International Communications in Heat and Mass Transfer 127 (2021) 105496

Contents lists available at ScienceDirect

International Communications in Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ichmt

An approximation of the finite line source solution to model thermal interactions between geothermal boreholes

Massimo Cimmino

Department of Mechanical Engineering, Polytechnique Montréal, C.P. 6079, succ. Centre-ville, Montreal, Quebec H3C 3A7, Canada

ARTICLE INFO

Keywords:
Boreholes
Geothermal
Finite line source
Thermal response factor
Ground heat exchangers

ABSTRACT

This paper presents an approximation of the finite line source (FLS) solution for the simulation of geothermal systems. The FLS solution requires the evaluation of an integral involving a product of the error function which cannot be solved analytically. An approximate solution of the FLS solution can be obtained using an approximation of the Gaussian *Q*-function in the form of a weighted sum of exponentials. The new approximation of the FLS solution is shown to be adequately accurate for simulations. Substantial gains in computational speed are obtained, in one case decreasing the computational time for 1000 evaluations of the FLS solution from 3.52 s down to 20 milliseconds.

1. Introduction

The finite line source (FLS) is used to predict ground temperature variations in geothermal bore fields comprising vertical geothermal boreholes. It's main usage is to evaluate thermal response factors, also called *g*-functions [1], that can later be used to design [2], simulate [3], and control [4] geothermal systems. It can also directly be superimposed in time and space to construct detailed models of geothermal systems [5–7].

The FLS solution was first proposed by Eskilson [1] to approximate the *g*-function of a single borehole. Zeng et al. [8] later used spatial superposition to obtain *g*-functions of bore fields comprising multiple boreholes. Their formulation required the evaluation of a double integral, which was later simplified to a single integral by Lamarche and Beauchamp [9] and by Claesson and Javed [10]. In both cases, the remaining integral has no analytical solution. Axial discretization along the boreholes is required to properly represent the evolution of temperatures and heat extraction rates in geothermal bore fields. Cimmino and Bernier [11] extended the finite line source solution to cover borehole segments of unequal lengths and at different depths below the surface. An integral also remains in their formulation.

Axial discretization of geothermal boreholes drastically increases the number of evaluations of the FLS solution required to model a geothermal bore field, which increases with the square of the number of segments (or boreholes). Various authors have introduced schemes to accelerate calculation times. Lazzarotto [12] developed a specialized quadrature scheme for the solution of the integral for discretized inclined boreholes. Lamarche [13] used a piecewise-linear profile for heat extraction along the length of boreholes to reduce the require number of segments along the axial discretization of boreholes. Cimmino [14] introduced the concept of similarities to reduce the number of required evaluations of the FLS solution for discretized boreholes. Dusseault et al. [15] used Chebyshev polynomials to approximate the integrand of the FLS solution in the case of non-discretized boreholes. Nguyen and Pasquier [16] interpolate between pre-calculated values of the FLS solution at different distances for non-discretized boreholes of equal lengths. In all aforementioned studies, the calculation of the FLS solution involves intricate numerical procedures.

The objective of this paper is to develop an approximation of the FLS solution. The approximation makes use of a recent approximation of the Gaussian *Q*-function (a function related to the error function) by Tanash and Riihonen [17]. This approximation is used in the field of communications where integrals of the *Q*-function need to be evaluated. The present paper shows that their approximation can also be used in the thermal simulation of geothermal systems, and perhaps other heat transfer applications.

2. Model

2.1. Finite line source solution

As shown on Fig. 1, the FLS solution gives the average temperature variation $\Delta T_{ij}(t)$ along a vertical line i of length H_i buried at a depth D_i from an isothermal surface in a semi-infinite medium of thermal

E-mail address: massimo.cimmino@polymtl.ca.

https://doi.org/10.1016/j.icheatmasstransfer.2021.105496

Available online 5 August 2021

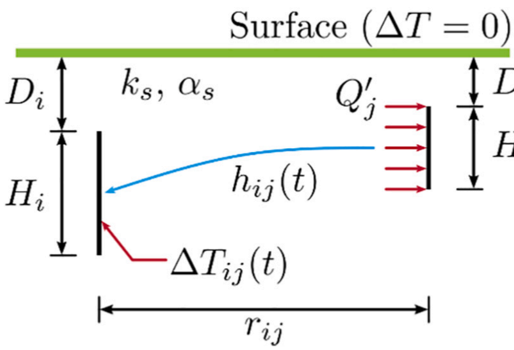

Fig. 1. Finite line source.

