Titre: Title:	Survival maximization for a Laguerre population
Auteurs: Authors:	Mario Lefebvre
Date:	2002
Type:	Article de revue / Article
Référence: Citation:	Lefebvre, M. (2002). Survival maximization for a Laguerre population. Mathematical Problems in Engineering, 8(6), 563-574. https://doi.org/10.1080/10241230210000061944

URL de PolyPublie: PolyPublie URL:	https://publications.polymtl.ca/3379/
Version:	Version officielle de l'éditeur / Published version Révisé par les pairs / Refereed
Conditions d'utilisation: Terms of Use:	CC BY

Document publié chez l'éditeur officiel
Document issued by the official publisher

Titre de la revue: Journal Title:	Mathematical Problems in Engineering (vol. 8, no. 6)
Maison d'édition: Publisher:	Hindawi
URL officiel: Official URL:	https://doi.org/10.1080/10241230210000061944
Mention légale: Legal notice:	

Mathematical Problems in Engineering, 2002, Vol. 8(6), pp. 563–574 Taylor & Francis
Taylor & Francis Group

Survival Maximization for a Laguerre Population

MARIO LEFEBVRE*

Département de mathématiques et de génie industriel, École Polytechnique de Montréal, C.P. 6079, Succursale Centre-ville, Montréal, Québec, Canada H3C 3A7

(Received 10 December 2001)

A population whose evolution is approximately described by a Laguerre diffusion process is considered. Let $Y(t)$ be the number of individuals alive at time t and $T(y, t_0)$ be the first time $Y(t)$ is equal to either 0 or $d(> 0)$, given that $Y(t_0) = y$ is in $(0, d)$. The aim is to minimize the expected value of a cost criterion in which the final cost is equal to 0 if $Y(T) = d$ and to ∞ if $Y(T) = 0$. The case when the final cost is 0 (respectively ∞) if T is greater than or equal to (resp. less than) a fixed constant s is also solved explicitly. In both cases, the risk sensitivity of the optimizer is taken into account.

Key words: Brownian motion; Diffusion processes; Stochastic control; Risk sensitivity; Hitting time; Stochastic differential equation

2000 Mathematics Subject Classification: Primary: 60J70; Secondary: 93E20

1 BIOLOGICAL BACKGROUND AND INTRODUCTION

1.1 Biological Background [See Ref. 1, pp. 176–182]

Assume that a population is composed of A-type and a-type individuals and that the size of the population is a constant N. Assume also that there are currently i A-type individuals and that before maturity *mutation* converts an A-type (respectively a-type) individual to an a-type (resp. A-type) with probability p_{Aa} (resp. p_{aA}). Then the expected proportion of mature A-type individuals before reproduction is given by

$$p_i = \frac{i(1 - p_{Aa}) + (N - i)p_{aA}}{N}.$$

Next, let X be the number of A-type individuals in the next generation; the *Wright-Fisher model* postulates that

$$P[X = j] = C_j^N p_i^j (1 - p_i)^{N-j}.$$

That is, it assumes that the probability that any individual will produce an A-type individual is equal to p_i, independently for each of the N individuals in the population. Therefore, the

* Tel.: 514-340-4711 ext. 4947; Fax: 514-340-4463; E-mail: mlefebvre@polymtl.ca

ISSN 1024-123X print; ISSN 1563-5147 online

population process evolves as a Markov chain. For N large, we can show that this process can be approximated by a diffusion process. In particular, if we set

$$p_{Aa} = \frac{\gamma_1}{N^d} \quad \text{and} \quad p_{aA} = \frac{\gamma_2}{N},$$

where $\gamma_i \in (0, \infty)$ for $i = 1, 2$ and $0 < d < 1$, then the limiting process (as N tends to ∞) is a diffusion process $\{X(t), t \geq 0\}$ with infinitesimal parameters

$$\mu_X(x) = \gamma_2 - \gamma_1 x \quad \text{and} \quad \sigma_X^2 = x$$

which is sometimes called a *Laguerre* diffusion process. This process is also used in branching theory. Notice that the orders of mutation rates are assumed to be different.

The model corresponds to the case when the number of A-type individuals is of the order N^d. The state space is the interval $[0, \infty)$ and is interpreted as follows: if $X(t)$ goes to infinity, then the number of A-types becomes larger than order N^d, whereas if $X(t)$ reaches zero, then this number becomes smaller than order N^d. Therefore, the survival of the A-type individuals in the population is endangered if d is small and $X(t)$ reaches zero.

