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Auteurs: Authors:	Philippe R. Richard, Michel Gagnon, Josep Maria Fortuny, Nicolas Leduc, & Michèle Tessier-Baillargeon
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Means of Choice for Interactive Management of Dynamic Geometry Problems Based on Instrumented Behaviour

Philippe R. Richard[1], Michel Gagnon[2], Josep Maria Fortuny[3], Nicolas Leduc[2], Michèle Tessier-Baillargeon[1]
[1]Département de didactique, Université de Montréal, Canada
[2]Département de génie informatique et génie logiciel, École Polytechnique de Montréal, Canada
[3]Departament de Didàctica de les Matemàtiques i de les Ciències Experimentals, Universitat Autònoma de Barcelona, Spain
Email: philippe.r.richard@umontreal.ca, michel.gagnon@polymtl.ca, josepmaria.fortuny@uab.es

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ABSTRACT

Our paper presents a project that involves two research questions: does the choice of a related problem by the tutorial system allow the problem solving process which is blocked for the student to be restarted? What information about learning do related problems returned by the system provide us? We answer the first question according to the didactic engineering, whose mode of validation is internal and based on the confrontation between an *a priori* analysis and an *a posteriori* analysis that relies on data from experiments in schools. We consider the student as a subject whose adaptation processes are conditioned by the problem and the possible interactions with the computer environment, and also by his knowledge, usually implicit, of the institutional norms that condition his relationship with geometry. Choosing a set of good problems within the system is therefore an essential element of the learning model. Since the source of a problem depends on the student's actions with the computer tool, it is necessary to wait and see what are the related to problems that are returned to him before being able to identify patterns and assess the learning. With the simultaneity of collecting and analysing interactions in each class, we answer the second question according to a grounded theory analysis. By approaching the problems posed by the system and the designs in play at learning blockages, our analysis links the characteristics of problems to the design components in order to theorize on the decisional, epistemological, representational, didactic and instrumental aspects of the subject-milieu system in interaction.

Keywords: Didactics of Mathematics; Competencies; Geometric Thinking; Tutorial System; Related Problems; Dynamic Geometry; Instrumented Behavior; Cognitive Interactions; Conceptions; Mathematical Work Space; Means of Choice; Didactic Contract

1. Foreword

In the third year of secondary school, two students tried to solve a problem of proof at the interface of an interactive tutorial system. It was to compare the area of two triangles in a parallelogram and to proof the assumption made. After reading the statement and constructing or moving the elements of the figure in the dynamic geometry module (**Figure 1**), the students quickly agree on equal areas. They began to create a mathematical proof on the tutorial system interface and were therefore delighted to see that Prof. Turing, an artificial tutor agent, indicated with a smiley that their first intuition was well founded. Even though they were good students, they sometimes got stuck in their mathematical proof. Happily, with his messages, Prof. Turing was always successful in reviving the solution process. Without replacing the teacher, this tutor agent has 69,000 potential solutions "in mind" and was quickly able to target the solution envisaged by the students, thereby providing personalized support. Prof. Turing also knows how to recognize a student's persistent difficulty and can suggest that he get help from his teacher. Furthermore, once he arrives, the teacher sees what has happened from the messages received but instead of insisting on their meaning in the context of the problem he rather asks that a new problem be solved. The students launched on paper without too much difficulty then one said to his companion: "look, I've got it... look, this is why it works!" And the solution to the original problem is relaunched. The use of a related problem therefore is a means of choice for this didactic system. Can they be made available to Prof. Turing?

2. Introduction

According to the theory of didactic situations, we know that the only way to "do" mathematics is to try and solve some specific problems and in this regard asking new questions. The teacher must therefore not communicate knowledge but pass on the right problem. If this transfer happens, the student plays the game and ends up winning, while learning takes place. But what if the student rejects or avoids the problem, or doesn't solve it? The teacher then has the social obligation to help him [7]. Set in didactics of mathematics, our research project is based on three key concepts: on the necessity of seeking and resolving specific problems for learning geometry in high school, on the assistance that makes up a transfer of the "right problems" in a context of instrumented learning, and on the voluntary but surprising action of the teacher who chooses to set a problem as a message to help a student whose solving of an initial problem remains blocked.

