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Auteurs: Authors:	Mohamed Menaa, & Aouni A. Lakis
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HYBRID FINITE ELEMENT METHOD APPLIED TO THE ANALYSIS OF FREE VIBRATION OF A SPHERICAL SHELL FILLED WITH FLUID

M. Menaa, A.A. Lakis
Département de Génie mécanique
École Polytechnique de Montréal

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par : Mohamed Menaa, Aouni A. Lakis
Département génie mécanique
École Polytechnique de Montréal

Abstract

In present study, a hybrid finite element method is applied to investigate the free vibration of spherical shell filled with fluid. The structural model is based on a combination of thin shell theory and the classical finite element method. It is assumed that the fluid is incompressible and has no free-surface effect. Fluid is considered as a velocity potential variable at each node of the shell element where its motion is expressed in terms of nodal elastic displacement at the fluid-structure interface. Numerical simulation is done and vibration frequencies for different filling ratios are obtained and compared with existing experimental and theoretical results. The dynamic behavior for different shell geometries, filling ratios and boundary conditions with different radius to thickness ratios is summarized. This proposed hybrid finite element method can be used efficiently for analyzing the dynamic behavior of aerospace structures at less computational cost than other commercial FEM software.

1. Introduction

Shells of revolution, particularly spherical shells are one of the primary structural elements in high speed aircraft. Their applications include the propellant tank or gas-deployed skirt of space crafts. Space shuttles need a large thrust within a short time interval; thus a large propellant tank is required. Dynamic behavior in the lightweight, thin-walled tank is an important aspect in its design. These liquid propelled space launch vehicles experience a significant longitudinal disturbance during thrust build up and also due to the effect of launch mechanism. Dynamic analysis of such a problem in the presence of fluid-structure interaction is one of the challenging subjects in aerospace engineering. Great care must be taken during the design of spacecraft vehicles to prevent dynamic instability.

Free vibration of spherical shell containing a fluid has been investigated by numerous researchers experimentally and analytically.

Rayleigh [1] solved the problem of axisymmetric vibrations of a fluid in a rigid spherical shell. The solution for vibrations of the fluid-filled spherical membrane appears in the work of Morse and Feshbach [2].

The fluid movement on the surface of fluid (sloshing) in non-deformable spherical shell has been investigated by many researchers as Budiansky [3], Stofan and Armsted [4], Chu[5], Karamanos et al.[6]. The oscillations of the fluid masses result from the lateral displacement or angular rotation of the spherical shell. Others researchers have studied particular cases like the case of a sphere filled with fluid.

Rand and Dimaggio[7] considered the free vibrations for axisymmetric, extensional, non-torsional of fluid-filled elastic spherical shells. Motivated by the fact that human head can be represented as a spherical shell filled by fluid, Engin and Liu[8] considered the free vibration of a thin homogenous spherical shell containing an inviscid irrotational fluid. Advani and Lee [9] investigated the vibration of the fluid-filled shell using higher-order shell theory including transverse shear and rotational inertia. Guarino and Elger [11] have looked at the frequency spectra of a fluid-filled sphere, both with and without a central solid sphere, in order to explore the use of auscultatory percussion as a clinical diagnostic tool. Free vibration of a thin spherical shell filled with a compressible fluid is investigated by Bai and Wu [12].The general non-axisymmetric free vibration of a spherically isotropic elastic spherical shell filled with a compressible fluid medium has been investigated by Chen and Ding [13]. Young [14] studied the free vibration of spheres composed of inviscid compressible liquid cores surrounded by spherical layers of linear elastic, homogeneous and isotropic materials.

The case of hemispherical shells filled with fluid was studied experimentally by Samoilov and Pavlov[15]. Hwang[16] investigated the case of the longitudinal sloshing of liquid in a flexible hemispherical tank supported along the edge, Chung and Rush[17] presented a rigorous and consistent formulation of dynamically coupled problems dealing with motion of a surface-fluid-shell system. A numerical example of a hemispherical bulkhead filled with fluid is modeled.

Komatsu [18][19] used a hybrid method with a fluid mass coefficient added to his system of equations. He also validated his model with experiments on hemispherical shells partially filled with fluid under two boundary conditions: a clamped boundary condition and a free boundary condition.

Recently, Ventsel et al. [20] used a combined formulation of the boundary elements method and finite elements method to study the free vibration of an isotropic simply supported hemispherical shell with different circumferential mode numbers.

For a spherical shell that is partially liquid-filled, if one wishes to consider the hydroelastic vibration developed as consequence of interaction between hydrodynamic pressure of liquid and elastic deformation of the shell, this is a complex problem. Numerical method such as the finite element method (FEM) are therefore used since they are powerful tools that can adequately describe the dynamic behavior of such system which contains complex structures, boundary conditions, materials and loadings. Some powerful commercial FEM software exists, such as ANSYS, ABAQUS and NASTRAN. When using these tools to model such a complex FSI problem, a large numbers of elements are required in order to get good convergence. The hybrid approach presented in this study provides very fast and precise convergence with less numerical cost compared to these commercial software packages.

In this work a combined formulation of shell theory and the standard finite element method (FEM) is applied to model the shell structure. Nodal displacements are found from exact solution of shell theory. This hybrid FEM has been applied to produce efficient and robust models during analysis of both cylindrical and conical shells. A spherical shell which has been filled partially with incompressible and inviscid is modeled in this study. The fluid is characterized as a velocity potential variable at each node of the shell finite element mesh; then fluid and structures are coupled through the linearized Bernoulli's equation and impermeable boundary condition at the fluid-structure interface. Dynamic analysis of the structure under various geometries, boundary conditions and filling ratios is analyzed

2. Finite element formulation

2.1 Structural modeling

In this study the structure is modeled using hybrid finite element method which is a combination of spherical shell theory and classical finite element method. In this hybrid finite element method, the displacement functions are found from exact solution of spherical shell theory rather approximated by polynomial functions done in classical finite element method. In the spherical coordinate system(R,θ,ϕ) shown in Fig. 1, five out of the six equations of equilibrium derived in reference for spherical shells under external load are written as follows :

$$
\begin{aligned}
&\frac{\partial N_\phi}{\partial \phi}+\frac{1}{\sin\phi}\frac{\partial N_{\phi\theta}}{\partial\theta}+\left(N_\phi-N_\theta\right)\cot\phi+Q_\phi=0\\
&\frac{\partial N_{\phi\theta}}{\partial \phi}+\frac{1}{\sin\phi}\frac{\partial N_{\theta}}{\partial\theta}+2N_{\phi\theta}\cot\phi+Q_\theta=0\\
&\frac{\partial Q_\phi}{\partial \phi}+\frac{1}{\sin\phi}\frac{\partial Q_{\theta}}{\partial\theta}+Q_\phi\cot\phi-(N_\phi+N_\theta)=0\\
&\frac{\partial M_\phi}{\partial \phi}+\frac{1}{\sin\phi}\frac{\partial M_{\phi\theta}}{\partial\theta}+\left(M_\phi-M_\theta\right)\cot\phi-RQ_\phi=0\\
&\frac{\partial M_{\phi\theta}}{\partial \phi}+\frac{1}{\sin\phi}\frac{\partial M_{\theta}}{\partial\theta}+2M_{\phi\theta}\cot\phi-RQ_\theta=0
\end{aligned}
\tag{1}
$$

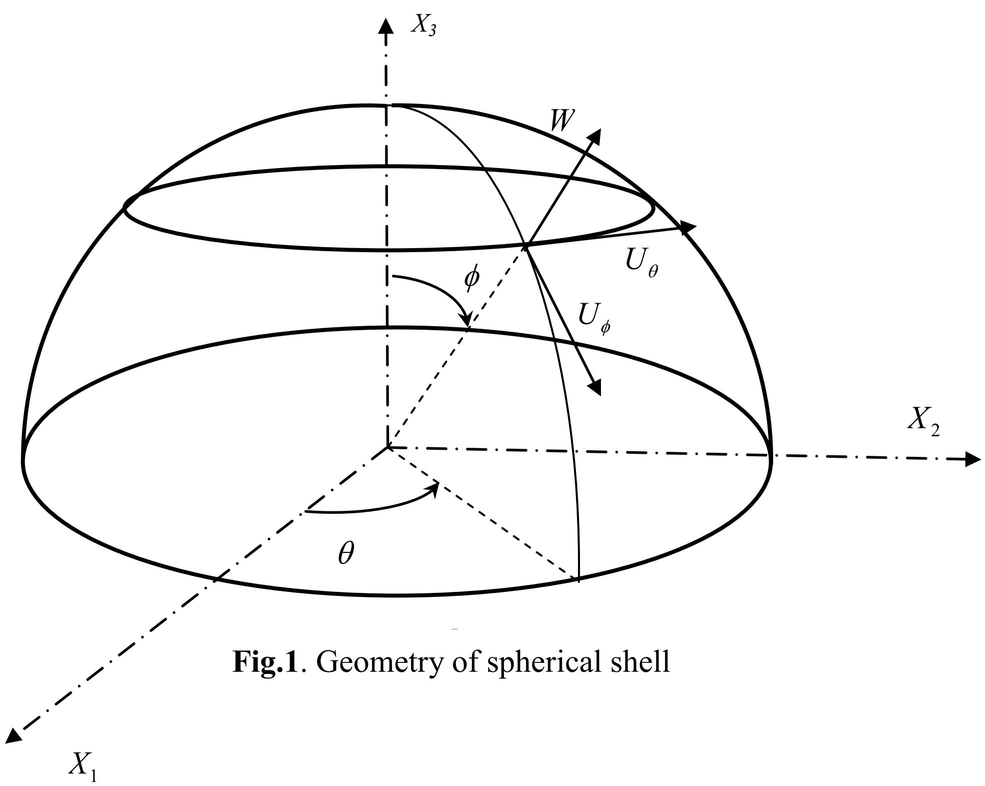

Fig.1. Geometry of spherical shell

Where N_ϕ, N_θ, $N_{\phi\theta}$ are membrane stress resultants; M_ϕ, M_θ, $M_{\phi\theta}$ the bending stress resultants and Q_ϕ, Q_θ the shear forces (Fig. 2). The sixth equation, which is an identity equation for spherical shells, is not presented here.

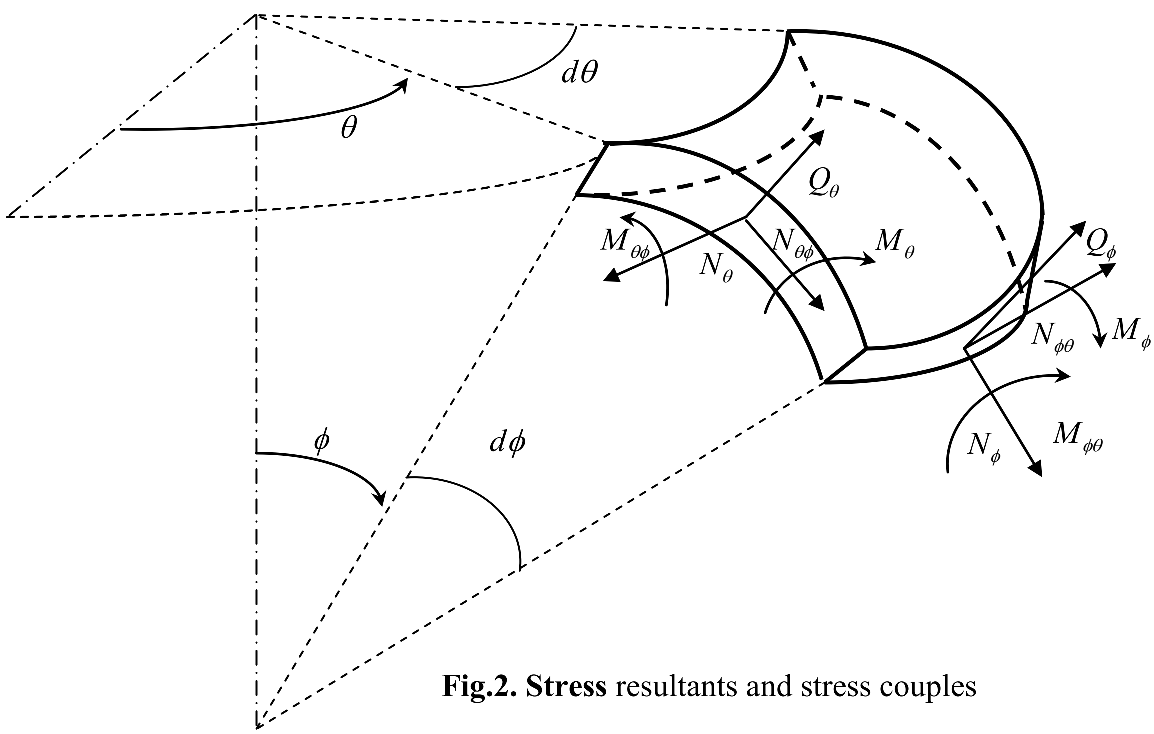

Fig.2. Stress resultants and stress couples

Strain and displacements for three displacements in axial U_ϕ, radial W and circumferential U_θ are related as follows:

