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An Extension of Lyapunov's First Method to Nonlinear Systems with Non-Continuously Differentiable Vector Fields

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***Abstract*—This paper investigates the extension of Lyapunov's first method to nonlinear systems in the case where the $\mathscr{C}^1$-regularity assumption, i.e., the underlying vector field is continuously differentiable, is not satisfied. It is shown that if this regularity assumption is not fulfilled, the Hurwitz nature of the Jacobian matrix, if it exists, does not guarantee the stability of the original nonlinear system. Under weaker assumptions than the $\mathscr{C}^1$-regularity, namely the existence of the directional derivatives of the vector field, conditions for guaranteeing the local exponential stability of the nonlinear system are derived.**

***Index Terms*—Lyapunov methods, Stability of nonlinear systems.**

I. INTRODUCTION

LYAPUNOV'S stability theory is one of the central topics in the field of dynamic systems and automatic control [1], [2], [3]. One of the most fundamental tools is Lyapunov's first method, also referred to as Lyapunov's indirect method (see, e.g., Theorem 4.7 in [1]), that ensures that the local stability of a nonlinear system with $\mathscr{C}^1$ vector field can be deduced from that of its tangent linearization evaluated at the considered equilibrium point. The necessity of the $\mathscr{C}^1$-regularity of the vector field in the proof of this theorem is twofold. First, it ensures that the vector field is locally Lipschitz continuous, guaranteeing, by virtue of the Cauchy-Lipschitz theorem [1, Th.3.1], the existence and the uniqueness of solutions to the ordinary differential equations. Second, it provides a first order Taylor expansion of the vector field at the equilibrium point, enabling to establish a link between the stability properties of the linearized system and the original nonlinear one.

In the context of nonsmooth analysis [4], numerous results have been reported regarding the application of Lyapunov's second method to the analysis of non-smooth systems using non $\mathscr{C}^1$ Lyapunov functions [2], [5], [6], [7], [8]. An alternative is to perform stability analysis in the framework of robust Lyapunov stability theory by embedding the system into an uncertain setting with a time-varying parameter [9], [10]. However, the embedding step induces, in general, conservatism regarding the original system. Very few results are concerned with a similar version of Lyapunov's first method for systems with non $\mathscr{C}^1$ vector fields. The most significant result is reported in [11]. It is shown that the local exponential stability of an ODE with non $\mathscr{C}^1$ vector fields can be deduced from a related differential inclusion, which reduces to the classic Jacobian linearization for $\mathscr{C}^1$ vector fields.

The main motivation of this paper is to show that if the $\mathscr{C}^1$-regularity assumption is not fulfilled, the Hurwitz nature of the Jacobian matrix, if it exists, does not guarantee in general the stability of the original nonlinear system. To tackle this issue, we first investigate the case where the vector fields admit directional derivatives [4]. Some conditions to guarantee the local exponential stability of the nonlinear system are then derived. Unlike the approach developed in [11], the proposed criteria reduces to checking if a given supremum over a compact set is strictly negative and hence, studying the global stability properties of a certain auxiliary differential inclusion is no longer necessary. Furthermore, we consider a class of vector fields which are not of class $\mathscr{C}^1$ but for which the local exponential stability can nevertheless be deduced from the study of the associated Jacobian matrix.

The remainder of the paper is organized as follows. Section II presents an example assessing the necessity of the $\mathscr{C}^1$-regularity to conclude the stability of a nonlinear system from its linearized dynamics. Local stability conditions for a nonlinear system are derived in Section III when the vector field admits directional derivatives. Then, a certain class of vector fields which are not of class $\mathscr{C}^1$ is investigated in Section IV. Concluding remarks are formulated in Section V.

II. MOTIVATING EXAMPLE

For any $\alpha,\beta>0$, let $f_{\alpha,\beta}:\mathbb{R}^2\rightarrow\mathbb{R}^2$ be defined by

$$\forall x=(x_1,x_2)\in\mathbb{R}^2,\; f_{\alpha,\beta}(x)=\begin{bmatrix}-\alpha x_1+\phi(x)\\-\beta x_2-\phi(x)\end{bmatrix},$$

where the function $\phi:\mathbb{R}^2\rightarrow\mathbb{R}$ is defined by

$$\forall (x_1,x_2)\in\mathbb{R}^2,\; \phi(x_1,x_2)=\begin{cases}\dfrac{x_1x_2^2}{x_1^2+x_2^2}, & \text{if } (x_1,x_2)\neq(0,0);\\ 0, & \text{if } (x_1,x_2)=(0,0).\end{cases}$$

The objective of this example is to determine the values of $\alpha,\beta>0$ such that the equilibrium point $(0,0)$ of the following nonlinear autonomous system is locally stable:

$$\dot{x}=f_{\alpha,\beta}(x). \tag{1}$$

A. Study of the function ϕ

To study the stability of (1), the properties of the function ϕ are first investigated. As the restriction of ϕ to the open

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set $\mathbb{R}^2\backslash\{(0,0)\}$ is a rational function with a non-vanishing denominator, $\phi|_{\mathbb{R}^2\backslash\{(0,0)\}} \in \mathscr{C}^1(\mathbb{R}^2\backslash\{(0,0)\};\mathbb{R})$. Furthermore, as for any $(x_1,x_2) \in \mathbb{R}^2\backslash\{(0,0)\}$,

$$|\phi(x_1,x_2) - \phi(0,0)| = \left|\frac{x_1x_2^2}{x_1^2+x_2^2}\right| \leq \frac{|x_2|}{2} \leq \frac{1}{2}\|(x_1,x_2)\|_2,$$

ϕ is continuous in $(0,0)$ and thus $\phi \in \mathscr{C}^0(\mathbb{R}^2;\mathbb{R})$. In addition, as $\phi(x_1,0) = \phi(0,x_2) = 0$, partial derivatives of ϕ at $(0,0)$ exist and are such that

$$\frac{\partial\phi}{\partial x_1}(0,0) = \frac{\partial\phi}{\partial x_2}(0,0) = 0. \tag{2}$$