Instrumented learning is based on the use by the student of a tutorial system created by our research team for learning geometry. This system supports the student in solving problems of proof, issuing messages as needed (verbal or iconic expressions) appropriate to the actions of the student in the internal logic of problems. During a validation phase of previous research (see next section), the introduction of a support structure that incorporates a set of related problems appeared necessary to acquire the means of choice that the teacher discusses with his students. Unlike existing approaches, these problems do not divide the original problem into sub-strategies. With a completely new approach, the new problems arise from the characteristics of relationships between problems and learning blockages, engendering new decision means for the tutorial system.

3. Research Program

In this section, key words are in bold.

Based on the **didactics of mathematics**, our project is a continuation of the project *a new approach to research on competential and instrumented learning of geometry in high school* (CRSH 410-2009-0179) and it renews the foundations laid down in the article *Didactic and theoretical-based perspectives in the experimental development of an intelligent tutorial system for the learning of geometry* [41]. These works were common to the design of geogebraTUTOR, a **tutorial system** which is intended to support the development of students' **mathematical competencies** [29, 46] and the **construction**

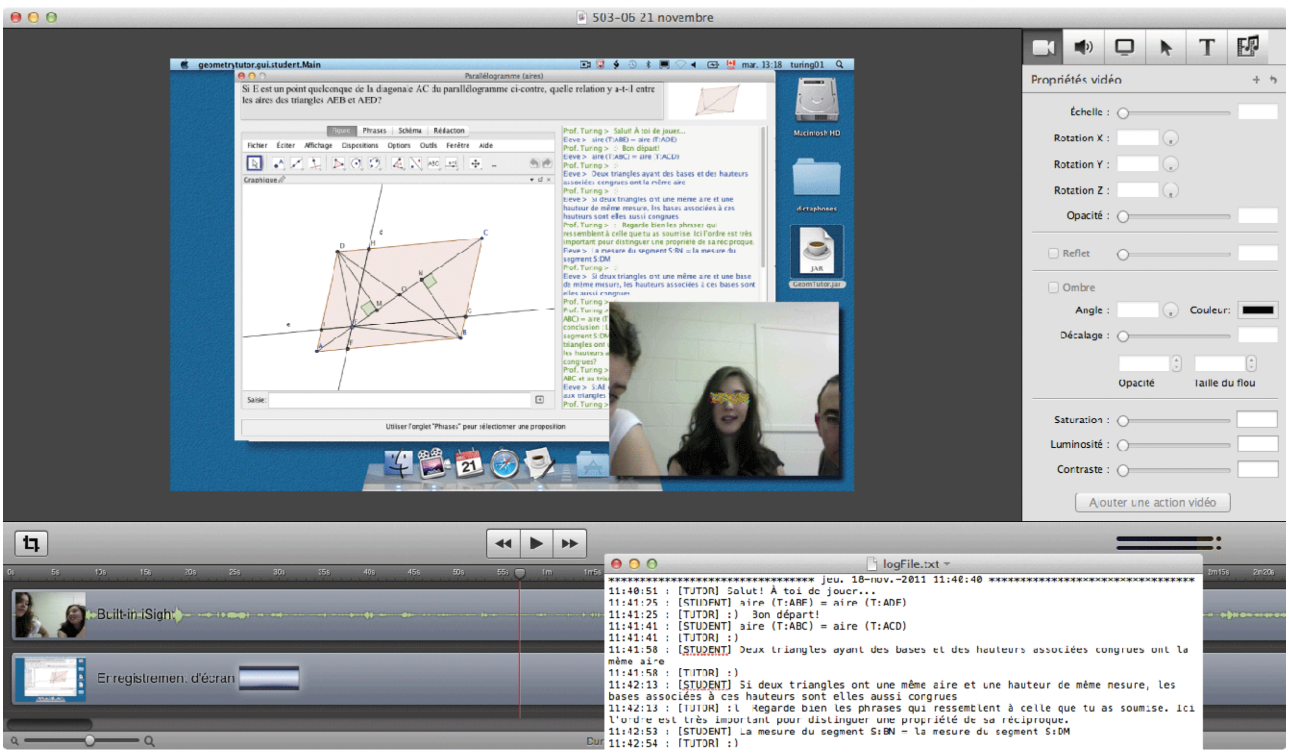

Figure 1. Analysis feature of interactions during solving of the parallelogram problem. In the background is the ScreenFlow software interface (recording sound, image and interaction on the screen) and in the foreground the log of the conversation between the student and the artificial tutor agent. On the student's screen, the "GeomTutor" Java applet launches the tutorial system and the "dictation" file saves the simultaneous recording of the teacher's intervention. The image used here shows the geometric (on the left) and discursive (on the right) modules but is hiding the modules for writing ("Statements" tab), structured arguments training ("Outline") and mathematical proof ("Writing"). For more information about the system's challenges, consult the video at http://www.matimtl.ca/evenements/evenement.jsp?id=106.

of geometric thinking [19, 22, 47]. It consists of two subsystems, Turing[1] (FQRSC 2005-AI-97435) and GeoGebra (http://www.geogebra.org/), a **dynamic geometry software** whose international influence is considerable in teaching mathematics and which includes a three dimensional geometry module in evolution [6]. By having to account for teacher intervention [49, 50], we have enriched our research program with assessment tools developed within Intergeo (http://i2geo.net), a consortium that manages a platform for sharing and assessment of the quality of resources to which our secondary mathematics teaching students