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STOCHASTIC BILEVEL PRICING PROBLEMS OVER A TRANSPORTATION NETWORK

SHAHROUZ MIRZA ALIZADEH
DÉPARTEMENT DE MATHÉMATIQUES ET DE GÉNIE INDUSTRIEL
ÉCOLE POLYTECHNIQUE DE MONTRÉAL

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Cette thèse intitulée :

STOCHASTIC BILEVEL PRICING PROBLEMS OVER A TRANSPORTATION NETWORK

présentée par : MIRZA ALIZADEH Shahrouz

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a été dûment acceptée par le jury d'examen constitué de :

M. ADJENGUE Luc-Désiré, Ph.D., président

M. SAVARD Gilles, Ph.D., membre et directeur de recherche

M. MARCOTTE Patrice, Ph.D., membre et codirecteur de recherche

M. GENDREAU Michel, Ph.D., membre

M. DUSSAULT Jean-Pierre, Ph.D., membre

- To all the special people in my life :

My parents who have dedicated their lives for their children

My brother and sisters who support me always

My dear lovely wife who taught me through love

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RÉSUMÉ

Dans cette thèse, nous étudions le problème de tarification sur un réseau sous des hypothèses d'incertitude (stochastiques). Elle comporte cinq chapitres. Les premier et deuxième chapitres sont une introduction générale et une introduction à la programmation biniveau, au modèle biniveau pour la tarification sur un réseau et à la programmation stochastique, respectivement. Mes deux articles soumis sont contenus dans les chapitres 3 et 4 et nous concluons cette thèse par le chapitre 5.

Dans le chapitre 3, nous considérons une extension stochastique à deux étapes du modèle de tarification biniveau introduit par Labbé et al. (1998). Dans la première étape, le meneur (leader), au niveau supérieur, fixe les tarifs sur un sous-ensemble d'arcs du réseau dans le but de maximiser son revenu, tandis qu'au niveau inférieur, les flots sont affectés aux chemins les moins onéreux du réseau de transport multiflots. Dans la deuxième étape, on introduit une incertitude sur la demande et les coûts (qui deviennent stochastiques) et ajoutons la contrainte que les nouveaux tarifs ne doivent pas être trop éloignés de ceux obtenus à la première étape. Nous considérons deux types de tolérances, une absolue et l'autre proportionnelle pour chaque arc tarifé du réseau, afin d'éviter le problème de tarifs excessifs. Nous reformulons notre modèle biniveau stochastique à deux étapes en un problème biniveau stochastique à une étape (standard) afin de déterminer une borne supérieure valide du revenu. Ceci nous a permis de mettre en évidence certaines propriétés de la fonction objectif tel que son caractère continu et linéaire par morceaux, dans le cas de la tolérance absolue. Nous appliquons notre approche à trois petits exemples de réseaux de transport aériens ayant des topologies, des commodités et des scénarios différents. Nous comparons la validité de notre modèle en comparant la solution stochastique obtenue en remplaçant les terms aléatoires par leur espérance. Enfin, nous présentons des résultats numériques pour des instances générées aléatoirement, sur du réseaux à 40 nœuds et 200 arcs, pour diverses formulations du modèle. L'analyse des résultats numériques montre que le modèle qui suppose des tarifs égaux sur

les deux étapes est plus complexe en raison de la limitation sévère sur les tarifs. Par ailleurs, le modèle où la demande est la seule variable aléatoire est moins complexe à puisque, plus court chemin du suiveur est le même dans les deux phases (étapes).

Au chapitre 4, nous présentons trois variantes du problème de tarification biniveau stochastique en définissant le temps de déplacement, la fiabilité du chemin et la capacité des lienes comme des variables aléatoires. Dans la première variante, nous introduisons des désutilités stochastiques au niveau du meneur. Ces dernières sont modélisées comme fonction des coûts fixes, des tarifs, des délais et de la fiabilité. Les deux dernières désutilités (délai et fiabilité) représentent la situation où les meneurs sont prêts à compromettre leur temps d'arrivée cible (deadline) pour une plus grande fiabilité. Pour ce faire, les termes de désutilité sont exprimés par des quotients du délai sur la fiabilité, où et le numérateur le dénominateur sont aléatoires. Le modèle considère l'espérance mathématique de la fonction objectif du leader et un comportement "wait and see" (attente - action) pour le suiveur. Un exemple illustre l'application du modèle à des réseaux de télécommunications ou des compagnies aériennes. Nous effectuons également une analyse de sensibilité à l'égard des variations des pénalités sur les délais et les dates limites (deadline) de différents ensembles de coûts fixes. Dans la deuxième variante, on considère un modèle qui prend en compte plusieurs caractéristiques de la première variante avec deux différences majeures : (1) une pénalité sur le délai est encourue par le leader et (2) la fiabilité d'un arc est maintenant fonction du flot qui le traverse, ainsi que et d'une capacité aléatoire. Une contrainte probabiliste, dont le rôle est d'éviter le débordement et d'assurer le fonctionnement fiable du système, est alors imposée au leader. Dans ce contexte, le choix du trajet du suiveur n'est pas influencé par la fiabilité des trajets car la contrainte imposée au niveau supérieur assure un niveau de service adéquat. Nous reformulons le modèle comme un problème de programmation linéaire mixte en nombres entiers, en transformant les contraintes probabilistes en contraintes linéaires. Nous montrons alors que la fonction du revenu est continu par rapport au vecteur de capacité. Nous illustrons l'application du modèle à un réseau de transport et nous montrons comment les changements dans le seuil de probabilité et le paramètre de proportion de la capacité de conception affectent le revenu. Enfin, la troisième variante considère la congestion, ce qui est un problème fréquent dans les transports urbains (résultant de conditions météorologiques, de travaux, de demande excédentaire) et dans les télécommunications (résultant du trafic et de la dégradation du réseau). En général la congestion est directement liée à la qualité de service. Contrairement à la deuxième variante où la qualité de service est imposée à l'aide d'une contrainte probabiliste, la troisième variante modélise explicitement la congestion par une fonction volume-délai du type BPR (Bureau of Public Roads). Nous présentons un exemple et une analyse de sensibilité du revenu en fonction des changements des paramètres de proportion de la capacité de conception et du seuil de probabilité.

ABSTRACT

This dissertation studies the network pricing problem (NPP) under uncertainty assumptions. It has five chapters. The first and second chapters give a general introduction to this thesis and the second chapter provides an introduction to bilevel programming (BP), the BP model of NPP, and stochastic programming. Chapters 3 and 4 contain my two submitted papers and we conclude this thesis in Chapter 5.

In Chapter 3, we consider a two-stage stochastic extension of the bilevel pricing model introduced by Labbé et al. (1998). In the first stage, the leader sets tariffs on a subset of arcs of a transportation network with the goal of maximizing profits, and at the lower level, flows are assigned to the cheapest paths of a multicommodity transportation network. In the second stage, we introduce uncertain information (stochastic demand and market prices) and the constraint that tariffs should not differ too greatly from those set in the first stage. We consider two forms of predetermined threshold restrictions (absolute restriction (AR) and proportional restriction (PR)) on each tariff arc of the network to avoid the excessive-tariff problem. We further provide a single-stage reformulation of the two-stage SBP to calculate a valid upper bound for the revenue. We derive a few propositions to show some properties of the value function of our model such as its continuity and piecewise linearity in the AR case. We present three small airline-network examples with different network topologies, numbers of commodities, and outcomes of the random variables. We also give the stochastic and the expected solution of the expected value (EEV) solutions to indicate the value of the model and the stochastic solution. Finally, we present numerical results for randomly generated instances with 40 nodes and 200 arcs for various formulations of the model. The analysis of the numerical results shows that the model that assumes equal tariffs for both stages is more complex because of the tight restriction on the tariffs. The model that assumes that the demand is the only random variable is less complex because of the shortest-path equality

for the follower problems in both stages.

In Chapter 4, we introduce three variations of the stochastic bilevel pricing problem by considering delay, path/link reliability, and link capacity as random variables. In the first variation, we introduce stochastic disutility at the user level. This is modeled as a function of fixed costs, tariffs, tardiness, and reliability, and represents the situation where users are ready to compromise their target arrival time (deadline) for higher reliability. To achieve this goal, the disutility terms are expressed as the ratio of tardiness to reliability, both the numerator and denominator being random. The model considers the expectation form of the leader's objective function and *wait and see* behavior for the follower. An example is presented to show the application of the model to telecom/airline networks. We also carry out a sensitivity analysis with respect to changes in the tardiness penalties and deadlines for different sets of fixed costs. In the second variation, we consider a framework that takes into account several features from the first variation. However, there are two differences. First, a tardiness penalty is incurred by the leader. Second, the reliability of an arc is now related to the flow that it carries and to a random capacity. A chance constraint, whose role is to prevent overflow and ensure the reliable performance of the system, is then imposed on the leader. In this setting, the path choice of the follower is not influenced by the reliability of the paths, since the constraint imposed at the upper level ensures a predetermined level of service. We reformulate the model as a linear MIP by transforming the chance constraints to linear constraints and show that the value function of the revenue is a continuous function with respect to the design-capacity proportion parameter. We illustrate the application of the model to telecom/transportation networks and show how changes in the probability level and the proportion parameter can change the revenue. Finally, the third variation considers congestion, which is a common issue in urban transportation (arising from construction, weather conditions, excess demand) and in telecommunications (arising from traffic, network degradation). It is directly related to the "quality of service." In contrast with the second variation, where the quality of service is enforced by a chance constraint, the third variation explicitly models congestion via a volume-delay function of the BPR (Bureau of Public Roads). We present an example and a sensitivity analysis to show the effects on the revenue of changes in the design-capacity proportion parameter and the probability level.

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LIST OF SYMBOLS AND ABBREVIATIONS

BCCP	Bilevel Chance-Constrained Programming
BP	Bilevel Programming
CCP	Chance-Constrained Programming
EEV	Expected solution of the Expected Value
MIP	Mixed-Integer Program
QoS	Quality of System
SBP	Stochastic Bilevel Programming
SP	Stochastic Programming
RM	Revenue Management

CHAPITRE 1

INTRODUCTION

1.1 Motivation

Nowadays, businesses are more competitive than ever before. Therefore, many companies, especially those which supply perishable products such as airline seats and hotel rooms, apply pricing policies and revenue management (RM). RM is the practice of maximizing expected revenues or profits by selling products or services to the right customers at the right time and the right price. This practice helps companies such as airlines, hotels, car rental firms, and even manufacturers to predict and influence market demand, to allocate limited resources to a variety of customers, and to optimize price availability in order to maximize the profit.

RM began at American Airlines, which developed in the 1960s the first totally automated computer-reservation system. In the 1970s, the airline industry began to pay more attention to the management of its products, by using information systems to store information about its products and customers. The RM concept then appeared in the Littlewood (1972) rule ; this rule is considered the basis of the RM decision that checks whether to accept or to reject a demand by the evaluation of the expected displacement costs. Research then began into RM perspectives and problems such as pricing, resource allocation, and forecasting. Industrial applications included the transportation, telecommunication, hotel, and car rental industries. Later, companies further developed their RM systems to cope with deregulated markets and balances of supply and demand especially under uncertainty conditions.

Today, pricing policies and decisions are considered fundamental business challenges, especially for service providers. They play a primary role at both the strategy and planning levels. This is probably because of the highly competitive business environment, where adjusting the price is considered the most effective way to influence the customer's motivation.

However, the evolution of information technology and the internet and a rise in unexpected events have forced managers to deal with pricing issues dynamically. They use stochastic models to consider the unexpected events that may affect decisions that must be taken in advance.

This thesis focuses on the network pricing problem (NPP) under uncertainty ; it belongs to the class of NP-hard problems. We deal with some market-uncertainty aspects such as demand, competitor's price, and delay in airline, transportation, and telecommunication networks to present pricing models that maximize revenue under market uncertainties. In general, for competitive markets, comprehensive pricing models must contain stochastic, dynamic, and game-theoretic elements. Customers emphasize price, and so suppliers try to offer prices that attract customers while maximizing revenue. Therefore, it is important to define a mathematical model of the pricing problem that considers the customer's behavior versus the prices and competition situation of the market. Labbé et al. (1998) present a bilevel programming (BP) model of the pricing problem ; it considers a game-theoretic approach between the leader who wants to maximize revenue and the follower who wants to minimize disutility. However, this model does not consider stochastic aspects. Our work aims to capture the stochastic aspects of the market to embed in the NPP presented by Labbé et al. (1998). Precisely, our research objective is to integrate stochastic programming (SP) approaches such as two-stage SP and chance-constrained programming (CCP) into the BP framework and to apply the algorithm to the NPP.

This thesis is organized as follows.

Chapter 2- This chapter provides the literature review and introduces the basic concepts including bilevel programming, stochastic programming, and NPP. Further we provide the basic notation used in this thesis.

Chapter 3- This chapter presents our first paper, in which we applied the SP approach to the NPP. Our main contributions are :

- we present a two-stage stochastic extension of a bilevel program and a generic pricing model ;
- we provide some properties of a general stochastic bilevel program (a two-stage stochastic bilevel program (SBP) with recourse in the first level) ;
- we present some properties of the two-stage stochastic bilevel pricing model ;
- we apply the model to a transportation network ; and
- we present numerical results to indicate the size of problems that can be solved in a reasonable time.

Chapter 4- This chapter presents our second paper where we again applied the SP approach to the NPP. Our main contributions are :

- we explore some random parameters relating to quality of service issues for telecom and transportation networks ;
- we model three variations of the stochastic NPP (one with the expected form of the first level's objective function and two with chance constraints in the first-level problem) ;
- we present some properties of the second variation ; and
- we discuss applications of the models to transportation and telecommunication networks.

Finally, in Chapter 5, we present conclusions and discuss possible future work.

CHAPITRE 2

BASIC NOTATIONS AND LITERATURE REVIEW

2.1 Bilevel Programming (BP)

In the real world, companies compete for the best positions from management and economic points of view. There are three known game strategies : win-win, win-fail, and fail-win. The Stackelberg game as a strategic game is the most interesting game between market players ; it is studied by Stackelberg (1952). The BP approach is the most useful tool to model these games. BP considers two players where each player seeks to optimize his objective function by controlling one set of actives (variables) subject to constraints. These problems consist of two programs where the second program is considered a constraint for the first. That is, some of the first-program variables are constrained to be an optimal solution of the second program. This structure of BP is closely related to a Stackelberg game.

The original formulation of BP was presented by Bracken and McGill (1974) as a hierarchical optimization problem involving two levels and formulated as the mathematical program

$$\begin{aligned} \max_{x \in X} \quad & F(x,y) \\ \text{s.t.} \quad & G(x,y) \leq 0, \\ \min_{y \in Y} \quad & f(x,y), \\ \text{s.t.} \quad & g(x,y) \leq 0, \end{aligned} \tag{2.1}$$

where the first-level and second-level problems are usually called the leader (or outer) and follower (or inner) problems respectively. We define these levels as follows :

$$\text{First level}\begin{cases}\min\limits_{(x,y)\in Z} & F(x,y)\\ \text{s.t.} & G(x,y)\leq 0,\end{cases}$$

and

$$\text{Second level}\begin{cases}y\in\arg\min\limits_{z\in Y} & f(x,z)\\ \qquad\quad\text{s.t.} & g(x,z)\leq 0,\end{cases}$$

where $Z = X \times Y$, $F : Z \to \Re$ and y is the solution of the follower problem for a given x. The BP problem gives the optimal value of the first-level problem by the solution of one set of variables allowed for an optimization problem (second level).

1. The feasible set (or constraint region) of BP is

$$\mathrm{FS} = \{(x,y) : (x,y)\in Z,\ G(x,y)\leq 0,\ g(x,y)\leq 0\}.$$

2. For each $x \in X$, the second-level feasible set is

$$\mathrm{FS}(x) = \{y : y\in Y,\ g(x,y)\leq 0\}.$$

3. For each x from the first level the set of optimal solutions (or rational reaction set) of the second level is

$$\mathrm{RR}(x) = \left\{y :\ y\in\arg\min_{z}\{f(x,z) : z\in \mathrm{FS}(x)\}\right\}.$$

4. Finally, the feasible set of BP is

$$\mathrm{IR} = \{(x,y) : (x,y)\in \mathrm{FS},\ y\in \mathrm{RR}(x)\}.$$

The feasible set of the BP problem is called the induced (inducible) region and is usually nonconvex. This set can be disconnected or even empty in the presence of leader constraints involving y. The compactness of the induced region is important for the existence of an optimal solution. This property can be guaranteed by the appropriate conditions. BP problems are usually nonconvex and nondifferentiable and therefore hard to solve. The main property of BP is that it is strongly NP hard even if all functions are linear (Vicente and Calamai, 1994; Ben-Ayed and Blair, 1990). Vicente and Calamai (1994) prove that obtaining a local optimum is also NP-hard. The optimal solution of BP also need not be Pareto optimal. For the continuous form of BP if the lower level problem is convex then it can be replaced by the KKT conditions under an appropriate constraint qualification. Generally, the convexity of the BP problem does not guarantee the convexity of the inducible region. In other words, restricting the upper and lower levels' objective and constraint functions to be continuous and bounded does not guarantee the existence of a solution.

BP algorithms use either continuous or combinatorial or both approaches, to find local and global solutions. Optimality conditions for BP problems have been studied by several researchers. Savard and Gauvin (1994) present necessary optimality conditions based on the steepest-descent direction and Bard (1998) gives detailed information on BP properties and solution methods. Generally, the solution methods and algorithms can be classified based on the different BP forms : linear, nonlinear, bilinear, or quadratic programs with continuous and/or discrete variables. Algorithms for BP problems may be classified into six classes : extreme-point approaches (for the linear case), branch-and-bound (B&B), complementarity pivot, descent methods, penalty function methods, and trust-region methods. For instance, Savard (1989) specified problems including linear BPs where at least one optimal solution can be located at an extreme point of the constraint region or feasible set without any particular assumptions over upper-level constraints.

2.1.1 BP applications : BP model of network pricing problem (NPP)

When the upper level decisions depend on the lower level decisions, the BP approach can be applied to model NPPs in different fields such as highway toll setting (Labbé et al. (1998), Brotcorne et al. (2000), Brotcorne et al. (2001)), airline revenue management (Côté et al. (2003)), network pricing (Bouhtou et al. (2003)), telecommunications (Altman and Wynter (2004), Bouhtou et al. (2006), Bouhtou et al. (2007a)), and supply chain pricing (Shouping and Baozhuang (2007), Gao et al. (2011)). Dempe et al. (2005) present a mixed-integer BP model with binary variables in the follower problem to model the problem of minimizing the cash-out penalties of a natural gas shipper, and Bard et al. (2000) applied the approach to model tax credits in biofuel production. Recently, Li et al. (2011) applied it to model the game behavior between government and companies in the trading of waterfront resources ; they presented a solution approach based on sensitivity analysis.

Consider a network represented by a graph $G(N, \Lambda)$ with node set N and arc set Λ. The arcs of graph G are divided into two sets, Λ_1 and Λ_2. Arc set Λ_1 is a set of tariff (taxed) arcs controlled by the leader and arc set Λ_2 is a set of tariff-free (untaxed) arcs. A set K of commodities models the demand. The goal is to determine the right tariffs on arc set Λ_1 to maximize the leader's revenue. Users (customers or followers) on the network choose their route from origin to destination according to the shortest path with respect to the total disutility costs. As far, the paths with the same total disutility are available then it is assumed that the users choose the path that is more profitable for the leader. This assumption can be satisfied by a small change in the fixed costs of the tariff arcs.

The arc formulation of the bilevel network pricing problem introduced by Labbé et al. (1998) is as follows :

$$\begin{aligned}
\max_{t} \quad & t \sum_{k \in K} x^k \\
\min_{x^k, y^k} \quad & (c+t) \sum_{k \in K} x^k + d \sum_{k \in K} y^k, \\
\text{s.t.} \quad & Ax^k + By^k = b^k, & \forall k \in K, \\
& x^k, y^k \geq 0, & \forall k \in K,
\end{aligned} \tag{2.2}$$

where t is the vector of tariff variables controlled by the leader, and x^k and y^k are the flow vectors of commodity k on the tariff and tariff-free arcs respectively. Vectors c and d are the fixed costs on the tariff and tariff-free arcs respectively. (A, B) is the node-arc incidence matrix that characterizes the flow-conservation constraints in the lower level, and b^k is the commodity demand vector defined as follows :

$$b_i^k = \begin{cases} n^k, & \text{if } i = O(k), \\ -n^k, & \text{if } i = D(k), \\ 0, & \text{otherwise,} \end{cases}$$

where n^k represents the amount of commodity k to be shipped from the origin $O(k)$ to the destination $D(k)$. We define the set of all paths for commodity k to be L^k, and the set of arcs included in path ρ by Λ^ρ. L_1^k is the subset of paths that include at least one tariff arc. Then the path reformulation of Program (2.2) is

$$\begin{aligned}
\max_{t,T} \quad & \sum_{k\in K}\sum_{\rho\in L_1^k} T_\rho f_\rho \\
\text{s.t.} \quad & \sum_{a\in\Lambda_1^\rho} t_a = T_\rho, && \forall k\in K, \forall \rho \in L_1^k, \\
\min_{r} \quad & \sum_{k\in K}\big[\sum_{\rho\in L_1^k} T_\rho f_\rho + \sum_{\rho\in L^k}\sum_{a\in\rho} c_a f_\rho\big], \\
\text{s.t.} \quad & \sum_{\rho\in L^k} f_\rho = n^k, && \forall k\in K, \\
& f_\rho \geq 0, && \forall k\in K, \forall \rho\in L^k,
\end{aligned} \tag{2.3}$$

where c is the vector of fixed costs. The leader's constraint refers to the relationship between the path and arc tariffs, and the first follower's constraint refers to the flow conservation constraints where variable f_ρ denotes the flow of commodity k assigned to path $\rho \in L^k$.

In this thesis we consider the following general assumptions for the above pricing model (see (Labbé et al., 1998)) :

1. There is at least one path for each user that is composed only of tariff-free (untaxed) arcs. This guarantees that the upper level of the leader's profit is bounded from above.
2. There does not exist a pricing procedure that makes a profit and has a negative cost cycle in the network. Thus, the lower level of the optimal solution corresponds to a set of shortest paths.

   We make the following assumption for the models presented in our second paper (3) :
3. The market prices are fixed and will not change when the operator sets its tariff. The client's demand is fixed and can be split between different paths.

Given the above assumptions Labbé et al. (1998) provided the feasible upper bound $\Gamma(\infty)-\Gamma(0)$ for the leader's profit ; it is the difference between the follower's optimal objectives corresponding to infinite and zero tariffs. In other words, $\Gamma(0)$ is the value of the follower's objective corresponding to a shortest path solution when $t = 0$, and $\Gamma(\infty)$ is the value of the follower's objective when the tariff is infinite. This upper bound is finite whenever the fixed costs are nonnegative.

Labbé et al. (1998) study the complexity of the BP of the pricing model (2.2) on a transportation network. They assume a single-user transportation network and show that this problem is NP-hard when the tariff is restricted by a given lower bound. Bouhtou et al. (2002) prove the NP-hardness of the pricing problem without a lower-bound assumption on the taxes by modifying the proof given by Labbé et al. (1998). Later, Grigoriev et al. (2007) and Roch et al. (2005) improved this NP-hardness result and presented a polynomial-time approximation approach to solve the NPP.

We now discuss solution procedures for the NPP and in particular exact methods, which are often based on the optimality conditions of the lower-level shortest paths. As mentioned before, if we replace the lower-level problem by the KKT conditions and linearize the complementarity slackness conditions, the BP model for the NPP becomes a mixed-integer program and can be solved by known methods. The best references for the exact methods are Dewez (2004) and Didi-Biha et al. (2006), where Dewez (2004) presented a method based on the cutting-plane approach and Didi-Biha et al. (2006) provided a method based on a path formulation of Program (2.2). Later Brotcorne et al. (2011) extended the exact method presented by Didi-Biha et al. (2006) by developing an efficient path generation method and applying column generation approach. Generally, solution methods based on the path formulation perform better than the mixed-integer formulation.

Brotcorne et al. (2001) and Bouhtou et al. (2007b) presented primal-dual heuristic procedures based on a single-level reformulation of NPP, and Brotcorne et al. (2012) developed an efficient tabu-search metaheuristic framework to solve large instances.