Nevertheless, ϕ is not Fréchet differentiable in $(0,0)$. Indeed, introducing a differentiable function $u : \mathbb{R} \to \mathbb{R}^2$ defined for any $x \in \mathbb{R}$ by $u(x) = (x,x)$, the differentiability of ϕ would imply based on (2) that $\mathrm{d}\phi_{(0,0)} = 0$ and thus $\mathrm{d}(\phi \circ u)_0 = \mathrm{d}\phi_{u(0)} \circ \mathrm{d}u_0 = 0$. However, $\phi(u(x)) = x/2$ implies that $\mathrm{d}(\phi\circ u)_0 = (1/2)\mathrm{d}x \neq 0$. By contradiction, ϕ is not Fréchet diffentiable in $(0,0)$ and hence, it does not admit a first order Taylor expansion at the origin. Finally, let us show that ϕ is Lipschitz continous. For any $(x_1,x_2) \neq (0,0)$,

$$\nabla\phi(x_1,x_2) = \begin{bmatrix} \frac{x_2^2(x_2^2-x_1^2)}{(x_1^2+x_2^2)^2} & \frac{2x_1^3x_2}{(x_1^2+x_2^2)^2} \end{bmatrix}^\top,$$

which implies that for any $(h,k) \in \mathbb{R}^2$,

$$\begin{aligned} \left|\mathrm{d}\phi_{(x_1,x_2)}(h,k)\right| &= \left|\nabla\phi(x_1,x_2)^\top \begin{bmatrix} h \\ k \end{bmatrix}\right| \\ &\leq \frac{x_2^2}{x_1^2+x_2^2}|h| + \frac{2|x_1x_2|}{x_1^2+x_2^2}|k| \leq |h|+|k| \leq 2\|(h,k)\|_2. \end{aligned}$$

Then, for any $(x_1,x_2) \neq (0,0)$, $|||\mathrm{d}\phi_{(x_1,x_2)}||| \leq 2$, implying by the mean value theorem that for any $x,y \in \mathbb{R}^2$ such that $(0,0) \notin [x,y] := \{(1-\lambda)x + \lambda y : \lambda \in [0,1]\}$,

$$|\phi(y) - \phi(x)| \leq 2\|y-x\|_2.$$

In the case where $(0,0) \in [x,y]$, one has $\|y\|_2 + \|x\|_2 = \|y-x\|_2$ and thus

$$|\phi(y)-\phi(x)| \leq \frac{1}{2}(\|y\|_2 + \|x\|_2) \leq \frac{1}{2}\|y-x\|_2.$$

This shows that ϕ is globally Lipschitz continuous.

B. Limitations of the Jacobian matrix-based stability assessment

As ϕ is Lipschitz continuous, it is straightforward to show that the vector field $f_{\alpha,\beta}$ is also globally Lipschitz continuous. Thus, for any given initial condition $x_0 \in \mathbb{R}^2$, the Cauchy-Lipschitz theorem [1, Th.3.1] guarantees the existence and the uniqueness of a global solution (i.e., defined for any $t \in \mathbb{R}_+$) of the ODE (1) with $x(0) = x_0$.

1) Stability of the associated linear system: As ϕ presents null partial derivatives in $(0,0)$, the components of $f_{\alpha,\beta} : \mathbb{R}^2 \to \mathbb{R}^2$ admit partial derivatives in $(0,0)$, providing the following Jacobian matrix:

$$A_{\alpha,\beta} := J(f_{\alpha,\beta})_{(0,0)} = \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix}.$$

As $\alpha,\beta > 0$, the matrix $A_{\alpha,\beta}$ is Hurwitz and hence, the linear system $\dot{\zeta} = A_{\alpha,\beta}\zeta$ is exponentially stable. It is then usually possible to conclude the local asymptotic stability of the original nonlinear system (1) at the equilibrium point $(0,0)$ by applying the following theorem.

Theorem 1: [1, Theorem 4.7] Let $0 \in D \subset \mathbb{R}^n$ be a domain[1]. Let $f : D \to \mathbb{R}^n$ be a $\mathscr{C}^1$ vector field such that $f(0) = 0$ and let $J(f)_0 := \partial f/\partial x|_0$ be the Jacobian matrix of f evaluated at 0. Then the equilibrium point 0 of $\dot{x} = f(x)$ is locally asymptotically stable if $J(f)_0$ is Hurwitz.

Remark 1: The $\mathscr{C}^1$ regularity assumption can be weakened by assuming that 1) f is locally Lipschitz continuous; 2) f is Fréchet differentiable in 0 [12].

However, this theorem cannot be applied to the nonlinear system (1). Indeed, the vector field $f_{\alpha,\beta}$ is not of class $\mathscr{C}^1$ in any neighborhood of the origin because ϕ is not Fréchet differentiable in $(0,0)$, as proved in Subsection II-A.

2) Instability of the nonlinear system for $\alpha = \beta \in (0,1/2)$: Let $\alpha = \beta > 0$ and $\xi_0 \in \mathbb{R}\backslash\{0\}$. We focus on the analysis of the following particular case of (1):

$$\dot{x} = f_{\alpha,\alpha}(x), \qquad x(0) = (\xi_0, -\xi_0).$$

Let $x = (x_1,x_2) : [0,+\infty) \to \mathbb{R}^2$ be the associated global solution and define $s := x_1 + x_2 : [0,+\infty) \to \mathbb{R}$. By the definition of $f_{\alpha,\beta}$, with $\alpha = \beta$, the function s satisfies:

$$\dot{s} = -\alpha s, \qquad s(0) = 0.$$

Thus $s \equiv 0$ and hence, $x_2 \equiv -x_1$. It implies that

$$\dot{x}_1 = -\alpha x_1 + \phi(x_1,-x_1) = (1/2-\alpha)x_1, \qquad x_1(0) = \xi_0,$$

and thus for any $t \in [0,+\infty)$, $x_1(t) = \exp((1/2-\alpha)t)\,\xi_0$ and $x_2(t) = -\exp((1/2-\alpha)t)\,\xi_0$. Hence, for any $\alpha \in (0,1/2)$,

$$\lim_{t\to+\infty} x_1(t) = \mathrm{sgn}(\xi_0)\cdot\infty, \ \lim_{t\to+\infty} x_2(t) = -\mathrm{sgn}(\xi_0)\cdot\infty.$$

As $\xi_0 \in \mathbb{R}\backslash\{0\}$ is arbitrary, the initial condition $(\xi_0,-\xi_0)$ can be selected arbitrary close to the equilibrium point $(0,0)$. One can conclude the instability of the equilibrium point $(0,0)$ of (1) for any $\alpha = \beta \in (0,1/2)$.