$$
\{\varepsilon\}=\begin{Bmatrix}\varepsilon_\phi\\ \varepsilon_\theta\\ 2\varepsilon_{\phi\theta}\\ \kappa_\phi\\ \kappa_\theta\\ 2\kappa_{\phi\theta}\end{Bmatrix}=\begin{Bmatrix}\frac{1}{R}\left(\frac{\partial U_\phi}{\partial\phi}+W\right)\\ \frac{1}{R}(\frac{1}{\sin\phi}\frac{\partial U_\theta}{\partial\theta}+U_\phi\cot\phi+W)\\ \frac{1}{R}(\frac{\partial U_\theta}{\partial\phi}+\frac{1}{\sin\phi}\frac{\partial U_\phi}{\partial\theta}-U_\theta\cot\phi)\\ \frac{1}{R^2}\left(\frac{\partial U_\phi}{\partial\phi}-\frac{\partial^2 W}{\partial\phi^2}\right)\\ \frac{1}{R^2}\left(\frac{1}{\sin\phi}\frac{\partial U_\theta}{\partial\theta}+U_\phi\cot\phi-\frac{1}{\sin^2\phi}\frac{\partial^2 W}{\partial\theta^2}-\cot\phi\frac{\partial W}{\partial\phi}\right)\\ \frac{1}{R^2}\left(\frac{\partial U_\theta}{\partial\phi}+\frac{1}{\sin\phi}\frac{\partial U_\phi}{\partial\theta}-U_\theta\cot\phi+2\frac{1}{\sin\phi}\cot\phi\frac{\partial W}{\partial\theta}-2\frac{1}{\sin\phi}\frac{\partial^2 W}{\partial\phi\partial\theta}\right)\end{Bmatrix} \tag{2}
$$

Displacements U , W and V in the global cartesian coordinate system are related to displacements $U_{\phi i}$, W_i and $U_{\theta i}$ indicated in Fig 3. by:

$$
\begin{Bmatrix}U\\ W\\ V\end{Bmatrix}=\begin{bmatrix}\sin\phi_i & -\cos\phi_i & 0\\ \cos\phi_i & \sin\phi_i & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}U_{\phi i}\\ W_i\\ U_{\theta i}\end{Bmatrix} \tag{3}
$$

The stress vector $\{\sigma\}$ is expressed as function of strain $\{\varepsilon\}$ by

$$
\{\sigma\}=[P]\{\varepsilon\} \tag{4}
$$

Where $[P]$ is the elasticity matrix for an anisotropic shell given by

$$
[P]=\begin{bmatrix}P_{11} & P_{12} & 0 & P_{14} & P_{15} & 0\\ P_{21} & P_{22} & 0 & P_{24} & P_{25} & 0\\ 0 & 0 & P_{33} & 0 & 0 & 0\\ P_{41} & P_{42} & 0 & P_{44} & P_{45} & 0\\ P_{51} & P_{52} & 0 & P_{54} & P_{55} & 0\\ 0 & 0 & P_{36} & 0 & 0 & P_{66}\end{bmatrix} \tag{5}
$$

Upon substitution of equations (2), (4) and (5) into equations (1), a system of equilibrium equations can be obtained as a function of displacements:

$$
\begin{aligned}
L_1\left(U_\phi, W, U_\theta, P_{ij}\right)=0\\
L_2\left(U_\phi, W, U_\theta, P_{ij}\right)=0\\
L_3\left(U_\phi, W, U_\theta, P_{ij}\right)=0
\end{aligned} \tag{6}
$$

These three linear partial differentials operators L_1, L_2 and L_3 are given in the Appendix, and P_{ij} are elements of the elasticity matrix which, for an isotopic thin shell with thickness h is given by:

$$[P]=\begin{bmatrix} D & \nu D & 0 & 0 & 0 & 0 \\ \nu D & D & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-\nu)D}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & K & \nu K & 0 \\ 0 & 0 & 0 & \nu K & K & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-\nu)K}{2} \end{bmatrix} \tag{7}$$

Where $D=\dfrac{Et}{1-\nu^2}$ is the membrane stiffness and $K=\dfrac{Et^3}{12\left(1-\nu^2\right)}$ is the bending stiffness.

The element is a circumferential spherical frustum shown in Fig. 3. It has two nodal circles with four degrees of freedom; axial, radial, circumferential and rotation at each node. This element type makes it possible to use thin shell equations easily to find the exact solution of displacement functions rather than an approximation with polynomial functions as done in classical finite element method. For motions associated with the nth circumferential wave number we may write:

$$\begin{Bmatrix} U_\phi(\phi,\theta) \\ W(\phi,\theta) \\ U_\theta(\phi,\theta) \end{Bmatrix}=\begin{bmatrix} \cos n\theta & 0 & 0 \\ 0 & \cos n\theta & 0 \\ 0 & 0 & \sin n\theta \end{bmatrix}\begin{Bmatrix} u_{\phi n}(\phi) \\ w_n(\phi) \\ u_{\theta n}(\phi) \end{Bmatrix}=[T]\begin{Bmatrix} u_{\phi n}(\phi) \\ w_n(\phi) \\ u_{\theta n}(\phi) \end{Bmatrix} \tag{8}$$

The transversal displacement $w_n(\phi)$ can be expressed as:

$$w_n(\phi)=\sum_{i=1}^{3} w_i^n \tag{9}$$

Where

$$w_i^n = A_i P_{\mu_i}^n(\cos\phi)+B_i Q_{\mu_i}^n(\cos\phi) \tag{10}$$

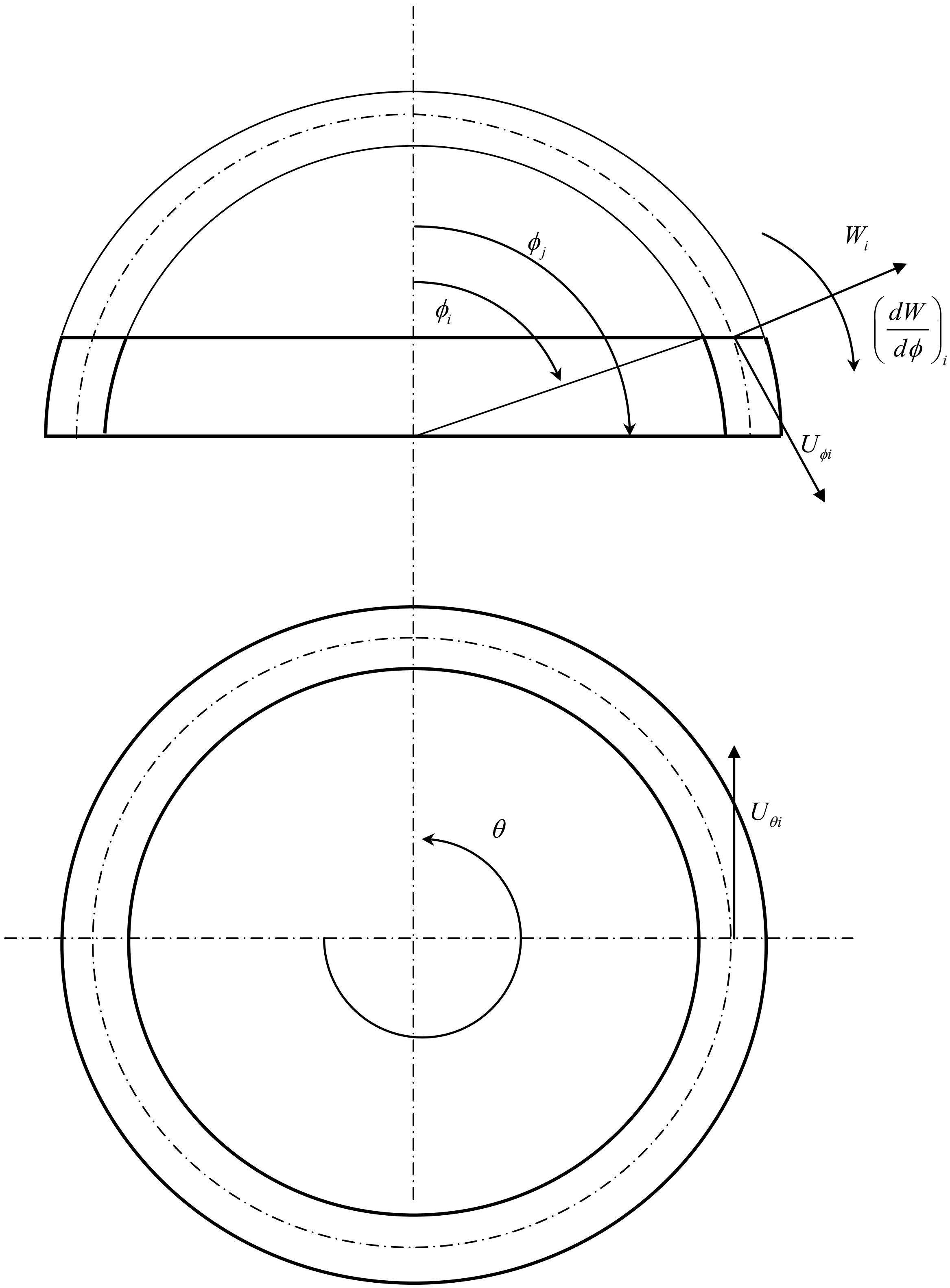

Fig. 3. Spherical frustum element

And where $P_{\mu_i}^{n}\left(\cos\phi\right), Q_{\mu_i}^{n}\left(\cos\phi\right)$ are the associated Legendre functions of the first and second kinds respectively of order n and degree μ_i.

The expression of the axial displacement $u_{\phi n}$ (ϕ) is:

$$u_{\phi n}\left(\phi\right)=\sum_{i=1}^{3}E_i\frac{dw_i^n}{d\phi}-\frac{n^2}{2\sin\phi}\psi\left(\phi\right) \tag{11}$$

Where the coefficient E_i is given by:

$$E_i=\frac{\lambda_i+k(1+\nu)-(1-\nu)}{\left(1+k\right)\left(\lambda_i-1+\nu\right)} \tag{12}$$

The auxiliary function ψ is given by the expression:

$$\psi\left(\phi\right)=A_4P_1^n\left(\cos\phi\right)+B_4Q_1^n\left(\cos\phi\right) \tag{13}$$

Finally the circumferential displacement $u_{\theta n}$ (ϕ) can be expressed as:

$$u_{\theta n}\left(\phi\right)=-n\sum_{i=1}^{3}\frac{1}{\sin\phi}E_iw_i^n+\frac{n}{2}\frac{d\psi}{d\phi} \tag{14}$$

The degree μ_i is obtained from the expression

$$\mu_i=\left(\frac{1}{4}+\lambda_i\right)^{1/2}-\frac{1}{2} \tag{15}$$

Where λ_i is one the roots of the cubic equation:

$$\lambda^3-h_1\lambda^2+h_2\lambda-h_3=0 \tag{16}$$

Where

$$\begin{aligned}h_1&=4\\h_2&=4+(1+k)(1-\nu^2)\\h_3&=2(1+k)(1-\nu^2)\end{aligned} \tag{17}$$

With $k=12\dfrac{R^2}{h^2}$

The above equation has three roots with one root is real and two other are complex conjugate roots.

