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Relative Stability in the Sup-norm and Input-to-state Stability in the Spatial Sup-norm for Parabolic PDEs

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***Abstract*— In this paper, we introduce the notion of *relative $\mathcal{K}$-equi-stability* (RKES) to characterize the uniformly continuous dependence of (weak) solutions on external disturbances for nonlinear parabolic PDE systems. Based on the RKES, we prove the input-to-state stability (ISS) in the spatial sup-norm for a class of nonlinear parabolic PDEs with either Dirichlet or Robin boundary disturbances. An example concerned with a super-linear parabolic PDE with Robin boundary condition is provided to illustrate the obtained ISS results. Besides, as an application of the notion of RKES, we conduct stability analysis for a class of parabolic PDEs in cascade coupled over the domain or on the boundary of the domain, in the spatial and time sup-norm, and in the spatial sup-norm, respectively. The technique of De Giorgi iteration is extensively used in the proof of the results presented in this paper.**

***Index Terms*— Nonlinear PDEs, relative stability, input-to-state stability, De Giorgi iteration, cascade of PDE systems.**

I. Introduction

ORIGINALLY introduced by Sontag in the late 1980s, the notion of *input-to-state stability* (ISS) has been proven to be a convenient tool for describing robust stability of finite dimensional systems with external inputs. The pioneering work on extending the application of ISS to infinite dimensional systems is owe to [3]–[5], [10], [18], [19], [23], [24], etc., where different methods were proposed for constructing ISS-Lyapunov functionals for parabolic PDEs, or hyperbolic PDES, or abstract equations in Banach spaces. Since then, the ISS of PDE systems has drawn much attention in the literature of PDE control. It is worth noting that applying the classical regularity theory of PDEs to ISS analysis of PDEs having only in-domain disturbances seems to be straightforward, while it is a challenge to establish the ISS for PDEs that have external disturbances distributed on the boundary of the domain.

In recent years, a great effort has been devoted to establishing the ISS for PDEs with boundary disturbances; see [13], [21] for comprehensive surveys on this topic, and [29], [30] for a summary of different approaches for establishing the ISS of PDEs with boundary disturbances. Among the existing literature, the ISS in L^1-norm and L^q-norm with $q \in [2, +\infty)$ has been well studied for PDEs with boundary disturbances via different methods; see, e.g., [8], [9], [11], [20], [22], [25], [27]–[30]. For example, in a recent work on the characterizations of the ISS for abstract infinite dimensional systems [8], explicit constructions of noncoercive Lyapunov functionals were presented for linear systems with unbounded admissible input operators. Consequences of the results obtained in [8] include that the ISS in the norm of Banach spaces, comprising the ISS in the spatial norm, can be characterized for a wide class of PDEs with boundary disturbances, and a noncoercive Lyapunov functional can be constructed for establishing the ISS in the norm of a Hilbert space for 1-D parabolic PDEs with Dirichlet boundary disturbances. However, few results are concerned with presenting explicit ISS estimates in L^∞-norm. For example, (i) the ISS in various norms, including weighted L^∞-norm, was proved for linear 1-D PDEs governed by Sturm-Liouville operators in [12] by exploiting the eigenfunction expansion and the finite difference scheme; (ii) the ISS-style estimates in the spatial sup-norm were established for classical solutions of nonlinear 1-D parabolic PDEs in [14] by using a novel ISS-style maximum principle and an ISS Lyapunov functional; (iii) characterizations of the ISS for nonlinear parabolic PDEs governed by monotone operators were provided in [20] by applying the monotonicity method, and the ISS in the spatial sup-norm was indicated as well; and (iv) an ISS estimate in L^∞-norm was established for a 1-D linear parabolic equation with a destabilizing term in [28] under an appropriate boundary feedback law and some compatibility conditions.

The aim of this paper is to provide a new method, which is different from those developed in the existing literature concerning classical solutions of 1-D PDEs, for establishing the ISS in the spatial sup-norm for weak solutions of a class of higher dimensional nonlinear parabolic PDEs with boundary disturbances. More precisely, in order to establish ISS estimates in the spatial sup-norm, we borrow first the notion of *relative stability* (RS) from [15], which was used to

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characterize a kind of relationship of stabilities for two control systems, to describe the uniformly continuous dependence on the external disturbances for weak solutions of nonlinear PDE systems. Then, based on the RS in the (spatial and time) sup-norm, we establish the ISS in the spatial sup-norm for the considered higher dimensional nonlinear PDEs with Dirichlet and Robin boundary disturbances. Moreover, we show how to apply the property of RS to characterize the stability in the sup-norm and the spatial sup-norm, respectively, for a class of PDE systems in cascade coupled via the boundary or over the domain.

The main tool exploited in this paper for the proof of various stability properties is the De Giorgi iteration, which has been used for the first time to establish ISS estimates for classical solutions of PDEs in [28]. It should be mentioned that in [28] the De Giorgi iteration was used for addressing the ISS of 1-D parabolic PDEs with Dirichlet boundary disturbances by combining the technique of splitting, while in this paper, the De Giorgi iteration is used not only for 1-D PDEs with Dirichlet boundary conditions, but also for higher dimensional PDEs with either Dirichlet or Robin boundary conditions. This represents an improvement of the previous work.

In addition, unlike [12], [14], [20], [28], where the continuity of the solutions along the boundary of the domain was used while establishing the ISS estimates of classical solutions, in this paper, we do not require any continuity of the solutions on the boundary of the domain, and we consider solutions in a weak sense. It is worth noting that for PDEs with Dirichlet boundary conditions, compatibility conditions are sufficient and necessary for the continuity of both classical (in any sense) and weak (in any sense) solutions along the boundary of the domain. As indicated in [29], if a compatibility condition is imposed in a certain case, e.g., when the technique of splitting is used for nonlinear PDEs, only ISS-like estimates can be obtained. Consequently, some nice properties, such as convergent input-convergent state property, may not hold anymore. On the contrary, as this paper deals with solutions that are not necessarily continuous on the boundary of the domain, the ISS estimates in a strict sense can be established.

In summary, the main contribution of this paper includes:

(i) introducing RKES to describe the uniformly continuous dependence of weak solutions on the external disturbances and establishing RKES estimates for a class of higher dimensional nonlinear parabolic PDEs;
(ii) establishing the ISS in the spatial sup-norm for weak solutions of higher dimensional PDEs with Dirichlet or Robin boundary disturbances by using the property of RKES;
(iii) establishing stability estimates in the spatial and time sup-norm and the spatial sup-norm, respectively for a class of parabolic systems in cascade, which are interconnected or coupled via the boundary of the domain;
(iv) extending the usage of De Giorgi iteration to ISS analysis of higher dimensional PDEs with Robin boundary conditions.

In the rest of the paper, we introduce first some basic notations. Section II presents the problem formulation, well-posedness, notions on relative stability, and the main results on RKES in the sup-norm and ISS in the spatial sup-norm for the considered PDE systems. Section III-A provides an example to illustrate the obtained ISS results. As an application of RKES presented in Section II, we show in Section III-B how to apply RKES to obtain stability estimates in the spatial and time sup-norm and the spatial sup-norm, respectively, for a class of parabolic systems in cascade connected over the domain or on the boundary of the domain. Some concluding remarks are given in Section IV.

Notations. $\mathbb{R}_{>0}$ denotes the set of positive real numbers and $\mathbb{R}_{\geq 0} := \{0\} \cup \mathbb{R}_{>0}$. Ω denotes a bounded domain in $\mathbb{R}^n (n \geq 1)$ of class C^2, that is, $\overline{\Omega}$ is an n-dimensional C^2-submanifold of $\mathbb{R}^n$ with boundary $\partial\Omega$. $|\Omega|$ denotes the n-dimensional Lebesgue measure of Ω. For any $T > 0$, $Q_T := \Omega \times (0,T)$, $\partial_l Q_T := \partial\Omega \times (0,T)$.

$\mathcal{K} := \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} |\ \gamma(0) = 0, \gamma$ is continuous, strictly increasing$\}$, $\mathcal{L} := \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} |\ \gamma$ is continuous, strictly decreasing, $\lim_{s\to\infty} \gamma(s) = 0\}$, $\mathcal{KL} := \{\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} |\ \beta(\cdot,t) \in \mathcal{K}, \forall t \in \mathbb{R}_{\geq 0}$, and $\beta(s,\cdot) \in \mathcal{L}, \forall s \in \mathbb{R}_{>0}\}$.

Throughout this paper, all notations on function spaces are standard, which can be found in, e.g., [6], [26]. For functions $a \in C^1(\overline{\Omega};\mathbb{R}_{>0}), c \in C(\overline{\Omega};\mathbb{R}), m \in C(\partial\Omega;\mathbb{R}_{>0})$, we always denote

$$\underline{a} := \min_{x\in\overline{\Omega}} a > 0, \underline{c} := \min_{x\in\overline{\Omega}} c, \underline{m} := \min_{x\in\partial\Omega} m > 0. \tag{1}$$

Furthermore, we always let p be a constant satisfying $p \geq 2$ and $p > n$. For a function $v \in W^{1,p}(\Omega)$, its "boundary value" along $\partial\Omega$ should be understood in the sense of trace; see, e.g., [6, Chap. 5].

II. Problem setting and main results

A. Problem formulation and well-posedness

For functions $a \in C^1(\overline{\Omega};\mathbb{R}_{>0})$, $c \in C(\overline{\Omega};\mathbb{R})$, $m \in C(\partial\Omega;\mathbb{R}_{>0})$, $h \in C^{0,1}((\overline{\Omega} \times \mathbb{R}_{\geq 0}) \times \mathbb{R};\mathbb{R})$, $f \in C(\overline{\Omega} \times \mathbb{R}_{\geq 0};\mathbb{R})$, $d \in C(\partial\Omega \times \mathbb{R}_{\geq 0};\mathbb{R})$, and $u^0 \in W^{1,p}(\Omega)$, we consider the stability of the following higher dimensional nonlinear parabolic system in a divergence form:

$$\mathscr{L}[u] + h(x,t,u) = f \quad \text{in } \Omega \times \mathbb{R}_{>0}, \tag{2a}$$
$$\mathscr{B}[u] = d \quad \text{on } \partial\Omega \times \mathbb{R}_{>0}, \tag{2b}$$
$$u(\cdot,0) = u^0(\cdot) \text{ in } \Omega, \tag{2c}$$

where $\mathscr{L}[u] := u_t - \text{div}\,(a\nabla u) + cu$, and

$$\mathscr{B}[u] := a\frac{\partial u}{\partial \boldsymbol{\nu}} + mu, \tag{3}$$

or

$$\mathscr{B}[u] := u, \tag{4}$$

represents the Robin boundary condition, or the Dirichlet boundary condition, respectively.

We always assume that for any $T > 0$, there exist a positive constant c_0, an increasing function $H : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a function $\Psi \in C(\mathbb{R}_{\geq 0};\mathbb{R}_{\geq 0})$ satisfying $\Psi(0) = 0$, such that

$$|h(x,t,\xi)| \leq c_0(1 + |\xi|^\lambda), |\partial_\xi h(x,t,\xi)| \leq H(|\xi|), \tag{5a}$$
$$|h(x,s,\xi) - h(x,t,\xi) - f(x,s) + f(x,t)| \leq H(|\xi|)\Psi(|s-t|), \tag{5b}$$

hold for all $x \in \overline{\Omega}, s, t \in [0,T], \xi \in \mathbb{R}$, where $\lambda \in (1, 1+\frac{2}{n}]$ is a constant.

