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A MANUFACTURED SOLUTION FOR A TWO-DIMENSIONAL STEADY WALL-BOUNDED INCOMPRESSIBLE TURBULENT FLOW

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A Manufactured Solution

for a Two-Dimensional Steady

Wall-Bounded

Incompressible Turbulent Flow

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Abstract

This report presents a Manufactured Solution for code and calculation verification of two-dimensional, steady, wall-bounded, incompressible, turbulent flows. Although the proposed solution does not reproduce exactly all the features of a near-wall turbulent flow, several of the numerical difficulties of the calculation of such flows are included in the present manufactured solution.

The specified flow field satisfies mass conservation, but it requires additional source terms in the momentum equations. Manufactured solutions are also proposed for the turbulence quantities of six eddy-viscosity turbulence models: the one-equation models of Spalart & Allmaras and Menter; the standard two-equation $k - \varepsilon$ model and the Low-Reynolds version proposed by Chien; the TNT and BSL versions of the $k - \omega$ model.

The report describes the prescribed flow field for all the flow quantities and it presents all the source functions required to guarantee that the manufactured solutions satisfy the momentum equations and the transport equations of the turbulence quantities used by each of the six turbulence models. The list of FORTRAN functions that include all the information required to solve numerically the present manufactured solution is presented in appendix.

Contents

1 Introduction

The first workshop on uncertainty analysis, held in Lisbon in 2004, was focused on two classical two-dimensional turbulent flow problems from the ER-COFTAC Classic database (Cases 18-A and 30), [1]. The participants were demanded to estimate the uncertainty of their solution by a grid-refinement study or alternative approach. The objective was to see if this would result in overlapping uncertainty bars in order to verify consistency of predictions across CFD codes and to assess the validity of error estimation techniques. Interesting conclusions were drawn from these comparisons. However, one of the main points raised during the final discussion was that the *solution* verification should have been preceded by a thorough *code* verification [2, 3]. Although well-known and fully documented, the flow problems considered do not have exact analytical solutions. Thus, the true errors could not be computed from the quantities studied. It was recommended that a possible next workshop should include a problem for which an exact solution holds.

This conclusion follows several Journal policy statements. Indeed, the process of determining the correctness of a code, known as *code verification*, can only be done by systematic grid convergence tests on a problem with a *benchmark* solution. The best standard of comparison is an exact analytical solution expressed in terms of simple mathematical functions such as *sin*, *exp*, *tanh*, etc. Infinite series are not desirable as they tend to be more trouble to evaluate accurately than the CFD solution itself. The benchmark solution should not only be exact, it should also exhibit a structure rich enough to ensure that all terms in the governing equations are exercised by the test, [4].

The Method of Manufactured Solution (MMS), [4, 5, 6, 7, 8, 9, 10], provides a general procedure for working with such analytical solutions. The procedure is very simple. A continuum solution is first picked. In general this solution will not satisfy the governing equations because of the arbitrary nature of the choice. An appropriate source term is defined to cancel out any imbalance in the partial-differential equation (PDE) caused by the choice of the continuum solution. Interestingly enough, this choice can often be made independently of the code or of the equations considered. That is, one can pick a solution and use it to verify an incompressible Navier-Stokes code, a Darcy flow model, a heat equation, a materials code, etc. The solution should be non-trivial in the sense that it exercises all derivatives in the PDE. The solution also defines the boundary conditions in all forms be they Dirichlet, Neumann or Robin. The chosen solution need

not have a physical meaning since Verification (of codes or of calculations) is a purely mathematical exercise. Also, a physically unrealistic solution can easily be chosen to strongly exercise all the terms in the governing equations, whereas a physically realistic solution usually will necessarily minimize some terms (e.g., those neglected in boundary-layer analyses); this approach is still recommended for general code Verification. On the other hand, choosing a physically realistic manufactured problem which has a closed form solution offers several advantages. First, it exercises each term involved in the PDE in a manner similar to that of a real problem so that similar difficulties in the solution and error estimation processes will arise. Secondly, using a physically realistic manufactured solution leads to smaller source terms so that the PDE does not tend towards a degenerate form controlled by the magnitude of the source terms. Finally, it makes the methodology more attractive for the engineering community.

This document presents a manufactured solution (MS) for the second Workshop on CFD Uncertainty Analysis. It is supposed to exercise most of the features of a near-wall, two-dimensional, steady incompressible turbulent flow. The intention of this MS is to perform verification of calculations, but also to serve as a code verification exercise, [6]. The effectiveness of the Calculation Verification/Uncertainty Estimation is problem-dependent, and therefore the Uncertainty Estimation methods will be evaluated on a MS that mimics a realistic solution.

The MS prescribes the main flow variables (velocity components and pressure) and two turbulence quantities: the eddy-viscosity, v_t, and the turbulence kinetic energy, k. In the most common eddy-viscosity models, all other turbulence quantities can be derived from v_t and k. Although the MS does not include all the complexity of a near-wall turbulent flow, it contains several features of this type of flows.

Six eddy-viscosity turbulence models have been considered:

- The Spalart & Allmaras one-equation model, [11].

- Menter's one-equation model, [12].

- The standard $k - \varepsilon$ two-equation model, [13].

- Chien's low-Reynolds $k - \varepsilon$ two-equation model, [14].

- The turbulent/non-turbulent (TNT) $k - \omega$ two-equation model, [15].

- Menter's baseline (BSL) $k - \omega$ two-equation model, [16].

The specification of v_t for the two one-equation models is not a good choice for the construction of the MS. In these models, the dependent variable of the models, $\tilde{v}$ or $\tilde{v}_t$, is related to the eddy-viscosity by a damping function. As we will illustrate in the present report, the second derivative of v_t with respect to $\tilde{v}$ and $\tilde{v}_t$ is very difficult to capture numerically due to large oscillations in the "near-wall" region. Therefore, the specification of v_t leads to second derivatives of $\tilde{v}$ and $\tilde{v}_t$ that are not easy to obtain numerically. On the other hand, if one specifies smooth functions for $\tilde{v}$ and $\tilde{v}_t$ there are no special difficulties for the solution of the momentum equations, because only the first-derivative of the eddy-viscosity is included in the momentum equations. Therefore, for the two one-equation models considered in this report, $\tilde{v}$ and $\tilde{v}_t$ are specified and the eddy-viscosity field is obtained from the turbulence model formulation. This means that the source functions added to the momentum equations will be dependent on the turbulence model selected. This does not represent a drawback for the MS, because there is no intention to compare or "classify" turbulence models.

In the shear-stress transport (SST) $k - \omega$ two-equation model proposed by Menter, [16], it is not possible to obtain ω from v_t and k. In this model, the eddy-viscosity definition guarantees that the shear-stress in a boundary-layer does not exceed $0.31k$. This is accomplished by replacing ω by the vorticity magnitude in the regions where the standard v_t definition would violate the limit on the shear-stress level. Therefore, in these regions v_t becomes independent of ω. For the SST $k - \omega$ model, it is possible to specify k and ω to construct a MS. However, the first derivatives of the eddy-viscosity will be discontinuous at the locations where the limiter is turned on and off.

The remainder of the document is organized as follows: sections 2 and 3 present the proposed computational domain, flow conditions and main flow variables. Section 4 presents the source terms of the two momentum equations, generated by the MS. The six turbulence models and their application in the MMS is described in section 5. The appendix A presents the list of FORTRAN functions with all the flow quantities and source terms of the MS.

2   Computational domain and Reynolds number

The proposed computational domain is a square of side 0.5L with $0.5L \leq X \leq$

L and $0 \le Y \le 0.5L$. The Reynolds number, Rn, is defined by

$$Rn = \frac{U_1 L}{\nu} \, , \tag{1}$$

where U_1 is the reference velocity, L the reference length and ν the kinematic viscosity.

In non-dimensional variables, (x, y), the computational domain is given by $0.5 \le x \le 1$ and $0 \le y \le 0.5$ and the proposed Reynolds number is $Rn = 10^6$, i.e.

$$\nu = 10^{-6} U_1 L \, . \tag{2}$$

All the quantities presented below are non-dimensional using L and U_1 as the reference length and velocity scales and $R_n = 10^6$.

3   Main flow variables

In the definition of the velocity components and pressure coefficient we will use the following quantity:

$$\eta = \frac{\sigma y}{x} \tag{3}$$

The suggested value for σ is $\sigma = 4$.

3.1   Velocity components

The velocity component in the x direction, u, is given by

$$u = \mathrm{erf}(\eta) \, . \tag{4}$$

The velocity component in the y direction, v, is given by

$$v = \frac{1}{\sigma \sqrt{\pi}} \left(1 - e^{-\eta^2} \right) \, . \tag{5}$$

With this choice of the velocity components, the mass conservation equation is satisfied identically. The isolines of u and v of the proposed solution are plotted in figure 1.

The derivatives of u with respect to x and y are:
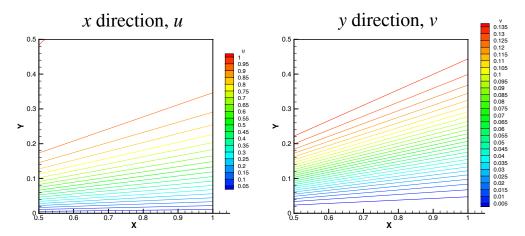
Figure 1: Isolines of the velocity components in the *x* direction, *u*, and *y* direction, *v*. $\sigma = 4$.

