POLYPUBLIE

Stochastic stability of linear time-delay system with Markovian jumping parameters

Khalid Benjelloun, El-Kébir Boukas

Document en libre accès dans PolyPublie
Open Access document in PolyPublie

Date	1997
Type	Article de revue / Article
Version	Version officielle de l'éditeur / Published version
URL de PolyPublie PolyPublie URL	https://publications.polymtl.ca/3374/
Évaluation par les pairs Peer reviewed	Révisé par les pairs / Refereed
Conditions d'utilisation Terms of use	Creative Commons Attribution 4.0 International (CC BY)
Référence Citation	Benjelloun, K., & Boukas, E.-K. (1997). Stochastic stability of linear time-delay system with Markovian jumping parameters. Mathematical Problems in Engineering, 3(3), 187-201. https://doi.org/10.1155/s1024123x97000525

Publication officielle
Official publication

Titre de la revue Journal title	Mathematical Problems in Engineering (vol. 3, no. 3)
Maison d'édition Publisher	Hindawi
URL officielle Official URL	https://doi.org/10.1155/s1024123x97000525
Mention légale Legal notice	

Ce fichier a été téléchargé à partir de PolyPublie, le dépôt institutionnel de Polytechnique Montréal
This file has been downloaded from PolyPublie, the institutional repository of Polytechnique Montréal
https://publications.polymtl.ca

Mathematical Problems in Engineering
Volume 3, pp. 187–201
Reprints available directly from the publisher
Photocopying permitted by license only

© 1997 OPA (Overseas Publishers Association)
Amsterdam B.V. Published in The Netherlands
under license by Gordon and Breach Science Publishers
Printed in Malaysia

Stochastic Stability of Linear Time-Delay System with Markovian Jumping Parameters

K. BENJELLOUN and E. K. BOUKAS*,†

Mechanical Engineering Department, École Polytechnique de Montréal, P.O. Box 6079, Station "centre-ville", Montréal, Québec, Canada, H3C 3A7

(Received 12 February 1996; Revised 30 April 1996)

This paper deals with the class of linear time-delay systems with Markovian jumping parameters (LTDSMJP). We mainly extend the stability results of the deterministic class of linear systems with time-delay to this class of systems. A delay-independent necessary condition and sufficient conditions for checking the stochastic stability are established. A sufficient condition is also given. Some numerical examples are provided to show the usefulness of the proposed theoretical results.

Keywords: stochastic stability; Markov process; linear systems with jumping parameters; time-delay

Classification Categories: 43E15

1 INTRODUCTION

This paper deals with stochastic stability of the class of linear time-delay systems with Markovian jumping parameters. The class of linear systems with Markovian jumping parameters (LSMJP) is a hybrid class of systems with two components in the vector state. The first component refers to the

*Corresponding author. E-mail: Boukas@anas.meca.polymtl.ca.

†Research of this author was supported by the Natural Sciences and Engineering Research Council of Canada under grants OGP0036444.

mode and the second one to the state. The mode is described by a continuous Markov process with finite state space. The state in each mode is represented by a differential equation.

This class of systems have received the attention of many researchers from both categories theoretical and practical. Many papers have been reported in the literature. Among these papers, we can quote Kats and Krasovskii [1], Sworder [2], Mariton and Bertrand [3], Ji and Chizeck [4], [5], [6], Kushner [7], Mariton [8], Feng et al [9], Boukas [10] and the references therein. In these cited papers the authors covered a large variety of problems such as stochastic stability, stochastic controllability, stochastic observability, optimal control, etc.

For the class of linear time-delay systems with Markovian jumping parameters, no work have been reported to the literature and the established results regarding the deterministic linear time-delay systems have not been extended yet to this class. For the deterministic linear time-delay systems there exists many papers in the literature. We quote Thowsen [11], Cheres [12], Ikeda [13], Lewis [14], Brierley [15], Mori [16] and the references therein.

The aim of this paper is to extend some results on the stability of the class of linear time-delay systems to our class of systems. We mainly establish a delay-independent sufficient condition to check the stochastic stability of LTDSMJP. A such condition is easy to check. A necessary condition which is easy to test is also established.

The paper is organized as follows: In section 2, the stability problem is stated for the class of systems under consideration. In section 3, a sufficient condition for stochastic stability of this class is established. Other conditions, (necessary, sufficient), are also established. In section 4, some examples are presented to show the usefulness of the proposed theoretical results.

