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Dehghan Mehdi, El-Kébir Boukas

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Date	2003
Type	Article de revue / Article
Version	Version officielle de l'éditeur / Published version
URL de PolyPublie PolyPublie URL	https://publications.polymtl.ca/3382/
Évaluation par les pairs Peer reviewed	Révisé par les pairs / Refereed
Conditions d'utilisation Terms of use	Creative Commons Attribution 4.0 International (CC BY)
Référence Citation	Mehdi, D., & Boukas, E.-K. (2003). Stability and stabilizability of dynamical systems with multiple time-varying delays: delay-dependent criteria. Mathematical Problems in Engineering, 2003(4), 137-152. https://doi.org/10.1155/s1024123x03206013

Publication officielle
Official publication

Titre de la revue Journal title	Mathematical Problems in Engineering (vol. 2003, no. 4)
Maison d'édition Publisher	Hindawi
URL officielle Official URL	https://doi.org/10.1155/s1024123x03206013
Mention légale Legal notice	

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STABILITY AND STABILIZABILITY OF DYNAMICAL SYSTEMS WITH MULTIPLE TIME-VARYING DELAYS: DELAY-DEPENDENT CRITERIA

D. MEHDI AND E. K. BOUKAS

Received 13 June 2002 and in revised form 13 January 2003

This paper deals with the class of uncertain systems with multiple time delays. The stability and stabilizability of this class of systems are considered. Their robustness are also studied when the norm-bounded uncertainties are considered. Linear matrix inequalities (LMIs) delay-dependent sufficient conditions for both stability and stabilizability and their robustness are established to check if a system of this class is stable and/or is stabilizable. Some numerical examples are provided to show the usefulness of the proposed results.

1. Introduction

It was shown in different studies that the presence of the time delay in the systems dynamics is the primary cause of instability and performance degradation. The class of dynamical systems with time delay has in fact attracted a lot of researchers, mainly from the control community. Many results on this class of systems have been reported in the literature. We refer the reader to [1, 7] and the references therein for more information.

In the present literature, there exist two techniques that can be used to study the stability and the stabilizability. The first one is based on the Lyapunov-Razumikhin technique and it consists of considering a Lyapunov function of the form $V(x_t) = x_t^\top P x_t$, with P a symmetric and positive-definite matrix with appropriate dimension and x_t the state vector of the system, to develop the conditions that can be used to check if the system under study is stable and/or stabilizable. This technique gives a condition that depends on the maximum value of the delay. The reader can consult [2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 15] and the references therein for more information.

The second technique is based on the Lyapunov-Krasovskii approach and it consists of considering a more complicated Lyapunov functional to determine the appropriate delay-dependent condition that in general depends on the upper bound of the first derivative of the delay when it is time-varying. This technique has been extensively used and the large number of references using it confirms this. See, for example, [1, 7] and the references therein for more information.

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Mathematical Problems in Engineering 2003:4 (2003) 137–152
2000 Mathematics Subject Classification: 34D20, 34K20
URL: http://dx.doi.org/10.1155/S1024123X03206013

But from the practical point of view, we are interested in conditions that depend on both, that is, the upper bound of the delay and the lower and the upper bounds of the first derivative of the time-varying delay. Since in practice the delay is in fact always time-varying, that can be usually represented by a function $h(t)$ and bounded by a constant $\bar{h}$, it is therefore desirable to have conditions that depend on the upper bound of the time-varying delay and on the lower and the upper bounds of the first derivative of the time-varying delay.

The goal of this paper consists of considering the class of uncertain linear systems with multiple time-varying delays and developing sufficient conditions for stability and stabilizability and their robustness that depend on the upper bounds of the delays and on the lower and upper bounds of the first derivative of the time-varying delays. The Lyapunov-Krasovskii approach is used in this paper.

Further, the proposed state feedback controller is obtained from some LMI conditions. The LMI problem we deal with introduces some slack variables in addition to the design of a state feedback controller. These slack variables, especially in the uncertain case, give more flexibility resulting in larger bounds.

The paper is organized as follows. In Section 2, the problem is stated and the required assumptions are formulated. Section 3 deals with the stability and the robust stability. Section 4 covers the stabilizability and the robust stabilizability of the class of systems under study. Section 5 presents some numerical examples to show the usefulness of the proposed results.

Notation 1.1. In the rest of this paper, the notation is standard unless it is specified otherwise. The notation $L > 0$ ($L < 0$) means that the matrix L is symmetric and positive-definite (symmetric and negative-definite) matrix and $\mathrm{Sym}(M) = M + M^{\top}$.

2. Problem statement

We consider the following class of systems with multiple time-varying delays:

$$\dot{x}_t = A(t)x_t + \sum_{j=1}^{p} A_{dj}(t)x_{t-h_j(t)} + B(t)u_t, \tag{2.1}$$

where x_t is the state vector, u_t is the control input, $h_j(t)$, $j = 1, 2, \ldots, p$, is the time-varying delay of the system and the matrices $A(t)$, $A_{dj}(t)$, and $B(t)$ are given by

$$\begin{aligned} A(t) &= A + DF(t)E, \\ A_{dj}(t) &= A_{dj} + D_j F_j(t) E_j, \quad \forall\, j = 1, 2, \ldots, p, \\ B(t) &= B + D_b F_b(t) E_b, \end{aligned} \tag{2.2}$$

with A, A_{dj}, $j = 1, 2, \ldots, p$, B, D, E, D_j, E_j, $j = 1, 2, \ldots, p$, D_b, and E_b given matrices with appropriate dimensions and $F(t)$, $F_j(t)$, $j = 1, 2, \ldots, p$, and $F_b(t)$ representing the system uncertainties satisfying the following assumption.

