**Titre:** A note on r-equitable k-colorings of trees  
**Auteurs:** Alain Hertz et Bernard Ries  
**Date:** 2014  
**Type:** Article de revue / Journal article  

---

**Document en libre accès dans PolyPublie**  
Open Access document in PolyPublie

**URL de PolyPublie:** https://publications.polymtl.ca/3627/  
**Version:** Version officielle de l'éditeur / Published version  
**Conditions d'utilisation:** CC BY-NC-SA

---

**Document publié chez l’éditeur officiel**  
Document issued by the official publisher

**Titre de la revue:** Yugoslav Journal of Operations Research (vol. 24, no 2)  
**Maison d'édition:** Faculty of Organizational Sciences, Belgrade, Mihajlo Pupin Institute, Belgrade, Faculty of Transport and Traffic Engineering, Belgrade, Faculty of Mining and Geology – Department of Mining, Belgrade, Mathematical Institute SANU, Belgrade  
**URL officiel:** https://doi.org/10.2298/yjor130704039h  
**Mention légale:** Ce fichier a été téléchargé à partir de PolyPublie, le dépôt institutionnel de Polytechnique Montréal

This file has been downloaded from PolyPublie, the institutional repository of Polytechnique Montréal

http://publications.polymtl.ca
A NOTE ON $R$-EQUITABLE $K$-COLORINGS OF TREES

Alain HERTZ
Ecole Polytechnique de Montréal and GERAD
Montréal, Canada
alain.hertz@gerad.ca

Bernard RIES
PSL, Université Paris-Dauphine
75775 Paris Cedex 16, France
CNRS, LAMSADE UMR 7243
bernard.ries@dauphine.fr

Received: July 2013 / Accepted: November 2013

Abstract: A graph $G = (V,E)$ is $r$-equitably $k$-colorable if there exists a partition of $V$ into $k$ independent sets $V_1, V_2, \ldots, V_k$ such that $|V_i| - |V_j| \leq r$ for all $i, j \in \{1, 2, \ldots, k\}$. In this note, we show that if two trees $T_1$ and $T_2$ of order at least two are $r$-equitably $k$-colorable for $r \geq 1$ and $k \geq 3$, then all trees obtained by adding an arbitrary edge between $T_1$ and $T_2$ are also $r$-equitably $k$-colorable.

Keywords: Trees, equitable coloring, independent sets.

MSC: 05C15, 05C69.

1 INTRODUCTION

All graphs in this paper are finite, simple and loopless. Let $G = (V,E)$ be a graph. We denote by $|G|$ its order, i.e., the number of vertices in $G$. For a vertex $v \in V$, let $N(v)$ denote the set of vertices in $G$ that are adjacent to $v$. $N(v)$ is called the neighborhood of $v$ and its elements are neighbors of $v$. The degree of vertex $v$, denoted by $\text{deg}(v)$, is the number of neighbors of $v$, i.e., $\text{deg}(v) = |N(v)|$. $\Delta(G)$ denotes the maximum degree of $G$, i.e., $\Delta(G) = \max\{|\text{deg}(v)| \mid v \in V\}$. For a set $V' \subseteq V$, we denote by $G - V'$ the graph obtained from $G$ by deleting all vertices in $V'$ as well as all edges incident to at least one vertex of $V'$.

An independent set in a graph $G = (V,E)$ is a set $S \subseteq V$ of pairwise nonadjacent vertices. The maximum size of an independent set in a graph $G = (V,E)$ is called the independence number of $G$ and denoted by $\alpha(G)$.

A $k$-coloring $c$ of a graph $G = (V,E)$ is a partition of $V$ into $k$ independent sets which we will denote by $V_1(c), V_2(c), \ldots, V_k(c)$ and refer to as color classes.
The cardinality of a largest color class with respect to a coloring $c$ will be denoted by $\text{Max}_c$. A graph $G$ is \textit{r-equitably $k$-colorable}, with $r \geq 1$ and $k \geq 2$, if there exists a $k$-coloring $c$ of $G$ such that $|V_i(c)| - |V_j(c)| \leq r$ for all $i, j \in \{1, 2, \ldots, k\}$. Such a coloring is called an \textit{r-equitable $k$-coloring of $G$}. A graph which is 1-equitably $k$-colorable is simply said to be \textit{equitably $k$-colorable}.