- First derivatives:

$$\frac{\partial u}{\partial x} = -\frac{2}{\sqrt{\pi}} \frac{\sigma y}{x^2} e^{-\eta^2}$$

$$\frac{\partial u}{\partial y} = \frac{2}{\sqrt{\pi}} \frac{\sigma}{x} e^{-\eta^2}$$

(6)

- Second derivatives:

$$\frac{\partial^2 u}{\partial x^2} = \frac{4}{\sqrt{\pi}} \frac{\eta}{x^2} e^{-\eta^2} \left(1 - \eta^2\right)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{4}{\sqrt{\pi}} \left(\frac{\sigma}{x}\right)^2 \eta \, e^{-\eta^2}$$

(7)

The derivatives of *v* with respect to *x* and *y* are:

- First derivatives:

$$\frac{\partial v}{\partial x} = -\frac{2}{\sqrt{\pi}} \frac{\sigma y^2}{x^3} e^{-\eta^2}$$

$$\frac{\partial v}{\partial y} = \frac{2}{\sqrt{\pi}} \frac{\sigma y}{x^2} e^{-\eta^2}$$

(8)

- Second derivatives:

$$
\begin{aligned}
\frac{\partial^2 v}{\partial x^2} &= \frac{2}{\sqrt{\pi}} \frac{\sigma y^2}{x^4} e^{-\eta^2} \left(3 - 2\eta^2 \right) \\
\frac{\partial^2 v}{\partial y^2} &= \frac{2}{\sqrt{\pi}} \frac{\sigma}{x^2} e^{-\eta^2} \left(1 - 2\eta^2 \right)
\end{aligned}
\tag{9}
$$

3.2 Pressure

The pressure is given by

$$
C_p = \frac{P}{\rho (U_1)^2} = 0.5 \ln \left(2x - x^2 + 0.25 \right) \ln \left(4y^3 - 3y^2 + 1.25 \right)
\tag{10}
$$

The derivatives of C_p with respect to x and y are:

$$
\begin{aligned}
\frac{\partial C_p}{\partial x} &= \frac{(1-x)}{2x - x^2 + 0.25} \ln \left(4y^3 - 3y^2 + 1.25 \right) \\
\frac{\partial C_p}{\partial y} &= \frac{3y(2y-1)}{4y^3 - 3y^2 + 1.25} \ln \left(2x - x^2 + 0.25 \right)
\end{aligned}
\tag{11}
$$

This choice of the pressure field leads to the following conditions at the boundaries:

- Bottom (y=0), $\dfrac{\partial C_p}{\partial y} = 0$.

- Inlet (x=0.5), $C_p = 0$.

- Top (y=0.5), $C_p = 0$ and $\dfrac{\partial C_p}{\partial y} = 0$.

- Outlet (x=1), $\dfrac{\partial C_p}{\partial x} = 0$.

The isolines of C_p are plotted in figure 2.
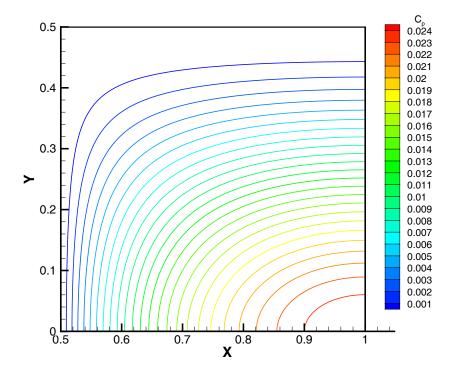
Figure 2: Isolines of the pressure coefficient, C_p. $\sigma = 4$.

3.3 Eddy-Viscosity, ν_t

3.3.1 Two-equation turbulence models

For the two-equation models, the eddy-viscosity is given by

$$\nu_t = 0.25 \nu_{max} \eta_v^4 e^{2-\eta_v^2} , \tag{12}$$

where

$$\eta_v = \frac{\sigma_v y}{x} \tag{13}$$

and $\sigma_v = 2.5\sigma$. The proposed ν_{max} is $10^3 \nu$.

Close to the bottom wall, ν_t varies with y^4 as in a near-wall turbulent flow. The isolines of ν_t are plotted in figure 3.
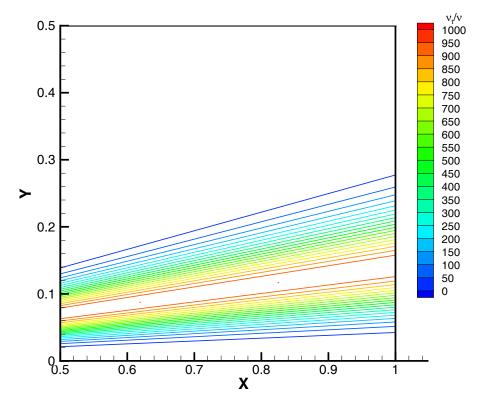
Figure 3: Isolines of the eddy-viscosity, v_t. $\sigma_v = 10$ and $v_{max} = 10^3 v$.

The first derivatives of v_t with respect to x and y are given by:

$$
\begin{aligned}
\frac{\partial v_t}{\partial x} &= 2\frac{v_t}{x}\left(\eta_v^2 - 2\right) \\
\frac{\partial v_t}{\partial y} &= 2\frac{v_t}{y}\left(2 - \eta_v^2\right)
\end{aligned}
\tag{14}
$$

3.3.2   One-equation turbulence models

We have considered two one-equation models that solve a transport equation for a dependent variable related to the eddy-viscosity by a damping function. In the Spalart & Allmaras model, [11], v_t is given by

$$
v_t = \tilde{v} f_{v1}
\tag{15}
$$

with

$$f_{v1} = \frac{\chi^3}{\chi^3 + c_{v1}^3} \tag{16}$$

and

$$\begin{aligned} \chi &= \frac{\tilde{v}}{v}, \\ c_{v1} &= 7.1. \end{aligned} \tag{17}$$

For the Menter model, [12], the eddy-viscosity is obtained from

$$v_t = D_2 \tilde{v}_t, \tag{18}$$

where

$$D_2 = 1 - e^{-\left(\frac{\tilde{v}_t}{A^+ \kappa v}\right)^2} \tag{19}$$

and

$$\begin{aligned} A^+ &= 13, \\ \kappa &= 0.41. \end{aligned} \tag{20}$$

If one specifies the eddy-viscosity and determines $\tilde{v}$ and $\tilde{v}_t$ from the definition equations (15 and 18), the first and second derivatives of $\tilde{v}$ and $\tilde{v}_t$ with respect to x and y will have to be determined implicitly. For example, for the dependent variable of the Spalart & Allmaras model, $\tilde{v}$, we have:

$$\begin{aligned} \frac{\partial \tilde{v}}{\partial x} &= \frac{\partial v_t}{\partial x} \frac{d\tilde{v}}{dv_t} \\ \frac{\partial \tilde{v}}{\partial y} &= \frac{\partial v_t}{\partial y} \frac{d\tilde{v}}{dv_t} \\ \frac{\partial^2 \tilde{v}}{\partial x^2} &= \frac{d\tilde{v}}{dv_t} \left[\frac{\partial^2 v_t}{\partial x^2} - \left(\frac{\partial v_t}{\partial x} \right)^2 \left(\frac{d\tilde{v}}{dv_t} \right)^2 \frac{d^2 v_t}{d\tilde{v}^2} \right] \\ \frac{\partial^2 \tilde{v}}{\partial y^2} &= \frac{d\tilde{v}}{dv_t} \left[\frac{\partial^2 v_t}{\partial y^2} - \left(\frac{\partial v_t}{\partial y} \right)^2 \left(\frac{d\tilde{v}}{dv_t} \right)^2 \frac{d^2 v_t}{d\tilde{v}^2} \right] \\ \frac{d\tilde{v}}{dv_t} &= \frac{1}{\frac{dv_t}{d\tilde{v}}} \end{aligned} \tag{21}$$

It is obvious that similar equations are obtained for $\tilde{v}_t$.

The damping functions, f_{v1} and D_2, and the first and second derivatives of v_t with respect to $\tilde{v}$ and $\tilde{v}_t$ are illustrated in figure 4. The second derivatives of v_t have been multiplied by v to fit in the plots of figure 4. The second derivatives of
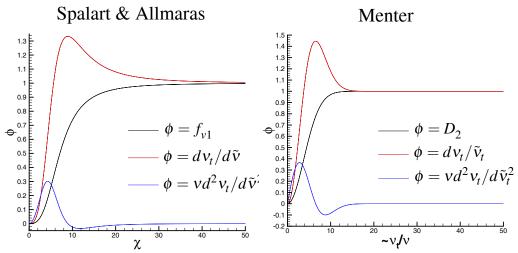
Figure 4: Damping function and first and second derivatives of v_t with respect to $\tilde{v}$ and $\tilde{v}_t$ for the the one-equation turbulence models of Spalart & Allmaras and Menter.

v_t with respect to $\tilde{v}$ and $\tilde{v}_t$ exhibit a very high peak value close to $\tilde{v} = \tilde{v}_t \simeq 4v$ (the plotted value has to be divided by v). Therefore, the second derivatives of $\tilde{v}$ and $\tilde{v}_t$ included in the diffusion terms of the turbulence quantities transport equations will be very difficult to capture numerically.
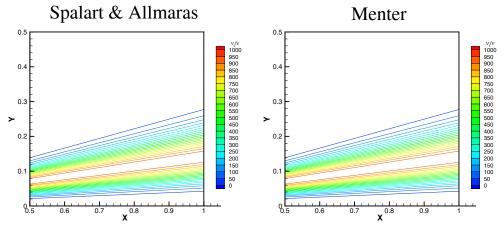
Figure 5: Isolines of the eddy-viscosity, v_t, for the two one-equation models. $\sigma_v = 10$ and $v_{max} = 10^3 v$.

On the other hand, if one specifies $\tilde{v}$ and $\tilde{v}_t$ from equation (12), the first and

second derivatives of $\tilde{v}$ and $\tilde{v}_t$ with respect to x and y will be perfectly smooth. The momentum equations include only the first derivatives of v_t with respect to x and y. Therefore, the manufactured solution for the eddy-viscosity will still be smooth because the first-derivative of v with respect to $\tilde{v}$ and $\tilde{v}_t$ does not exhibit any special difficulties.

Figure 5 presents the manufactured eddy-viscosity fields for the two one-equation models using equation (12) to define $\tilde{v}$ and $\tilde{v}_t$ and equations (15) and (18) to obtain v_t.

The first-derivatives of the eddy-viscosity required for the calculation of the manufactured source terms of the x and y momentum equations are given by:

- Spalart & Allmaras one-equation turbulence model.

$$
\begin{aligned}
\frac{\partial v_t}{\partial x} &= \left(f_{v1} + \frac{3\chi^3 c_{v1}^3}{\left(\chi^3 + c_{v1}^3\right)^2} \right) \frac{\partial \tilde{v}}{\partial x} \\
\frac{\partial v_t}{\partial y} &= \left(f_{v1} + \frac{3\chi^3 c_{v1}^3}{\left(\chi^3 + c_{v1}^3\right)^2} \right) \frac{\partial \tilde{v}}{\partial y}
\end{aligned}
\tag{22}
$$

- Menter one-equation turbulence model.

$$
\begin{aligned}
\frac{\partial v_t}{\partial x} &= \left(D_2 + 2\left(\frac{\tilde{v}_t}{A^+ \kappa v}\right)^2 e^{-\left(\frac{\tilde{v}_t}{A^+ \kappa v}\right)^2} \right) \frac{\partial \tilde{v}_t}{\partial x} \\
\frac{\partial v_t}{\partial y} &= \left(D_2 + 2\left(\frac{\tilde{v}_t}{A^+ \kappa v}\right)^2 e^{-\left(\frac{\tilde{v}_t}{A^+ \kappa v}\right)^2} \right) \frac{\partial \tilde{v}_t}{\partial y}
\end{aligned}
\tag{23}
$$

Logically, the first derivatives of $\tilde{v}$ and $\tilde{v}_t$ with respect to x and y are given by equations (14).