Assumption 2.1. We assume that the following hold:

$$F^{\top}(t)RF(t) \leq R, \qquad F_d^{\top}(t)R_dF_d(t) \leq R_d, \qquad F_b^{\top}(t)R_bF_b(t) \leq R_b, \tag{2.3}$$

where $R_d(t)$ and $F_d(t)$ are diagonal matrices given by

$$F_d(t) = \begin{bmatrix} F_1(t) & & \\ & \ddots & \\ & & F_p(t) \end{bmatrix}, \qquad R_d(t) = \begin{bmatrix} R_1 & & \\ & \ddots & \\ & & R_p \end{bmatrix}, \tag{2.4}$$

with $R, R_1, \ldots, R_p$, and R_b given matrices with appropriate dimensions.

Remark 2.2. The uncertainties that satisfy (2.3) will be referred to as admissible uncertainties. Notice that the uncertainties $F(t)$, $F_j(t)$, $j = 1, 2, \ldots, p$, and $F_b(t)$ can be chosen dependent on the system state and the developed results will remain valid. However, in the present paper, we will consider only the case of time-varying uncertainties.

Assumption 2.3. For each $j = 1, 2, \ldots, p$, the time-varying delay $h_j(t)$ is assumed to satisfy

$$0 \leq h_j(t) \leq \bar{h}_j < \infty, \qquad \underline{l}_j \leq \dot{h}_j(t) \leq \bar{l}_j < 1, \tag{2.5}$$

where $\bar{h}_j$, $\underline{l}_j$, and $\bar{l}_j$ are given positive constants.

We define $\bar{\tau}$ as $\bar{\tau} = \max(\bar{h}_1, \ldots, \bar{h}_p)$ and $\mathbf{x}_t$ as $\mathbf{x}_t(s) = x_{t+s}$, $t - \bar{\tau} \leq s \leq t$. In the rest of the paper, we will use $\mathbf{x}_t$ instead of $\mathbf{x}_t(s)$.

In our investigation of a robust controller, we need to bound some uncertain terms to derive sufficient conditions for robust stability or stabilizability, and Lemma 2.4 below will be very useful for this purpose.

Lemma 2.4 [6]. *Let Z, E, F, R, and Δ be matrices of appropriate dimensions. Assume that Z is symmetric, R is symmetric and positive definite, and $\Delta^{\top}R\Delta \leq R$. Then*

$$Z + E\Delta F + F^{\top}\Delta^{\top}E^{\top} < 0 \tag{2.6}$$

if and only if there exists a scalar $\lambda > 0$ satisfying

$$Z + E(\lambda R)E^{\top} + F^{\top}(\lambda R)^{-1}F < 0. \tag{2.7}$$

In the next section, we deal with the stability problem and its robustness and establish LMI conditions that we can use to check if a system of the class we are studying is stable.

3. Stability and robust stability

The goal of this section consists of establishing what will be the sufficient conditions that can be used to check if the class of systems under study is stable. We are also interested in the robust stability of this class of systems. These two problems will be discussed in the following subsections.

3.1. Stability. We now suppose that the control is equal to zero, that is, $u_t = 0$, for all $t \geq 0$, and that the system does not contain uncertainties, a matter which gives the following dynamics:

$$\dot{x}_t = Ax_t + \sum_{j=1}^{p} A_{dj} x_{t-h_j(t)}. \tag{3.1}$$

The goal of this subsection consists of developing a condition that can be used to check if the class of systems under study is stable. The condition we are looking for should depend on the upper bound of the delay and on the lower and upper bounds of the first derivative of the time-varying delays given in Assumption 2.3. The following theorem states such result.

Theorem 3.1. *Assume that Assumption 2.3 is satisfied. If there exist* F_j, $j = 1, 2, 3, 4$, $P > 0$, $Q_j > 0$, $W_j > 0$, X_j, Y_j, *and* Z_j, *for* $j = 1, 2, \ldots, p$, *such that the following hold:*

$$\mathscr{Z}_j = \begin{bmatrix} Z_j & Y_j \\ Y_j^\top & X_j \end{bmatrix} > 0, \tag{3.2}$$

$$(\bar{l}_j - \underline{l}_j) X_j + (\bar{l}_j - 1) W_j < 0, \tag{3.3}$$

$$\begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & 0 \\ 0 & 0 & -\mathscr{W} & \mathscr{W}\mathbb{I}^\top \\ P & 0 & \mathbb{I}\mathscr{W} & 0 \end{bmatrix} + \operatorname{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \begin{bmatrix} A & A_d & 0 & -I \end{bmatrix} \right\} < 0, \tag{3.4}$$

where

$$\begin{aligned} \mathbb{I} &= \begin{bmatrix} I & \cdots & I \end{bmatrix}, \\ A_d &= \begin{bmatrix} A_{d1} & \cdots & A_{dp} \end{bmatrix}, \\ \mathscr{W} &= \operatorname{diag}(h_1 W_1, \ldots, h_p W_p), \\ \Psi_1 &= \sum_{j=1}^{p} \left[Q_j + (\bar{l}_j - \underline{l}_j)(\bar{h}_j Z_j + Y_j + Y_j^\top) \right], \\ \Psi_3 &= \left[(\bar{l}_1 - \underline{l}_1) Y_1 \cdots (\bar{l}_p - \underline{l}_p) Y_p \right], \\ \Psi_2 &= \operatorname{diag}((1 - \bar{l}_1) Q_1, \ldots, (1 - \bar{l}_p) Q_p), \end{aligned} \tag{3.5}$$

then the system under study is asymptotically stable.

