*Mathematical Problems in Engineering*
Volume 4, pp. 269–288
Reprints available directly from the publisher
Photocopying permitted by license only



# Design of Robust Controller for Linear Systems with Markovian Jumping Parameters

K. BENJELLOUN [a], E.K. BOUKAS [a,*], O.L.V. COSTA [b,†] and P. SHI [c,‡]

[a] *Mechanical Engineering Department, École Polytechnique de Montréal, P. O. Box 6079, Station "Centre-Ville", Montréal, Que., Canada H3C 3A7;* [b] *University of Sao Paulo, Department of Electronic Engineering, 05508-900 Sao Paulo, SP, Brazil;* [c] *Centre for Industrial and Applicable Mathematics, The University of South Australia, SA 5095, Australia*

*(Received 15 May 1997; Revised 11 August 1997)*

This paper deals with the robustness of the class of uncertain linear systems with Markovian jumping parameters (ULSMJP). The uncertainty is taken to be time-varying norm bounded. Under the assumptions of the boundedness of the uncertainties and the complete access to the system's state and its modes, a sufficient condition for stochastic stabilizability of this class of systems is established. An example is provided to demonstrate the usefulness of the proposed theoretical results.

*Keywords:* Linear systems with Markovian jumping parameters; Stochastic stability; Markov process; Norm bounded uncertainties

## 1 INTRODUCTION

This paper deals with the class of systems with Markovian jumping parameters. In the last two decades this class of systems has been

* Corresponding author. Fax: 514-340-5867. E-mail: boukas@anas.meca.polymtl.ca.
Research of this author was supported by the Natural Sciences and Engineering Research Council of Canada under grants OGP0036444.

† Research of this author was supported by CNPq (Brazilian National Research Council), FAPESP (Research Council of the state of Sao Paulo).
E-mail: oswaldo@lac.usp.br.

‡ Research of this author was supported by the Australian Research Council under Grant A4932206. E-mail: matsp@zarniwoop.levels.unisa.edu.au.

extensively studied. Theoretical and practical achievements have been reported in the literature. Without any intention of being exhaustive here, we mention [2–4,7–11,15–18,24,26–29]. Other references dealing with this class of systems can be found in the quoted references.

Most of the quoted references, except [5,6,12], have considered certain systems with Markovian jumping parameters. Nowadays, it is well known that these nominal systems may not describe the real systems appropriately and the uncertainties have to be considered if at least the robustness stability is required. The readers are referred to [14] for the different possibilities to model uncertainties.

In [5,6], the uncertain linear and nonlinear piecewise deterministic systems have been investigated. Under some matching conditions, with a state feedback control law, sufficient conditions, which guarantee the stochastic stabilizability robustness of this class of systems have been established. In the work of De souza and Fragoso [12], the disturbance rejection problem has been studied for finite and infinite horizons.

For the deterministic class of linear systems, different approaches to design robust linear controllers have been reported in the literature. Most of these approaches are based on the Lyapunov equation or Riccati equation (see [20,21]). These approaches which are proposed for the design of robust controller for the deterministic class of systems, however, cannot be applied to the class of systems under consideration in this paper. The goal of this paper is to study the stabilizability robustness of the class of uncertain linear systems with Markovian jumping parameters. A feedback control which will assure a good tracking of a given reference and attenuate the effect of the time-varying norm bounded uncertainties is designed. Mainly, we establish a sufficient condition which guarantees the stochastic stability robustness of the class of systems with Markovian jumping parameters under norm bounded uncertainties and we will give an iterative algorithm to determine the parameters of the proposed controller. The paper is organized as follows: in Section 2, we give a brief description of the class of linear systems with Markovian jumping parameters (LSMJP) and recall the definition of the stochastic stabilizability of this class of systems. In Section 3, we establish the sufficient condition for the robust stabilizability of the linear systems with Markovian jumping parameters under some time-varying norm bounded uncertainties.

Section 4 presents a numerical example and the paper is concluded in Section 5 with some final comments.

## 2 PROBLEM STATEMENT

Before presenting the class of linear systems with Markovian jumping parameters let us consider some examples of this class of systems. Consider an industrial application where we control a dc servomotor driving a given mechanism. The objective is to assure the control of the mechanism's position (linear or angular). Let us assume that the sense of rotation of the actuator is random, which corresponds to a random position of the mechanism, and it is described by a stochastic continuous Markov process with discrete finite state space (two states). Consider also that the complete system (dc servomotor and the mechanism) has two different models, one for each sense of rotation. Let us also assume that in each sense of rotation the system can be described by a linear model with some bounded uncertainties. The problem now is how we can control the system to assure the required performances and eventually assure the robustness of the stability of the closed-loop.