3.4   Turbulence kinetic energy, k

The square root of the turbulence kinetic energy, k, is the turbulence velocity scale of most of the two-equation eddy-viscosity turbulence models. The field of k for the MS is generated using the following equation:

$$k = k_{max}\eta_v^2 e^{1-\eta_v^2} . \tag{24}$$

Close to the bottom of the domain ($y = 0$), i.e. in the "near-wall" region, k varies with y^2 and the maximum of k occurs closer to the bottom than the maximum of v_t. The proposed value of k_{max} is 0.01.[1] The isolines of k are plotted in figure 6.
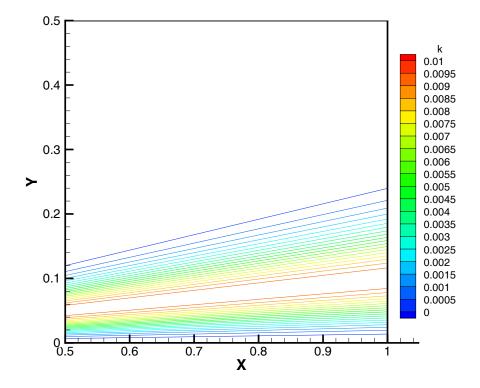
Figure 6: Isolines of the turbulence kinetic energy, k.

The derivatives of k with respect to x and y are

---

[1]As a first attempt, we have tried to define k from the Bradshaw's hypothesis for turbulence in equilibrium, written for an eddy-viscosity approximation.

$$k = \frac{v_t}{0.3} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

However, we found serious numerical problems for the solution of the k transport equation close to the bottom using this option.

- First derivatives:

$$\frac{\partial k}{\partial x} = \frac{2k}{x}\left(\eta_v^2 - 1\right)$$

$$\frac{\partial k}{\partial y} = \frac{2k}{y}\left(1 - \eta_v^2\right)$$

(25)

- Second derivatives:

$$\frac{\partial^2 k}{\partial x^2} = \frac{2k}{x^2}\left(2\eta_v^4 - 7\eta_v^2 + 3\right)$$

$$\frac{\partial^2 k}{\partial y^2} = \frac{2k}{y^2}\left(2\eta_v^4 - 5\eta_v^2 + 1\right)$$

(26)

4 Source terms of the momentum equations

The specified solution for the flow variables implies that the mass conservation equation is satisfied, but that balancing source terms must be added to the momentum equations.

The source terms are computed in the following way:

$$\begin{aligned}
f_x &= T_{px} + T_{cx} + T_{dx} \\
f_y &= T_{py} + T_{cy} + T_{dy}
\end{aligned}$$

(27)

T_p stands for the pressure term, T_c for the contribution of convection and T_d for diffusion.

$$\begin{aligned}
T_{px} &= \frac{\partial C_p}{\partial x} \\
T_{cx} &= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\
T_{dx} &= -(\nu + \nu_t)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - 2\frac{\partial \nu_t}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial \nu_t}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
T_{py} &= \frac{\partial C_p}{\partial y} \\
T_{cy} &= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \\
T_{dy} &= -(\nu + \nu_t)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - 2\frac{\partial \nu_t}{\partial y}\frac{\partial v}{\partial y} - \frac{\partial \nu_t}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned}$$

(28)

For the present MS, the diffusion terms depend on the turbulence model selected. The following sections present the manufactured source functions and the convective, pressure and diffusion terms of the *x* and *y* momentum equation.

Table 1 presents the maximum and minimum values of the convective and pressure terms of the *x* and *y* momentum equations, which are common to all the turbulence models. Figure 7 illustrates the isolines of the convective and pressure terms.

Term	*x* momentum		*y* momentum	
	Minimum	Maximum	Minimum	Maximum
T_c	-0.407	0	-0.107	0
T_p	0	0.112	-0.075	0

Table 1: Minimum and maximum values of the convective and pressure terms of the *x* and *y* momentum equations.
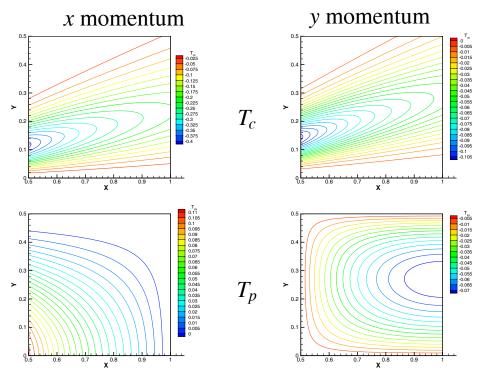
Figure 7: Convection and pressure terms of the *x* and *y* momentum equations.

4.1 Two-equation turbulence models

Table 2 presents the maximum and minimum values of the diffusion terms and of the manufactured source functions of the x and y momentum equations.

Term	x momentum		y momentum	
	Minimum	Maximum	Minimum	Maximum
T_d	-0.209	0.162	-0.029	0.023
f	-0.278	0.112	-0.131	0

Table 2: Minimum and maximum values of the diffusion terms and of the manufactured source term of the x and y momentum equations. Two-equation turbulence models.

The isolines of T_d and f for the x and y momentum equations are presented in figure 8.

Figure 8: Source term, convection, diffusion and pressure terms of the x momentum equation. Two-equation turbulence models.

4.2   One-equation turbulence models

The maximum and minimum values of the diffusion terms and of the manufactured source functions of the x and y momentum equations for the eddy-viscosity fields of the one-equation turbulence models are presented in tables 3 and 4.

Term	x momentum		y momentum	
	Minimum	Maximum	Minimum	Maximum
T_d	-0.209	0.162	-0.029	0.023
f	-0.277	0.112	-0.131	0

Table 3: Minimum and maximum values of the diffusion terms and of the manufactured source term of the x and y momentum equations. Spalart & Allmaras one-equation turbulence model.

Figure 9: Source term, convection, diffusion and pressure terms of the x momentum equation. Spalart & Allmaras one-equation turbulence model.

The maximum and minimum values of the diffusion terms and of the manufactured source functions are almost identical to the ones of the two-equation models

eddy-viscosity field. Nevertheless, figures 9 and 10 present the isolines of T_d and f for the x and y momentum equations with the manufactured eddy-viscosity field of the one-equation models. The comparison of figures 8, 9 and 10 shows that the changes in the source functions of the momentum equations introduced by specifying the dependent variable in the one-equation turbulence models instead of the eddy-viscosity are impossible to detect graphically.

Term	x momentum		y momentum	
	Minimum	Maximum	Minimum	Maximum
T_d	-0.209	0.162	-0.029	0.023
f	-0.277	0.112	-0.131	0

Table 4: Minimum and maximum values of the diffusion terms and of the manufactured source term of the x and y momentum equations. Menter one-equation turbulence model.
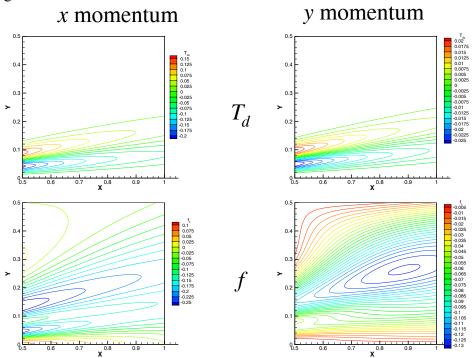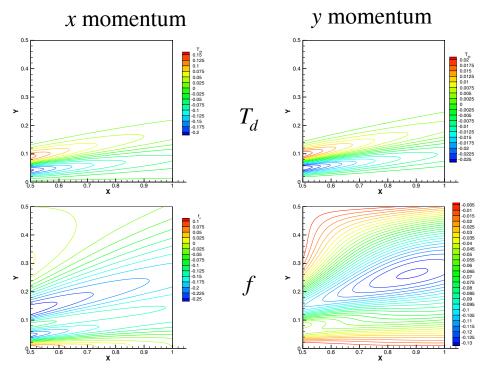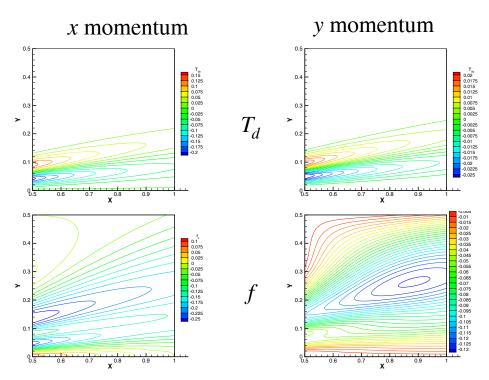

Figure 10: Source term, convection, diffusion and pressure terms of the x momentum equation. Menter one-equation turbulence model.

5   Turbulence models

5.1   One-equation models

5.1.1   Spalart & Allmaras

The Spalart & Allmaras model proposed in [11] solves the following transport equation :

$$
u\frac{\partial \tilde{v}}{\partial x} + v\frac{\partial \tilde{v}}{\partial y} = c_{b1}\tilde{S}\tilde{v} + \frac{1}{\sigma_s}\left[\nabla \cdot ((v+\tilde{v})\nabla \tilde{v}) + c_{b2}(\nabla \tilde{v} \cdot \nabla \tilde{v})\right] - c_{w1}f_w\left[\frac{\tilde{v}}{d}\right]^2,
$$
(29)

where

$$
\begin{aligned}
S_\Omega &= \left|\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right| \\
\tilde{S} &= S_\Omega + \frac{\tilde{v}}{\kappa^2 d^2}f_{v2} \\
d &= y \\
f_w &= g\left[\frac{1+c_{w3}^6}{g^6+c_{w3}^6}\right]^{\frac{1}{6}} \\
f_{v2} &= 1 - \frac{\chi}{1+\chi f_{v1}} \\
g &= r + c_{w2}(r^6 - r) \\
r &= \frac{\tilde{v}}{\tilde{S}\kappa^2 d^2} \\
\chi &= \frac{\tilde{v}}{v} \\
f_{v1} &= \frac{\chi^3}{\chi^3 + c_{v1}^3}
\end{aligned}
$$
(30)

The eddy-viscosity is obtained from

$$
v_t = \tilde{v}f_{v1}
$$
(31)

The model constants are :

$$
\begin{array}{lll}
\kappa = 0.41, & c_{b1} = 0.1355, & c_{b2} = 0.622 \\
c_{w1} = 3.2391, & c_{w2} = 0.3, & c_{w3} = 2 \\
c_{v1} = 7.1, & \sigma_s = \frac{2}{3}.
\end{array}
$$

The MS is obtained specifying $\tilde{v}$ from equation (12). The first derivatives of $\tilde{v}$ with respect to x and y are given by equations (14). The transport equation of $\tilde{v}$ includes also the second derivatives of $\tilde{v}$ with respect to x and y, which are given by

$$\begin{aligned}
\frac{\partial^2 \tilde{v}}{\partial x^2} &= 2\frac{\tilde{v}}{x^2}\left(2\eta_v^4 - 11\eta_v^2 + 10\right) \\
\frac{\partial^2 \tilde{v}}{\partial y^2} &= 2\frac{\tilde{v}}{y^2}\left(2\eta_v^4 - 9\eta_v^2 + 6\right) ,
\end{aligned} \tag{32}$$

where η_v is defined by equation (13).