2.2 Stochastic Programming (SP)

Stochastic programming (SP) is a mathematical programming framework used to model problems under uncertainty. These programs are more difficult to formulate and solve than general deterministic mathematical programs. The main advantage of using SP is the ability to perform optimality analysis under uncertainty ; sensitivity analysis must be used to review the impact of uncertainty in general mathematical programs. Stochastic models are mainly introduced for economic models subject to uncertainty in demand and price changes. However, these models are also used in the engineering sciences, such as civil, mechanical, and aerospace engineering. Bilevel and dynamic programming approaches are tools used for modeling problems with two or more stages, but the stochastic approach is more important for long-term planning because it uses the information and data that form the basis of future events. Managers and decision-makers can make decisions by considering the risk of all scenarios and forming an overall optimal strategy.

A generic stochastic program is as follows :

$$\begin{aligned} \max_{x \in X} \quad & F(x,\omega) \\ \text{s.t.} \quad & G(x,\omega) \leq 0, \end{aligned} \tag{2.4}$$

where $X \subseteq R^n$ and it is assumed that the functions F and G are not accurately known. These functions depend on a pair of variables $(x, \xi(\omega))$ where ω is a random experiment vector or a possible generalization of ξ, and $\boldsymbol{\xi}$ is a real random-variable vector that varies over a support set Ξ in a probability space $(\Omega, \mathcal{F}, P)$. Ω denotes the set of all random events, $\mathcal{F}$ is a set of random events, and P is the set of probabilities. Further, we assume that the probability distribution $p \in P$ is given and is independent of x, so that for all x, $F(x, \cdot) : \Xi \to R$ and $G(x, \cdot) : \Xi \to R$ are random variables.

2.2.1 Two-stage and multi-stage SP

The two-stage SP problems or SP problems with recourse are the most well-developed models. The decision-maker can take decisions before or after realizing outcomes in the first-stage and second-stage respectively. In other words, second-stage decisions are made in response to the realized outcomes and the term "recourse" refers to the possibility of choosing a solution given specific realizations of the random variables. Decisions that are taken before are called first-stage decisions and those taken after are second-stage decisions. Normally the first-stage and second-stage decision variables are considered proactive and reactive, corresponding to the planning and operating decisions. According to the above definition, the two-stage SBP is as follows :

$$\begin{aligned} \max_{x} \quad & F_1(x) + Q(x) \\ \text{s.t.} \quad & G_1(x) \leq 0, \end{aligned} \tag{2.5}$$

where

$$Q(x) = E_{\boldsymbol{\xi}}\left[\Phi(x, \xi(\omega))\right], \ \boldsymbol{\xi} : \Omega \to \Re^r,$$

is the recourse and, for any outcome $\xi = \xi(\omega) \in \Xi$,

$$\begin{aligned} \Phi(x, \xi(\omega)) = \ \max_{x'(\omega)} \quad & F_2(x'(\omega), \omega) \\ \text{s.t.} \quad & G_2(x, x'(\omega), \omega) \leq 0. \end{aligned}$$

where x is known as the first-stage decision and x' is the second-stage variable determined after outcome $\xi(\omega)$ is realized. In the linear case of Program (2.6), a special form of the recourse program is called the fixed-recourse program ; here the matrix of coefficients in the second-stage program is fixed, i.e., it is not subject to uncertainty. Further, if it is assumed that the second-stage coefficients matrix is an identity then two-stage recourse is called simple recourse, and if the second-stage program is feasible for every first-stage feasible decision then the two-stage recourse is called relatively complete recourse. A general approach to solve a linear two-stage SP is the L-shaped method, which was developed by Slyke and Wets (1969) and extended by Birge and Louveaux (1988) by applying a multicut generation procedure.

In some problems the outcomes are realized sequentially. The problem can then be divided into multiple stages over time and the outcomes. Program (2.6) can be extended to a multi-stage SP problem for applications such as long-term planning in project management where success is sensitive and depends on information change and future events. In a multi-stage SP model, the scenario information and data can be organized into a tree structure.

The two-stage recourse problem to define a multi-stage stochastic program (MSP) as follows :

$$\begin{aligned} \max_{x_1} \quad & F_1(x_1) + Q(x_1) \\ \text{s.t.} \quad & G_1(x_1) \leq 0, \end{aligned} \tag{2.6}$$

where

$$Q(x_1) = E_{\boldsymbol{\xi_1}}\left[\Phi(x_1, \xi_1(\omega_1))\right], \, \boldsymbol{\xi_1} : \Omega_1 \to \Re^{r_1},$$

and, for any outcome $\xi_1 = \xi_1(\omega_1) \in \Xi$, we replace the recourse program of Program (2.6) to generalize the two stages to S stages as follows :

$$\begin{aligned} \Phi(x_{s-1}, \xi_s(\omega_s)) = \quad \max_{x'_s(\omega_s)} \quad & F_2(x'_s(\omega_s), \omega_s) + E_{\boldsymbol{\xi_{s+1}}|\boldsymbol{\xi^s}}\left[\Phi_{s+1}(x_s, \xi_{s+1}(\omega_{s+1}))\right] \\ \text{s.t.} \quad & G_{2,s}(x_{s-1}, x'_s(\omega_s), \omega_s) \leq 0, \end{aligned}$$

where $s = 2, ..., S$ and $\boldsymbol{\xi_s} : \Omega_s \to \Re^{r_s}$. Further, for each realization ω_s of ξ_s, $\Phi(x_S, \cdot) = 0$ and ξ^s is the history of the random variables up to time s. We define the history process by $\xi^s \stackrel{def}{=} \{\xi_1, ...\xi_s\}$. Also, the $E_{\boldsymbol{\xi_{s+1}}|\boldsymbol{\xi^s}}$ term is the expected value according to the conditional of ξ_{s+1} on ξ^s. Birge (1985) presented an extension of the L-shaped method to multi-stage SP problems.

2.2.2 Chance-constrained programming (CCP)

One of the main consequences of uncertainty in the context of decision-making is the possibility of infeasibility in the future. In many instances, we must make our decision before the future realization. Decision problems that consider risk-aversion issues use chance (or probabilistic) constraints to express the feasibility of the problem. Therefore, CCP problems are SP problems that consider chance constraints. Such models are known as anticipative models.

We consider a form of Program (2.4) with no random parameters in the objective function and then we introduce the chance constrained program as follows :

$$\begin{aligned} \max_{x \in X} \quad & F(x) \\ \text{s.t.} \quad & p(x) \geq \alpha, \end{aligned} \tag{2.7}$$

where again $X \subseteq R^n$,

$$p(x) = \Pr\{G(x, \omega) \leq 0\},$$

and $\alpha \in [0, 1]$ denotes the probability/reliability level and the choice of α is left to the decision-maker. The complement $1 - p(x)$ refers to the risk of infeasibility associated with x and the values $\alpha = 0$ and $\alpha = 1$ correspond to extremely risky and conservative attitudes.

Program (2.7) is called a joint chance constrained (JCC) program when there may be multiple inequalities in the system $G(x, \omega) \leq 0$. The separate (or individual) chance constrained (SCC) program, for $\alpha_i \in [0, 1]$, $i = 1, ..., m$, is as follows :

$$\begin{aligned} \max_{x \in X} \quad & F(x) \\ \text{s.t.} \quad & \Pr\{G_i(x, \omega) \leq 0\} \geq \alpha_i, \quad \forall i = 1, ..., m. \end{aligned} \tag{2.8}$$

Normally a suitable feasible solution of the JCC problem can be obtained by solving the SCC problem and choosing $\alpha_i = 1 - \dfrac{1 - \alpha}{m}$. The feasible sets of the JCC and SCC problems are as follows :

$$C(\alpha) = \{x \in R^n : p(x) \geq \alpha, \},\ \alpha \in [0,1],$$

$$C_i(\alpha_i) = \{x \in R^n : p_i(x) \geq \alpha_i\},\ \alpha_i \in [0,1],\ i = 1, ..., m,$$

where

$$C(\alpha) = \bigcap_{i=1}^{m} C_i(\alpha_i),$$

for $\alpha = (\alpha_1, ...\alpha_m) \in [0,1]^m$.

More details on JCC, SCC, and solution methods based on the properties of the distribution function F and feasible set $C(\alpha)$ have been provided by Kall and Wallace (1994), Birge and Louveaux (1997), and Kall and Mayer (2010).

2.3 Notations

We now summarize the notation, basic variables, and parameters used in Chapters 3 and 4. We denote all random variables, parameters, and the second-stage variables of the two-stage SP by adding the prime symbol ($'$) to the deterministic parameters and the first-stage variables of the two-stage SP. We use the following notation :

Sets :

N	set of nodes of graph G ;
Λ	set of arcs/links of graph G ;
Λ_1	set of tariff arcs of graph G ;
Λ_2	set of tariff-free arcs of graph G ;
K	set of commodities ;
L^k	set of paths available to commodity k ;
L^k_1	set of paths available to commodity k that contain at least one tariff arc ;
L^k_2	set of paths available to commodity k that contain only tariff-free arcs ;

Λ^ρ set of arcs common to set Λ and path ρ ;

Π set of constraints corresponding to threshold values ;

$(\Omega, \mathcal{F}, P)$ probability space ;

Ω set of all random events ;

$\mathcal{F}$ set of all subsets of Ω ;

Ξ set of outcomes of the random variable $\boldsymbol{\xi}$, i.e., support of the random variable $\boldsymbol{\xi}$;

$L^k(\xi)$ set of paths available to commodity k corresponding to outcome ξ ;

$L_1^k(\xi)$ set of paths available to commodity k that contain at least one tariff arc corresponding to outcome ξ ;

$L_2^k(\xi)$ set of paths available to commodity k that contain only tariff-free arcs corresponding to outcome ξ ;

Indices :

$a \in \Lambda$ arc index ;

$k \in K$ commodity index ;

$\rho \in L^k$ path index for each commodity k ;

$\xi = \xi(\omega) \in \Xi$ outcome index ;

$\rho' \in L^k(\xi)$ path index for each commodity k corresponding to outcome ξ ;

Matrix and vectors :

(A, B) node-arc incidence matrix that characterizes flow conservation constraints ;

c vector of fixed costs on tariff arcs (in Chapter 3, we refer to c as the vector of fixed costs on tariff and tariff-free arcs) ;

d vector of fixed costs on tariff-free arcs ;

C vector of design (or target) capacity of tariff arcs ;

δ vector of threshold values on tariffs to avoid unplanned tariff increases (or decreases) ;

θ	vector of proportion of maximum protection of tariff arcs ;
b	vector of demands for commodities ;
τ^{arc}	vector of free flow travel time on arcs ;
τ^{node}	vector of free flow travel time on nodes ;
$\bar{p}$	vector of penalty costs for one unit of tardiness of commodities ;
$d^{\text{arc}}(\boldsymbol{\xi})$	vector of random delay variables on arcs corresponding to outcome ξ ;
$d^{\text{node}}(\boldsymbol{\xi})$	vector of random delay variables on nodes corresponding to outcome ξ ;
$\boldsymbol{\xi}$	vector of random variables and $\boldsymbol{\xi} : \Omega \to \Re^r$;
$C'(\xi)$	vector of random capacity of tariff arcs corresponding to outcome ξ ;

Parameters :

n^k	number of users of commodity k ;
$O(k)$	origin node of commodity k ;
$D(k)$	destination node of commodity k ;
Δ	small positive number ;
σ	small positive number less than one ;
M	arbitrary large constant number ;
$\Gamma(t)$	lower-level optimal value for given tariff vector t ;
α, β	reliability/probability levels ;
H	vector of travel time limits or preferred arrival times at the destination for commodities ;
$\Gamma'_{\xi}(t')$	lower-level optimal value for given tariff vector t' corresponding to outcome ξ ;
$g(\xi)$	vector of random travel time functions on paths corresponding to outcome ξ ;
$h(\xi)$	vector of random availability of paths corresponding to outcome ξ ;

Variables :

t vector of arc tariff variables controlled by leader ;

T vector of path tariff variables ;

x^k vector of arc flow variables of commodity k on tariff arcs (in Chapter 3, we refer to x as the vector of flow variables on tariff and tariff-free arcs) ;

y^k vector of arc flow variables of commodity k on tariff-free arcs ;

r_ρ vector of path choice variables of commodity k ;

f vector of path flow variables of commodity k ;

λ vector of dual variables associated with the first-stage follower constraints.

The following variables are defined as the above variables corresponding to outcome ξ :

$t'(\xi)$ vector of arc tariff variables controlled by leader ;

$T'(\xi)$ vector of path tariff variables ;

$x'^k(\xi)$ vector of arc flow variables of commodity k on tariff arcs ;

$y'^k(\xi)$ vector of arc flow variables of commodity k on tariff-free arcs ;

$r'_{\rho'}(\xi)$ vector of path choice variables of commodity k ;

$\lambda'(\xi)$ vector of dual variables associated with the second-stage follower constraints ;

$u(\xi)$ vector of path tardiness variables ;

Functions :

$U(\cdot)$ first-stage lower-level disutility function ;

$Q(\cdot)$ expected value of recourse ;

$U'(\cdot, \xi)$ second-stage stochastic lower-level disutility function ;

$\Phi(\cdot, \xi)$ second-stage value.

CHAPITRE 3

TWO-STAGE STOCHASTIC BILEVEL PROGRAMMING OVER A TRANSPORTATION NETWORK

RÉSUMÉ

Nous considérons une extension stochastique sur deux étapes (ou périodes) du modèle biniveau pour la tarification de réseau introduit par Labbé et al. (1998). À la première étape, le meneur (leader) fixe les tarifs sur un sous-ensemble d'arcs du réseau dans le but de maximiser son revenu, tandis qu'au second niveau les flots sont affectés sur les chemins les moins chers du réseau de transport multiflots. À la deuxième étape (ou période), la situation se répète sous la contrainte que les tarifs de la seconde période sont contraints à ne pas varier plus que d'un certain pourcentage préétabli de ceux de la première période. Enfin nous analysons les propriétés théoriques du modèle et présentons quelques résultats numériques.

Two-stage stochastic bilevel programming over a transportation network

Shahrouz Mirza Alizadeh, Patrice Marcotte and Gilles Savard

ABSTRACT

We consider a two-stage stochastic extension of the bilevel pricing model introduced by Labbé et al. (1998). In the first-stage, the leader sets tariffs on a subset of arcs of a transportation network, with the aim of maximizing profits while, at the lower level, flows are assigned to cheapest paths of a multicommodity transportation network. In the second-stage, the situation repeats itself under the constraint that tariffs should not differ too widely from those set at the first-stage. We analyze properties of the model and provide numerical illustrations

Key words : Revenue Management, Pricing, Bilevel Programming, Stochastic Programming

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3.1 Introduction

Designing an efficient pricing policy can significantly improve the position of a product and/or service provider. The key to successful revenue maximization actually rests on the knowledge of customers' options vis-à-vis the products (or services) supplied by the firm and its competitors. These features are well captured by the bilevel pricing model introduced by Labbé et al. (1998), where a revenue-maximizing leader anticipates the reaction to its decisions of cost-minimizing followers. The focus of the present work is the extension of this model to a stochastic environment characterized by market uncertainties.

The bilevel programming paradigm has been adapted to pricing issues in various fields, such as highway toll setting (Labbé et al. (1998), Brotcorne et al. (2000)), airline revenue management (Côté et al. (2003)), network pricing (Bouhtou et al. (2003)) and telecommunications (Altman and Wynter (2004), Bouhtou et al. (2006), Bouhtou et al. (2007a)). Actually a stochastic programming extension of bilevel programming, whose underlying principles have been laid out by Patriksson and Wynter (1999), has been proposed by Patriksson and Wynter (1997) for addressing a structural optimization problem, and by Christiansen et al. (2001) to formulate a topology optimization model in structural mechanics. More recently, Patriksson (2008) extended the scope of bilevel traffic models by taking explicitly into account stochastic data fluctuations. Closer to our application, Fampa et al. (2008) used stochastic bilevel programming (SBP) to model the strategic-bidding process that takes place in a wholesale energy market. The model assumes that the economic payment of each provider depends on the ability of its management to yield price and quantity bids. This SBP maximizes the expected profit at the upper level, and minimizes operational costs at the lower level. In contrast with the Nash equilibrium approach adopted by Hobbs et al. (2000), the bidding process is based on scenarios embedded within a Stackelberg (bilevel) framework. Carrion et al. (2009) present an SBP where a retailer optimizes its medium-term revenue at a given risk level, assuming that pool prices, demand, and competitor prices are random. Always in the realm of energy modeling, a bilevel multi-stage stochastic programming model has been presented by Kalashnikov et al. (2010) to formulate the natural gas cash-out problem. More recently, Cooper et al. (2012) assessed the performance of strategies that are oblivious to competition, in the context of a duopoly, focusing on the dynamic estimation of prices and demand parameters by both players.

The aim of this paper is to understand the properties of the network bilevel pricing problem and to estimate the loss of revenue due to neglecting randomness. It is structured as follows. In Section 3.2, we provide a preliminary view of two-stage stochastic programming. In Section 3.3, we introduce the two-stage network pricing model, whose mathematical properties are investigated in Section 3.4. In Section 3.5 we illustrate the various concepts through two examples, while numerical tests on a larger instance are presented and analyzed in Section 3.6. In a concluding section, we open avenues for further research.

3.2 Two-stage stochastic bilevel programming

Bilevel programming (BP) allows the natural modelling of hierarchical situations where a subset of decision variables is not under the control of the main optimizer (leader/upper level) but is controlled by a follower (lower level) who optimizes its own objective function with respect to the parameters set by the leader. Mathematically, it is expressed as

$$\begin{aligned} \max_{x,y} \quad & F_1(x,y) \\ \text{s.t.} \quad & G_1(x,y) \le 0, \\ & y \in \arg\min_{\hat{y}} \quad f_1(x,\hat{y}), \\ & \qquad\qquad \text{s.t.} \quad g_1(x,\hat{y}) \le 0. \end{aligned} \tag{3.1}$$

In the sequel, and in order to simplify notation, we will only specify the programs of the leader and the follower, since they contain all the information relevant to the bilevel program.

We now introduce a two-stage stochastic model, where the first-stage decisions are made before observing the random outcome at the second-stage. The second-stage decision corresponds to a "recourse", once all randomness has been removed. Notationwise, we consider a random vector $\boldsymbol{\xi}$ with realizations ξ and support Ξ, in a probability space $(\Omega, \mathcal{F}, P)$, where Ω denotes the set of all random events and $\mathcal{F}$ the set of all subsets of Ω. The two-stage SBP is formulated as follows :

$$\begin{aligned}
\max_{x} \quad & F_1(x,y) + Q(x,y) \\
\text{s.t.} \quad & G_1(x,y) \leq 0, \\
\min_{y} \quad & f_1(x,y) \\
\text{s.t.} \quad & g_1(x,y) \leq 0,
\end{aligned} \tag{3.2}$$

where

$$Q(x,y) = E_{\boldsymbol{\xi}}\left[\Phi(x,y,\xi)\right], \ \boldsymbol{\xi} : \Omega \rightarrow \Re^r,$$

and, for any outcome $\xi(\omega) \in \Xi$ $(\omega \in \Omega)$,

$$\begin{aligned}
\Phi(x,y,\xi(\omega)) = \quad \max_{x'(\omega)} \quad & F_2(x'(\omega), y'(\omega), \omega) \\
\text{s.t.} \quad & G_2(x'(\omega), y'(\omega), x, y, \omega) \leq 0, \\
\min_{y'(\omega)} \quad & f_2(x'(\omega), y'(\omega), \omega) \\
\text{s.t.} \quad & g_2(x'(\omega), y'(\omega), x, \omega) \leq 0.
\end{aligned}$$

It is easy to show that the above program can be reformulated as the "standard" bilevel program

$$
\begin{aligned}
\max_{x,x'} \quad & F_1(x,y) + E_{\boldsymbol{\xi}}\left[F_2(x'(\xi), y'(\xi), \xi)\right] \\
\text{s.t.} \quad & G_1(x,y) \leq 0, \\
& G_2(x'(\omega), y'(\omega), x, y, \omega) \leq 0, \quad \forall \omega \in \Omega, \\
\min_{y,y'} \quad & f_1(x,y) + E_{\boldsymbol{\xi}}\left[f_2(x'(\xi), y'(\xi), \xi)\right] \\
\text{s.t.} \quad & g_1(x,y) \leq 0, \\
& g_2(x'(\omega), y'(\omega), x, \omega) \leq 0, \qquad \forall \omega \in \Omega.
\end{aligned}
\tag{3.3}
$$

Alternatively, if all lower level problems are convex and regular (assuming some constraint qualification is satisfied) and if all functions involved are continuously differentiable, then the SBP can be equivalently stated as the "standard" stochastic program

$$
\begin{aligned}
\max_{\substack{x,y \\ \mu}} \quad & F_1(x,y) + Q(x,y) \\
\text{s.t.} \quad & G_1(x,y) \leq 0, \\
& \nabla_y f_1(x,y) + \mu \nabla_y g_1(x,y) = 0, \\
& \mu g_1(x,y) = 0, \\
& g_1(x,y) \leq 0, \\
& \mu \geq 0,
\end{aligned}
\tag{3.4}
$$

where

$$
Q(x,y) = E_{\boldsymbol{\xi}}\left[\Phi(x,y,\xi)\right], \; \boldsymbol{\xi} : \Omega \to \Re^r,
$$

and, for any outcome $\xi(\omega) \in \Xi$,

$$\begin{aligned}
\Phi(x, y, \xi(\omega)) = \max_{\substack{x'(\omega), y'(\omega) \\ \mu'(\omega)}} \quad & F_2(x'(\omega), y'(\omega), \omega) \\
\text{s.t.} \quad & G_2(x'(\omega), y'(\omega), x, y, \omega) \leq 0, \\
& \nabla_{y'(\omega)} f_2(x'(\omega), y'(\omega), \omega) + \mu'(\omega) \nabla_{y'(\omega)} g_2(x'(\omega), y'(\omega), x, \omega) = 0, \\
& \mu'(\omega) g_2(x'(\omega), y'(\omega), x, \omega) = 0, \\
& g_2(x'(\omega), y'(\omega), x, \omega) \leq 0, \\
& \mu'(\omega) \geq 0,
\end{aligned}$$

where μ and $\mu'(\omega)$ (for each $\omega \in \Omega$) are the multipliers associated with the first- and second-stage follower subproblems, respectively. Under suitable assumptions, such as the finiteness of the set Ω (i.e. discrete-distribution assumption), the two-stage SP (3.4) can be expressed as the single level program

$$\begin{aligned}
\max_{\substack{x, y, \mu \\ x', y', \mu'}} \quad & F_1(x, y) + E_{\boldsymbol{\xi}} \left[F_2(x'(\xi), y'(\xi), \xi) \right] \\
\text{s.t.} \quad & G_1(x, y) \leq 0, \\
& \nabla_y f_1(x, y) + \mu \nabla_y g_1(x, y) = 0, \\
& \mu g_1(x, y) = 0, \\
& g_1(x, y) \leq 0, \\
& \mu \geq 0, \\
& \left. \begin{aligned}
& G_2(x'(\omega), y'(\omega), x, y, \omega) \leq 0, \\
& \nabla_{y'(\omega)} f_2(x'(\omega), y'(\omega), \omega) + \mu'(\omega) \nabla_{y'(\omega)} g_2(x'(\omega), y'(\omega), x, \omega) = 0, \\
& \mu'(\omega) g_2(x'(\omega), y'(\omega), x, \omega) = 0, \\
& g_2(x'(\omega), y'(\omega), x, \omega) \leq 0, \\
& \mu'(\omega) \geq 0,
\end{aligned} \right\} \forall \omega \in \Omega.
\end{aligned} \tag{3.5}$$