The Legendre functions $P_{\mu_1}^{n}$, $P_{\mu_1}^{n-1}, Q_{\mu_1}^{n}$ and $Q_{\mu_1}^{n-1}$ are a real functions whereas $P_{\mu_i}^{n}$, $P_{\mu_i}^{n-1}, Q_{\mu_i}^{n}$ and $Q_{\mu_i}^{n-1}$ (i = 2, 3) are complex functions so we can put:

$$
\begin{aligned}
P^n_{\mu_2} &= \mathrm{Re}(P^n_{\mu_2}) + i\,\mathrm{Im}(P^n_{\mu_2}) \\
P^n_{\mu_3} &= \mathrm{Re}(P^n_{\mu_2}) - i\,\mathrm{Im}(P^n_{\mu_2}) \\
Q^n_{\mu_2} &= \mathrm{Re}(Q^n_{\mu_2}) + i\,\mathrm{Im}(Q^n_{\mu_2}) \\
Q^n_{\mu_3} &= \mathrm{Re}(Q^n_{\mu_2}) - i\,\mathrm{Im}(Q^n_{\mu_2}) \\
P^{n-1}_{\mu_2} &= \mathrm{Re}(P^{n-1}_{\mu_2}) + i\,\mathrm{Im}(P^{n-1}_{\mu_2}) \\
P^{n-1}_{\mu_3} &= \mathrm{Re}(P^{n-1}_{\mu_2}) - i\,\mathrm{Im}(P^{n-1}_{\mu_2}) \\
Q^{n-1}_{\mu_2} &= \mathrm{Re}(Q^{n-1}_{\mu_2}) + i\,\mathrm{Im}(Q^{n-1}_{\mu_2}) \\
Q^{n-1}_{\mu_3} &= \mathrm{Re}(P^{n-1}_{\mu_2}) - i\,\mathrm{Im}(Q^{n-1}_{\mu_2})
\end{aligned}
\tag{18}
$$

Setting

$$
\begin{aligned}
(n-\mu_1-1)\left(n+\mu_1\right) &= c_1 \\
(n-\mu_2-1)\left(n+\mu_2\right) &= c_2 + ic_3 \\
(n-\mu_3-1)\left(n+\mu_3\right) &= c_2 - ic_3
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
E_1 &= e_1 \\
E_2 &= e_2 - ie_3 \\
E_3 &= e_2 + ie_3
\end{aligned}
\tag{20}
$$

Substituting equations (18), (19) and (20) in equations (9), (11) and (14) we have:

$$
\begin{aligned}
u_{n\phi}\left(\phi\right) = &\left(-ne_1 \cot\phi P^n_{\mu_1} + e_1 c_1 P^{n-1}_{\mu_1}\right) A_1 \\
&+\left[-ne_2 \cot\phi \,\mathrm{Re}(P^n_{\mu_2}) - ne_3 \cot\phi\,\mathrm{Im}(P^n_{\mu_2}) + (e_2c_2 + e_3c_3)\,\mathrm{Re}(P^{n-1}_{\mu_2}) + (e_3c_2 - e_2c_3)\,\mathrm{Im}(P^{n-1}_{\mu_2})\right](A_2 + A_3) \\
&+\left[\ \ ne_3 \cot\phi \,\mathrm{Re}(P^n_{\mu_2}) - ne_2 \cot\phi\,\mathrm{Im}(P^n_{\mu_2}) - (e_3c_2 - e_2c_3)\,\mathrm{Re}(P^{n-1}_{\mu_2}) + (e_2c_2 + e_3c_3)\,\mathrm{Im}(P^{n-1}_{\mu_2})\right]i(A_2 - A_3) \\
&+\left[-\frac{n^2}{2\sin\phi} P^n_1\right] A_4 \\
&+\left(-ne_1 \cot\phi Q^n_{\mu_1} + e_1 c_1 Q^{n-1}_{\mu_1}\right) B_1 \\
&+\left[-ne_2 \cot\phi \,\mathrm{Re}(Q^n_{\mu_2}) - ne_3 \cot\phi\,\mathrm{Im}(Q^n_{\mu_2}) + (e_2c_2 + e_3c_3)\,\mathrm{Re}(Q^{n-1}_{\mu_2}) + (e_3c_2 - e_2c_3)\,\mathrm{Im}(Q^{n-1}_{\mu_2})\right](B_2 + B_3) \\
&+\left[\ \ ne_3 \cot\phi \,\mathrm{Re}(Q^n_{\mu_2}) - ne_2 \cot\phi\,\mathrm{Im}(Q^n_{\mu_2}) - (e_3c_2 - e_2c_3)\,\mathrm{Re}(Q^{n-1}_{\mu_2}) + (e_2c_2 + e_3c_3)\,\mathrm{Im}(Q^{n-1}_{\mu_2})\right]i(B_2 - B_3) \\
&+\left[-\frac{n^2}{2\sin\phi} Q^n_1\right] B_4
\end{aligned}
$$

$$w_n(\phi) = P_{\mu_1}^n A_1 + \mathrm{Re}(P_{\mu_2}^n)(A_2 + A_3) + \mathrm{Im}(P_{\mu_2}^n) i(A_2 - A_3) + Q_{\mu_1}^n B_1 + \mathrm{Re}(Q_{\mu_2}^n)(B_2 + B_3) + \mathrm{Im}(Q_{\mu_2}^n) i(B_2 - B_3)$$

$$\begin{aligned}
u_{n\theta}(\phi) = & -ne_1 P_{\mu_1}^n \frac{1}{\sin\phi} A_1 \\
& -\left[ne_2 \frac{1}{\sin\phi} \mathrm{Re}(P_{\mu_2}^n) + ne_3 \frac{1}{\sin\phi} \mathrm{Im}(P_{\mu_2}^n) \right](A_2 + A_3) + \left[ne_3 \frac{1}{\sin\phi} \mathrm{Re}(P_{\mu_2}^n) - ne_2 \frac{1}{\sin\phi} \mathrm{Im}(P_{\mu_2}^n) \right] i(A_2 - A_3) \\
& + \left[-\frac{n^2}{2} \cot\phi P_1^n + \frac{n}{2}(n-2)(n+1) P_1^{n-1} \right] A_4 \\
& -ne_1 Q_{\mu_1}^n \frac{1}{\sin\phi} B_1 \\
& -\left[ne_2 \frac{1}{\sin\phi} \mathrm{Re}(Q_{\mu_2}^n) + ne_3 \frac{1}{\sin\phi} \mathrm{Im}(Q_{\mu_2}^n) \right](B_2 + B_3) + \left[ne_3 \frac{1}{\sin\phi} \mathrm{Re}(Q_{\mu_2}^n) - ne_2 \frac{1}{\sin\phi} \mathrm{Im}(Q_{\mu_2}^n) \right] i(B_2 - B_3) \\
& + \left[-\frac{n^2}{2} \cot\phi Q_1^n + \frac{n}{2}(n-2)(n+1) Q_1^{n-1} \right] B_4
\end{aligned} \tag{21}$$

In deriving the above relation we used the recursive relations:

$$\begin{aligned}
\frac{dP_{\mu_i}^n}{d\phi} &= -n\cot\phi P_{\mu_i}^n + (n - \mu_i - 1)(n + \mu_i) P_{\mu_i}^{n-1} \\
\frac{dQ_{\mu_i}^n}{d\phi} &= -n\cot\phi Q_{\mu_i}^n + (n - \mu_i - 1)(n + \mu_i) Q_{\mu_i}^{n-1}
\end{aligned} \tag{22}$$

Using matrix formulation, the displacement functions can be expressed as follows:

$$\begin{Bmatrix} U_\phi(\phi,\theta) \\ W(\phi,\theta) \\ U_\theta(\phi,\theta) \end{Bmatrix} = [T] \begin{Bmatrix} u_{\phi n}(\phi) \\ w_n(\phi) \\ u_{\theta n}(\phi) \end{Bmatrix} = [T][R]\{C\} \tag{23}$$

The vector $\{C\}$ is given by the expression:

$$\{C\}^T = \{ A_1 \quad A_2 + A_3 \quad i(A_2 - A_3) \quad A_4 \quad B_1 \quad B_2 + B_3 \quad i(B_2 - B_3) \quad B_4 \} \tag{24}$$

The elements of matrix $[R]$ are given in the Appendix.

In the finite element method, the vector C is eliminated in favor of displacements at elements nodes. At each finite element node, the three displacements (axial, transversal and circumferential) and the rotation are applied. The displacement of node *i* are defined by the vector:

$$\{\delta_i\} = \left\{ u_{\phi n}^i \quad w_n^i \quad \left(\frac{dw_n}{dx}\right)^i \quad u_{\theta n}^i \right\}^T \tag{25}$$

The element in Fig. 3 with two nodal lines (*i* and *j*) and eight degrees of freedom will have the following nodal displacement vector:

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \left\{ u_{\phi n}^i \quad w_n^i \quad \left(\frac{dw_n}{d\phi}\right)^i \quad u_{\theta n}^i \quad u_{\phi n}^j \quad w_n^j \quad \left(\frac{dw_n}{d\phi}\right)^j \quad u_{\theta n}^j \right\}^T = [A]\{C\} \tag{26}$$

With

$$\begin{aligned}
\frac{dw_n}{d\phi} &= \left(-n\cot\phi P_{\mu_1}^n + c_1 P_{\mu_1}^{n-1}\right)A_1 + \left[-n\cot\phi\,\mathrm{Re}(P_{\mu_2}^n) + c_2\,\mathrm{Re}(P_{\mu_2}^{n-1}) - c_3\,\mathrm{Im}(P_{\mu_2}^{n-1})\right](A_2 + A_3) \\
&+ \left[-n\cot\phi\,\mathrm{Im}(P_{\mu_2}^n) + c_3\,\mathrm{Re}(P_{\mu_2}^{n-1}) + c_2\,\mathrm{Im}(P_{\mu_2}^{n-1})\right]i(A_2 - A_3) + \left(-n\cot\phi Q_{\mu_1}^n + c_1 Q_{\mu_1}^{n-1}\right)B_1 \\
&+ \left[-n\cot\phi\,\mathrm{Re}(Q_{\mu_2}^n) + c_2\,\mathrm{Re}(Q_{\mu_2}^{n-1}) - c_3\,\mathrm{Im}(Q_{\mu_2}^{n-1})\right](B_2 + B_3) \\
&+ \left[-n\cot\phi\,\mathrm{Im}(Q_{\mu_2}^n) + c_3\,\mathrm{Re}(Q_{\mu_2}^{n-1}) + c_2\,\mathrm{Im}(Q_{\mu_2}^{n-1})\right]i(B_2 - B_3)
\end{aligned} \tag{27}$$

The terms of matrix $[A]$ are obtained from the values of matrix $[R]$ and $\frac{dw_n}{dx}$ are given in the appendix.

Now, pre-multiplying by $[A]^{-1}$ equation (26) one obtains the matrix of the constant C_i as a function of the degree of freedom:

$$\{C\} = [A]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{28}$$

Finally, one substitutes the vector $\{C\}$ into equation and obtains the displacement functions as follows:

$$\begin{Bmatrix} U_\phi(\phi,\theta) \\ W(\phi,\theta) \\ U_\theta(\phi,\theta) \end{Bmatrix} = [T][R][A]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{29}$$

The strain vector $\{\varepsilon\}$ can be determined from the displacement functions U_ϕ, U_θ, W and the deformation –displacement as:

$$\{\varepsilon\}=\begin{bmatrix}[T] & [0]\\ [0] & [T]\end{bmatrix}[Q]\{C\}=\begin{bmatrix}[T] & [0]\\ [0] & [T]\end{bmatrix}[Q][A]^{-1}\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix}=[B]\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix} \tag{30}$$

Where matrix $[Q]$ is given in the appendix.