have already contributed [51]. On the basis of these achievements, the current project still aims to improve learning but it now innovates by the original consideration of a structure of **related problems** which meets these student learning blockages, with a view to **instrumented behaviour** and whose reference geometry allows adaptation to actual class didactic contracts. Our reference to the decision-making theory of Schoenfeld [48] sheds light on both the resources, goals and orientations of the teacher intervention and the tutorial action, and the notions of **conceptions** [5] and **mathematical work space** [21] pose an epistemological, semiotic and instrumental view of cognitive interactions that emerge from the student **cognitive interaction** with the milieu. The notion of **means of choice** generalises the teacher's judgements and the decisions of the tutorial system when it returns a related problem following a learning blockage by the student, and that of a **didactic contract** designates the most frequently implicit expectations that there are for the student and the teacher respectively concerning mathematical knowledge.

4. Objectives

In seeking to better understand the means of choice for interactive management of dynamic geometry problems based on instrumented behaviour in high school, our research project has the objectives in the **Table 1**.

The idea of means of choice, as a voluntary action to consider one problem rather than another, transposes into decision making means within the tutorial system. We consider the instructional model (also known as instructional design) on two levels, that of didactics, the responsibility of active educational members and those in training, and that of the technology of computer programming and deployment, the responsibility of members with technological training (see the section *Research team, latest results and student training*). The achievement of our research objectives will have a direct affect on teacher training (**Table 2**).

[1] French acronym for TUtoRiel Intelligent en Géométrie and a nod to the engineer and mathematician Alan Mathison Turing.

5. Background

Well beyond the establishment of a simple online exerciser or a learning guide in a deterministic MOOC structure [16], our research project is based on a computerised environment for complex human learning which is in an international dynamic geometry movement where community development and sharing of reference activities are carried out among experts and teachers from different traditions. The idea of a tutorial system that supports students in problems of proofs is not new. Among first generation achievements, we can mention the Geometry Proofs Tutor [3], the Tigre-Mentoniezh project [32] and Géométrix (http://geometrix.free.fr/ by Jacques Gressier). All these systems are based on formal geometry models that, despite an evident advantage of computer programming, presuppose development in geometric thought

Table 1. The three general objectives.