1.2 Introduction

In this paper, we consider the problem of optimally controlling a diffusion process $\{X(t), t \geq 0\}$ with infinitesimal parameters

$$\mu_X(x) = \alpha + \beta x \quad \text{and} \quad \sigma_X^2 = \sigma^2 x,$$

where $\sigma > 0$ and where α and β can take any real values. Hence, the Laguerre process is the special case when $\sigma = 1, \alpha > 0$ and $\beta < 0$.

More precisely, we consider the controlled diffusion process $\{Y(t), t \geq 0\}$ defined by the stochastic differential equation

$$\mathrm{d}Y(t) = (\alpha + \beta Y(t))\mathrm{d}t + b[Y(t)]u(t)\mathrm{d}t + \sigma Y^{1/2}(t)\mathrm{d}W(t), \tag{1}$$

where $\{W(t), t \geq 0\}$ is a standard Brownian motion and the controlled variable $u(t)$ must be chosen so as to minimize the cost criterion

$$C(\theta, y) := -\frac{1}{\theta} \ln E[e^{-\theta J(y)}] \tag{2}$$

in which

$$J(y) := \int_{t_0}^{T(y)} \frac{1}{2} u^2(t)\, \mathrm{d}t + K[T, Y(T)], \tag{3}$$

where

$$T(y, t_0) := \inf\{t > t_0 \colon Y(t) = 0 \text{ or } d\ (>0) | Y(t_0) = y \in (0, d)\} \tag{4}$$

and $K(\cdot, \cdot)$ is a general termination cost function. Furthermore, in (2) θ is a parameter that takes the risk sensitivity of the optimizer into account [see Ref. 2, p. 4]. If θ is positive (respectively negative), the optimizer is said to be risk-seeking (resp. risk-averse), whereas we have:

$$\lim_{\theta \to 0} C(\theta, y) = E[J(y)],$$

which is the risk-neutral case.

Our objective is to maximize the survival time of the population (of A-type individuals), so that we would like $Y(T)$ to take on the value d (and *not* 0). To do so, we will consider two special cases for the termination cost function K. In Section 2, we will take

$$K_1[T, Y(T)] = \begin{cases} 0 & \text{if } Y(T) = d, \\ \infty & \text{if } Y(T) = 0. \end{cases} \tag{5}$$

Actually, we could be less extreme and set

$$K_1[T, Y(T)] = \begin{cases} c_1 & \text{if } Y(T) = d, \\ c_2 & \text{if } Y(T) = 0, \end{cases}$$

with $c_2 \gg c_1$. However, here we prefer to treat the limiting case above.

Next, in Section 3, we will let d tend to ∞ and choose

$$K_2[T, Y(T)] = \begin{cases} 0 & \text{if } T \geq s, \\ \infty & \text{if } T < s, \end{cases} \tag{6}$$

where s is a fixed constant. That is, we will force the process $\{Y(t), t \geq 0\}$ to remain above 0 at least until time s. Here again we could replace 0 and ∞ by c_1 and c_2 respectively.

Finally, some concluding remarks will be made in Section 4.

2 FORCING THE POPULATION TO REACH *d* BEFORE 0

First we prove the following lemma.

LEMMA 2.1 *For the diffusion process $\{X(t), t \geq 0\}$ with infinitesimal parameters $\mu_X(x) = \alpha + \beta x$ and $\sigma_X^2(x) = \sigma^2 x$, the origin is*

$$\begin{cases} \textit{an exit boundary} & \textit{if } \alpha \leq 0, \\ \textit{a regular boundary} & \textit{if } 0 < \alpha < \dfrac{\sigma^2}{2}, \\ \textit{an entrance boundary} & \textit{if } \alpha \geq \dfrac{\sigma^2}{2}. \end{cases}$$

Proof The results above follow from the formulae [see Ref. 3, p. 279]

$$\sigma_1 := \int_0^b \int_y^b \frac{2}{\sigma^2 x} \left(\frac{x}{y}\right)^{2\alpha/\sigma^2} \exp\left[\frac{2\beta(x-y)}{\sigma^2}\right] \mathrm{d}x\, \mathrm{d}y = \infty \Leftrightarrow \alpha \geq \frac{\sigma^2}{2}$$

and

$$\mu_1 := \int_0^b \int_y^b \frac{2}{\sigma^2 y} \left(\frac{y}{x}\right)^{2\alpha/\sigma^2} \exp\left[\frac{2\beta(y-x)}{\sigma^2}\right] \mathrm{d}x\, \mathrm{d}y = \infty \Leftrightarrow \alpha \leq 0.$$