In order to prove Theorem 3.1, we give the following lemma.

Lemma 3.2. *Condition (3.4) is equivalent to*

$$\begin{bmatrix} A^\top P + PA + \Psi_1 & PA_d - \Psi_3 & A^\top \mathbb{I}^\wedge W \\ (PA_d - \Psi_3)^\top & -\Psi_2 & A_d^\top \mathbb{I}^\wedge W \\ {}^\wedge W \mathbb{I}^\top A & {}^\wedge W \mathbb{I}^\top A_d & -{}^\wedge W \end{bmatrix} < 0, \tag{3.6}$$

Proof. The proof is a direct application of the elimination lemma; that is, notice that

$$\mathcal{N}_{\text{right}} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ A & A_d & 0 \end{bmatrix} \tag{3.7}$$

satisfies

$$\begin{bmatrix} A & A_d & 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ A & A_d & 0 \end{bmatrix} = 0. \tag{3.8}$$

Then condition (3.4) is equivalent to

$$\mathcal{N}_{\text{right}}^\top \begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & 0 \\ 0 & 0 & -{}^\wedge W & {}^\wedge W \mathbb{I}^\top \\ P & 0 & \mathbb{I}^\wedge W & 0 \end{bmatrix} \mathcal{N}_{\text{right}} = \begin{bmatrix} A^\top P + PA + \Psi_1 & PA_d - \Psi_3 & A^\top \mathbb{I}^\wedge W \\ (PA_d - \Psi_3)^\top & -\Psi_2 & A_d^\top \mathbb{I}^\wedge W \\ {}^\wedge W \mathbb{I}^\top A & {}^\wedge W \mathbb{I}^\top A_d & -{}^\wedge W \end{bmatrix} < 0, \tag{3.9}$$

and this ends the proof of Lemma 3.2. □

Proof of Theorem 3.1. Let the Lyapunov functional be defined by

$$V(\mathbf{x}_t) = V_1(\mathbf{x}_t) + V_2(\mathbf{x}_t) + V_3(\mathbf{x}_t) + V_4(\mathbf{x}_t), \tag{3.10}$$

where

$$\begin{aligned}
V_1(\mathbf{x}_t) &= x_t^\top P x_t,\\
V_2(\mathbf{x}_t) &= \sum_{j=1}^{p}\int_{t-h_j(t)}^{t}\int_{s}^{t}\dot{x}_z^\top W_j \dot{x}_z\, dz\, ds,\\
V_3(\mathbf{x}_t) &= \sum_{j=1}^{p}\int_{t-h_j(t)}^{t} x_s^\top Q_j x_s\, ds,\\
V_4(\mathbf{x}_t) &= \sum_{j=1}^{p}\int_{0}^{t}(\bar{l}_j-\dot{h}_j(z))\int_{z-h_j(z)}^{z}\begin{bmatrix}x_z^\top & \dot{x}_s\end{bmatrix}\begin{bmatrix}Z_j & Y_j\\ Y_j^\top & X_j\end{bmatrix}\begin{bmatrix}x_z\\ \dot{x}_s\end{bmatrix} ds\, dz.
\end{aligned} \tag{3.11}$$

After taking the derivative of these functionals and some algebraic manipulations, we get

$$\dot{V}(\mathbf{x_t}) = \xi_t^\top M \xi_t + \sum_{j=1}^{p}\int_{t-h_j(t)}^{t}\dot{x}_s^\top\left[(\bar{l}_j-\underline{l}_j)X_j+(\bar{l}_j-1)W_j\right]\dot{x}_s\, ds, \tag{3.12}$$

with

$$\begin{aligned}
\xi_t^\top &= \begin{bmatrix}x_t^\top x_{t-h_1(t)}^\top & \cdots & x_{t-h_p(t)}^\top\end{bmatrix},\\
M &= \begin{bmatrix}M_{11} & M_{12}\\ M_{12}^\top & M_{22}\end{bmatrix},
\end{aligned} \tag{3.13}$$

where M_{11}, M_{12}, and M_{22} are given by

$$\begin{aligned}
M_{11} &= A^\top P + PA + A^\top \mathbb{I}^\circledcirc \mathcal{W} \mathbb{I}^\top A + \Psi_1,\\
M_{12} &= PA_d - \Psi_3 + A^\top \mathbb{I}^\circledcirc \mathcal{W} \mathbb{I}^\top A_d,\\
M_{22} &= A_d^\top \mathbb{I}^\circledcirc \mathcal{W} \mathbb{I}^\top A_d - \Psi_2.
\end{aligned} \tag{3.14}$$