As a second example, let us consider a dynamical system subject to random failures and repairs, so that at each moment it can be in one of the following states: *good functioning*, *intermediate functioning*, *bad functioning* and *failure* for many reasons. The failure mode is not interesting to us since the system cannot be operated. For the other modes, the performances of the system will be different in each mode and the ones obtained in the good functioning mode are certainly better than the ones of the intermediate mode etc. The mode of the system will evolve in a random way between these states. The question which arises is what we can do to assure that the performances of the system will remain the same or at least how we can assure the robustness of the stability of the closed-loop.

These examples fit in the formalism of the class of systems with Markovian jumping parameters which represents a rich class of systems. Other examples of this class of systems can be found in different areas such as manufacturing systems, economics, etc.; see, for example [25].

### 2.1 Linear Systems with Markovian Jumping Parameters

To describe the class of linear systems with Markovian jumping parameters, let us fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. This class of systems owns a hybrid state vector. The first component vector is continuous, the second one is discrete and is always referred to as the mode. Let $\mathbf{x}(t) \in \mathbb{R}^{n_1}$ be the continuous component state vector at time $t$, $\mathbf{u}(t) \in \mathbb{R}^m$ be the control vector at time $t$, $\mathbf{y}(t) \in \mathbb{R}^p$ be the system's controlled output, and $\{\mathbf{r}(t),\ t \in [0, T]\}$ be the homogeneous Markov process with right continuous trajectories and taking values on the finite set $\mathcal{B} = \{0, 1, \ldots, N\}$. We assume also that $\mathbf{P}_t := (P_t^1, P_t^2, \ldots, P_t^N)$, with $P_t^i := Prob(\mathbf{r}(t) = i)$; $i = 1, 2, \ldots, N$ satisfies the forward Kolmogorov equation, i.e.

$$\frac{\mathrm{d}\mathbf{P}_t}{\mathrm{d}t} = \Lambda \mathbf{P}_t, \quad 0 \leq t \leq T, \tag{1}$$

$$\mathbf{P}_0 = \bar{\mathbf{P}}, \tag{2}$$

where $\bar{\mathbf{P}}$ is the initial vector probability of the process $\{\mathbf{r}(t), t \in [0, T]\}$ and $\Lambda := [q_{ij}]$ is the stationary transition rate matrix of the process $\{\mathbf{r}(t), t \in [0, T]\}$ with $q_{ij} \geq 0$, and $q_{ii} = -\sum_{j=1,\, j \neq i} q_{ij}$ (see [13]).

The class of linear systems with Markovian jumping parameters is described by the following dynamics:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{r}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{r}(t))\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{3}$$

with controlled output

$$\mathbf{y}(t) = \mathbf{C}(\mathbf{r}(t))\mathbf{x}(t) + \mathbf{D}(\mathbf{r}(t))\mathbf{u}(t), \tag{4}$$

where for each $\mathbf{r}(t) = i$, $\mathbf{A}(i)$, $\mathbf{B}(i)$, $\mathbf{C}(i)$ and $\mathbf{D}(i)$ are constant matrices with appropriate dimensions.

For the stability of this class of systems, there exist many definitions in the literature (see [15,25] and references therein). In the rest of this paper, we will use the following one.

Let $\mathbf{x}(t, \mathbf{x}_0, i)$ represent the corresponding solution of system (3) at time $t$ when the control $\mathbf{u}(\cdot) = 0$ is used and the initial conditions are respectively $\mathbf{x}_0$ and $i$.

DEFINITION 2.1 [17] For system (3)–(4), the equilibrium point 0 is stochastically stable, if for every initial state $(\mathbf{x}_0, \mathbf{r}_0)$ the following holds:

$$\mathbf{E}\left\{\int_0^\infty \|\mathbf{x}(t, \mathbf{x}_0, \mathbf{r}_0)\|^2 \mathrm{d}t \mid \mathbf{x}_0, \mathbf{r}_0\right\} < \infty. \tag{5}$$

Let $\mathbf{x}(t, \mathbf{x}_0, i, \mathbf{u})$ represent the corresponding solution of system (3) at time $t$ when the control $\mathbf{u}(\cdot)$ is used and the initial conditions are respectively $\mathbf{x}_0$ and $i$.