Objective 1 Instructional model	• Design, index, implement and test a structure of related problems in a tutorial system (geogebraTUTOR) which is based on the means of usual choice for teacher intervention and the instrumented behaviour of the student during solving of fundamental problems involving proofs.
Objective 2 Interpretation and theorizing	• Interpret and theorize on the decisional, epistemological, representational, didactic and instrumental aspects of the subject-milieu system in interaction, with reference to the student's conceptions and the mathematical workspace.
Objective 3 Assessment and control	• Assess the consistency of the subject-milieu system in interaction, with reference to the development of mathematical competencies, construction of geometric thought and the student's learning in an instrumented perspective.

Table 2. The two training objectives.

Initial training	• Support, by trainees (university students), of a part of the cognitive, heuristic, semiotic and metamathematical means made available to students during simulated situations, to develop their ability to identify with what the student knows and, reciprocally, to test their teaching action.
Continuing education	Development of disciplinary competencies in geometry, as professional successor, critiques and interprets from its subjects or culture, in the exercise of his functions.

- Richard, Cobo, Fortuny and Hohenwarter [40]
- Trgalová, Richard and Soury-Lavergne [52]

by adherence to an axiomatic approach. Assisting student learning blockages is therefore formal. The same applies to second generation systems, although the interface, communication with the user and processing of significant actions are more developed. Among systems similar to geogebraTUTOR [42] should be mentioned the Advanced Geometry Tutor [28], the Baghera project [23], the Cabri-Euclide microworld [25] and the Geometry Explanation Tutor [1]. There is also the Andes Physics Tutor [54], for which some situations-problems are already premodeled in geometry.

Among the precursor achievements to our current system are AgentGeom [10] and Turing [39]. Unlike the systems which relate to formal axiomatic geometry, AgentGeom and Turing were developed on cognitive geometry models that lie between natural geometry and the axiomatic natural geometry of Kuzniak [20]. This allows full originality when considering situations-problems that bring together physical sciences to the process of discovery in mathematics [12], as Clairaut [9] appeared to desire in the Enlightenment by stating "this presumed induction carries its demonstration with it" (p. 64), following the representation on paper of a "dynamic" geometric figure [43]. In addition, although cognitive geometry is essential to considering reference geometry that is effectively practiced in the classroom, it allows for constitution of a structure of related problems that respond to informal learning blockages.

6. References Axes of the Conceptual Framework

6.1. Epistemological Axis

In the process of mathematical discovery, the epistemological dialectic of proofs and refutations of [24] considers the criticisms that arise with counter-examples in discussions between students and teacher. These criticisms are likely to require an adjustment of the conjecture, the proof or the counter-example itself, and also of knowledge and the problem. Although they can be seen as "breakpoints" [27] in solving a problem of proof, this is because like Lakatos we believe that the steps of proof are only summarised with formal or deductive approaches. In addition, we know that the supporting role of these criticisms can be incorporated into the steps of instrumented reasoning [18], while allowing the creation of a geometric workspace in continuity with the development of mathematical competencies inherited from elementary school [11]. Our desire to bring together the epistemology of mathematics into training programs, so the student can perform his work in geometry, is an attempt for subtle adaptation between an actual state of his mathematical competencies and the intrinsic requirement for performing geometry in class.

6.2. Semiotic Axis

When a study examines dynamic geometry and instrumented reasoning, questions on communications, processing of cognitive representations and objectivities of virtual representations are essential. The Duval's theory of language functions [13] sets out the conditions for learning based on the coordination of representation registers, of which the register of figures [14], and the functional-structural approach of Richard and Sierpinska [44] insists on the traditional semiotic means serving the quality of communications. When the student acts on a dynamic drawing, he is also acting on the system of representation, possibly for the communication of inductive reasoning [45]. This action can be the source of a learning blockage and, although the figural representations convey reasoning [36, 37] or simulate movement [2], they cannot generate it. In this context, a dynamic figure is also a kind of problem.