Therefore, the system is then asymptotically stable if the following hold:

$$\begin{aligned}
&M < 0,\\
&\left[(\bar{l}_j-\underline{l}_j)X_j+(\bar{l}_j-1)W_j\right] < 0, \quad \forall\, j = 1,\ldots,p.
\end{aligned} \tag{3.15}$$

Notice that matrix M can be expressed as follows:

$$\begin{aligned}
M &= \begin{bmatrix}A^\top P + PA + \Psi_1 & PA_d - \Psi_3\\ (PA_d-\Psi_3)^\top & -\Psi_2\end{bmatrix} + \begin{bmatrix}A^\top \mathbb{I}^\circledcirc \mathcal{W} \mathbb{I}^\top A & A^\top \mathbb{I}^\circledcirc \mathcal{W} \mathbb{I}^\top A_d\\ (A^\top \mathbb{I}^\circledcirc \mathcal{W} \mathbb{I}^\top A_d)^\top & A_d^\top \mathbb{I}^\circledcirc \mathcal{W} \mathbb{I}^\top A_d\end{bmatrix}\\
&= \begin{bmatrix}A^\top P + PA + \Psi_1 & PA_d - \Psi_3\\ (PA_d-\Psi_3)^\top & -\Psi_2\end{bmatrix} + \begin{bmatrix}A^\top \mathbb{I}^\circledcirc \mathcal{W}\\ A_d^\top \mathbb{I}^\circledcirc \mathcal{W}\end{bmatrix}(^\circledcirc\mathcal{W})^{-1}\begin{bmatrix}A^\top \mathbb{I}^\circledcirc \mathcal{W}\\ A_d^\top \mathbb{I}^\circledcirc \mathcal{W}\end{bmatrix}^\top.
\end{aligned} \tag{3.16}$$

Using Schur complement, we conclude that M is negative definite if and only if condition (3.6) is satisfied. Furthermore, since condition (3.6) is equivalent to (3.4) according to Lemma 3.2 and since (3.4) is verified by assumption as well as conditions (3.3) and (3.2), then the system under study is asymptotically stable. This ends the proof of the theorem. □

Remark 3.3. The results of Theorem 3.1 are only sufficient and, therefore, if these conditions are not verified, we cannot claim that the system under study is not stable.

3.2. Robust stability. We now assume that the control is still equal to zero for all time and assume that the system has uncertainties on all the matrices, that is,

$$\dot{x}_t = [A + DF(t)E]x_t + \sum_{j=1}^{p} [A_{dj} + D_j F_j(t) E_j]x_{t-h_j(t)}, \tag{3.17}$$

where all the terms keep the same meaning as before.

We introduce the following notations:

$$\begin{aligned} \tilde{A} &= A + DF(t)E, \\ \tilde{A}_d &= \begin{bmatrix} A_{d1} + D_1 F_1(t) E_1 & \cdots & A_{dp} + D_p F_p(t) E_p \end{bmatrix} = A_d + D_d F_d E_d, \end{aligned} \tag{3.18}$$

where E_d and D_d are given by

$$D_d = \begin{bmatrix} D_1 & \cdots & D_p \end{bmatrix}, \qquad E_d = \operatorname{diag}(E_1, \ldots, E_p). \tag{3.19}$$

Note that conditions (3.2) and (3.3) do not depend on the system matrices so they do not need to be adapted to the uncertain case. Besides, we have to replace A and A_d by $\tilde{A}$ and $\tilde{A}_d$, respectively, in condition (3.4) to get a condition for the robust case which is stated by Theorem 3.4.

Theorem 3.4. *Assume that Assumptions 2.1 and 2.3 are satisfied. If there exist F_1, F_2, F_3, F_4, $P > 0$, $Q_j > 0$, $W_j > 0$, X_j, Y_j, Z_j, for $j = 1, 2, \ldots, p$, and λ such that conditions (3.2), (3.3), and*

$$\begin{bmatrix} \alpha_{11} & * & * & * & * & * \\ A_d^T F_1^T + F_2 A - \Psi_3^\top & \alpha_{22} & * & * & * & * \\ F_3 A & F_3 A_d & -\mathbb{W} & * & * & * \\ F_4 A + P - F_1^\top & F_4 A_d - F_2^\top & \mathbb{W} - F_3^\top & -F_4 - F_4^\top & * & * \\ D^\top F_1^\top & D^\top F_2^\top & D^\top F_3^\top & D^\top F_4^\top & -\lambda R & * \\ D_d^\top F_1^\top & D_d^\top F_2^\top & D_d^\top F_3^\top & D_d^\top F_4^\top & 0 & -\lambda R_d \end{bmatrix} < 0 \tag{3.20}$$

hold, with

$$\begin{aligned}\alpha_{11} &= \Psi_1 + A^T F_1^T + F_1 A + \lambda E^\top R E,\\ \alpha_{22} &= A_d^T F_2^T + F_2 A_d - \Psi_2 + \lambda E_d^\top R_d E_d,\end{aligned} \tag{3.21}$$

then the uncertain system under study is asymptotically stable for all admissible uncertainties.

Proof of Theorem 3.4. As we have said before, the robust stability is achieved, according to Theorem 3.1, if conditions (3.2), (3.3), and

$$\begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & \\ 0 & 0 & -\mathcal{W} & \mathcal{W}\mathbb{I}^\top \\ P & 0 & \mathbb{I}\mathcal{W} & 0 \end{bmatrix} + \operatorname{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{A}_d & 0 & -I \end{bmatrix} \right\} < 0 \tag{3.22}$$

are satisfied. Since conditions (3.2) and (3.3) remain unchanged in the presence of uncertainty, we have to work out only condition (3.22).