This relation can be used to find the stress vector, equation (4), in terms of the nodal degrees of freedom vector:

$$\{\sigma\}=[P][B]\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix} \tag{31}$$

Based on the finite element formulation, the local stiffness and mass matrices are:

$$\begin{aligned}[k]_{loc} &= \iint_A [B]^T[P][B]\,dA\\ [m]_{loc} &= \rho\, h\iint_A [N]^T[N]\,dA\end{aligned} \tag{32}$$

Where ρ the density and h is the thickness of shell.

The surface element of the shell wall is $dA=R^2\sin\phi d\phi d\theta$. After integrating over θ, the preceding equations become

$$\begin{aligned}[k]_{loc} &= \left[A^{-1}\right]^T\left(\pi R^2\int_{\phi_i}^{\phi_j}[Q]^T[P][Q]\sin\phi d\phi\right)\left[A^{-1}\right]=\left[A^{-1}\right]^T[G]\left[A^{-1}\right]\\ [m]_{loc} &= \rho h\left[A^{-1}\right]^T\left(\pi R^2\int_{\phi_i}^{\phi_j}[R]^T[R]\sin\phi d\phi\right)\left[A^{-1}\right]=\rho h\left[A^{-1}\right]^T[S]\left[A^{-1}\right]\end{aligned} \tag{33}$$

In the global system the element stiffness and mass matrices are

$$\begin{aligned}[k] &= [LG]^T\left[A^{-1}\right]^T[G]\left[A^{-1}\right][LG]\\ [m] &= \rho h[LG]^T\left[A^{-1}\right]^T[S]\left[A^{-1}\right][LG]\end{aligned} \tag{34}$$

Where

$$
[LG] = \begin{bmatrix} \sin\phi_i & -\cos\phi_i & 0 & 0 & 0 & 0 & 0 & 0 \\ \cos\phi_i & \sin\phi_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\phi_j & -\cos\phi_j & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\phi_j & \sin\phi_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{35}
$$

From these equations, one can assemble the mass and stiffness matrix for each element to obtain the mass and stiffness matrices for the whole shell: $[M]$ and $[K]$. Each elementary matrix is 8x8, therefore the final dimensions of $[M]$ and $[K]$ will be 4(N+1) where N is the number of elements of the shell.

2.2 Fluid modeling

The Laplace equation satisfied by velocity potential for inviscid, incompressible and irrotational fluid in the spherical system is written as:

$$
\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\varphi}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2\varphi}{\partial\theta^2} = 0 \tag{36}
$$

Where the velocity components are:

$$
V_\phi = U_f + \frac{1}{r}\frac{\partial\varphi}{\partial\phi} \qquad V_r = \frac{\partial\varphi}{\partial r} \qquad V_\theta = \frac{1}{r\sin\phi}\frac{\partial\varphi}{\partial\theta} \tag{37}
$$

Using the Bernouilli equation, hydrodynamic pressure in terms of velocity potentiel φ and fluid density ρ_f is found as :

$$
P_f = -\rho_f\left(\frac{\partial\varphi}{\partial t} + \frac{U_f}{r}\frac{\partial\varphi}{\partial\phi}\right)_{r=R} \tag{38}
$$

The impermeability condition, which ensures contact between the shell surface and the peripheral fluid is written as:

$$
V_r\big|_{r=R} = \frac{\partial\varphi}{\partial r}\bigg|_{r=R} = \left(\frac{\partial W}{\partial t} + \frac{U_f}{r}\frac{\partial W}{\partial\phi}\right)\bigg|_{r=R} \tag{39}
$$

With

$$
W = \sum_{j=1}^{3}\left(A_j P_{\mu_j}^n(\cos\phi) + B_j Q_{\mu_j}^n(\cos\phi)\right)\cos n\theta e^{i\omega t} \tag{40}
$$

Method of separation of variables for the velocity potential solution can be done as follows :

$$\varphi(\phi, r, \theta) = \sum_{j=1}^{3} R_j(r) S_j(\phi, \theta, t)$$

Placing this relation into the impermeability condition (39), we can find the function $S_j(\phi, \theta, t)$ in term of radial displacement:

$$S_j(\phi, \theta, t) = \frac{1}{R_j^{'}(R)} \left(\frac{\partial W}{\partial t} + \frac{U_f}{r} \frac{\partial W}{\partial \phi} \right) \Bigg|_{r=R} \tag{41}$$

Hence the equation becomes

$$\varphi(\phi, r, \theta) = \sum_{j=1}^{3} \frac{R_j(r)}{R_j^{'}(R)} \left(\frac{\partial W}{\partial t} + \frac{U_f}{r} \frac{\partial W}{\partial \phi} \right) \Bigg|_{r=R} \tag{42}$$

With substitution of the above equation into Laplace equation (36), the following second order equation in terms of $R_j(r)$ is obtained

$$R_j^{''}(r) + \frac{2}{r} R_j^{'}(r) - \frac{\mu_j \left(\mu_j + 1\right)}{r^2} R_j(r) = 0 \tag{43}$$

Solution of the above differential equation yields the following:

$$R_j(r) = A_j r^{\mu_j} + \frac{B_j}{r^{\mu_j}} \tag{44}$$

For internal flow $B_j = 0$

Finally, the hydrodynamic pressure in terms of radial displacement is written:

$$P_f = -\rho_f \sum_{j=1}^{3} \frac{R}{\mu_j} \left[\ddot{W}_j + 2 \frac{U_f}{R} \dot{W}_j' + \frac{U_f^2}{R^2} W_j'' \right] \tag{45}$$

We put :

$$\begin{aligned} \frac{R}{\mu_1} &= f_1 \\ \frac{R}{\mu_2} &= f_2 - i f_3 \\ \frac{R}{\mu_3} &= f_2 + i f_3 \end{aligned} \tag{46}$$

And the pressure loading in terms of nodal degrees of freedom is written as:

$$\{P\} = \begin{Bmatrix} 0 \\ P_f \\ 0 \end{Bmatrix} = -\rho_f [T] \left[R_1^f \right] \left[A^{-1} \right] \begin{Bmatrix} \ddot{\delta}_i \\ \ddot{\delta}_j \end{Bmatrix} - 2\rho_f \frac{U_f}{R} [T] \left[R_2^f \right] \left[A^{-1} \right] \begin{Bmatrix} \dot{\delta}_i \\ \dot{\delta}_j \end{Bmatrix} - 2\rho_f \frac{U_f^2}{R^2} [T] \left[R_3^f \right] \left[A^{-1} \right] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{47}$$

Where matrix $\left[R_1^f \right]$ is given by :

$$\left[R_1^f\right]=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{\mu_1}^n & \mathrm{Re}(P_{\mu_2}^n) & \mathrm{Im}(P_{\mu_2}^n) & 0 & Q_{\mu_1}^n & \mathrm{Re}(Q_{\mu_2}^n) & \mathrm{Im}(Q_{\mu_2}^n) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\left[F\right] \tag{48}$$

Where $\left[F\right]$ is expressed as:

$$\left[F\right]=\begin{bmatrix} f_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_2 & -f_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_3 & f_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f_2 & -f_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & f_3 & f_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{49}$$

The matrix $\left[R_2^f\right]$ is given by :

$$\begin{aligned}\left[R_2^f\right]=-n\cot\phi&\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{\mu_1}^n & \mathrm{Re}(P_{\mu_2}^n) & \mathrm{Im}(P_{\mu_2}^n) & 0 & Q_{\mu_1}^n & \mathrm{Re}(Q_{\mu_2}^n) & \mathrm{Im}(Q_{\mu_2}^n) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\left[F\right] \\ &+\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{\mu_1}^{n-1} & \mathrm{Re}(P_{\mu_2}^{n-1}) & \mathrm{Im}(P_{\mu_2}^{n-1}) & 0 & Q_{\mu_1}^{n-1} & \mathrm{Re}(Q_{\mu_2}^{n-1}) & \mathrm{Im}(Q_{\mu_2}^{n-1}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\left[F\right]\left[C\right]\end{aligned} \tag{50}$$

Where matrix $\left[C\right]$ is given by:

$$\left[C\right]=\begin{bmatrix} c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_2 & c_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_3 & c_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_2 & c_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c_3 & c_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{51}$$

The matrix $\left[R_3^f\right]$ is given by :

$$\left[R_3^f\right]=n\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{\mu_1}^n & \mathrm{Re}(P_{\mu_2}^n) & \mathrm{Im}(P_{\mu_2}^n) & 0 & Q_{\mu_1}^n & \mathrm{Re}(Q_{\mu_2}^n) & \mathrm{Im}(Q_{\mu_2}^n) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}[F]$$

$$+\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{\mu_1}^n & \mathrm{Re}(P_{\mu_2}^n) & \mathrm{Im}(P_{\mu_2}^n) & 0 & Q_{\mu_1}^n & \mathrm{Re}(Q_{\mu_2}^n) & \mathrm{Im}(Q_{\mu_2}^n) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}[F][C] \tag{52}$$

$$-\cot\phi\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ P_{\mu_1}^{n-1} & \mathrm{Re}(P_{\mu_2}^{n-1}) & \mathrm{Im}(P_{\mu_2}^{n-1}) & 0 & Q_{\mu_1}^{n-1} & \mathrm{Re}(Q_{\mu_2}^{n-1}) & \mathrm{Im}(Q_{\mu_2}^{n-1}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}[F][C]$$

The general force vector due the fluid pressure loading is given by:

$$\left\{F_p\right\}=\iint_A [N]^T\{P\}\,dA \tag{53}$$

After substituting for pressure field vector and matrix $[N]$ in the above equation, the local matrix $\left[m_f\right]$ can be found from the following:

$$\left[m_f\right]_{loc}=-\rho_f\left[A^{-1}\right]^T\left(\pi R^2\int_{\phi_i}^{\phi_j}[R]^T\left[R_1^f\right]\sin\phi d\phi\right)\left[A^{-1}\right]=-\rho_f\left[A^{-1}\right]^T\left[S_f\right]\left[A^{-1}\right] \tag{54}$$

The local damping matrix is given by:

$$\left[c_f\right]_{loc}=-2\rho_f\frac{U_f}{R}\left[A^{-1}\right]^T\left(\pi R^2\int_{\phi_i}^{\phi_j}[R]^T\left[R_2^f\right]\sin\phi d\phi\right)\left[A^{-1}\right]=-2\rho_f\frac{U_f}{R}\left[A^{-1}\right]^T\left[D_f\right]\left[A^{-1}\right] \tag{55}$$

Finally the local stiffness matrix is given by:

$$\left[k_f\right]_{loc}=-\rho_f\frac{U_f^2}{R^2}\left[A^{-1}\right]^T\left(\pi R^2\int_{\phi_i}^{\phi_j}[R]^T\left[R_3^f\right]\sin\phi d\phi\right)\left[A^{-1}\right]=-\rho_f\frac{U_f^2}{R^2}\left[A^{-1}\right]^T\left[G_f\right]\left[A^{-1}\right] \tag{56}$$

In the global system the element stiffness and mass matrices are

$$\begin{aligned}\left[m_f\right]&=-\rho_f[LG]^T\left[A^{-1}\right]^T\left[S_f\right]\left[A^{-1}\right][LG]\\ \left[c_f\right]&=-2\rho_f\frac{U_f}{R}[LG]^T\left[A^{-1}\right]^T\left[D_f\right]\left[A^{-1}\right][LG]\\ \left[k_f\right]&=-\rho_f\frac{U_f^2}{R^2}[LG]^T\left[A^{-1}\right]^T\left[G_f\right]\left[A^{-1}\right][LG]\end{aligned} \tag{57}$$

From these equations, one can assemble the mass and stiffness matrix for each element to obtain the mass and stiffness matrices for the whole shell: $\left[M_f\right]$ and $\left[K_f\right]$.