3.3 A Two-Stage bilevel pricing model

Let us consider a multicommodity transportation network built around a graph $G(N, \Lambda, K)$ with node set N, arc set Λ, and commodity set K, each commodity (origin-destination pair) k being endowed with a demand n^k. The set Λ is partitioned into the subsets Λ_1 and Λ_2 of tariff and tariff-free arcs, respectively. The bilevel network pricing problem introduced by Labbé et al. (1998) consists in maximizing the revenue raised from tariffs, knowing that user flows are assigned to cheapest paths. Since large tariffs will drive users towards tariff-free paths, the optimal trade-off is achieved by solving the bilevel mathematical program

$$
\begin{aligned}
\max_{t} \quad & t \sum_{k \in K} x^k \\
\min_{x,y} \quad & (c+t) \sum_{k \in K} x^k + d \sum_{k \in K} y^k \\
\text{s.t.} \quad & Ax^k + By^k = b^k, && \forall k \in K, \\
& x^k, y^k \geq 0, && \forall k \in K,
\end{aligned}
\tag{3.6}
$$

where t is the vector of tariff variables controlled by the leader, x^k and y^k are the flows of commodity k on the tariff and tariff-free arcs, the vectors c and d are the fixed costs on the tariff and tariff-free arcs, and (A, B) denotes the node-arc incidence matrix that characterizes the flow-conservation constraints at the lower level. The vectors b^k, which express nodal balance, are defined as

$$
b_i^k = \begin{cases} n^k, & \text{if } i = O(k), \\ -n^k, & \text{if } i = D(k), \\ 0, & \text{otherwise.} \end{cases}
$$

We now extend this model to a two-stage stochastic framework where all scenarios share a common network structure, but may differ in the values of the cost and demand parameters. As frequently occurs in practice, we impose that tariff increases (or decreases) cannot exceed a predetermined threshold δ_a on each toll arc a of the network. Under a risk-neutrality assumption, this yields the mathematical program

$$
\begin{aligned}
\max_{t} \quad & t \sum_{k \in K} x^k + Q(t) \\
\min_{x,y} \quad & (c+t) \sum_{k \in K} x^k + d \sum_{k \in K} y^k \\
\text{s.t.} \quad & Ax^k + By^k = b^k, && \forall k \in K, \\
& x^k, y^k \geq 0, && \forall k \in K,
\end{aligned}
\tag{3.7}
$$

where

$$Q(t) = E_{\boldsymbol{\xi}} \left[\Phi(t, \xi) \right], \ \boldsymbol{\xi} : \Omega \to \Re^2,$$

and, for any outcome $\xi(\omega) \in \Xi$, the recourse takes the form of the bilevel program

$$
\begin{aligned}
\Phi(t, \xi(\omega)) = \quad \max_{t'(\omega)} \quad & t'(\omega) \sum_{k \in K} x'^k(\omega) \\
\text{s.t.} \quad & (t, t'(\omega)) \in \Pi, \\
\min_{x',y'} \quad & (c'(\omega) + t'(\omega)) \sum_{k \in K} x'^k(\omega) + d'(\omega) \sum_{k \in K} y'^k(\omega) \\
\text{s.t.} \quad & Ax'^k(\omega) + By'^k(\omega) = b'^k(\omega), && \forall k \in K, \\
& x'^k(\omega), y'^k(\omega) \geq 0, && \forall k \in K,
\end{aligned}
$$

where the set Π is defined as either

$$\Pi = \{ (t, t'(\omega)) : |t'_a(\omega) - t_a| \leq \delta_a, \forall a \in \Lambda_1 \} \tag{3.8}$$

if tariff changes are limited in absolute values (absolute restriction, AR in short) or

$$\Pi = \{ (t, t'(\omega)) : |t'_a(\omega) - t_a| \leq \delta_a \left| t_a \right|, \forall a \in \Lambda_1 \} \tag{3.9}$$

if tariff changes are limited proportionally (proportional restriction, PR in short). It is clear that, if $\delta = \infty$, the first- and second-stage programs can be solved independently of each other, and that the formulation is devoid of interest. Also, due to its structure, the problem can alternatively be formulated as the single-stage bilevel program

$$\begin{aligned}
\max_{t,t'} \quad & t \sum_{k\in K} x^k + E_{\boldsymbol{\xi}}\big[t'(\xi) \sum_{k\in K} x'^k(\xi)\big] & \\
\text{s.t.} \quad & (t, t'(\omega)) \in \Pi, & \forall \omega \in \Omega, \\
\min_{\substack{x,x'\\ y,y'}} \quad & U(t,x,y) + E_{\boldsymbol{\xi}}\left[U'(t'(\xi), x'(\xi), y'(\xi), \xi)\right] & \\
\text{s.t.} \quad & Ax^k + By^k = b^k, & \forall k \in K, \\
& Ax'^k(\omega) + By'^k(\omega) = b'^k(\omega), & \forall k \in K,\ \forall \omega \in \Omega, \\
& x^k, y^k, x'^k(\omega), y'^k(\omega) \geq 0, & \forall k \in K,\ \forall \omega \in \Omega.
\end{aligned} \tag{3.10}$$

where

$$U(t,x,y) = (c+t)\sum_{k\in K} x^k + d \sum_{k\in K} y^k$$

and

$$U'(t'(\xi), x'(\xi), y'(\xi), \xi) = (c'(\omega) + t'(\xi)) \sum_{k\in K} x'^k(\xi) + d'(\xi) \sum_{k\in K} y'^k(\xi).$$

An interesting instance of Program (3.7) occurs when tariffs are not allowed to vary from one stage to the next ($\delta = 0$), and the cost vectors $c'(\omega)$ and $d'(\omega)$ on the tariff and tariff-free arcs assume common values c and d for all scenarios ω, respectively. If both these conditions are realized, the shortest paths for the first- and second-stage follower problems will agree for any realization $\omega \in \Omega$, and the two-stage SBP (3.7) reduces to the single stage bilevel program

$$
\begin{aligned}
\max_{t} \quad & t \sum_{k \in K} x^k \\
\min_{x,y} \quad & (c+t) \sum_{k \in K} x^k + d \sum_{k \in K} y^k \\
\text{s.t.} \quad & A x^k + B y^k = b^k + E_{\boldsymbol{\xi}} \left[b'^k(\xi) \right], \quad && \forall k \in K, \\
& x^k, y^k \geq 0, && \forall k \in K.
\end{aligned}
\tag{3.11}
$$

We close this section with a path reformulation that will prove useful in the sequel. To this aim, we denote by L^k the set of paths available to commodity k, by Λ_1^ρ the set of tariff arcs belonging to path ρ and by r_ρ the indicator variable that takes value one if the flow of commodity k is assigned to path $\rho \in L^k$, and zero otherwise. At the second-stage, $r'_{\rho'}(\omega)$ is the path choice variable associated with scenario ω for every path $\rho' \in L'^k(\omega)$ and commodity k. This yields the bilevel formulation

$$
\begin{aligned}
R(\delta) = \quad \max_{t,t'} \quad & \sum_{k \in K} n^k \sum_{\rho \in L^k, a \in \Lambda_1^\rho} t_a r_\rho + E_{\boldsymbol{\xi}} \Bigg[\sum_{k \in K} n'^k(\xi) \sum_{\rho' \in L'^k(\xi), a \in \Lambda_1^{\rho'}} t'_a(\xi) r'_{\rho'}(\xi) \Bigg] \\
\text{s.t.} \quad & (t, t'(\omega)) \in \Pi(\delta), && \forall \omega \in \Omega, \\
\min_{r,r'} \quad & U(t,r) + E_{\boldsymbol{\xi}} \left[U'(t'(\xi), r'(\xi), \xi) \right] \\
\text{s.t.} \quad & \sum_{\rho \in L^k} r_\rho = 1, && \forall k \in K, \\
& \sum_{\rho' \in L'^k(\omega)} r'_{\rho'}(\omega) = 1, && \forall k \in K, \forall \omega \in \Omega, \\
& r_\rho \geq 0, r'_{\rho'}(\omega) \geq 0, && \begin{cases} \forall k \in K, \forall \rho \in L^k, \\ \forall \omega \in \Omega, \\ \forall \rho' \in L'^k(\omega), \end{cases}
\end{aligned}
\tag{3.12}
$$

where

$$U(t,r)=\sum_{k\in K,\rho\in L^k} n^k\Bigg(\sum_{a\in\Lambda_1^\rho}(c_a+t_a)r_\rho+\sum_{a\in\Lambda_2^\rho}d_a r_\rho\Bigg)$$

and

$$U'(t'(\xi),r'(\xi),\xi)=\sum_{k\in K,\rho'\in L'^k(\xi)} n'^k(\xi)\Bigg(\sum_{a\in\Lambda_1^{\rho'}}(c'_a(\xi)+t'_a(\xi))r'_{\rho'}(\xi)+\sum_{a\in\Lambda_2^{\rho'}}d'_a(\xi)r'_{\rho'}(\xi)\Bigg).$$

Replacing, at each stage, the lower level linear programs by their primal-dual optimality conditions yields the bilinear program

$$\begin{array}{lll}
\max\limits_{\substack{t,r,\lambda\\ t',r',\lambda'}} & \sum\limits_{k\in K} n^k \sum\limits_{\rho\in L^k,a\in\Lambda_1^\rho} t_a r_\rho + E_{\boldsymbol{\xi}}\Bigg[\sum\limits_{k\in K} n'^k(\xi) \sum\limits_{\rho'\in L'^k(\xi),a\in\Lambda_1^{\rho'}} t'_a(\xi) r'_{\rho'}(\xi)\Bigg] & \\
\text{s.t.} & (t_a,t'_a(\omega))\in\Pi(\delta), & \forall\omega\in\Omega,\forall a\in\Lambda_1,\\
& \sum\limits_{\rho\in L^k} r_\rho = 1, & \forall k\in K,\\
& \sum\limits_{\rho'\in L'^k(\omega)} r'_{\rho'}(\omega)=1, & \forall k\in K,\forall\omega\in\Omega,\\
& \lambda^k\le n^k\sum\limits_{a\in\Lambda_1^\rho}(c_a+t_a), & \forall k\in K,\forall\rho\in L^k,\\
& \lambda^k\le n^k\sum\limits_{a\in\Lambda_2^\rho} d_a, & \forall k\in K,\forall\rho\in L^k,\\
& \lambda'^k(\omega)\le n'^k(\omega)\sum\limits_{a\in\Lambda_1^{\rho'}}(c'_a(\omega)+t'_a(\omega)), & \left\{\begin{array}{l}\forall k\in K,\forall\omega\in\Omega,\\ \forall\rho'\in L'^k(\omega),\end{array}\right.\\
& \lambda'^k(\omega)\le n'^k(\omega)\sum\limits_{a\in\Lambda_2^{\rho'}} d'_a(\omega), & \left\{\begin{array}{l}\forall k\in K,\forall\omega\in\Omega,\\ \forall\rho'\in L'^k(\omega),\end{array}\right.\\
& U(t,r)+E_{\boldsymbol{\xi}}\left[U'(t'(\xi),r'(\xi),\xi)\right]=\sum\limits_{k\in K}(\lambda^k+E_{\boldsymbol{\xi}}\left[\lambda'^k(\xi)\right]), & \\
& r_\rho\ge 0, r'_{\rho'}(\omega)\ge 0, & \left\{\begin{array}{l}\forall k\in K,\forall\rho\in L^k,\\ \forall\omega\in\Omega,\\ \forall\rho'\in L'^k(\omega).\end{array}\right.
\end{array} \tag{3.13}$$

Note that, for a limited number of scenarios and admissible paths, the solution to the above program can, through a reformulation proposed for the deterministic case (see Labbé et al. (1998)), be obtained from an off-the-shelf MIP solver.

3.4 Model properties

Some properties of Program (3.7), such as NP-hardness, are directly inherited from the deterministic case (Roch et al. (2005)). Indeed, denoting by $\Gamma(t)$ the lower level optimal value for a given tariff vector t, $\Gamma(\infty) - \Gamma(0)$ is an upper bound on the firm's revenue. Since similar bounds $\Gamma'_\omega(\infty) - \Gamma'_\omega(0)$ apply to each scenario ω, it follows that the revenue of the stochastic bilevel program is bounded above by

$$\Gamma(\infty) - \Gamma(0) + E_{\boldsymbol{\xi}}\left[\Gamma'_\xi(\infty) - \Gamma'_\xi(0)\right].$$

The remainder of the section is devoted to a sensitivity analysis of the revenue function with respect to the scalar parameter δ. Notation wise, it will be useful to refer explicitly to this parameter when denoting sets and solutions : $\Pi(\delta)$, $(t(\delta),\, t'(\omega,\delta))$, $(r_\rho(\delta),\, r'_{\rho'}(\omega,\delta))$. The use of a star next to toll vectors will refer to optimal tolls.

It is straightforward that the revenue is an increasing function of δ. Our main results will be concerned with continuity and piecewise linearity under the AR (absolute restriction) case.

Proposition 3.4.1 *Under the AR case, the value function $R(\delta)$ of Program (3.12) is a continuous and piecewise linear function of the parameter δ.*

Proof. Let $\hat{\delta}$ be an arbitrary positive number. We shall prove that, for an arbitrary positive number ϵ, there exists a positive number Δ such that

$$\|\delta - \hat{\delta}\|_\infty \leq \Delta \Rightarrow |R(\delta) - R(\hat{\delta})| \leq \epsilon,$$

where $\|\cdot\|_\infty$ denotes the infinity (or maximum) norm.

Continuity from the left. Let $\delta_a = \hat{\delta}_a - \Delta$. We create a feasible δ-solution by multiplying the optimal $\hat{\delta}$-solution by $1-\sigma$, where σ is a positive number less than one :

$$\begin{aligned} t(\delta) &\leftarrow (1-\sigma)t^*(\hat{\delta}) \\ t'(\omega,\delta) &\leftarrow (1-\sigma)t'^*(\omega,\hat{\delta}). \end{aligned}$$

Since path tariffs all decrease in a proportional fashion, even though the shortest path of each commodity may change, the optimal revenue under new tariffs will be at least $(1-\sigma)R(\hat{\delta})$. Let us define, for each $\rho \in L^k$ and $k \in K$, $T_\rho = \sum_{a\in\rho} t_a$ and $U_{\rho^*} = C_{\rho^*} + T_{\rho^*}$ where ρ^* denotes the shortest path of commodity k with respect to the optimal $\hat{\delta}$-solution. Similar definitions can also be considered for each $\rho' \in L'^k(\omega)$ ($k \in K$ and $\omega \in \Xi$). Then, following four partitions of each set L^k are considered to discuss about changes of shortest path and total revenue under the new tariffs.

First partition is a subset of set L^k so that for each path ρ from this subset $T_\rho = T_{\rho^*}$ and $U_\rho = U_{\rho^*}$. Therefore all paths with these properties are dominated and by decreasing all tariffs proportionally the revenue corresponding to commodity k will decrease proportionally under new tariffs. Besides, as far as the total revenue is the linear function of the commodities' revenues, the total revenue will decrease proportionally and its optimal value will be at least $(1-\sigma)R(\hat{\delta})$.

Second and third partitions are subsets of set L^k so that for each path ρ from this subset $T_\rho < T_{\rho^*}$ and $U_\rho = U_{\rho^*}$, or $T_\rho \le T_{\rho^*}$ and $U_{\rho^*} < U_\rho$. It is clear that, all paths of these partitions are extremely dominated and the revenue corresponding to commodity k will decrease proportionally as long as all tariffs decrease proportionally. However, according to the linearity of the revenue function, the total revenue will decrease proportionally and the optimal revenue will be at least $(1-\sigma)R(\hat{\delta})$.

Fourth partition is a subset of set L^k so that for each path ρ from this subset $T_{\rho^*} < T_\rho$ and $U_{\rho^*} < U_\rho$. So, under the new tariffs, the paths with these properties can dominate the shortest path of commodity k for certain values of σ. So, even if the shortest path of commodity k is dominated by a path ρ under new tariffs, the revenue corresponding to path ρ is greater than the revenue corresponding to the shortest path ρ^* because $T_{\rho^*} < T_\rho$. Consequently, according to the linearity of the revenue function, the total revenue will decrease proportionally under new tariffs and its optimal value will be at least $(1-\sigma)R(\hat{\delta})$ under the new tariffs.

Moreover since, for every scenario ω there holds

$$\begin{aligned} |t'_a(\omega,\delta) - t_a(\delta)| &= (1-\sigma)|t'^*_a(\omega,\hat{\delta}) - t^*_a(\hat{\delta})| \\ &\leq (1-\sigma)\hat{\delta}, \end{aligned}$$

it follows that the perturbed solution is feasible and achieves a revenue equal to $(1-\sigma)R(\hat{\delta})$, and thus the optimal revenue associated with δ is at least $(1-\sigma)R(\hat{\delta})$. By setting $\sigma = \Delta/\hat{\delta}$ and $\Delta = (\hat{\delta}/R(\hat{\delta}))\epsilon$, straightforward algebra yields $R(\delta) \geq (1-\Delta/\hat{\delta})R(\hat{\delta}) = R(\hat{\delta}) - \epsilon$, from which the conclusion follows.

Continuity from the right. If an arbitrary small increase from $\hat{\delta}$ to δ yields a jump in the revenue function, then a contradiction is obtained by working backwards from δ the preceding argument, i.e., through a proportional decrease of tariffs.

Let $t^*(\delta)$ and $t'^*(\omega,\delta)$ denote an optimal δ-solution, and assume that the corresponding difference in revenues $M = R(\delta) - R(\hat{\delta})$ is larger than some positive number ϵ. A feasible $\hat{\delta}$-solution is then obtained by multiplying the optimal δ-solution by $1-\sigma$, where σ is a proper positive number less than one :

$$\begin{aligned} t(\hat{\delta}) &\leftarrow (1-\sigma)t^*(\delta) \\ t'(\omega,\hat{\delta}) &\leftarrow (1-\sigma)t'^*(\omega,\delta). \end{aligned}$$

Since the path tariffs all decrease in a proportional fashion, as already discussed, even though the shortest path of each commodity may change the revenue will be at least $(1-\sigma)R(\hat{\delta})$. Moreover since, for every scenario ω and tariff arc a there holds

$$\begin{aligned} |t'_a(\omega,\hat{\delta}) - t_a(\hat{\delta})| &= (1-\sigma)|t'^*_a(\omega,\delta) - t^*_a(\delta)| \\ &\leq (1-\sigma)(\hat{\delta}+\Delta), \end{aligned}$$

it follows that the perturbed solution is feasible and achieves a revenue equal to $(1-\sigma)R(\delta)$, and thus the optimal revenue associated with $\hat{\delta}$ is at least $(1-\sigma)R(\delta)$. By setting $\sigma = \Delta/(\hat{\delta}+\Delta)$ and $\Delta = \hat{\delta}\epsilon/(R(\delta)-\epsilon)$ from left hand-side continuity at point δ, we have

$$\begin{aligned} R(\delta) - R(\hat{\delta}) \leq & \frac{\Delta}{\hat{\delta}+\Delta}R(\delta) \\ \leq & \epsilon \end{aligned}$$

which contradicts our assumption and concludes the proof.

Piecewise linearity. Whenever the lower level shortest paths are unique, the bilinear formulation (3.13) reduces to a linear program in the tariff variables t and t'. It follows from standard results in linear programming that the value function is piecewise linear concave, with its slope possibly shifting downwards at points where the lower level solution is not unique. □

The extension of the previous result to the PR case is not straightforward. While it is true that the revenue function varies continuously with the parameter δ when the lower level solution is unique, we could only prove continuity over the whole range of values of δ for problems involving but a single tariff arc (we omit the proof). As regards the piecewise linearity, the result actually does not hold. Indeed, as we will be shown in the next section, continuity pieces are hyperbolic.

3.5 Two illustrative examples

In this section, we illustrate the model through two small examples that involve a firm optimizing over a two-period horizon where it is assumed that $c'(\omega) = c$ for every scenario ω. We also compare the optimal value of the recourse problem (RP) versus that obtained from a deterministic model based on the expected values of the random parameters (EEV), where the latter is obtained in two phases. First, we solve the expected value problem corresponding to Program (3.2)

$$\max_{x,y} F(x, y, \bar{\omega}), \tag{3.14}$$

where $\bar{\omega} = E[\boldsymbol{\xi}]$ and

$$F(x, y, \bar{\omega}) = F_1(x, y) + \Phi(x, y, \bar{\omega}).$$

Let $(\bar{x}, \bar{y})$ be a first-stage optimal solution of Program (3.14). Then EEV is defined as

$$\text{EEV} = \quad E_{\boldsymbol{\xi}} \left[F(\bar{x}, \bar{y}, \xi) \right]. \tag{3.15}$$

Throughout this paper, a mixed-integer reformulation of Program (3.12) is obtained by replacing the follower problem with its primal-dual (KKT) optimality conditions and solved using CPLEX 11.0 under both the AR and PR cases.

3.5.1 First example

Let us consider the network of Figure 3.1, where fixed costs are displayed next to the corresponding arcs. The demands associated with commodities $a-c$ and $a-d$ are set to $n^1 = 2$ and $n^2 = 3$ respectively.

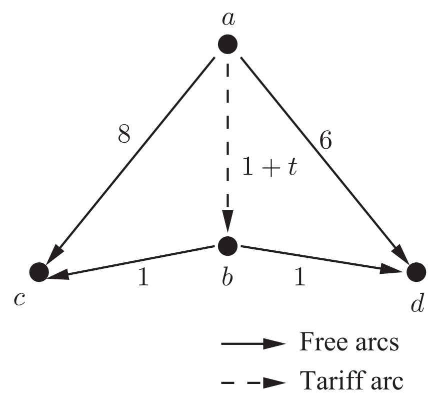

Figure 3.1 Transportation network (first example).

The data corresponding to each of two scenarios are displayed in Table 3.1 below. Throughout this paper, the abbreviations SC, OD, PROB and REV will refer to scenario, commodity, probability of a scenario and revenue, respectively. Table 3.2 provides available paths of each commodity. The parameter δ is set to 0.2.

Table 3.1 Random data for the two scenarios (first example).

SC	PROB	d'_{ac}	d'_{ad}	d'_{bc}	d'_{bd}	n'_1	n'_2
1	0.30	7.50	5.20	2.04	1.00	1	3
2	0.70	7.40	7.00	1.00	3.00	3	2

Table 3.2 Available paths (first example).

OD	PATH	OD	PATH
$a-c$	$\rho_1^1 : a-c$	$a-d$	$\rho_1^2 : a-d$
	$\rho_2^1 : a-b-c$		$\rho_2^2 : a-b-d$

The optimal solutions for the absolute and proportional restrictions are displayed in Table 3.3. Columns STAGE I and STAGE II refer to the first- and second-stage solutions, respectively, while columns SC I and SC II refer to the solutions of the second-stage problem with respect to the first and second scenarios respectively. For this example, the added value of the stochastic solution for the AR and PR cases are 0.26 and 0, respectively.

Table 3.3 RP and EEV solutions under the AR and PR cases (first example).

			STAGE I		STAGE II				
RES	SOL	OD		Tariff	SC I	Tariff	SC II	Tariff	REV
AR	RP	$a-c$	ρ_2^1	3.20	ρ_2^1	3.20	ρ_2^1	3.00	30.34
		$a-d$	ρ_2^2		ρ_2^2		ρ_2^2		
	EEV	$a-c$	ρ_2^1	4.00	ρ_2^1	4.20	ρ_2^1	4.20	30.08
		$a-d$	ρ_2^2		ρ_1^2		ρ_1^2		
PR	RP	$a-c$	ρ_2^1	4.00	ρ_2^1	3.20	ρ_2^1	4.80	33.92
		$a-d$	ρ_2^2		ρ_2^2		ρ_1^2		
	EEV	$a-c$	ρ_2^1	4.00	ρ_2^1	3.20	ρ_2^1	4.80	33.92
		$a-d$	ρ_2^2		ρ_2^2		ρ_1^2		

More interesting than raw data is the sensitivity information with respect to the strategic parameter δ that is shown in Table 3.4 and, Figures 3.2 and 3.3. Note that, under the PR case, the revenue function is piecewise continuous, each piece corresponding to a hyperbola. Under either restriction, the revenue function is *not* concave and converges finitely to the value 35.18, as δ increases.

Table 3.4 Revenue, tariff, and path changes for different values of δ under the AR and PR cases (first example).