Remarks

(1) A boundary $x = b$ is said to be *accessible* if the probability of reaching it in finite time is strictly positive. Both exit and regular boundaries are accessible, whereas entrance (and natural) boundaries are *inaccessible*. Furthermore, a boundary is accessible if and only if the quantity σ_1 in the proof above is finite. Then, the boundary is *regular* if and only if $\mu_1 < \infty$.
(2) Notice that the type of boundary at the origin does not depend on the constant β.

Now, if the constant α in the stochastic differential equation (1) is greater than or equal to $\sigma^2/2$, we can write that the optimal control is $u^* \equiv 0$ because the controlled process $\{Y(t), t \geq 0\}$ is then identical to the uncontrolled diffusion process $\{X(t), t \geq 0\}$ and, according to the previous lemma, it cannot reach the origin. Hence, we can write that $C(\theta, y) \equiv 0$ if $u(t) \equiv 0$. Since $C(\theta, y)$ cannot be negative, we have indeed $u^* \equiv 0$. Therefore, we will assume from now on that $\alpha < \sigma^2/2$.

When we choose the termination cost function defined in (5), the optimal control problem is time-invariant, so that we can assume that $t_0 = 0$. Let $F(y)$ be the minimum expected cost incurred from the initial state $y\,(=y(0))$. That is,

$$F(y) := \inf_{u(t), 0 \leq t \leq T(y)} C(\theta, y).$$

Assuming that $F(y)$ exists and is twice differentiable, we can show that it satisfies the dynamic programming equation

$$\inf_u \left[\frac{1}{2}u^2 + (\alpha + \beta y + b(y)u)F'(y) - \theta\frac{\sigma^2}{2}y[F'(y)]^2 + \frac{\sigma^2}{2}yF''(y)\right] = 0,$$

where $u = u(0)$. This non-linear ordinary differential equation is valid for $0 < y < d$; the boundary condition is

$$F(y) = K_1(y) \quad \text{if } y = 0 \text{ or } d.$$

The minimizing u is given by

$$u^* = -b(y)F'(y), \tag{7}$$

so that

$$(\alpha + \beta y)F'(y) - \frac{1}{2}b^2(y)[F'(y)]^2 - \theta\frac{\sigma^2}{2}y[F'(y)]^2 + \frac{\sigma^2}{2}yF''(y) = 0. \tag{8}$$

PROPOSITION 2.1 *If $\alpha < \sigma^2/2$ and $b[Y(t)]$ is equal to $Y^{1/2}(t)$, then the optimal control u^* is given by*

$$u^* = \frac{y^{1/2}}{\gamma}\frac{y^{-2\alpha/\sigma^2}e^{-2\beta y/\sigma^2}}{\int_0^y z^{-2\alpha/\sigma^2}e^{-2\beta z/\sigma^2}\,dz}$$

for $0 < y < d$, where

$$\gamma = \frac{1}{\sigma^2} + \theta$$

is assumed to be positive.

Proof Notice that we have:

$$\gamma\sigma^2 y = b^2(y) + \theta\sigma^2 y \tag{9}$$

when $b(y) = y^{1/2}$. Then the transformation [see Ref. 2, p. 223]

$$G(y) = e^{-\gamma F(y)} \tag{10}$$

linearizes the ordinary differential equation (8) to

$$(\alpha + \beta y)G'(y) + \frac{1}{2}\sigma^2 y G''(y) = 0 \tag{11}$$

for $0 < y < d$. The boundary condition is

$$G(y) = e^{-\gamma K_1(y)} = \begin{cases} 1 & \text{if } y = d, \\ 0 & \text{if } y = 0. \end{cases} \tag{12}$$

Remark The function $G(y)$ can be interpreted as

$$G(y) = E[e^{-\gamma K_1[X(T)]}|X(0) = y] = P[X(T) = d|X(0) = y],$$

where $\{X(t), t \geq 0\}$ is the *uncontrolled* process corresponding to $\{Y(t), t \geq 0\}$ [see Ref. 2, p. 224]. This probability does indeed satisfy the linear ordinary differential equation (11), subject to the boundary condition (12) [see Ref. 4, p. 231].