The governing equation which accounts for fluid-shell interaction in the presence of axial internal pressure is derived as:

$$\left[\left[M_s\right]-\left[M_f\right]\right]\begin{Bmatrix}\ddot{\delta}_i\\ \ddot{\delta}_j\end{Bmatrix}-\left[C_f\right]\begin{Bmatrix}\dot{\delta}_i\\ \dot{\delta}_j\end{Bmatrix}+\left[\left[K_s\right]-\left[K_f\right]\right]\begin{Bmatrix}\delta_i\\ \delta_j\end{Bmatrix}=0$$

Where subscripts *s* and *f* refer to shells in vacuum and fluid respectively.

3. Results and discussion

In this section numerical results are presented and compared with existing experimental, analytical and numerical data.

3.1 Validation and comparison

The main advantages of this proposed hybrid is its fast and precise convergence; 12 elements were required for the convergence of the frequency for a clamped spherical shell.

For the cases investigated in the present paper, the predicted dimensionless frequencies are expressed by the following relation:

$$\Omega=\omega R\left(\frac{\rho}{E}\right)^{\frac{1}{2}} \tag{58}$$

Where:

ω is the natural angular frequency.

R is the radius of the reference surface.

ρ is the density.

E is the modulus of elasticity.

Results for different filling ratios and modes numbers compared to experimental, theoretical and numerical analyses are presented.

3.2 case of a spherical shell with ϕ_0 =60°

The case considered here is a simply supported spherical shell with ϕ_0 =60° with the following characteristics and studied by Komatsu [18]: the material density ρ= 2270 kg/m^3, the Poisson coefficient ν=0.3, Young modulus of elasticity : E=70 GPa, the radius to thickness ratio R/h=243

The figure 5 shows that when the shell is partially filled, the dimensionless frequency initially drops sharply, Then as the shell becomes fuller, the frequency drops less quickly. Free-surface effects of the liquid surface and sloshing of the fluid are not taken into account in this study. This assumption relies on the fact that the sloshing frequencies have a period of vibration that is much longer than the period of vibration of the spherical shell. As can be seen, there is perfect agreement between both methods.

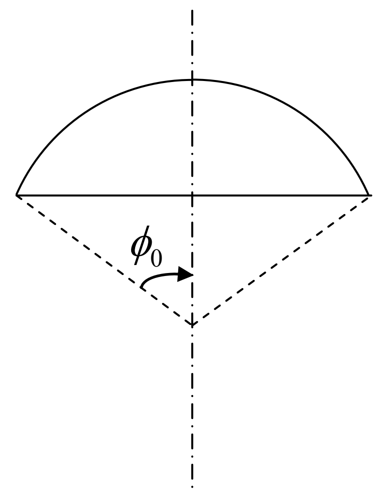

Fig.4. Definition of angle ϕ_0

The comparatively good accuracy of our method can be explained by that fact that the formulation used is a combination of the finite element method and classical shell theory where the displacement functions are derived from exact solutions of shell equations. On the other hand, integrations of all matrices (solid and fluid) are calculated numerically over the solid–fluid element. This numerical model can easily be used to study partially filled spherical shells by imposing a null density of fluid for the circumferential finite elements which are not submerged.

The third study we carried out is on the effect of radius to thickness ratio R/h on the natural frequencies in both the case where the shell is empty or full. Figure 6 shows as the mass of shell is greater when the shell is thick, the effect of fluid is less important in a thick shell than for a thin shell. The same figure shows that ratio of natural frequencies of an empty shell and full shell is of order 10 for R/h=1000 . But this ratio was 3 for R/h=243.

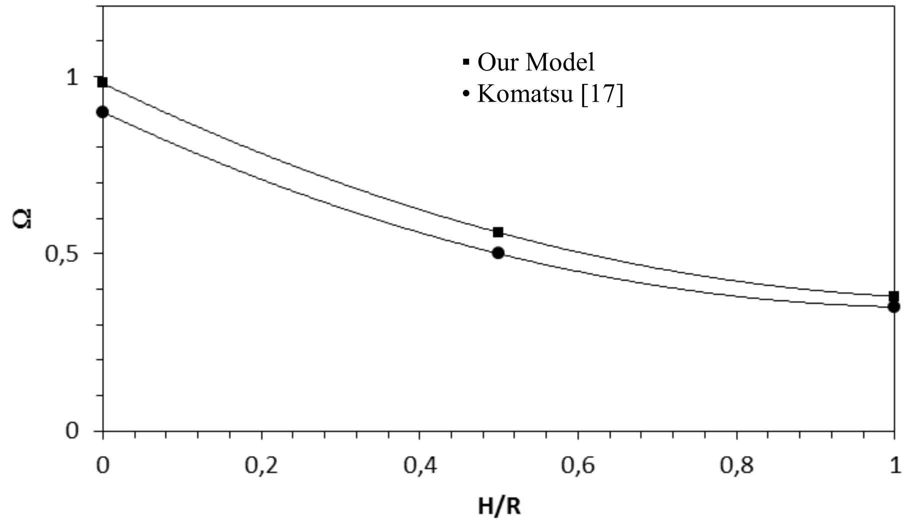

Fig. 5. Dimensionless frequency as function of liquid depth in simply supported spherical shell of with ϕ_0 =60°

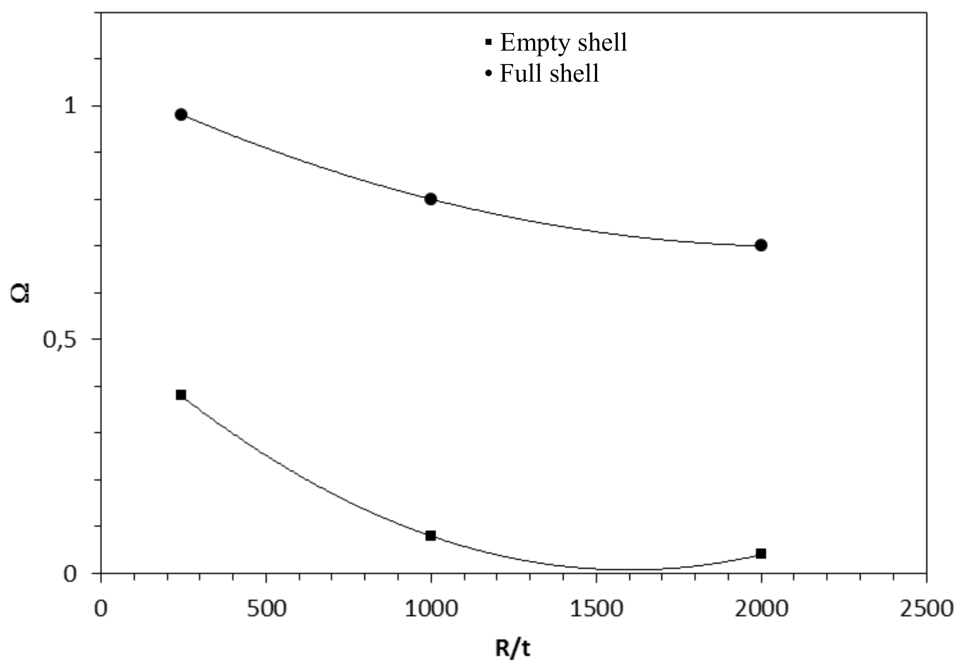

Fig. 6. Dimensionless frequency as function of radius to thickness ratio R/h in simply supported spherical shell of with ϕ_0 =60°

3.2 case of a hemispherical shell

This case was investigated by many authors**.** The problem of a hemispherical shell completely filled with water was investigated by Ventsel et Al [20]**.** The solution of the problem of hydroelastic vibrations has been obtained using the methods of the boundary element (BEM) and the finite element (FEM). Thes data are obtained by applying the simply supported condition which is an adequate condition for liquid storage tanks. The hemispherical shell is considered empty or filled with fluid and having the following parameters: the shell radius R=5.08m, the thickness h=0.0254m, the modulus of elasticity E=70GPa, Poisson ratio ν=0.3, the material density ρ = 2270 kg/m^3, the fluid density ρ_f = 1000 kg/m^3. Very good agreement can be seen. The ratio of empty shell frequency and completely filled shell frequency is 4.5 for the first axial mode.

	Ventsel et al[20]		Present theory	
(n, m)	H/R =0	H/R=1	H/R=0	H/R=1
2,1	0.8987	0.2004	0.9057	0.2134
2,2	0.9611	0.2579	0.9658	0.2604
2,3	0.9838	0.3020	0.9901	0.3102

Table 1. Dimensionless frequencies for a simply supported hemispherical shell

The case of a hemispherical shell completely filled with water clamped along equator was investigated experimentally by Samoilov and Pavlov. The characteristics of the shell were as follows: the shell radius R=0.133m, the thickness h=0.0007m, the modulus of elasticity E=4.016GPa, Poisson ratio ν=0.4, the material density ρ = 1180kg/m^3

The table 2 shows the dimensionless frequencies obtained by these authors and compared to the frequencies obtained by our model.

m	Samoilov and Pavlov[15]	Present Theory
1	0.0978	0.1038
2	0.1382	0.1451
3	0.1676	0.1789

Table 2. Dimensionless frequencies for a clamped hemispherical shell completely filled with fluid

Data resulting from experiments conducted by Kana and Nagy [10] on a clamped hemispherical shell filled with water are shown in table 3. The shell has a density of 2.59 10^{-4} lb-s^2/in^4 and has a radius 5 in and a thickness of 0.03 in. The elastic modulus is 10^7 lb/in^2 and the Poisson coefficient is 0.3.

m	Kana and Nagy[10]	Present Theory
1	0.1199	0.1239
2	0.1919	0.2036
3	0.2398	0.2436

Table 3. Dimensionless frequencies for a clamped hemispherical shell completely filled with fluid

The fourth example is the case of a clamped hemispherical shell that was studied experimentally by Hwang [16]. The shell is made of aluminum with density of 2.59 10^{-4} lb-s^2/in^4 and has a radius 200 in and a thickness of 0.1 in. The elastic modulus is 10^7 lb/in^2 and the Poisson coefficient is 0.3. The fluid inside the shell is liquid oxygen with a density of 1.06 10^{-4} lb- s^2/in^4. This example of a hemispherical bulkhead filled with liquid oxygen was modeled by Chung and Rush [17] and investigated numerically. The same study was conducted by Komatsu and Matsuhima [19] experimentally. The results are presented in table 4.

Present model	Hwang[14]	Chung and Rush [15]	Komatsu [17]
0.066	0.0689	0.0625	0.065

Table 4. Dimensionless frequencies for a clamped hemispherical shell completely filled with liquid oxygen

4. Conclusion

The problem of free vibration of a partially liquid-filled spherical shell under different shell geometries, filling ratios and boundary conditions with different radius to thickness ratios is investigated. An efficient hybrid finite element method is presented to analyze the dynamic behavior of liquid- filled spherical shell. Shell theory of spherical shell is coupled Laplace equation of an inviscid fluid to account for hydrodynamic pressure of an internal fluid. This theoretical approach is much more precise than usual finite element methods because the displacement functions are derived from exact solutions of shell equilibrium equations for spherical shells. The mass and stiffness matrices are determined by numerical integration. The velocity potential and Bernoulli's equation are adopted to express the fluid pressure acting on the structure which yields three forces (inertial, centrifugal Coriolis) in the case of flowing fluid.

The results obtained for conical shells with various geometric configurations and different boundary conditions are compared with results available in the literature. Very good agreement was found. This approach resulted in a very precise element that leads to fast convergence and less numerical difficulties from the computational point of view.