conductivity k_s and thermal diffusivity α_s, due to a uniform heat extraction rate Q_j' along a vertical line j of length H_j buried at a depth D_j and at a distance r_{ij} from line i. The FLS solution is given by [10]:

$$\Delta T_{ij}(t) = \frac{Q_j'}{2\pi k_s} \bullet h_{ij}(t), \tag{1}$$

$$h_{ij}(t) = \frac{1}{2H_i} \int_{1/\sqrt{4\alpha_s t}}^{\infty} \frac{1}{s^2} exp\left(-r_{ij}^2 s^2\right) F_{ij}(s) ds, \tag{2}$$

$$\begin{aligned} F_{ij}(s) =& \mathrm{erfint}\left((D_i - D_j + H_i)s\right) - \mathrm{erfint}\left((D_i - D_j)s\right) + \mathrm{erfint}\left((D_i - D_j - H_j)s\right) \\ &- \mathrm{erfint}\left((D_i - D_j + H_i - H_j)s\right) + \mathrm{erfint}\left((D_i + D_j + H_i)s\right) - \mathrm{erfint}\left((D_i + D_j)s\right) \\ &+ \mathrm{erfint}\left((D_i + D_j + H_j)s\right) - \mathrm{erfint}\left((D_i + D_j + H_i + H_j)s\right), \end{aligned} \tag{3}$$

$$\mathrm{erfint}(x) = \int_0^x erf(x')dx' = xerf(x) - \frac{1}{\sqrt{\pi}}\left(1 - exp\left(-x^2\right)\right), \tag{4}$$

where $h_{ij}(t)$ is the finite line source solution, erf(x) is the error function and erfint(x) is the integral of the error function.

Spatial and temporal superposition can be applied to obtain the total temperature variation due to variable heat extraction at multiple lines j:

$$\Delta T_i(t_k) = \sum_j \sum_{p=1}^{k} \frac{Q_j'(t_p)}{2\pi k_s} h_{ij}(t_k - t_{p-1}) \tag{5}$$

2.2. Approximation of the error function

The integral in eq. (2) cannot be solved analytically due to the product of the form $\exp(-ux^2)$ erf $(vx)/x$. However, an approximation of the FLS solution can be obtained by substituting the error function by a suitable approximation. Such an approximation was proposed by Tanash and Riihonen [17] for the Gaussian Q-function which is closely related to the error function. The Gaussian Q-function is defined by:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} exp\left(-\frac{1}{2}x^2\right) dx. \tag{6}$$

Tanash and Riihonen [17] approximated the Gaussian Q-function by a weighted sum of exponentials globally minimizing the absolute or relative error:

$$Q(x) \cong \sum_{n=1}^{N} a_{Q,n} exp\left(-b_{Q,n} x^2\right) \text{ for } x \geq 0, \tag{7}$$

where $a_{Q,n}$ and $b_{Q,n}$ are real positive coefficients. In this paper, the sets of coefficients that minimize the absolute error (up to $N = 25$) are used to approximate the error function. The reader is referred to the original paper of Tanash and Riihonen [17] and their supplementary dataset for the values of coefficients.

The error function is defined by:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x exp\left(-x^2\right) dx = 1 - 2 \cdot Q\left(\sqrt{2} \cdot x\right) \tag{8}$$