It should be mentioned that PDEs in divergence forms arise in diffusion theory, such as the study of heat conduction in an isotropic medium, fluid flow through porous media, chemical reaction, etc., see, e.g., [2], [7]. It is also worth noting that for nonlinear parabolic PDEs in divergence forms, the structural condition (5), as well as its general forms, are often used for establishing the existence of weak solutions, or generalized solutions (and certain smooth solutions); see, e.g., [1] and [16, Chap. VI], respectively. A typical form of h is given by $h(x,t,u) := k_1(x,t)|u|^{\sigma_1-1}u + k_2(x,t)|u|^{\sigma_2} + k_3(x,t)$, where $\sigma_1 \in [1,\lambda], \sigma_2 \in (1,\lambda]$ are constants, and k_1, k_2, k_3 are continuous functions w.r.t. (x,t) and, particularly, Hölder continuous in t with exponents $\alpha_1, \alpha_2, \alpha_3 \in (0,1]$, respectively.

We provide the definition of a (weak) solution of the system (2).

Definition 2.1: (i) We say that u is a weak solution of the system (2) with the Robin boundary condition (3), if for any $T > 0$:

$$u \in C([0,T]; W^{1,p}(\Omega)), u(\cdot,0) = u^0(\cdot) \text{ in } \Omega,$$

and the equality

$$\begin{aligned} &-\int_0^T\int_\Omega u\eta_t \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega a\nabla u \nabla\eta \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T\int_\Omega (cu + h(x,t,u))\eta \,\mathrm{d}x\,\mathrm{d}t \\ =&\int_0^T\int_\Omega f\eta \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\partial\Omega}(d - mu)\eta \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_\Omega u^0(x)\eta(x,0)\,\mathrm{d}x \end{aligned}$$

holds true for any $\eta \in C([0,T]; (W^{1,p}(\Omega))') \cap C^1((0,T); L^{p'}(\Omega))$ with $\eta(\cdot,T) = 0$ in Ω, where $p' = \frac{p}{p-1}$, and $(W^{1,p}(\Omega))'$ is the dual space of $W^{1,p}(\Omega)$.

(ii) We say that u is a weak solution of the system (2) with the Dirichlet boundary condition (4), if for any $T > 0$:

$$\begin{aligned} &u \in C([0,T]; W^{1,p}(\Omega)), \\ &u = d \text{ in } \partial_l Q_T,\ u(\cdot,0) = u^0(\cdot) \text{ in } \Omega, \end{aligned}$$

and the equality

$$\begin{aligned} &-\int_0^T\int_\Omega u\eta_t \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega a\nabla u \nabla\eta \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T\int_\Omega (cu + h(x,t,u))\eta \,\mathrm{d}x\,\mathrm{d}t \\ =&\int_0^T\int_\Omega f\eta \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u^0(x)\eta(x,0)\,\mathrm{d}x \end{aligned}$$

holds true for any $\eta \in C([0,T]; (W_0^{1,p}(\Omega))') \cap C^1((0,T); L^{p'}(\Omega))$ with $\eta(\cdot,T) = 0$ in Ω, where $p' = \frac{p}{p-1}$, and $(W_0^{1,p}(\Omega))'$ is the dual space of $W_0^{1,p}(\Omega)$.

For the well-posedness of the considered problem, we have the following result.

Proposition 2.1: System (2) with either the Robin boundary condition (3), or the Dirichlet boundary condition (4), admits a unique weak solution belonging to $C([0,T]; W^{1,p}(\Omega)) \cap C^1((0,T); L^{p'}(\Omega))$ for any $T > 0$, where $p' = \frac{p}{p-1}$.

Proof: For any $T > 0$, by [1, Theorem 14.5], the system (2) with the Robin boundary condition (3), or the Dirichlet boundary condition (4), admits a unique maximal weak solution[1] $u \in C([0,T_0]; W^{1,p}(\Omega)) \cap C^1((0,T_0); L^{p'}(\Omega))$ with some $T_0 \in (0,T]$. Furthermore, according to [1, Theorem 15.2(i)], if there exists a positive constant C such that the following *a priori* estimate holds true:

$$\|u[t]\|_{L^1(\Omega)} \leq C, \forall t \in (0,T_0), \tag{6}$$

then, such a maximal solution u must exist globally on $[0,T]$. Therefore, it suffices to prove that (6) holds true for the maximal solution of the system (2) with the Robin boundary condition (3), or the Dirichlet boundary condition (4), respectively.

Indeed, for the system (2) with the Robin boundary condition (3), the estimate in (6) is guaranteed by [30, Theorem 3.1]. For system (2) with the Dirichlet boundary condition (4), the estimate in (6) is guaranteed by applying the estimate given in Theorem 2.4(ii) to (2) defined over Q_{T_0}. ∎

Remark 2.1: The growth conditions on the nonlinear term h appearing in (5) are only used for guaranteeing the existence and uniqueness of a weak solution. In particular, as indicated in [1], the exponent $1+\frac{2}{n}$ in (5a) is optimal for the existence of a global weak solution. However, regarding the well-posedness, if a certain compatibility condition is imposed and a smooth solution is considered, then the growth conditions on the nonlinear term h in (5) can be relaxed; see [30, Proposition 2.1] and [16, Chap. VI].

B. Notion on relative stability

It is known that both the initial value and enforced terms (external disturbances) have a deep effect on the stability (and well-posedness) of PDE systems. In order to describe the influence induced by these data, we define some stability characteristics for PDE systems. More precisely, based on the notion of *relative stability* (RS) given by [15], which is an extension of the concept for finite dimensional systems, e.g. [17], to general control systems, we define several properties of relative stability for the considered PDE systems.

To emphasize the dependence of the solutions on the initial value and external disturbances, we denote by $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ the system (2) with data $(u^0, f, d) \in W^{1,p}(\Omega) \times C(\overline{\Omega} \times \mathbb{R}_{\geq 0}; \mathbb{R}) \times C(\partial\Omega \times \mathbb{R}_{\geq 0}; \mathbb{R})$. Note that if $v \in W^{1,p}(\Omega)$ with $p > n$, then by the embedding $W^{1,p}(\Omega) \hookrightarrow C^{1-\frac{n}{p}}(\overline{\Omega})$, we have $v \in C^{1-\frac{n}{p}}(\overline{\Omega})$; see, e.g. [26, Theorem 1.3.2]. Thus, $\sup_{x\in\Omega}|v(x)| < +\infty$ and hence, $\sup_{x\in\Omega}|v(x)|$ is well-defined. Analogously, if $v \in C([0,T]; W^{1,p}(\Omega))$, then

$$\begin{aligned} \sup_{(x,t)\in Q_T}|v(x,t)| \leq& \max_{t\in[0,T]}\sup_{x\in\Omega}|v(x,t)| \\ \leq& \max_{t\in[0,T]}\|v[t]\|_{C^{1-\frac{n}{p}}(\overline{\Omega})} \\ \leq& C(n,p,\Omega)\max_{t\in[0,T]}\|v[t]\|_{W^{1,p}(\Omega)} \\ =& C(n,p,\Omega)\|v\|_{C([0,T];W^{1,p}(\Omega))}, \end{aligned}$$

[1]A solution u is said to be maximal if there is no solution which is a proper extension of u; see [1].

where $C(n,p,\Omega)$ is a constant depending only on n,p,Ω. Therefore, $\sup_{(x,t)\in Q_T}|v(x,t)|$ is also well-defined. In addition, it should be noticed that, according to the definition of Hölder spaces, a function v belonging to some Hölder space $C^{\alpha}(\overline{\Omega})$ with $\alpha\in(0,1)$ is not necessarily continuous on $\partial\Omega$; see, e.g., [6], [26].

Definition 2.2: The system $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ is said to be relatively equi-stable (RES) in the sup-norm with respect to (w.r.t.) in-domain and boundary disturbances in $C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, if for every constant $\varepsilon>0$, there exists a positive constant δ depending only on ε, such that the following implication

$$\begin{aligned}&\sup_{(x,t)\in Q_T}|f_1(x,t)-f_2(x,t)|\\&+\sup_{(x,t)\in\partial_l Q_T}|d_1(x,t)-d_2(x,t)|<\delta\\&\Rightarrow\sup_{(x,t)\in Q_T}|u_1(x,t)-u_2(x,t)|<\varepsilon\end{aligned}$$

holds true for all $(f_1,d_1),(f_2,d_2)\in C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$ and all $T>0$, where u_i is the solution of the system $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ corresponding to the data $(u^0,f_i,d_i),i=1,2$.

Definition 2.3: The system $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ is said to be relatively $\mathcal{K}$-equi-stable (RKES) in the sup-norm w.r.t. in-domain and boundary disturbances in $C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, if there exist functions $\gamma_d,\gamma_f\in\mathcal{K}$ such that

$$\begin{aligned}&\sup_{(x,t)\in Q_T}|u_1(x,t)-u_2(x,t)|\\\leq&\gamma_f\left(\sup_{(x,t)\in Q_T}|f_1(x,t)-f_2(x,t)|\right)\\&+\gamma_d\left(\sup_{(x,t)\in\partial_l Q_T}|d_1(x,t)-d_2(x,t)|\right),\forall T>0,\end{aligned}$$

where u_i is the solution of the system $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ corresponding to the data $(u^0,f_i,d_i),i=1,2$.

Particularly, the system $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ is said to be relatively Lipschitz-equi-stable (RLES) in the sup-norm w.r.t. in-domain and boundary disturbances in $C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, if $\gamma_f(s)=L_f s,\gamma_d(s)=L_d s$ for any $s\geq 0$, where L_f and L_d are certain positive constants.

Remark 2.2: We provide some comments on RES.

(i) It should be noticed that there is a slight difference for RES between the definition given in [15] and the one introduced in this paper. Indeed, RES defined in [15] is mainly used to describe the relationship of stabilities between two systems, namely it is in term of multiple different systems, while RES defined in this paper is mainly used to characterize the uniformly continuous dependence of the solution on the external disturbances for a certain PDE, namely it is in term of one system (i.e., the nominal dynamics of the PDEs are governed by the same differential operator) with different external inputs.

(ii) Note that a system with external disturbances may be RES w.r.t. external disturbances while not being asymptotically stable. For example, we consider the following systems:

$$\begin{aligned}(u_k)_t-(u_k)_{xx}=&f_k(x,t),\ (x,t)\in\left(0,\frac{\pi}{2}\right)\times\mathbb{R}_{\geq 0},\\u_k(0,t)=&0,\qquad t\in\mathbb{R}_{>0},\\u_k\left(\frac{\pi}{2},t\right)=&d_k(t),\qquad t\in\mathbb{R}_{>0},\\u_k(x,0)=&u^0(x),\quad x\in\left(0,\frac{\pi}{2}\right),\end{aligned}$$

where $u^0(x):=0,\ f_k(x,t):=\sqrt{2}k\sin x\cos\left(t-\frac{\pi}{4}\right)$, $d_k(t):=k\sin t$, and $k\in\mathbb{R}_{>0}$. It is clear that $u_k(x,t)=k\sin t\sin x$ is the unique solution, which is bounded for any fixed k. Since there exists a point (x_0,t_0) such that $u_k(x_0,t_0)>\frac{k}{2}$, u_k is unbounded as $k\to+\infty$. However, noting that

$$\begin{aligned}&\sup_{(x,t)\in\left(0,\frac{\pi}{2}\right)\times(0,T)}|f_k(x,t)-f_l(x,t)|\\=&\sqrt{2}|k-l|\sup_{(x,t)\in\left(0,\frac{\pi}{2}\right)\times(0,T)}\left|\sin x\cos\left(t-\frac{\pi}{4}\right)\right|,\forall T>0,\end{aligned}$$

and

$$\sup_{t\in(0,T)}|d_k(t)-d_l(t)|=|k-l|\sup_{t\in(0,T)}|\sin t|,\forall T>0,$$

for all $k,l\in\mathbb{R}_{>0}$, then the system is RLES, having the estimate for all $k,l\in\mathbb{R}_{>0}$:

$$\begin{aligned}&\sup_{(x,t)\in\left(0,\frac{\pi}{2}\right)\times(0,T)}|u_k(x,t)-u_l(x,t)|\\=&|k-l|\sup_{(x,t)\in\left(0,\frac{\pi}{2}\right)\times(0,T)}|\sin t\sin x|\\\leq&\sup_{(x,t)\in\left(0,\frac{\pi}{2}\right)\times(0,T)}|f_k(x,t)-f_l(x,t)|\\&+\sup_{t\in(0,T)}|d_k(t)-d_l(t)|,\forall T>0.\end{aligned}$$

(iii) It is obvious that RLES $\Rightarrow$ RKES $\Rightarrow$ RES.