			STAGE I			STAGE II			
RES	δ	OD		Tariff	SC I	Tariff	SC II	Tariff	REV
AR	$[0.00, 0.10]$	$a-c$	ρ_2^1	4.00	ρ_2^1	$4+\delta$	ρ_2^1	$4+\delta$	$29.6+2.4\delta$
		$a-d$	ρ_2^2		ρ_1^2		ρ_1^2		
	$[0.10, 0.662]$	$a-c$	ρ_2^1	$3+\delta$	ρ_2^1	3.20	ρ_2^1	3.00	$29.34+5\delta$
		$a-d$	ρ_2^2		ρ_2^2		ρ_2^2		
	$[0.662, 0.80]$	$a-c$	ρ_2^1	$3.2+\delta$	ρ_2^1	3.20	ρ_2^1	$3.2+2\delta$	$26.56+9.2\delta$
		$a-d$	ρ_2^2		ρ_2^2		ρ_1^2		
	$[0.80, 1.40]$	$a-c$	ρ_2^1	4.00	ρ_2^1	3.20	ρ_2^1	$4+\delta$	$32.24+2.1\delta$
		$a-d$	ρ_2^2		ρ_2^2		ρ_1^2		
	$\delta \geq 1.40$	$a-c$	ρ_2^1	4.00	ρ_2^1	3.20	ρ_2^1	5.40	35.18
		$a-d$	ρ_2^2		ρ_2^2		ρ_1^2		
PR	$[0.00, 0.043]$	$a-c$	ρ_2^1	4.00	ρ_2^1	$4(1+\delta)$	ρ_2^1	$4(1+\delta)$	$29.6+9.6\delta$
		$a-d$	ρ_2^2		ρ_1^2		ρ_1^2		
	$[0.043, 0.16]$	$a-c$	ρ_2^1	$\dfrac{3}{1-\delta}$	ρ_2^1	3.20	ρ_2^1	3.00	$\dfrac{29.34-14.34\delta}{1-\delta}$
		$a-d$	ρ_2^2		ρ_2^2		ρ_2^2		
	$[0.16, 0.20]$	$a-c$	ρ_2^1	$\dfrac{3.2}{1-\delta}$	ρ_2^1	3.20	ρ_2^1	$\dfrac{3.2(1+\delta)}{1-\delta}$	$\dfrac{26.56+2.88\delta}{1-\delta}$
		$a-d$	ρ_2^2		ρ_2^2		ρ_1^2		
	$[0.20, 0.35]$	$a-c$	ρ_2^1	4.00	ρ_2^1	3.20	ρ_2^1	$4(1+\delta)$	$32.24+8.4\delta$
		$a-d$	ρ_2^2		ρ_2^2		ρ_1^2		
	$\delta \geq 0.35$	$a-c$	ρ_2^1	4.00	ρ_2^1	3.20	ρ_2^1	5.40	35.18
		$a-d$	ρ_2^2		ρ_2^2		ρ_1^2		

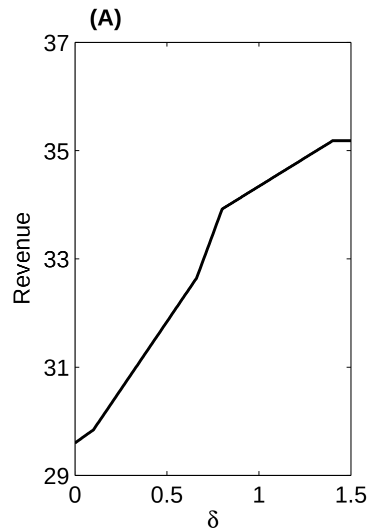

Figure 3.2 Sensitivity with respect to δ (first example) : (**A**) The AR case (**B**) Zoom in.

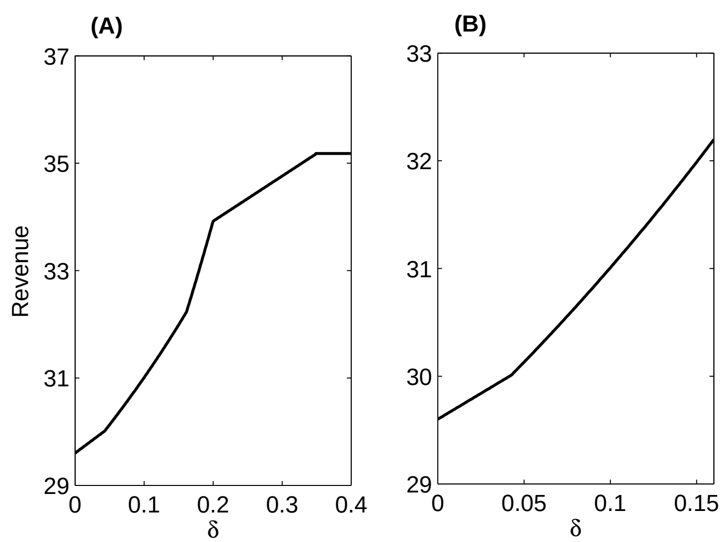

Figure 3.3 Sensitivity with respect to δ (first example) : (**A**) The PR case (**B**) zoom in.

3.5.2 Second example

Our second example involves the network of Figure 3.4, that involves the single commodity $a - f$ with demand 8, three tariff arcs (dotted arcs on the picture), the two scenarios of Table 3.5, and the vector of δ-values set to $(0.25, 0.10, 0.20)$. Travel costs are shown next to the corresponding arcs. Available paths are specified in Table 3.6.

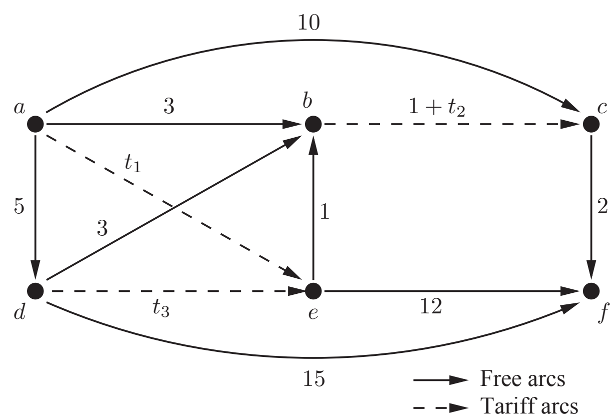

Figure 3.4 Network diagram (second example).

Table 3.5 Data for the two scenarios (second example).

SC	PROB	d'_{ab}	d'_{ac}	d'_{ad}	d'_{cf}	d'_{db}	d'_{df}	d'_{eb}	d'_{ef}	n'_{af}
1	0.50	3.00	9.20	5.90	1.50	4.50	14.60	1.70	11.90	8
2	0.50	4.10	9.70	6.00	3.20	3.00	13.90	0.60	13.30	3

Tables 3.7 and 3.8 display the stochastic and expected solutions under the AR and PR cases for the unconstrained and nonnegative tariff assumptions, respectively. Note that, although the tariffs are different, yet the revenues are equal for both restrictions.

Table 3.6 Available paths for commodity $a-f$ (second example).

$\rho_1 : a-c-f$

$\rho_2 : a-b-c-f$

$\rho_3 : a-e-b-c-f$

$\rho_4 : a-e-f$

$\rho_5 : a-d-b-c-f$

$\rho_6 : a-d-e-b-c-f$

$\rho_7 : a-d-e-f$

$\rho_8 : a-f$

Table 3.7 RP and EEV solutions under the AR and PR cases (second example).

		STAGE I			STAGE II			
RES	SOL		Tariff	SC I	Tariff	SC II	Tariff	REV
AR	RP	ρ_3	$(-0.95, 7.80, 2.00)$	ρ_3	$(-1.20, 7.70, 2.20)$	ρ_3	$(-0.70, 7.90, 2.20)$	91.60
	EEV	ρ_3	$(2.00, 5.65, 0.00)$	ρ_1	$(1.75, 5.55, 0.20)$	ρ_3	$(2.25, 5.75, 0.20)$	73.20
PR	RP	ρ_3	$(1.73, 5.77, 0.00)$	ρ_3	$(1.30, 5.20, 0.00)$	ρ_3	$(1.74, 6.35, 0.00)$	98.24
	EEV	ρ_3	$(2.00, 6.00, 0.00)$	ρ_1	$(1.50, 5.40, 0.00)$	ρ_3	$(1.50, 6.60, 0.00)$	76.15

Table 3.8 RP and EEV solutions under the AR and PR cases and the nonnegative tariffs (second example).

		STAGE I			STAGE II			
RES	SOL		Tariff	SC I	Tariff	SC II	Tariff	REV
AR	RP	ρ_3	$(0.25, 6.60, 0.00)$	ρ_3	$(0.00, 6.50, 0.00)$	ρ_3	$(0.50, 6.70, 0.00)$	91.60
	EEV	ρ_3	$(2.00, 5.65, 0.00)$	ρ_1	$(1.75, 5.55, 0.20)$	ρ_3	$(2.25, 5.75, 0.20)$	73.20
PR	RP	ρ_3	$(1.73, 5.77, 0.00)$	ρ_3	$(1.30, 5.20, 0.00)$	ρ_3	$(2.16, 5.93, 0.00)$	98.24
	EEV	ρ_3	$(2.00, 6.00, 0.00)$	ρ_1	$(1.50, 5.40, 0.00)$	ρ_3	$(1.50, 6.60, 0.00)$	76.15

Table 3.9 provides the complete stochastic and expected optimal solutions under the AR and PR cases for different values of the vector δ (all of its entries are equal). At critical points 0.152, 0.4, and 0.75, the solution of the lower level is not unique (AR case). Nonuniqueness occurs at critical points 0.037, 0.0625, 0.08, 0.12, and 0.187 under the PR case. In Figure 3.5, the value of the revenue is plotted against the value of δ, in both the AR and PR cases. A more complex extension of this example, that involves two commodities and four outcomes, is thoroughly analyzed in 4.6.

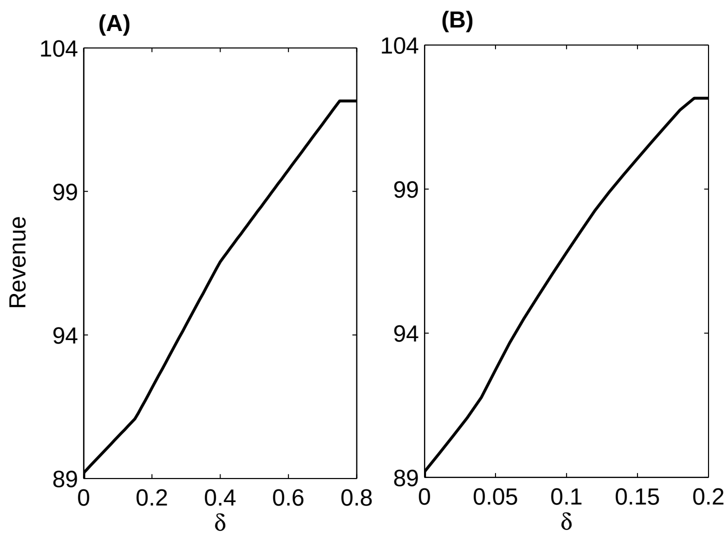

Figure 3.5 Sensitivity with respect to δ (second example) : (**A**) The AR case (**B**) The PR case.

Table 3.9 Revenue, tariff, and paths changes for different values of δ under the AR and PR cases (second example).

RES	δ	STAGE I	Tariff	SC	STAGE II	Tariff	REV
AR	$[0.00, 0.152]$	ρ_1	$(2.00, 5.2+\delta, 0.00)$	1	ρ_2	$(2-\delta, 5.20, 0.00)$	$89.2 + 12.5\delta$
				2	ρ_1	$(2+\delta, 5.2+2\delta, 0.00)$	
	$[0.152, 0.40]$	ρ_1	$(1.3+\delta, 5.2+\delta, 0.00)$	1	ρ_1	$(1.30, 5.20, 0.00)$	$87.75 + 22\delta$
				2	ρ_1	$(1.3+2\delta, 5.2+2\delta, 0.00)$	
	[0.40,0.75]	ρ_1	$(1.3+\delta, 5.2+\delta, 0)$	1	ρ_1	$(1.30, 5.20, 0.00)$	$90.15 + 16\delta$
				2	ρ_1	$(2.10, 6.00, 0.00)$	
	$\delta \geq 0.75$	ρ_1	$(0.00, 8.00, 0.00)$	1	ρ_1	$(1.30, 5.20, 0.00)$	102.15
				2	ρ_1	$(0.00, 8.10, 0.00)$	
PR	$[0.00, 0.038]$	ρ_1	$(2.00, \frac{5.20}{1-\delta}, 0.00)$	1	ρ_2	$(2(1-\delta), 5.20, 0.00)$	$39.8 + 3\delta + \frac{41.60}{1-\delta}$
				2	ρ_1	$(2(1+\delta), 5.2\frac{1+\delta}{1-\delta}, 0.000)$	$+7.8\frac{1+\delta}{1-\delta}$
	$[0.038, 0.0625]$	ρ_1	$(\frac{8\delta-1.5}{1+\delta}, 8.00, 0.00)$	1	ρ_1	$(8\delta-1.5, 8(1-\delta), 0.00)$	$102 + (12\delta - 2.25)\frac{1-\delta}{1+\delta}$
				2	ρ_1	$((8\delta-1.5)\frac{1-\delta}{1+\delta}, 8(1+\delta), 0.00)$	$+12\delta + \frac{64.00\delta-12.00}{1+\delta}$
	$[0.625, 0.08]$	ρ_1	$(\frac{-2.5+15\delta}{(1+\delta)^2}, \frac{8.50}{1+\delta}, 0.00)$	1	ρ_1	$(\frac{15\delta-2}{1+\delta}, 8.5\frac{1-\delta}{1+\delta}, 0.00)$	$102.3 + \frac{52}{1+\delta} + 9.75\frac{1-\delta}{1+\delta}$
				2	ρ_1	$(\frac{(15\delta-2)(1-\delta)}{(1+\delta)^2}, 8.50, 0.00)$	$-68\frac{1-\delta}{(1+\delta)^2} - 12.75\frac{(1-\delta)^2}{(1+\delta)^2}$
	$[0.08, 0.12]$	ρ_1	$(\frac{8\delta-1.5}{1+\delta}, 8.00, 0.00)$	1	ρ_1	$(8\delta-1.5, 8(1-\delta), 0.00)$	$102.75 + \frac{64\delta-12}{1+\delta}$
				2	ρ_1	$((8\delta-1.5)\frac{1-\delta}{1+\delta}, 8.50, 0.00)$	$+(12\delta - 2.25)\frac{1-\delta}{1+\delta}$
	$[0.12, 0.187]$	ρ_1	$(\frac{8\delta-1.5}{1+\delta}, 8.00, 0.00)$	1	ρ_1	$(8\delta-1.5, 8(1-\delta), 0.00)$	$102.15 + \frac{64\delta-12}{1+\delta}$
				2	ρ_1	$((8\delta-1.5)\frac{1-\delta}{1+\delta}, 8.1-(8\delta-1.5)\frac{1-\delta}{1+\delta}, 0.00)$	
	$\delta \geq 0.187$	ρ_1	$(0.00, 8.00, 0.00)$	1	ρ_1	$(1.30, 5.20, 0.00)$	102.15
				2	ρ_1	$(0.00, 8.10, 0.00)$	

3.6 Larger instances

In this section, we present numerical results pertaining to randomly generated instances involving 40 nodes and 200 arcs. Similar to Didi-Biha et al. (2006), random arcs are appended to a "backbone" cycle, ensuring that there always exist alternative tariff-free paths, and that the network is connected. For the sake of comparison, we have investigated several situations, each one corresponding to a specific set of assumptions :

- Case 1 : Base case.
- Case 2 : Nonnegative tariffs.
- Case 3 : $\delta = \infty$ (decoupled stages).
- Case 4 : Nonnegative tariffs and $\delta = \infty$ (decoupled stages).
- Case 5 : $\delta = 0$ (equal tariffs at both stages).
- Case 6 : Nonnegative tariffs and $\delta = 0$ (equal tariffs at both stages).
- Case 7 : $\delta = 0$ (equal tariffs at both stages) and deterministic cost structure.
- Case 8 : Nonnegative tariffs, $\delta = 0$ (equal tariffs at both stages) and deterministic cost structure.

In Tables 3.10–3.12, labels '%Tariff', '#OD', and '#SC' refer to the percentage of tariff arcs, the number of commodities, and the number of scenarios, respectively. Computational time was limited to six hours and the average CPU time corresponding to five random instances is reported in seconds. Columns 'CPU-T' and '#Opt' display the average CPU time and the number of instances solved to optimality within the allotted time, respectively.

Since large values of the parameter δ yield a decoupled system, one could expect that the corresponding instances are numerically easier to solve. However, the tables tell a different story, i.e., the complexity of Program (3.10) may or may not increase with the value of δ. It was also observed that including nonnegativity constraints did not impact the computational time significantly.

At an another extreme, when δ is set to zero, one might expect that the smaller number of variables resulting from equal tariffs at both stages eases the computational burden. Again, and contrary to intuition, this is not the case. Indeed, according to Tables 3.11 and 3.12, one can observe that Cases 5 and 6 are as challenging as Cases 1 and 2. In contrast, the assumption of deterministic and equal fixed travel costs at both stages of the stochastic programs (Cases 7 and 8), implies that shortest paths are the same at both stages, which makes for instances that scale much better.

In order to assess the maximal size of problems that could be addressed to optimality, we varied the percentage of tariff arcs, the number of commodities and the number of scenarios. According to our tests, at most one instance can be solved for problems involving 5% tariff arcs, 50 commodities, and two outcomes. In Cases 5 and 6, the maximum number of commodities and outcomes that could be addressed is 40 and 20, respectively. If we increase the percentage of tariff arcs, the maximum size (with the notable exceptions of Cases 7 and 8) involves 30 commodities and 2 outcomes for 10% tariff arcs, 20 commodities and 2 outcomes for 15% tariff arcs, and 20 commodities and 2 outcomes for 20% tariff arcs.

Table 3.10 Results for Didi Network (larger instances).

			AR case				PR case			
			Case 1		Case 2		Case 1		Case 2	
% Tariff	# OD	# SC	CPU-T	#Opt	CPU-T	#Opt	CPU-T	#Opt	CPU-T	#Opt
5	5	2	1.87	5	0.75	5	0.26	5	0.26	5
5	5	4	26.91	5	1.14	5	0.32	5	0.31	5
5	5	8	351.56	5	3.93	5	1.26	5	1.25	5
5	10	2	49.71	5	9.17	5	3.72	5	3.28	5
5	10	4	83.41	5	11.48	5	2.86	5	2.82	5
5	10	8	875.27	3	86.72	5	105.61	5	92.59	5
5	20	2	137.05	3	615.40	5	335.45	5	322.64	5
5	20	4	149.61	1	153.29	3	70.49	3	67.16	3
5	20	8	960.60	1	1007.41	3	2474.48	3	2234.36	3
5	30	2	9961.13	2	899.28	3	2678.31	3	3024.45	3
5	30	4	-	0	-	0	1203.89	1	1016.32	1
5	30	8	-	0	-	0	-	0	-	0
5	40	2	-	0	6419.89	2	2284.95	1	2271.55	1
5	40	4	-	0	-	0	-	0	-	0
5	40	8	-	0	-	0	-	0	-	0
5	50	2	-	0	495.61	1	293.15	1	259.56	1
5	50	4	-	0	-	0	-	0	-	0
10	5	2	496.28	5	310.36	5	725.97	5	354.31	5
10	5	4	20.29	4	13.58	4	96.69	4	1.11	4
10	5	8	1056.18	4	987.39	4	787.50	3	4.20	4
10	10	2	48.11	3	32.87	3	30.70	3	23.79	4
10	10	4	588.35	3	1436.34	3	1945.78	3	321.95	4
10	10	8	2187.39	2	2610.53	2	20251.69	1	646.30	4
10	20	2	190.97	2	241.45	2	639.65	2	22.73	2
10	20	4	-	0	15221.37	1	-	0	2171.06	2
10	20	8	-	0	-	0	-	0	-	0
10	30	2	-	0	-	0	4537.59	1	425.68	1
10	30	4	-	0	-	0	-	0	-	0
15	5	2	2043.35	5	180.79	4	9.70	4	952.23	5
15	5	4	963.56	3	2954.89	3	82.13	4	7.33	4
15	5	8	3750.32	1	3689.99	1	396.38	2	364.27	3
15	10	2	45.40	2	65.52	2	126.59	3	8365.98	4
15	10	4	-	0	201.35	1	-	0	-	0
15	10	8	-	0	1179.94	1	-	0	-	0
15	20	2	-	0	-	0	503.71	1	991.27	2
20	5	2	35.07	4	2554.77	4	330.44	5	708.64	5
20	5	4	84.19	1	84.67	1	369.80	2	1648.94	2
20	5	8	-	0	-	0	-	0	508.88	2
20	10	2	5599.52	4	5425.45	4	644.59	4	2647.49	3
20	10	4	275.36	1	-	0	-	0	7334.49	2
20	10	8	-	0	-	0	-	0	-	0

Table 3.11 Results for Didi Network : 5% and 10% Tariff (larger instances).

%	#	#	Case 3		Case 4		Case 5		Case 6		Case 7		Case 8	
Tariff	OD	SC	CPU-T	#Opt	CPU-T	#Opt	CPU-T	#Opt	CPU-T	#Opt	CPU-T	#Opt	CPU-T	#Opt
5	5	2	0.19	5	0.15	5	0.58	5	0.64	5	0.05	5	0.04	5
5	5	4	0.40	5	0.31	5	85.90	5	48.38	5	0.05	5	0.04	5
5	5	8	0.97	5	0.77	5	5.50	3	5.09	3	0.05	5	0.03	5
5	10	2	0.94	5	0.81	5	27.80	5	48.10	5	0.18	5	0.18	5
5	10	4	1.83	5	2.25	5	195.80	5	72.44	5	0.21	5	0.17	5
5	10	8	81.70	5	61.36	5	909.64	3	1263.23	3	0.17	5	0.16	5
5	20	2	179.52	5	889.77	5	50.27	3	248.96	3	1.05	5	0.88	5
5	20	4	55.14	3	22.90	3	747.60	1	119.23	1	1.09	5	0.94	5
5	20	8	456.21	3	240.10	2	-	0	14216.75	1	1.26	5	0.93	5
5	30	2	110.50	2	2676.81	3	9890.69	1	20453.06	1	25.62	5	33.67	5
5	30	4	6656.48	1	3390.63	1	-	0	-	0	22.02	5	22.62	5
5	30	8	-	0	-	0	-	0	-	0	27.09	5	24.80	5
5	40	2	7580.56	2	5000.96	1	6555.57	1	-	0	2432.28	5	2870.64	5
5	40	4	-	0	14401.36	1	-	0	-	0	1662.55	5	1877.03	5
5	40	8	-	0	-	0	-	0	-	0	3879.25	5	2400.55	5
5	50	2	752.10	1	403.45	1	-	0	-	0	22.05	3	37.25	3
5	50	4	-	0	-	0	-	0	-	0	5181.36	4	63.48	3
5	50	8	-	0	-	0	-	0	-	0	2367.14	4	47.48	3
10	5	2	1178.15	5	45.13	5	619.60	5	139.12	5	0.14	5	0.11	5
10	5	4	0.47	4	636.34	5	8.03	4	7.12	4	0.18	5	0.12	5
10	5	8	1.46	4	0.98	4	4015.20	3	7949.49	3	0.15	5	0.14	5
10	10	2	2.71	3	14.66	4	99.60	4	79.21	4	1.57	5	2.21	5
10	10	4	171.06	3	82.17	3	804.80	3	2236.51	3	1.93	5	2.93	5
10	10	8	58.18	3	49.96	3	644.90	1	1777.32	1	1.13	5	1.19	5
10	20	2	17.07	2	36.47	2	720.58	2	319.84	2	31.85	5	36.09	5
10	20	4	109.43	2	908.16	2	-	0	-	0	41.90	5	106.03	5
10	20	8	3961.97	2	-	0	-	0	-	0	54.36	5	148.61	5
10	30	2	458.18	1	158.58	1	-	0	-	0	1378.95	4	1484.43	4
10	30	4	2614.47	1	2124.28	1	-	0	-	0	1299.18	5	1403.52	5
10	30	8	-	0	-	0	-	0	-	0	2189.90	5	1299.01	4
10	40	2	-	0	-	0	-	0	-	0	731.59	3	1399.34	3
10	40	4	-	0	-	0	-	0	-	0	287.81	3	491.54	3
10	40	8	-	0	-	0	-	0	-	0	4193.81	4	324.07	3
10	50	2	-	0	-	0	-	0	-	0	13100.62	1	-	0
10	50	4	-	0	-	0	-	0	-	0	-	0	-	0
10	50	8	-	0	-	0	-	0	-	0	-	0	-	0

Table 3.12 Results for Didi Network : 15% and 20% Tariff (larger instances).