First, notice that in condition (3.22) the second term of the left side can be split into two parts to yield

$$\begin{aligned}&\begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & 0 \\ 0 & 0 & -\mathcal{W} & \mathcal{W}\mathbb{I}^\top \\ P & 0 & \mathbb{I}\mathcal{W} & 0 \end{bmatrix} + \operatorname{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \begin{bmatrix} A & A_d & 0 & -I \end{bmatrix} \right\} \\ &\quad + \operatorname{Sym}\left\{ \begin{bmatrix} F_1 D & F_1 D_d \\ F_2 D & F_4 D_d \\ F_3 D & F_3 D_d \\ F_4 D & F_4 D_d \end{bmatrix} \begin{bmatrix} F(t) & \\ & F_d(t) \end{bmatrix} \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E_d & 0 & 0 \end{bmatrix} \right\} < 0.\end{aligned} \tag{3.23}$$

According to Lemma 2.4, the previous inequality is satisfied if there exists $\lambda > 0$ such that

$$\begin{aligned}&\begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & 0 \\ 0 & 0 & -\mathcal{W} & \mathcal{W}\mathbb{I}^\top \\ P & 0 & \mathbb{I}\mathcal{W} & 0 \end{bmatrix} + \operatorname{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \begin{bmatrix} A & A_d & 0 & -I \end{bmatrix} \right\} \\ &\quad + \begin{bmatrix} F_1 D & F_1 D_d \\ F_2 D & F_4 D_d \\ F_3 D & F_3 D_d \\ F_4 D & F_4 D_d \end{bmatrix} \begin{bmatrix} \lambda R & \\ & \lambda R_d \end{bmatrix}^{-1} \begin{bmatrix} F_1 D & F_1 D_d \\ F_2 D & F_4 D_d \\ F_3 D & F_3 D_d \\ F_4 D & F_4 D_d \end{bmatrix}^\top \\ &\quad + \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E_d & 0 & 0 \end{bmatrix}^\top \begin{bmatrix} \lambda R & \\ & \lambda R_d \end{bmatrix} \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E_d & 0 & 0 \end{bmatrix} < 0,\end{aligned} \tag{3.24}$$

and using a Schur complement operation yields

$$\begin{bmatrix} f_{11} & * & * & * & * & * \\ -\Psi_3^\top & f_{22} & * & * & * & * \\ 0 & 0 & -\mathcal{W} & * & * & * \\ P & 0 & \mathbb{I}\mathcal{W} & 0 & * & * \\ 0 & 0 & 0 & 0 & -\lambda R & * \\ 0 & 0 & 0 & 0 & 0 & -\lambda R_d \end{bmatrix} + \mathrm{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} A & A_d & 0 & -I & D & D_d \end{bmatrix} \right\} < 0, \tag{3.25}$$

with

$$f_{11} = \Psi_1 + \lambda E^\top R E, \qquad f_{22} = -\Psi_2 + \lambda E_d^\top R_d E_d. \tag{3.26}$$

It is worth noting that (3.25) can be rewritten as condition (3.20) and this ends the proof. This condition, (3.2), and (3.3) represent the sufficient conditions for robust stability of the class of systems under consideration. □

The next section deals with the stabilizability and the robust stabilizability of the class of systems under study.

3.3. Stabilizability. This section deals with the stabilizability problem, and we try to design a controller that stabilizes the closed-loop system. We restrict ourselves to the class of memoryless state feedback controller.

Thus the state feedback controller is of the form

$$u(t) = Kx(t). \tag{3.27}$$

Substituting (3.27) in the plant model and taking $A^{cl} = (A + BK)$, we get the closed-loop dynamics

$$\dot{x}_t = A^{cl} x_t + \sum_{j=1}^{p} A_{dj}(t) x_{t-h_j(t)}. \tag{3.28}$$

We note that only condition (3.4) must be adapted to the stabilizability case. We replace A by A^{cl} in (3.4) to get

$$\mathcal{M}^{cl} = \begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & 0 \\ 0 & 0 & -\mathcal{W} & \mathcal{W}\mathbb{I}^\top \\ P & 0 & \mathbb{I}\mathcal{W} & 0 \end{bmatrix} + \mathrm{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \begin{bmatrix} A^{cl} & A_d & 0 & -I \end{bmatrix} \right\} < 0. \tag{3.29}$$

The problem of robust stabilization by state feedback is stated by Theorem 3.5.