3) Conclusion of the motivating example: It has been shown that the equilibrium point $(0,0)$ of (1) for any $\alpha = \beta \in (0,1/2)$ is unstable while the associated linear system is exponentially stable. This result shows that the $\mathscr{C}^1$-regularity assumption of the vector field in the vicinity of the considered equilibrium point is paramount in Theorem 1 to conclude the local stability of the nonlinear system from the stability of the linear one.

Remark 2: Note that eventually, one can try to apply the main result of [11]. Nevertheless, as the vector field $f_{\alpha,\beta}$ is homogeneous (i.e., for all $x \in \mathbb{R}^2$ and all $\lambda > 0$, $f_{\alpha,\beta}(\lambda x) = \lambda f_{\alpha,\beta}(x)$) and continuous, the sufficient condition provided in [11, Th. 1.2] for assessing the local exponential stability of (1) boils down to verifying that all the solutions of (1) converge to zero as $t \to +\infty$. Such an analysis does not provide, in this case, a straightforward conclusion.

[1]Throughout the paper, a domain is an open and connected subset of $\mathbb{R}^n$.

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III. STABILITY CRITERIA FOR VECTOR FIELDS ADMITING DIRECTIONAL DERIVATIVES

Lyapunov's first method relies mainly on the existence of a first order Taylor expansion of the vector field $f : D \to \mathbb{R}^n$ at the considered equilibrium point. The existence of such an expansion is usually guaranteed by the $\mathscr{C}^1$-regularity of the vector field f. As demonstrated by the motivating example, this result does not hold in general when the regularity assumption is not satisfied. In this section, assuming the existence and the equicontinuity of the directional derivatives of the vector field f at the considered equilibrium point, it is shown that a directional Taylor expansion of f can be considered to assess the local exponential stability of the original nonlinear system.

In this section, for $n \in \mathbb{N}^* := \mathbb{N}\backslash\{0\}$, $\mathbb{S}^{n-1}$ denotes the unitary hypersphere of $\mathbb{R}^n$, i.e., $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ with $\|x\|_2 := \sqrt{x^\top x}$. The sets $\mathscr{S}_n^{+}(\mathbb{R})$, $\mathscr{S}_n^{+*}(\mathbb{R})$, and $\mathscr{S}_n^{-*}(\mathbb{R})$ are the subsets of the real symmetric matrices $\mathscr{S}_n(\mathbb{R})$ which are positive, positive-definite, and negative-definite, respectively. For a given norm $\|\cdot\|$ on $\mathbb{R}^n$, and for any $a \in \mathbb{R}^n$ and $r > 0$, $\mathrm{D}_{\|\cdot\|}(a,r) = \{x \in \mathbb{R}^n : \|x-a\| < r\}$, or simply denoted by $\mathrm{D}(a,r)$ when there is no ambiguity on the associated norm. Similarly $\mathrm{D}_{\|\cdot\|}(a,r] = \{x \in \mathbb{R}^n : \|x-a\| \leq r\}$. The symbols o and $\mathscr{O}$ denote the classic Bachmann-Landau notations.

A. Directional Taylor expansion

Lemma 1: Let $0 \in D \subset \mathbb{R}^n$ be a domain and $f : D \to \mathbb{R}^n$ with $f(0) = 0$. Let $R > 0$ such that $\mathrm{D}(0,R) \subset D$. Let the family of functions $(g_v : [0,R) \to \mathbb{R}^n)_{v\in\mathbb{S}^{n-1}}$ be defined for any $v \in \mathbb{S}^{n-1}$ and $t \in [0,R)$ by $g_v(t) = f(tv)$. Assume that:

(A1) there exists $r_0 > 0$ such that for all $v \in \mathbb{S}^{n-1}$, $g_v|_{[0,r_0)}$ is of class $\mathscr{C}^1$;

(A2) denoting by $g_v|'_{[0,r_0)}$ the derivative of $g_v|_{[0,r_0)}$, the family $\left(g_v|'_{[0,r_0)}\right)_{v\in\mathbb{S}^{n-1}}$ is equicontinuous in 0, i.e., for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$|\tau| \leq \eta \Rightarrow \forall v \in \mathbb{S}^{n-1}, \left\| g_v|'_{[0,r_0)}(\tau) - g_v|'_{[0,r_0)}(0) \right\|_2 \leq \varepsilon.$$

Then there exists $h : D \to \mathbb{R}^n$ such that $h(x) \underset{\|x\|_2\to 0}{\longrightarrow} 0$ and

$$\forall x \in D,\ f(x) = g_{v_x}|'_{[0,r_0)}(0)\,\|x\|_2 + h(x)\,\|x\|_2,$$

where for any $x \in \mathbb{R}^n\backslash\{0\}$, $v_x = x/\|x\|_2 \in \mathbb{S}^{n-1}$ and, by convention, v_0 is an arbitrary vector of $\mathbb{S}^{n-1}$.

