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A NOTE ON R-EQUITABLE K-COLORINGS OF TREES

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Abstract: A graph $G = (V, E)$ is r-equitably k-colorable if there exists a partition of V into k independent sets $V_1, V_2, \cdots, V_k$ such that $|\,|V_i| - |V_j|\,| \leq r$ for all $i, j \in \{1, 2, \cdots, k\}$. In this note, we show that if two trees T_1 and T_2 of order at least two are r-equitably k-colorable for $r \geq 1$ and $k \geq 3$, then all trees obtained by adding an arbitrary edge between T_1 and T_2 are also r-equitably k-colorable.

Keywords: Trees, equitable coloring, independent sets.

MSC: 05C15, 05C69.

1 INTRODUCTION

All graphs in this paper are finite, simple and loopless. Let $G = (V, E)$ be a graph. We denote by $|G|$ its order, i.e, the number of vertices in G. For a vertex $v \in V$, let $N(v)$ denote the set of vertices in G that are adjacent to v. $N(v)$ is called the *neighborhood* of v and its elements are *neighbors* of v. The *degree* of vertex v, denoted by $deg(v)$, is the number of neighbors of v, i.e., $deg(v) = |N(v)|$. $\Delta(G)$ denotes the *maximum degree* of G, i.e., $\Delta(G) = \max\{deg(v)|\ v \in V\}$. For a set $V' \subseteq V$, we denote by $G - V'$ the graph obtained from G by deleting all vertices in V' as well as all edges incident to at least one vertex of V'.

An *independent set* in a graph $G = (V, E)$ is a set $S \subseteq V$ of pairwise nonadjacent vertices. The maximum size of an independent set in a graph $G = (V, E)$ is called the *independence number* of G and denoted by $\alpha(G)$.

A *k-coloring* c of a graph $G = (V, E)$ is a partition of V into k independent sets which we will denote by $V_1(c), V_2(c), \cdots, V_k(c)$ and refer to as *color classes.*

The cardinality of a largest color class with respect to a coloring c will be denoted by Max_c. A graph G is *r-equitably k-colorable*, with $r \geq 1$ and $k \geq 2$, if there exists a k-coloring c of G such that $|\,|V_i(c)| - |V_j(c)|\,| \leq r$ for all $i, j \in \{1, 2, \cdots, k\}$. Such a coloring is called an *r-equitable k-coloring of G*. A graph which is 1-equitably k-colorable is simply said to be *equitably k-colorable.*

The notion of equitable colorability was introduced in [8] and has been studied since then by many authors (see [2, 3, 4, 5, 6, 7, 9]). In [3], the authors gave a complete characterization of trees which are equitably k-colorable. This result was then generalized to forests in [2]. More precisely, for a forest $F = (V, E)$, let $\alpha^*(F) = \min\{\alpha(F - N[v])|\, v \in V \text{ and } deg(v) = \Delta(F)\}$

Theorem 1.1 *([2]) Suppose $F = (V, E)$ is a forest and $k \geq 3$ is an integer. Then F is equitably k-colorable if and only if $k \geq \lceil \frac{|F|+1}{\alpha^*(F)+2} \rceil$.*

This result can easily be generalized to r-equitable k-colorings.

Theorem 1.2 *([1]) Suppose $F = (V, E)$ is a forest and $r \geq 1, k \geq 3$ are two integers. Then F is r-equitably k-colorable if and only if $k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil$.*

Proof: Suppose F is r-equitably k-colorable for $r \geq 1$ and $k \geq 3$. Let v be a vertex in F such that $deg(v) = \Delta(F)$ and $\alpha(F - N[v]) = \alpha^*(F)$. Clearly, for such a coloring, there are at most $\alpha^*(F) + 1$ vertices in the color class that contains v. It follows that all other color classes contain at most $\alpha^*(F) + r + 1$ vertices. Thus $|F| \leq \alpha^*(F) + 1 + (k-1)(\alpha^*(F) + r + 1) = k(\alpha^*(F) + r + 1) - r$, and we therefore have $k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil$.

Conversely, let $k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil$. Consider the forest $F' = (V', E')$ obtained from F by adding $r - 1$ new isolated vertices. Then $|F'| = |F| + r - 1$ and $\alpha^*(F') = \alpha^*(F) + r - 1$. Thus $k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil = \lceil \frac{|F'|+1}{\alpha^*(F')+2} \rceil$. By Theorem 1.1, F' is equitably k-colorable. Restricting the color classes to V gives an r-equitable k-coloring of F.