2 PROBLEM STATEMENT

In this paper, we consider the class of linear time-delay systems with Markovian jumping parameters where the dynamics is described by the following systems of differential equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0(\mathbf{r}(t))\mathbf{x}(t) + \sum_{q=1}^{N} \mathbf{A}_q(\mathbf{r}(t))\mathbf{x}(t - \tau_q) \tag{1}$$

$$\mathbf{x}(t) = \Psi(t), t \in [t_0 - \tau_m, t_0] \tag{2}$$

where t_0 is the initial time (in the rest of the paper we will take $t_0 = 0$) and $\tau_1, \tau_2, \ldots, \tau_N$ are the system's constant delays and $\tau_m = \max_{q=1,\ldots,N}\{\tau_q\}$, $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector at time t, $\Psi(t) \in \mathcal{C}([-\tau_m, 0] \to \mathbb{R}^n)$ is a continuous vector-valued initial function, ($\mathcal{C}([-\tau_m, 0] \to \mathbb{R}^n)$ is the space of continuous functions mapping $[-\tau_m, 0]$ into $\mathbb{R}^n$) and $\mathbf{A}_0(\mathbf{r}(t)) \in \mathbb{R}^{n\times n}$ and $\mathbf{A}_q(\mathbf{r}(t)) \in \mathbb{R}^{n\times n}$, $q = 1, \ldots, N$ are matrices with real values. These matrices are functions of the random parameter $\mathbf{r}(t)$.

The parameter $\mathbf{r}(t)$ in Eq. (1) represents a continuous-time discrete-state Markov process with values in a finite set $\mathcal{S} = \{1, 2, \ldots, s\}$ with transition probability rate matrix $\Pi = [\pi_{ij}]_{i,j=1,s}$. The transition probability from mode i to mode j is given by:

$$P\{\mathbf{r}(t + \Delta) = j | \mathbf{r}(t) = i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta) & i \neq j \\ 1 + \pi_{ii}\Delta + o(\Delta) & i = j \end{cases} \tag{3}$$

where $\Delta > 0$ and $\lim_{\Delta\to 0} \dfrac{o(\Delta)}{\Delta}$.

In this Eq. (3), ($\pi_{ij} \geq 0$) is the transition probability rate from mode i to mode j ($i \neq j$) and

$$\sum_{j=1\, j\neq}^{s} \pi_{ij} = -\pi_{ii} = \pi_i \quad \text{for } i = 1, \ldots, s$$

Remarks 2.1

- The system described by Eqs. (1)–(3) is a hybrid system with state vector $(\mathbf{r}(t), \mathbf{x}(t))$;
- For each mode i, the system described by Eqs. (1)–(3) is linear in $\mathbf{x}(t)$, $\mathbf{x}(t - \tau_1), \ldots, \mathbf{x}(t - \tau_q)$.

To simplify the study of the class of systems under consideration, we will assume that the system delays are constant and not dependent on the stochastic process but depend on the system structure. Let the initial val-

ues $\{\Psi(t)\}_{-\tau_m \le t \le 0}$ and $\{\rho(t) = \mathbf{r}(t)\}_{-\tau_m \le t \le 0}$ be independent random functions and fixed. Furthermore, we assume that the system has the same dimension at each mode and the Markov process is irreducible.

The question we address in this paper is how to check the stability of this class of systems. In the rest of this paper, we will establish the necessary or sufficient condition for the stochastic stability, and some examples are given to illustrate our results.

When the mode of the system is i, i.e. $\mathbf{r}(t) = i$, we will use the notation $\mathbf{A}_{qi}$ to represent the matrix $\mathbf{A}_q(\mathbf{r}(t))$ with $q = 0, 1, \ldots, N$. $\|\mathbf{x}\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ (where x_i, for $i = 1, \ldots, n$, denotes the ith element of $\mathbf{x}(t)$) represents the Euclidean norm of vector $\mathbf{x}$. The induced Euclidean norm of matrix $\|\mathbf{M}\|$ is given by $\|\mathbf{M}\| = [\lambda_{max}(\mathbf{M}'\mathbf{M})]^{1/2}$, where $\mathbf{M}'$ denotes the transpose of matrix $\mathbf{M}$ and $\lambda_{min}(\mathbf{M})$ and $\lambda_{max}(\mathbf{M})$ denote respectively the minimum eigenvalue and the maximum eigenvalue of matrix $(\mathbf{M})$. $\mathbf{M} > 0$ means that the matrix $\mathbf{M}$ is positive-definite and $\mathbf{M}^{1/2}$ is the matrix square root.