Proof: As $g_v(0) = f(0) = 0$, it yields for any $x \in \mathrm{D}(0,r)$,

$$f(x) = f(\|x\|_2\, v_x) = g_{v_x}(\|x\|_2) = \int_0^{\|x\|_2} g_{v_x}|'_{[0,r_0)}(\tau)\mathrm{d}\tau.$$

Then, for any $x \in \mathrm{D}(0,r)$,

$$\begin{aligned} &\left\| f(x) - g_{v_x}|'_{[0,r_0)}(0)\,\|x\|_2 \right\|_2 \\ &= \left\| \int_0^{\|x\|_2} g_{v_x}|'_{[0,r_0)}(\tau) - g_{v_x}|'_{[0,r_0)}(0)\mathrm{d}\tau \right\|_2 \\ &\leq \|x\|_2 \sup_{\tau\in[0,\|x\|_2]} \left\| g_{v_x}|'_{[0,r_0)}(\tau) - g_{v_x}|'_{[0,r_0)}(0) \right\|_2. \end{aligned}$$

Let $\varepsilon > 0$. By equicontinuity of $\left(g_v|'_{[0,r_0)}\right)_{v\in\mathbb{S}^{n-1}}$ in 0, there exists $\eta \in (0,r_0)$ such that for any $|\tau| \leq \eta$ and $v \in \mathbb{S}^{n-1}$, $\left\| g_v|'_{[0,r_0)}(\tau) - g_v|'_{[0,r_0)}(0) \right\|_2 \leq \varepsilon$. It implies that

$$\|x\|_2 \leq \eta \Rightarrow \left\| f(x) - g_{v_x}|'_{[0,r_0)}(0)\,\|x\|_2 \right\|_2 \leq \varepsilon\,\|x\|_2.$$

Introducing, independently of $\varepsilon > 0$, $h : D \to \mathbb{R}^n$ defined by

$$\forall x \in D, h(x) = \begin{cases} \dfrac{f(x) - g_{v_x}|'_{[0,r_0)}(0)\,\|x\|_2}{\|x\|_2}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases}$$

it yields $\|x\|_2 \leq \eta \Rightarrow \|h(x)\|_2 \leq \varepsilon$. Thus $h(x) \underset{\|x\|_2\to 0}{\longrightarrow} 0$ and

$$\forall x \in D,\ f(x) = g_{v_x}|'_{[0,r_0)}(0)\,\|x\|_2 + h(x)\,\|x\|_2.$$

□

Remark 3: Lemma 1 provides a generalized first order Taylor expansion under a "directional" form. Note that if f is of class $\mathscr{C}^1$ in a neighborhood of 0 with $f(0) = 0$, the result boils down to the usual first order Taylor expansion given by:

$$f(x) = J(f)_0 x + o(\|x\|_2).$$

Indeed, it can be shown that $g_{v_x}|'_{[0,r_0)}(0) = J(f)_0 v_x$ and thus $g_{v_x}|'_{[0,r_0)}(0)\,\|x\|_2 = J(f)_0 x$. Note also that Assumption (A1) of Lemma 1 can be interpreted in the context of directional derivatives as $g_v|'_{[0,r_0)}(0)$ corresponds to the directional derivative of f at 0 in the direction v [4].

B. Sufficient condition for local exponential stability

Based on the above first order directional Taylor expansion, a criterion is derived in this section for guaranteeing the local exponential stability of $\dot{x} = f(x)$ at the origin.

Theorem 2: Let $0 \in D \subset \mathbb{R}^n$ be a domain. Let $f : D \to \mathbb{R}^n$ be locally Lipschitz continuous such that $f(0) = 0$ and Assumptions (A1) and (A2) hold. If there exists $P \in \mathscr{S}_n^{+*}$ such that

$$\sup_{v\in\mathbb{S}^{n-1}} v^\top P\, g_v|'_{[0,r_0)}(0) = -\gamma < 0, \tag{3}$$

then $\dot{x} = f(x)$ is locally exponentially stable at the origin.

Proof: As $P \in \mathscr{S}_n^{+*}$, it admits a unique square-root, denoted by $\sqrt{P}$, such that $\sqrt{P} \in \mathscr{S}_n^{+*}$. Let the norm $\|\cdot\|_{\sqrt{P}} : \mathbb{R}^n \to \mathbb{R}_+$ be defined for any $x \in \mathbb{R}$ by $\|x\|_{\sqrt{P}} = \left\|\sqrt{P}x\right\|_2$. Introducing $c_1 = \left\|\sqrt{P}\right\|_2^{-1} > 0$ and $c_2 = \left\|\left(\sqrt{P}\right)^{-1}\right\|_2 > 0$, for any $x \in \mathbb{R}^n$, $c_1\,\|x\|_{\sqrt{P}} \leq \|x\|_2 \leq c_2\,\|x\|_{\sqrt{P}}$. Let $V : D \to \mathbb{R}$ be a Lyapunov candidate function such that for any $x \in D$, $V(x) = \|x\|_{\sqrt{P}}^2/2 = x^\top Px/2$. The Lie derivative of V along the vector field f is given by:

$$\begin{aligned} \mathscr{L}_f V(x) = x^\top P f(x) &= x^\top P\, g_{v_x}|'_{[0,r_0)}(0)\,\|x\|_2 + x^\top Ph(x)\,\|x\|_2 \\ &= v_x^\top P\, g_{v_x}|'_{[0,r_0)}(0)\,\|x\|_2^2 + x^\top Ph(x)\,\|x\|_2 \\ &\leq -2\gamma c_1^2 V(x) + x^\top Ph(x)\,\|x\|_2. \end{aligned}$$

As D is an open set, we can take $r > 0$ such that $\mathrm{D}_{\|\cdot\|_2}(0,r] \subset D$. Let $\varepsilon \in (0, \gamma c_1^2/(\|P\|_2\, c_2^2))$. There exists $\eta \in (0,r)$ such that

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$\|x\|_2 < \eta$ implies $\|h(x)\|_2 < \varepsilon$. Then for any $x \in \mathrm{D}_{\|\cdot\|_2}(0,\eta)$, and in particular for any $x \in \mathrm{D}_{\|\cdot\|_{\sqrt{P}}}(0,\eta/c_2) \subset \mathrm{D}_{\|\cdot\|_2}(0,\eta)$,

$$\begin{aligned}\mathscr{L}_f V(x) &\leq -2\gamma c_1^2 V(x) + \varepsilon \|P\|_2 \|x\|_2^2 \\ &\leq -2\underbrace{(\gamma c_1^2 - \varepsilon \|P\|_2 c_2^2)}_{>0} V(x).\end{aligned}$$