%	#	#	Case 3		Case 4		Case 5		Case 6		Case 7		Case 8	
Tariff	OD	SC	CPU-T	#Opt	CPU-T	#Opt	CPU-T	#Opt	CPU-T	#Opt	CPU-T	#Opt	CPU-T	#Opt
15	5	2	1.11	4	1.16	4	78.18	4	2275.32	5	0.44	5	0.31	5
15	5	4	10.73	4	7.69	4	4986.55	3	7.42	2	0.29	5	0.25	5
15	5	8	543.31	4	811.75	3	-	0	-	0	0.27	5	0.22	5
15	10	2	338.47	4	2150.81	3	28.47	2	42,59	2	3.12	5	3.65	5
15	10	4	112.89	3	2371.82	3	292.66	1	104.40	1	3.88	5	3.01	5
15	10	8	52.83	2	32.96	1	-	0	-	0	2.91	5	2.97	5
15	20	2	35.67	2	952.47	2	-	0	-	0	344.71	5	761.94	5
15	20	4	274.50	1	-	0	-	0	-	0	144.29	5	156.79	4
15	20	8	6726.40	1	-	0	-	0	-	0	149.53	5	2274.63	5
15	30	2	-	0	-	0	-	0	-	0	920.71	4	4924.55	3
15	30	4	-	0	-	0	-	0	-	0	458.90	3	51.82	2
15	30	8	-	0	-	0	-	0	-	0	131.29	3	722.94	3
15	40	2	-	0	-	0	-	0	-	0	6780.25	3	783.33	1
15	40	4	-	0	-	0	-	0	-	0	476.67	2	5789.84	2
15	40	8	-	0	-	0	-	0	-	0	4535.04	3	6580.84	2
15	50	2	-	0	-	0	-	0	-	0	21551.69	1	-	0
15	50	4	-	0	-	0	-	0	-	0	15065.39	1	-	0
15	50	8	-	0	-	0	-	0	-	0	-	0	-	0
20	5	2	8.00	5	45.47	5	42.40	4	189.17	3	0.22	5	0.26	5
20	5	4	1.45	3	78.15	3	140.49	1	2209.54	1	0.22	5	0.24	5
20	5	8	8.83	3	9.79	1	-	0	-	0	0.22	5	0.29	5
20	10	2	1027.07	5	11.28	2	1231.89	2	9706.06	2	7.54	5	7.74	4
20	10	4	446.97	3	5.30	1	1956.53	1	1913.78	1	3.98	5	3165.36	5
20	10	8	-	0	-	0	-	0	-	0	4.61	5	3.69	4
20	20	2	454.37	1	1744.76	1	-	0	-	0	1205.51	4	3316.91	4
20	20	4	-	0	-	0	-	0	-	0	83.10	4	25.95	3
20	20	8	-	0	-	0	-	0	-	0	233.10	4	23.58	3
20	30	2	-	0	-	0	-	0	-	0	6311.75	3	1853.35	2
20	30	4	-	0	-	0	-	0	-	0	4147.70	3	2958.07	2
20	30	8	-	0	-	0	-	0	-	0	5695.37	3	2225.56	2
20	40	2	-	0	-	0	-	0	-	0	4714.99	1	6497.55	1
20	40	4	-	0	-	0	-	0	-	0	12304.76	1	19658.04	1
20	40	8	-	0	-	0	-	0	-	0	13733.04	1	5639.26	1
20	50	2	-	0	-	0	-	0	-	0	-	0	-	0
20	50	4	-	0	-	0	-	0	-	0	-	0	-	0
20	50	8	-	0	-	0	-	0	-	0	-	0	-	0

3.7 Conclusion and Future Work

In this paper, we have analyzed a two-stage stochastic bilevel pricing problem and its reformulation as a single-stage SBP, focusing on sensitivity analysis with respect to the constraints linking the tariffs at the two stages of the stochastic program. Several avenues for future research are open. They include the theoretical analysis of the situation involving continuous random variables, together with the development of a suitable numerical approach that takes advantage of the network structure, and does not simply relies on a straightforward extension of techniques developed in the deterministic case. On the modelling side, the extension of the model to an arbitrary number of stages, either in closed loop or open loop ("feedback Stackelberg") poses formidable challenges, both from the theoretical and computational points of view, and requires investigation.

Appendix

Appendix A : Example 2 revisited

Let us consider a two-commodity extension of the network depicted in Figure 3.4 where origin-destination pairs $a-c$ and $d-f$ are endowed with respective demands $n^1 = 8$ and $n^2 = 5$. Table 3.13 lists each random market price d'_j ($j = \{1, ..., 8\}$) and random demand n'^k ($k = 1, 2$). These are generated according to discrete uniform distributions $[d_j(1-\tau), d_j(1+\tau)]$ and $\left[n^k(1-\eta), n^k(1+\eta)\right]$, respectively. Note that the first outcome represents the most favorable case with respect to the parameters τ and η, and the fourth outcome the least favorable.

Table 3.13 Random market prices and demands.

SC	PROB	τ	η	d'_{ab}	d'_{ac}	d'_{ad}	d'_{cf}	d'_{db}	d'_{df}	d'_{eb}	d'_{ef}	n'_{ac}	n'_{df}
1	0.20	0.10	0.20	3.00	9.10	4.80	1.90	3.30	14.00	1.10	12.80	8	5
2	0.30	0.25	0.30	2.40	10.60	4.00	1.50	2.60	11.90	1.10	14.70	7	5
3	0.30	0.25	0.30	3.40	8.60	5.40	1.80	2.70	11.60	1.10	10.40	9	4
4	0.20	0.40	0.40	4.00	6.30	6.80	2.30	1.80	18.20	0.90	16.40	11	7

Table 3.14 lists paths for the first and second commodities ($a-c$ and $d-f$) associated with the second example network (Figure 3.4).

Table 3.14 Available paths for commodities $a-c$ and $d-f$.

OD	PATH	OD	PATH
$a-c$	$\rho_1^1 : a-c$	$d-f$	$\rho_1^2 : d-b-c-f$
	$\rho_2^1 : a-b-c$		$\rho_2^2 : d-e-b-c-f$
	$\rho_3^1 : a-e-b-c$		$\rho_3^2 : d-e-f$
	$\rho_4^1 : a-d-b-c$		$\rho_4^2 : d-f$
	$\rho_5^1 : a-d-e-b-c$		

3.7.1 Results under the AR case

Tables 3.15–3.16 provide the stochastic and expected optimal solutions under the AR case, for both unconstrained or nonnegative tariffs. In tables 3.15, 3.16, 3.19, and 3.20, label 'EV' refers to the expected revenue corresponding to four scenarios. The values of the stochastic solutions are 23.85 and 22.27, respectively, and show that the added cost of solving the stochastic model is reasonable. It can also be observed from Tables 3.16 and 3.16 that, somewhat counterintuitively, a larger revenue can be obtained by restricting oneself to nonnegative tariffs, when stochasticity is ignored.

Table 3.15 RP solution under the AR case : unrestricted/nonnegative tariffs.

	SC	OD		Tariff	REV	EV
STAGE I		$a-c$	ρ_3^1	$(0.25, 6.60, 1.40)$	94.80	
		$d-f$	ρ_2^2			
STAGE II	1	$a-c$	ρ_3^1	$(0.30, 6.70, 1.60)$	97.50	
		$d-f$	ρ_2^2			
	2	$a-c$	ρ_3^1	$(0.50, 6.70, 1.50)$	91.40	
		$d-f$	ρ_2^2			83.09
	3	$a-c$	ρ_3^1	$(0.00, 6.50, 1.20)$	89.30	
		$d-f$	ρ_2^2			
	4	$a-c$	ρ_1^1	$(0.00, 6.70, 1.20)$	46.90	
		$d-f$	ρ_1^2			

Total REV (STAGE I+STAGE II) $= 177.89$

Table 3.16 EEV solution under the AR case

			Unrestricted tariffs				Nonnegative tariffs			
	SC	OD		Tariff	REV	EV		Tariff	REV	EV
STAGE I		$a-c$	ρ_3^1	$(-0.87, 8.00, 1.75)$	105.79		ρ_3^1	$(0.25, 6.88, 1.75)$	100.22	
		$d-f$	ρ_2^2				ρ_2^2			
STAGE II	1	$a-c$	ρ_3^1	$(-1.05, 8.05, 1.95)$	106.00	48.25	ρ_3^1	$(0.02, 6.98, 1.95)$	100.65	55.40
		$d-f$	ρ_2^2				ρ_2^2			
	2	$a-c$	ρ_3^1	$(-0.62, 8.10, 1.55)$	52.36		ρ_3^1	$(0.50, 6.80, 1.55)$	85.01	
		$d-f$	ρ_4^2				ρ_1^2			
	3	$a-c$	ρ_1^1	$(-1.12, 7.90, 1.95)$	0.00		ρ_1^1	$(0.00, 6.78, 1.55)$	0.00	
		$d-f$	ρ_4^2				ρ_4^2			
	4	$a-c$	ρ_1^1	$(-0.62, 8.10, 1.55)$	56.70		ρ_1^1	$(0.00, 6.98, 1.55)$	48.86	
		$d-f$	ρ_1^2				ρ_1^2			

Total REV (STAGE I+STAGE II) = 154.04 Total REV (STAGE I+STAGE II) = 155.62

Figure 3.6 **(A)** illustrates the revenue growth under the AR case, as δ increases, with Figure 3.6 **(B)** zooming in on the interval $[0, 1]$. Table 3.17 displays the optimal tariffs, second-stage revenue, and total revenue for different values of δ under the AR case.

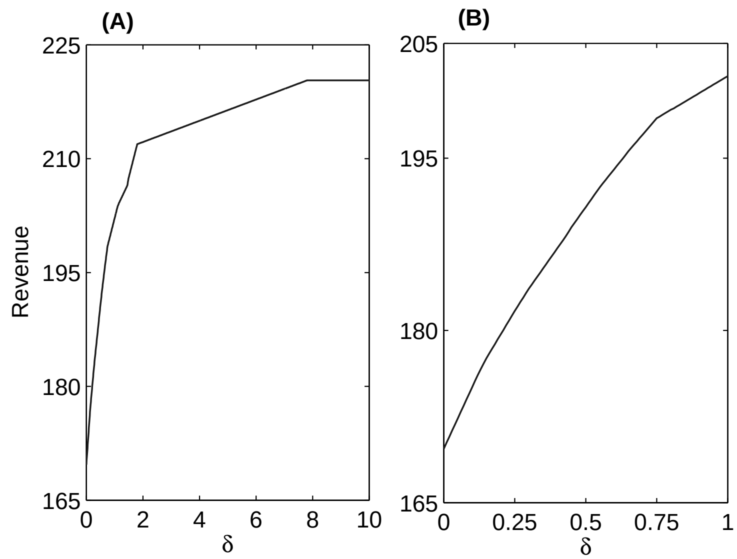

Figure 3.6 Sensitivity with respect to δ : (A) The AR case (B) Zoom in.

Table 3.17 Revenue and tariff changes for different values of δ under the AR case.

δ	Tariff	Q	REV
$[0.00, 0.125]$	$(\delta, 6.50+\delta, 1.20+\delta)$	$79.19+27.60\delta$	$169.69+53.60\delta$
$[0.125, 0.15]$	$(\delta, 6.50+\delta, 1.20+\delta)$	$79.99+21.20\delta$	$170.49+47.20\delta$
$[0.15, 0.25]$	$(\delta, 6.50+\delta, 1.20+\delta)$	$80.89+15.20\delta$	$171.39+41.20\delta$
$[0.25, 0.30]$	$(\delta, 6.50+\delta, 1.20+\delta)$	$81.39+13.20\delta$	$171.89+39.20\delta$
$[0.30, 0.428]$	$(\delta, 6.50+\delta, 1.20+\delta)$	$82.65+9.00\delta$	$173.15+35.00\delta$
$[0.428, 0.45]$	$(\delta, 6.50+\delta, 0.90+\delta)$	$82.35+13.20\delta$	$171.35+39.20\delta$
$[0.45, 0.55]$	$(\delta, 6.50+\delta, 0.90+\delta)$	$84.24+9.00\delta$	$173.24+35.00\delta$
$[0.55, 0.636]$	$(\delta, 6.50+\delta, 0.90+\delta)$	$86.55+4.80\delta$	$175.55+30.80\delta$
$[0.636, 0.65]$	$(\delta, 6.50+\delta, 1.20+\delta)$	$83.65+7.00\delta$	$174.15+33.00\delta$
$[0.65, 0.75]$	$(\delta, 6.50+\delta, 1.20+\delta)$	$86.38+2.80\delta$	$176.88+28.80\delta$
$[0.75, 0.80]$	$(1.50-\delta, 6.50+\delta, 1.20+\delta)$	$84.88+4.80\delta$	$187.38+14.80\delta$
$[0.80, 0.13]$	$(1.50-\delta, 6.50+\delta, 2)$	$83.20+6.90\delta$	$189.70+11.90\delta$
$[0.13, 1.11]$	$(1.50-\delta, 6.50+\delta, 1.20+\delta)$	$86.35+4.80\delta$	$187.35+14.80\delta$
$[1.11, 1.15]$	$(1.50-\delta, 6.50+\delta, 2.00)$	$86.35+4.80\delta$	$192.85+9.80\delta$
$[1.15, 1.45]$	$(1.50-\delta, 6.50+\delta, 2.00)$	$88.65+2.80\delta$	$195.15+7.80\delta$
$[1.45, 1.50]$	$(-2.10+\delta, 6.50+\delta, 2.00)$	98.33	$176.03+21.00\delta$
$[1.50, 1.80]$	$(-3.60+2\delta, 8.00, 2.00)$	$100.43-1.40\delta$	$185.63+14.60\delta$
$[1.80, 7.80]$	$(0.00, 8.00, 2.00)$	$95.39+1.40\delta$	$209.39+1.40\delta$
$\delta \geq 7.80$	$(0.00, 8.00, 2.00)$	106.31	220.31

Table 3.18 Optimal paths for different values of δ under the AR case.

	SC	OD	$\delta \in [0.00, 0.13]$	$\delta \in [0.13, 1.45]$	$\delta \in [1.45, \infty)$
STAGE I		$a-c$	ρ_3^1	ρ_3^1	ρ_3^1
		$d-f$	ρ_2^2	ρ_2^2	ρ_2^2
STAGE II	1	$a-c$	ρ_3^1	ρ_3^1	ρ_3^1
		$d-f$	ρ_2^2	ρ_2^2	ρ_2^2
	2	$a-c$	ρ_3^1	ρ_3^1	ρ_3^1
		$d-f$	ρ_2^2	ρ_2^2	ρ_2^2
	3	$a-c$	ρ_3^1	ρ_3^1	ρ_3^1
		$d-f$	ρ_2^2	ρ_2^2	ρ_2^2
	4	$a-c$	ρ_1^1	ρ_1^1	ρ_3^1
		$d-f$	ρ_1^2	ρ_2^2	ρ_1^2

3.7.2 Results under the PR case

Tables 3.19–3.20 provide the stochastic and expected optimal solutions under the PR case. However according to the inequality RP $\geq$ EEV, the value of the stochastic solution shows the expected benefit obtained by solving the stochastic model instead of its deterministic counterpart. As observed earlier, restriction to nonnegative tariffs can actually yield higher revenues, when stochasticity is ignored (see Tables 3.19 and 3.20).

Table 3.19 RP solution under the PR case : unrestricted/nonnegative tariffs.

	SC	OD		Tariff	REV	EV
STAGE I		$a-c$	ρ_3^1	(0.40,6.89,1.87)	102.13	
		$d-f$	ρ_2^2			
STAGE II	1	$a-c$	ρ_3^1	$(0.30, 6.70, 2.20)$	100.50	85.28
		$d-f$	ρ_2^2			
	2	$a-c$	ρ_3^1	$(0.50, 6.80, 1.50)$	92.60	
		$d-f$	ρ_2^2			
	3	$a-c$	ρ_3^1	$(0.30, 6.20, 1.50)$	89.30	
		$d-f$	ρ_2^2			
	4	$a-c$	ρ_1^1	$(0.30, 7.58, 1.50)$	53.04	
		$d-f$	ρ_1^2			

Total REV (STAGE I+STAGE II) = 187.41

Table 3.20 EEV solution under the PR case

			Unrestricted tariffs				Nonnegative tariffs			
	SC	OD		Tariff	REV	EV		Tariff	REV	EV
STAGE I		$a-c$	ρ_3^1	$(-0.33, 8.00, 1.93)$	111.01		ρ_3^1	$(0.00, 7.53, 1.93)$	107.62	
		$d-f$	ρ_2^2				ρ_2^2			
STAGE II	1	$a-c$	ρ_3^1	$(-0.42, 7.42, 2.20)$	104.10		ρ_3^1	$(0.00, 7.00, 2.20)$	102.00	
		$d-f$	ρ_2^2				ρ_2^2			
	2	$a-c$	ρ_3^1	$(-0.25, 8.75, 1.55)$	59.50		ρ_3^1	$(0.00, 6.80, 1.55)$	81.60	
		$d-f$	ρ_4^2			50.99	ρ_4^2			56.48
	3	$a-c$	ρ_1^1	$(-0.42, 7.20, 1.55)$	0.00		ρ_1^1	$(0.00, 6.78, 1.55)$	0.00	
		$d-f$	ρ_4^2				ρ_4^2			
	4	$a-c$	ρ_1^1	$(-0.42, 8.80, 1.55)$	61.60		ρ_1^1	$(0.00, 8.28, 1.55)$	58.00	
		$d-f$	ρ_1^2				ρ_1^2			

Total REV (STAGE I+STAGE II) = 162.00 Total REV (STAGE I+STAGE II) = 164.10

Figure 3.7 **(A)** presents the revenue for different values of δ under the PR case. Figure 3.7 **(B)** zooms in on the interval $[0, 8]$.

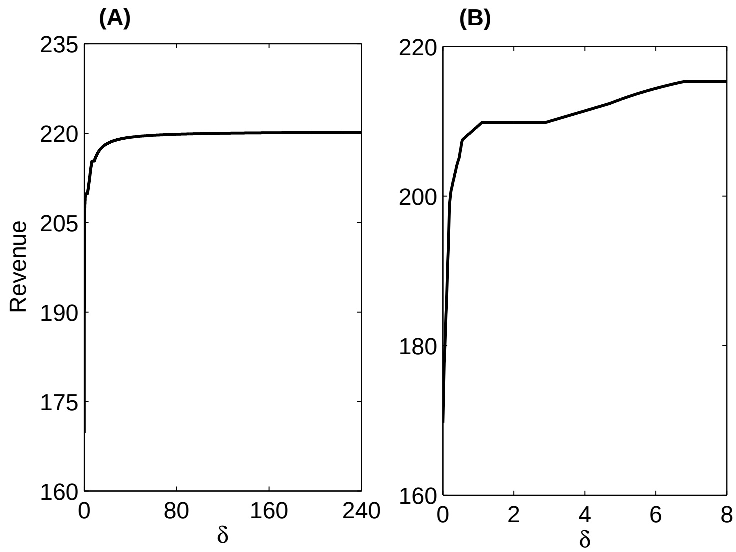

Figure 3.7 Sensitivity with respect to δ : (A) The PR case (B) Zoom in.

For simple examples, it is possible to obtain closed form expressions for the revenue, expressed as a function of the key parameter δ. This is achived in Table 3.21 for the PR case. Nonuniqueness of the lower level paths can be observed at the critical points corresponding to δ-values such as $\delta = 0.462, 2.88$ or 8.80.

Table 3.21 Revenue and tariff changes for different values of δ under the PR case.

δ	Tariff	Q	REV
$[0.00, 0.03]$	$(0, \frac{6.50}{1-\delta}, \frac{1.20}{1-\delta})$	$\frac{79.19+25.61\delta}{1-\delta}$	$\frac{169.69+25.61\delta}{1-\delta}$
$[0.03, 0.05]$	$(0, \frac{6.50}{1-\delta}, \frac{1.20}{1-\delta})$	$82.43 \uparrow 83.73$	$175.73 \uparrow 17.89$
$[0.05, 0.18]$	$(0, \frac{6.50}{1-\delta}, \frac{1.20}{1-\delta})$	$\frac{82.65-62.05\delta}{1-\delta}$	$\frac{173.15-62.05\delta}{1-\delta}$
$[0.18, 0.23]$	$(0, 7.92 \uparrow 8, 1.46 \uparrow 1.94)$	$87.17 \uparrow 86.88$	$197.53 \uparrow 200.62$
$[0.23, 0.25]$	$(0, 8, \frac{1.50}{1-\delta})$	$84.31 + 11.2\delta$	$\frac{180.81-177.11\delta-11.2\delta^2}{1-\delta}$
$[0.25, 0.36]$	$(0, 8, 2)$	$81.91 + 20.8\delta$	$195.91 + 20.80\delta$
$[0.36, 0.37]$	$(0, 8, 2)$	$73.48 + 44.2\delta$	$187.486 + 44.2\delta$
$[0.37, 0.38]$	$(0, 8, 2)$	$86.8 + 8.2\delta$	$200.806 + 8.2\delta$
$[0.38, 0.462]$	$(0, 8, 2)$	$84.07 + 15.4\delta$	$198.07 + 15.4\delta$
$[0.462, 0.525]$	$(0,8,\frac{0.90}{1-\delta})$	$87.64 + 11.20\delta$	$\frac{196.14-180.44\delta-11.2\delta^2}{1-\delta}$
$[0.525, 0.55]$	$(0,8,\frac{0.90}{1-\delta})$	93.52	$\frac{202.02-197.52\delta}{1-\delta}$
$[0.55, 1.10]$	$(0, 8, 2)$	$91.21 + 4.20\delta$	$205.21 + 4.2\delta$
$[1.10, 2.88]$	$(0, 8, 2)$	95.83	209.83
$[2.88, 4.71]$	$(-1, 9, 3)$	$96.79 + 1.40\delta$	$205.79 + 1.4\delta$
$[4.71, 6.80]$	$(-\frac{7.80}{1+\delta}, 9.00, 3.00)$	106.31	$\frac{160.91+223.31\delta}{1+\delta}$
$[6.80, 8.80]$	$(-1.00, 9.00, 3.00)$	106.31	215.31
$\delta \geq 8.80$	$(-\frac{7.8}{1-\delta}, 8 + \frac{7.80}{1-\delta}, 2)$	106.31	$\frac{259.31-220.31\delta}{1-\delta}$

Table 3.22 Optimal paths for different values of δ under the PR case.

	SC	OD	$\delta \in [0.00, 0.462]$	$\delta \in [0.462, 2.88]$	$\delta \in [2.88, 8.80]$	$\delta \in [8.80, \infty)$
STAGE I		$a-c$	ρ_3^1	ρ_3^1	ρ_3^1	ρ_3^1
		$d-f$	ρ_2^2	ρ_2^2	ρ_1^2	ρ_2^2
STAGE II	1	$a-c$	ρ_3^1	ρ_3^1	ρ_3^1	ρ_3^1
		$d-f$	ρ_2^2	ρ_2^2	ρ_2^2	ρ_2^2
	2	$a-c$	ρ_3^1	ρ_3^1	ρ_3^1	ρ_3^1
		$d-f$	ρ_2^2	ρ_2^2	ρ_2^2	ρ_2^2
	3	$a-c$	ρ_3^1	ρ_3^1	ρ_3^1	ρ_3^1
		$d-f$	ρ_2^2	ρ_2^2	ρ_2^2	ρ_2^2
	4	$a-c$	ρ_1^1	ρ_1^1	ρ_3^1	ρ_1^1
		$d-f$	ρ_1^2	ρ_2^2	ρ_1^2	ρ_2^2

CHAPITRE 4

STOCHASTIC NETWORK PRICING : A THEME AND THREE VARIATIONS

RÉSUMÉ

Dans cet article, nous poursuivons l'étude de la tarification biniveau stochastique sur deux étapes (ou période) introduite dans Alizadeh et al. (2012), où les péages doivent être déterminés sur un sous-ensemble d'arcs du réseau dans le but de maximiser les profits. Nous considérons trois variations. Dans la première variation, nous supposons que la désutilité des navetteurs intègre le délai et les conditions de fiabilité. Dans la seconde variation, nous introduisons des contraintes aléatoires au niveau du meneur dont la pertinence se justifie par des applications au domaine du transport ou des télécommunications. Dans la troisième variation du modèle, nous intégrons la congestion associée aux capacités aléatoires sur les arcs du réseau. Enfin pour chaque modèle, nous proposons une formulation en programmation mathématique et présentons quelques résultats numériques.