Figure 11 presents the isolines of $\tilde{v}$ (which are equal to the isolines of the manufactured v_t for the two-equation models) for the Spalart & Allmaras model and the damping function f_{v1} of the proposed solution. The damping function is active close to the bottom as in a near-wall turbulent flow. However, the thickness of the region with f_{v1} less than 1 is much larger than in the typical viscous sub-layer of a turbulent boundary-layer.
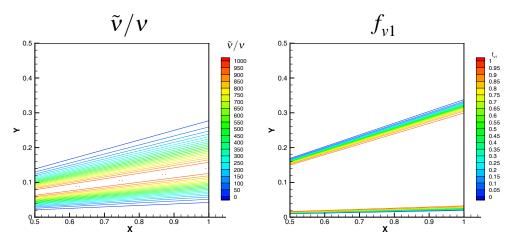
Figure 11: Isolines of the dependent variable, $\tilde{v}$, and damping function, f_{v1}, of the Spalart & one-equation turbulence model.

The source function to be added to the right-hand side of the transport equation of $\tilde{v}$, equation (29), is given by:

$$f_{spal} = T_{cs} + T_{ds} + T_{ps} + T_{dis} \tag{33}$$

where

$$T_{cs} = u\frac{\partial \tilde{v}}{\partial x} + v\frac{\partial \tilde{v}}{\partial y}$$

$$T_{ds} = -\frac{1}{\sigma_s}\left[(v+\tilde{v})\left(\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2}\right) + (1+c_{b2})\left(\left(\frac{\partial \tilde{v}}{\partial x}\right)^2 + \left(\frac{\partial \tilde{v}}{\partial y}\right)^2\right)\right]$$

$$T_{ps} = -c_{b1}\tilde{S}\tilde{v}$$

$$T_{dis} = c_{w1}f_w\left[\frac{\tilde{v}}{d}\right]^2$$

$$(34)$$

Figure 12: Convection, diffusion, production and dissipation terms of the $\tilde{v}$ transport equation. Spalart & Allmaras turbulence model.

Figure 12 presents the T_{cs}, T_{ds}, T_{ps} and T_{dis} fields. f_{spal} is illustrated in figure 13. The maximum and minimum values are given in table 5.
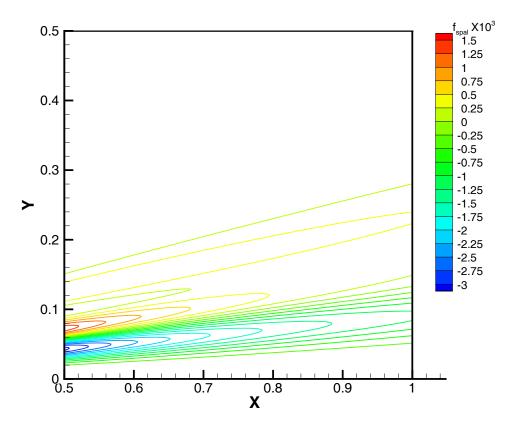
Figure 13: Source function of the eddy-viscosity transport equation of the Spalart & Allmaras turbulence model.

Term	$\sigma_v = 10, v_{max} = 1000v$	
	Minimum	Maximum
$T_{cs} \times 10^3$	-0.618	1.794
$T_{ds} \times 10^3$	-2.155	2.466
$T_{ps} \times 10^3$	-0.925	0
$T_{dis} \times 10^3$	0	0.107
$f_{spal} \times 10^3$	-3.072	1.662

Table 5: Minimum and maximum values of the different terms of the $\tilde{v}$ transport equation including the extra source term. Spalart & Allmaras turbulence model.

5.1.2 Menter

The one-equation model proposed by Menter in [12] solves the following transport equation :

$$u\frac{\partial \tilde{v}_t}{\partial x} + v\frac{\partial \tilde{v}_t}{\partial y} = c_1 D_1 \tilde{v}_t \sqrt{S} + \nabla \cdot \left((v + \frac{\tilde{v}_t}{\sigma})\nabla \tilde{v}_t \right) - c_2 E_{1e}. \tag{35}$$

The eddy-viscosity is given by

$$v_t = D_2 \tilde{v}_t \tag{36}$$

and

$$
\begin{aligned}
S &= 2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \right) + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \\
D_1 &= \frac{v_t + v}{\tilde{v}_t + v} \\
E_{1e} &= c_3 E_{BB}\tanh\left(\frac{E_{k-\varepsilon}}{c_3 E_{BB}} \right) \\
D_2 &= 1 - e^{-\left(\frac{\tilde{v}_t}{A^+ \kappa v}\right)^2} \\
E_{k-\varepsilon} &= \tilde{v}_t^2 \left(\frac{\nabla\sqrt{S}\cdot\nabla\sqrt{S}}{S} \right) \\
E_{BB} &= \nabla\tilde{v}_t \cdot \nabla\tilde{v}_t
\end{aligned}
\tag{37}
$$

The model constants are :

$$
\begin{array}{lll}
c_1 = 0.144 \,, & c_2 = 1.862 \,, & c_3 = 7 \\
\kappa = 0.41 \,, & \sigma = 1 \,, & A^+ = 13 \,.
\end{array}
$$

As for the Spalart & Allmaras turbulence model, $\tilde{v}_t$ is defined by equation (12) and the first derivatives of $\tilde{v}_t$ with respect to x and y are given by equations (14). The second derivatives of $\tilde{v}_t$ with respect to x and y are equal to the ones of $\tilde{v}$, which have been presented in equations (32).

Figure 14 presents the isolines of $\tilde{v}_t$ (once more equal to the eddy-viscosity isolines of the two-equation models) and D_2 of the proposed solution. As expected, the damping function field is very similar to the one of the Spalart & Allmaras turbulence model.
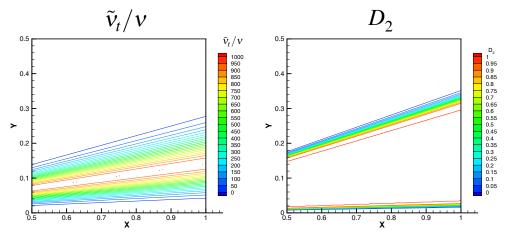
$$\tilde{v}_t/v \qquad\qquad D_2$$

Figure 14: Isolines of the dependent variable, $\tilde{v}_t$, and damping function, D_2, of the Menter one-equation turbulence model.

The source function to be added to the right-hand side of the transport equation of $\tilde{v}_t$, equation (35), is given by:

$$f_{mnt} = T_{cm} + T_{dm} + T_{pm} + T_{dim} \tag{38}$$

where

$$
\begin{aligned}
T_{cm} &= u\frac{\partial \tilde{v}_t}{\partial x} + v\frac{\partial \tilde{v}_t}{\partial y} \\[2mm]
T_{dm} &= -\left[\left(v + \frac{\tilde{v}_t}{\sigma}\right)\left(\frac{\partial^2 \tilde{v}_t}{\partial x^2} + \frac{\partial^2 \tilde{v}_t}{\partial y^2}\right) + \frac{1}{\sigma}\left(\left(\frac{\partial \tilde{v}_t}{\partial x}\right)^2 + \left(\frac{\partial \tilde{v}_t}{\partial y}\right)^2\right)\right] \\[2mm]
T_{pm} &= -c_1 D_1 \sqrt{S}\,\tilde{v}_t \\[2mm]
T_{dim} &= c_2 E_{1e}
\end{aligned}
\tag{39}
$$

$$\nabla\sqrt{S}\cdot\nabla\sqrt{S} = \frac{1}{4S}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right]$$

with

$$
\begin{aligned}
\frac{\partial S}{\partial x} &= 4\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial x\partial y}\right) + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left(\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 v}{\partial x^2}\right) \\[2mm]
\frac{\partial S}{\partial y} &= 4\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2}\right) + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\partial y}\right)
\end{aligned}
$$
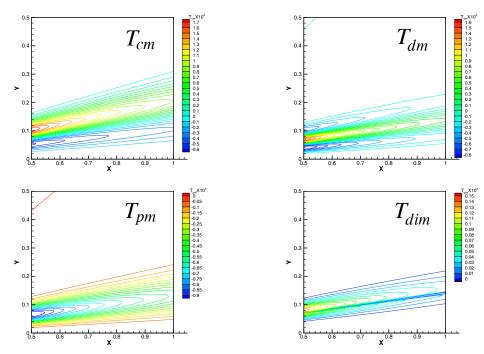
Figure 15: Convection, diffusion, production and dissipation terms of the $\tilde{v}$ transport equation. Menter one-equation turbulence model.

The calculation of the derivatives of S requires the cross-derivatives of the velocity components, which are given by:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x \partial y} &= \frac{2}{\sqrt{\pi}} \frac{\sigma}{x^2} e^{-\eta^2} \left(2\eta^2 - 1 \right) \\
\frac{\partial^2 v}{\partial x \partial y} &= \frac{4}{\sqrt{\pi}} \frac{\sigma y}{x^3} e^{-\eta^2} \left(\eta^2 - 1 \right)
\end{aligned}
\tag{40}
$$

Figure 15 presents the T_{cm}, T_{dm}, T_{pm} and T_{dim} fields. f_{mnt} is illustrated in figure 16. The maximum and minimum values are given in table 6.
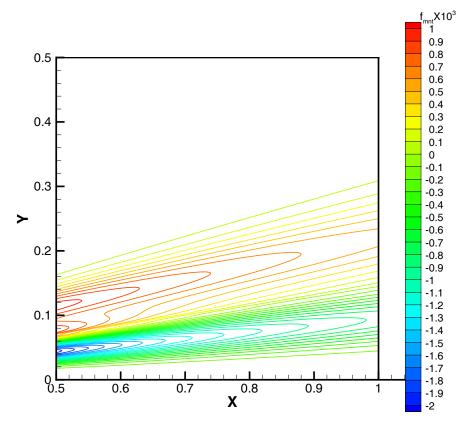
Figure 16: Source function of the eddy-viscosity transport equation of the Menter one-equation turbulence model.