Therefore, for any given $x_0 \in \mathrm{D}_{\|\cdot\|_{\sqrt{P}}}(0,\eta/c_2) \subset D$, the maximal solution of $\dot{x} = f(x)$ associated to the initial condition $x(0) = x_0$ is defined on $[0,t_1)$ where $t_1 \in (0,+\infty]$ and satisfies $x([0,t_1)) \subset \mathrm{D}_{\|\cdot\|_{\sqrt{P}}}(0,\eta/c_2)$ with:

$$\forall t \in [0,t_1),\ \|x(t)\|_{\sqrt{P}} \leq \exp(-(\gamma c_1^2 - \varepsilon \|P\|_2 c_2^2)t)\, \|x_0\|_{\sqrt{P}}.$$

Then, for any $x_0 \in \mathrm{D}_{\|\cdot\|_2}(0,\eta c_1/c_2) \subset \mathrm{D}_{\|\cdot\|_{\sqrt{P}}}(0,\eta/c_2)$, and any $t \in [0,t_1)$,

$$\|x(t)\|_2 \leq \frac{c_2}{c_1} \exp(-(\gamma c_1^2 - \varepsilon \|P\|_2 c_2^2)t)\, \|x_0\|_2.$$

As the solution is bounded on $[0,t_1)$ in norm $\|\cdot\|_2$ by $\frac{c_2}{c_1}\|x_0\|_2 < \eta < r$, $x([0,t_1)) \subset \mathrm{D}_{\|\cdot\|_2}(0,r) \subset D$, where $\mathrm{D}_{\|\cdot\|_2}(0,r]$ is a compact set. It yields $t_1 = +\infty$ (Theorem 2, Section 2.4, [13]), which concludes the proof. □

A more convenient form of condition (3) can be obtained by a parametrization of the unitary hypersphere based on the spherical coordinate system. Let $E_{n-1} = [0,\pi]^{n-2} \times [0,2\pi]$ and for any $\Theta = (\theta_1,\theta_2,\ldots,\theta_{n-1}) \in E_{n-1}$, $u_\Theta = (u_{\Theta,1},u_{\Theta,2},\ldots,u_{\Theta,n}) \in \mathbb{R}^n$ with

$$u_{\Theta,i} = \left(\prod_{k=1}^{i-1} \sin(\theta_k)\right)\cos(\theta_i),\ \ \forall i \in \{1,\ldots,n-1\},$$

$$u_{\Theta,n} = \prod_{k=1}^{n-1} \sin(\theta_k).$$

Then the unitary hypersphere $\mathbb{S}^{n-1}$ is parametrized by u_Θ : $\mathbb{S}^{n-1} = \{u_\Theta, \Theta \in E_{n-1}\}$.

Lemma 2: Condition (3) of Theorem 2 is equivalent to

$$\sup_{\Theta \in E_{n-1}} u_\Theta^\top P\, g_{u_\Theta}\big|'_{[0,r_0)}(0) < 0. \tag{4}$$

Remark 4: Assuming that the vector field f is of class $\mathscr{C}^1$ in the neighborhood of 0, it yields $g_v|'_{[0,r_0)}(0) = J(f)_0 v$. Therefore, the condition (3) of Theorem 2 consists in finding a matrix $P \in \mathscr{S}_n^{+*}$ such that

$$\sup_{v \in \mathbb{S}^{n-1}} v^\top P J(f)_0 v = \max_{v \in \mathbb{S}^{n-1}} v^\top P J(f)_0 v < 0,$$

which is equivalent to the fact that for any $x \in \mathbb{R}^n \backslash \{0\}$, $x^\top P J(f)_0 x < 0$, i.e., to find a matrix $P \in \mathscr{S}_n^{+*}$ such that $PJ(f)_0 + J(f)_0^\top P \in \mathscr{S}_n^{-*}$, which is the usual condition for the stability of linear time invariant systems [1, Th.4.6].

The following result provides a sufficient condition for assessing the instability of $\dot{x} = f(x)$ at the origin.

Theorem 3: Let $0 \in D \subset \mathbb{R}^n$ be a domain. Let $f : D \to \mathbb{R}^n$ be locally Lipschitz continuous such that $f(0) = 0$ and Assumptions (A1) and (A2) hold. Let $\{0\} \subsetneq M \subset D$ be a path-connected set which is positively invariant[2]. For any $R > 0$, let $S_R = \{x/\|x\|_2 : x \in (M \backslash \{0\}) \cap \mathrm{D}_{\|\cdot\|_2}(0,R)\} \subset \mathbb{S}^{n-1}$. If there exist $R_0 > 0$ and $P \in \mathscr{S}_n^{+*}$ such that

$$\inf_{v \in S_{R_0}} v^\top P\, g_v|'_{[0,r_0)}(0) = \gamma > 0, \tag{5}$$

then $\dot{x} = f(x)$ is unstable at the origin.

Proof: The proof is similar to the one of Theorem 2 and is omitted due to space limitation. □

[2] i.e., for all $x_0 \in M$, the maximal solution of $\dot{x} = f(x)$ such that $x(0) = x_0$ is defined over $\mathbb{R}_+$ and satisfies $x(t) \in M$ for all $t \in \mathbb{R}_+$.

C. Necessary conditions

A weak converse result of Theorem 2 is provided below.