THEOREM 3.5. *Assume that Assumption 2.3 is satisfied. If there exist* F_1, F_2, F_3, F_4, $P > 0$, $Q_i > 0$, $W_i > 0$, X_i, Y_i, Z_i *(for* $i = 1, \ldots, p$*),* L*, and* G *such that*

$$\begin{bmatrix} \bar{Z}_i & \bar{Y}_i \\ \bar{Y}_i^\top & \bar{X}_i \end{bmatrix} > 0, \tag{3.30}$$

$$(\bar{l}_i - \underline{l}_i)\bar{X}_i + (\bar{l}_i - 1)\bar{W}_i < 0, \tag{3.31}$$

$$\begin{bmatrix} \Psi_1 + A_o^T F_1^T + F_1 A_o & * & * & * & * \\ A_{d_o}^T F_1^T + F_2 A_o - \Psi_3^\top & A_{d_o}^T F_2^T + F_2 A_{d_o} - \Psi_2 & * & * & * \\ F_3 A_o & F_3 A_{d_o} & -\mathbb{W} & * & * \\ F_4 A_o + P - F_1^\top & F_4 A_{d_o} - F_2^\top & \mathbb{I}\mathbb{W} - F_3^\top & -F_4 - F_4^\top & * \\ B^T F_1^T + L - G K_o & B^T F_2^T - G K_{d_o} & B^T F_3^T & B^T F_4^T & -G - G^T \end{bmatrix} < 0 \tag{3.32}$$

hold for $i = 1, \ldots, p$ *for given gains* K_o *and* K_{d_o} *that make the matrices* $A_o = (A + BK_o)$ *and* $A_{d_o} = (A_d + BK_{d_o})$ *stable, then the closed loop system is asymptotically stable with the stabilizing feedback gain given by*

$$K = G^{-1}L. \tag{3.33}$$

Proof of Theorem 3.5. The closed loop matrix A^{cl} can also be rewritten as

$$\begin{aligned} A^{cl} &= A + BK = A + BK_o + B(K - K_o) = A_o + BS_o, \\ A_d &= A_d + BK_{d_o} - BK_{d_o} = A_{d_o} + BS_{d_o}, \end{aligned} \tag{3.34}$$

where the gain K_o and K_{d_o} are chosen in such a way that $A + BK_o$ and $A_d + BK_{d_o}$ are stable. This allows us to rewrite $\mathcal{M}^{cl}$ as

$$\begin{aligned} \mathcal{M}^{cl} = &\begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P \\ -\Psi_3^\top & -\Psi_2 & 0 & 0 \\ 0 & 0 & -\mathbb{W} & \mathbb{W}\mathbb{I}^\top \\ P & 0 & \mathbb{I}\mathbb{W} & 0 \end{bmatrix} \\ &+ \operatorname{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \begin{bmatrix} A_o & A_{d_o} & 0 & -I \end{bmatrix} \right\} + \operatorname{Sym}\left\{ \begin{bmatrix} F_1 B \\ F_2 B \\ F_3 B \\ F_4 B \end{bmatrix} \begin{bmatrix} S_o & S_{d_o} & 0 & 0 \end{bmatrix} \right\} < 0, \end{aligned} \tag{3.35}$$

and using similar arguments as in the proof of Lemma 3.2, we introduce a new variable G to get the condition

$$\mathcal{M}_{SF} = \begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P & 0 \\ -\Psi_3^\top & -\Psi_2 & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{W} & \mathcal{W}\mathbb{I}^\top & 0 \\ P & 0 & \mathbb{I}\mathcal{W} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \text{Sym}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ 0 \end{bmatrix} \begin{bmatrix} A_o & A_{d_o} & 0 & -I & B \end{bmatrix} \right\} + \text{Sym}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix} G \begin{bmatrix} S_o & S_{d_o} & 0 & 0 & -I \end{bmatrix} \right\} < 0 \tag{3.36}$$

which is in fact condition (3.32) where we have introduced the change of variable $L = GK$.

Indeed, notice that

$$\begin{bmatrix} I & 0 & 0 & 0 & -S_o^\top \\ 0 & I & 0 & 0 & -S_{d_o}^\top \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix} \mathcal{M}_{SF} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ S_o & S_{d_o} & 0 & 0 \end{bmatrix} = \mathcal{M}^{cl} < 0. \tag{3.37}$$

The introduction of the new variable G allows us to perform a decoupling between the matrices B and S_o and hence between B and the state feedback gain K. □

3.4. Robust stabilizability. In this subsection, we are concerned with robust stabilizability of the uncertain system under the control law (3.27). The closed loop system is then given by

$$\dot{x}_t = [A + BK + DF(t)E + D_b F_b(t) E_b K] x_t + \sum_{j=1}^{p} [A_{dj} + D_j F_j(t) E_j] x_{t-h_j(t)}, \tag{3.38}$$

where all the terms keep the same meaning as previously. Taking account of the uncertainties in (3.32), we get

$$\tilde{\mathcal{M}}_{SF} = \begin{bmatrix} \Psi_1 & -\Psi_3 & 0 & P & 0 \\ -\Psi_3^\top & -\Psi_2 & 0 & 0 & 0 \\ 0 & 0 & -\mathcal{W} & \mathcal{W}\mathbb{I}^\top & 0 \\ P & 0 & \mathbb{I}\mathcal{W} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \text{Sym}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix} G \begin{bmatrix} S_o & S_{d_o} & 0 & 0 & -I \end{bmatrix} \right\}$$

$$+\mathrm{Sym}\left\{\begin{bmatrix}F_1\\F_2\\F_3\\F_4\\0\end{bmatrix}\begin{bmatrix}\tilde{A}^{cl} & \tilde{A}_d^{cl} & 0 & -I & \tilde{B}\end{bmatrix}\right\}$$