Term	$\sigma_v = 10$, $v_{max} = 1000v$	
	Minimum	Maximum
$T_{cm} \times 10^3$	-0.618	1.794
$T_{dm} \times 10^3$	-0.957	1.646
$T_{pm} \times 10^3$	-0.983	0
$T_{dim} \times 10^3$	0	0.151
$f_{mnt} \times 10^3$	-1.946	1.051

Table 6: Minimum and maximum values of the different terms of the $\tilde{v}$ transport equation including the extra source term. Menter one-equation turbulence model.

5.2   Two-equation models

5.2.1 $k - \varepsilon$

The standard two-equation $k - \varepsilon$ model proposed in [13] is supposed to be valid only in fully-turbulent regions. The damping functions of the two one-equation models suggest that the present manufactured solution is not fully-turbulent close to the bottom. Nevertheless, being the application of the MMS a purely mathematical exercise, the present MS can be used for the $k - \varepsilon$ model as well. However, its usefulness for Calculation Verification becomes questionable.

The eddy-viscosity is obtained from

$$\nu_t = c_\mu \frac{k^2}{\varepsilon} \ . \tag{41}$$

The k and ε transport equations are :

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = \nu_t S + \nabla \cdot \left(\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right) - \varepsilon \tag{42}$$

and

$$u\frac{\partial \varepsilon}{\partial x} + v\frac{\partial \varepsilon}{\partial y} = C_{\varepsilon 1}\frac{\varepsilon}{k}\nu_t S + \nabla \cdot \left(\left(\nu + \frac{\nu_t}{\sigma_\varepsilon} \right) \nabla \varepsilon \right) - C_{\varepsilon 2}\frac{\varepsilon^2}{k} \ . \tag{43}$$

The model constants are

$$c_\mu = 0.09 \quad , \quad C_{\varepsilon 1} = 1.44 \quad , \quad C_{\varepsilon 2} = 1.92 \ ,$$
$$\sigma_k = 1 \qquad , \quad \sigma_\varepsilon = 1.3 \qquad ,$$

The dissipation rate ε follows from the manufactured ν_t and k:

$$\varepsilon = 0.36\frac{k_{max}^2}{\nu_{max}}e^{-\eta_v^2}. \tag{44}$$

Figure 17 presents the isolines of ε. At the bottom boundary, ε is equal to $0.36k_{max}^2/\nu_{max}$. The first and second derivatives of ε with respect to x and y are given by:

- First derivatives:
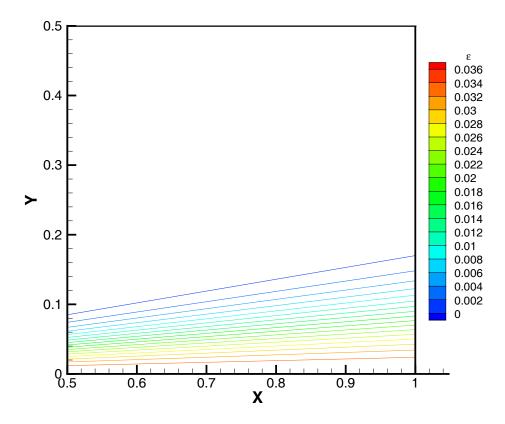
Figure 17: Isolines of ε for the $k - \varepsilon$ two-equation turbulence model.

$$\frac{\partial \varepsilon}{\partial x} = \frac{2\eta_v^2}{x}\varepsilon$$

$$\frac{\partial \varepsilon}{\partial y} = -\frac{2\eta_v^2}{y}\varepsilon$$

$$(45)$$

- Second derivatives:

$$\frac{\partial^2 \varepsilon}{\partial x^2} = \frac{2\eta_v^2}{x^2}\varepsilon\left(2\eta_v^2 - 3\right)$$

$$\frac{\partial^2 \varepsilon}{\partial y^2} = \frac{2\eta_v^2}{y^2}\varepsilon\left(2\eta_v^2 - 1\right)$$

$$(46)$$

Figure 18: Convection, diffusion, production and dissipation terms of the k transport equation. $k - \varepsilon$ two-equation turbulence model.

In order to satisfy the manufactured solution, the source terms to be added to the k and ε transport equations are:

$$
\begin{aligned}
f_{kst} &= T_{ckst} + T_{dkst} + T_{pkst} + T_{dikst} \\
f_{\varepsilon st} &= T_{c\varepsilon st} + T_{d\varepsilon st} + T_{p\varepsilon st} + T_{di\varepsilon st}
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
T_{ckst} &= u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} \\[2mm]
T_{dkst} &= -\left[\left(v + \frac{v_t}{\sigma_k}\right)\left(\frac{\partial^2 k}{\partial x^2} + \frac{\partial^2 k}{\partial y^2}\right) + \frac{1}{\sigma_k}\left(\frac{\partial v_t}{\partial x}\frac{\partial k}{\partial x} + \frac{\partial v_t}{\partial y}\frac{\partial k}{\partial y}\right)\right] \\[2mm]
T_{pkst} &= -v_t S \\[2mm]
T_{dikst} &= \varepsilon
\end{aligned}
\tag{48}
$$

$$T_{c\varepsilon st} = u\frac{\partial \varepsilon}{\partial x} + v\frac{\partial \varepsilon}{\partial y}$$

$$T_{d\varepsilon st} = -\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\left(\frac{\partial^2 \varepsilon}{\partial x^2} + \frac{\partial^2 \varepsilon}{\partial y^2}\right) + \frac{1}{\sigma_\varepsilon}\left(\frac{\partial \nu_t}{\partial x}\frac{\partial \varepsilon}{\partial x} + \frac{\partial \nu_t}{\partial y}\frac{\partial \varepsilon}{\partial y}\right)\right]$$

$$T_{p\varepsilon st} = -C_{\varepsilon 1}\frac{\varepsilon}{k}\nu_t S$$

$$T_{di\varepsilon st} = C_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

$$(49)$$

Figures 18 and 19 present the four contributions to the source terms of the k and ε transport equations. The isolines of the source terms, f_{kst} and $f_{\varepsilon st}$, are displayed in Figure 20. Table 7 presents the maximum and minimum values of the different terms.
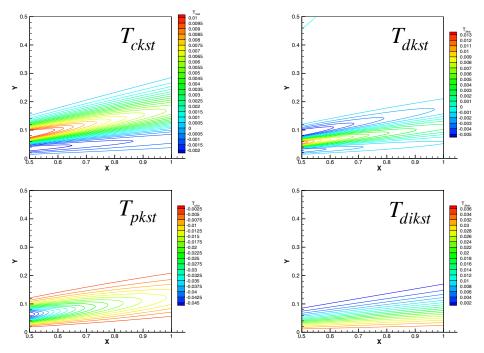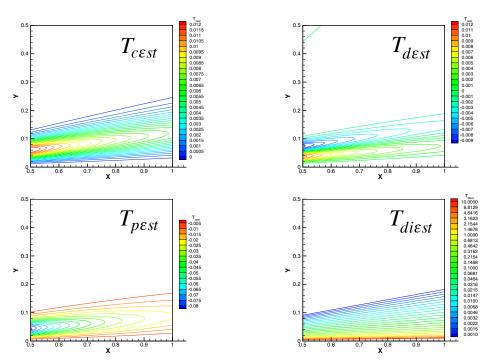

Figure 19: Convection, diffusion, production and dissipation terms of the ε transport equation. $k - \varepsilon$ two-equation turbulence model.
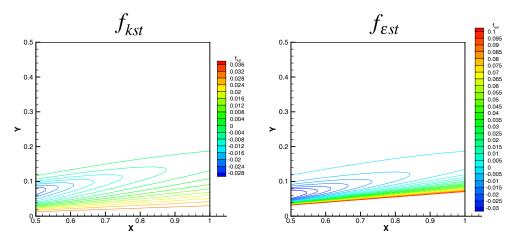
Figure 20: Isolines of the source functions of the k and ε transport equation, f_{kst} and $f_{\varepsilon st}$. $k - \varepsilon$ two-equation turbulence model.

Term	k equation		ε equation	
	Minimum	Maximum	Minimum	Maximum
T_c	-0.0026	0.012	0	0.013
T_d	-0.0075	0.013	-0.0093	0.012
T_p	-0.048	0.	-0.081	0
T_{di}	0	0.036	0	—
f	-0.028	0.036	-0.039	—

Table 7: Minimum and maximum values of the different terms of the k and ε transport equations including the extra source term. $k - \varepsilon$ two-equation turbulence model.

At the bottom of the computational domain, $(y = 0)$, the source term of the ε transport equation tends to infinity, due to the behaviour of the dissipation term, $T_{di\varepsilon st}$. This result would also be present in the application of the standard $k - \varepsilon$ model in the near-wall region of a turbulent flow. In the present context of a manufactured solution, this will lead to a transport equation for ε driven by the forcing source term in the "near-wall" region.

5.2.2 Chien's $k - \varepsilon$

In the low Reynolds $k - \varepsilon$ model proposed by Chien, [14], the eddy-viscosity is obtained from

$$\nu_t = c_\mu f_\mu \frac{k^2}{\tilde{\varepsilon}} \ . \tag{50}$$

The k and $\tilde{\varepsilon}$ transport equations of this model are :

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = \nu_t S + \nabla \cdot \left(\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right) - \left(\tilde{\varepsilon} + \frac{2\nu k}{y^2} \right) \tag{51}$$

and

$$u\frac{\partial \tilde{\varepsilon}}{\partial x} + v\frac{\partial \tilde{\varepsilon}}{\partial y} = C_1 \frac{\tilde{\varepsilon}}{k} \nu_t S + \nabla \cdot \left(\left(\nu + \frac{\nu_t}{\sigma_\varepsilon} \right) \nabla \tilde{\varepsilon} \right) - \frac{\tilde{\varepsilon}}{k} \left[C_2 f_2 \tilde{\varepsilon} + \frac{2\nu k e^{-C_4 y^+}}{y^2} \right] \ . \tag{52}$$

The damping functions are given by :

$$f_\mu = 1 - e^{-C_3 y^+} \ , \tag{53}$$

$$f_2 = 1 - \frac{0.4}{1.8} e^{-\left(\frac{k^2}{6\nu\tilde{\varepsilon}} \right)^2} \tag{54}$$

and

$$y^+ = \frac{u_\tau y}{\nu} \tag{55}$$

with

$$u_\tau = \sqrt{\frac{\tau_w}{\rho}} = \sqrt{\nu \left(\frac{\partial u}{\partial y} \right)_{y=0}} \tag{56}$$

The model constants are

$$c_\mu = 0.09 \quad , \quad C_1 = 1.35 \quad , \quad C_2 = 1.8 \quad , \quad C_3 = 0.0115 \ ,$$
$$\sigma_k = 1 \quad , \quad \sigma_\varepsilon = 1.3 \quad , \quad C_4 = 0.5$$

The variable $\tilde{\varepsilon}$ is based on the manufactured ν_t and k and can be easily obtained using equation (44) that defines ε for the standard $k - \varepsilon$ model.