3 STOCHASTIC STABILITY OF LTDSMJP

The concept of stochastic stability for linear and nonlinear systems was used by many authors. Among these authors, we quote the research of Kats and Krasovskii [1], Mariton [8], Ji and Chizeck [6] and the references therein. In this paper, we will use the following definition:

DEFINITION Let E $\{.\}$ be the expectation, $\mathbf{x}_0 = \mathbf{x}(0)$ be the initial state of the system and $\mathbf{x}(t, \Psi, \rho)$ be the corresponding solution of system (1)–(3) at time t when the initial conditions are respectively Ψ and ρ. The system (1)–(3) is stochastically stable if for all finite $\Psi(t) \in \mathcal{C}([-\tau_m, 0] \to \mathbb{R}^n)$, and $\rho([-\tau_m, 0] \to \mathcal{S}$, there exists a constant $\tilde{c}$ such that

$$\lim_{T \to \infty} E\left\{\int_0^T \mathbf{x}'(t, \Psi, \mathbf{r}_0)\mathbf{x}(t, \Psi, \rho)dt | \Psi\, \mathbf{r}_0\right\} \le \tilde{c}\|\mathbf{x}_0\|^2 \tag{4}$$

is satisfied.

The following assumption establishes a relation between the system state at time t and the one at time $t + \eta$, $\eta \in [-\tau_m, 0]$.

ASSUMPTION A *(Kolmanovskii [17]) For all* $\eta \in [-\tau_m, 0]$, *there exists a scalar* $h > 1$ *such that*

$$\|\mathbf{x}(t+\eta)\| \leq h\|\mathbf{x}(t)\| \tag{5}$$

where $\mathbf{x}(t)$ *is the state of system at time* t.

Based on the results of Ji and Chizeck [6] for the class of linear systems with Markovian jumps and the ones of Gregory and Nazaroff [18] for the class of linear systems with delay, we can establish the following delay-independent sufficient condition for stochastic stability:

THEOREM 1 *The system (1)–(3) is stochastically stable if for any given positive-definite symmetric matrix* $\mathbf{Q}_i$, $\forall i \in \mathcal{S}$ *and a constant positive-definite symmetric matrix* $\mathbf{Q}$, *the unique set of solutions,* $\mathbf{P}_i$, $\forall i \in \mathcal{S}$, *of the coupled system:*

$$\mathbf{A}'_{0i}\mathbf{P}_i + \mathbf{P}_i\mathbf{A}_{0i} + \mathbf{P}_i\left[\sum_{q=1}^{N} \mathbf{A}_{qi}\mathbf{Q}^{-1}\mathbf{A}'_{qi}\right]\mathbf{P}_i + \sum_{j=1}^{s} \pi_{ij}\mathbf{P}_j + N\mathbf{Q} = -\mathbf{Q}_i \tag{6}$$

is positive-definite symmetric.

Proof Let the mode at time t be i, i.e. $\mathbf{r}(t) = i \in \mathcal{S}$; then Eq. (1) becomes:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_{0i}\mathbf{x}(t) + \sum_{q=1}^{N} \mathbf{A}_{qi}\mathbf{x}(t-\tau_q) \tag{7}$$

Based on the results of Kushner [7] and Gregory and Nazaroff [18], we can choose the following stochastic Lyapunov functional $V(.)$: $\mathbb{R}^n \times \mathcal{S} \rightarrow \mathbf{R}_+$:

$$V(\mathbf{x}(t), \mathbf{r}(t) = i) = V(\mathbf{x}, i)$$

$$= \mathbf{x}'(t)\mathbf{P}_i\mathbf{x}(t) + \sum_{q=1}^{N} \int_{t-\tau_q}^{t} \mathbf{x}'(\eta)\mathbf{Q}\mathbf{x}(\eta)d\eta \tag{8}$$

The infinitesimal operator $\tilde{\mathcal{A}}$ of the random process $\{(\mathbf{x}(t), \mathbf{r}(t)), t \geq 0\}$ is given by:

$$\tilde{\mathscr{A}}V(\mathbf{x}(t), \mathbf{r}(t)) = \lim_{\Delta\to 0}\frac{1}{\Delta}[E\{V(\mathbf{x}(t+\Delta), \mathbf{r}(t+\Delta)|\mathbf{x}(t), \mathbf{r}(t) = i\} - V(\mathbf{x}(t), \mathbf{r}(t) = i)]$$

$$= \mathbf{x}'(t)\left[\mathbf{A}_{0i}'\mathbf{P}_i + \mathbf{P}_i\mathbf{A}_{0i} + \sum_{j=1}^{s}\pi_{ij}\mathbf{P}_j + N\mathbf{Q}\right]\mathbf{x}(t)$$

$$+2\sum_{q=1}^{N}\mathbf{x}'(t)\mathbf{P}_i\mathbf{A}_{qi}\mathbf{x}(t-\tau_q) - \sum_{q=1}^{N}\mathbf{x}'(t-\tau_q)\mathbf{Q}\mathbf{x}(t-\tau_q) \tag{9}$$