Theorem 4: Let $0 \in D \subset \mathbb{R}^n$ be a domain. Let $f : D \to \mathbb{R}^n$ be locally Lipschitz continuous such that $f(0) = 0$ and Assumptions (A1) and (A2) hold. If the ODE $\dot{x} = f(x)$ satisfies one of the two following assumptions:

1) the origin is locally quadratically stable, i.e., there exists $P \in \mathscr{S}_n^{+*}$ such that $V(x) = x^\top P x/2$ is a Lyapunov function;
2) there exists a Lyapunov function $V : D \to \mathbb{R}$ of class $\mathscr{C}^2$ in the neighborhood of the origin,

then there exists a matrix $P \in \mathscr{S}_n^{+*}$ (resp. $P \in \mathscr{S}_n^{+}$) such that

$$\sup_{v \in \mathbb{S}^{n-1}} v^\top P\, g_v|'_{[0,r_0)}(0) \leq 0.$$

Proof: In the first case, there exists a $P \in \mathscr{S}_n^{+*}$ such that $V(x) = x^\top P x/2$ is a Lyapunov function, i.e., there exists $\eta > 0$ such that for any $x \in \mathrm{D}(0,\eta) \subset D$, $\mathscr{L}_f V(x) \leq 0$. Based on Lemma 1, for any $x \in D$,

$$\mathscr{L}_f V(x) = x^\top P f(x) = v_x^\top P\, g_{v_x}|'_{[0,r_0)}(0)\, \|x\|_2^2 + x^\top P h(x)\, \|x\|_2.$$

Thus, in the neighborhood of 0,

$$\mathscr{L}_f V(x) = v_x^\top P\, g_{v_x}|'_{[0,r_0)}(0)\, \|x\|_2^2 + o(\|x\|_2^2).$$

Assume that there exists $v \in \mathbb{S}^{n-1}$ such that $v^\top P\, g_v|'_{[0,r_0)}(0) > 0$. For $x = \varepsilon v$ with $\varepsilon > 0$, it yields

$$\mathscr{L}_f V(\varepsilon v) = \varepsilon^2 \{v^\top P\, g_v|'_{[0,r_0)}(0) + o(1)\},$$

which, for ε arbitrarily close to 0^+, contradicts the condition $\|x\|_2 < \eta \Rightarrow \mathscr{L}_f V(x) \leq 0$. Therefore, for any vector $v \in \mathbb{S}^{n-1}$, $v^\top P\, g_v|'_{[0,r_0)}(0) \leq 0$.

In the second case, V admits at the origin a second order Taylor expansion:

$$V(x) = V(0) + \nabla V(0)^\top x + \frac{1}{2} x^\top \mathscr{H} V(0) x + o(\|x\|_2^2), \tag{6}$$

where $\nabla V(0) \in \mathbb{R}^n$ and $\mathscr{H} V(0) \in \mathbb{R}^{n \times n}$ denote respectively the gradient and the Hessian matrix of V at 0. As V is a Lyapunov function, $V(0) = 0$ and there exists $\eta > 0$ such that $0 < \|x\|_2 < \eta \Rightarrow V(x) > 0$. Then $\nabla V(0) = 0$ (take $x = -\varepsilon \nabla V(0)$ in the Taylor expansion (6) with ε arbitrarily close to 0^+). It yields

$$V(x) = \frac{1}{2} x^\top \mathscr{H} V(0) x + o(\|x\|_2^2),$$

with, because V is of class $\mathscr{C}^2$ in the neighborhood of 0, $\mathscr{H} V(0) \in \mathscr{S}_n(\mathbb{R})$ by the Schwarz theorem. Let λ be an eigenvalue of $\mathscr{H} V(0)$, which is real because $\mathscr{H} V(0) \in \mathscr{S}_n(\mathbb{R})$, and $x_\lambda \in \mathbb{R}^n$ be an associated unitary eigenvector. For any $\varepsilon > 0$,

$$V(\varepsilon x_\lambda) = \varepsilon^2 \{\lambda/2 + o(1)\}.$$

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Using again the fact that $0 < \|x\|_2 < \eta \Rightarrow V(x) > 0$, it yields $\lambda \geq 0$. Hence $\mathscr{H}V(0) \in \mathscr{S}_n^+(\mathbb{R})$. Now, based on Lemma 1, for any $x \in D$,

$$\begin{aligned}\mathscr{L}_f V(x) &= \nabla V(x)^\top f(x) \\ &= (\mathscr{H}V(0)x + o(\|x\|_2))^\top \\ &\quad \times \left(g_{v_x}|'_{[0,r_0)}(0)\, \|x\|_2 + o(\|x\|_2) \right).\end{aligned} \tag{7}$$

Assume there exists $v \in \mathbb{S}^{n-1}$ such that $v^\top \mathscr{H}V(0)\, g_v|'_{[0,r_0)}(0) > 0$. Injecting $x = \varepsilon v$ in the Taylor expansion (7) with $\varepsilon > 0$, it gives

$$\begin{aligned}\mathscr{L}_f V(\varepsilon v) &= \varepsilon\, (\mathscr{H}V(0)v + o(1))^\top \times \varepsilon \left(g_v|'_{[0,r_0)}(0) + o(1) \right) \\ &= \varepsilon^2 \{ v^\top \mathscr{H}V(0)\, g_v|'_{[0,r_0)}(0) + o(1) \}.\end{aligned}$$

There exists $\eta' > 0$ such that $0 < \varepsilon < \eta' \Rightarrow \mathscr{L}_f V(\varepsilon v) > 0$, which contradicts the assumption that V is a Lyapounov function for the equilibrium point 0. Therefore, it yields that for any $v \in \mathbb{S}^{n-1}$, $v^\top \mathscr{H}V(0)\, g_v|'_{[0,r_0)}(0) \leq 0$. □

D. Application to the motivating example

The result of Theorem 2 is applied to the motivating example (1) to determine the couples $(\alpha, \beta) \in (\mathbb{R}_+^*)^2$ for which the system is locally exponentially stable at the origin. In this case,

$$\forall v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{S}^1,\ \forall t \geq 0,\ g_v(t) = f_{\alpha,\beta}(tv) = \begin{bmatrix} -\alpha t v_1 + t v_1 v_2^2 \\ -\beta t v_2 - t v_1 v_2^2 \end{bmatrix}.$$