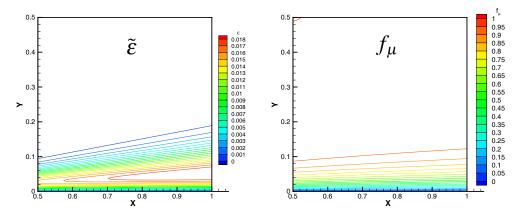$$\tilde{\varepsilon} = f_\mu \varepsilon. \tag{57}$$

Figure 21: Isolines of $\tilde{\varepsilon}$ and of the damping function, f_μ, of Chien's $k - \varepsilon$ two-equation turbulence model.

The damping function f_μ refers to y^+, which for the present manufactured solution is given by

$$y^+ = y\sqrt{\frac{2\sigma}{x\nu\sqrt{\pi}}} \ . \tag{58}$$

Figure 21 presents the isolines of $\tilde{\varepsilon}$ and of the damping function f_μ.

There is a clear difference between the fields of $\tilde{\varepsilon}$ and ε close to the bottom, where the damping function, f_μ, of the low-Reynolds version is active.

The first and second derivatives of $\tilde{\varepsilon}$ with respect to x and y can be computed from:

- First derivatives:

$$
\begin{aligned}
\frac{\partial \tilde{\varepsilon}}{\partial x} &= \frac{\partial f_\mu}{\partial x}\varepsilon + f_\mu \frac{\partial \varepsilon}{\partial x} \\
\frac{\partial \tilde{\varepsilon}}{\partial y} &= \frac{\partial f_\mu}{\partial y}\varepsilon + f_\mu \frac{\partial \varepsilon}{\partial y}
\end{aligned}
\tag{59}
$$

- Second derivatives:

$$
\begin{aligned}
\frac{\partial^2 \tilde{\varepsilon}}{\partial x^2} &= \frac{\partial^2 f_\mu}{\partial x^2}\varepsilon + 2\frac{\partial f_\mu}{\partial x}\frac{\partial \varepsilon}{\partial x} + f_\mu\frac{\partial^2 \varepsilon}{\partial x^2} \\
\frac{\partial^2 \tilde{\varepsilon}}{\partial y^2} &= \frac{\partial^2 f_\mu}{\partial y^2}\varepsilon + 2\frac{\partial f_\mu}{\partial y}\frac{\partial \varepsilon}{\partial y} + f_\mu\frac{\partial^2 \varepsilon}{\partial y^2}
\end{aligned}
\tag{60}
$$

The calculation of the derivatives of $\tilde{\varepsilon}$ requires the first and second order derivatives of the damping function f_μ with respect to x and y.

- First derivatives:

$$
\begin{aligned}
\frac{\partial f_\mu}{\partial x} &= -C_3 \frac{y^+}{2x} e^{-C_3 y^+} \\
\frac{\partial f_\mu}{\partial y} &= C_3 \frac{y^+}{y} e^{-C_3 y^+}
\end{aligned}
\tag{61}
$$

- Second derivatives:

$$
\begin{aligned}
\frac{\partial^2 f_\mu}{\partial x^2} &= C_3 \frac{y^+}{4x^2} e^{-C_3 y^+} \left(3 - C_3 y^+\right) \\
\frac{\partial^2 f_\mu}{\partial y^2} &= -\left(C_3 \frac{y^+}{y}\right)^2 e^{-C_3 y^+}
\end{aligned}
\tag{62}
$$

The source terms of the k and $\tilde{\varepsilon}$ transport equations, required to satisfy the manufactured solution, are given by:

$$
\begin{aligned}
f_{kch} &= T_{ckch} + T_{dkch} + T_{pkch} + T_{dikch} \\
f_{\tilde{\varepsilon}ch} &= T_{c\tilde{\varepsilon}ch} + T_{d\tilde{\varepsilon}ch} + T_{p\tilde{\varepsilon}ch} + T_{di\tilde{\varepsilon}ch}
\end{aligned}
\tag{63}
$$

$$
\begin{aligned}
T_{ckch} &= u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} \\
T_{dkch} &= -\left[\left(v + \frac{v_t}{\sigma_k}\right)\left(\frac{\partial^2 k}{\partial x^2} + \frac{\partial^2 k}{\partial y^2}\right) + \frac{1}{\sigma_k}\left(\frac{\partial v_t}{\partial x}\frac{\partial k}{\partial x} + \frac{\partial v_t}{\partial y}\frac{\partial k}{\partial y}\right)\right] \\
T_{pkch} &= -v_t S \\
T_{dikch} &= \left(\tilde{\varepsilon} + \frac{2vk}{y^2}\right)
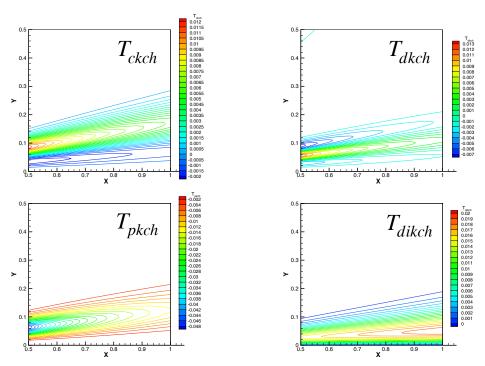\end{aligned}
\tag{64}
$$

Figure 22: Convection, diffusion, production and dissipation terms of the k transport equation. Chien's $k - \varepsilon$ two-equation turbulence model.

$$
\begin{aligned}
T_{c\tilde{\varepsilon}ch} &= u\frac{\partial \tilde{\varepsilon}}{\partial x} + v\frac{\partial \tilde{\varepsilon}}{\partial y} \\[2mm]
T_{d\tilde{\varepsilon}ch} &= -\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\left(\frac{\partial^2 \tilde{\varepsilon}}{\partial x^2} + \frac{\partial^2 \tilde{\varepsilon}}{\partial y^2}\right) + \frac{1}{\sigma_\varepsilon}\left(\frac{\partial \nu_t}{\partial x}\frac{\partial \tilde{\varepsilon}}{\partial x} + \frac{\partial \nu_t}{\partial y}\frac{\partial \tilde{\varepsilon}}{\partial y}\right)\right] \\[2mm]
T_{p\tilde{\varepsilon}ch} &= -C_1\frac{\tilde{\varepsilon}}{k}\nu_t S \\[2mm]
T_{di\tilde{\varepsilon}ch} &= \frac{\tilde{\varepsilon}}{k}\left[C_2 f_2 \tilde{\varepsilon} + \frac{2\nu k e^{-C_4 y^+}}{y^2}\right]
\end{aligned}
\tag{65}
$$

Figures 22 and 23 present the four contributions to the source terms of the k and $\tilde{\varepsilon}$ transport equations. The isolines of the source terms, f_{kch} and $f_{\tilde{\varepsilon}st}$, are plotted in Figure 24. Table 8 presents the maximum and minimum values of the different terms.
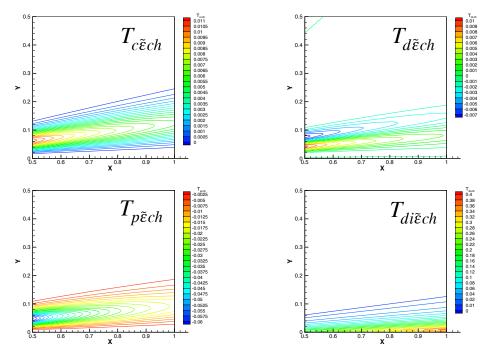
Figure 23: Convection, diffusion, production and dissipation terms of the $\tilde{\varepsilon}$ transport equation. Chien's $k - \varepsilon$ two-equation turbulence model.
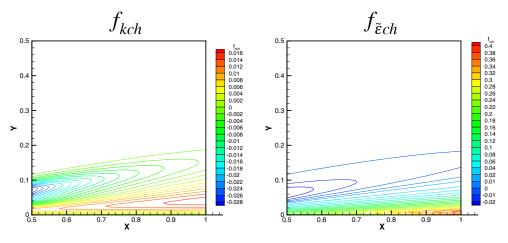

Figure 24: Isolines of the source functions of the k and ε transport equation, f_{kch} and $f_{\tilde{\varepsilon}ch}$. Chien's $k - \varepsilon$ two-equation turbulence model.

Term	k equation		ε equation	
	Minimum	Maximum	Minimum	Maximum
T_c	-0.0026	0.012	0	0.013
T_d	-0.0075	0.013	-0.0075	0.012
T_p	-0.048	0.	-0.061	0
T_{di}	0	0.020	0	0.391
f	-0.028	0.017	-0.031	0.391

Table 8: Minimum and maximum values of the different terms of the k and ε transport equations including the extra source term. $k-\varepsilon$ two-equation turbulence model.

In this low-Reynolds version of the $k-\varepsilon$ turbulence model, there is no singular behaviour of the $\tilde{\varepsilon}$ transport equation at the bottom of the computational domain.