Stochastic network pricing and three variations

Shahrouz Mirza Alizadeh, Patrice Marcotte and Gilles Savard

ABSTRACT

Pursuing on the theme of two-stage bilevel stochastic pricing introduced in Alizadeh et al. (2012), where profit-maximizing tariffs must be determined on a subset of arcs of a transportation network, we consider three variations. In the first one, it is assumed that the disutility of commuters incorporates tardiness and reliability terms. The second variation involves chance constraints at the leader level, and is relevant to the realm of transportation/telecommunication. The third model embeds congestion associated with random capacities along the arcs of the transportation network. For each model, we provide a mathematical programming formulation, and illustrate their features through numerical examples.

Key words : Pricing, Bilevel Programming, Stochastic Programming, Mixed Integer Programming

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4.1 Introduction

In competitive markets, service providers must cope with different aspects of uncertainty or risk, some of them related to an imperfect knowledge of demand, or of their competitors' policies. With respect to demand, a firm would like to base a pricing policy on the sensitivity of its potential customers to service delay or service reliability, taking into account the time-varying and game-theoretic issues that impact the decision process. In recent years, researchers have realized that this process fits the bilevel programming paradigm, whereby a 'leader' firm takes explicitly into account the rational reaction of customers within its optimization framework.

Our starting point is the pricing model of Labbé et al. (1998), where profit-maximizing tariffs have to be set on a prescribed subset of arcs of a multicommodity transportation network, knowing that demand is assigned to shortest paths with respect to the sum of initial costs and tariffs. In a series of ensuing works, this modeling approach was adapted by Bouhtou et al. (2002), Altman and Wynter (2004), Bouhtou et al. (2006), and Bouhtou et al. (2007a) to the pricing of telecommunication networks, and by Shouping and Baozhuang (2007) and Gao et al. (2011) in the realm of supply chains.

While the above mentioned studies took place in a deterministic setting, stochastic bilevel programs have been the topic of articles by Patriksson and Wynter (1997), Christiansen et al. (2001), Wynter et al. (2002), and Werner (2005), with respective applications in structural optimization, network planning, and telecommunications. Also relevant to our study is the work of Audestad et al. (2006), who applied the stochastic programming paradigm to the analysis of competitive telecommunication markets. More precisely, the authors analyze a game that takes place between several agents, namely the owner of the network, the customers, and virtual operators who lease capacity from the network owner, with the objective of maximizing the joint profit of the market owner and the virtual operators, under either a deterministic or stochastic setting. One can also mention Kosuch et al. (2012), who consider the problem of simultaneously determining product prices, together with capacities that arise as stochastic right-hand-sides of multiknapsack constraints, and where chance constraints are applied to each capacity. The role of the lower level is to enforce the purchase of cheapest products.

More recently, Alizadeh et al. (2012) introduced a two-stage stochastic model for network pricing, where each stage is endowed with a bilevel structure. In this model, second-stage costs and demands may be random, and constraints link the price levels at both stages of the program. They develop mathematical properties of the model, and provide a reformulation as a standard bilevel program that can be numerically addressed by 'standard' techniques.

In the telecommunication industry, the pricing issue must be addressed by network operators who offer end-to-end services, while facing competition, uncertainty, risk averse clients, and other quality of service (QoS) concerns. In this realm, customers select a QoS that best suits their needs with respect to delay (related to congestion), reliability, costs, or other relevant metrics. Since the trade-off values of each customer with respect to these features, or even the measure of these features themselves, cannot be evaluated with great accuracy, it is natural to cast the pricing problem within a stochastic framework. Besides, a traditional and large source of randomness is demand for each service per se. Therefore, stochastic optimization is relevant to a large variety of telecommunication network problems, especially at the design, planning and support levels. The interested reader is referred to Gaivoronski (2005), who illustrates the power of stochastic programming tools for addressing problems that arise in the telecommunication industry at the technological, network and enterprise levels.

Motivated by the above-mentioned frameworks of application, the present work focuses on three stochastic variations on the network pricing theme, each one embedding aspects that are relevant to real-life applications in transportation or telecommunications. These incorporate features such as link capacity, path delay, punctuality, and network reliability. Throughout, we assume that delays are random and that links may become unavailable, due to congestion or random connection failures. For each model, we specify how randomness enters the mathematical formulation, and provide numerical illustrations based on mixed integer linear reformulations of the original bilevel programs.

The structure of the paper is as follows. The first section introduces a two-stage stochastic bilevel model, and more specifically its pricing version. Each of the following three sections is devoted to one variation, including its formulation and a numerical illustration. The first variation deals with stochastic disutilities that involve a tardiness term on top of the actual travel delay, as arises in daily commuting from home to work. The second variation addresses delay and reliability issues, two features that are common in transportation/telecom networks. In this model, tardiness enters the leader's objective through a penalty term, while chance constraints are used to restrict overcapacity. The third variation, which mostly applies to road networks, explicitly considers congestion, through the introduction of volume-delay curves. Following a conclusion, and in order not to disrupt the presentation, the appendix discusses very shortly the computational complexity of the problem, provides upper bounds that are valid for the second and third variations, and mentions a continuity result.

4.2 Stochastic Bilevel programming

Bilevel programming allows the natural modelling of hierarchical situations where a subset of decision variables is not under the direct control of the main optimizer (leader, or upper level) but is controlled by a follower (user, or lower level) who optimizes its own objective function with respect to parameters set by the leader. Mathematically, it is expressed as

$$\begin{aligned}
\max_{x,y} \quad & F_1(x,y) \\
\text{s.t.} \quad & G_1(x,y) \le 0, \\
& y \in \arg\min_{\hat{y}} \quad f_1(x,\hat{y}), \\
& \qquad\quad \text{s.t.} \quad g_1(x,\hat{y}) \le 0.
\end{aligned} \tag{4.1}$$

In the sequel, and in order to simplify the notation, we will only specify the programs of the leader and the follower, since they contain all the information relevant to the bilevel program.

We now introduce the stochastic bilevel programming (SBP) frameworks that will be used to formulate the three variations. The first one involves random parameters at the follower level, while the leader's objective function involves an expectation. As a general rule, we use the 'prime' notation to indicate the actual value of random variable associated with an outcome ω. This yields :

$$\begin{aligned} &\max_{x} \quad E_{\boldsymbol{\xi}}\left[F_1(x, y'(\xi))\right] \\ &\text{s.t.} \quad G_1(x, y'(\omega)) \leq 0, \qquad\qquad \forall \omega \in \Omega, \\ &\left.\begin{aligned} &\min_{y'(\omega)} \quad f_1(x, y'(\omega), \omega), \\ &\text{s.t.} \quad g_1(x, y'(\omega), \omega) \leq 0, \end{aligned}\right\} \quad \forall \omega \in \Omega, \end{aligned} \tag{4.2}$$

where $\boldsymbol{\xi}$ ($\boldsymbol{\xi} : \Omega \to \Re^r$) is a continuous random vector with realizations ξ ($\xi = \xi(\omega)$) in some probability space $(\Omega, \mathcal{F}, P)$, with Ω denoting the set of all random events, and $\mathcal{F}$ the set of all subsets of Ω. In this 'wait and see' framework, the follower solves a deterministic program, while the risk-neutral leader must factor randomness within its optimization process.

The second SBP takes the form of a chance-constrained bilevel program. Such constraints are introduced as a measure of quality, and also as a means to limit exposure to risk, and may be embedded either at the upper or lower level, or at both, as formulated below :

$$
\begin{aligned}
\max_{x} \quad & F(x,y) \\
\text{s.t.} \quad & G_1(x,y) \le 0, \\
& \Pr\{G_2(x,\boldsymbol{\xi}) \ge H(\boldsymbol{\xi})\} \ge \alpha, \\
\min_{y} \quad & f_1(x,y), \\
\text{s.t.} \quad & g_1(x,y) \le 0, \\
& \Pr\{g_2(y,\boldsymbol{\xi}) \ge h(\boldsymbol{\xi})\} \ge \beta,
\end{aligned}
\tag{4.3}
$$

where $G_2(x,\boldsymbol{\xi})$ and $g_2(y,\boldsymbol{\xi})$ are frequently assumed, to be linear functions with respect to x and y, respectively. The threshold α and β are reliability parameters belonging to $(0,1)$, which can also be interpreted as the risk levels deemed acceptable by the decision makers. Values close to zero correspond to a risky behaviour, and values close to one to a conservative attitude.

Let us now focus our attention on a multicommodity transportation network built around a graph $G(N,\Lambda,K)$ with node set N, arc set Λ, and commodity set K, each commodity k (origin-destination pair) being endowed with a demand n^k. The set Λ is partitioned into the subsets Λ_1 and Λ_2 of tariff and tariff-free arcs, respectively. The set of paths available for commodity k is denoted by L^k, the set of tariff arcs belonging to path ρ by Λ_1^ρ, and the set of paths including at least one tariff arc by L_1^k. The bilevel network pricing problem introduced by Labbé et al. (1998) consists in maximizing the revenue raised from tariffs, knowing that user flows are assigned to cheapest paths. Since large tariffs will drive users to tariff-free paths, the optimal trade-off is achieved by solving the bilevel mathematical program

$$
\begin{aligned}
\max_{t,T} \quad & \sum_{k\in K} n^k \sum_{\rho\in L_1^k} T_\rho r_\rho \\
\text{s.t.} \quad & T_\rho = \sum_{a\in\Lambda_1^\rho} t_a, && \forall k\in K, \forall \rho\in L_1^k, \\
\min_{r} \quad & \sum_{k\in K} n^k \big[\sum_{\rho\in L_1^k} T_\rho r_\rho + \sum_{\rho\in L^k}\sum_{a\in\rho} c_a r_\rho\big], \\
\text{s.t.} \quad & \sum_{\rho\in L^k} r_\rho^k = 1, && \forall k\in K, \\
& r_\rho \geq 0, && \forall k\in K, \forall\rho\in L^k,
\end{aligned}
\tag{4.4}
$$

where t is the vector of tariff variables controlled by the leader, T the vector of revenue raised from the various paths, and c the vector of fixed costs along the arcs. The leader constraint establishes compatibility between tariffs and revenues, while the follower's constraints consist in flow conservation and flow nonnegativity. The proportion of flow from commodity k assigned to path ρ is denoted by r_ρ^k, and it has been shown that, without loss of generality, this variable can be restricted to the values zero or one, i.e., it can be considered as an indicator function, in the absence of capacities. Throughout, we will make the following basic assumptions :

1. For each commodity, there exists at least one path composed entirely of tariff-free arcs, and whose availability (probability of being available) is strictly positive.
2. There does not exist a pricing procedure that yields positive returns and simultaneously generates a negative-cost cycle. This assumption ensures that the lower level flows are assigned to shortest paths.
3. Competition prices are fixed and are not influenced by the leader's tariffs.
4. Demand is fixed and can be split between the paths of the network.

Note that the first two assumptions imply that the upper level revenue is bounded from above. In the following sections, we extend Program (4.4) to the stochastic framework introduced at the beginning of this section.

4.3 First variation : Stochastic disutility

In the first variation, we introduce stochastic disutility at the user level. Disutility is modelled as a function of fixed costs, tariffs, tardiness and reliability, and represents a situation where users are ready to compromise their target arrival time for higher reliability. To achieve this goal, the disutility terms are expressed as the ratio of tardiness over reliability, both the numerator and denominator being random.

In this model, $d_a^{\text{arc}}(\boldsymbol{\xi})$ and $d_j^{\text{node}}(\boldsymbol{\xi})$ denote random delays along the arcs and nodes, respectively, H^k refers to the target time (deadline) to destination $D(k)$ associated with commodity k, and $\bar{p}^k$ is the penalty associated with one unit of tardiness, the latter being equal to

$$\max\left\{0, g_\rho(\omega) - H^k\right\} \tag{4.5}$$

where the total random delay along a given path $\rho \in L^k$ is given by

$$g_\rho(\boldsymbol{\xi}) \;=\; \sum_{a\in\rho} d_a^{\text{arc}}(\boldsymbol{\xi}) + \sum_{j\in\rho|j\neq D(k)} d_j^{\text{node}}(\boldsymbol{\xi}). \tag{4.6}$$

As mentioned previously, network links may become unavailable, due to unforeseen failures. We assume that the leader has perfect knowledge about the state of its own tariff links, but not those of the competition. In order to model this situation, we introduce an exogenous measure of the availability $h_a(\boldsymbol{\xi}) \in (0,1]$ of a tariff-free link. Assuming independence between the links, the availability of a path is thus given by

$$h'_\rho(\boldsymbol{\xi}) = \prod_{a\in\rho} h_a(\boldsymbol{\xi}) \text{ for } \forall k \in K, \rho \in L^k.$$

The first variation is then expressed as the mathematical program

$$
\begin{aligned}
\max_{t,T} \quad & E_{\boldsymbol{\xi}}\Big[\sum_{k\in K} n^k \sum_{\rho\in L_1^k} T_\rho r'_\rho(\xi(\omega))\Big] \\
\text{s.t.} \quad & \sum_{a\in\Lambda_1^\rho} t_a = T_\rho, \qquad\qquad \forall k\in K, \forall\rho\in L_1^k, \\
& \left.\begin{aligned}
\min_{r} \quad & \sum_{k\in K} n^k \sum_{\rho\in L^k} \big[(T_\rho + \sum_{a\in\rho} c_a + \bar{p}^k \max\big\{0, g_\rho(\omega) - H^k\big\})/h'_\rho(\omega)\big] r'_\rho(\omega), \\
\text{s.t.} \quad & \sum_{\rho\in L^k} r'_\rho(\omega) = 1, \qquad \forall k\in K, \\
& r'_\rho(\omega) \geq 0, \qquad \forall k\in K,
\end{aligned}\right\} \forall\omega\in\Omega,
\end{aligned} \tag{4.7}
$$

where $r'_\rho(\omega)$ is the indicator variable corresponding to outcome ω, that takes value one if the flow of commodity k is assigned to path $\rho \in L^k$, and zero otherwise. As mentioned before, the 'prime' symbol refers to the actual value of the random variable associated with an outcome ω. For instance, $c'_\rho(\omega)$ denotes the fixed cost of path $\rho \in L^k$ for any realization $\omega \in \Omega$, i.e.,

$$
c'_\rho(\omega) = (\sum_{a\in\rho} c_a + \bar{p}^k \max\big\{0, g_\rho(\omega) - H^k\big\})/h'_\rho(\omega).
$$

In the standard fashion, we rewrite Programs (4.7) as a mixed-integer program. To achieve this, we introduce a variable $\hat{t}_{a,\rho}(\omega)$ for each path $\rho \in L^k$, tariff arc $a \in \Lambda_1$ and outcome $\omega \in \Omega$, which takes value t_a if and only if commodity k is assigned to path $\rho \in L^k$ going through tariff arc a, and is equal to zero otherwise. We obtain

$$
\begin{aligned}
\max_{\substack{r',t\\ \hat{t}}} \quad & E_{\boldsymbol{\xi}}\Big[\sum_{\substack{k\in K\\ \rho\in L_1^k}}\sum_{a\in\Lambda_1^\rho} n^k \hat{t}_{a,\rho}(\xi(\omega))\Big] && \\
\text{s.t.} \quad & \sum_{\rho\in L^k} r'_\rho(\omega) = 1, && \forall\omega\in\Omega, \forall k\in K,\\
& \left.\begin{aligned} & (1/h'_\rho(\omega)) \sum_{a\in\Lambda_1^\rho} t_a + c'_\rho(\omega) \geq \\ & \qquad \sum_{\nu\in L_1^k}\sum_{a\in\Lambda_1^\nu}(1/h'_\nu(\omega))\hat{t}_{a,\nu}(\omega) + \sum_{\nu\in L^k} c'_\nu(\omega) r_\nu(\omega), \end{aligned}\right\} && \begin{aligned}&\forall\omega\in\Omega, \forall k\in K,\\ &\forall\rho\in L^k,\end{aligned}\\
& \left.\begin{aligned} & -M_\omega^{k^*}(1-r'_\rho(\omega)) \leq \hat{t}_{a,\rho}(\omega) - t_a \leq M_\omega^{k^*}(1-r'_\rho(\omega)),\\ & -M_\omega^{k^*} r'_\rho(\omega) \leq \hat{t}_{a,\rho}(\omega) \leq M_\omega^{k^*} r'_\rho(\omega), \end{aligned}\right\} && \begin{aligned}&\forall\omega\in\Omega, \forall k\in K,\\ &\forall\rho\in L_1^k, \forall a\in\Lambda_1^\rho,\end{aligned}\\
& r'_\rho(\omega)\in\{0,1\}, \quad\Big\} && \begin{aligned}&\forall\omega\in\Omega, \forall k\in K,\\ &\forall\rho\in L^k.\end{aligned}
\end{aligned}
\tag{4.8}
$$

The first set of constraints forces, without loss of optimality, each commodity to be assigned to a single path, while the second set ensures that only paths of minimum disutility carry nonzero flow. A suitable value for the 'big-M' parameter $M_\omega^{k^*}$, i.e., one that does not cut off potentially optimal solutions, is given by :

$$
M_\omega^{k^*} = \max_{k\in K}\{\Gamma'^k_\omega(\infty) - \Gamma'^k_\omega(0)\}
$$

where $\Gamma'(t)$ denotes the lower level optimal value corresponding to a given tariff vector t, and $\Gamma'_\omega(\infty) - \Gamma'_\omega(0)$ is an upper bound on the firm's revenue for outcome ω.

Remark 4.3.1 *If negative tariffs are disallowed, then one can a priori (in a preprocessing phase) remove path $\nu\in L^k$ if there exists a path $\rho\in L^k$ such that $\Lambda_1^\rho\subseteq\Lambda_1^\nu$ and $c'_\rho(\omega)\leq c'_\nu(\omega)$ for every realization $\omega\in\Omega$. We then say that path ρ dominates path ν.*

4.3.1 A numerical example

In this section, we illustrate the model through an example inspired from the airline/telecom network shown in Figure 4.1, where fixed costs are displayed next to the corresponding arcs. The demand associated with commodities $a-c$ and $d-f$ is set to $n^1 = 8$ and $n^2 = 5$ respectively, and the available paths for each commodity are displayed in Table 4.1. The vector of deadlines at destination (target arrival times) is $H = (5.5, 6.3)$, and the vector of unit tardiness penalties is set to $\bar{p} = (0.2, 0.5)$. Table 4.2 contains the three distinct sets of fixed cost data that were considered.

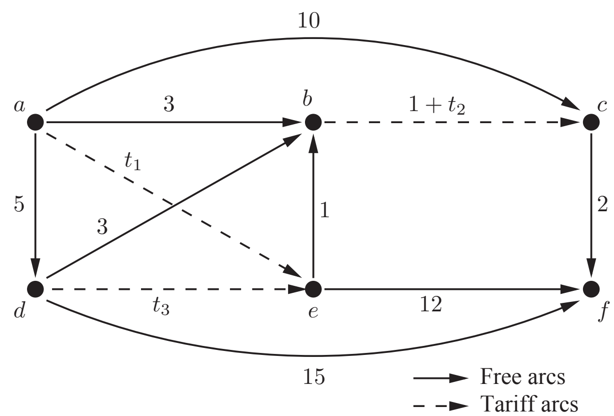

Figure 4.1 Example network

Table 4.1 Available paths for commodities $a-c$ and $d-f$

OD	PATH	OD	PATH
$a-c$	$\rho_1^1 : a-c$	$d-f$	$\rho_1^2 : d-f$
	$\rho_2^1 : a-b-c$		$\rho_2^2 : d-e-f$
	$\rho_3^1 : a-e-b-c$		$\rho_3^2 : d-e-b-c-f$
	$\rho_4^1 : a-d-b-c$		$\rho_4^2 : d-b-c-f$
	$\rho_5^1 : a-d-e-b-c$		

Table 4.2 Fixed costs along the arcs

#	ae	bc	de	ab	ac	ad	cf	db	df	eb	ef
1	0	1	0	3	10	5	2	3	15	1	12
2	0	0	0	7	4	4	1	11	32	2	20
3	2	1	1	2	6	2	0	4	15	2	18

In the realm of air transportation, arc and node delays allow the representation of air travel times and ground delays, respectively. Those can be influenced by weather conditions or technical problems, among other factors, and have a ripple effect throughout the network. In the world of telecommunications, arc delays are related to transmission along the lines of the network, while queueing occurs at the nodes.

Tables 4.3-4.5 list the fixed part of node delays d_j^{node} ($j = \{a, b, c, d, e, f\}$), link delays d_a^{arc} ($a \in \{ae, bc, de, ab, ac, ad, cf, db, df, eb, ef\}$), and random link reliabilities of tariff-free link h_a ($a \in \{ab, ac, ad, cf, db, df, eb, ef\}$), respectively. These are generated according to discrete uniform distributions $\left[\upsilon_j^{\text{node}}(1-\eta), \upsilon_j^{\text{node}}(1+\eta)\right]$, $[\upsilon_a^{\text{arc}}(1-\eta), \upsilon_a^{\text{arc}}(1+\eta)]$, and $[\varkappa_a(1-\eta), \varkappa_a(1+\eta)]$, respectively, where vectors υ^{arc}, υ^{node}, and refer to the free flow travel/transmission time on the arcs and through the nodes, respectively, and $\varkappa$ refers to the free flow reliability on the arcs. Vectors υ^{arc} and υ^{node} are given as

$$\upsilon^{\text{arc}} = (5.34, 0.97, 4.46, 1.36, 9.95, 4.63, 1.29, 1.30, 5.78, 2.68, 3.35),$$

$$\upsilon^{\text{node}} = (0.32, 0.82, 1.28, 0.35, 1.78, 1.41),$$

and

$$\varkappa = (0.95, 0.90, 0.75, 0.80, 0.70, 0.90, 0.85, 0.70).$$

Note that, with respect to the parameter η, the first and fourth outcomes are the most and least favorable, respectively. Finally, throughout this paper, the abbreviations SC and PROB will refer to a scenario and its associated reliability.

Table 4.3 Random node delays

SC	PROB	η	d_a^{node}	d_b^{node}	d_c^{node}	d_d^{node}	d_e^{node}	d_f^{node}
1	0.20	0.10	0.30	0.77	1.20	0.33	1.67	1.33
2	0.30	0.25	0.29	0.74	1.15	0.31	1.60	1.27
3	0.35	0.25	0.30	0.76	1.18	0.32	1.65	1.30
4	0.15	0.40	0.23	0.59	0.92	0.25	1.28	1.02

Table 4.4 Random link delays

SC	d_{ae}^{arc}	d_{bc}^{arc}	d_{de}^{arc}	d_{ab}^{arc}	d_{ac}^{arc}	d_{ad}^{arc}	d_{cf}^{arc}	d_{db}^{arc}	d_{df}^{arc}	d_{eb}^{arc}	d_{ef}^{arc}
1	5.02	0.91	4.19	1.28	9.35	4.35	1.21	1.22	5.43	2.52	3.15
2	4.81	0.87	4.01	1.22	8.95	4.17	1.16	1.17	5.20	2.41	3.01
3	4.94	0.90	4.13	1.26	9.20	4.28	1.19	1.20	5.35	2.48	3.10
4	3.84	0.70	3.21	0.98	7.16	3.33	0.93	0.94	4.16	1.93	2.41

Table 4.5 Random link reliability of tariff-free links

SC	h_{ab}	h_{ac}	h_{ad}	h_{cf}	h_{db}	h_{df}	h_{eb}	h_{ef}
1	0.89	0.85	0.71	0.75	0.66	0.85	0.80	0.66
2	0.85	0.81	0.67	0.72	0.63	0.81	0.77	0.63
3	0.88	0.83	0.69	0.74	0.65	0.83	0.79	0.65
4	0.68	0.65	0.54	0.58	0.50	0.65	0.61	0.50

In order to assess the loss of performance due to neglecting randomness, we will compare the above revenues with the ones obtained from a deterministic approximation based on the expected values of the random parameters (EEV). Those are obtained in two phases. First, we solve the expected value problem corresponding to Program (4.2)

$$
\begin{aligned}
&\max_{x} \quad F_1(x,y)\\
&\text{s.t.} \quad G_1(x,y) \le 0,\\
&\min_{y} \quad f_1(x,y,\bar{\omega}),\\
&\text{s.t.} \quad g_1(x,y,\bar{\omega}) \le 0,
\end{aligned}
\tag{4.9}
$$

where $\bar{\omega} = E[\boldsymbol{\xi}]$. Next, letting $\bar{x}$ denote a first stage optimal solution of Program (4.9), we define EEV as

$$
\begin{aligned}
&\max \quad E_{\boldsymbol{\xi}}\left[F_1(\bar{x}, y'(\xi))\right]\\
&\text{s.t.} \quad G_1(\bar{x}, y'(\omega)) \le 0, \qquad\qquad \forall \omega \in \Omega,\\
&\left.\begin{aligned}
&\min_{y'(\omega)} \quad f_1(\bar{x}, y'(\omega), \omega),\\
&\text{s.t.} \quad g_1(\bar{x}, y'(\omega), \omega) \le 0,
\end{aligned}\right\} \quad \forall \omega \in \Omega.
\end{aligned}
\tag{4.10}
$$

Tables 4.6 and 4.7 present the stochastic and EEV solutions of Program (4.8), corresponding to the given data and different sets of the fixed costs. The abbreviations OD and REV refer to commodity and revenue, respectively, and columns ‘I’, ‘II’, ‘III’ and, ‘IV’ refer to the scenario indices.

