately), it is very unlikely that the uncontrolled process will hit the boundary at $d = 15$ before time $t_1 = 5$. Therefore, the optimal control is close to 0. Notice that $\lim_{d\to\infty} u^* = 0$ (for a finite value of x). Indeed, we can write that $\lim_{d\to\infty} P[\tau_1(x) > t_1] = 1$. Hence, we deduce from (17) that $E[\tau(x)] = t_1$, which implies that $G(x) = 1 + t_1$, and thus $u^*_{d=\infty} = 0$.

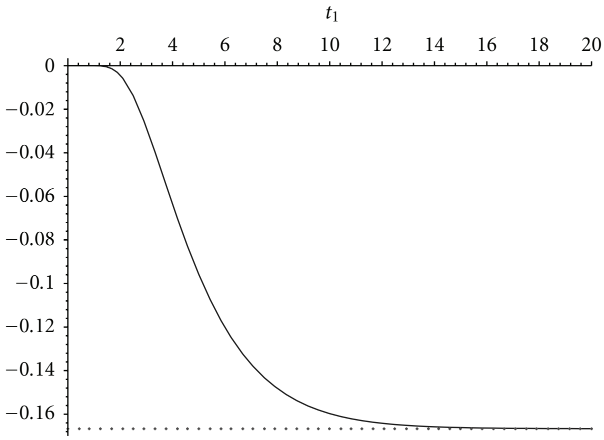

FIGURE 2: Optimal controls $u^*_{t_1=\infty} = -1/6$ (dotted line) and u^* (solid line) when $x = 0$, $b_0 = q_0 = \sigma_0 = \mu = 1$, $d = 5$, and $t_1 \in (0, 20)$.

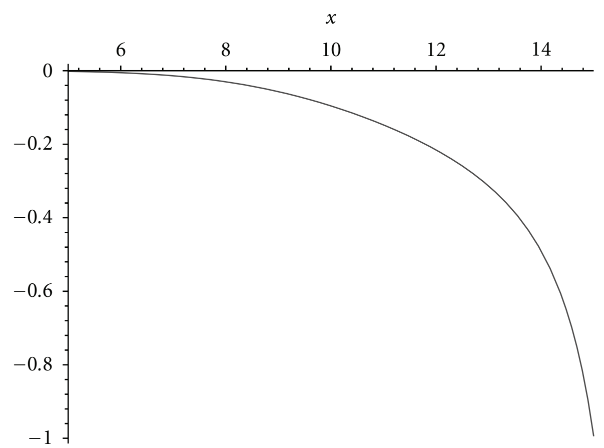

FIGURE 3: Optimal control u^* when $x \in (5, 15)$, $b_0 = q_0 = \sigma_0 = \mu = 1$, $d = 15$, and $t_1 = 5$.

3. Conclusion

We have considered LQG homing problems for which the optimizer controls the diffusion process in the time interval $(0, T(x)]$, where the random variable $T(x)$ is smaller than or equal to a fixed constant t_1. Moreover, the termination cost function was chosen so that the optimal control was expressed in terms of the expected value of a first-passage time for the corresponding uncontrolled process.

An application of this type of problem is the following: suppose that $X(t)$ denotes the wear of a machine. The optimizer wants to maximize the lifetime of the machine. However, it is natural to assume that the machine will be replaced after a certain time, even if it is still in working order, because it might become obsolete.

To obtain a more realistic model for the wear of a device, we could use a degenerate two-dimensional diffusion process, as in Lefebvre [7]. The most difficult problem would then be to compute the probability density function of the first-passage time τ_1.

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