In this note, we are interested in a different sufficient condition for a tree to be r-equitably k-colorable. More precisely, given a tree $T = (V, E)$ and an edge $e \in E$ such that its removal from T creates two trees T_1 and T_2 of order at least two, we show that if both T_1 and T_2 are r-equitably k-colorable, for $r \geq 1$ and $k \geq 3$, then T is also r-equitably k-colorable. We also explain why $|T_1|, |T_2| \geq 2$ and $k \geq 3$ are necessary conditions.

2 A SUFFICIENT CONDITION

Consider a tree T and two integers $r \geq 1$ and $k \geq 3$. Let c be an arbitrary r-equitable k-coloring of the vertex set of T such that $|V_1(c)| \geq |V_2(c)| \geq \cdots \geq |V_k(c)|$. Then there may be vertices in T which are forced to be colored with color k. Indeed, if for instance T is a star on $(k-1)r + k$ vertices, then the vertex v of degree > 1 necessarily belongs to $V_k(c)$ and actually $V_k(c) = \{v\}$. Furthermore, we have $|V_i(c)| = r + 1$ for $i \in \{1, 2, \cdots, k-1\}$. It turns out that this is no longer true for colors $1, 2, \cdots, k-1$, as shown in the following property.

Lemma 2.1 *Consider an r-equitably k-colorable tree T of order at least two, where $r \geq 1$ and $k \geq 3$. Also, let ℓ be any element in $\{1, 2, \cdots, k-1\}$. Then, for any vertex u in T, there exists an r-equitable k-coloring c of T with $|V_i(c)| \geq |V_j(c)|$ for all $1 \leq i < j \leq k$ such that $u \notin V_\ell(c)$.*

Proof: Suppose the lemma is false. We then clearly have $|T| \geq 3$. Let c be an r-equitable k-coloring of T with $|V_i(c)| \geq |V_j(c)|$ for all $1 \leq i < j \leq k$. Among all such colorings we choose one such that, for each $t = 1, 2, \cdots, k$, there is no r-equitable k-coloring c' of T with $|V_i(c)| = |V_i(c')|$ for $i = 1, 2, \cdots, t-1$ and $\max_{i=t}^{k}\{|V_i(c')|\} < |V_t(c)|$. In other words, $Max_c = |V_1(c)|$ is minimum among all r-equitable k-colorings of T, $|V_2(c)|$ is mininum among all r-equitable k-colorings c' of T with $Max_{c'} = Max_c$, and so on.

Let $\ell \in \{1, 2, \cdots, k-1\}$ be an integer for which the lemma does not hold. We define $x = 1,\ y = 2,\ z = 3$ if $\ell = 1$, and $x = \ell - 1,\ y = \ell,\ z = \ell + 1$ if $\ell > 1$. Since we assume that the lemma is false, it follows that $u \in V_\ell(c)$, which means that $u \in V_x(c)$ if $\ell = 1$ and $u \in V_y(c)$ if $\ell > 1$. Then $|V_x(c)| > |V_y(c)|$, otherwise we could assign color x to all vertices in $V_y(c)$ and color y to all vertices in $V_x(c)$ to obtain an r-equitable k-coloring c' with $u \notin V_\ell(c')$, a contradiction. Similarly, we must have $|V_y(c)| > |V_z(c)|$ when $\ell > 1$ since otherwise we could assign color y to all vertices in $V_z(c)$ and color z to all vertices in $V_y(c)$, and thus the lemma would hold.

We define F as the subgraph of T induced by $V_x(c) \cup V_y(c) \cup V_z(c)$. If F is disconnected, we add some edges to make F become a tree T' such that no two adjacent vertices have the same color with respect to c; otherwise we set $T' = F$. Let V' denote the vertex set of T'. Moreover, for $q = y$ or z, we denote $\overline{q} = y + z - q$. This implies that $\overline{q} = z$ if $q = y$ and $\overline{q} = y$ if $q = z$. We start by proving the following two claims.

Claim 1: There exists no r-equitable 3-coloring c' of T' (using colors x, y, z) with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_q(c')| = |V_q(c)| + 1$ and $|V_{\overline{q}}(c')| = |V_{\overline{q}}(c)|$ for $q = y$ or z.

Indeed, if such a coloring c' exists, then the assumption on c implies $|V_q(c')| = |V_x(c)| > |V_x(c')|$. Now we can obtain an r-equitable k-coloring c^* of T by letting $V_x(c^*) = V_q(c')$, $V_q(c^*) = V_x(c')$, and $V_i(c^*) = V_i(c')$ if $i \neq x, q$. We distinguish two cases:

- If $\ell = 1$, we have $|V_1(c^*)| > \max_{i=2}^{k}\{|V_i(c^*)|\}$ and $u \notin V_1(c^*)$.
- If $\ell > 1$, we have $q = y$ since otherwise $|V_z(c')| = |V_z(c)| + 1 = |V_x(c)|$ which contradicts $|V_x(c)| > |V_y(c)| > |V_z(c)|$. Then $|V_1(c^*)| \geq \cdots \geq |V_{\ell-1}(c^*)| > |V_\ell(c^*)| \geq |V_{\ell+1}(c^*)| \geq \cdots \geq |V_k(c^*)|$ and $u \in V_{\ell-1}(c^*)$.