Definition 2.4 ([22]): The system $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ is said to be globally asymptotically stable at zero uniformly w.r.t. the state (0-UGAS w.r.t. the state) in the spatial sup-norm if the in-domain and boundary disturbances in $C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$ are set to 0, and there exists a function $\beta\in\mathcal{KL}$ such that

$$\sup_{x\in\Omega}|u(x,T)|\leq\beta\left(\sup_{x\in\Omega}|u^0(x)|,T\right),\forall u^0\in W^{1,p}(\Omega),\forall T>0,$$

where u is the solution of the system $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ corresponding to the data $(u^0,0,0)$.

Definition 2.5: The system $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ is said to be input-to-state stable (ISS) in the spatial sup-norm w.r.t. in-domain and boundary disturbances in $C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, if there exist functions $\beta\in\mathcal{KL}$ and $\gamma_f,\gamma_d\in\mathcal{K}$ such that for all $(u^0,f,d)\in W^{1,p}(\Omega)\times C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$:

$$\begin{aligned}\sup_{x\in\Omega}|u(x,T)|\leq&\beta\left(\sup_{x\in\Omega}|u^0(x)|,T\right)+\gamma_f\left(\sup_{(x,t)\in Q_T}|f(x,t)|\right)\\&+\gamma_d\left(\sup_{(x,t)\in\partial_l Q_T}|d(x,t)|\right),\forall T>0,\end{aligned}\tag{7}$$

where u is the solution of the system $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ corresponding to the data $(u^0,f,d)\in W^{1,p}(\Omega)\times C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$.

Furthermore, the system $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ is said to be exponentially input-to-state stable (EISS) in the spatial sup-norm w.r.t. in-domain and boundary disturbances in $C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, if there exist constants $M,\sigma>0$ such that $\beta(r,t)=Mr\,\mathrm{e}^{-\sigma t}$ in (7) for all $r\geq 0$.

Proposition 2.2: If the system $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ is RKES in the sup-norm w.r.t. in-domain and boundary disturbances in $C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$ and 0-UGAS w.r.t. the state in the spatial sup-norm, then it is ISS in the spatial sup-norm w.r.t. in-domain and boundary disturbances in $C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$.

Proof: For any $(u^0,f,d)\in W^{1,p}(\Omega)\times C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, let u,v be the solutions of the system $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ with data (u^0,f,d) and $(u^0,0,0)$, respectively. For simplicity, we write $|g|_{\infty,\omega}:=\sup_{y\in\omega}|g(y)|$ for a function g defined on a domain ω of $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$. Since $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ is RKES in the sup-norm w.r.t. in-domain and boundary disturbances, and u and v are continuous in T (see Proposition 2.1), there exist functions $\gamma_d,\gamma_f\in\mathcal{K}$ such that

$$\begin{aligned}&|u[T]-v[T]|_{\infty,\Omega}\\ \leq&|u-v|_{\infty,Q_T}\\ \leq&\gamma_f\big(|f-0|_{\infty,Q_T}\big)+\gamma_d\big(|d-0|_{\infty,\partial_l Q_T}\big)\\ =&\gamma_f\big(|f|_{\infty,Q_T}\big)+\gamma_d\big(|d|_{\infty,\partial_l Q_T}\big),\ \forall T>0.\end{aligned}$$

Since $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ is 0-UGAS w.r.t. the state in the spatial sup-norm, there exists a function $\beta\in\mathcal{KL}$ such that

$$|v[T]|_{\infty,\Omega}\leq\beta(|u^0|_{\infty,\Omega)},T),\ \forall T>0.$$

It follows that

$$\begin{aligned}|u[T]|_{\infty,\Omega}\leq&|u[T]-v[T]|_{\infty,\Omega}+|v[T]|_{\infty,\Omega}\\ \leq&\beta(|u^0|_{\infty,\Omega},T)+\gamma_f\big(|f|_{\infty,Q_T}\big)\\ &+\gamma_d\big(|d|_{\infty,\partial_l Q_T}\big),\ \forall T>0,\end{aligned}$$

which implies that $\Sigma(\mathbb{U},\mathbb{F},\mathbb{D})$ is ISS in the spatial sup-norm w.r.t. in-domain and boundary disturbances in $C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$. ■

Remark 2.3: In general, for a nonlinear system, it is not an easy task to establish the ISS w.r.t. boundary disturbances directly. Proposition 2.2 provides an alternative for ISS analysis of 0-UGAS systems, which amounts to only assessing the properties of RKES and may be more easily obtained.

C. Main stability results

Assume further that

$$h(x,t,\xi_1)-h(x,t,\xi_2)\geq -L(\xi_1-\xi_2)\tag{8}$$

for all $x\in\overline{\Omega},t\in\mathbb{R}_{\geq 0}$ and all $\xi_1,\xi_2\in\mathbb{R}$ satisfying $\xi_1\geq\xi_2$, where L is a constant.

The first main result is on the RKES in the sup-norm w.r.t. in-domain and boundary disturbances, whose proof is provided in Appendix.

Theorem 2.3: The following statements hold true.

(i) Assume that $L<\underline{c}$ in (8). System (2) with the Robin boundary condition (3) is RLES in the sup-norm w.r.t. in-domain and boundary disturbances in $C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, having the estimate:

$$\begin{aligned}&\sup_{(x,t)\in Q_T}|u_1(x,t)-u_2(x,t)|\\ \leq&C_R|\Omega|^{\frac{q-2}{q}}2^{\frac{3q-4}{2q-4}}\sup_{(x,t)\in Q_T}|f_1(x,t)-f_2(x,t)|\\ &+\frac{1}{\underline{m}}\sup_{(x,t)\in\partial_l Q_T}|d_1(x,t)-d_2(x,t)|,\forall T>0,\end{aligned}\tag{9}$$

for all $(f_i,d_i)\in C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R}),i=1,2$, where u_i is the solution of the system corresponding to the data $(u^0,f_i,d_i)\in W^{1,p}(\Omega)\times C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R}),i=1,2$, $C_R:=\frac{2C_S^2}{\min\{\underline{a},\underline{c}-L\}}>0$, and q and C_S are constants specified in Lemma A.1(i).

(ii) Assume that $L\leq\underline{c}$ in (8). System (2) with the Dirichlet boundary condition (4) is RLES in the sup-norm w.r.t. in-domain and boundary disturbances in $C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, having the estimate:

$$\begin{aligned}&\sup_{(x,t)\in Q_T}|u_1(x,t)-u_2(x,t)|\\ \leq&C_D|\Omega|^{\frac{q-2}{q}}2^{\frac{3q-4}{2q-4}}\sup_{(x,t)\in Q_T}|f_1(x,t)-f_2(x,t)|\\ &+\sup_{(x,t)\in\partial_l Q_T}|d_1(x,t)-d_2(x,t)|,\forall T>0,\end{aligned}\tag{10}$$

for all $(f_i,d_i)\in C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R}),i=1,2$, where u_i is the solution of the system corresponding to the data $(u^0,f_i,d_i)\in W^{1,p}(\Omega)\times C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R}),i=1,2$, $C_D:=\frac{C_P^2}{\underline{a}}>0$, and q and C_P are constants specified in Lemma A.1(ii).

Furthermore, if $L<\underline{c}$ in (8), it holds that

$$\begin{aligned}&\sup_{(x,t)\in Q_T}|u_1(x,t)-u_2(x,t)|\\ \leq&C_0|\Omega|^{\frac{q-2}{q}}2^{\frac{3q-4}{2q-4}}\sup_{(x,t)\in Q_T}|f_1(x,t)-f_2(x,t)|\\ &+\sup_{(x,t)\in\partial_l Q_T}|d_1(x,t)-d_2(x,t)|,\forall T>0,\end{aligned}\tag{11}$$

for all $(f_i,d_i)\in C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R}),i=1,2$, where u_i is the solution of the system corresponding to the data $(u^0,f_i,d_i)\in W^{1,p}(\Omega)\times C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R}),i=1,2$, $C_0:=\min\{C_R,C_D\}$, and q,C_S,C_P are constants specified in Lemma A.1.

As an application of RKES, we have the second main result on the ISS in the spatial sup-norm for the system (2) with in-domain and boundary disturbances, whose proof is also provided in Appendix.

Theorem 2.4: The following statements hold true.

(i) Assume that $L<\underline{c}$ in (8). System (2) with the Robin boundary condition (3) is EISS in the spatial sup-norm w.r.t. in-domain and boundary disturbances in

$C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, having the estimate:

$$\begin{aligned}\sup_{x\in\Omega}|u(x,T)| \leq& \sup_{x\in\Omega}|u^0(x)|\,\mathrm{e}^{-(\underline{c}-L)T}\\&+C_R|\Omega|^{\frac{q-2}{q}}2^{\frac{3q-4}{2q-4}}\sup_{(x,t)\in Q_T}|f(x,t)|\\&+\frac{1}{\underline{m}}\sup_{(x,t)\in\partial_l Q_T}|d(x,t)|,\forall T>0,\end{aligned}\quad(12)$$

where the constants C_R, q are the same as in Theorem 2.3(i).

(ii) Assume that $L<\underline{c}$ in (8). System (2) with the Dirichlet boundary condition (4) is EISS in the spatial sup-norm w.r.t. in-domain and boundary disturbances in $C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, having the estimate:

$$\begin{aligned}\sup_{x\in\Omega}|u(x,T)| \leq& \sup_{x\in\Omega}|u^0(x)|\mathrm{e}^{-(\underline{c}-L)T}\\&+C_0|\Omega|^{\frac{q-2}{q}}2^{\frac{3q-4}{2q-4}}\sup_{(x,t)\in Q_T}|f(x,t)|\\&+\sup_{(x,t)\in\partial_l Q_T}|d(x,t)|,\forall T>0,\end{aligned}\quad(13)$$

where the constants C_0, q are the same as in Theorem 2.3(ii).