It is thus possible to approximate the error function using the coefficients of the Gaussian Q-function:

$$erf(x) \cong \sum_{n=0}^{N} a_n exp\left(-b_n x^2\right) \text{ for } x \geq 0, \tag{9}$$

$$a_0 = 1, a_n = -2a_{Q,n}, b_0 = 0, b_n = 2b_{Q,n}. \tag{10}$$

2.3. Approximation of the finite line source solution

The integral of the finite line source solution can be solved analytically using the approximation of the error function. A compact expression is introduced for the function $F_{ij}(s)$ in eq. (3):

$$F_{ij}(s) = \sum_{m=1}^{8} c_m \mathrm{erfint}(d_m s), \tag{11}$$

$$c_m = (-1)^{m+1}, \tag{12}$$

$$\begin{aligned} \{d_m\} = \{&(D_i - D_j + H_i), (D_i - D_j), (D_i - D_j - H_j), (D_i - D_j + H_i - H_j), \\ &(D_i + D_j + H_i), (D_i + D_j), (D_i + D_j + H_j), (D_i + D_j + H_i + H_j)\}, \end{aligned} \tag{13}$$

Substituting the integral of the error function by its definition in eq. (4):

$$F_{ij}(s) = \sum_{m=1}^{8} c_m \left[d_m s erf(d_m s) - \frac{1}{\sqrt{\pi}}\left(1 - exp\left(-d_m^2 s^2\right)\right)\right], \tag{14}$$

and then substituting the error function by its approximation in eq. (9):

$$\widetilde{F}_{ij}(s) = \sum_{n=0}^{N}\left[a_n \sum_{m=1}^{8} c_m |d_m| s exp\left(-b_n d_m^2 s^2\right)\right] - \sum_{m=1}^{8}\left[\frac{c_m}{\sqrt{\pi}}\left(1 - exp\left(-d_m^2 s^2\right)\right)\right] \tag{15}$$

In eq. (15), the absolute value of the coefficient d_m is introduced since the approximation is only valid for positive arguments of the error function, F_{ij} is an even function, and s is positive.

An approximation of the FLS solution $\widetilde{h}_{ij}(t)$ is then given by a sum of three integrals:

$$\widetilde{h}_{ij}(t) = \frac{1}{2H_i} \sum_{n=0}^{N} a_n \sum_{m=1}^{8} c_m |d_m| G_{1,mn}(t) + \frac{-1}{2\sqrt{\pi} H_i} G_2(t) \sum_{m=1}^{8} c_m + \frac{1}{2\sqrt{\pi} H_i} \sum_{m=1}^{8} c_m G_{3,m}(t), \tag{16}$$

$$G_{1,mn}(t) = \int_{1/\sqrt{4\alpha_s t}}^{\infty} \frac{1}{s} exp\left(-\left(r_{ij}^2 + b_n d_m^2\right) s^2\right) ds, \tag{17}$$

$$G_2(t) = \int_{1/\sqrt{4\alpha_s t}}^{\infty} \frac{1}{s^2} exp\left(-r_{ij}^2 s^2\right) ds, \tag{18}$$

$$G_{3,m}(t) = \int_{1/\sqrt{4\alpha_s t}}^{\infty} \frac{1}{s^2} exp\left(-\left(r_{ij}^2 + d_m^2\right) s^2\right) ds. \tag{19}$$

The three integrals, $G_{1,mn}(t)$, $G_2(t)$ and $G_{3,m}(t)$ have analytical solutions:

$$G_{1,mn}(t) = \frac{1}{2} \mathrm{E}_1\left(\frac{r_{ij}^2 + b_n d_m^2}{4\alpha_s t}\right), \tag{20}$$

Table 1
Geometry of test cases.

Case	H_i [m]	D_i [m]	H_j [m]	D_j [m]	r_{ij} [m]
A	150	4	150	4	0.075
B	10	144	10	4	0.075
C	10	4	10	144	95.46

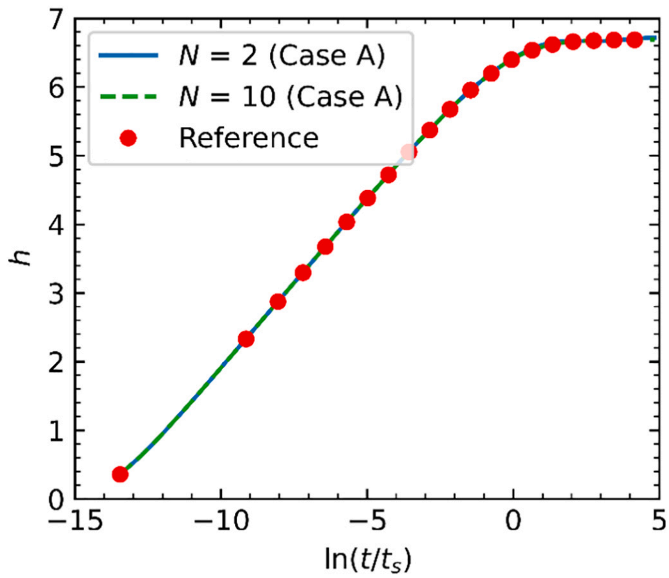

Fig. 2. Thermal response factor (Case A).