The applicability conditions of Lemma 1 are satisfied as for any $v \in \mathbb{S}^1$, the function g_v is of class $\mathscr{C}^1$ on $\mathbb{R}_+$ with

$$\forall v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{S}^1,\ \forall t \geq 0,\ g_v'(t) = \begin{bmatrix} -\alpha v_1 + v_1 v_2^2 \\ -\beta v_2 - v_1 v_2^2 \end{bmatrix},$$

and the family of functions $(g_v')_{v \in \mathbb{S}^{n-1}}$ is equicontinuous in 0 (because for any $v \in \mathbb{S}^{n-1}$, g_v' is a constant function). Considering the parametrization of $\mathbb{S}^1$ with $v_1 = c_\theta := \cos(\theta)$ and $v_2 = s_\theta := \sin(\theta)$ for $\theta \in [0, 2\pi]$, the applicability condition of Theorem 2 consists in finding $P \in \mathscr{S}_2^{+*}(\mathbb{R})$ such that

$$\max_{\theta \in [0,2\pi]} \begin{bmatrix} c_\theta \\ s_\theta \end{bmatrix}^\top \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} -\alpha c_\theta + c_\theta s_\theta^2 \\ -\beta s_\theta - c_\theta s_\theta^2 \end{bmatrix} < 0,$$

where the entries of matrix P must satisfy $P_{11} > 0$ and $P_{11}P_{22} - P_{12}^2 > 0$.

The obtained results are depicted in Fig. 1. The application of Theorem 2 with $P = I_2$ enables to assess the local stability at the origin for multiple values of the couple (α, β) (blue area in Fig. 1(a)). Nevertheless, the application of Theorem 2 for a specific $P \in \mathscr{S}^{+*}$ provides conservative results. Thus, the application of Theorem 2 with $P = \mathrm{diag}(0.2, 1)$ enables to obtain a new area over which the local stability of the equilibrium point is guaranteed including couples for which the stability cannot be assessed with $P = I_2$ (Fig. 1(b)); e.g., $\alpha = 0.2$ and $\beta = 0.6$. In order to reduce the conservatism and to merge the obtained results, Theorem 2 has been applied for 400 matrices $P \in \mathscr{S}_2^{+*}(\mathbb{R})$. The obtained results are depicted in Fig. 1(c). These results are in accordance with the conclusion of Section II that the system is unstable for $0 < \alpha = \beta < 1/2$. Nonlinear simulations tend to indicate that the results depicted in Fig. 1(c) provide a tight delimitation of the stability area.

IV. NON $\mathscr{C}^1$ VECTOR FIELDS ADMITTING A FIRST ORDER TAYLOR EXPANSION

In this section, we investigate a family of vector fields that are not of class $\mathscr{C}^1$ but are locally Lipschitz continuous and Fréchet differentiable at the equilibrium point. Such vector fields are of interest because the classic stability criterion applies in the sense that the stability of the linearized system implies the local stability of the nonlinear one. This is because the $\mathscr{C}^1$-regularity assumption is a sufficient condition for the existence of a first order Taylor expansion, but it is not a necessary one, as illustrated by the following example.

A. Illustrative example

Let us consider the vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\forall x = (x_1, x_2) \in \mathbb{R}^2,\ f(x) = \begin{bmatrix} -x_1 + |x_1| x_2 \\ -x_2 \end{bmatrix}. \tag{8}$$

The function f is locally Lipschitz continuous and does not admit partial derivatives with respect to x_1 at the points $(0, x_2)$ with $x_2 \neq 0$. Hence, f is not of class $\mathscr{C}^1$ in any neighborhood of the equilibrium point 0. One could apply Theorem 2 with $P = I_2$ to conclude the local stability of the origin for $\dot{x} = f(x)$ because for any $v \in \mathbb{S}^1$, $v^\top g_v|'_{[0,r_0)}(0) = -1$. However, the conclusion simply follows from the fact that f is Fréchet differentiable at $(0,0)$. Indeed, for any $x = (x_1, x_2) \in \mathbb{R}^2$,

$$\frac{\|f(x) + I_2 x\|_2}{\|x\|_2} = \frac{|x_1 x_2|}{\|x\|_2} \leq \frac{1}{2} \|x\|_2 \underset{\|x\|_2 \to 0}{\longrightarrow} 0.$$

Thus f admits a first order Taylor expansion in $(0,0)$ and the associated Jacobian matrix $J(f)_0 = -I_2$ is Hurwitz. It is then possible, following the same argument as in the proof of Theorem 4.7 in [1], to conclude the local exponential stability of $\dot{x} = f(x)$ at the origin. The general situation is investigated in the next subsection.

B. General setting

For $n \in \mathbb{N}^*$, let $\mathscr{U}^n = \{-1, +1\}^n$. One can associate to $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathscr{U}^n$ the sets $\mathbb{R}^n_\mu = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \forall i \in \{1, \ldots, n\},\ \mu_i x_i \geq 0\}$ and for any $r > 0$, $\mathrm{D}_\mu(0, r) = \mathrm{D}(0, r) \cap \mathbb{R}^n_\mu$. Let e_i be the i-th vector of the canonical basis of $\mathbb{R}^n$.

Lemma 3: Let $0 \in D \subset \mathbb{R}^n$ be a domain. Let $f, g : D \to \mathbb{R}$ be two continuous functions with g of class $\mathscr{C}^1$ in an open neighborhood of the origin and $g(0) = 0$. Assume there exists $r > 0$ with $\mathrm{D}(0, r] \subset D$ such that for any $\mu \in \mathscr{U}^n$:

(B1) $f|_{\mathrm{D}_\mu(0,r)}$, denoted by f_μ, is of class $\mathscr{C}^1$ over $\mathrm{D}_\mu(0, r) \backslash \{0\}$ with bounded Jacobian matrix;
(B2) for any $j \in \{1, \ldots, n\}$, the function $(0, r) \ni t \to \partial f_\mu / \partial x_j (t \mu_j e_j)$ admits a limit in 0^+.