Table 4.6 Stochastic optimal solutions of Program (4.8)

#	OD	I	II	III	IV	Tariff	REV
1	$a-c$	ρ_2^1	ρ_2^1	ρ_2^1	ρ_3^1	$(-0.16, 7.35, -0.76)$	58.61
	$d-f$	ρ_1^2	ρ_1^2	ρ_1^2	ρ_1^2		
2	$a-c$	ρ_3^1	ρ_3^1	ρ_3^1	ρ_3^1	$(-9.89, 11.32, 4.46)$	81.85
	$d-f$	ρ_3^2	ρ_3^2	ρ_3^2	ρ_2^2		
3	$a-c$	ρ_2^1	ρ_2^1	ρ_2^1	ρ_2^1	$(-3.32, 3.73, -0.52)$	43.48
	$d-f$	ρ_3^2	ρ_3^2	ρ_3^2	ρ_1^2		

Table 4.7 EEV optimal solutions of Program (4.8)

#	OD	I	II	III	IV	Tariff	REV
1	$a-c$	ρ_2^1	ρ_2^1	ρ_2^1	ρ_3^1	$(-0.21, 7.32, -1.61)$	57.10
	$d-f$	ρ_1^2	ρ_1^2	ρ_1^2	ρ_2^2		
2	$a-c$	ρ_1^1	ρ_3^1	ρ_1^1	ρ_3^1	$(-4.39, 5.83, 9.91)$	29.96
	$d-f$	ρ_4^2	ρ_4^2	ρ_4^2	ρ_1^2		
3	$a-c$	ρ_2^1	ρ_2^1	ρ_2^1	ρ_3^1	$(-3.77, 4.10, -0.86)$	37.19
	$d-f$	ρ_3^2	ρ_1^2	ρ_3^2	ρ_1^2		

From Tables 4.6 and 4.7, we observe that the loss of revenue associated with the deterministic approximation can be in excess of 10%, which confirms the relevance of the stochastic formulation. Tables 4.8 and 4.9 provide sensitivity information with respect to changes in tardiness penalty $\bar{p}$ and deadline H, for the first data set, illustrating how optimal paths vary. A graphical representation of the sensitivity is provided in Figures 4.2 and 4.3.

From an economic standpoint, one expects that higher penalty sensitivity will drive commuters to more reliable paths, usually (partially) controlled by the leader, who can then take advantage of the situation to increase tariffs and revenues (see Figure 4.2). The impact of the variation of the deadline H is less straightforward. Indeed, increasing H may render attractive a previously unattractive of even infeasible tariff path, and consequently open new opportunities for increasing tariffs and revenues. This can be observed in the two upper graphs of Figure 4.3 . Alternatively, large values of H can render attractive a longer tariff-free path, and then force the leader to lower its tariffs, leading to lower revenue, as can be seen on the lower graph of Figure 4.3. A priori, and without solving the problem, one cannot predict which of the two situations will occur.

Table 4.8 Sensitivity analysis with respect to $\bar{p}$ for the first data set (#1)

$\bar{p}$	I	II	III	IV
$[0.00, 0.02]$	(ρ_3^1, ρ_3^2)	(ρ_3^1, ρ_3^2)	(ρ_3^1, ρ_3^2)	(ρ_3^1, ρ_1^2)
$[0.02, 0.15]$	(ρ_3^1, ρ_3^2)	(ρ_3^1, ρ_3^2)	(ρ_3^1, ρ_3^2)	(ρ_3^1, ρ_2^2)
$[0.15, 0.31]$	(ρ_2^1, ρ_3^2)	(ρ_2^1, ρ_3^2)	(ρ_2^1, ρ_3^2)	(ρ_3^1, ρ_2^2)
$\bar{p} \geq 0.31$	(ρ_2^1, ρ_1^2)	(ρ_2^1, ρ_1^2)	(ρ_2^1, ρ_1^2)	(ρ_3^1, ρ_1^2)

Table 4.9 Sensitivity analysis with respect to H for the first data set (#1)

H	I	II	III	IV
$[0.00, 6.30]$	(ρ_2^1, ρ_1^2)	(ρ_2^1, ρ_1^2)	(ρ_2^1, ρ_1^2)	(ρ_3^1, ρ_1^2)
$[6.30, 6.84]$	(ρ_2^1, ρ_1^2)	(ρ_3^1, ρ_1^2)	(ρ_2^1, ρ_1^2)	(ρ_3^1, ρ_1^2)
$[6.84, 7.14]$	(ρ_2^1, ρ_1^2)	(ρ_3^1, ρ_1^2)	(ρ_3^1, ρ_1^2)	(ρ_3^1, ρ_1^2)
$[7.14, 7.47]$	(ρ_3^1, ρ_1^2)	(ρ_3^1, ρ_1^2)	(ρ_3^1, ρ_1^2)	(ρ_3^1, ρ_1^2)
$[7.47, 8.86]$	(ρ_3^1, ρ_3^2)	(ρ_3^1, ρ_3^2)	(ρ_3^1, ρ_3^2)	(ρ_3^1, ρ_2^2)
$H \geq 8.86$	(ρ_3^1, ρ_3^2)	(ρ_3^1, ρ_3^2)	(ρ_3^1, ρ_3^2)	(ρ_3^1, ρ_1^2)

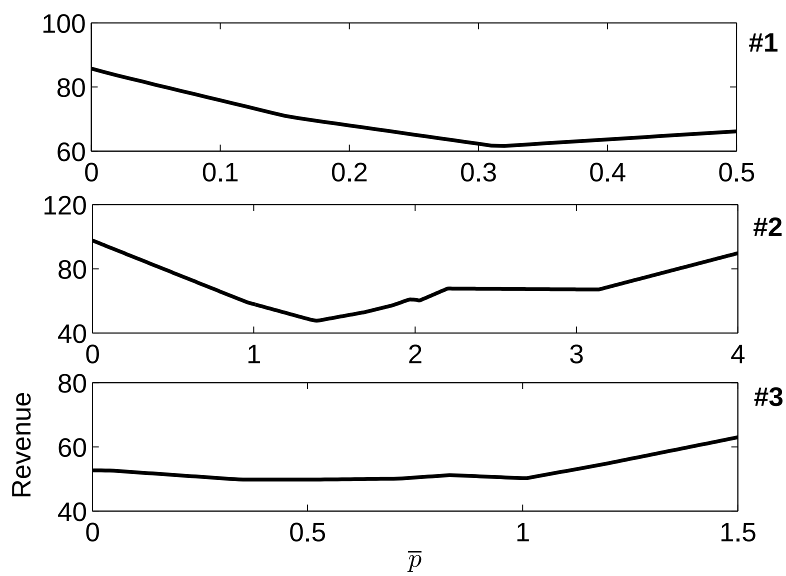

Figure 4.2 Sensitivity with respect to the tardiness penalty $\bar{p}$

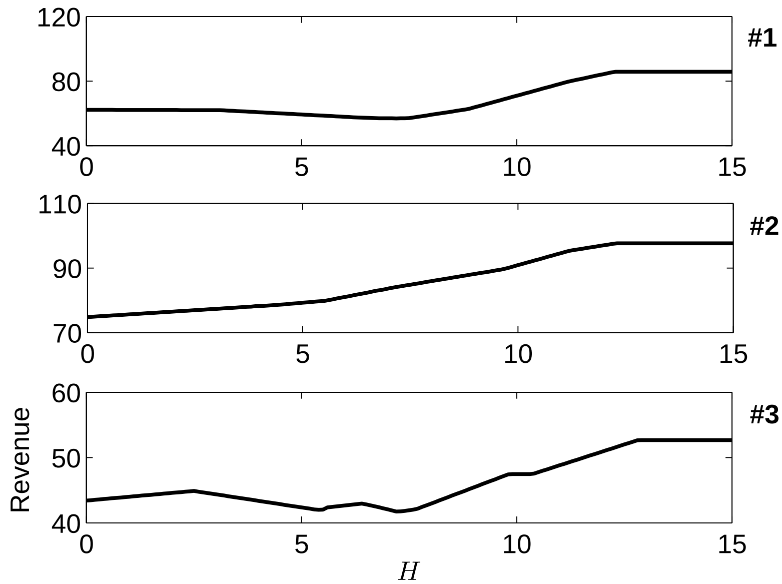

Figure 4.3 Sensitivity with respect to the deadline H

4.4 Second variation : Chance constraints

We now introduce the framework of the second variation, which shares several features with the first one. However, the differences are twofold. First, the tardiness penalty is incurred by the leader. Second, reliability of an arc is now related to the flow that it carries, and to a random target capacity. A chance constraint, whose role is to prevent overflow and ensure a reliable performance level of the system, is then imposed to the leader. In this setting, the path choice of the follower is not influenced by the reliability of the paths, since the constraint imposed at the upper level ensures a predetermined and satisfactory level of service.

Notation-wise, the model involves a random vector of link capacities $C'(\boldsymbol{\xi})$ $(\boldsymbol{\xi} : \Omega' \to \Re^{|A_1|})$ at the upper level. The role of chance constraints is then to ensure that the probability of not meeting these target values lies below some predetermined threshold. According to the above considerations, the corresponding model takes the form of the chance constrained bilevel program

$$
\begin{aligned}
\max_{t,T} \quad & \sum_{k\in K}\sum_{\rho\in L_1^k}\left(T_\rho - \bar{p}^k E_{\boldsymbol{\xi}}\left[\max\left\{0, g_\rho(\xi) - H^k\right\}\right]\right) f_\rho \\
\text{s.t.} \quad & \sum_{a\in\Lambda_1^\rho} t_a = T_\rho, && \forall k\in K, \forall\rho\in L_1^k, \\
& x_a = \sum_{k\in K}\sum_{\rho\in L^k | a\in\rho} f_\rho^k, && \forall a\in\Lambda_1, \\
& \Pr\{x_a \geq C_a'(\boldsymbol{\xi})\} \leq \alpha_a, && \forall a\in\Lambda_1, \\
\min_f \quad & \sum_{k\in K}\Big[\sum_{\rho\in L_1^k} T_\rho f_\rho^k + \sum_{\rho\in L^k}\sum_{a\in\rho} c_a f_\rho^k\Big], \\
\text{s.t.} \quad & \sum_{\rho\in L^k} f_\rho^k = n^k, && \forall k\in K, \\
& f_\rho^k \geq 0, && \forall k\in K, \forall\rho\in L^k,
\end{aligned}
\tag{4.11}
$$

where f is the vector of path flows and $g_\rho(\boldsymbol{\xi})$, defined in Equation (4.6), represents the total delay. The parameter α_a specifies the maximum risk of overloading, and may be interpreted as a 'protection level' associated with link a.

According to the form of the chance constraints, setting the value of the tolerance parameter α close to one reflects a risky attitude with respect to reliability requirements, but might yield high profit. At the other end of the spectrum, setting α close to zero corresponds to a risk averse leader. It will ensure high reliability (low congestion), at the expense of revenue. Actually, if $\alpha = 0$, then no flow at all (except for the case of infinite capacity) can be assigned to the arcs that are in control of the leader, hence revenue (profit) is null.

In order to deal numerically with chance constraints without resorting to simulation, the latter being computationally expensive, we make the assumption that arc capacities are distributed according to uniform variates $U[\theta_a C_a, C_a]$, where the parameter θ is an indicator of the reliability of the link : the higher θ_a, the higher the probability that the target capacity C_a be fully available. Low values of θ_a reflect a situation where disruptions may occur with high probability, and thus that the link become partially or totally unavailable. Denoting by

F_a the cumulative distribution function of $U[\theta_a C_a, C_a]$, we can write

$$\begin{array}{rrcl} & \Pr\{x_a \geq C'_a(\boldsymbol{\xi})\} & \leq & \alpha_a, \\ \Longleftrightarrow & x_a & \leq & F_a^{-1}(\alpha_a), \\ \Longleftrightarrow & x_a & \leq & (\alpha_a + (1-\alpha_a)\theta_a)C_a, \end{array}$$

and it follows that Program (4.11) can be reformulated as

$$\begin{array}{rll} R(\theta) = & \max\limits_{t,T} & \sum\limits_{k\in K}\sum\limits_{\rho\in L_1^k} \left(T_\rho - \bar{p}^k E_{\boldsymbol{\xi}}\left[\max\left\{0, g_\rho(\xi(\omega)) - H^k\right\}\right]\right) f_\rho \\ & \text{s.t.} & \sum\limits_{a\in\Lambda_1^\rho} t_a = T_\rho, \qquad \forall k\in K, \forall\rho\in L_1^k, \\ & & x_a = \sum\limits_{k\in K}\sum\limits_{\rho\in L^k|a\in\rho} f_\rho^k, \qquad \forall a\in\Lambda_1, \\ & & x_a \leq (\alpha_a + (1-\alpha_a)\theta_a)C_a, \qquad \forall a\in\Lambda_1, \\ & \min\limits_{f} & \sum\limits_{k\in K}\Big[\sum\limits_{\rho\in L_1^k} T_\rho f_\rho + \sum\limits_{\rho\in L^k}\sum\limits_{a\in\rho} c_a f_\rho^k\Big], \\ & \text{s.t.} & \sum\limits_{\rho\in L^k} f_\rho^k = n^k, \qquad \forall k\in K, \\ & & f_\rho^k \geq 0, \qquad \forall k\in K, \forall\rho\in L^k. \end{array} \tag{4.12}$$

4.4.1 A numerical example

Let us again consider the network depicted in Figure 4.1, where the relevant information is left unchanged, together with the additional data

$$C = (10, 20, 25),\ \ \theta = (0.15, 0.20, 0.25),\ \ \alpha = (0.05, 0.03, 0.04).$$

Table 4.10 Second variation : optimal solutions

#	OD	Flow	Tariff	REV
1	$a-c$	$(8,0,0,0,0)$	$(-1,9,3)$	41.22
	$d-f$	$(0,0.52,0,4.48)$		
2	$a-c$	$(8,0,0,0,0)$	$(-15,17,9)$	107.44
	$d-f$	$(0,0.52,4.48,0)$		
3	$a-c$	$(8,0,0,0,0)$	$(-9,10,1)$	44.80
	$d-f$	$(0.52,0,0,4.48)$		

Table 4.10 displays the optimal solutions of Program (4.12) (link flows, tariffs and revenue) for the three data sets introduced earlier for the first variation. While flow is assigned to a single path for the OD pair $a-c$, this is not the case for OD pair $d-f$, where flow is split between two paths, in order to meet the capacity requirements set by the leader. Since path flows at the lower level must be assigned to shortest paths, it follows that all paths that carry positive flow are of equal and minimal costs. Moreover, since the same values of the parameters α and θ have been uses across the three data sets, the path flow proportions do not vary, as can be observed in Table 4.10, where we assumed identical capacities for each data set. Sensitivity with respect to parameters α and θ is provided, for the first data set, in Figures 4.4 and 4.5, as well as in Tables 4.11 and 4.12. When sensitivity is performed with respect to a vector (versus one of its components), it is understood that all components vary simultaneously and assume a common value. Note that when the parameters α_2 and θ_2 of tariff arc 2 are set to their initial values, varying the parameters of the two other tariff arcs does not result in a revenue increase. The reason of this occurrence is because of less role of the tariff arcs ae and de in the paths that the users are interested. Notwithstanding, increasing *simultaneously* the three parameters results in a faster revenue increase.

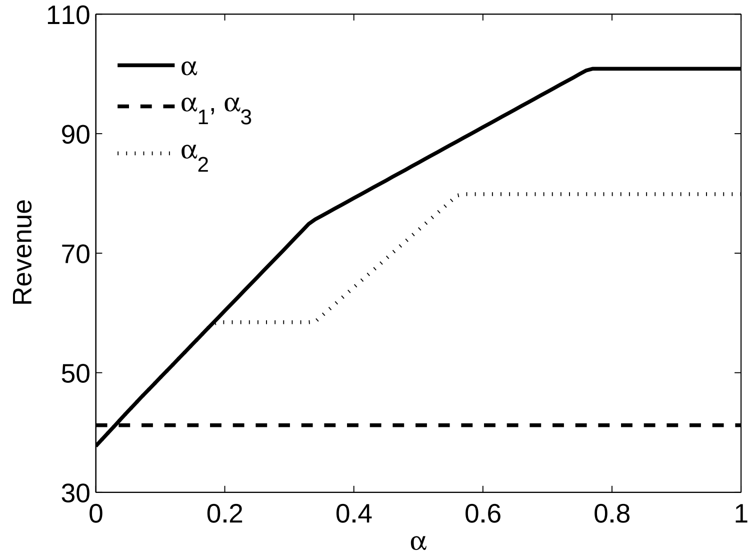

Figure 4.4 Second variation : sensitivity of revenue with respect to α

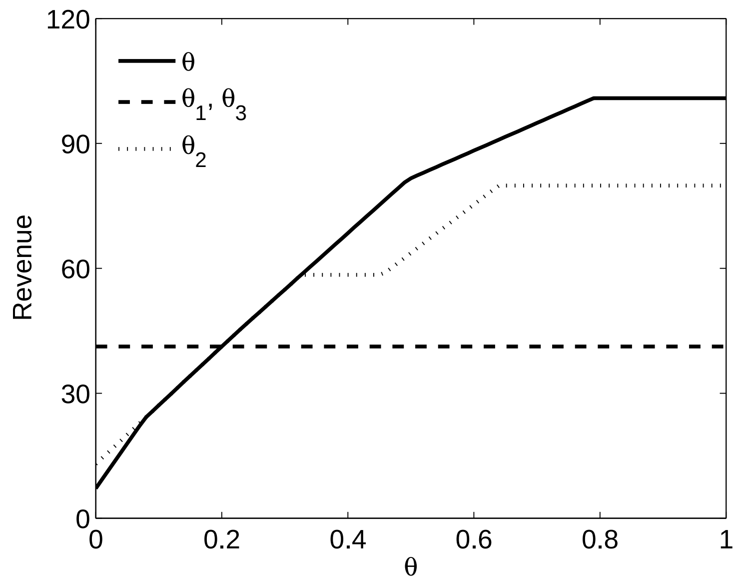

Figure 4.5 Second variation : sensitivity of revenue with respect to θ

Table 4.11 Second variation : sensitivity of revenue with respect to α

α	OD	Flow	REV
$[0.000, 0.062]$	$a-c$	$(8, 0, 0, 0, 0)$	$37.73 + 116.35\alpha$
	$d-f$	$(0, 1-16\alpha, 0, 4+16\alpha)$	
$[0.062, 0.334]$	$a-c$	$(9-16\alpha, 0, -1+16\alpha, 0, 0)$	$38.02 + 111.69\alpha$
	$d-f$	$(0, 0, 5, 0)$	
$[0.334, 0.764]$	$a-c$	$(6.5-8.5\alpha, 0, 1.5+8.5\alpha, 0, 0)$	$55.50 + 59.34\alpha$
	$d-f$	$(0, 0, 5, 0)$	
$\alpha \geq 0.764$	$a-c$	$(0, 0, 8, 0, 0)$	100.85
	$d-f$	$(0, 0, 5, 0)$	

Table 4.12 Second variation : sensitivity of revenue with respect to θ

θ	OD	Flow	REV
$[0.000, 0.078]$	$a-c$	$(8, 0, 0, 0, 0)$	$7.13 + 216.07\theta$
	$d-f$	$(3.4 - 43.4\theta, 1 + 24\theta, 0, 0.6 + 19.4\theta)$	
$[0.078, 0.107]$	$a-c$	$(8, 0, 0, 0, 0)$	$12.93 + 141.73\theta$
	$d-f$	$(0, 4.4 - 19.4\theta, 0, 0.6 + 19.4\theta)$	
$[0.107, 0.123]$	$a-c$	$(9.285 - 22.25\theta, 0, -1.285 + 22.25\theta, 0, 0)$	$13 + 141.08\theta$
	$d-f$	$(0, 4.9 - 9.9\theta, 0, 0.1 + 9.9\theta)$	
$[0.123, 0.334]$	$a-c$	$(7.5 - 9.5\theta, 0, 0.5 + 9.5\theta, 0, 0)$	$13.35 + 138.23\theta$
	$d-f$	$(0, 4.9 - 9.9\theta, 0, 0.1 + 9.9\theta)$	
$[0.334, 0.495]$	$a-c$	$(12.4 - 19.4\theta, 0, -4.4 + 19.4\theta, 0, 0)$	$14.29 + 135.42\theta$
	$d-f$	$(0, 0, 5, 0)$	
$[0.495, 0.788]$	$a-c$	$(7.5 - 9.5\theta, 0, 0.5 + 9.5\theta, 0, 0)$	$48.41 + 66.54\theta$
	$d-f$	$(0, 0, 5, 0)$	
$\theta \geq 0.788$	$a-c$	$(0, 0, 8, 0, 0)$	100.85
	$d-f$	$(0, 0, 5, 0)$	

Let us focus on the first data set. In Table 4.11, we observe that the lower level solution is not unique at the critical values 0.062, 0.334, and 0.764 of the parameter α. The same situation occurs for θ at points 0.078, 0.107, 0.123, 0.334, 0.495, and 0.788 (see Table 4.12). It is interesting to note that the corresponding sensitivity curves (see Figure 4.4 and Figure 4.5) are continuous, increasing and piecewise linear, but not necessarily concave, as might have been expected. The two dotted curves, corresponding to fixed values for α_2 and θ_2, respectively, illustrate the role played by the second tariff arc bc. Indeed, higher values of the parameter α relate to an increase in path reliability. Figure 4.6 shows the variation of the revenue function, jointly in α and θ, where common values of α and θ are shown along the abscissa.

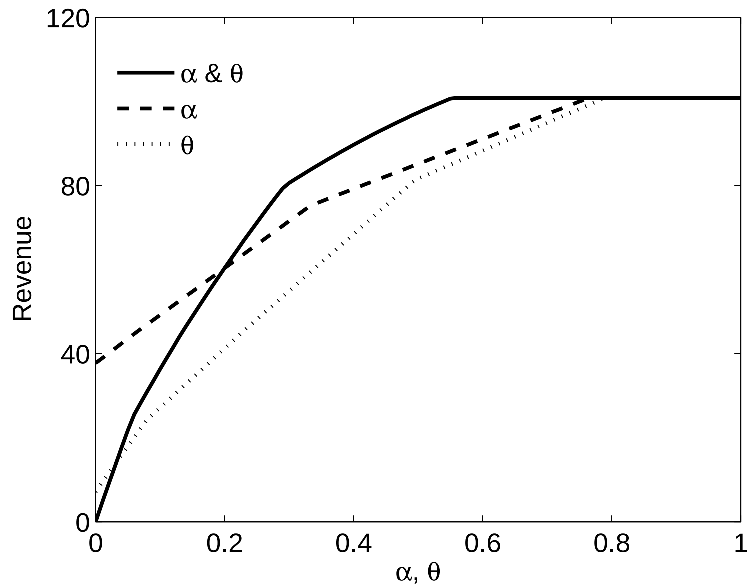

Figure 4.6 Revenue with respect to simultaneous changes in α and θ

We close this section with a depiction of sensitivity analysis with respect to tardiness and deadline arrival. Since the penalty is incurred on the leader, it is clear that revenue is a decreasing (actually continuous, piecewise linear and convex) function of $\bar{p}$. If tariff links are the most reliable, large values of $\bar{p}$ force the leader to reduce its tariffs in order to prevent the lower level flows to be assigned to less reliable paths. In contrast, revenue increases when deadline H increases. The 'max' operator involved in the tardiness term makes it nonlinear, but it need not be concave. As H goes beyond some threshold value, it becomes irrelevant, and revenue stabilizes to some maximum value. This is illustrated in Figure 4.7.