$$G_2(t) = \sqrt{4\alpha_s t}\cdot exp\left(\frac{r_{ij}^2}{4\alpha_s t}\right) - r_{ij}\sqrt{\pi}\cdot erfc\left(\frac{r_{ij}}{\sqrt{4\alpha_s t}}\right), \tag{21}$$

$$G_{3,m}(t) = \sqrt{4\alpha_s t}\cdot exp\left(\frac{r_{ij}^2 + d_m^2}{4\alpha_s t}\right) - \sqrt{r_{ij}^2 + d_m^2}\cdot\sqrt{\pi}\cdot erfc\left(\sqrt{\frac{r_{ij}^2 + d_m^2}{4\alpha_s t}}\right). \tag{22}$$

Note that the function $G_2(t)$ does not need to be evaluated since the sum $\sum_{m=1}^{8} c_m$ in eq. (16) is equal to zero.

3. Results

The approximation of the FLS solution is tested on three cases that arise in the simulation of geothermal bore fields. The geometrical parameters of the test cases are presented in Table 1. The thermal diffusivity is $\alpha_s = 10^{-6}$ m^2/s. Case A corresponds to the temperature variation at the wall of a borehole of length $H = 150$ m and radius $r_b = 0.075$ m buried at a depth $D = 4$ m from the ground surface due to heat extracted from the same borehole (in this case, $r_{ij} = r_b$). Case B corresponds to the temperature variation at the wall of the bottom 10 m of the borehole due to heat extracted from the first 10 m of the borehole. Case C corresponds to the temperature variation at the top 10 m of a borehole due to heat extracted from the bottom 10 m of another borehole at a radial distance $r_{ij} = 95.46$ m (corresponding to the furthest positioned boreholes in a square grid of 10 × 10 boreholes with a spacing of 7.5 m between adjacent boreholes).

Eqs. (2) and (16) are implemented into Python, using SciPy [18] for the evaluation of the integral in Eq. (2). The FLS solution is evaluated at 1000 exponentially spaced time steps, ranging from $t_1 = 1$ h to $t_{1000} =$ 10,000 years. Figs. 2-4 present the FLS solution (as a function of the dimensionless time $\ln(t/t_S)$, with $t_s = H_j^2/9\alpha_s$) for the three cases using the approximation with sets of coefficients corresponding to $N = 2$ and $N = 10$. It is shown that the approximation is accurate for sufficiently large N. As observed on Figs. 3 and 4 (and barely visible on Fig. 2), the

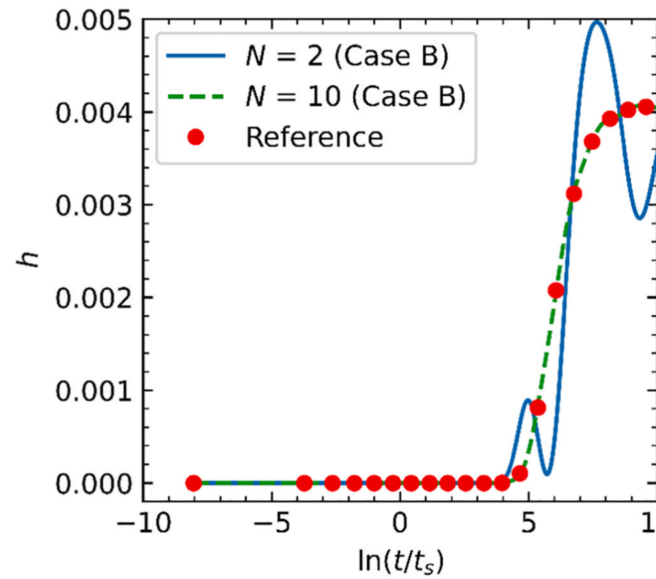

Fig. 3. Thermal response factor (Case B).