To the best of the authors' knowledge, this paper reports the first comparison made between works which deal with spherical shells subjected to internal fluid effects. The proposed hybrid finite element method provides the capability to analyze cases involving application of different complex boundaries and loading patterns for spherical shell

Appendix

$$L_1(U_\phi, U_\theta, W) = \left(\frac{P_{11}}{R} + \frac{P_{41}}{R^2}\right)\left[\frac{\partial}{\partial\phi}\left(\frac{\partial U_\phi}{\partial\phi} + W\right) + \left(\frac{\partial U_\phi}{\partial\phi} + W\right)\cot(\phi)\right]$$

$$+\left(\frac{P_{12}}{R} + \frac{P_{42}}{R^2}\right)\left[\frac{\partial}{\partial\phi}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta} + U_\phi\cot(\phi) + W\right) + \left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta} + U_\phi\cot(\phi) + W\right)\cot(\phi)\right]$$

$$+\frac{1}{R}\left(\frac{P_{14}}{R} + \frac{P_{44}}{R^2}\right)\left[\frac{\partial}{\partial\phi}\left(\frac{\partial U_\phi}{\partial\phi} - \frac{\partial^2 W}{\partial\phi^2}\right) + \left(\frac{\partial U_\phi}{\partial\phi} - \frac{\partial^2 W}{\partial\phi^2}\right)\cot(\phi)\right]$$

$$+\frac{1}{R}\left(\frac{P_{15}}{R} + \frac{P_{45}}{R^2}\right)\left[\frac{\partial}{\partial\phi}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta} + U_\phi\cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial\theta^2} - \frac{\partial W}{\partial\phi}\cot(\phi)\right) + \left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta} + U_\phi\cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial\theta^2} - \frac{\partial W}{\partial\phi}\cot(\phi)\right)\cot(\phi)\right]$$

$$-\left(\frac{P_{21}}{R} + \frac{P_{51}}{R^2}\right)\left(\frac{\partial U_\phi}{\partial\phi} + W\right)\cot(\phi)$$

$$-\left(\frac{P_{22}}{R} + \frac{P_{52}}{R^2}\right)\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta} + U_\phi\cot(\phi) + W\right)\cot(\phi)$$

$$-\frac{1}{R}\left(\frac{P_{24}}{R} + \frac{P_{54}}{R^2}\right)\left(\frac{\partial U_\phi}{\partial\phi} - \frac{\partial^2 W}{\partial\phi^2}\right)\cot(\phi)$$

$$-\frac{1}{R}\left(\frac{P_{25}}{R} + \frac{P_{55}}{R^2}\right)\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta} + U_\phi\cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial\theta^2} - \frac{\partial W}{\partial\phi}\cot(\phi)\right)\cot(\phi)$$

$$+\left(\frac{P_{33}}{R} + \frac{P_{63}}{R^2}\right)\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\left(\frac{\partial U_\theta}{\partial\phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial\theta} - U_\theta\cot(\phi)\right)$$

$$+\frac{1}{R}\left(\frac{P_{36}}{R} + \frac{P_{66}}{R^2}\right)\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\left(\frac{\partial U_\theta}{\partial\phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial\theta} - U_\theta\cot(\phi) + 2\frac{\cot(\phi)}{\sin(\phi)}\frac{\partial W}{\partial\theta} - \frac{2}{\sin(\phi)}\frac{\partial^2 W}{\partial\phi\,\partial\theta}\right)$$

$$L_2(U_\phi, U_\theta, W) = \left(\frac{P_{21}}{R} + \frac{P_{51}}{R^2}\right)\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\left(\frac{\partial U_\phi}{\partial\phi} + W\right)$$

$$+\left(\frac{P_{22}}{R} + \frac{P_{52}}{R^2}\right)\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta} + U_\phi\cot(\phi) + W\right)$$

$$+\frac{1}{R}\left(\frac{P_{24}}{R} + \frac{P_{54}}{R^2}\right)\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\left(\frac{\partial U_\phi}{\partial\phi} - \frac{\partial^2 W}{\partial\phi^2}\right)$$

$$+\frac{1}{R}\left(\frac{P_{25}}{R} + \frac{P_{55}}{R^2}\right)\frac{1}{\sin(\phi)}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial\theta} + U_\phi\cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial\theta^2} - \frac{\partial W}{\partial\phi}\cot(\phi)\right)$$

$$+\left(\frac{P_{33}}{R} + \frac{P_{63}}{R^2}\right)\left[\frac{\partial}{\partial\phi}\left(\frac{\partial U_\theta}{\partial\phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial\theta} - U_\theta\cot(\phi)\right) + 2\left(\frac{\partial U_\theta}{\partial\phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial\theta} - U_\theta\cot(\phi)\right)\cot(\phi)\right]$$

$$+\frac{1}{R}\left(\frac{P_{36}}{R} + \frac{P_{66}}{R^2}\right)\left[\frac{\partial}{\partial\phi}\left(\frac{\partial U_\theta}{\partial\phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial\theta} - U_\theta\cot(\phi) + 2\frac{\cot(\phi)}{\sin(\phi)}\frac{\partial W}{\partial\theta} - \frac{2}{\sin(\phi)}\frac{\partial^2 W}{\partial\phi\,\partial\theta}\right) + 2\left(\frac{\partial U_\theta}{\partial\phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial\theta} - U_\theta\cot(\phi) + 2\frac{\cot(\phi)}{\sin(\phi)}\frac{\partial W}{\partial\theta} - \frac{2}{\sin(\phi)}\frac{\partial^2 W}{\partial\phi\,\partial\theta}\right)\cot t(\phi)\right]$$

$$L_3(U_\phi, U_\theta, W) = -\left(\frac{P_{11}}{R} + \frac{P_{21}}{R}\right)\left(\frac{\partial U_\phi}{\partial \phi} + W\right)$$

$$-\left(\frac{P_{12}}{R} + \frac{P_{22}}{R}\right)\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) + W\right)$$

$$-\left(\frac{P_{14}}{R^2} + \frac{P_{24}}{R^2}\right)\left(\frac{\partial U_\phi}{\partial \phi} - \frac{\partial^2 W}{\partial \phi^2}\right)$$

$$-\left(\frac{P_{15}}{R^2} + \frac{P_{25}}{R^2}\right)\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial \theta^2} - \frac{\partial W}{\partial \phi}\cot(\phi)\right)$$

$$+\frac{P_{41} - P_{51}}{R^2}\left(\cot(\phi)\frac{\partial}{\partial \phi}\left(\frac{\partial U_\phi}{\partial \phi} + W\right) - \left(\frac{\partial U_\phi}{\partial \phi} + W\right)\right)$$

$$+\frac{1}{R^2 \sin(\phi)}\left[P_{41}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial}{\partial \phi}\left(\frac{\partial U_\phi}{\partial \phi} + W\right)\right) + P_{51}\frac{\partial}{\partial \theta}\left(\frac{\partial U_\phi}{\partial \phi} + W\right)\right]$$

$$+\frac{P_{42} - P_{52}}{R^2}\left(\cot(\phi)\frac{\partial}{\partial \phi}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) + W\right) - \left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) + W\right)\right)$$

$$+\frac{1}{R^2 \sin(\phi)}\left[P_{42}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial}{\partial \phi}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) + W\right)\right) + P_{52}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) + W\right)\right]$$

$$+\frac{P_{44} - P_{54}}{R^3}\left(\cot(\phi)\frac{\partial}{\partial \phi}\left(\frac{\partial U_\phi}{\partial \phi} - \frac{\partial^2 W}{\partial \phi^2}\right) - \left(\frac{\partial U_\phi}{\partial \phi} - \frac{\partial^2 W}{\partial \phi^2}\right)\right)$$

$$+\frac{1}{R^3 \sin(\phi)}\left[P_{44}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial}{\partial \phi}\left(\frac{\partial U_\phi}{\partial \phi} - \frac{\partial^2 W}{\partial \phi^2}\right)\right) + P_{54}\frac{\partial}{\partial \theta}\left(\frac{\partial U_\phi}{\partial \phi} - \frac{\partial^2 W}{\partial \phi^2}\right)\right]$$

$$+\frac{P_{45} - P_{55}}{R^3}\left(\cot(\phi)\frac{\partial}{\partial \phi}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial \theta^2} - \frac{\partial W}{\partial \phi}\cot(\phi)\right) - \left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial \theta^2} - \frac{\partial W}{\partial \phi}\cot(\phi)\right)\right)$$

$$+\frac{1}{R^3 \sin(\phi)}\left[P_{45}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial}{\partial \phi}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial \theta^2} - \frac{\partial W}{\partial \phi}\cot(\phi)\right)\right) + P_{55}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin(\phi)}\frac{\partial U_\theta}{\partial \theta} + U_\phi \cot(\phi) - \frac{1}{\sin^2(\phi)}\frac{\partial^2 W}{\partial \theta^2} - \frac{\partial W}{\partial \phi}\cot(\phi)\right)\right]$$

$$+\frac{P_{63}}{R^2 \sin(\phi)}\left[\frac{\partial^2}{\partial \phi \partial \theta}\left(\frac{\partial U_\theta}{\partial \phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial \theta} - U_\theta \cot(\phi)\right) + 3\cot(\phi)\left(\frac{\partial}{\partial \theta}\frac{\partial U_\theta}{\partial \phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial \theta} - U_\theta \cot(\phi)\right)\right]$$

$$+\frac{P_{66}}{R^3 \sin(\phi)}\left[\frac{\partial^2}{\partial \phi \partial \theta}\left(\frac{\partial U_\theta}{\partial \phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial \theta} - U_\theta \cot(\phi) + 2\frac{\cot(\phi)}{\sin(\phi)}\frac{\partial W}{\partial \theta} - \frac{2}{\sin(\phi)}\frac{\partial^2 W}{\partial \phi\, \partial \theta}\right) + 3\cot(\phi)\left(\frac{\partial U_\theta}{\partial \phi} + \frac{1}{\sin(\phi)}\frac{\partial U_\varphi}{\partial \theta} - U_\theta \cot(\phi) + 2\frac{\cot(\phi)}{\sin(\phi)}\frac{\partial W}{\partial \theta} - \frac{2}{\sin(\phi)}\frac{\partial^2 W}{\partial \phi\, \partial \theta}\right)\right]$$

$$R(1,1)=\left(-e_1 n\cot\phi P_{\mu_1}^n+e_1c_1P_{\mu_1}^{n-1}\right)$$

$$R(1,2)=-ne_2\cot\phi\,\mathrm{Re}(P_{\mu_2}^n)-ne_3\cot\phi\,\mathrm{Im}(P_{\mu_2}^n)+(e_2c_2+e_3c_3)\,\mathrm{Re}(P_{\mu_2}^{n-1})+(e_3c_2-e_2c_3)\,\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$R(1,3)=ne_3\cot\phi\,\mathrm{Re}(P_{\mu_2}^n)-ne_2\cot\phi\,\mathrm{Im}(P_{\mu_2}^n)-(e_3c_2-e_2c_3)\,\mathrm{Re}(P_{\mu_2}^{n-1})+(e_2c_2+e_3c_3)\,\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$R(1,4)=-\frac{n^2}{2\sin\phi}P_1^n$$

$$R(1,5)=\left(-e_1 n\cot\phi Q_{\mu_1}^n+e_1c_1Q_{\mu_1}^{n-1}\right)$$

$$R(1,6)=-ne_2\cot\phi\,\mathrm{Re}(Q_{\mu_2}^n)-ne_3\cot\phi\,\mathrm{Im}(Q_{\mu_2}^n)+(e_2c_2+e_3c_3)\,\mathrm{Re}(Q_{\mu_2}^{n-1})+(e_3c_2-e_2c_3)\,\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$R(1,7)=ne_3\cot\phi\,\mathrm{Re}(Q_{\mu_2}^n)-ne_2\cot\phi\,\mathrm{Im}(Q_{\mu_2}^n)-(e_3c_2-e_2c_3)\,\mathrm{Re}(Q_{\mu_2}^{n-1})+(e_2c_2+e_3c_3)\,\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$R(1,8)=-\frac{n^2}{2\sin\phi}Q_1^n$$

$$R(2,1)=P_{\mu_1}^n$$

$$R(2,2)=\mathrm{Re}(P_{\mu_2}^n)$$

$$R(2,3)=\mathrm{Im}(P_{\mu_2}^n)$$

$$R(2,4)=0$$

$$R(2,5)=Q_{\mu_1}^n$$

$$R(2,6)=\mathrm{Re}(Q_{\mu_2}^n)$$

$$R(2,7)=\mathrm{Im}(Q_{\mu_2}^n)$$

$$R(2,8)=0$$

$$R(3,1)=-ne_1\frac{1}{\sin\phi}P_{\mu_1}^n$$

$$R(3,2)=-ne_2\frac{1}{\sin\phi}\mathrm{Re}(P_{\mu_2}^n)-ne_3\frac{1}{\sin\phi}\mathrm{Im}(P_{\mu_2}^n)$$

$$R(3,3)=ne_3\frac{1}{\sin\phi}\mathrm{Re}(P_{\mu_2}^n)-ne_2\frac{1}{\sin\phi}\mathrm{Im}(P_{\mu_2}^n)$$

$$R(3,4)=-\frac{n^2}{2}\cot\phi P_1^n+\frac{n}{2}(n-2)(n+1)P_1^{n-1}$$

$$R(3,5)=-ne_1\frac{1}{\sin\phi}Q_{\mu_1}^n$$

$$R(3,6) = -ne_2 \frac{1}{\sin\phi}\operatorname{Re}(Q_{\mu_2}^n) - ne_3 \frac{1}{\sin\phi}\operatorname{Im}(Q_{\mu_2}^n)$$

$$R(3,7) = ne_3 \frac{1}{\sin\phi}\operatorname{Re}(Q_{\mu_2}^n) - ne_2 \frac{1}{\sin\phi}\operatorname{Im}(Q_{\mu_2}^n)$$

$$R(3,8) = -\frac{n^2}{2}\cot\phi Q_1^n + \frac{n}{2}\left(n-2\right)\left(n+1\right)Q_1^{n-1}$$