The notion of equitable colorability was introduced in [8] and has been studied since then by many authors (see [2, 3, 4, 5, 6, 7, 9]). In [3], the authors gave a complete characterization of trees which are equitably $k$-colorable. This result was then generalized to forests in [2]. More precisely, for a forest $F = (V, E)$, let $\alpha^*(F) = \min\{\alpha(F - N[v])|v \in V \text{ and } \deg(v) = \Delta(F)\}$

**Theorem 1.1** ([2]) Suppose $F = (V, E)$ is a forest and $k \geq 3$ is an integer. Then $F$ is equitably $k$-colorable if and only if $k \geq \lceil \frac{|F|+1}{\alpha^*(F)+2} \rceil$.

This result can easily be generalized to $r$-equitable $k$-colorings.

**Theorem 1.2** ([1]) Suppose $F = (V, E)$ is a forest and $r \geq 1, k \geq 3$ are two integers. Then $F$ is $r$-equitably $k$-colorable if and only if $k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil$.

**Proof:** Suppose $F$ is $r$-equitably $k$-colorable for $r \geq 1$ and $k \geq 3$. Let $v$ be a vertex in $F$ such that $\deg(v) = \Delta(F)$ and $\alpha(F - N[v]) = \alpha^*(F)$. Clearly, for such a coloring, there are at most $\alpha^*(F) + 1$ vertices in the color class that contains $v$. It follows that all other color classes contain at most $\alpha^*(F) + r + 1$ vertices. Thus $|F| \leq \alpha^*(F) + 1 + (k - 1)(\alpha^*(F) + r + 1) = k(\alpha^*(F) + r + 1) - r$, and we therefore have $k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil$.

Conversely, let $k \geq \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil$. Consider the forest $F' = (V', E')$ obtained from $F$ by adding $r - 1$ new isolated vertices. Then $|F'| = |F| + r - 1$ and $\alpha^*(F') = \alpha^*(F) + r - 1$. Thus $k \geq \lceil \frac{|F'|+r}{\alpha^*(F')+r+1} \rceil = \lceil \frac{|F|+r}{\alpha^*(F)+r+1} \rceil$. By Theorem 1.1, $F'$ is equitably $k$-colorable. Restricting the color classes to $V$ gives an $r$-equitable $k$-coloring of $F$.

In this note, we are interested in a different sufficient condition for a tree to be $r$-equitably $k$-colorable. More precisely, given a tree $T = (V, E)$ and an edge $e \in E$ such that its removal from $T$ creates two trees $T_1$ and $T_2$ of order at least two, we show that if both $T_1$ and $T_2$ are $r$-equitably $k$-colorable, for $r \geq 1$ and $k \geq 3$, then $T$ is also $r$-equitably $k$-colorable. We also explain why $|T_1|, |T_2| \geq 2$ and $k \geq 3$ are necessary conditions.

## 2 A SUFFICIENT CONDITION

Consider a tree $T$ and two integers $r \geq 1$ and $k \geq 3$. Let $c$ be an arbitrary $r$-equitable $k$-coloring of the vertex set of $T$ such that $|V_i(c)| \geq |V_j(c)| \geq \cdots \geq |V_k(c)|$. Then there may be vertices in $T$ which are forced to be colored with color $k$. Indeed, if for instance $T$ is a star on $(k - 1)r + k$ vertices, then the vertex $v$ of degree $> 1$ necessarily belongs to $V_k(c)$ and actually $V_k(c) = \{v\}$. Furthermore, we have $|V_i(c)| = r + 1$ for $i \in \{1, 2, \ldots, k - 1\}$. It turns out that this is no longer true for colors $1, 2, \ldots, k - 1$, as shown in the following property.