Remark 2.4: For Theorem 2.4, it is possible to weaken the condition $(u^0,f,d)\in W^{1,p}(\Omega)\times C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$ to $(u^0,f,d)\in L^\infty(\Omega)\times L^\infty_{loc}(\mathbb{R}_{\geq 0};L^\infty(\Omega))\times L^\infty_{loc}(\mathbb{R}_{\geq 0};L^\infty(\Omega))$, and hence obtain

$$\begin{aligned}\|u[T]\|_{L^\infty(\Omega)}\leq&\|u^0\|_{L^\infty(\Omega)}\,\mathrm{e}^{-(\underline{c}-L)T}\\&+C_R|\Omega|^{\frac{q-2}{q}}2^{\frac{3q-4}{2q-4}}\|f\|_{L^\infty(Q_T)}\\&+\frac{1}{\underline{m}}\|d\|_{L^\infty(Q_T)},\forall T>0,\end{aligned}$$

and

$$\begin{aligned}\|u[T]\|_{L^\infty(\Omega)}\leq&\|u^0\|_{L^\infty(\Omega)}\mathrm{e}^{-\left(\underline{c}-L+\frac{a}{C_P^2}\right)T}\\&+C_0|\Omega|^{\frac{q-2}{q}}2^{\frac{3q-4}{2q-4}}\|f\|_{L^\infty(Q_T)}\\&+\|d\|_{L^\infty(Q_T)},\forall T>0,\end{aligned}$$

for a weak solution (in a certain sense differing from Definition 2.1) of the system (2) with the Robin boundary condition (3), and the Dirichlet boundary condition (4), respectively. Indeed, letting $\{u^0_n\},\{f_n\},\{d_n\}$ be sequences of sufficiently smooth functions, which satisfy $(u^0_n,f_n,d_n)\to(u^0,f,d)$ in $L^\infty(\Omega)\times L^\infty(Q_T)\times L^\infty(Q_T)$ as $n\to+\infty$, we consider the approximating equation (2) with data (u^0_n,f_n,d_n), and establish uniform *a priori* estimates of strong (or smooth) solutions $\{u_n\}$ as in [31]. For the existence of a weak solution u, we may prove by using the uniform *a priori* estimates of $\{u_n\}$ and taking limits in appropriate functional spaces. For the ISS in the spatial L^∞-norm of u_n, we may prove as in the proof of Theorem 2.4; see Appendix. Then, by taking limits of (u_n,u^0_n,f_n,d_n) within the ISS estimates of $\{u_n\}$, we may obtain the aforementioned ISS estimates of u.

III. An Illustrative Example of ISS and Applications of RKES

In this section, we illustrate one of the ISS results stated in Theorem 2.4 of Section II and present applications of the introduced notion of *RKES* by establishing different stability estimates for a class of parabolic systems in cascade coupled over the domain or on the boundary of the domain.

A. The ISS in the spatial sup-norm for a super-linear parabolic equation

Consider the following super-linear parabolic equation:

$$u_t-\Delta u+cu+u\ln(1+u^2)=f\quad\text{in }\Omega\times\mathbb{R}_{>0},\quad(14a)$$

$$\frac{\partial u}{\partial\boldsymbol{\nu}}+mu=d\quad\text{on }\partial\Omega\times\mathbb{R}_{>0},\quad(14b)$$

$$u(\cdot,0)=u^0(\cdot)\text{ in }\Omega,\quad(14c)$$

where Ω is an open bounded domain in $\mathbb{R}^n (n\geq 3)$ with a smooth boundary $\partial\Omega$, Δ is the Laplace operator, c,m are positive constants with $c\geq 1$, $u^0\in W^{1,p}(\Omega)$, $d\in C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, and $f\in C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})$. We have the following result.

Proposition 3.1: Assume that there exists a point $x^0\in\partial\Omega$ such that, near x^0, $\partial\Omega$ lies in the plane $\{x:=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n|\ x_i=0\}$ for some $i\in\{1,2,\dots,n\}$, and $f(x,t)$ is Lipschitz continuous in $t\in\mathbb{R}_{\geq 0}$ for all $x\in\overline{\Omega}$. Then, the system (14) is EISS in the spatial sup-norm, having the estimate:

$$\begin{aligned}\sup_{x\in\Omega}|u(x,T)|\leq&\sup_{x\in\Omega}|u^0(x)|\,\mathrm{e}^{-cT}\\&+\frac{9(n-1)^2}{(n-2)^2}|\Omega|^{\frac{2}{n}}2^8\sup_{(x,t)\in Q_T}|f(x,t)|\\&+\frac{1}{m}\sup_{(x,t)\in\partial_l Q_T}|d(x,t)|,\ \forall T>0.\end{aligned}$$

Proof: Setting $h(x,t,u):=u\ln(1+u^2)$, it is easy to verify that h and f satisfy the conditions proposed in Section II (with $L=0$). Then, according to Theorem 2.4(i), the system (14) is EISS in the spatial sup-norm, having the estimate:

$$\begin{aligned}\sup_{x\in\Omega}|u(x,T)|\leq&\sup_{x\in\Omega}|u^0(x)|\,\mathrm{e}^{-cT}\\&+C_S^2|\Omega|^{\frac{q-2}{q}}2^{\frac{5q-8}{2q-4}}\sup_{(x,t)\in Q_T}|f(x,t)|\\&+\frac{1}{m}\sup_{(x,t)\in\partial_l Q_T}|d(x,t)|,\forall T>0,\end{aligned}\quad(15)$$

where q and C_S are constants specified in Lemma A.1(i).

Note that, near x^0, $\partial\Omega$ lies in the plane $\{x\in\mathbb{R}^n|\ x_i=0\}$. Thus, in the following Sobolev inequality

$$\|v\|_{L^{2^*}(\Omega)}\leq C_1(\|v\|_{L^2(\Omega)}+\|\nabla v\|_{L^2(\Omega)}),\forall v\in W^{1,2}(\Omega),\quad(16)$$

the constant C_1 can be chosen as $C_1:=\frac{2(n-1)}{n-2}\times(1+3+4\times 2)=\frac{24(n-1)}{n-2}$; see [6, Theorem 2, §5.6.1], whose proof is based on Step 2 of the proof of [6, Theorem 1, §5.6.1], and Steps 1-4 of the proof of [6, Theorem 1, §5.4].

For any $q \in (2, 2^*)$ with $2^* := \frac{2n}{n-2}$, it follows from the Hölder's inequality that

$$\|v\|^q_{L^q(\Omega)} \leq \|v\|^q_{L^{2^*}(\Omega)} \|1\|_{L^{\frac{2^*}{2^*-q}}(\Omega)} = \|v\|^q_{L^{2^*}(\Omega)} |\Omega|^{\frac{2^*-q}{2^*}},$$

which along with (16) gives for $v \in W^{1,2}(\Omega)$:

$$\|v\|_{L^q(\Omega)} \leq \frac{24(n-1)}{n-2} |\Omega|^{\frac{2^*-q}{2^* q}} (\|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}).$$

Thus, C_S in Lemma A.1(i) can be chosen as $C_S := \frac{24(n-1)}{n-2} |\Omega|^{\frac{2^*-q}{2^* q}}$. Finally, (15) becomes

$$\begin{aligned} &\sup_{x\in\Omega} |u(x, T)| \\ \leq &\sup_{x\in\Omega} |u^0(x)| \, \mathrm{e}^{-cT} \\ &+ \frac{24^2(n-1)^2}{(n-2)^2} |\Omega|^{\frac{q-2}{q} + 2\left(\frac{2^*-q}{2^* q}\right)} 2^{\frac{5q-8}{2q-4}} \sup_{(x,t)\in Q_T} |f(x, t)| \\ &+ \frac{1}{m} \sup_{(x,t)\in\partial_l Q_T} |d(x, t)|, \ \forall T > 0. \end{aligned} \quad (17)$$

Note that (17) holds true for an arbitrary constant $q \in (2, 2^*)$, and

$$\lim_{q\to 2^*} \left(|\Omega|^{\frac{q-2}{q} + 2\left(\frac{2^*-q}{2^* q}\right)} 2^{\frac{5q-8}{2q-4}} \right) = |\Omega|^{\frac{2^*-2}{2^*}} 2^{\frac{5\times 2^* - 8}{2\times 2^* - 4}} = 4|\Omega|^{\frac{2}{n}}.$$

Then, letting $q \to 2^*$ in (17), we obtain the desired result. ■

Remark 3.1: Note that $\frac{24(n-1)}{n-2} |\Omega|^{\frac{2^*-q}{2^* q}}$ is not the best embedding constant in the Sobolev inequality given in Lemma A.1(i).

B. Applications of RKES to cascades of nonlinear parabolic systems

In this subsection, as an application of the main results presented in Section II-C, we show how to apply the RKES to establish stability estimates in the sup-norm, or the spatial sup-norm, for a class of nonlinear parabolic systems in cascade connected either on the boundary or in the domain. Specifically, for a fixed integer $k \geq 2$ and $j \in \{1, 2, \ldots, k\}$, given functions a_j, c_j, m_j, h_j, ϕ_j, f, d, we consider the following systems coupled on the boundary as shown in Fig. 1:

$$(\Sigma_j) \begin{cases} \mathscr{L}_j[u_j] + h_j(x, t, u_j) = 0 & \text{in } \Omega \times \mathbb{R}_{>0}, \\ a_j \dfrac{\partial u_j}{\partial \boldsymbol{\nu}} + m_j u_j = d_j & \text{on } \partial\Omega \times \mathbb{R}_{>0}, \\ u_j(\cdot, 0) = \phi_j(\cdot) & \text{in } \Omega, \end{cases}$$

where $\mathscr{L}_j[u] := u_t - \text{div}\,(a_j \nabla u) + c_j u$, and for $(x, t) \in \partial\Omega \times \mathbb{R}_{>0}$:

$$d_1(x, t) := d(x, t), d_j(x, t) := u_{j-1}(x, t), \forall j \in [2, k], \quad (18)$$

or

$$d_1(x, t) := u_k(x, t), d_j(x, t) := u_{j-1}(x, t), \forall j \in [2, k]. \quad (19)$$

We also consider the following systems coupled over the domain as shown in Fig. 2:

$$(\Sigma'_j) \begin{cases} \mathscr{L}_j[u_j] + h_j(x, t, u_j) = f_j & \text{in } \Omega \times \mathbb{R}_{>0}, \\ u_j = d_j & \text{on } \partial\Omega \times \mathbb{R}_{>0}, \\ u_j(\cdot, 0) = \phi_j(\cdot) & \text{in } \Omega, \end{cases}$$

where for $(x, t) \in \Omega \times \mathbb{R}_{>0}$:

$$f_1(x, t) := f(x, t), f_j(x, t) := u_{j-1}(x, t), \forall j \in [2, k], \quad (20)$$

or

$$f_1(x, t) := u_k(x, t), f_j(x, t) := u_{j-1}(x, t), \forall j \in [2, k]. \quad (21)$$

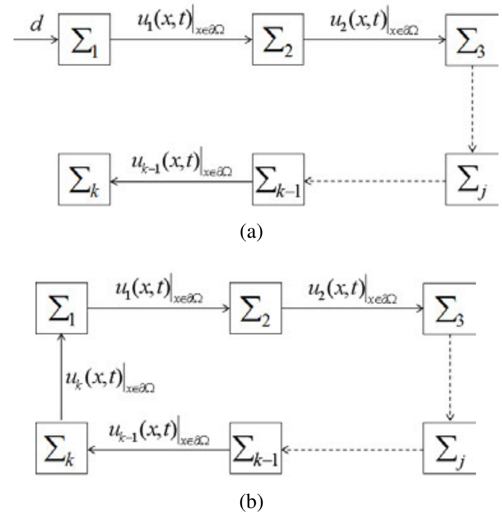

Fig. 1. Systems coupled on the boundary: (a) Systems coupled via (18); (b) Systems coupled via (19).

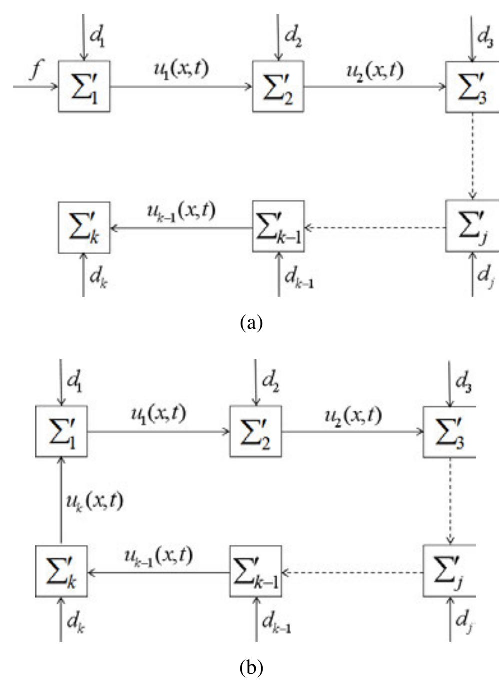

Fig. 2. Systems coupled over the domain: (a) Systems coupled via (20); (b) Systems coupled via (21).