$A(1,j) = R(1,j)$, $A(2,j) = R(2,j)$, $A(3,j) = \dfrac{dw_n}{d\phi}(j)$, $A(4,j) = R(3,j)$ with $\phi = \phi_i$ $A(5,j) = R(1,j)$,

$A(6,j) = R(2,j)$; $A(7,j) = \dfrac{dw_n}{d\phi}(j)$, $A(8,j) = R(3,j)$ with $\phi = \phi_j$

j=1,…,8

$$Q(1,1)=\frac{1}{R}\left[e_1c_1+ne_1(\frac{1}{\sin^2\phi}+n\cot^2\phi)+1\right]P_{\mu_1}^{n}-\frac{e_1}{r}c_1\cot\phi P_{\mu_1}^{n-1}$$

$$Q(1,2)=\frac{1}{R}\left[\left(e_2c_2+e_3c_3\right)+ne_2(\frac{1}{\sin^2\phi}+n\cot^2\phi)+1\right]\mathrm{Re}(P_{\mu_2}^{n})$$

$$+\frac{1}{R}\left[\left(e_3c_2-e_2c_3\right)+ne_3(\frac{1}{\sin^2\phi}+n\cot^2\phi)\right]\mathrm{Im}(P_{\mu_2}^{n})$$

$$-\frac{1}{R}\left(e_2c_2+e_3c_3\right)\cot\phi\,\mathrm{Re}(P_{\mu_2}^{n-1})-\frac{1}{R}\left(e_3c_2-e_2c_3\right)\cot\phi\,\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$Q(1,3)=-\frac{1}{R}\left[\left(e_3c_2-e_2c_3\right)+ne_3(\frac{1}{\sin^2\phi}+n\cot^2\phi)\right]\mathrm{Re}(P_{\mu_2}^{n})$$

$$+\frac{1}{R}\left[\left(e_2c_2+e_3c_3\right)+ne_2(\frac{1}{\sin^2\phi}+n\cot^2\phi)+1\right]\mathrm{Im}(P_{\mu_2}^{n})$$

$$+\frac{1}{R}\left(e_3c_2-e_2c_3\right)\cot\phi\,\mathrm{Re}(P_{\mu_2}^{n-1})-\frac{1}{R}\left(e_2c_2+e_3c_3\right)\cot\phi\,\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$Q(1,4)=\frac{n^2}{2R}(n+1)\frac{1}{\sin\phi}\cot\phi P_1^{n}-\frac{n^2}{2R}(n-2)(n+1)\frac{1}{\sin\phi}P_1^{n-1}$$

$$Q(1,5)=\frac{1}{R}\left[e_1c_1+ne_1(\frac{1}{\sin^2\phi}+n\cot^2\phi)+1\right]Q_{\mu_1}^{n}-\frac{e_1}{r}c_1\cot\phi Q_{\mu_1}^{n-1}$$

$$Q(1,6)=\frac{1}{R}\left[\left(e_2c_2+e_3c_3\right)+ne_2(\frac{1}{\sin^2\phi}+n\cot^2\phi)+1\right]\mathrm{Re}(Q_{\mu_2}^{n})$$

$$+\frac{1}{R}\left[\left(e_3c_2-e_2c_3\right)+ne_3(\frac{1}{\sin^2\phi}+n\cot^2\phi)\right]\mathrm{Im}(Q_{\mu_2}^{n})$$

$$-\frac{1}{R}\left(e_2c_2+e_3c_3\right)\cot\phi\,\mathrm{Re}(Q_{\mu_2}^{n-1})-\frac{1}{R}\left(e_3c_2-e_2c_3\right)\cot\phi\,\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$Q(1,7)=-\frac{1}{R}\left[\left(e_3c_2-e_2c_3\right)+ne_3(\frac{1}{\sin^2\phi}+n\cot^2\phi)\right]\mathrm{Re}(Q_{\mu_2}^{n})$$

$$+\frac{1}{R}\left[\left(e_2c_2+e_3c_3\right)+ne_2(\frac{1}{\sin^2\phi}+n\cot^2\phi)+1\right]\mathrm{Im}(Q_{\mu_2}^{n})$$

$$+\frac{1}{R}\left(e_3c_2-e_2c_3\right)\cot\phi\,\mathrm{Re}(Q_{\mu_2}^{n-1})-\frac{1}{R}\left(e_2c_2+e_3c_3\right)\cot\phi\,\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$Q(1,8)=\frac{n^2}{2R}(n+1)\frac{1}{\sin\phi}\cot\phi Q_1^{n}-\frac{n^2}{2R}(n-2)(n+1)\frac{1}{\sin\phi}Q_1^{n-1}$$

$$Q(2,1)=\frac{1}{R}\left[1-ne_1(n\frac{1}{\sin^2\phi}+\cot^2\phi)\right]P_{\mu_1}^n+\frac{e_1}{r}c_1\cot\phi P_{\mu_1}^{n-1}$$

$$Q(2,2)=\frac{1}{R}\left[1-ne_2(n\frac{1}{\sin^2\phi}+\cot^2\phi)\right]\mathrm{Re}(P_{\mu_2}^n)-\frac{ne_3}{R}(n\frac{1}{\sin^2\phi}+\cot^2\phi)\,\mathrm{Im}(P_{\mu_2}^n)$$
$$+\frac{1}{R}(e_2c_2+e_3c_3)\cot\phi\,\mathrm{Re}(P_{\mu_2}^{n-1})+\frac{1}{R}(e_3c_2-e_2c_3)\cot\phi\,\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$Q(2,3)=\frac{ne_3}{R}(n\frac{1}{\sin^2\phi}+\cot^2\phi)\,\mathrm{Re}(P_{\mu_2}^n)+\frac{1}{R}\left[1-ne_2(n\frac{1}{\sin^2\phi}+\cot^2\phi)\right]\mathrm{Im}(P_{\mu_2}^n)$$
$$-\frac{1}{R}(e_3c_2-e_2c_3)\cot\phi\,\mathrm{Re}(P_{\mu_2}^{n-1})+\frac{1}{R}(e_2c_2+e_3c_3)\cot\phi\,\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$Q(2,4)=-\frac{n^2}{2R}\left(n+1\right)\frac{1}{\sin\phi}\cot\phi P_1^n+\frac{n^2}{2R}(n-2)\left(n+1\right)\frac{1}{\sin\phi}P_1^{n-1}$$

$$Q(2,5)=\frac{1}{R}\left[1-ne_1(\frac{1}{\sin^2\phi}+n\cot^2\phi)\right]Q_{\mu_1}^n+\frac{e_1}{r}c_1\cot\phi Q_{\mu_1}^{n-1}$$

$$Q(2,6)=\frac{1}{R}\left[1-ne_2(n\frac{1}{\sin^2\phi}+\cot^2\phi)\right]\mathrm{Re}(Q_{\mu_2}^n)-\frac{ne_3}{R}(n\frac{1}{\sin^2\phi}+\cot^2\phi)\,\mathrm{Im}(Q_{\mu_2}^n)$$
$$+\frac{1}{R}(e_2c_2+e_3c_3)\cot\phi\,\mathrm{Re}(Q_{\mu_2}^{n-1})+\frac{1}{R}(e_3c_2-e_2c_3)\cot\phi\,\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$Q(2,7)=\frac{ne_3}{R}(n\frac{1}{\sin^2\phi}+\cot^2\phi)\,\mathrm{Re}(Q_{\mu_2}^n)+\frac{1}{R}\left[1-ne_2(n\frac{1}{\sin^2\phi}+\cot^2\phi)\right]\mathrm{Im}(Q_{\mu_2}^n)$$
$$-\frac{1}{R}(e_3c_2-e_2c_3)\cot\phi\,\mathrm{Re}(Q_{\mu_2}^{n-1})+\frac{1}{R}(e_2c_2+e_3c_3)\cot\phi\,\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$Q(2,8)=-\frac{n^2}{2R}\left(n+1\right)\frac{1}{\sin\phi}\cot\phi Q_1^n+\frac{n^2}{2R}(n-2)\left(n+1\right)\frac{1}{\sin\phi}Q_1^{n-1}$$

$$Q(3,1)=\frac{2n}{R}e_1(n+1)\frac{1}{\sin\phi}\cot\phi P_{\mu_1}^{n}-\frac{2n}{R}e_1c_1\frac{1}{\sin\phi}P_{\mu_1}^{n-1}$$

$$Q(3,2)=\frac{2n}{R}e_2(n+1)\frac{1}{\sin\phi}\cot\phi\,\mathrm{Re}(P_{\mu_2}^{n})+\frac{2n}{R}e_3(n+1)\frac{1}{\sin\phi}\cot\phi\,\mathrm{Im}(P_{\mu_2}^{n})$$
$$-\frac{2n}{R}(e_2c_2+e_3c_3)\frac{1}{\sin\phi}\mathrm{Re}(P_{\mu_2}^{n-1})-\frac{2n}{R}(e_3c_2-e_2c_3)\frac{1}{\sin\phi}\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$Q(3,3)=-\frac{2n}{R}e_3(n+1)\frac{1}{\sin\phi}\cot\phi\,\mathrm{Re}(P_{\mu_2}^{n})+\frac{2n}{R}e_2(n+1)\frac{1}{\sin\phi}\cot\phi\,\mathrm{Im}(P_{\mu_2}^{n})$$
$$+\frac{2n}{R}(e_3c_2-e_2c_3)\frac{1}{\sin\phi}\mathrm{Re}(P_{\mu_2}^{n-1})-\frac{2n}{R}(e_2c_2+e_3c_3)\frac{1}{\sin\phi}\mathrm{Im}(P_{\mu_2}^{n-1})$$

$$Q(3,4)=\frac{n}{2R}(n+1)\left[(n-2)+n\left(\frac{1}{\sin^2\phi}+\cot^2\phi\right)\right]P_1^{n}-\frac{n}{R}(n-2)(n+1)\cot\phi P_1^{n-1}$$

$$Q(3,5)=\frac{2n}{R}e_1(n+1)\frac{1}{\sin\phi}\cot\phi Q_{\mu_1}^{n}-\frac{2n}{R}c_1\frac{1}{\sin\phi}Q_{\mu_1}^{n-1}$$