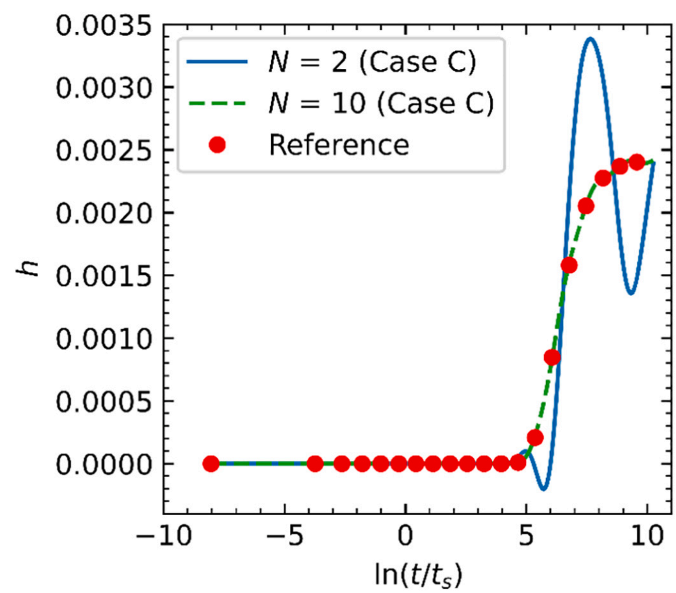

Fig. 4. Thermal response factor (Case C).

approximation of the FLS presents an oscillatory behavior for low values of N. This is in line with the results of Tanash and Riihonen [17] that showed that the error on their approximation of the Gaussian Q-function presents an oscillatory behavior.

Figs. 5-7 present the absolute error for the three cases. It is shown that the absolute error decreases with increasing values of N. The maximum errors are found at larger values of time, where the values of the FLS solution are also maximal. Fig. 8 shows the variation of the maximum absolute error for all three cases for all values of N. For reference, the maximum absolute errors using $N = 10$ are 1.495×10^{-5}, 2.842×10^{-5} and 2.842×10^{-5} for cases A, B and C, respectively. The maximum absolute errors using $N = 25$ are 3.544×10^{-8}, 1.609×10^{-7} and 1.609×10^{-7}. In all cases, the calculation times for the evaluation of the approximation of the FLS at all 1000 time steps are 20 ms and 49 ms for $N = 10$ and $N = 25$, respectively. Meanwhile, the calculation times for the evaluation of Eq. (2) are 4.02 s, 3.66 s and 3.52 s, respectively.

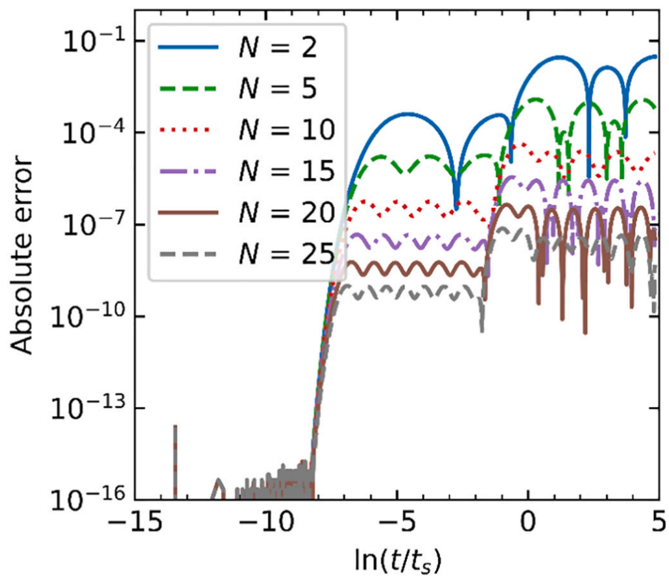

Fig. 5. Absolute error (Case A).