6.3. Situational Axis

In Brousseau's theory of didactical situations in mathematics [7], the main intervention of the teacher (arrow 1, **Figure 2**) occurs within a system which is itself in interaction, the student-milieu system (2), but with a didactic milieu which brings a tutor agent to him the role of arrows 1 and 2 is transposed with 6 and 7. According to [26], Brousseau will consider the subject-milieu interaction as the smallest unit of cognitive interactions. A state of equilibrium for this interaction defines a state of knowledge, the subject-milieu imbalance producing new knowledge (search for a new equilibrium)". However, the theory of didactic situations characterises each item of knowledge by situations that are specific to it, and the knowledge model of Balacheff and Margolinas [5] locates conceptions in the subject-milieu interaction, while first characterising a conception by the problems in which it is involved. If it results in a strong conceptual relationship between a moment of learning blockage and a related problem, it is also because a learning blockage is a breach of contract with what is expected in the context of the root problem and that the appearance of a new problem, in addition to relaunching the solving process, is not a sophisticated response suggestion which at its core considers the objective of the root problem.

6.4. Instrumental Axis

In Rabardel's theory of instrumentation [34], the instrument is the mode of action or thought constructed by the subject when he uses a tool; the manner in which the instrument is formed in the subject is the instrumental genesis. According to [21], instrumental genesis constitutes the geometric workspace and, when it is considered in the process of student-milieu interaction which creates its own space, instrumental genesis operates both during stages of discovery and validation [11]. Since the processes of instrumentation relate to the emergence and development of patterns of use and instrumented action, the progressive discovery of the tool's intrinsic properties by students, for whom the appearance of a related problem following a learning blockage is accompanied by the accommodation of their patterns and also changes in meaning of the instrument, results in association of the tool with new patterns [35]. The notions of conception and instrument occupy dual places in modelling of the subject-milieu [5].

6.5. Decisional Axis

The transposition of the teacher's intervention in the educational environment is necessarily accompanied by a transposition of means of the choice. Although these means can be interpreted in respect of a didactic contract, the implicitly shared significance that it supposes complicates understanding of the decisional process in all of the fundamental relationships. According to [7], the didactic contract is not really a true contract, since it is not explicit or voluntary and because neither the conditions for breach nor sanctions can be given in advance due to their didactic nature, which depends significantly on a knowledge of students that is as yet unknown. In Schoenfeld's decision making theory [48], if we know enough about what someone's resources, goals and orientations are, teacher or student, we can even come to understand, explain and model actions and decisions that seem unusual or abnormal. In the Introduction paragraph, the illustration of the sudden appearance of a new request for solving problem, while it was the first solving processed that was blocked, may surprise the observer but not the students, who are used to reacting to this type of requirement from their teacher.

7. Methodology

7.1. General Direction

As with our previous projects, we take advantage of a similar experimental effort to "verify a state or a change" and "develop accordingly". In fact, the illustration in the Foreword comes from an experiment (see the link below **Figure 1** and [50] which allowed, firstly, verification of the validity of a structure of messages from the tutor agent when two classes of high school students resolved problems of proof and, secondly, identification of the means of choice of their teachers with the use of related problems after indexation of conceptual, heuristic, semiotic and met mathematical criteria [42]. This requirement invites us to consider together two paradigms on the epistemological didactic level. On the conceptual side, the learning models that we claim identify with the use of our tutorial system based on the preceding axes. On the methodological side, the approach implemented combines the didactic engineering of Artigue [4] and the grounded theory analysis of Glaser and Strauss [17]. Since the functional integration of a structure of related problems in the instructional model is in line with our project (see *Acknowledgements*), we can achieve objectives 1 and 3 with a confirmatory model [53] of the hypothesis-deduction type [4]. But the intervention of these problems depends on the instrumented behaviour, and since they generate a non-deterministic learning itinerary (following related problems), the emergence of learning models adapted to the use of an advanced tutorial system commits us to achieving objective 2 using a comparative-inductive type model [17].