Based on Eq. (6), we obtain:

$$\tilde{\mathscr{A}}V(\mathbf{x}, i) = -\mathbf{x}'(t)\mathbf{Q}_i\mathbf{x}(t) - \mathbf{x}'(t)\mathbf{P}_i\left[\sum_{q=1}^{N}\mathbf{A}_{qi}\mathbf{Q}^{-1}\mathbf{A}_{qi}'\right]\mathbf{P}_i\mathbf{x}(t)$$

$$+2\sum_{q=1}^{N}\mathbf{x}'(t)\mathbf{P}_i\mathbf{A}_{qi}\mathbf{x}(t-\tau_q) - \sum_{q=1}^{N}\mathbf{x}'(t-\tau_q)\mathbf{Q}\mathbf{x}(t-\tau_q)$$

$$= -\sum_{q=1}^{N}\{(\mathbf{Q}^{1/2}\mathbf{x}(t-\tau_q) - \mathbf{Q}^{1/2}\mathbf{A}_{qi}'\mathbf{P}_i\mathbf{x}(t))'\{(\mathbf{Q}^{1/2}\mathbf{x}(t-\tau_q) - \mathbf{Q}^{-1/2}\mathbf{A}_{qi}'\mathbf{P}_i\mathbf{x}(t))\} - \mathbf{x}'(t)\mathbf{Q}_i\mathbf{x}(t)$$

Notice that $\{Q^{1/2}\mathbf{x}(t-\tau_q) - \mathbf{Q}^{-1/2}\mathbf{A}_{qi}'\mathbf{P}_i\mathbf{x}(t))'\{\mathbf{Q}^{1/2}\mathbf{x}(t-\tau_q) - \mathbf{Q}^{-1/2}\mathbf{A}_{qi}'\mathbf{P}_i\mathbf{x}(t)\}$ is a positive number. Then we have:

$$\tilde{\mathscr{A}}V(\mathbf{x}, i) \leq -\mathbf{x}'(t)\mathbf{Q}_i\mathbf{x}(t) \tag{10}$$

On the other side, for $\mathbf{x} \neq 0$ and for each mode i, we have:

$$V(\mathbf{x}, i) \geq \mathbf{x}'\mathbf{P}_i\mathbf{x} > 0 \tag{11}$$

Thus,

$$\frac{\tilde{\mathcal{A}}V(\mathbf{x}, i)}{V(\mathbf{x}, i)} \leq -\frac{\mathbf{x}'\mathbf{Q}_i\mathbf{x}}{\mathbf{x}'\mathbf{P}_i\mathbf{x}} \quad \text{for } \mathbf{x} \neq 0$$

Using the fact that $\mathbf{Q}_i > 0$ and $\mathbf{P}_i > 0$, we can write

$$\frac{\tilde{\mathcal{A}}V(\mathbf{x}, i)}{V(\mathbf{x}, i)} \leq -\beta := -\min_{i \in \mathcal{S}} \left\{ \frac{\lambda_{min}(\mathbf{Q}_i)}{\lambda_{max}(\mathbf{P}_i)} \right\} \tag{12}$$

Using now Dynkin's formula [7], the Fubini's theorem and the Gronwall-Bellman lemma [19], we get for each $i \in \mathcal{S}$:

$$E\{V(\mathbf{x}(t), \mathbf{r}(t) = i) | \mathbf{x}_0, \mathbf{r}_0 = i\} \leq e^{-\beta t} V(\mathbf{x}_0, i)$$

Thus,

$$E\{V(\mathbf{x}, i) | \mathbf{x}_0, \mathbf{r}_0 = i\}$$

$$= E\left\{ \mathbf{x}'(t)\mathbf{P}_i\mathbf{x}(t) + \sum_{q=1}^{N} \int_{t-\tau_q}^{t} \mathbf{x}'(\eta)\mathbf{Q}\mathbf{x}(\eta) d\eta | \mathbf{x}_0, \mathbf{r}_0 = i \right\}$$

$$= E\{\mathbf{x}'(t)\mathbf{P}_i\mathbf{x}(t) | \mathbf{x}_0, \mathbf{r}_0 = i\} + E\left\{ \sum_{q=1}^{N} \int_{t-\tau_q}^{t} \mathbf{x}'(\eta)\mathbf{Q}\mathbf{x}(\eta) d\eta | \mathbf{x}_0, \mathbf{r}_0 = i \right\}$$