**Lemma 2.1** Consider an $r$-equitably $k$-colorable tree $T$ of order at least two, where $r \geq 1$ and $k \geq 3$. Also, let $\ell$ be any element in $\{1, 2, \ldots, k - 1\}$. Then, for any vertex $u$ in $T$, there exists an $r$-equitable $k$-coloring $c$ of $T$ with $|V_i(c)| \geq |V_j(c)|$ for all $1 \leq i < j \leq k$ such that $u \notin V_i(c)$. 

Proof: Suppose the lemma is false. We then clearly have $|T| \geq 3$. Let $c$ be an $r$-equitable $k$-coloring of $T$ with $|V_i(c)| \geq |V_j(c)|$ for all $1 \leq i < j \leq k$. Among all such colorings we choose one such that, for each $t = 1, 2, \ldots, k$, there is no $r$-equitable $k$-coloring $c'$ of $T$ with $|V_i(c')| = |V_i(c')|$ for $i = 1, 2, \ldots, t-1$ and $\max_{i=1}^{k}(|V_i(c')|) < |V_i(c)|$. In other words, $\max_{c'} = |V_1(c)|$ is minimum among all $r$-equitable $k$-colorings of $T$, $|V_2(c)|$ is minimum among all $r$-equitable $k$-colorings $c'$ of $T$ with $\max_{c'} = \max_{c_0}$, and so on.

Let $\ell \in \{1, 2, \ldots, k-1\}$ be an integer for which the lemma does not hold. We define $x = 1$, $y = 2$, $z = 3$ if $\ell = 1$, and $x = \ell - 1$, $y = \ell$, $z = \ell + 1$ if $\ell > 1$. Since we assume that the lemma is false, it follows that $u \in V_i(c)$, which means that $u \in V_2(c)$ if $\ell = 1$ and $u \in V_\ell(c)$ if $\ell > 1$. Then $|V_2(c)| > |V_\ell(c)|$, otherwise we could assign color $x$ to all vertices in $V_2(c)$ and color $y$ to all vertices in $V_\ell(c)$ to obtain an $r$-equitable $k$-coloring $c'$ with $u \notin V_i(c')$, a contradiction. Similarly, we must have $|V_\ell(c)| > |V_2(c)|$ when $\ell > 1$ since otherwise we could assign color $y$ to all vertices in $V_2(c)$ and color $z$ to all vertices in $V_\ell(c)$, and thus the lemma would hold.

We define $F$ as the subgraph of $T$ induced by $V_2(c) \cup V_\ell(c) \cup V_z(c)$. If $F$ is disconnected, we add some edges to make $F$ become a tree $T'$ such that no two adjacent vertices have the same color with respect to $c$; otherwise we set $T' = F$. Let $V'$ denote the vertex set of $T'$. Moreover, for $q = y$ or $z$, we denote $\eta = y + z - q$. This implies that $\eta = z$ if $q = y$ and $\eta = y$ if $q = z$. We start by proving the following two claims.

Claim 1: There exists no $r$-equitable 3-coloring $c'$ of $T'$ (using colors $x, y, z$) with $c'(u) = c(u)$, $|V_2(c')| = |V_2(c)| - 1$, $|V_\ell(c')| = |V_\ell(c)| + 1$, and $|V_\eta(c')| = |V_\eta(c)|$ for $q = y$ or $z$.