$$=\mathcal{M}_{SF_o}+\mathrm{Sym}\left\{\begin{bmatrix}F_1\\F_2\\F_3\\F_4\\0\end{bmatrix}\begin{bmatrix}D & D_b & D_d & D_b & D_b\end{bmatrix}\mathcal{F}(t)\begin{bmatrix}E & 0 & 0 & 0 & 0\\ E_bK_o & 0 & 0 & 0 & 0\\ 0 & E_bK_{d_o} & 0 & 0 & 0\\ 0 & E_d & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & E_b\end{bmatrix}\right\}<0, \tag{3.39}$$

with

$$\begin{aligned}\mathcal{F}(t) &= \mathrm{diag}\begin{bmatrix}F(t) & F_b(t) & F_d(t) & F_b(t) & F_b(t)\end{bmatrix},\\ \tilde{A}^{cl} &= \tilde{A}+\tilde{B}K_o,\\ \tilde{A}_d^{cl} &= \tilde{A}_d+\tilde{B}K_{d_o},\end{aligned} \tag{3.40}$$

and $\mathcal{M}_{SF_o}$ the part of $\tilde{\mathcal{M}}_{SF}$ that contains only the non-uncertain terms. Using Lemma 2.4 as previously, we get

$$\mathcal{M}_{SFR}=\mathrm{Sym}\left\{\begin{bmatrix}F_1\\F_2\\F_3\\F_4\\0\\0\\0\\0\\0\\0\end{bmatrix}\begin{bmatrix}A_o & A_{d_o} & 0 & -I & B & D & D_b & D_d & D_b & D_b\end{bmatrix}\right\}$$

$$+\begin{bmatrix}\beta_{11} & * & * & * & * & * & * & * & * & *\\ -\Psi_3^\top & \beta_{22} & * & * & * & * & * & * & * & *\\ 0 & 0 & -\mathcal{W} & * & * & * & * & * & * & *\\ P & 0 & \mathbb{I}\mathcal{W} & 0 & * & * & * & * & * & *\\ 0 & 0 & 0 & 0 & \lambda E_b^\top R_b E_b & * & * & * & * & *\\ 0 & 0 & 0 & 0 & 0 & -\lambda R & * & * & * & *\\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_b & * & * & *\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_d & * & *\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_b & *\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda R_b\end{bmatrix}$$

$$+\operatorname{Sym}\left\{\begin{bmatrix}0\\0\\0\\0\\I\\0\\0\\0\\0\\0\end{bmatrix} G\begin{bmatrix}S_o & S_{d_o} & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0\end{bmatrix}\right\} < 0, \tag{3.41}$$

with

$$\begin{aligned}\beta_{11} &= \Psi_1 + \lambda\left(E^\top RE + K_o^\top E_b^\top R_b E_b K_o\right),\\ \beta_{22} &= -\Psi_2 + \lambda\left(E_d^\top R_d E_d + K_{d_o}^\top E_b^\top R_b E_b K_{d_o}\right).\end{aligned} \tag{3.42}$$

In condition (3.41), we have to proceed to the change of variable $L = GK$ and then we get an LMI problem from which when it is feasible, we get the feedback gain as

$$K = G^{-1}L. \tag{3.43}$$

The following theorem summarizes the result of robust stability.

Theorem 3.6. *Assume that Assumptions 2.1 and 2.3 are satisfied. If there exist F_1, F_2, F_3, F_4, $P > 0$, $Q_i > 0$, $W_i > 0$, X_i, Y_i, Z_i (for $i = 1,\ldots,p$), L, G, and λ such that the LMI problem composed of the three conditions (3.30), (3.31), and (3.41) is feasible, the robust stabilizing state feedback gain is given by (3.43) and the uncertain closed loop system under study is asymptotically stable for the set of all admissible uncertainties.*

4. Example

To show the usefulness of our results, we consider some numerical examples.

Example 4.1. The goal of the example is in fact to show the flexibility introduced by the slack variables F_i, $i = 1,\ldots,4$, in condition (3.4) in comparison with the LMI problem using condition (3.6) as in [9].

For this purpose, we consider that the system under study has one time delay and we solve the LMI problem resulting from Theorem 3.4.

Thus, we assume that the dynamics are described by [9]

$$\dot{x}(t) = \left(A + DF(t)E\right)x(t) + \left(A_d + D_d F_d(t) E_d\right)x\left(t - h(t)\right), \tag{4.1}$$

where $F(t)$ and $F_d(t)$ are uncertain matrices of appropriate dimension satisfying condition (2.3).

Table 4.1

$\dot{h}(t)$	Maximal value of $\bar{h}$ [9]	Maximal value of $\bar{h}$ (Theorem 3.4)
$\dot{h}(t) \le 0.9$	0.1621	0.225
$\dot{h}(t) \le 0.8$	0.3802	0.49
$\dot{h}(t) \le 0.6$	1.0662	1.425
$\dot{h}(t) \le 0.4$	7.1784	∞

The system matrices are as follows:

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -0.2 & 0.1 \\ -0.3 & -0.1 \end{bmatrix}, \\ D = D_d = 0.2I, \qquad E = E_d = I, \qquad R = R_d = I. \tag{4.2}$$

The delay satisfies

$$0 < h(t) \le \bar{h}, \tag{4.3}$$

where the bound $\bar{h}$ is obtained as the maximal value yielding feasibility for the LMI problem associated with Theorem 3.4. This maximal value is a function of the upper bound of the first derivative of the delay as shown in Table 4.1.