Thus, in both cases, c^* is an r-equitable k-coloring of T such that $|V_i(c^*)| \geq |V_j(c^*)|$ for all $1 \leq i < j \leq k$ and $u \notin V_\ell(c^*)$, a contradiction.

Claim 2: No leaf of T', except possibly u, is in $V_x(c)$.

Indeed, assume T' has a leaf $v \neq u$ in $V_x(c)$ and let w be its unique neighbor in T'. We can change the color of v from x to $\overline{c(w)}$ to obtain an r-equitable 3-coloring c' of T' with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(w)}}(c')| = |V_{\overline{c(w)}}(c)| + 1$ and $|V_{c(w)}(c')| = |V_{c(w)}(c)|$, contradicting Claim 1.

Let **vec**T be the oriented rooted tree obtained from T' by orienting the edges from root u to the leaves. Let us partition the vertices in $V_x(c)$ into subsets $U_1, \cdots, U_p$ such that U_q $(q = 1, 2, \cdots, p)$ contains all vertices in $V_x(c)$ having no successor in $V_x(c) - \bigcup_{j=1}^{q-1} U_j$. For a vertex $v \in U_1$, let $L(v)$ denote the set of leaves in **vec**T having v as predecessor.

If $|L(v)| = 1$ for some $v \in U_1$, then let $P = v \to s_1 \to \cdots \to s_a$ denote the path from v to the leaf s_a in $L(v)$. If $v = u$ (and hence $\ell = 1$ since $u \in V_x(c)$) then T' is a chain with only one vertex in $V_x(c)$, which means that $V_y(c) = V_z(c) = \emptyset$ since $|V_x(c)| > |V_y(c)| \geq |V_z(c)|$. Thus T' has only one vertex, namely u, and since $u \in V_1(c)$ this implies that T has only one vertex, a contradiction. Hence $v \neq u$. Let w be the predecessor of v in **vec**T:

- if $c(w) = c(s_1)$, we change the color of v to $\overline{c(w)}$ to obtain an r-equitable 3-coloring c' of T' with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(w)}}(c')| = |V_{\overline{c(w)}}(c)| + 1$ and $|V_{c(w)}(c')| = |V_{c(w)}(c)|$, contradicting Claim 1;
- if $c(w) \neq c(s_1)$, we assign color $c(s_1)$ to v, color $c(s_{j+1})$ to s_j $(j = 1, 2, ..., a-1)$, and color x to s_a; we obtain an r-equitable 3-coloring c' of T' with $|V_i(c')| = |V_i(c)|$ $(i = x, y, z)$, $c'(u) = c(u)$ and a leaf $s_a \in V_x(c')$. But this contradicts Claim 2.

We therefore conclude that $|L(v)| \geq 2$ for all $v \in U_1$. By denoting $W_1 = \bigcup_{v \in U_1} L(v)$, we get $|W_1| \geq 2|U_1|$. For each set U_q, with $q > 1$, we will now construct a set W_q containing vertices in $V_y(c) \cup V_z(c)$ that are successors of vertices in U_q but not successors of vertices in U_{q-1}. So let v be any vertex in U_q $(q > 1)$. If v has at least 2 immediate successors in **vec**T, we add two of them to W_q. If v has a unique immediate successor in **vec**T, then let $P = v \to s_1 \to \cdots \to s_a \to v'$ denote a path from v to a vertex $v' \in U_{q-1}$. If $a > 1$, we add s_1 and s_2 to W_q. If $a = 1$ and s_1 has an immediate successor $w \notin V_x(c)$, then we add s_1 and w to W_q. Assume now that $a = 1$ and all the immediate successors of s_1 are in $V_x(c)$. We will prove that such a case is not possible.

- If $v \neq u$, then v has a predecessor w in **vec**T. We must have $c(w) = \overline{c(s_1)}$, otherwise we could assign color $\overline{c(s_1)}$ to v to obtain an r-equitable 3-coloring c' of T' with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c)| + 1$ and $|V_{c(s_1)}(c')| = |V_{c(s_1)}(c)|$, contradicting Claim 1. But now we can assign color $c(s_1)$ to v and assign color $\overline{c(s_1)}$ to s_1 to obtain an r-equitable 3-coloring c' of T' with $c'(u) = c(u)$, $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c)| + 1$ and $|V_{c(s_1)}(c')| = |V_{c(s_1)}(c)|$, contradicting Claim 1.
- If $v = u$, then $\ell = 1$ since $u \in V_x(c)$. By assigning color $c(s_1)$ to u and color $\overline{c(s_1)}$ to s_1, we obtain an r-equitable 3-coloring c' of T' with $|V_x(c')| = |V_x(c)| - 1$, $|V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c)| + 1$ and $|V_{c(s_1)}(c')| = |V_{c(s_1)}(c)|$. It follows from the assumptions on c that $|V_{\overline{c(s_1)}}(c')| = |V_x(c)| > |V_{c(s_1)}(c)| = |V_{c(s_1)}(c')|$. Thus the lemma would hold, a contradiction.