$$\leq e^{-\beta t} V(\mathbf{x}_0, i)$$

Notice that $\mathrm{E}\left\{\sum_{q=1}^{\mathrm{N}} \int_{t-\tau_q}^{t} \mathbf{x}'(\eta)\mathbf{Q}\mathbf{x}(\eta) d\eta | \mathbf{x}_0, \mathbf{r}_0 = i\right\}$ is a positive number, then

$$E\{\mathbf{x}'(t)\mathbf{P}_i\mathbf{x}(t) | \mathbf{x}_0, \mathbf{r}_0 = i\} \leq e^{-\beta t} V(\mathbf{x}_0, i) \tag{13}$$

Let $\lambda_{\mathrm{m}} := \min_{\mathrm{i} \in \mathcal{S}}\{\lambda_{min}(\mathbf{P}_i)\}$, $\lambda_M := \max_{i \in \mathcal{S}}\{\lambda_{max}(\mathbf{P}_i)\}$ and using once more Fubini's theorem we have:

$$E\left\{\int_0^T \mathbf{x}'(t, \psi, \rho)\mathbf{P}_i\mathbf{x}(t, \psi, \rho)dt|\mathbf{x}_0, \mathbf{r}_0 = i\right\} \leq \left(\int_0^T e^{-\beta t}dt\right) V(\mathbf{x}_0, i)$$

$$\lambda E\left\{\int_0^T \mathbf{x}'(t, \psi, \rho)\mathbf{x}(t, \psi, \rho)dt|\psi, \mathbf{r}_0 = i\right\} \leq -\frac{1}{\beta}[e^{-\beta T} - 1]\, V(\mathbf{x}_0, i)$$

Let $\tau = \Sigma_{q=1}^{N} \tau_q$, using the assumption **A** and taking the limit as $T \to \infty$, we obtain:

$$\lim_{T\to\infty} E\left\{\int_0^T \mathbf{x}'(t, \psi, \rho)\mathbf{x}(t, \psi, \rho)dt|\mathbf{x}_0, \mathbf{r}_0 = i\right\} \leq \frac{1}{\lambda_m\beta}\mathbf{x}_o'[\mathbf{P}_i + \boldsymbol{\tau}\mathbf{h}^2\boldsymbol{\lambda}_{max}(\mathbf{Q})\mathbf{I}]\mathbf{x}_0$$
$$\leq \tilde{c}\|\mathbf{x}_0\|^2$$

where

$$\tilde{c} := \frac{1}{\lambda_m\beta}[\lambda_{max}(\mathbf{P}_i) + \mathbf{h}^2\boldsymbol{\tau}\boldsymbol{\lambda}_{max}(\mathbf{Q})]$$

This completes the proof of the Theorem 1. □

Remarks 3.1

- If the system has only a single mode, then condition (6) is reduced to the condition of the deterministic stability of linear time-delay system.
- When the time delay is equal to 0, we obtain the result established by Ji and Chizeck [6]. The Eq. (6) can be rewritten as:

$$\left(\mathbf{A}_{0i} + \sum_{q=1}^{N}\mathbf{A}_{qi}\right)\mathbf{P}_i + \mathbf{P}_i\left(\mathbf{A}_{0i} + \sum_{q=1}^{N}\mathbf{A}_{qi}\right) + \sum_{j=1}^{s} q_{ij}\mathbf{P}_j = -\mathbf{Q}_i \qquad (14)$$

- Notice that the proposed result doesn't depend on the delay of the system.

- Notice that the condition of Theorem 1 given by Eq. (6) is similar to the following conditions:

$$\begin{cases} \mathbf{A}_{0i}'\mathbf{P}_i + \mathbf{P}_i\mathbf{A}_{0i} + \sum_{j=1}^{s}\pi_{ij}\mathbf{P}_j + -N\mathbf{Q} - \mathbf{Q}_i \\ \mathbf{Q}_i - \mathbf{P}_i[\sum_{q=1}^{N}\mathbf{A}_{qi}\mathbf{Q}^{-1}\mathbf{A}_{qi}']\mathbf{P}_i > \mathbf{0} \end{cases} \qquad (15)$$

The first equation of the system (17) gives the condition of the stochastic stability of linear system with Markovian jumping parameters (LSMJP) given by (1)–(3) when $\mathbf{A}_{qi} = \mathbf{0}$ for $q = 1,\ldots,N$ and $i = 1,\ldots,s$.