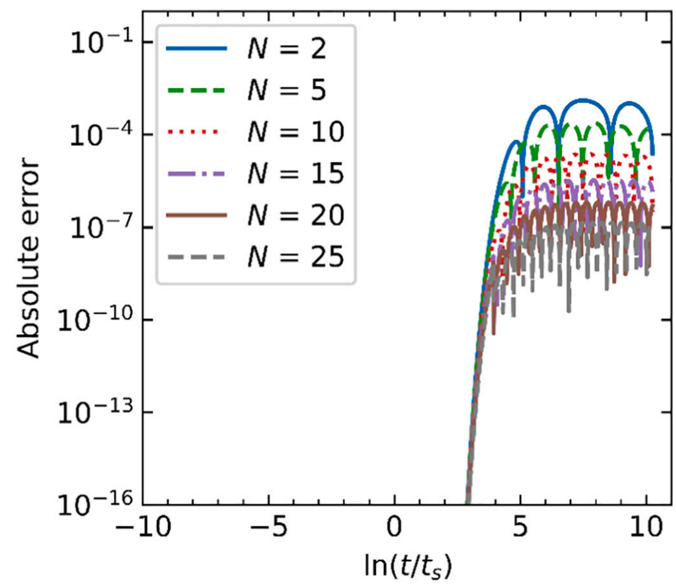

Fig. 7. Absolute error (Case C).

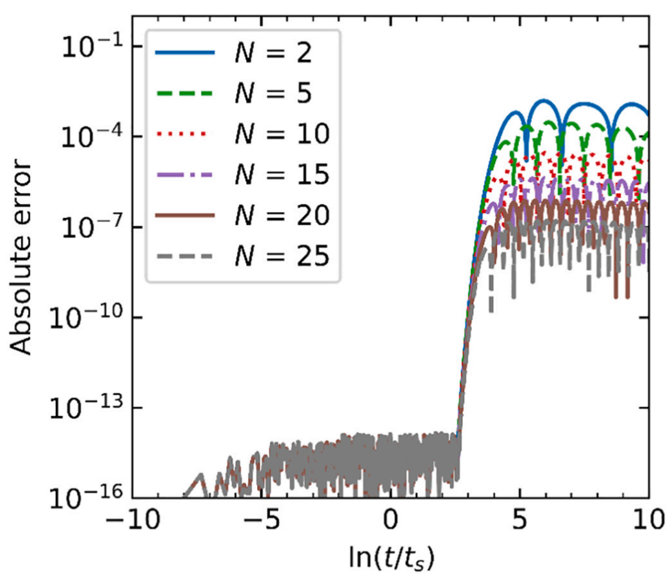

Fig. 6. Absolute error (Case B).

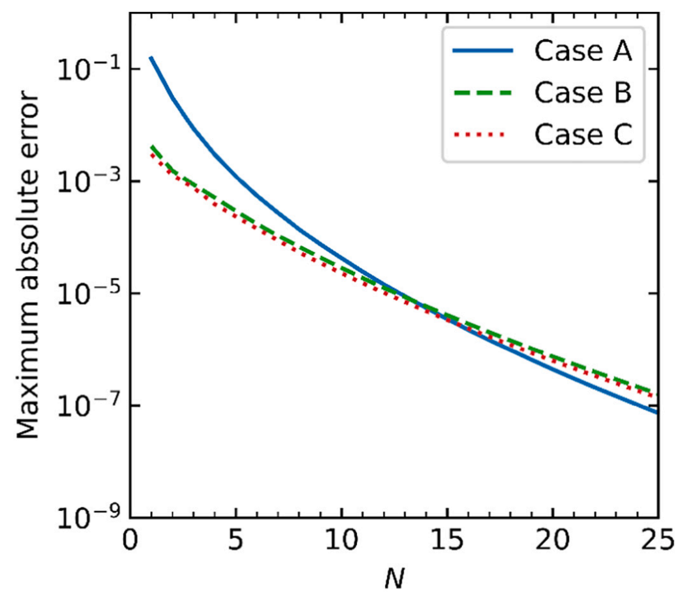

Fig. 8. Maximum absolute error.