Indeed, if such a coloring $c'$ exists, then the assumption on $c$ implies $|V_\ell(c')| = |V_\ell(c)| > |V_2(c')|$. Now we can obtain an $r$-equitable $k$-coloring $c^*$ of $T$ by letting $V_2(c^*) = V_2(c')$, $V_\ell(c^*) = V_\ell(c')$, and $V_\eta(c^*) = V_\eta(c')$ if $i \neq x, q$. We distinguish two cases:

- If $\ell = 1$, we have $|V_1(c^*)| > \max_{i=2}^{k}(|V_i(c^*)|)$ and $u \notin V_1(c^*)$.
- If $\ell > 1$, we have $q = y$ since otherwise $|V_\ell(c')| = |V_\ell(c)| + 1 = |V_2(c)|$ which contradicts $|V_\ell(c)| > |V_y(c)| > |V_2(c)|$. Then $|V_1(c^*)| \geq \cdots \geq |V_{\ell-1}(c^*)| > |V_{\ell}(c^*)| \geq |V_{\ell+1}(c^*)| \geq \cdots \geq |V_k(c^*)|$ and $u \in V_{\ell-1}(c^*)$.

Thus, in both cases, $c^*$ is an $r$-equitable $k$-coloring of $T$ such that $|V_1(c^*)| \geq |V_j(c^*)|$ for all $1 \leq i < j \leq k$ and $u \notin V_i(c^*)$, a contradiction.

Claim 2: No leaf of $T'$, except possibly $u$, is in $V_\ell(c)$.

Indeed, assume $T'$ has a leaf $v \neq u$ in $V_{\ell}(c)$ and let $w$ be its unique neighbor in $T'$. We can change the color of $v$ from $x$ to $c(w)$ to obtain an $r$-equitable 3-coloring $c'$ of $T'$ with $c'(u) = c(u)$, $|V_2(c')| = |V_2(c)| - 1$, $|V_{\ell}(c')| = |V_{\ell}(c)| + 1$, and $|V_{\eta}(c')| = |V_{\eta}(c')|$, contradicting Claim 1.

Let $\text{vec}T$ be the oriented rooted tree obtained from $T'$ by orienting the edges from root $u$ to the leaves. Let us partition the vertices in $V_2(c)$ into subsets $U_1, \ldots, U_p$ such that $U_q (q = 1, 2, \ldots, p)$ contains all vertices in $V_2(c)$ having no successor in $V_2(c) \setminus \bigcup_{j=1}^{q-1} U_j$. For a vertex $v \in U_1$, let $L(v)$ denote the set of leaves in $\text{vec}T$ having $v$ as predecessor.
Hence if \(|L(v)| = 1\) for some \(v \in U_1\), then let \(P = v \rightarrow s_1 \rightarrow \cdots \rightarrow s_{a}\) denote the path from \(v\) to the leaf \(s_a\) in \(L(v)\). If \(v = u\) (and hence \(\ell = 1\) since \(u \in V_2(c)\)) then \(T'\) is a chain with only one vertex in \(V_2(c)\), which means that \(V_2(c) = V_2(c) = \emptyset\) since \(|V_2(c)| > |V_2(c)| \geq |V_2(c)|\). Thus \(T'\) has only one vertex, namely \(u\), and since \(u \in V_1(c)\) this implies that \(T\) has only one vertex, a contradiction. Hence \(v \neq u\).

Let \(w\) be the predecessor of \(v\) in \(\text{vec}T\):

- if \(c(w) = c(s_1)\), we change the color of \(v\) to \(c(w)\) to obtain an \(r\)-equitable 3-coloring \(c'\) of \(T'\) with \(c'(u) = c(u), |V_2(c')| = |V_2(c')| - 1, |V_{\text{vec}(w)}(c')| = |V_{\text{vec}(w)}(c')| + 1\) and \(|V_{\text{vec}(w)}(c')| = |V_{\text{vec}(w)}(c')|\), contradicting Claim 1;

- if \(c(w) \neq c(s_1)\), we assign color \(c(s_1)\) to \(v\), color \(c(s_{j+1})\) to \(s_j\) \((j = 1, 2, \ldots, a-1)\), and color \(x\) to \(s_a\); we obtain an \(r\)-equitable 3-coloring \(c'\) of \(T'\) with \(|V_2(c')| = |V_2(c')|\) \((i = x, y, z)\), \(c'(u) = c(u)\) and a leaf \(s_a \in V_2(c')\). But this contradicts Claim 2.