5.2.3   TNT $k-\omega$

The TNT version of the two-equation $k-\omega$ model is proposed in [15]. The eddy-viscosity is obtained from

$$\nu_t = \frac{k}{\omega} \tag{66}$$

and the k and ω transport equations are :

$$u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y} = \nu_t S + \nabla \cdot \left(\left(\nu + \frac{\nu_t}{\sigma_k} \right) \nabla k \right) - \beta^* \omega k \tag{67}$$

and

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \alpha S + \nabla \cdot \left(\left(\nu + \frac{\nu_t}{\sigma_\omega} \right) \nabla \omega \right) - \beta \omega^2 + F_\omega \frac{1}{\omega} \nabla k \cdot \nabla \omega \ . \tag{68}$$

The model constants are

$$\beta^* = 0.09 \quad , \quad \beta = 0.075 \quad , \quad \alpha = \frac{\beta}{\beta^*} - \frac{\kappa^2}{\sigma_\omega \sqrt{\beta^*}} \ ,$$

$$\sigma_k = 1.5 \quad , \quad \sigma_\omega = 2. \quad , \quad \kappa = 0.41$$

$$F_\omega = 0.5 \quad \text{if} \quad \nabla k \cdot \nabla \omega > 0$$

$$F_\omega = 0. \quad \text{if} \quad \nabla k \cdot \nabla \omega < 0$$
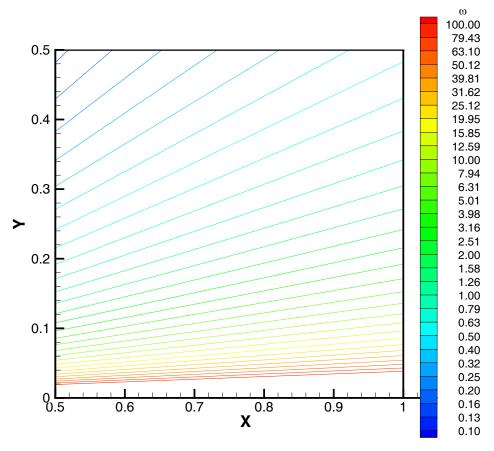
Figure 25: Isolines of ω for the TNT $k - \omega$ two-equation turbulence model.

ω is based on the manufactured v_t and k

$$\omega = 4\frac{k_{max}}{v_{max}}e^{-1}\eta_v^{-2} = 4\frac{k_{max}}{v_{max}}e^{-1}\frac{x^2}{\sigma_v^2 y^2} \ . \tag{69}$$

The behaviour of ω in the "near-wall" region is identical to what happens in a real turbulent flow in so far that it varies with y^{-2}. Thus the manufactured solution of ω raises a problem which also exists in near-wall turbulent flows: ω goes to infinity at the bottom of the computational domain. Figure 25 presents the isolines of ω.

The first and second derivatives of ω with respect to x and y are given by:

- First derivatives:

$$\begin{aligned}
\frac{\partial \omega}{\partial x} &= \frac{2\omega}{x} \\
\frac{\partial \omega}{\partial y} &= -\frac{2\omega}{y}
\end{aligned} \tag{70}$$

- Second derivatives:

$$\begin{aligned}
\frac{\partial^2 \omega}{\partial x^2} &= \frac{2\omega}{x^2} \\
\frac{\partial^2 \omega}{\partial y^2} &= \frac{6\omega}{y^2}
\end{aligned} \tag{71}$$

Figure 26: Convection, diffusion, production and dissipation terms of the k transport equation. TNT $k - \omega$ two-equation turbulence model.

The satisfaction of the manufactured solution requires source terms to be added to the k and ω transport equations, which in this case are:

$$\begin{aligned}
f_{k\omega t} &= T_{ck\omega t} + T_{dk\omega t} + T_{pk\omega t} + T_{dik\omega t} \\
f_{\omega t} &= T_{c\omega t} + T_{d\omega t} + T_{p\omega t} + T_{di\omega t}
\end{aligned} \tag{72}$$

$$T_{ck\omega t} = u\frac{\partial k}{\partial x} + v\frac{\partial k}{\partial y}$$

$$T_{dk\omega t} = -\left[\left(\nu + \frac{\nu_t}{\sigma_k}\right)\left(\frac{\partial^2 k}{\partial x^2} + \frac{\partial^2 k}{\partial y^2}\right) + \frac{1}{\sigma_k}\left(\frac{\partial \nu_t}{\partial x}\frac{\partial k}{\partial x} + \frac{\partial \nu_t}{\partial y}\frac{\partial k}{\partial y}\right)\right]$$

$$T_{pk\omega t} = -\nu_t S$$

$$T_{dik\omega t} = \beta^* \omega k$$

(73)

$$T_{c\omega t} = u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}$$

$$T_{d\omega t} = -\left[\left(\nu + \frac{\nu_t}{\sigma_\omega}\right)\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) + \frac{1}{\sigma_\omega}\left(\frac{\partial \nu_t}{\partial x}\frac{\partial \omega}{\partial x} + \frac{\partial \nu_t}{\partial y}\frac{\partial \omega}{\partial y}\right)\right]$$

$$T_{p\omega t} = -\alpha S$$

$$T_{di\omega t} = \beta \omega^2 - F_\omega \frac{1}{\omega}\nabla k \cdot \nabla \omega$$

(74)

Figures 26 and 27 present the four contributions to the source terms of the k and ω transport equations. The isolines of the source terms, $f_{k\omega t}$ and $f_{\omega t}$, are depicted in Figure 28. Table 9 presents the maximum and minimum values of the different terms.

Term	k equation		ω equation	
	Minimum	Maximum	Minimum	Maximum
T_c	-0.0026	0.012	0.50	—
T_d	-0.0050	0.0087	—	5.04
T_p	-0.048	0.	-45.1	0
T_{di}	0	0.036	-0.12	—
f	-0.029	0.036	-18.5	—

Table 9: Minimum and maximum values of the different terms of the k and ω transport equations including the extra source term. TNT $k - \omega$ two-equation turbulence model.
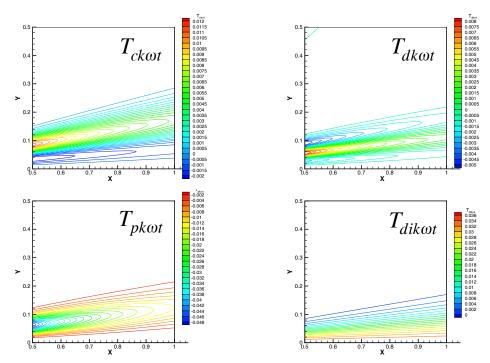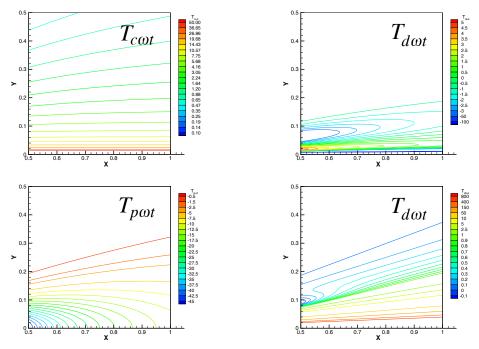
Figure 27: Convection, diffusion, production and dissipation terms of the ω transport equation. TNT $k - \omega$ two-equation turbulence model.
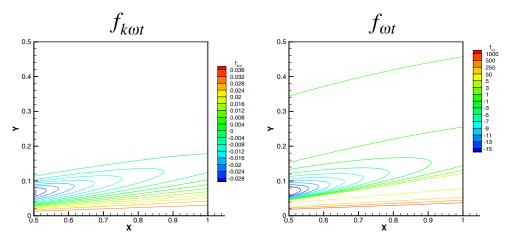

Figure 28: Isolines of the source functions of the k and ε transport equation, $f_{k\omega t}$ and $f_{\omega t}$. TNT $k - \omega$ two-equation turbulence model.

With the "near-wall" behaviour of the $k - \omega$ model, all the terms of the ω transport equation tend to infinity at $y = 0$ excepting the production term, which remains finite.

5.2.4   BSL $k - \omega$

The BSL version of the two-equation $k - \omega$ model is proposed in [16]. This version of the $k - \omega$ model is similar to the TNT version presented above. The calculation of the eddy-viscosity and the k and ω transport equations are identical to the ones of the TNT version (equations (66), (67) and (68)), but the constants of the model are obtained in a different way:

$$\alpha = F_1\alpha_1 + (1 - F_1)\alpha_2 \qquad \alpha = F_1\beta_1 + (1 - F_1)\beta_2$$
$$\sigma_k = F_1\sigma_{k1} + (1 - F_1)\sigma_{k2} \qquad \sigma_\omega = F_1\sigma_{\omega1} + (1 - F_1)\sigma_{\omega2}$$
$$\beta^* = 0.09 \qquad \kappa = 0.41$$
$$F_\omega = \frac{2(1 - F_1)}{\sigma_{\omega2}}$$

$$\alpha_1 = \frac{\beta_1}{\beta^*} - \frac{\kappa^2}{\sigma_{\omega1}\sqrt{\beta^*}} \qquad \alpha_2 = \frac{\beta_2}{\beta^*} - \frac{\kappa^2}{\sigma_{\omega2}\sqrt{\beta^*}}$$
$$\beta_1 = 0.075 \qquad \beta_2 = 0.0828$$
$$\sigma_{k1} = 2 \qquad \sigma_{k2} = 1$$
$$\sigma_{\omega1} = 2 \qquad \sigma_{\omega2} = \frac{1}{0.856}$$

The blending function F_1 is given by

$$F_1 = \tanh\left(arg_1^4\right) \tag{75}$$

where

$$arg_1 = \min\left[\max\left(\frac{\sqrt{k}}{0.09\omega y}, \frac{500\nu}{\omega y^2}\right), \frac{4k}{\sigma_{\omega2}CD_{k\omega}y^2}\right] \tag{76}$$

and

$$CD_{k\omega} = \max\left(\frac{2}{\sigma_{\omega2}\omega}\nabla k \cdot \nabla\omega, 10^{-20}\right) \tag{77}$$

As in the TNT model, ω is defined by equation (69) and its isolines are depicted in figure 25.

The required source terms for k and ω transport equations are given by equations (72), (73) and (74), which were presented above for the TNT model. The

main difference between the two models is the behaviour of the cross-diffusion term ($\nabla k \cdot \nabla \omega$) which has a continuous definition in the BSL model, whereas the TNT model includes an on/off (step function in space) contribution. Nevertheless, figures 26, 27 and 28 and table 9 are representative of the source terms obtained for the BSL version of the $k - \omega$ turbulence model.

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Acknowledgement

A FORTRAN functions with the Manufactured Solution

A.1 General

All the functions have been written in FORTRAN 90 with double precision (REAL*8) variables. The input arguments of all the functions are the Cartesian coordinates x and y. However, for the BSL $k - \omega$ model there is one function that includes an extra argument, which is an output parameter.

The function that computes the blending function, F_1 (equation (75)), of the BSL version of the $k - \omega$ model has an extra argument that contains the contribution to the cross-diffusion term,

$$\frac{1}{\omega} \nabla k \cdot \nabla \omega .$$

The argument of the damping functions of the one-equation models is the dependent variable of the model, $\tilde{v}$ or $\tilde{v}_t$.