In summary, we have $|W_q| \geq 2|U_q|$. Since all sets W_q are disjoint, we have

$$|V_y(c)| + |V_z(c)| \geq \sum_{q=1}^{p} |W_q| \geq \sum_{q=1}^{p} 2|U_q| = 2|V_x(c)|.$$

Hence $|V_y(c)|$ or $|V_z(c)|$ is larger than or equal to $|V_x(c)|$, a contradiction.

Lemma 2.1 allows us to show our main result.

Theorem 2.2 *Let T_1 and T_2 be two trees or order at least two. If both T_1 and T_2 are r-equitably k-colorable for $r \geq 1$ and $k \geq 3$, then a tree T obtained by adding an arbitrary edge between T_1 and T_2 is also r-equitably k-colorable.*

Proof: Consider an r-equitable k-coloring c of T_1 and an r-equitable k-coloring c' of T_2 such that $|V_i(c)| \geq |V_j(c)|$ and $|V_i(c')| \geq |V_j(c')|$ for all $1 \leq i < j \leq k$. Let u be a vertex in T_1 and v a vertex in T_2, and let T be the tree obtained by adding an edge which joins u and v. According to Lemma 2.1, we may assume that $v \notin V_1(c')$. Hence $v \in V_{k-\ell+1}(c')$ for some $\ell \in \{1, 2, \cdots, k-1\}$ and it follows from Lemma 2.1 that we may assume that $u \notin V_\ell(c)$. We can therefore construct a k-coloring c^* of T such that $V_i(c^*) = V_i(c) \cup V_{k-i+1}(c')$, $i = 1, 2, \cdots, k$. For $i > j$, we have :

$$\begin{aligned} |V_i(c^*)| - |V_j(c^*)| &= |V_i(c)| + |V_{k-i+1}(c')| - (|V_j(c)| + |V_{k-j+1}(c')|) \\ &= (|V_i(c)| - |V_j(c)|) + (|V_{k-i+1}(c')| - |V_{k-j+1}(c')|). \end{aligned}$$

Since $V_j(c) \geq |V_i(c)|$ and $|V_{k-j+1}(c')| \leq |V_{k-i+1}(c')|$, we have :

- $|V_i(c^*)| - |V_j(c^*)| \geq |V_i(c)| - |V_j(c)| \geq -r$;
- $|V_i(c^*)| - |V_j(c^*)| \leq |V_{k-i+1}(c')| - |V_{k-j+1}(c')| \leq r$.

This proves that the considered k-coloring c^* of T is r-equitable.

Note that the condition $k \geq 3$ in Theorem 2.2 is necessary. Indeed, if both T_1 and T_2 are isomorphic to a star on 3 vertices (with u being the vertex of degree two in T_1 and v a leaf in T_2) then clearly T_1 and T_2 are 1-equitably 2-colorable. But by adding an edge which joins u and v, we obtain a tree T which is not 1-equitably 2-colorable.

Note also that the condition in Theorem 2.2 on the number of vertices in each tree is necessary. Indeed, if T_1 is an r-equitably k-colorable tree for some $k \geq 3$ and $r \geq 1$, and if T_2 contains a single vertex v, then the tree T' obtained by adding an edge which joins v and a vertex u of T_1 is possibly not r-equitably k-colorable. For example, if u is the vertex of degree four in the star T_1 on five vertices, and if we add a neighbor v (the single vertex in T_2) to u, we obtain a star T' on six vertices. While T_1 and T_2 are clearly 1-equitably 3-colorable, T' is not 1-equitably 3-colorable. It is however not difficult to prove that if T is an r-equitably k-colorable tree for some $k \geq 2$ and $r \geq 1$, then the tree T' obtained by adding a new vertex v and making it adjacent to some vertex u of T is $(r+1)$-equitably k-colorable. Indeed, given an r-equitable k-coloring c of T, we can extend it to a k-coloring c' of T' by assigning any color $j \neq c(u)$ to v with $j \in \{1, 2, \cdots, k\}$. If $|V_j(c)| \geq |V_i(c)|$ for all $i \neq j$, then c' is $(r+1)$-equitable, otherwise c' is r-equitable.

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