We therefore conclude that \(|L(v)| \geq 2\) for all \(v \in U_1\). By denoting \(W_1 = \bigcup_{v \in U_1} L(v)\), we get \(|W_1| \geq 2|U_1|\). For each set \(U_q\), with \(q > 1\), we will now construct a set \(W_q\) containing vertices in \(V_2(c) \cup V_2(c)\) that are successors of vertices in \(U_q\) but not successors of vertices in \(U_{q-1}\). So let \(v\) be any vertex in \(U_q\) \((q > 1)\). If \(v\) has at least 2 immediate successors in \(\text{vec}T\), we add two of them to \(W_q\). If \(v\) has a unique immediate successor in \(\text{vec}T\), then let \(P = v \rightarrow s_1 \rightarrow \cdots \rightarrow s_{a} \rightarrow v'\) denote a path from \(v\) to a vertex \(v' \in U_{q-1}\). If \(a > 1\), we add \(s_1\) and \(s_2\) to \(W_q\). If \(a = 1\) and \(s_1\) has an immediate successor \(w \notin V_2(c)\), then we add \(s_1\) and \(w\) to \(W_q\). Assume now that \(a = 1\) and all the immediate successors of \(s_1\) are \(V_2(c)\). We will prove that such a case is not possible.

- If \(v \neq u\), then \(v\) has a predecessor \(w\) in \(\text{vec}T\). We must have \(c(w) = \overline{c(s_1)}\), otherwise we could assign color \(\overline{c(s_1)}\) to \(v\) to obtain an \(r\)-equitable 3-coloring \(c'\) of \(T'\) with \(c'(u) = c(u), |V_2(c')| = |V_2(c')| - 1, |V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c')| + 1\) and \(|V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c')|\), contradicting Claim 1. But now we can assign color \(c(s_1)\) to \(v\) and assign color \(c(s_1)\) to \(s_1\) to obtain an \(r\)-equitable 3-coloring \(c'\) of \(T'\) with \(c'(u) = c(u), |V_2(c')| = |V_2(c')| - 1, |V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c')| + 1\) and \(|V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c')|\), contradicting Claim 1.

- If \(v = u\), then \(\ell = 1\) since \(u \in V_2(c)\). By assigning color \(c(s_1)\) to \(u\) and color \(\overline{c(s_1)}\) to \(s_1\), we obtain an \(r\)-equitable 3-coloring \(c'\) of \(T'\) with \(|V_2(c')| = |V_2(c')| - 1, |V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c')| + 1\) and \(|V_{\overline{c(s_1)}}(c')| = |V_{\overline{c(s_1)}}(c')|\). It follows from the assumptions on \(c\) that \(|V_{\overline{c(s_1)}}(c')| = |V_2(c')| > |V_{c(s_1)}(c')| = |V_{c(s_1)}(c')|\).

Thus the lemma would hold, a contradiction.

In summary, we have \(|W_q| \geq 2|U_q|\). Since all sets \(W_q\) are disjoint, we have

\[
|V_2(c)| + |V_2(c)| \geq \sum_{q=1}^{p} |W_q| \geq \sum_{q=1}^{p} 2|U_q| = 2|V_2(c)|.
\]

Hence \(|V_2(c)|\) or \(|V_2(c)|\) is larger than or equal to \(|V_2(c)|\), a contradiction.

Lemma 2.1 allows us to show our main result.