In this subsection, for any $T > 0$, we intend to establish respectively:

(i) the estimate in the sup-norm, i.e., $\sup_{(x,t)\in Q_T} |u_j(x, t)|$; and

(ii) the estimate in the spatial sup-norm, i.e., $\sup_{x\in\Omega}|u_j(x,T)|$,

for the considered cascade of systems, where u_j is the solution of the j-th subsystem.

For $j\in[1,k]$, we always assume that $a_j\in C^1(\overline{\Omega};\mathbb{R}_{>0})$, $c_j\in C(\overline{\Omega};\mathbb{R})$, $m_j\in C(\partial\Omega;\mathbb{R}_{>0})$, $h_j\in C^{0,1}((\overline{\Omega}\times\mathbb{R}_{\geq 0})\times\mathbb{R};\mathbb{R})$, $\phi_j\in W^{1,p}(\Omega)$. Let $\underline{a_j},\underline{m_j}$ and $\underline{c_j}$ be defined in the same way as (1). Let

$$\begin{aligned}\Phi_j:=&\max\left\{\sup_{x\in\Omega}|\phi_1(x)|,\dots,\sup_{x\in\Omega}|\phi_j(x)|\right\},\\ \mathcal{M}_j:=&\min\{\underline{m_1},\dots,\underline{m_j}\},\\ \mathcal{A}_j:=&\frac{1}{|\Omega|^{\frac{q-2}{q}}2^{\frac{3q-4}{2q-4}}\min\{\tau_1,\dots,\tau_j\}},\end{aligned}$$

where

$$\tau_j:=\begin{cases}\dfrac{C_P^2}{\underline{a_j}}, & \text{if } \underline{c_j}-L_j=0\\ \min\left\{\dfrac{2C_S^2}{\min\{\underline{a_j},\underline{c_j}-L_j\}},\dfrac{C_P^2}{\underline{a_j}}\right\}, & \text{if } \underline{c_j}-L_j>0\end{cases},$$

and constants q,C_S, and C_P are specified in Lemma A.1. We have the following two propositions.

Proposition 3.2: For the system (Σ_j), assume that $d\in C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, and the structural conditions (5) and (8) are satisfied with $(h,f)=(h_j,0)$, and $h=h_j,L=L_j$, respectively. In addition, assume that $\underline{c_j}-L_j\geq 0$ for all $j\in[1,k]$. Then the following statements hold true:

(i) For $j\in[1,k]$, if u_j is the solution of the system (Σ_j) with the Robin boundary condition given by (18), then

$$\begin{aligned}\sup_{(x,t)\in Q_T}|u_j(x,t)|\leq&\frac{\mathcal{M}_k}{\mathcal{M}_k-1}\left(1-\frac{1}{\mathcal{M}_k^j}\right)\Phi_j\\&+\frac{1}{\mathcal{M}_k^j}\sup_{(x,t)\in\partial_l Q_T}|d(x,t)|,\forall T>0.\end{aligned}\tag{22}$$

Furthermore, if $\underline{c_j}-L_j>0$, then

$$\begin{aligned}\sup_{x\in\Omega}|u_j(x,T)|\leq&\frac{\mathcal{M}_k}{\mathcal{M}_k-1}\left(1-\frac{1}{\mathcal{M}_k^j}\right)\Phi_j\,\mathrm{e}^{-(\underline{c_j}-L_j)T}\\&+\frac{1}{\mathcal{M}_k^j}\sup_{(x,t)\in\partial_l Q_T}|d(x,t)|,\forall T>0.\end{aligned}\tag{23}$$

(ii) Assume further that $\mathcal{M}_k>1$. For $j\in[1,k]$, if u_j is the solution of the system (Σ_j) with the Robin boundary condition given by (19), then

$$\sup_{(x,t)\in Q_T}|u_j(x,t)|\leq\frac{\mathcal{M}_k}{\mathcal{M}_k-1}\Phi_k,\forall T>0.\tag{24}$$

Furthermore, if $\underline{c_j}-L_j>0$, then

$$\sup_{x\in\Omega}|u_j(x,T)|\leq\frac{\mathcal{M}_k}{\mathcal{M}_k-1}\Phi_k\,\mathrm{e}^{-(\underline{c_j}-L_j)T},\forall T>0.\tag{25}$$

Proposition 3.3: For the system (Σ_j'), assume that $f\in C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})$, $d_j\in C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, and the structural condition (5) is satisfied with $(h,f)=(h_j,f)$ for $j=1$, and $(h,f)=(h_j,0)$ for $j\in[2,k]$. In addition, assume that the structural condition (8) is satisfied with $h=h_j$ and $\underline{c_j}-L_j\geq 0$ for all $j\in[1,k]$. Then the following statements hold true:

(i) For $j\in[1,k]$, if u_j is the solution of the system (Σ_j') with (20), then

$$\begin{aligned}&\sup_{(x,t)\in Q_T}|u_j(x,t)|\\ \leq&\frac{\mathcal{A}_k}{\mathcal{A}_k-1}\left(1-\frac{1}{\mathcal{A}_k^j}\right)\Phi_j+\frac{1}{\mathcal{A}_k^j}\sup_{(x,t)\in Q_T}|f(x,t)|\\&+\sum_{i=1}^{j}\frac{1}{\mathcal{A}_k^{j-i}}\sup_{(x,t)\in\partial_l Q_T}|d_i(x,t)|,\forall T>0.\end{aligned}\tag{26}$$

Furthermore, if $\underline{c_j}-L_j>0$, then

$$\begin{aligned}&\sup_{x\in\Omega}|u_j(x,T)|\\ \leq&\frac{\mathcal{A}_k}{\mathcal{A}_k-1}\left(1-\frac{1}{\mathcal{A}_k^j}\right)\Phi_j\mathrm{e}^{-(\underline{c_j}-L_j)T}\\&+\frac{1}{\mathcal{A}_k^j}\sup_{(x,t)\in Q_T}|f(x,t)|\\&+\sum_{i=1}^{j}\frac{1}{\mathcal{A}_k^{j-i}}\sup_{(x,t)\in\partial_l Q_T}|d_i(x,t)|,\forall T>0.\end{aligned}\tag{27}$$

(ii) Assume further that $\mathcal{A}_k>1$. For $j\in[1,k]$, if u_j is the solution of the system (Σ_j') with (21), then

$$\begin{aligned}&\sup_{(x,t)\in Q_T}|u_j(x,t)|\\ \leq&\frac{\mathcal{A}_k}{\mathcal{A}_k-1}\Phi_k\\&+\frac{\mathcal{A}_k^k}{\mathcal{A}_k^k-1}\sum_{i=1}^{k}\frac{1}{\mathcal{A}_k^{k-i}}\sup_{(x,t)\in\partial_l Q_T}|d_i(x,t)|,\forall T>0,\end{aligned}\tag{28}$$

Furthermore, if $\underline{c_j}-L_j>0$, then

$$\begin{aligned}&\sup_{x\in\Omega}|u_j(x,T)|\\ \leq&\frac{\mathcal{A}_k}{\mathcal{A}_k-1}\Phi_k\mathrm{e}^{-(\underline{c_j}-L_j)T}\\&+\frac{\mathcal{A}_k^k}{\mathcal{A}_k^k-1}\sum_{i=1}^{k}\frac{1}{\mathcal{A}_k^{k-i}}\sup_{(x,t)\in\partial_l Q_T}|d_i(x,t)|,\forall T>0.\end{aligned}\tag{29}$$

Remark 3.2: Note that the systems considered in this subsection are essentially networks of finite connections. Nevertheless, under suitable conditions, the stability of the systems composed of $(\Sigma_1),\dots,(\Sigma_k)$ (or $(\Sigma_1'),\dots,(\Sigma_k')$) with different boundary conditions may be independent of the number of the subsystems, i.e., k. For example:

(i) If there exist some constants $B>0,M>1$ and $N>0$ such that

$$\underline{m_j}\geq M,|\phi_j|\leq N,\forall k,\forall j\in[1,k],\tag{30}$$

and

$$\left(\underline{c_{j+1}} - L_{j+1}\right) - \left(\underline{c_j} - L_j\right) \geq B, \forall k, \forall j \in [1, k-1], \tag{31}$$

then (23) along with (30) implies the following estimate:

$$\sum_{j=1}^{k} \sup_{x\in\Omega} |u_j(x,T)| \leq \frac{MN}{M-1} \sum_{j=1}^{k} \mathrm{e}^{-(\underline{c_j}-L_j)T} + \frac{1}{M-1} \sup_{(x,t)\in\partial_l Q_T} |d(x,t)|, \forall T > 0,$$

while (25) along with (30) implies that

$$\sum_{j=1}^{k} \sup_{x\in\Omega} |u_j(x,T)| \leq \frac{MN}{M-1} \sum_{j=1}^{k} \mathrm{e}^{-(\underline{c_j}-L_j)T}, \forall T > 0.$$

Noting that the condition (31) and the d'Alembert's test guarantee the existace of $\lim_{k\to+\infty} \sum_{j=1}^{k} \mathrm{e}^{-(\underline{c_j}-L_j)T}$ for any fixed $T > 0$, we conclude that $\sum_{j=1}^{k} \sup_{x\in\Omega} |u_j(x,T)|$ is independent of k for any fixed $T > 0$.

(ii) Analogously, under appropriate assumptions on $\mathcal{A}_j, \underline{c_j} - L_j, \Phi_j$, we may also conclude that the stability of the system composed of $(\Sigma'_1), \ldots, (\Sigma'_k)$ is independent of k.

Remark 3.3: It should be mentioned that in [4], the ISS and a small-gain theorem were established for a class of interconnected systems, provided that ISS-Lyapunov functions of the subsystems are known and a small-gain condition holds. As an application of the obtained results, small-gain conditions for guaranteeing the 0-UGAS in the spatial L^2-norm were proposed for a class of linear, and nonlinear, interconnected parabolic PDEs with homogeneous Dirichlet boundary conditions, respectively. For general interconnected reaction-diffusion systems, it is reasonable to believe that such small-gain conditions depend on the coefficients of the reaction and diffusion terms; see [4] for two special cases. For the interconnected system (Σ'_j) coupled via (19), the small-gain condition is characterized by $\mathcal{A}_k > 1$. While, for the system (Σ_j) coupled on the boundary given in (21), the small-gain condition is characterized solely by $\mathcal{M}_k > 1$. Moreover, the small-gain conditions proposed in this paper can be used for guaranteeing not only the 0-UGAS, but also the ISS, in the spatial sup-norm, for the considered systems with either Robin or Dirichlet boundary conditions.

Proof of Proposition 3.2: The proof is based on using RKES repeatedly and composed of 4 steps.