Solving equation (11) subject to (12) is an easy task. We find that

$$G(y) = \frac{\int_0^y z^{-2\alpha/\sigma^2} e^{-2\beta z/\sigma^2}\,\mathrm{d}z}{\int_0^d z^{-2\alpha/\sigma^2} e^{-2\beta z/\sigma^2}\,\mathrm{d}z} \tag{13}$$

for $0 \leq y \leq d$. The optimal solution u^* is then deduced from (10) and (7).

Remarks

(1) Suppose that the cost function $J(y)$ defined in (3) is generalized to

$$J(y) = \int_{t_0}^{T(y)} \frac{1}{2} q[Y(t)]u^2(t)\,\mathrm{d}t + K[T, Y(T)],$$

where $q(\cdot)$ is positive. Then the transformation used to linearize the ordinary differential equation (8) will work if there exists a constant γ such that

$$\gamma\sigma^2 y = \frac{b^2(y)}{q(y)} + \theta\sigma^2 y.$$

For instance, the problem has the same optimal solution if $b[Y(t)] \equiv 1$ and $q[Y(t)] = 1/Y(t)$. These choices for the functions b and q imply that the smaller the value of $Y(t)$ is, the more expensive it is to control the process.

(2) If the risk parameter θ is equal to $-1/\sigma^2$, then the ordinary differential equation (8) becomes

$$(\alpha + \beta y)F'(y) + \frac{\sigma^2}{2} yF''(y) = 0,$$

which is the same differential equation as $G(y)$ satisfies. However, the boundary condition is

$$F(y) = \begin{cases} 0 & \text{if } y = d, \\ \infty & \text{if } y = 0. \end{cases} \tag{14}$$

Therefore, we can conclude that the function $F(y)$ does not exist when $\theta = -1/\sigma^2$. Actually, $F(y)$ does not exist if $\theta \leq -1/\sigma^2$. This can be deduced from the probabilistic interpretation

$$e^{-\gamma F(y)} = G(y) = P[X(T) = d | X(0) = y] \in (0, 1) \quad \text{for } 0 < y < d$$

given above. Since $F(y)$ is non-negative, the constant γ cannot be negative.

Going back to the interpretation of the risk parameter θ given in Section 1, we can state that the optimizer must not be too risk-averse or pessimistic. On the other hand, we can allow the optimizer to be as risk-seeking or optimistic as we want. If θ increases, then the optimal control u^* decreases. Hence, a very risk-seeking optimizer is willing to take the risk of not using much control. If the controlled process $\{Y(t), t \geq 0\}$ hits the boundary d first, then the total cost is small. In the case of a risk-averse optimizer, he/she is afraid of receiving an infinite penalty for finishing at $y = 0$. Therefore, he/she is ready to use as large a control as needed (u^* tends to ∞ if θ decreases to $-1/\sigma^2$) to avoid this penalty.

(3) If $\beta \neq 0$, the definite integral that appears in the function $G(y)$ and/or u^* can be expressed in terms of the incomplete gamma function.

Particular Cases

(1) If $\alpha = \beta = 0$, we have:

$$G(y) = \frac{y}{d} \quad \text{and} \quad u^* = \frac{1}{\gamma y^{1/2}} \quad \text{for } 0 < y < d.$$

That is, $u^*(t) = [\gamma Y^{1/2}(t)]^{-1}$ and it follows that the optimally controlled process $\{Y^*(t), t \geq 0\}$ obeys the stochastic differential equation

$$\mathrm{d}Y^*(t) = \gamma^{-1}\mathrm{d}t + \sigma\{(Y^*)^{1/2}(t)\}\mathrm{d}W(t).$$

According to Lemma 2.1, this process cannot hit the origin if

$$\frac{1}{\gamma} \geq \frac{\sigma^2}{2} \Leftrightarrow \gamma \leq \frac{2}{\sigma^2}.$$

Thus if $(-\sigma^2 <)\, \theta \leq \sigma^{-2}$, the optimal control assures the optimizer not to receive the infinite penalty incurred for finishing at $y = 0$. When the optimizer is more risk-seeking, he/she is willing to take the risk of receiving this infinite penalty. Using (13) with $\alpha = 1/\gamma$ and $\beta = 0$, we find that this risk is given by

$$1 - G(y) = 1 - \left(\frac{y}{d}\right)^{1-2/(\gamma\sigma^2)}$$