**Theorem 2.2** Let \(T_1\) and \(T_2\) be two trees or order at least two. If both \(T_1\) and \(T_2\) are \(r\)-equitably \(k\)-colorable for \(r \geq 1\) and \(k \geq 3\), then a tree \(T\) obtained by adding an arbitrary edge between \(T_1\) and \(T_2\) is also \(r\)-equitably \(k\)-colorable.
Consider an $r$-equitable $k$-coloring $c$ of $T_1$ and an $r$-equitable $k$-coloring $c'$ of $T_2$ such that $|V_i(c)| \geq |V_j(c')|$ and $|V_i(c')| \geq |V_j(c)|$ for all $1 \leq i < j \leq k$. Let $u$ be a vertex in $T_1$ and $v$ a vertex in $T_2$, and let $T$ be the tree obtained by adding an edge which joins $u$ and $v$. According to Lemma 2.1, we may assume that $v \notin V_1(c')$. Hence $v \in V_{k-i+1}(c')$ for some $\ell \in \{1, 2, \ldots, k-1\}$ and it follows from Lemma 2.1 that we may assume that $u \notin V_\ell(c)$. We can therefore construct a $k$-coloring $c^*$ of $T$ such that $V_i(c^*) = V_i(c) \cup V_{k-i+1}(c')$, $i = 1, 2, \ldots, k$. For $i > j$, we have:

$$|V_i(c^*) - |V_j(c^*)| = |V_i(c)| + |V_{k-i+1}(c')| - (|V_j(c)| + |V_{k-j+1}(c')|)$$

$$= (|V_i(c)| - |V_j(c)|) + (|V_{k-i+1}(c')| - |V_{k-j+1}(c')|).$$

Since $V_j(c) \geq |V_i(c)|$ and $|V_{k-j+1}(c')| \leq |V_{k-i+1}(c')|$, we have:

- $|V_i(c^*) - |V_j(c^*)| \geq |V_i(c)| - |V_j(c)| \geq r$;
- $|V_i(c^*) - |V_j(c^*)| \leq |V_{k-i+1}(c')| - |V_{k-j+1}(c')| \leq r$.

This proves that the considered $k$-coloring $c^*$ of $T$ is $r$-equitable.

Note that the condition $k \geq 3$ in Theorem 2.2 is necessary. Indeed, if both $T_1$ and $T_2$ are isomorphic to a star on 3 vertices (with $u$ being the vertex of degree two in $T_1$ and $v$ a leaf in $T_2$) then clearly $T_1$ and $T_2$ are 1-equitably 2-colorable. But by adding an edge which joins $u$ and $v$, we obtain a tree $T$ which is not 1-equitably 2-colorable.

Note also that the condition in Theorem 2.2 on the number of vertices in each tree is necessary. Indeed, if $T_1$ is an $r$-equitably $k$-colorable tree for some $k \geq 3$ and $r \geq 1$, and if $T_2$ contains a single vertex $v$, then the tree $T''$ obtained by adding an edge which joins $v$ and a vertex $u$ of $T_1$ is possibly not an $r$-equitably $k$-colorable. For example, if $u$ is the vertex of degree four in the star $T_1$ on five vertices, and if we add a neighbor $v$ (the single vertex in $T_2$) to $u$, we obtain a star $T''$ on six vertices. While $T_1$ and $T_2$ are clearly 1-equitably 3-colorable, $T''$ is not 1-equitably 3-colorable. It is however not difficult to prove that if $T$ is an $r$-equitably $k$-colorable tree for some $k \geq 2$ and $r \geq 1$, then the tree $T''$ obtained by adding a new vertex $v$ and making it adjacent to some vertex $u$ of $T$ is $(r+1)$-equitably $k$-colorable. Indeed, given an $r$-equitable $k$-coloring $c$ of $T$, we can extend it to a $k$-coloring $c'$ of $T''$ by assigning any color $j \neq c(u)$ to $v$ with $j \in \{1, 2, \ldots, k\}$. If $|V_j(c)| \geq |V_i(c)|$ for all $i \neq j$, then $c'$ is $(r+1)$-equitable, otherwise $c'$ is $r$-equitable.

ACKNOWLEDGEMENT

This note was written while the first author was visiting LAMSADE at the Université Paris-Dauphine and while the second author was visiting GERAD and Ecole Polytechnique de Montréal. The support of both institutions is gratefully acknowledged.

REFERENCES