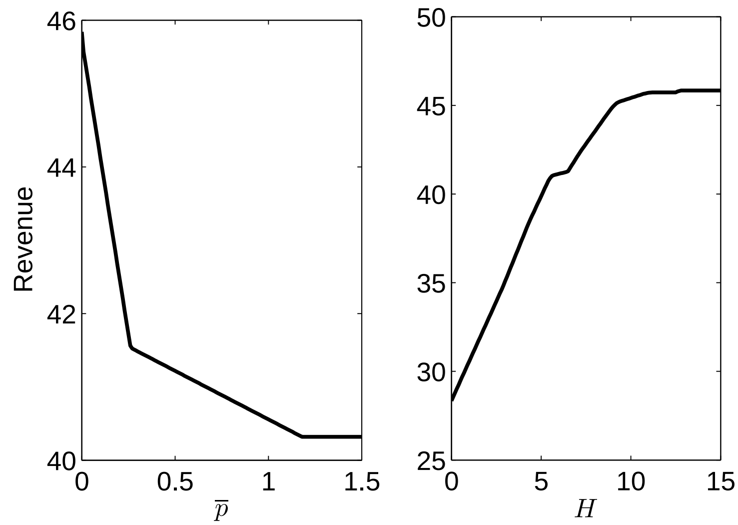

Figure 4.7 Sensitivity with respect to tardiness penalty $\bar{p}$ and deadline H

4.5 Third variation : Congestion

Congestion is a prevalent issue in urban transportation (road works, weather conditions, excess demand), as well as in telecommunications (traffic, network degradation), where it is directly related to the 'quality of service'. In contrast with the second variation, where quality of service was enforced through a chance constraint, the third variation explicitly models congestion via a volume-delay function of the BPR (Bureau of Public Roads) type :

$$v_a^{\text{arc}} \left[1 + \left(\frac{x_a}{C'_a(\boldsymbol{\xi})} \right)^m \right]$$

where, for every link a, v_a^{arc} denotes the 'free-flow' travel time, x_a the flow along link a, $C'_a(\boldsymbol{\xi})$ the random capacity of the link, and m is an exponent frequently set to the value 4. Assuming a value of one for the value-of-time parameter that converts delays into monetary units, the travel delay on path ρ takes the form

$$\begin{aligned} g_\rho(x,\boldsymbol{\xi}) &= \sum_{a\in\Lambda_1^\rho} \upsilon_a^{\text{arc}}\big[1+\big(\frac{x_a}{C'_a(\boldsymbol{\xi})}\big)^m\big] + \sum_{a\in\Lambda_2^\rho} \upsilon_a^{\text{arc}} \\ &= \sum_{a\in\Lambda_1^\rho} \upsilon_a^{\text{arc}}\big(\frac{x_a}{C'_a(\boldsymbol{\xi})}\big)^m + \sum_{a\in\Lambda^\rho} \upsilon_a^{\text{arc}}. \end{aligned} \tag{4.13}$$

Assuming that customers only integrate costs (not congestion) within their objective leads to the following bilevel formulation of the stochastic pricing problem over a network :

$$\begin{aligned} \max_{t,T} \quad & \sum_{k\in K}\sum_{\rho\in L_1^k} T_\rho f_\rho \\ \text{s.t.} \quad & \sum_{a\in\Lambda_1^\rho} t_a = T_\rho, && \forall k\in K, \forall\rho\in L_1^k, \\ & \Pr\{g_\rho(x,\boldsymbol{\xi}) \le H^k\} \ge \beta^k, && \forall k\in K, \forall\rho\in L_1^k, \\ & x_a = \sum_{k\in K}\sum_{\rho\in L^k|a\in\rho} f_\rho, && \forall a\in\Lambda_1, \\ \min_f \quad & \sum_{k\in K}\big[\sum_{\rho\in L_1^k} T_\rho f_\rho + \sum_{\rho\in L^k}\sum_{a\in\rho} c_a f_\rho\big], \\ \text{s.t.} \quad & \sum_{\rho\in L^k} f_\rho = n^k, && \forall k\in K, \\ & f_\rho \ge 0, && \forall k\in K, \forall\rho\in L^k, \end{aligned} \tag{4.14}$$

where the leader's chance constraints guarantee that travel time on the paths involving at least one tariff arc respect to the target time, with some tolerance to tardiness β^k that depends on the commodity index k. Such requirement can be satisfied by attracting flow to the reliable paths in control of the leader, striking the right balance between low tariffs (that attract flow), and tariffs sufficiently large to prevent high congestion levels that would lead to violations of the chance constraints.

In the model, we assume that the capacity of link a is distributed according to a discrete uniform random variable over the interval $[\theta C, C]$. Since the number of outcomes is finite, it is then possible to replace the chance constraints by linear constraints. To this aim, we introduce the auxiliary binary variables $u_\rho(\omega)$ for each $\omega \in \Omega'$, which is equal to 0 if and only if $g_\rho(x,\omega) \leq H^k$, together with a 'big-M' constant M. One can then enforce the tardiness constraints, for every commodity k, through the inequalities

$$g_\rho(x,\omega) - M u_\rho(\omega) \leq H^k, \tag{4.15}$$

$$E_{\boldsymbol{\xi}}\left[u_\rho(\omega)\right] \leq 1 - \beta^k. \tag{4.16}$$

This yields the mixed integer nonlinear bilevel program

$$\begin{array}{lll}
\max\limits_{u,t,T} & \sum\limits_{k\in K}\sum\limits_{\rho\in L_1^k} T_\rho f_\rho & \\
\text{s.t.} & x_a = \sum\limits_{k\in K}\sum\limits_{\rho\in L^k|a\in\rho} f_\rho, & \forall a \in \Lambda_1, \\
& \left.\begin{array}{l} \sum\limits_{a\in\Lambda_1^\rho} t_a = T_\rho, \\ E_{\boldsymbol{\xi}}\left[u_\rho(\xi(\omega))\right] \leq 1-\beta^k, \end{array}\right\} & \forall k\in K, \forall \rho\in L_1^k, \\
& \left.\begin{array}{l} g_\rho(x,\omega) - M u_\rho(\omega) \leq H^k, \\ u_\rho(\omega) \in \{0,1\}, \end{array}\right\} & \forall k\in K, \forall\rho\in L_1^k, \forall\omega\in\Omega', \\
\min\limits_{f} & \sum\limits_{k\in K}\big[\sum\limits_{\rho\in L_1^k} T_\rho f_\rho + \sum\limits_{\rho\in L^k}\sum\limits_{a\in\rho} c_a f_\rho\big], & \\
\text{s.t.} & \sum\limits_{\rho\in L^k} f_\rho = n^k, & \forall k\in K, \\
& f_\rho \geq 0, & \forall k\in K, \forall\rho\in L^k.
\end{array} \tag{4.17}$$

In order to address Program (4.17) numerically, we replace the convex and link-separable functions g_ρ by piecewise linear lower approximations over the intervals $[0,(UB_a)^m]$, where the upper bound UB_a is set to

$$UB_a = \sum_{k\in K}\sum_{\rho\in L^k|a\in\rho} n^k$$

for each $a \in \Lambda_1$. This leads to the linear constraints

$$X_a \geq q_a^s x_a + w_a^s,$$

where q_a^s and w_a^s are the coefficients of the tangent line to the function $(x_a)^m$ at the point x_a^s, that is, $q_a^s = m(x_a^s)^{m-1}$ and $w_a^s = (1-m)(x_a^s)^m$ for $a \in \Lambda_1$ and $s = 1, ..., S_a$. It follows from the convexity of g_ρ that this approximation is valid, i.e., it yields points (x_a, X_a) that lie on the graph and not strictly above the approximating curve (x_a, X_a), at optimality. This leads to the mixed integer program

$$\begin{aligned}
\max_{t,T} \quad & \sum_{k\in K}\sum_{\rho\in L_1^k} T_\rho f_\rho & & \\
\text{s.t.} \quad & X_a \geq q_a^s x_a + w_a^s, & & \forall a \in \Lambda_1, s = 1, .., S_a, \\
& \left.\begin{aligned} & x_a = \sum_{k\in K}\sum_{\rho\in L^k | a\in\rho} f_\rho, \\ & X_a \leq (UB_a)^m, \end{aligned}\right\} & & \forall a \in \Lambda_1, \\
& \left.\begin{aligned} & \sum_{a\in\Lambda_1^\rho} t_a = T_\rho, \\ & E_{\boldsymbol{\xi}}\left[u_\rho(\xi(\omega))\right] \leq 1 - \beta^k, \end{aligned}\right\} & & \forall k \in K, \forall \rho \in L_1^k, \\
& \left.\begin{aligned} & \sum_{a\in\Lambda_1^\rho} V_a(\omega) X_a - M u_\rho(\omega) \leq H^k, \\ & u_\rho(\omega) \in \{0,1\}, \end{aligned}\right\} & & \begin{aligned} & \forall k \in K, \forall \rho \in L_1^k, \\ & \forall \omega \in \Omega', \end{aligned} \\
\min_{f} \quad & \sum_{k\in K}\Big[\sum_{\rho\in L_1^k} T_\rho f_\rho + \sum_{\rho\in L^k}\sum_{a\in\rho} c_a f_\rho\Big], & & \\
\text{s.t.} \quad & \sum_{\rho\in L^k} f_\rho = n^k, & & \forall k \in K, \\
& f_\rho \geq 0, & & \forall k \in K, \forall \rho \in L^k,
\end{aligned} \tag{4.18}$$

where

$$V_a(\omega) = \upsilon_a^{\text{arc}} / \big(C'_a(\omega)\big)^m.$$

Program (4.18) can be solved through an iterative procedure based on iteratively refining the approximation of g_ρ. The algorithm is stopped as soon as the difference between to consecutive values of the objective is less than some predetermined threshold ϵ. Note that, at each iteration, the mixed integer approximation yields an upper bound on the actual revenue. Symmetrically, a piecewise linear over-approximation of the BPR curves, yielding an interpolation based on cords of the original power function, would have yielded a lower bound on the actual revenue. In both cases, the iterative solutions converge to the solution of the original solution, as the width of the subintervals involved in the partition tend to zero.

4.5.1 A numerical example

Let us consider the network and data used for the second variation. The vectors H and β are set to $(15, 11)$ and $(0.90, 0.85)$, respectively, while the BPR exponent is set to 2 (quadratic volume-delay curve). Ten outcomes of random link capacities $C'_a(\xi)$ are generated according to a uniform distribution $[\theta C_a, C_a]$, with associated probability vector

$$(0.05, 0.04, 0.10, 0.15, 0.01, 0.20, 0.15, 0.05, 0.10, 0.15).$$

Table 4.13 displays the optimal solutions of Program (4.18) for the three data sets.

Table 4.13 Third variation : optimal solutions

	OD	Flow	Tariff	REV
1	$a-c$	$(3.91, 0, 4.09, 0, 0)$	$(0, 8, 2)$	77.92
	$d-f$	$(0, 0.60, 4.40, 0)$		
2	$a-c$	$(4.51, 0, 3.49, 0, 0)$	$(-15, 17, 9)$	136.98
	$d-f$	$(0, 0, 5, 0)$		
3	$a-c$	$(4.51, 0, 3.49, 0, 0)$	$(-9, 10, 1)$	58.49
	$d-f$	$(0, 0, 5, 0)$		

For the first data set, sensitivity analysis of revenue with respect to parameters β and θ is provided in Figures 4.8, 4.9, and Table 4.14.

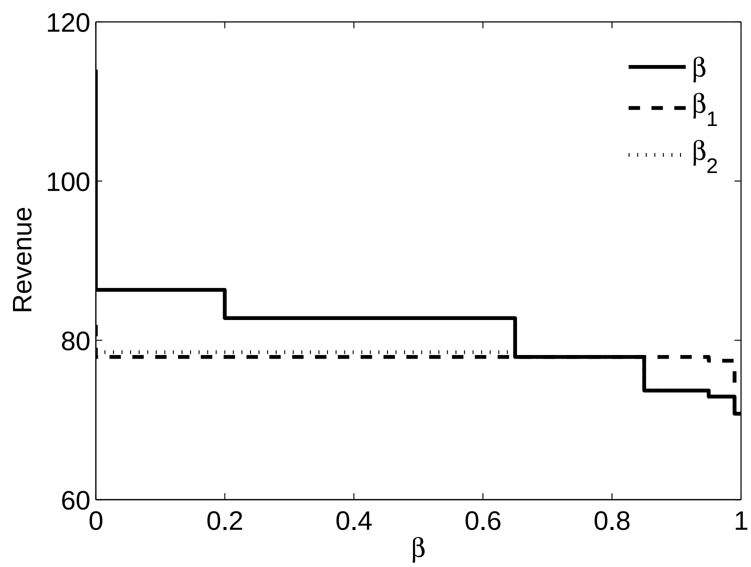

Figure 4.8 Third variation : sensitivity with respect to β

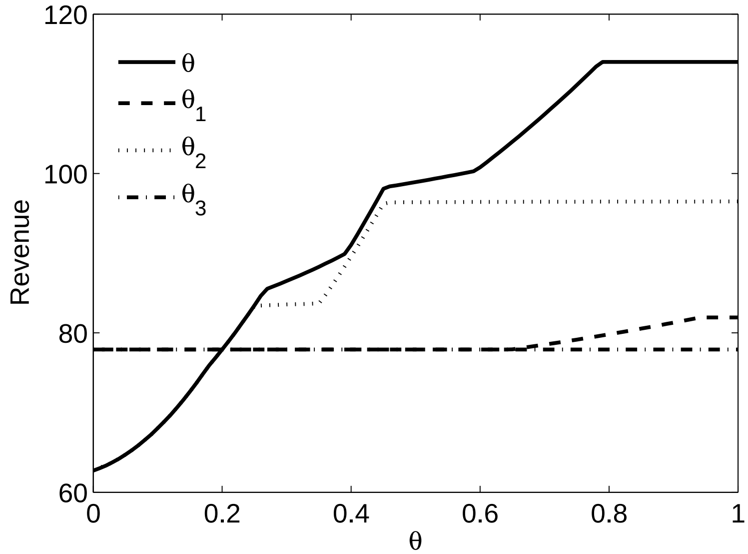

Figure 4.9 Third variation : sensitivity with respect to θ

Table 4.14 Third variation : sensitivity with respect to β (detailed flows) for the first data set (#1)

β	OD	Flow	REV
0.00	$a-c$	$(0,0,8,0,0)$	114
	$d-f$	$(0,0,5,0)$	
$(0.00,0.2]$	$a-c$	$(3.46,0,4.54,0,0)$	86.34
	$d-f$	$(0,0,5,0)$	
$(0.20,0.65]$	$a-c$	$(3.9,0,4.1,0,0)$	82.79
	$d-f$	$(0,0,5,0)$	
$(0.65,0.85]$	$a-c$	$(3.71,0,4.29,0,0)$	77.92
	$d-f$	$(0,0,5,0)$	
$(0.85,0.95]$	$a-c$	$(5.13,0,2.87,0,0)$	72.93
	$d-f$	$(0,0,5,0)$	
$(0.95,0.99]$	$a-c$	$(5.13,0,2.87,0,0)$	72.93
	$d-f$	$(0,0,5,0)$	
$(0.99,1.00]$	$a-c$	$(5.4,0,2.6,0,0)$	70.78
	$d-f$	$(0,0,5,0)$	

The revenue is a decreasing, discontinuous and piecewise linear function of the probability level β, while it is increasing and piecewise linear with respect to θ. This confirms our intuition that high values of β tighten the chance constraints related to deadlines and allow to increase tariffs on the links controlled by the leader.

Figure 4.10 illustrates the sensitivity of the revenue function with respect to simultaneous variations of α and β, both parameters taking identical values specified on the abscissa. For some critical values, such as 0.20 or 0.65, the revenue may exhibit discontinuities downwards, similar to what was observed when only β was increased. However, the rightmost jump shown on the dotted curve does not translate into a jump in the solid curve since, before the abscissa value 0.8 is reached, the constraint imposed by β is not active, and θ is large enough to allow the maximum revenue 114 to be achieved. In other words, θ 'offsets' β, chance constraints are not active and do not impact flows in Program (4.14), and the number of discontinuities in the revenue function is reduced (by one in our example).

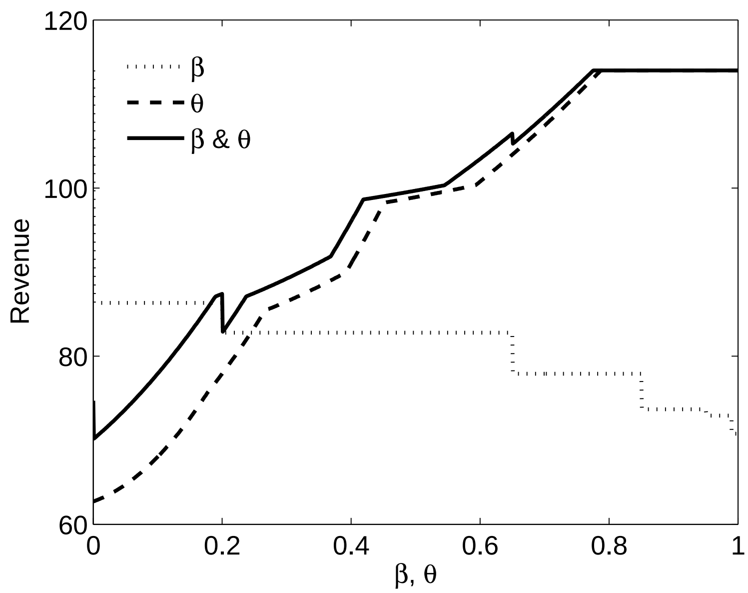

Figure 4.10 Third variation : sensitivity with respect to simultaneous changes in β and θ

Finally, Figure 4.11 illustrates the sensitivity of the revenue with respect to parameters H and C, individually, where revenue is clearly an increasing function of either of these parameters, and stabilizes at the value 114. We observe that, while the slope corresponding to the C-function is the smaller, the latter eventually overtakes the H-function, and reaches the maximal value quickly.

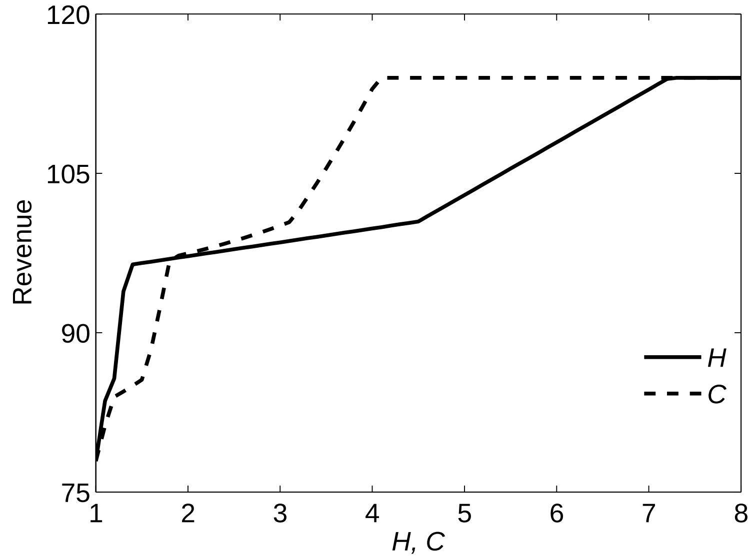

Figure 4.11 Third variation : sensitivity with respect to H and C, individually

4.6 Conclusion and Future Work

In this paper, we have considered three stochastic variations on the theme of bilevel network pricing. Through reformulations as mixed integer programs, we have been able to solve the problems for their global optima, as well as perform sensitivity analyses of the revenue function with respect to key parameters pertaining to link capacities and tardiness to destination. Since the MIP formulations do not scale well with problem size, future research will focus on algorithmic approaches, either exact or heuristic, that take advantage of the problem structure, as has been achieved for deterministic variants of bilevel pricing.

Appendix

Appendix A : Upper bounds

Two properties of the stochastic network pricing problems considered in this paper are worth mentioning.

1. Program (4.11) inherits NP-hardness from the basic deterministic case (Roch et al. (2005)). Also, the upper bound on the leader's profit derived in the deterministic case (Program (4.4)) by Labbé et al. (1998) and extended to a stochastic environment by Alizadeh et al. (2012), can also apply to the second variation (4.12). Precisely, it can be computed as $\Gamma'(\infty) - \Gamma'(0)$, where Γ' is defined as

$$\Gamma' = \sum_{k \in K} \left[\sum_{\rho \in L_1^k} \big(\sum_{a \in \rho} c_a + T_\rho'^k + \bar{p}^k E_{\boldsymbol{\xi}} \left[\max\left\{0, g_\rho(\xi) - H^k\right\}\right]\big) f_\rho + \sum_{\substack{\rho \in L^k \\ a \in \rho}} c_a f_\rho \right]$$

with

$$T_\rho' = T_\rho - \bar{p}^k E_{\boldsymbol{\xi}} \left[\max\left\{0, g_\rho(\xi) - H^k\right\}\right].$$

A related but slightly more complex upper bound can also be derived for the third variation.

2. Variations two and three are reformulated as deterministic bilevel programs. It follows (the proof is straightforward) that their objectives are continuous functions of the target capacity vector C.

CHAPITRE 5

GENERAL DISCUSSION and CONCLUSION

In this thesis, we studied the network pricing problem with a BP structure under uncertainty. First, we presented (in Chapter 3) a two-stage stochastic bilevel pricing problem and its reformulation as a single-stage SBP. We focused on sensitivity analysis with respect to the constraints linking the tariffs at the two stages of the stochastic program. We considered two forms of predetermined threshold restrictions (absolute restriction (AR) and proportional restriction (PR)) on each tariff arc of the network in the link constraints of the first- and second-stage tariffs. We showed that the value function of our model is a continuous and piecewise linear function in the AR case and a continuous piecewise hyperbolic in the PR case. The numerical results show that the randomness of the fixed costs plays a more important role than the randomness of the demands in the difficulty of different variations of the model. There are several avenues for future research. They include the theoretical analysis of the situation involving continuous random variables, and the development of a numerical approach that takes advantage of the network structure and does not rely on a straightforward extension of the techniques developed in the deterministic case. On the modeling side, the extension of the model to an arbitrary number of stages or to a two-stage SB pricing model with capacity constraints, either in closed loop or open loop ("Stackelberg feedback"), poses formidable challenges, both from the theoretical and computational points of view.

Second, we introduced (in Chapter 4) three variations of stochastic bilevel pricing problems based on real-life features, extending a standard framework that has been the topic of several studies. Through reformulation as mixed integer programs, we have been able to find global optima for these problems, and to perform sensitivity analyses of the revenue function with respect to key parameters pertaining to link capacities and tardiness at destination. Theoretical and numerical results showed that the value function of the second variation is a continuous function with respect to the design-capacity proportion parameter. We showed that increasing the design-capacity proportion parameters, especially for those links that play important roles in attracting users, increases the revenue significantly. Moreover, the value function of the second variation is a discontinuous function with respect to the probability level when we assume a discrete distribution for the random capacity variables to reformulate the chance constraints as linear constraints. Finally, since the MIP formulations do not scale well with problem size, future research will have to focus on algorithmic approaches, either exact or heuristic (such as primal-dual procedures), that take advantage of the problem structure, as has been done for deterministic variants of bilevel pricing.

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