$$Q(3,6)=\frac{2n}{R}e_2(n+1)\frac{1}{\sin\phi}\cot\phi\,\mathrm{Re}(Q_{\mu_2}^{n})+\frac{2n}{R}e_3(n+1)\frac{1}{\sin\phi}\cot\phi\,\mathrm{Im}(Q_{\mu_2}^{n})$$
$$-\frac{2n}{R}(e_2c_2+e_3c_3)\frac{1}{\sin\phi}\mathrm{Re}(Q_{\mu_2}^{n-1})-\frac{2n}{R}(e_3c_2-e_2c_3)\frac{1}{\sin\phi}\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$Q(3,7)=-\frac{2n}{R}e_3(n+1)\frac{1}{\sin\phi}\cot\phi\,\mathrm{Re}(Q_{\mu_2}^{n})+\frac{2n}{R}e_2(n+1)\frac{1}{\sin\phi}\cot\phi\,\mathrm{Im}(Q_{\mu_2}^{n})$$
$$+\frac{2n}{R}(e_3c_2-e_2c_3)\frac{1}{\sin\phi}\mathrm{Re}(Q_{\mu_2}^{n-1})-\frac{2n}{R}(e_2c_2+e_3c_3)\frac{1}{\sin\phi}\mathrm{Im}(Q_{\mu_2}^{n-1})$$

$$Q(3,8)=\frac{n}{2R}(n+1)\left[(n-2)+n\left(\frac{1}{\sin^2\phi}+\cot^2\phi\right)\right]Q_1^{n}-\frac{n}{R}(n-2)(n+1)\cot\phi Q_1^{n-1}$$

$$Q(4,1)=\frac{1}{R^2}\left[\left(e_1-1\right)c_1+n\left(e_1-1\right)\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]P_{\mu_1}^{n}-\frac{e_1-1}{R^2}c_1\cot\phi P_{\mu_1}^{n-1}$$

$$\begin{aligned}Q(4,2)=&\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3+n\left(e_2-1\right)\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Re}\left(P_{\mu_2}^{n}\right)\\&+\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3+ne_3\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Im}\left(P_{\mu_2}^{n}\right)\\&-\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\,\mathrm{Re}\left(P_{\mu_2}^{n-1}\right)-\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\,\mathrm{Im}\left(P_{\mu_2}^{n-1}\right)\end{aligned}$$

$$\begin{aligned}Q(4,3)=&-\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3+ne_3\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Re}\left(P_{\mu_2}^{n}\right)\\&+\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3+n\left(e_2-1\right)\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Im}\left(P_{\mu_2}^{n}\right)\\&+\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\,\mathrm{Re}\left(P_{\mu_2}^{n-1}\right)-\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\,\mathrm{Im}\left(P_{\mu_2}^{n-1}\right)\end{aligned}$$

$$Q(4,4)=\frac{n^2}{2R^2}\left(n+1\right)\frac{1}{\sin\phi}\cot\phi P_1^{n}-\frac{n^2}{2R^2}(n-2)(n+1)\frac{1}{\sin\phi}P_1^{n-1}$$

$$Q(4,5)=\frac{1}{R^2}\left[\left(e_1-1\right)c_1+n\left(e_1-1\right)\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]Q_{\mu_1}^{n}-\frac{e_1-1}{R^2}c_1\cot\phi Q_{\mu_1}^{n-1}$$

$$\begin{aligned}Q(4,6)=&\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3+n\left(e_2-1\right)\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Re}\left(Q_{\mu_2}^{n}\right)\\&+\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3+ne_3\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Im}\left(Q_{\mu_2}^{n}\right)\\&-\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\,\mathrm{Re}\left(Q_{\mu_2}^{n-1}\right)-\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\,\mathrm{Im}\left(Q_{\mu_2}^{n-1}\right)\end{aligned}$$

$$\begin{aligned}Q(4,7)=&-\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3+ne_3\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Re}\left(Q_{\mu_2}^{n}\right)\\&+\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3+n\left(e_2-1\right)\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]\mathrm{Im}\left(Q_{\mu_2}^{n}\right)\\&+\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\,\mathrm{Re}\left(Q_{\mu_2}^{n-1}\right)-\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\,\mathrm{Im}\left(Q_{\mu_2}^{n-1}\right)\end{aligned}$$

$$Q(4,8)=\frac{n^2}{2R^2}\left(n+1\right)\frac{1}{\sin\phi}\cot\phi Q_1^{n}-\frac{n^2}{2R^2}(n-2)(n+1)\frac{1}{\sin\phi}Q_1^{n-1}$$

$$Q(5,1)=\frac{n}{R^2}\left(1-e_1\right)\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)P_{\mu_1}^{n}+\frac{(e_1-1)}{R^2}c_1\cot\phi P_{\mu_1}^{n-1}$$

$$Q(5,2)=\frac{n}{R^2}\left(1-e_2\right)\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\operatorname{Re}\left(P_{\mu_2}^{n}\right)-\frac{ne_3}{R^2}\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\operatorname{Im}\left(P_{\mu_2}^{n}\right)$$

$$+\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\operatorname{Re}\left(P_{\mu_2}^{n-1}\right)+\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\operatorname{Im}\left(P_{\mu_2}^{n-1}\right)$$

$$Q(5,3)=\frac{ne_3}{R^2}\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\operatorname{Re}\left(P_{\mu_2}^{n}\right)+\frac{n}{R^2}\left(1-e_2\right)\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\operatorname{Im}\left(P_{\mu_2}^{n}\right)$$

$$-\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\operatorname{Re}\left(P_{\mu_2}^{n-1}\right)+\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\operatorname{Im}\left(P_{\mu_2}^{n-1}\right)$$

$$Q(5,4)=-\frac{n}{2R^2}\left(n+1\right)\frac{1}{\sin\phi}\cot\phi P_1^{n}+\frac{n}{2R^2}\left(n-2\right)\left(n+1\right)\frac{1}{\sin\phi}P_1^{n-1}$$

$$Q(5,5)=\frac{n}{R^2}\left(1-e_1\right)\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)Q_{\mu_1}^{n}+\frac{(e_1-1)}{R^2}c_1\cot\phi Q_{\mu_1}^{n-1}$$

$$Q(5,6)=\frac{n}{R^2}\left(1-e_2\right)\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\operatorname{Re}\left(Q_{\mu_2}^{n}\right)-\frac{ne_3}{R^2}\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\operatorname{Im}\left(Q_{\mu_2}^{n}\right)$$

$$+\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\operatorname{Re}\left(Q_{\mu_2}^{n-1}\right)+\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\operatorname{Im}\left(Q_{\mu_2}^{n-1}\right)$$

$$Q(5,7)=\frac{ne_3}{R^2}\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\operatorname{Re}\left(Q_{\mu_2}^{n}\right)+\frac{n}{R^2}\left(1-e_2\right)\left(\cot^2\phi+n\frac{1}{\sin^2\phi}\right)\operatorname{Im}\left(Q_{\mu_2}^{n}\right)$$

$$-\frac{1}{R^2}\left[e_3c_2-\left(e_2-1\right)c_3\right]\cot\phi\operatorname{Re}\left(Q_{\mu_2}^{n-1}\right)+\frac{1}{R^2}\left[\left(e_2-1\right)c_2+e_3c_3\right]\cot\phi\operatorname{Im}\left(Q_{\mu_2}^{n-1}\right)$$

$$Q(5,8)=-\frac{n}{2R^2}\left(n+1\right)\frac{1}{\sin\phi}\cot\phi Q_1^{n}+\frac{n}{2R^2}\left(n-2\right)\left(n+1\right)\frac{1}{\sin\phi}Q_1^{n-1}$$

$$Q(6,1)=\frac{2n}{R^2}(n+1)(e_1-1)\frac{1}{\sin\phi}\cot\phi P_{\mu_1}^n+\frac{2n}{R^2}(1-e_1)c_1\frac{1}{\sin\phi}P_{\mu_1}^{n-1}$$

$$Q(6,2)=\frac{2n(n+1)}{R^2}(e_2-1)\frac{1}{\sin\phi}\cot\phi\,\mathrm{Re}\left(P_{\mu_2}^n\right)+\frac{2n(n+1)}{R^2}e_3\frac{1}{\sin\phi}\cot\phi\,\mathrm{Im}\left(P_{\mu_2}^n\right)$$

$$-\frac{2n}{R^2}\left[(e_2-1)c_2+e_3c_3\right]\frac{1}{\sin\phi}\mathrm{Re}\left(P_{\mu_2}^{n-1}\right)-\frac{2n}{R^2}\left[e_3c_2-(e_2-1)c_3\right]\frac{1}{\sin\phi}\mathrm{Im}\left(P_{\mu_2}^{n-1}\right)$$

$$Q(6,3)=-\frac{2n(n+1)}{R^2}e_3\frac{1}{\sin\phi}\cot\phi\,\mathrm{Re}\left(P_{\mu_2}^n\right)+\frac{2n(n+1)}{R^2}(e_2-1)\frac{1}{\sin\phi}\cot\phi\,\mathrm{Im}\left(P_{\mu_2}^n\right)$$

$$+\frac{2n}{R^2}\left[e_3c_2-(e_2-1)c_3\right]\frac{1}{\sin\phi}\mathrm{Re}\left(P_{\mu_2}^{n-1}\right)-\frac{2n}{R^2}\left[(e_2-1)c_2+e_3c_3\right]\frac{1}{\sin\phi}\mathrm{Im}\left(P_{\mu_2}^{n-1}\right)$$

$$Q(6,4)=\frac{n}{2R^2}(n+1)\left[(n-2)+n\left(\frac{1}{\sin^2\phi}+\cot^2\phi\right)\right]P_1^n-\frac{n}{R^2}(n-2)(n+1)\cot\phi P_1^{n-1}$$

$$Q(6,5)=\frac{2n}{R^2}(n+1)(e_1-1)\frac{1}{\sin\phi}\cot\phi Q_{\mu_1}^n+\frac{2n}{R^2}(1-e_1)c_1\frac{1}{\sin\phi}Q_{\mu_1}^{n-1}$$

$$Q(6,6)=\frac{2n(n+1)}{R^2}(e_2-1)\frac{1}{\sin\phi}\cot\phi\,\mathrm{Re}\left(Q_{\mu_2}^n\right)+\frac{2n(n+1)}{R^2}e_3\frac{1}{\sin\phi}\cot\phi\,\mathrm{Im}\left(Q_{\mu_2}^n\right)$$

$$-\frac{2n}{R^2}\left[(e_2-1)c_2+e_3c_3\right]\frac{1}{\sin\phi}\mathrm{Re}\left(Q_{\mu_2}^{n-1}\right)-\frac{2n}{R^2}\left[e_3c_2-(e_2-1)c_3\right]\frac{1}{\sin\phi}\mathrm{Im}\left(Q_{\mu_2}^{n-1}\right)$$

$$Q(6,7)=-\frac{2n(n+1)}{R^2}e_3\frac{1}{\sin\phi}\cot\phi\,\mathrm{Re}\left(Q_{\mu_2}^n\right)+\frac{2n(n+1)}{R^2}(e_2-1)\frac{1}{\sin\phi}\cot\phi\,\mathrm{Im}\left(Q_{\mu_2}^n\right)$$

$$+\frac{2n}{R^2}\left[e_3c_2-(e_2-1)c_3\right]\frac{1}{\sin\phi}\mathrm{Re}\left(Q_{\mu_2}^{n-1}\right)-\frac{2n}{R^2}\left[(e_2-1)c_2+e_3c_3\right]\frac{1}{\sin\phi}\mathrm{Im}\left(Q_{\mu_2}^{n-1}\right)$$

$$Q(6,8)=\frac{n}{2R^2}(n+1)\left[(n-2)+n\left(\frac{1}{\sin^2\phi}+\cot^2\phi\right)\right]Q_1^n-\frac{n}{R^2}(n-2)(n+1)\cot\phi Q_1^{n-1}$$

In deriving the above relation we used the recursive relations:

$$\frac{d^2P_\mu^n}{d\phi^2}=\left[(n-\mu-1)(n+\mu)+n\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]P_\mu^n-\cot\phi(n-\mu-1)(n+\mu)P_\mu^{n-1}$$

$$\frac{d^2Q_\mu^n}{d\phi^2}=\left[(n-\mu-1)(n+\mu)+n\left(\frac{1}{\sin^2\phi}+n\cot^2\phi\right)\right]Q_\mu^n-\cot\phi(n-\mu-1)(n+\mu)Q_\mu^{n-1}$$

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