Then $f \times g$ admits a first order Taylor expansion in 0 given by:

$$f(x)g(x) = f(0) \nabla g(0)^\top x + o(\|x\|_2).$$

Proof: Take an arbitrary $\mu \in \mathscr{U}^n$. Assumption (B2) guarantees that the function $[0, r) \ni t \to f_\mu(t \mu_j e_j)$ admits a derivative at 0 with a derivative which is, based on Assumption (B1), continuous over $[0, r)$. Thus f_μ admits partial derivatives at 0 and are such that $(\partial f_\mu / \partial x_j)\big|_{[0,r)\mu_j e_j}$ is continuous and thus

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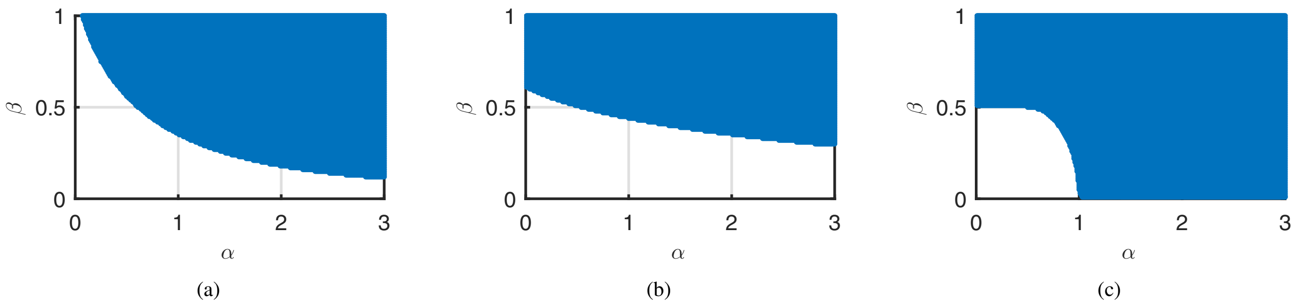

Fig. 1. Couples (α,β) for which the stability of the equilibrium 0 for the system $\dot{x} = f_{\alpha,\beta}(x)$ has been established (area in blue): Application of Theorem 2 with (a) $P = I_2$; (b) $P = \mathrm{diag}(0.2, 1)$; (c) 400 matrices $P \in \mathscr{S}_2^{+*}$.

bounded over the compact set $[0, r/2]\mu_j e_j \subset \mathbb{R}^n_\mu$. Then, for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_\mu$ such that $\|x\|_2 < r$,

$$f(x) - f(0) = \sum_{j=1}^{n} \left\{ f\left(\sum_{k=n+1-j}^{n} x_k e_k \right) - f\left(\sum_{k=n+2-j}^{n} x_k e_k \right) \right\}$$
$$= \sum_{j=1}^{n} \int_0^{x_{n+1-j}} \left(\frac{\partial f_\mu}{\partial x_{n+1-j}} \right) \left(\tau e_{n+1-j} + \sum_{k=n+2-j}^{n} x_k e_k \right) \mathrm{d}\tau.$$

Assumption (B1) guarantees that the partial derivatives of f_μ are bounded over $\mathrm{D}_\mu(0,r)\backslash\{0\}$. Moreover, it was shown that the partial derivatives are also bounded along the semi-axis (i.e., $\mathbb{R}_+\mu_j e_j$ where $j \in \{1,\ldots,n\}$) for $\|x\|_2 < r/2$. We can then deduce that there exists a constant $M_\mu \geq 0$ such that

$$\forall x \in \mathbb{R}^n_\mu,\ \|x\|_2 < r/2 \Rightarrow |f(x) - f(0)| \leq M_\mu \|x\|_2 .$$

As this result holds for any $\mu \in \mathscr{U}^n$, we obtain that for any $x \in \mathbb{R}^n$ such that $\|x\|_2 < r/2$, $|f(x) - f(0)| \leq M\|x\|_2$, where $M = \max\{M_\mu : \mu \in \mathscr{U}^n\}$ (which exists because $\mathrm{Card}(\mathscr{U}^n) = 2^n < +\infty$). Hence, in the neighborhood of 0:

$$f(x) = f(0) + \mathscr{O}(\|x\|_2).$$

As the function g is of class $\mathscr{C}^1$ in the vicinity of 0 with $g(0) = 0$, it admits the following first order Taylor expansion:

$$g(x) = \nabla g(0)^\top x + o(\|x\|_2).$$

Finally, the claim holds true because

$$f(x)g(x) = f(0)\nabla g(0)^\top x + \underbrace{\left\{ \mathscr{O}(\|x\|_2) \times \nabla g(0)^\top x + o(\|x\|_2) \right\}}_{=o(\|x\|_2)}.$$

□

The Hadamard product of two matrices $U, V \in \mathbb{R}^{n\times m}$, denoted by $U \cdot V \in \mathbb{R}^{n\times m}$, is the matrix whose entries are such that $(U\cdot V)_{i,j} = (U)_{i,j}(V)_{i,j}$.

Theorem 5: Let $0 \in D$ be a domain. Let $f, g, h : D \to \mathbb{R}^n$ be three functions with $g(0) = h(0) = 0$ and such that for any $i \in \{1,\ldots,n\}$, the i-th components f_i and g_i of functions f and g satisfy the assumptions of Lemma 3. Furthermore, assume that $f\cdot g + h$ is locally Lipschitz continuous, h is of class $\mathscr{C}^1$ in the neighborhood of 0, and the matrix

$$A := J(f(0)\cdot g + h)_0 = J(f(0)\cdot g)_0 + J(h)_0$$

is Hurwitz. Then $\dot{x} = f(x)\cdot g(x) + h(x)$ is locally exponentially stable at the origin.

Proof: Based on Lemma 3, the function $f\cdot g + h$ admits the following first order Taylor expansion at the origin:

$$f(x)\cdot g(x) + h(x) = Ax + o(\|x\|_2).$$

The rest of the proof is identical to the one of Theorem 4.7 in [1]. □

Remark 5: In particular, Theorem 5 applies to the example of Subsection IV-A. More generally, any function f under the form $f(x_1, x_2, \ldots, x_n) = \psi(|x_1|, |x_2|, \ldots, |x_n|)$, where ψ is a continuously differentiable function, satisfies the regularity assumptions of Theorem 5.

V. CONCLUSION

This paper investigated the necessity of the $\mathscr{C}^1$-regularity assumption of the vector fields of a nonlinear dynamic system to deduce its local stability from its Jacobian linearization. For vector fields admitting directional derivatives, conditions that extend the classic Lyapunov's first method have been derived.

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