A.2 Main flow variables

A.2.1 *u* velocity component

Name	Arguments	Output
UMS	x, y	Horizontal velocity component, u
DUDXMS	x, y	Derivative of u with respect to x, $\frac{\partial u}{\partial x}$
DUDYMS	x, y	Derivative of u with respect to y, $\frac{\partial u}{\partial y}$
DUDX2MS	x, y	Second derivative of u with respect to x, $\frac{\partial^2 u}{\partial x^2}$
DUDY2MS	x, y	Second derivative of u with respect to y, $\frac{\partial^2 u}{\partial y^2}$
DUDXYMS	x, y	Second-order cross-derivative of u, $\frac{\partial^2 u}{\partial x \partial y}$

A.2.2   *v* **velocity component**

Name	Arguments	Output
VMS	x, y	Vertical velocity component, v
DVDXMS	x, y	Derivative of v with respect to x, $\frac{\partial v}{\partial x}$
DVDYMS	x, y	Derivative of v with respect to y, $\frac{\partial v}{\partial y}$
DVDX2MS	x, y	Second derivative of v with respect to x, $\frac{\partial^2 v}{\partial x^2}$
DVDY2MS	x, y	Second derivative of v with respect to y, $\frac{\partial^2 v}{\partial y^2}$
DVDXYMS	x, y	Second-order cross-derivative of v, $\frac{\partial^2 v}{\partial x \partial y}$

A.2.3   **Pressure,** C_p

Name	Arguments	Output
PMS	x, y	Pressure coefficient, $C_p = \frac{p - p_{ref}}{\rho U_{ref}^2}$
DPDXMS	x, y	Derivative of C_p with respect to x, $\frac{\partial C_p}{\partial x}$
DPDYMS	x, y	Derivative of C_p with respect to y, $\frac{\partial C_p}{\partial y}$

A.2.4   **Eddy-Viscosity,** ν_t

- Two-equation Turbulence Models

Name	Arguments	Output
EDDYMS	x, y	Eddy-Viscosity, ν_t
DEDXMS	x, y	Derivative of ν_t with respect to x, $\frac{\partial \nu_t}{\partial x}$
DEDYMS	x, y	Derivative of ν_t with respect to y, $\frac{\partial \nu_t}{\partial y}$

- One-equation turbulence model

  - Spalart & Allmaras

Name	Arguments	Output
EDDYSAMS	x, y	Eddy-Viscosity, ν_t
DESADXMS	x, y	Derivative of ν_t with respect to x, $\frac{\partial \nu_t}{\partial x}$
DESADYMS	x, y	Derivative of ν_t with respect to y, $\frac{\partial \nu_t}{\partial y}$

  - Menter

Name	Arguments	Output
EDDYMTMS	x, y	Eddy-Viscosity, ν_t
DEMTDXMS	x, y	Derivative of ν_t with respect to x, $\frac{\partial \nu_t}{\partial x}$
DEMTDYMS	x, y	Derivative of ν_t with respect to y, $\frac{\partial \nu_t}{\partial y}$

A.2.5 Turbulence kinetic energy, k

Name	Arguments	Output
TKMS	x, y	Turbulence kinetic energy, k
DKDXMS	x, y	Derivative of k with respect to x, $\frac{\partial k}{\partial x}$
DKDYMS	x, y	Derivative of k with respect to y, $\frac{\partial k}{\partial y}$
DKDX2MS	x, y	Second derivative of k with respect to x, $\frac{\partial^2 k}{\partial x^2}$
DKDY2MS	x, y	Second derivative of k with respect to y, $\frac{\partial^2 k}{\partial y^2}$

A.2.6 Auxiliary variables

Name	Arg.	Output
VORTMS	x, y	Magnitude of Vorticity, $S_\Omega = \left\lvert \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right\rvert$
STRAINMS	x, y	Strain-rate, $\sqrt{S} = \sqrt{2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right) + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2}$

A.3   Source terms of the momentum equations

A.3.1   Two-equation turbulence models

Name	Arguments	Output
SMXMS	x, y	Source function of the x momentum equation, f_x
SMYMS	x, y	Source function of the y momentum equation, f_y

A.3.2   One-equation turbulence models

- Spalart & Allmaras

Name	Arguments	Output
SMXSAMS	x, y	Source function of the x momentum equation, f_x
SMYSAMS	x, y	Source function of the y momentum equation, f_y

- Menter

Name	Arguments	Output
SMXMTMS	x, y	Source function of the x momentum equation, f_x
SMYMTMS	x, y	Source function of the y momentum equation, f_y

A.4 Turbulence models

A.4.1 Spalart & Allmaras one-equation model

Name	Arguments	Output
SSAMS	x, y	Source function of the $\tilde{v}$ transport equation, f_{spal}
EDDYMS	x, y	Dependent variable of the turbulence model, $\tilde{v}$
DEDXMS	x, y	Derivative of $\tilde{v}$ with respect to x, $\frac{\partial \tilde{v}}{\partial x}$
DEDYMS	x, y	Derivative of $\tilde{v}$ with respect to y, $\frac{\partial \tilde{v}}{\partial y}$
DEDX2MS	x, y	Second derivative of $\tilde{v}$ with respect to x, $\frac{\partial^2 \tilde{v}}{\partial x^2}$
DEDY2MS	x, y	Second derivative of $\tilde{v}$ with respect to y, $\frac{\partial^2 \tilde{v}}{\partial y^2}$
FV1SAMS	$\tilde{v}$	Damping function of the model
DFV1SAMS	$\tilde{v}$	Derivative of the damping function with respect to $\tilde{v}$

A.4.2 Menter one-equation model

Name	Arguments	Output
SMTMS	x, y	Source function of the $\tilde{v}_t$ transport equation, f_{mnt}
EDDYMS	x, y	Dependent variable of the turbulence model, $\tilde{v}_t$
DEDXMS	x, y	Derivative of $\tilde{v}_t$ with respect to x, $\frac{\partial \tilde{v}_t}{\partial x}$
DEDYMS	x, y	Derivative of $\tilde{v}_t$ with respect to y, $\frac{\partial \tilde{v}_t}{\partial y}$
DEDX2MS	x, y	Second derivative of $\tilde{v}_t$ with respect to x, $\frac{\partial^2 \tilde{v}_t}{\partial x^2}$
DEDY2MS	x, y	Second derivative of $\tilde{v}_t$ with respect to y, $\frac{\partial^2 \tilde{v}_t}{\partial y^2}$
D2MTMS	$\tilde{v}_t$	Damping function of the model
DD2MTMS	$\tilde{v}_t$	Derivative of the damping function with respect to $\tilde{v}_t$

A.4.3 Standard $k - \varepsilon$ two-equation model

Name	Arguments	Output
SKSTMS	x, y	Source function of the k transport equation, f_{kst}
SESTMS	x, y	Source function of the ε transport equation, $f_{\varepsilon st}$
EPSSTMS	x, y	Second dependent variable of the turbulence model, ε
DESDXMS	x, y	First derivative of ε with respect to x, $\frac{\partial \varepsilon}{\partial x}$
DESDYMS	x, y	First derivative of ε with respect to y, $\frac{\partial \varepsilon}{\partial y}$
DESDX2MS	x, y	Second derivative of ε with respect to x, $\frac{\partial^2 \varepsilon}{\partial x^2}$
DESDY2MS	x, y	Second derivative of ε with respect to y, $\frac{\partial^2 \varepsilon}{\partial y^2}$

A.4.4 Chien's $k - \varepsilon$ two-equation model

Name	Arguments	Output
SKCHMS	x, y	Source function of the k transport equation, f_{kch}
SECHMS	x, y	Source function of the $\tilde{\varepsilon}$ transport equation, $f_{\tilde{\varepsilon}ch}$
EPSCHMS	x, y	Second dependent variable of the turbulence model, $\tilde{\varepsilon}$
DECDXMS	x, y	First derivative of $\tilde{\varepsilon}$ with respect to x, $\frac{\partial \tilde{\varepsilon}}{\partial x}$
DECDYMS	x, y	First derivative of $\tilde{\varepsilon}$ with respect to y, $\frac{\partial \tilde{\varepsilon}}{\partial y}$
DECDX2MS	x, y	Second derivative of $\tilde{\varepsilon}$ with respect to x, $\frac{\partial^2 \tilde{\varepsilon}}{\partial x^2}$
DECDY2MS	x, y	Second derivative of $\tilde{\varepsilon}$ with respect to y, $\frac{\partial^2 \tilde{\varepsilon}}{\partial y^2}$
FUCHMS	x, y	Damping function of the turbulence model, f_μ
DFUDXMS	x, y	First derivative of f_μ with respect to x, $\frac{\partial f_\mu}{\partial x}$
DFUDYMS	x, y	First derivative of f_μ with respect to y, $\frac{\partial f_\mu}{\partial y}$

A.4.5 TNT $k - \omega$ two-equation model

Name	Arguments	Output
SKWTMS	x, y	Source function of the k transport equation, $f_{k\omega t}$
SWTMS	x, y	Source function of the ω transport equation, $f_{\omega t}$
WSTMS	x, y	Second dependent variable of the turbulence model, ω
DWSDXMS	x, y	First derivative of ω with respect to x, $\frac{\partial \omega}{\partial x}$
DWSDYMS	x, y	First derivative of ω with respect to y, $\frac{\partial \omega}{\partial y}$
DWSDX2MS	x, y	Second derivative of ω with respect to x, $\frac{\partial^2 \omega}{\partial x^2}$
DWSDY2MS	x, y	Second derivative of ω with respect to y, $\frac{\partial^2 \omega}{\partial y^2}$

A.4.6 BSL $k - \omega$ two-equation model

Name	Arguments	Output
SKWBMS	x, y	Source function of the k transport equation, $f_{k\omega t}$
SWBMS	x, y	Source function of the ω transport equation, $f_{\omega t}$
WSTMS	x, y	Second dependent variable of the turbulence model, ω
F1KWMS	$x, y, \frac{1}{\omega}\nabla k \cdot \nabla \omega$	Blending function of the model, F_1
DWSDXMS	x, y	First derivative of ω with respect to x, $\frac{\partial \omega}{\partial x}$
DWSDYMS	x, y	First derivative of ω with respect to y, $\frac{\partial \omega}{\partial y}$
DWSDX2MS	x, y	Second derivative of ω with respect to x, $\frac{\partial^2 \omega}{\partial x^2}$
DWSDY2MS	x, y	Second derivative of ω with respect to y, $\frac{\partial^2 \omega}{\partial y^2}$

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