In this example, we proceed to a comparison with the result given in [9] and we show that in the robust case, Theorem 3.4 of the present paper provides better results than [9, Theorem 3.2]. Notice that for the case where the first derivative of the delay satisfies the condition $\dot{h}(t) \le 0.4$, Theorem 3.4 says that the system is robustly stable for unbounded delay (a very large value) whereas in [9], the maximal value was $\bar{h} = 7.1884$.

It is worth noting that for the nominal system we get similar bounds.

Example 4.2. In this example, we consider the robust stabilizability problem. For this purpose, we consider the single time-varying delay system:

$$\dot{x}(t) = \big(A + DF(t)E\big)x(t) + \big(A_d + D_d F_d(t) E_d\big)x\big(t - h(t)\big) + \big(B + D_b F_b(t) E_b\big)u(t), \tag{4.4}$$

where $F(t)$, $F_d(t)$, and $F_b(t)$ are uncertain matrices of appropriate dimension satisfying condition (2.3), with

$$A = \begin{bmatrix} 2.0 & 0.0 \\ 1.0 & 3.0 \end{bmatrix}, \qquad D = 0.2I, \qquad E = I, \qquad R = I, \\ B = \begin{bmatrix} 1.0 & 2.0 \\ 1.0 & 0.0 \end{bmatrix}, \qquad D_b = 0.2I, \qquad E_b = I, \qquad R_b = I, \\ A_d = \begin{bmatrix} -0.1 & 0.0 \\ -0.8 & -1.0 \end{bmatrix}, \qquad D_d = 0.2I, \qquad E_d = I, \qquad R_d = I. \tag{4.5}$$

The characteristics of the first derivative of the delay are as follows:

$$\underline{l} = 0.0, \qquad \bar{l} = 0.825. \tag{4.6}$$

The application of Theorem 3.6 leads to the following results:

$$X = \begin{bmatrix} 30.1136 & -3.9990 \\ -3.9990 & 0.5311 \end{bmatrix}, \qquad Y = \begin{bmatrix} -3.3436 & 0.4440 \\ 4.4596 & -0.5922 \end{bmatrix},$$

$$Z = \begin{bmatrix} 0.3712 & -0.4950 \\ -0.4950 & 0.6626 \end{bmatrix}, \qquad W = \begin{bmatrix} 141.9641 & -18.8526 \\ -18.8526 & 2.5036 \end{bmatrix},$$

$$P = \begin{bmatrix} 8999.5211 & -28.5684 \\ -28.5684 & 37679.0852 \end{bmatrix}, \qquad Q = \begin{bmatrix} 7942.9421 & 17142.9757 \\ 17142.9757 & 303352.7590 \end{bmatrix},$$

$$F1 = \begin{bmatrix} 8946.4080 & 152.8018 \\ -88.7054 & 37661.7437 \end{bmatrix}, \qquad F2 = \begin{bmatrix} -100.1079 & 85.6633 \\ -1447.3920 & 38.6734 \end{bmatrix},$$

$$F3 = \begin{bmatrix} 350.8231 & -40.7311 \\ -46.5886 & 5.4090 \end{bmatrix}, \qquad F4 = \begin{bmatrix} 3449.8341 & -375.8000 \\ -500.6067 & 161.2343 \end{bmatrix}, \tag{4.7}$$

$$L = \begin{bmatrix} -223754.6705 & 19190.1742 \\ -258925.2826 & 120109.2273 \end{bmatrix}, \qquad G = \begin{bmatrix} 37765.6548 & 53117.4530 \\ 42764.2230 & 62293.1025 \end{bmatrix},$$

$$K_o = \begin{bmatrix} -0.5000 & -6.2506 \\ -3.8114 & 6.2506 \end{bmatrix}, \qquad K_{d_o} = \begin{bmatrix} 0.4000 & 0.4697 \\ -0.3486 & -0.4697 \end{bmatrix},$$

$$\lambda = 816.2566.$$

The stabilizing state feedback gain is then

$$K = \begin{bmatrix} -2.2826 & -63.9955 \\ -2.5896 & 45.8611 \end{bmatrix}. \tag{4.8}$$

These results were obtained for

$$\bar{h} = 26.9650. \tag{4.9}$$

The parameter $\bar{h}$ has been found by search, and it is worth noting that this value does not correspond to the maximal value and one can improve this result by choosing adequately the parameters K_o and K_{d_o}. Based on the results of the previous theorem, we conclude that the system under study in this example is robustly stable for all admissible uncertainties.

5. Conclusion

This paper has dealt with a class of dynamical linear uncertain systems with multiple time-varying delays in the system state. Delay-dependent sufficient conditions have been developed to check if a system of this class of systems is stable and/or stabilizable. A memoryless state feedback controller with consequent parameters has been used to stabilize the system. The LMI technique is used in all the development.

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D. Mehdi: Laboratoire d'Automatique et d'Informatique Industielle LAII-ESIP, Université de Poitiers, 40 avenue du Recteur Pineau, 86022 Poitiers Cedex, France

E-mail address: mehdi@esip.univ-poitiers.fr

E. K. Boukas: Département de Génie Mecanique, École Polytechnique de Montréal, C.P. 6079, Succursale Centre-Ville, Montréal, Québec, Canada H3C 3A7

E-mail address: el-kebir.boukas@polymtl.ca