Step 1: proof of (22). Let v_j be the solution of the following system:

$$\begin{aligned} \mathscr{L}_j[v_j] + h_j(x,t,v_j) &= 0 \quad \text{in } \Omega \times \mathbb{R}_{>0}, \\ a_j \frac{\partial v}{\partial \boldsymbol{\nu}} + m_j v_j &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}_{>0}, \\ v_j(\cdot,0) &= \phi_j(\cdot) \quad \text{in } \Omega. \end{aligned}$$

The maximum estimate of v_j is given by (see (49) in Appendix)

$$\sup_{(x,t)\in Q_T} |v_j(x,t)| \leq \sup_{x\in\Omega} |\phi_j(x)| \leq \Phi_j, \forall T > 0. \tag{32}$$

For $T > 0$, we deduce from Theorem 2.3(i) and (32) that

$$\begin{aligned}
&\sup_{(x,t)\in Q_T} |u_j(x,t)| \\
\leq& \sup_{(x,t)\in Q_T} |v_j(x,t)| + \sup_{(x,t)\in Q_T} |u_j(x,t) - v_j(x,t)| \\
\leq& \Phi_j + \frac{1}{\underline{m_j}} \sup_{(x,t)\in\partial_l Q_T} |d_j(x,t) - 0| \\
=& \Phi_j + \frac{1}{\underline{m_j}} \sup_{(x,t)\in\partial_l Q_T} |u_{j-1}(x,t)| \\
\leq& \Phi_j + \frac{1}{\underline{m_j}} \sup_{(x,t)\in Q_T} |u_{j-1}(x,t)| \\
\leq& \Phi_j + \frac{1}{\underline{m_j}} \left(\sup_{(x,t)\in Q_T} |v_{j-1}(x,t)| \right. \\
&\left. + \sup_{(x,t)\in Q_T} |u_{j-1}(x,t) - v_{j-1}(x,t)| \right) \\
\leq& \Phi_j + \frac{1}{\underline{m_j}} \left(\Phi_j + \frac{1}{\underline{m_{j-1}}} \sup_{(x,t)\in\partial_l Q_T} |d_{j-1}(x,t)| \right) \\
=& \Phi_j \left(1 + \frac{1}{\underline{m_j}} \right) + \frac{1}{\underline{m_j} \cdot \underline{m_{j-1}}} \sup_{(x,t)\in\partial_l Q_T} |u_{j-2}(x,t)| \\
\leq& \Phi_j \left(1 + \frac{1}{\underline{m_j}} \right) + \frac{1}{\underline{m_j} \cdot \underline{m_{j-1}}} \sup_{(x,t)\in Q_T} |u_{j-2}(x,t)| \\
\leq& \cdots \\
\leq& \Phi_j \left(1 + \frac{1}{\underline{m_j}} + \frac{1}{\underline{m_j} \cdot \underline{m_{j-1}}} + \cdots + \frac{1}{\underline{m_j} \cdots \underline{m_3}} \right) \\
&+ \frac{1}{\underline{m_j} \cdots \underline{m_2}} \sup_{(x,t)\in Q_T} |u_1(x,t)| \qquad (33) \\
\leq& \Phi_j \left(1 + \frac{1}{\underline{m_j}} + \frac{1}{\underline{m_j} \cdot \underline{m_{j-1}}} + \cdots + \frac{1}{\underline{m_j} \cdots \underline{m_2}} \right) \\
&+ \frac{1}{\underline{m_j} \cdots \underline{m_1}} \sup_{(x,t)\in\partial_l Q_T} |d(x,t)| \\
\leq& \Phi_j \left(1 + \frac{1}{\mathcal{M}_k} + \frac{1}{\mathcal{M}_k^2} + \cdots + \frac{1}{\mathcal{M}_k^{j-1}} \right) \\
&+ \frac{1}{\mathcal{M}_k^j} \sup_{(x,t)\in\partial_l Q_T} |d(x,t)| \\
=& \frac{\mathcal{M}_k}{\mathcal{M}_k - 1} \left(1 - \frac{1}{\mathcal{M}_k^j} \right) \Phi_j + \frac{1}{\mathcal{M}_k^j} \sup_{(x,t)\in\partial_l Q_T} |d(x,t)|,
\end{aligned}$$

which gives (22).

Step 2: proof of (23). For any constant $\delta \in (0, \underline{c_j} - L_j)$, let $w_j := u_j \mathrm{e}^{\delta t}, \widetilde{c}_j := c_j - \delta, \widetilde{h}_j(x,t,w_j) := h_j(x,t,w_j\mathrm{e}^{-\delta t})\mathrm{e}^{\delta t}, \widetilde{d}_j(x,t) := \mathrm{e}^{\delta t} d_j(x,t)$, and $\widehat{\mathscr{L}_j}[w_j] := (w_j)_t - \mathrm{div}\,(a_j \nabla w_j) + \widetilde{c}_j w_j$. By direct computations, we have

$$(\widetilde{\Sigma}_j) \begin{cases} \widehat{\mathscr{L}_j}[w_j] + \widetilde{h}_j(x,t,w_j) = 0 & \text{in } \Omega \times \mathbb{R}_{>0}, \\ a_j \dfrac{\partial w_j}{\partial \boldsymbol{\nu}} + m_j w_j = \widetilde{d}_j & \text{on } \partial\Omega \times \mathbb{R}_{>0}, \\ w_j(\cdot,0) = \phi_j(\cdot) & \text{in } \Omega. \end{cases}$$

Note that $\min_{x\in\overline{\Omega}} \widetilde{c}_j = \underline{c_j} - L_j - \delta > 0$. Then, applying (22) to the system $(\widetilde{\Sigma}_j)$, we obtain

$$\sup_{(x,t)\in Q_T}|w_j(x,t)| \leq \frac{\mathcal{M}_k}{\mathcal{M}_k-1}\left(1-\frac{1}{\mathcal{M}_k^j}\right)\Phi_j + \frac{1}{\mathcal{M}_k^j}\sup_{(x,t)\in\partial_l Q_T}|\widetilde{d}_1(x,t)|, \forall T>0,$$

which implies

$$\sup_{x\in\Omega}|u_j(x,T)| \leq \frac{\mathcal{M}_k}{\mathcal{M}_k-1}\left(1-\frac{1}{\mathcal{M}_k^j}\right)\Phi_j\,\mathrm{e}^{-\delta T} + \frac{1}{\mathcal{M}_k^j}\sup_{(x,t)\in\partial_l Q_T}|d(x,t)|, \forall T>0.$$

Letting $\delta\to \underline{c_j}-L_j$, we obtain (23).

Step 3: proof of (24). Indeed, defining $u_0 := u_k$, we deduce from (33), (19), and Theorem 2.3(i) that

$$\begin{aligned}
&\sup_{(x,t)\in Q_T}|u_j(x,t)|\\
\leq&\Phi_k\left(1+\frac{1}{\mathcal{M}_k}+\cdots+\frac{1}{\mathcal{M}_k^{k-2}}\right)\\
&+\frac{1}{\mathcal{M}_k^{k-1}}\sup_{(x,t)\in Q_T}|u_{j-1}(x,t)|\\
\leq&\Phi_k\left(1+\frac{1}{\mathcal{M}_k}+\cdots+\frac{1}{\mathcal{M}_k^{k-2}}\right)\\
&+\frac{1}{\mathcal{M}_k^{k-1}}\left(\Phi_k+\frac{1}{\mathcal{M}_k}\sup_{(x,t)\in\partial_l Q_T}|u_j(x,t)|\right)\\
\leq&\frac{\mathcal{M}_k}{\mathcal{M}_k-1}\left(1-\frac{1}{\mathcal{M}_k^k}\right)\Phi_k+\frac{1}{\mathcal{M}_k^k}\sup_{(x,t)\in Q_T}|u_j(x,t)|, \qquad (34)
\end{aligned}$$

which along with $\mathcal{M}_k>1$ implies (24).

Step 4: proof of (25). Using transformation as in Step 2, considering $(\widetilde{\Sigma}_j)$ with the boundary conditions given by (19), and applying (24), we get (25). ■

Proof of Proposition 3.3: Indeed, for $j\in[1,k]$, let v_j be the solution of the following system:

$$\begin{aligned}
\mathscr{L}_j[v_j]+h_j(x,t,v_j)=&0 && \text{in } \Omega\times\mathbb{R}_{>0},\\
v_j=&0 && \text{on } \partial\Omega\times\mathbb{R}_{>0},\\
v_j(\cdot,0)=&\phi_j(\cdot) && \text{in } \Omega,
\end{aligned}$$

Analogous to (32), the maximum estimate of v_j is given by

$$\sup_{(x,t)\in Q_T}|v_j(x,t)|\leq\sup_{x\in\Omega}|\phi_j(x)|, \forall T>0. \qquad (35)$$

First, we prove (22) For $T>0$, we deduce from Theorem 2.3(ii) and (35) that

$$\begin{aligned}
&\sup_{(x,t)\in Q_T}|u_j(x,t)|\\
\leq&\sup_{(x,t)\in Q_T}|v_j(x,t)|+\sup_{(x,t)\in Q_T}|u_j(x,t)-v_j(x,t)|\\
\leq&\Phi_j+\frac{1}{\mathcal{A}_k}\sup_{(x,t)\in Q_T}|f_j(x,t)-0|+\sup_{(x,t)\in\partial_l Q_T}|d_j(x,t)-0|\\
\leq&\Phi_j+\frac{1}{\mathcal{A}_k}\sup_{(x,t)\in Q_T}|u_{j-1}(x,t)|+\sup_{(x,t)\in\partial_l Q_T}|d_j(x,t)|\\
\leq&\Phi_j+\frac{1}{\mathcal{A}_k}\left(\sup_{(x,t)\in Q_T}|v_{j-1}(x,t)|\right.\\
&\left.+\sup_{(x,t)\in Q_T}|u_{j-1}(x,t)-v_{j-1}(x,t)|\right)+\sup_{(x,t)\in\partial_l Q_T}|d_j(x,t)|\\
\leq&\Phi_j+\frac{1}{\mathcal{A}_k}\left(\Phi_j+\frac{1}{\mathcal{A}_k}\sup_{(x,t)\in Q_T}|f_{j-1}(x,t)|\right.\\
&\left.+\sup_{(x,t)\in\partial_l Q_T}|d_{j-1}(x,t)|\right)+\sup_{(x,t)\in\partial_l Q_T}|d_j(x,t)|\\
\leq&\cdots\\
\leq&\frac{\mathcal{A}_k}{\mathcal{A}_k-1}\left(1-\frac{1}{\mathcal{A}_k^j}\right)\Phi_j+\frac{1}{\mathcal{A}_k^j}\sup_{(x,t)\in Q_T}|f(x,t)|\\
&+\sum_{i=1}^{j}\frac{1}{\mathcal{A}_k^{j-i}}\sup_{(x,t)\in\partial_l Q_T}|d_i(x,t)|, \qquad (36)
\end{aligned}$$

which gives (26). Then, by (26) and using the technique of transformation as in the proof of Proposition 3.2, we obtain (27).

Now we prove (28). Without loss of generality, we consider the case of $j=k$. Analogous to (36) (see also (34)), we have

$$\begin{aligned}
&\sup_{(x,t)\in Q_T}|u_k(x,t)|\\
\leq&\frac{\mathcal{A}_k}{\mathcal{A}_k-1}\left(1-\frac{1}{\mathcal{A}_k^k}\right)\Phi_k+\frac{1}{\mathcal{A}_k^k}\sup_{(x,t)\in Q_T}|f_1(x,t)|\\
&+\sum_{i=1}^{k}\frac{1}{\mathcal{A}_k^{k-i}}\sup_{(x,t)\in\partial_l Q_T}|d_i(x,t)|,\\
=&\frac{\mathcal{A}_k}{\mathcal{A}_k-1}\left(1-\frac{1}{\mathcal{A}_k^k}\right)\Phi_k+\sum_{i=1}^{k}\frac{1}{\mathcal{A}_k^{k-i}}\sup_{(x,t)\in\partial_l Q_T}|d_i(x,t)|\\
&+\frac{1}{\mathcal{A}_k^k}\sup_{(x,t)\in Q_T}|u_k(x,t)|,
\end{aligned}$$

which along with $\mathcal{A}_k>1$ gives (28).