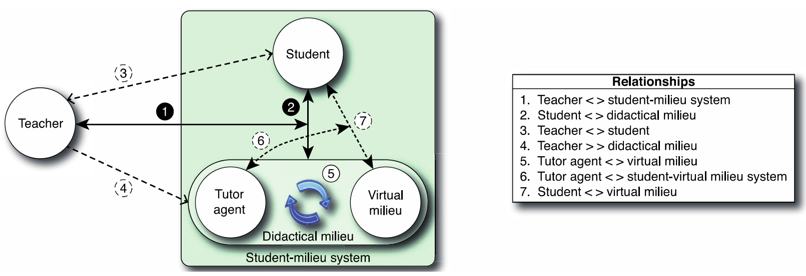

Figure 2. Fundamental relationships in the research system. It is an adaptation of [7] on the relationship of 1 to 4 between the teacher, the student and the milieu [41].

7.2. Description of Key Activities and Specific Procedures

Here is a brief description of phases of the research in relation to the objectives, for trimester Q.

Phase 1 (Q_{1-2}) ↓ Conception of the instructional model– preparation of objective 1 (1^{st} part)

•From the 34 problems based on [33] and adaptation of these to the curriculum in effect in the regions of the participating researchers, characterisation of problems according to the identity of problems [38] and indexation of them according to the above criteria.

•Implementation in geogebraTUTOR of a structure of related problems and testing experts on the functioning of the didactic milieu according to the learning model of [41].

Phase 2 (Q_{3-4}) ↓ 1st experiment in the schools – achievement of objective 1 (1st part)

To validate by cross-referencing primitive sources – or "triangulation" [15]:

•Collection of student-milieu interactions and teacher interventions in 3 classes of second cycle of high school in three different regions when, in their normal courses, students solve five problems of proof using the system interface (qualitative data, [30]).

•Request to teachers to reconstruct their interventions and compare the means of choice of the tutorial system with the didactic contract of the course [31, 8] during 50 minute explanatory discussions [55].

In addition to the log files, there will be "ScreenFlow" records (see **Figure 1**) for a sample of 24 volunteer students (4 teams of 2 per class), as well as audio recordings for the teachers of each class (intervention in class and explanatory discussions), in accordance with the ethical rules in force – ditto for entering and processing data in all phases.

Phase 3 (Q_{5-6}) ↓ Analysis and interpretation – preparation of objectives 2 and 3 (1st part)

•Analysis of times of learning blockage – or breakpoints – according to the knowledge model of Balacheff and Margolinas [5] by characterising conceptions **C** by a defining set of problems (P) for which solving tools (R) are provided based on representation systems (L) and a control structure (Σ) which permits judgements and decisions.

•Interpretation of times of learning blockage by joining the student's conceptions **C** = (P, R, L, Σ) with the characteristics of the related problems in play (of the system, phase 1; of teachers, phase 2).

Phase 4 (Q_{7-8}) ↓ Cross-referencing, validation and fine-tuning – preparation for objective 1 (2nd part) and achievement of objective 2 (1st part)

•Pooling of the results of phase 3 between researchers and identification of patterns.

•To improve the instructional model and better understand the common or invariant characteristics of the subject-milieu system in interaction, expert validation of the combination of phase 3.

•Optimisation of the articulation and continuity between the conception of the tutorial system and pursuit of the conception during solving by the students (or "conception in use", [34]).

Phase 5 (Q_{9-10}) ↓ 2nd experiment in the schools – preparation of objective 3 (2nd part) and achievement of objective 1 (2nd part)

•Resumption of the procedures for phase 2, by this time asking teachers to assess the solutions (without noting them) with a view to assessing competencies [38].

Phase 6 (Q_{11-12}) ↓ Modelling, synthesis, theorisation – achievement of objectives 2 (2nd part) and 3

•Modelling of an ontology on the basis of didactic contracts and instrumented behaviour.

•Summary of the non-deterministic learning itineraries and underlying means of choice.

•Theorisation on the fundamental relationships of the instrumented didactic situation (**Figure 2**).