DEFINITION 2.2 System (3)–(4) is said to be stochastically stabilizable if, for all finite $\mathbf{x}_0 \in \mathbb{R}^{n_1}$ and $i \in \mathcal{B}$, there exists a state feedback control, $\mathbf{u}(\cdot)$, such that

$$\lim_{T\to\infty} E_{\mathbf{u}(\cdot)}\left\{\int_0^T \mathbf{x}'(t, \mathbf{x}_0, i, \mathbf{u})\mathbf{x}(t, \mathbf{x}_0, i, \mathbf{u})\,\mathrm{d}t \mid \mathbf{x}_0,\ \mathbf{r}(0) = \mathbf{r}_0\right\} \le \mathbf{x}'_0\tilde{\mathbf{P}}\mathbf{x}_0, \tag{6}$$

where $\tilde{\mathbf{P}}$ is a symmetric positive-definite matrix.

*Remark 2.1* Notice that the upper bound in this definition depends on the initial conditions of the system.

The following theorem was established by Ji and Chizeck [17]. It states the necessary and sufficient conditions for stochastic stabilizability of this class of systems. For the proof, the reader is referred to this reference.

THEOREM 2.1 [17] *The system above is stochastically stabilizable if and only if there exists a control law* $\mathbf{u}(t) = -\mathbf{K}(\mathbf{r}(t))\mathbf{x}(t)$ *such that, for any given positive-definite and symmetric matrix* $\mathbf{Q}(i)$, *the unique set of solutions*, $\mathbf{P}(i)$, *of the N coupled equations*

$$\begin{aligned}&[\mathbf{A}(i) - \mathbf{B}(i)\mathbf{K}(i)]'\mathbf{P}(i) + \mathbf{P}(i)[\mathbf{A}(i) - \mathbf{B}(i)\mathbf{K}(i)] + \sum_{j\in\mathcal{B}} q_{ij}\mathbf{P}(j)\\ &\quad = -\mathbf{Q}(i), \quad \forall i \in \mathcal{B},\end{aligned} \tag{7}$$

*are positive-definite symmetric.*

*Proof* See [17].

In the standard formulation of the control problem, we often use a nominal model of the system for the design procedure. But in real life, the matrices $\mathbf{A}(i)$, $\mathbf{B}(i)$, $\mathbf{C}(i)$ and $\mathbf{D}(i)$ for each $\mathbf{r}(t)=i$ with value in $\mathcal{B}$ are not precisely known for many reasons, well known by the control community, and we always retrieve a discrepancy between the used nominal model and the real process. These uncertainties, which can be divided into two categories, i.e., matched and mismatched uncertainties, can make the feedback controller inefficient and in worst case, the real system can become unstable. In order to avoid these problems, when designing the controller, we need to take into account the system's uncertainties. From the practical point of view, the purpose of each used controller consists in assuring the asymptotic tracking and the disturbance rejection. In the next section, we will propose a control law which guarantees simultaneously the asymptotic tracking and the disturbance rejection for the class of systems under study.

*Remark 2.2* [17] The autonomous linear system with Markovian jumping parameters is said to be stable if and only if for any given positive-definite and symmetric matrix $\mathbf{Q}(i)$ the unique set of solutions, $\mathbf{P}(i)$, of the $N$ coupled equations:

$$\mathbf{A}'(i)\mathbf{P}(i)+\mathbf{P}(i)\mathbf{A}(i)+\sum_{j\in\mathcal{B}} q_{ij}\mathbf{P}(j)=-\mathbf{Q}(i), \quad \forall i\in\mathcal{B} \tag{8}$$

are positive-definite and symmetric.

*Remark 2.3* If for some mode $i$ (for any $i\in\mathcal{B}$) the jump rate $q_{ij}$ is 0, the system will remain at this mode for ever. The mode is called *absorbing mode* and the robustness condition is similar to the one used in the deterministic case.

In this paper, the adopted Euclidean norm of vector $\mathbf{x}$, denoted by $\|\mathbf{x}\|$ is $\|\mathbf{x}\|=(x_1^2+x_2^2+\cdots+x_n^2)^{1/2}$ where $x_i$, for $i=1,\ldots,n$, denotes the $i$th element of the vector $\mathbf{x}$. The induced Euclidean norm of matrix $\|\mathbf{M}\|$ is given by $\|\mathbf{M}\|=[\lambda_{\max}(\mathbf{M}'\mathbf{M})]^{1/2}$, where $\mathbf{M}'$ denotes the transpose of matrix $\mathbf{M}$ and $\lambda_{\min}(\mathbf{M})$ and $\lambda_{\max}(\mathbf{M})$ denote respectively the minimum eigenvalue and the maximum eigenvalue of matrix $(\mathbf{M})$.

In the remainder of this paper, we assume that the system has the same dimension at each mode, and that the mode $\mathbf{r}(t)$ and the continuous state $\mathbf{x}(t)$ are available for controller at each time $t$.