Finally, by (28), and using the technique of transformation as in the proof of Proposition 3.2, we obtain (29). ■

IV. Concluding remarks

This paper proposed a new method for establishing the ISS in the spatial sup-norm for nonlinear parabolic PDEs with boundary and in-domain disturbances. More precisely, we introduced the notion of *RKES* to describe the uniform dependence of solutions on the external disturbances. Based on RKES in the (spatial and time) sup-norm, we proved the ISS in the spatial sup-norm for a class of higher dimensional nonlinear PDEs with Dirichlet and Robin boundary disturbances, respectively. One example was provided to illustate the obtained ISS results. In addition, as an application of the introduced notion of *RKES*, we also established stability estimates in the sup-norm and spatial sup-norm for a class of parabolic systems in cascade coupled over the domain and on the boundary of the domain, respectively.

It should be mentioned that the approach presented in this paper is well suited for ISS analysis of weak solutions to higher dimensional nonlinear PDEs with Dirichlet or Robin boundary conditions. However, it seems to be difficult to apply the proposed method to obtain the ISS in the spatial sup-norm for PDEs with Neumann boundary disturbances due to the usage of De Giorgi iteration. Therefore, there is a need to overcome this obstacle and establish ISS estimates in the spatial sup-norm for a wider class of PDEs with various boundary disturbances by a unified approach, which will be considered in our future work.

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Appendix A
Proofs of main results

We present some basic Sobolev embedding inequalities that are used in the proofs of stabilities.

Lemma A.1 (Theorem 1.3.2 and 1.3.4 of [26]): Let Ω be a bounded, open subset of $\mathbb{R}^n (n \geq 1)$, and suppose that $\partial\Omega$ is C^1. For $n = 1, 2$ and $q \in (2, +\infty)$, or $n \geq 3$ and $q \in (2, \frac{2n}{n-2})$, the following inequalities hold true:

(i) $\|v\|_{L^q(\Omega)} \leq C_S(\|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}), \forall v \in W^{1,2}(\Omega)$,

(ii) $\|v\|_{L^q(\Omega)} \leq C_P\|\nabla v\|_{L^2(\Omega)}, \forall v \in W_0^{1,2}(\Omega)$,

where C_S and C_P are positive constants depending only on q, n, and Ω.

Proof of Theorem 2.3: We prove first Theorem 2.3(i). Let u_i be the solution of the system $\Sigma(\mathbb{U}, \mathbb{F}, \mathbb{D})$ corresponding to the data $(u^0, f_i, d_i) \in W^{1,p}(\Omega) \times C(\overline{\Omega} \times \mathbb{R}_{\geq 0}; \mathbb{R}) \times C(\partial\Omega \times \mathbb{R}_{\geq 0}; \mathbb{R}), i = 1, 2$.

Consider $w = u_1 - u_2$, which satisfies:

$$\mathscr{L}[w] + h(x, t, u_1) - h(x, t, u_2) = \widetilde{f} \quad \text{in } \Omega \times \mathbb{R}_{>0}, \tag{37a}$$

$$a\frac{\partial w}{\partial \boldsymbol{\nu}} + mw = \widetilde{d} \quad \text{on } \partial\Omega \times \mathbb{R}_{>0}, \tag{37b}$$

$$w(\cdot, 0) = 0 \quad \text{in } \Omega, \tag{37c}$$

with $\widetilde{d} := d_1 - d_2$ and $\widetilde{f} := f_1 - f_2$.

We proceed by De Giorgi iteration. Specifically, for any $T > 0$, let $k_0 := \max\{0, \frac{1}{\underline{m}} \sup_{\partial_l Q_T} \widetilde{d}\}$. For $k \geq k_0$ and $0 < t_1 < t_2 < T$, let $\eta(x, t) := (w(x, t) - k)_+ \chi_{[t_1,t_2]}(t)$, where $s_+ := \max\{s, 0\}$ for $s \in \mathbb{R}$, and $\chi_{[t_1,t_2]}(t)$ is the character function on $[t_1, t_2]$. By virtue of Proposition 2.1, and that $W^{1,p}(\Omega) \hookrightarrow W^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow (W^{1,2}(\Omega))' \hookrightarrow (W^{1,p}(\Omega))'$ for $p \geq 2$, we have $\eta \in L^\infty((0, T); (W^{1,p}(\Omega))')$ with $\eta_t \in L^\infty((0, T); L^{p'}(\Omega))$. Then, η can be chosen as a test function for (37).

By the Fubini's theorem and integrating by parts, we have

$$\begin{aligned}
&-\int_0^T \int_\Omega w\eta_t \mathrm{d}x\mathrm{d}t \\
=&-\int_\Omega w(x, T)\eta(x, T)\mathrm{d}x + \int_\Omega w(x, 0)\eta(x, 0)\mathrm{d}x \\
&+\int_0^T \int_\Omega w_t\eta \mathrm{d}x\mathrm{d}t \\
=&\int_0^T \int_\Omega w_t\eta \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_0^T \int_\Omega (w - k)_t(w - k)_+\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t \\
&-\int_0^T \int_{\partial\Omega} (\widetilde{d} - mw)(w - k)_+\chi_{[t_1,t_2]}(t)\mathrm{d}S\mathrm{d}t \\
&+\int_0^T \int_\Omega a|\nabla(w - k)_+|^2\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t \\
&+\int_0^T \int_\Omega cw(w - k)_+\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t \\
&+\int_0^T \int_\Omega (h(x, t, u_1) - h(x, t, u_2))(w - k)_+\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t \\
=&\int_0^T \int_\Omega \widetilde{f}(w - k)_+\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{38}$$

Note that for $w \geq k \geq k_0 \geq 0$, we have $-mw \leq -mk_0 \leq -\underline{m}k_0 \leq -\sup_{\partial_l Q_T} \widetilde{d}$, which implies that

$$\begin{aligned}
&\int_0^T \int_{\partial\Omega} (\widetilde{d} - mw)(w - k)_+\chi_{[t_1,t_2]}(t)\mathrm{d}S\mathrm{d}t \\
\leq&\int_0^T \int_{\partial\Omega} (\widetilde{d} - \sup_{\partial_l Q_T} \widetilde{d})(w - k)_+\chi_{[t_1,t_2]}(t)\mathrm{d}S\mathrm{d}t \\
\leq&0.
\end{aligned} \tag{39}$$

In addition, for $w \geq k \geq k_0 \geq 0$, it follows that $u_1 = u_2 + w \geq u_2$, which and (8) give

$$\begin{aligned}
&\int_0^T\int_\Omega (h(x, t, u_1) - h(x, t, u_2))(w - k)_+\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t \\
\geq&-\int_0^T\int_\Omega Lw(w - k)_+\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{40}$$

It is clear that

$$\begin{aligned}
&\int_0^T \int_\Omega (c - L)w(w - k)_+\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t \\
=&\int_0^T \int_\Omega (c - L)((w - k)_+)^2\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t \\
&+\int_0^T \int_\Omega (c - L)k(w - k)_+\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}s \\
\geq&(\underline{c} - L)\int_0^T \int_\Omega ((w - k)_+)^2\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{41}$$

Then, by (38), (39), (40), and (41), we obtain

$$\begin{aligned}&\int_0^T\int_\Omega (w-k)_t(w-k)_+\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t\\&+\underline{a}\int_0^T\int_\Omega |\nabla(w-k)_+|^2\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t\\&+(\underline{c}-L)\int_0^T\int_\Omega ((w-k)_+)^2\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t\\\le&\int_0^T\int_\Omega \widetilde{f}(w-k)_+\chi_{[t_1,t_2]}(t)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{1}{2}\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega((w-k)_+)^2\mathrm{d}x\mathrm{d}t+\underline{a}\int_{t_1}^{t_2}\int_\Omega|\nabla(w-k)_+|^2\mathrm{d}x\mathrm{d}t\\&+(\underline{c}-L)\int_{t_1}^{t_2}\int_\Omega((w-k)_+)^2\mathrm{d}x\mathrm{d}t\\\le&\int_{t_1}^{t_2}\int_\Omega\widetilde{f}(w-k)_+\mathrm{d}x\mathrm{d}t,\end{aligned}$$

i.e.,

$$\begin{aligned}&\frac{1}{2}(I_k(t_2)-I_k(t_1))+\underline{a}\int_{t_1}^{t_2}\int_\Omega|\nabla(w-k)_+|^2\mathrm{d}x\mathrm{d}t\\&+(\underline{c}-L)\int_{t_1}^{t_2}\int_\Omega((w-k)_+)^2\mathrm{d}x\mathrm{d}t\\\le&\int_{t_1}^{t_2}\int_\Omega\widetilde{f}(w-k)_+\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $I_k(t):=\int_\Omega((w(x,t)-k)_+)^2\mathrm{d}x$.

Suppose that $I_k(t_0)=\max_{t\in[0,T]}I_k(t)$ with some $t_0\in[0,T]$. Due to $I_k(0)=0$ and $I_k(t)\ge 0$, we can assume that $t_0\in(0,T]$ without loss of generality.

If $t_0=T$, then $I_k'(T)\ge 0$. Thus $I_k'(t)\ge 0$ on $(T-\delta,T]$ for some $\delta>0$. Then, there exists a sufficiently small constant $\varepsilon>0$ such that $I_k(T-\varepsilon)-I_k(T-2\varepsilon)\ge 0$. Taking $t_2=T-\varepsilon$ and $t_1=T-2\varepsilon>0$, we obtain

$$\begin{aligned}&\frac{\underline{a}}{\varepsilon}\int_{T-2\varepsilon}^{T-\varepsilon}\int_\Omega|\nabla(w-k)_+|^2\mathrm{d}x\mathrm{d}t\\&+\frac{\underline{c}-L}{\varepsilon}\int_{T-2\varepsilon}^{T-\varepsilon}\int_\Omega((w-k)_+)^2\mathrm{d}x\mathrm{d}t\\\le&\frac{1}{\varepsilon}\int_{T-2\varepsilon}^{T-\varepsilon}\int_\Omega|\widetilde{f}|(w-k)_+\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Letting $\varepsilon\to 0^+$, we get for such $t_0:=T$:

$$\begin{aligned}&\underline{a}\int_\Omega|\nabla(w(x,t_0)-k)_+|^2\mathrm{d}x\\&+(\underline{c}-L)\int_\Omega((w(x,t_0)-k)_+)^2\mathrm{d}x\\\le&\int_\Omega|f(x,t_0)|(w(x,t_0)-k)_+\mathrm{d}x.\end{aligned}\tag{42}$$

If $t_0\in(0,T)$, we can take $t_2=t_0$ and $t_1=t_0-\varepsilon>0$ for a small $\varepsilon>0$. Analogously, we can obtain (42). Thus, (42) holds true whenever $t_0\in(0,T]$.

Using Lemma A.1(i), we have

$$\begin{aligned}&\|(w(x,t_0)-k)_+\|^2_{L^q(\Omega)}\\\le&2C_S^2\Big(\|(w(x,t_0)-k)_+\|^2_{L^2(\Omega)}\\&+\|\nabla(w(x,t_0)-k)_+\|^2_{L^2(\Omega)}\Big),\end{aligned}\tag{43}$$

where q and C_S are the same as in Lemma A.1(i).