8. Knowledge Mobilisation Plan

As a human sciences discipline, the teaching of mathematics has a scientific side and a professional side, including the initial and continuing training of teachers in the field. When a project involves not only teaching theories but also construction of computer environments for human learning, questions arise in a very practical way, which leads to seeking a functional modelling of knowledge by making distinctions that are useful feedback for the teaching of mathematics as a whole. Our plan for knowledge mobilisation, similar to that which is grounded in our research program, aims to continue the multidirectional exchange of knowledge between researchers, teachers and other persons involved in the world of teaching mathematics, in a collaborative spirit of sharing which includes quality, integration and popularisation. We can summarise the overall mobilisation plan in terms of *places*[2]:

•Publication in journals for the quality of the research (ESM, IJCML, ZDM, etc.), its integration (ADSC, REC, PME, etc.) and its popularisation (AMQ, PME, UNO, etc.);

•Organisation and participation in international symposia on teaching mathematics (EMF, ETM, CERME, etc.) as well as mathematics classroom technologies (INTERGEO, CADGME, E-LEARN, etc.);

•Initial and continuing training sessions for teachers (UDM, UAB, GRMS, etc.);

•Participation as appropriate in advisory bodies on training or technology programs (UDM, CCPÉ, MATI, etc.).

and of *means*, thanks to advanced technological skilled of team members:

•Community and sharing development platform for research quality (Turing, cK¢ wikibook, etc.), its integration (I2GÉO, NTLMP, etc.) and its popularisation

[2]We list acronyms and websites in the section References.

(GIC-IGC, GeoGebraTube, etc.).

We are also planning to organise a workshop during phase 3 in collaboration with other research groups, so that researchers and teachers associated with a project can share their experiences following the first experimental phase. The formula proposed is a symposium, like the one we organised most recently in Montreal http://turing.scedu.umontreal.ca/etm/, in the collaborative spirit typical of the *Congress of European Research in Mathematics Education* http://www.cerme8.metu.edu.tr, which balances quality (of seasoned researchers) and integration (of young researchers).

9. Results Expected

In the *Knowledge mobilisation plan* section, we insisted on the fact that the mutual contribution of didactics and computers generates research advantages and impacts within the university environment, with teachers and other stakeholders in the educational world, through interactions and increased access during the research itself.

By creating an "in use" tutorial system (within the meaning of [34]), our research approach is empirically based on the articulation and continuity between the institutional system design processes and the pursuit of the design in problem solving by the student. Since it was designed to produce a class of effects (support for learning through messages, problems and controls), implementation of the system, under the conditions provided for each phase of the project (see *Detailed description* section), allows updating of these effects following usage noted during experimental phases. In other words, if the cognitive outcome constitutes the design of the tutorial system, it is the source of its own existence by an expert anticipation of interactions of a of a changing student-milieu system. Unlike existing tutorial systems (see *Background* section), our choice of cognitive geometry is a significant mark of originality since it lets us both adapt the means of choice of the student's instrumented behaviour and integrate the specifics of authentic didactic contracts.

The idea of meeting a student's learning blockage by providing timely related problems to solve is an effective solution to one of the major difficulties in teaching: avoiding giving answers (discursive messages) at the same time as the questions (root problems). In this sense, our project theoretically answers the first didactic paradox of [7]: everything the teacher does to produce the expected behaviours by students tends to reduce the student's uncertainty and thereby deprives him of the necessary conditions for understanding and learning the intended concept; if the teacher tells or signifies what he wants from the student he can no longer get it other than as performance of an instruction and not by the exercise of his knowledge and judgement.

Apart from the institutional requirement for student training, the dissemination of knowledge and the influence on the community of researchers in the field, the integration of multidisciplinary doctoral research and the effective collaboration of the school institution remains a strategic advantage of the project, as is its influence on teachers practise and training. Whether first to improve students' geometrical skills, including deductive (reasoning, arguing), visual (observing, exploring), figural (modelling, conjecturing, defining) and operational (instrumentation, manipulation) skills, the potential for development of the tutorial system then allows the teacher to adapt a port of his pedagogical engineering according to the division of his responsibilities with those of the student. In initial training, these same arrangements sharpen students' abilities to simulate the effect of their teaching activities and to identify student's behaviour, since the anticipation of solutions up to planning (and not programming) learning itineraries adapted to the student.

The material benefits of our project are intended for public use in schools.

10. Conclusions: Four Centres of Originality

Although the current project is a continuation of the founding projects, it remains profoundly original in relation to it. We conclude by noting here four centres of originality.

A first centre relates to the organisation of a structure of related problems that responds to times of learning blockage by the student. Although desirable, this type of structure is unusual in math classes, since it is difficult to put in place in a paper-pencil environment and even if the use of related problems occasionally happens with some teachers, the choice of the problem remains limited by the environment. In addition, we know of no geometry tutorial system that integrates such organisation to restart a block problem solving process.

A second centre affects the joint consideration of the approaches to mathematical discovery and proof which, in the same context as instrumented learning, links the epistemology of mathematics with training programs. Attached to a reference geometry based first on the meaning of objects that it models and which makes as such an approximation possible, geometry becomes a means of learning and not longer its object, as is found with tutorial systems which develop geometric thinking by taking on an axiomatic approach.

A third centre relates to the functional modelling of knowledge in our tutorial system. Most of the time, a human learning computer environment is validated by comparing the results of a pre-test and a post-test, while requiring the user to comply with the system as it was designed. This attitude undoubtedly leads fairly quickly to concrete accomplishments, but it is necessary for these accomplishments to be effective learning aids. Although our tutorial system aims for effectiveness of the tutorial activity by first considering a modelling of human behaviour and designing a computer device which takes this model account, it is due to the structure of related problems is part of a learning model, distinct from the model of assessing mathematical competencies in an instrumented perspective.

A fourth centre looks at the non-deterministic character of the system and considers a large number of solutions. When a related problem is chosen, it is not so much because we know exactly why the student was blocked but because we suppose that the student knows what is expected of him and in return the teacher or the tutorial system knows the logic of the problem. The question of correlation between related a problem and a learning blockage is not deterministic since the system does not indicate how to proceed. It follows the student in his reasoning (by comparison with expert solutions generally in the range of 50-100,000), regardless if it belongs to a moment of discovery or proof and it invites him to remain in the logic of the situation, simultaneously offering personalised assistance based on the instrumented behaviour of each student in a single class.

11. Acknowledgements

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GeoGebraTube	http://www.geogebratube.org
GDM	http://turing.scedu.umontreal.ca/gdm/
GIC-IGC	http://www.geogebracanada.org/home
I2GÉO	http://i2geo.net (access to activities for registrants)
Turing	http://turing.scedu.umontreal.ca (access to Moodle for registrants)

Acronym reference

AC-GGB	Associació Catalana de geogebra
ADSC	Annales de didactique et de sciences cognitives
AMQ	Bulletin of the Association mathématique du Québec
CADGME	Computer Algebra and Dynamic Geometry Systems in Mathematics Education
CCPÉ	Advisory Committee on the curriculum of the Quebec Ministry of Education
CERME	Congress of European Research in Mathematics Education
CRSH	Conseil de recherche en sciences humaines du Canada
E-LEARN	World Conference on E-Learning in Corporate, Government, Healthcare & Higher Education
EMF	Espace mathématique francophone Symposium
ESM	Educational Studies in Mathematics – An International Journal
ETM	Symposium Espace de travail mathématique
GRMS	Group of leaders in high school mathematics
IJCML	International Journal of Computers for Mathematical Learning
INTERGEO	Interoperable Interactive Geometry Conference
MATI	Roland-Giguère training and learning technologies House
NTLMP	International Newsletter on the Teaching and Learning of Mathematical Proof
PME	Publicaciones del Ministerio de Educación de España
REC	Revista enseñanza de las ciencias – Investigación y experiencias didácticas
UAB	Universitat Autònoma de Barcelona
UDM	Université de Montréal
UNO	Revista Uno – Didáctica de las matemáticas
ZDM	Zentralblatt für Didaktik der Mathematik – The International Journal on Mathematics Education