Let $A_k(t):=\{x\in\Omega;w(x,t)>k\}$. By (42), (43), $\underline{a}>0$, and $\underline{c}-L>0$, we have

$$\begin{aligned}&\frac{\min\{\underline{a},\underline{c}-L\}}{2C_S^2}\left(\int_{A_k(t_0)}|w(x,t_0)-k|^q\mathrm{d}x\right)^{\frac{2}{q}}\\=&\frac{\min\{\underline{a},\underline{c}-L\}}{2C_S^2}\|(w(x,t_0)-k)_+\|^2_{L^q(\Omega)}\\\le&\underline{a}\|\nabla(w(x,t_0)-k)_+\|^2_{L^2(\Omega)}\\&+(\underline{c}-L)\|(w(x,t_0)-k)_+\|^2_{L^2(\Omega)}\\\le&\int_\Omega|\widetilde{f}(x,t_0)|(w(x,t_0)-k)_+\mathrm{d}x.\end{aligned}\tag{44}$$

By the Hölder's inequality, we have

$$\begin{aligned}&\int_\Omega|\widetilde{f}(x,t_0)|(w(x,t_0)-k)_+\mathrm{d}x\\\le&\left(\int_{A_k(t_0)}|\widetilde{f}(x,t_0)|^{q'}\mathrm{d}x\right)^{\frac{1}{q'}}\left(\int_{A_k(t_0)}|w(x,t_0)-k|^q\mathrm{d}x\right)^{\frac{1}{q}},\end{aligned}$$

which along with (44) gives

$$\begin{aligned}&\left(\int_{A_k(t_0)}|w(x,t_0)-k|^q\mathrm{d}x\right)^{\frac{1}{q}}\\\le&\frac{2C_S^2}{\min\{\underline{a},\underline{c}-L\}}\left(\int_{A_k(t_0)}|\widetilde{f}(x,t_0)|^{q'}\mathrm{d}x\right)^{\frac{1}{q'}}\\\le&\frac{2C_S^2}{\min\{\underline{a},\underline{c}-L\}}\|\widetilde{f}\|_{L^\infty(Q_T)}|A_k(t_0)|^{\frac{1}{q'}}\\\le&\frac{2C_S^2}{\min\{\underline{a},\underline{c}-L\}}\|\widetilde{f}\|_{L^\infty(Q_T)}\mu_k^{\frac{1}{q'}},\end{aligned}$$

where $q':=\frac{q}{q-1}$, $\mu_k:=\sup_{t\in(0,T)}|A_k(t)|$, and $|A_k(t_0)|$ denotes the n-dimensional Lebesgue measure of $A_k(t_0)$. Then, we may proceed exactly as in the proof of [26, Theorem 4.2.1] to obtain

$$w\le k_0+\frac{2C_S^2}{\min\{\underline{a},\underline{c}-L\}}|\Omega|^{\frac{q-2}{q}}2^{\frac{3q-4}{2q-4}}\|\widetilde{f}\|_{L^\infty(Q_T)}\quad\text{a.e. in } Q_T,$$

which along with the continuity of w and f yields

$$\begin{aligned}w\le&\max\left\{0,\frac{1}{\underline{m}}\sup_{\partial_l Q_T}\widetilde{d}\right\}\\&+\frac{2C_S^2}{\min\{\underline{a},\underline{c}-L\}}|\Omega|^{\frac{q-2}{q}}2^{\frac{3q-4}{2q-4}}\sup_{Q_T}|\widetilde{f}|\quad\text{in } Q_T.\end{aligned}\tag{45}$$

We need to prove the lower boundedness of w. Indeed, it suffices to set $\overline{w}:=-w=u_2-u_1$, and consider the equation

$$\begin{aligned}\mathscr{L}[\overline{w}]+h(x,t,u_2)-h(x,t,u_1)=&-\widetilde{f}\text{ in }\Omega\times\mathbb{R}_{>0},\\a\frac{\partial\overline{w}}{\partial\boldsymbol{\nu}}+m\overline{w}=&-\widetilde{d}\text{ on }\partial\Omega\times\mathbb{R}_{>0},\\\overline{w}(\cdot,0)=&0\quad\text{ in }\Omega.\end{aligned}$$

Let $\overline{k}_0 = \max\{0, \frac{1}{\underline{m}} \sup_{\partial_l Q_T}(-\widetilde{d})\}$. Proceeding as above, we obtain

$$\begin{aligned}-w =&\overline{w}\\ \leq&\overline{k}_0 + \frac{2C_S^2}{\min\{\underline{a}, \underline{c}-L\}}|\Omega|^{\frac{q-2}{q}} 2^{\frac{3q-4}{2q-4}} \sup_{Q_T}|\widetilde{f}| \text{ in } Q_T.\end{aligned} \tag{46}$$

Finally, by (45) and (46), we have

$$\sup_{Q_T}|w| \leq \frac{1}{\underline{m}} \sup_{\partial_l Q_T}|\widetilde{d}| + \frac{2C_S^2}{\min\{\underline{a}, \underline{c}-L\}}|\Omega|^{\frac{q-2}{q}} 2^{\frac{3q-4}{2q-4}} \sup_{Q_T}|\widetilde{f}|,$$

which gives the result stated in Theorem 2.3(i).

Now we prove Theorem 2.3(ii). Consider (37) by replacing (37b) with

$$w = \widetilde{d} \text{ on } \partial\Omega \times \mathbb{R}_{>0}.$$

For any $T > 0$, let $k_0 := \max\{0, \sup_{\partial_l Q_T} \widetilde{d}\}$. For $k \geq k_0$ and $0 < t_1 < t_2 < T$, let $\eta(x,t) := (w(x,t)-k)_+ \chi_{[t_1,t_2]}(t)$. It suffices to apply De Giorgi iteration as in the proof of Theorem 2.3 (i).

Indeed, if $L \leq \underline{c}$, (42) can be reduced to

$$\underline{a}\int_\Omega |\nabla(w(x,t_0)-k)_+|^2 \mathrm{d}x \leq \int_\Omega |f(x,t_0)|(w(x,t_0)-k)_+ \mathrm{d}x.$$

By Lemma A.1(ii), (43) can be reduced to

$$\|(w(x,t_0)-k)_+\|^2_{L^q(\Omega)} \leq C_P^2 \|\nabla(w(x,t_0)-k)_+\|^2_{L^2(\Omega)},$$

where q and C_P are the same as in Lemma A.1(ii). Hence, (44) becomes

$$\begin{aligned}&\frac{\underline{a}}{C_P^2}\left(\int_{A_k(t_0)} |w(x,t_0)-k|^q \mathrm{d}x\right)^{\frac{2}{q}}\\ \leq&\int_\Omega |\widetilde{f}(x,t_0)|(w(x,t_0)-k)_+ \mathrm{d}x.\end{aligned} \tag{47}$$

Then, analogous to (45), we obtain the following estimate:

$$w \leq \max\left\{0, \sup_{\partial_l Q_T} \widetilde{d}\right\} + \frac{C_P^2}{\underline{a}}|\Omega|^{\frac{q-2}{q}} 2^{\frac{3q-4}{2q-4}} \sup_{Q_T}|\widetilde{f}| \text{ in } Q_T.$$

The lower boundedness of w can be estimated in the similar way, and the boundedness of w specified in (10) is guaranteed.

Now for $\underline{c} - L > 0$, we shall determine an appropriate coefficient of $(\int_{A_k(t_0)} |w(x,t_0)-k|^q \mathrm{d}x)^{\frac{2}{q}}$. Indeed, for $v \in W_0^{1,2}(\Omega)$, we also have $v \in W^{1,2}(\Omega)$. Thus, (47) and (44) hold true at the same time. Then, we obtain

$$\begin{aligned}&\left(\int_{A_k(t_0)} |w(x,t_0)-k|^q \mathrm{d}x\right)^{\frac{2}{q}}\\ \leq&C_0\int_\Omega |\widetilde{f}(x,t_0)|(w(x,t_0)-k)_+ \mathrm{d}x,\end{aligned}$$

where $C_0 := \min\left\{\frac{2C_S^2}{\min\{\underline{a},\underline{c}-L\}}, \frac{C_P^2}{\underline{a}}\right\}$. Finally, (11) is guaranteed. ■

Proof of Theorem 2.4: We only prove Theorem 2.4(i), since the proof of Theorem 2.4(ii) can be proceeded in the same way.

We first prove that the system (2) with the Robin boundary condition (3) is 0-UGAS w.r.t. the state in the spatial sup-norm. Indeed, let v be the solution of the following equation:

$$\mathscr{L}[v] + h(x,t,v) = 0 \quad \text{in } \Omega \times \mathbb{R}_{>0}, \tag{48a}$$
$$a\frac{\partial v}{\partial \boldsymbol{\nu}} + mv = 0 \quad \text{on } \partial\Omega \times \mathbb{R}_{>0}, \tag{48b}$$
$$v(\cdot,0) = u^0(\cdot) \text{ in } \Omega. \tag{48c}$$

For any constant $\delta \in (0, \underline{c}-L)$, let $w := v\,\mathrm{e}^{\delta t}, \widetilde{c} := c - \delta, \widetilde{h}(x,t,w) := h(x,t,w\mathrm{e}^{-\delta t})\,\mathrm{e}^{\delta t}, \widetilde{f}(x,t) := \widetilde{d}(x,t) := 0$. By direct computations, we have

$$\begin{aligned}w_t - \text{div }(a\nabla w) + \widetilde{c}w + \widetilde{h}(x,t,w) =&\widetilde{f} \quad \text{in } \Omega \times \mathbb{R}_{>0},\\ a\frac{\partial w}{\partial \boldsymbol{\nu}} + mw =&\widetilde{d} \quad \text{on } \partial\Omega \times \mathbb{R}_{>0},\\ w(\cdot,0) =&u^0(\cdot) \text{ in } \Omega.\end{aligned}$$

Note that $\widetilde{h}(x,t,\xi_1) - \widetilde{h}(x,t,\xi_2) \geq -L(\xi_1 - \xi_2)$ for all $x \in \overline{\Omega}, t \in \mathbb{R}_{\geq 0}, \xi_1, \xi_2 \in \mathbb{R}$, and $\min_{x\in\overline{\Omega}} \widetilde{c} = \underline{c} - \delta > L$. Thus, we can apply De Giorgi iteration as in the proof of Theorem 2.3(i) and obtain

$$\begin{aligned}&\sup_{(x,t)\in Q_T} |w(x,t)|\\ \leq&\max\Bigg\{\sup_{x\in\Omega}|u^0(x)|, \frac{1}{\underline{m}}\sup_{(x,t)\in\partial_l Q_T}|\widetilde{d}(x,t)|,\\ &\frac{2C_S^2}{\min\{\underline{a},\underline{c}-L\}}|\Omega|^{\frac{q-2}{q}} 2^{\frac{3q-4}{2q-4}} \sup_{(x,t)\in Q_T}|\widetilde{f}(x,t)|\Bigg\}\\ =&\sup_{x\in\Omega}|u^0(x)|, \forall T > 0,\end{aligned} \tag{49}$$

which along with the continuity of w in $t = T$ implies that

$$\sup_{x\in\Omega}|w(x,T)| \leq \sup_{(x,t)\in Q_T}|w(x,t)| \leq \sup_{x\in\Omega}|u^0(x)|.$$

It follows that

$$\sup_{x\in\Omega}|v(x,T)| \leq \mathrm{e}^{-\delta T}\sup_{x\in\Omega}|u^0(x)|.$$

Letting $\delta \to \underline{c} - L$, we have

$$\sup_{x\in\Omega}|v(x,T)| \leq \mathrm{e}^{-(\underline{c}-L)T}\sup_{x\in\Omega}|u^0(x)|. \tag{50}$$

Finally, by (9), (50), and Proposition 2.2, we conclude that the system (2) with the Robin boundary condition (3) is EISS in the spatial sup-norm w.r.t. in-domain and boundary disturbance (f,d) in $C(\overline{\Omega}\times\mathbb{R}_{\geq 0};\mathbb{R})\times C(\partial\Omega\times\mathbb{R}_{\geq 0};\mathbb{R})$, having the estimate (12). ■

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