4. Conclusion

An approximation of the finite line source (FLS) solution is presented, making use of the approximation of the Gaussian *Q*-function by Tanash and Riihonen [17] to simplify the integral and allow for an analytical solution. The approximation of the FLS is fast and accurate for a sufficiently large number of terms *N*. The approximation of the FLS solution could readily replace the integral formulation of the FLS for the simulation of geothermal systems and lead to substantial gains in computational speed. In the future, the approximation (or other similar approximations) of the error function should be considered in other FLS-type solutions arising in the simulation of geothermal boreholes, such as the moving finite line source and the inclined finite line source.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

References

[1] P. Eskilson, Thermal Analysis of Heat Extraction Boreholes, University of Lund, 1987.
[2] M. Ahmadfard, M. Bernier, A review of vertical ground heat exchanger sizing tools including an inter-model comparison, Renew. Sust. Energ. Rev. 110 (2019) 247–265, https://doi.org/10.1016/j.rser.2019.04.045.
[3] M.S. Mitchell, J.D. Spitler, Characterization, testing, and optimization of load aggregation methods for ground heat exchanger response-factor models, Sci. Technol. Built Environ. 25 (2019) 1036–1051, https://doi.org/10.1080/23744731.2019.1648936.
[4] I. Cupeiro Figueroa, M. Cimmino, L. Helsen, A methodology for long-term model predictive control of hybrid geothermal systems: the shadow-cost formulation, Energies. 13 (2020) 6203, https://doi.org/10.3390/en13236203.
[5] A. Lazzarotto, A network-based methodology for the simulation of borehole heat storage systems, Renew. Energy 62 (2014) 265–275, https://doi.org/10.1016/j.renene.2013.07.020.
[6] L. Lamarche, Mixed arrangement of multiple input-output borehole systems, Appl. Therm. Eng. (2017), https://doi.org/10.1016/j.applthermaleng.2017.06.060.

[7] M. Cimmino, A finite line source simulation model for geothermal systems with series- and parallel-connected boreholes and independent fluid loops, J. Build. Perform. Simul. 11 (2018) 414–432, https://doi.org/10.1080/19401493.2017.1381993.
[8] H.Y. Zeng, N.R. Diao, Z.H. Fang, A finite line-source model for boreholes in geothermal heat exchangers, Heat Transf. - Asian Res. 31 (2002) 558–567, https://doi.org/10.1002/htj.10057.
[9] L. Lamarche, B. Beauchamp, A new contribution to the finite line-source model for geothermal boreholes, Energy Build. 39 (2007) 188–198, https://doi.org/10.1016/j.enbuild.2006.06.003.
[10] J. Claesson, S. Javed, An analytical method to calculate borehole fluid temperatures for time-scales from minutes to decades, ASHRAE Trans. 117 (2011) 279–288.
[11] M. Cimmino, M. Bernier, A semi-analytical method to generate g-functions for geothermal bore fields, Int. J. Heat Mass Transf. 70 (2014) 641–650, https://doi.org/10.1016/j.ijheatmasstransfer.2013.11.037.
[12] A. Lazzarotto, A methodology for the calculation of response functions for geothermal fields with arbitrarily oriented boreholes – part 1, Renew. Energy 86 (2016) 1380–1393, https://doi.org/10.1016/j.renene.2015.09.056.
[13] L. Lamarche, g-function generation using a piecewise-linear profile applied to ground heat exchangers, Int. J. Heat Mass Transf. 115 (2017) 354–360, https://doi.org/10.1016/j.ijheatmasstransfer.2017.08.051.
[14] M. Cimmino, Fast calculation of the g-functions of geothermal borehole fields using similarities in the evaluation of the finite line source solution, J. Build. Perform. Simul. 11 (2018) 655–668, https://doi.org/10.1080/19401493.2017.1423390.
[15] B. Dusseault, P. Pasquier, D. Marcotte, A block matrix formulation for efficient g-function construction, Renew. Energy 121 (2018) 249–260, https://doi.org/10.1016/J.RENENE.2017.12.092.
[16] A. Nguyen, P. Pasquier, A successive flux estimation method for rapid g-function construction of small to large-scale ground heat exchanger, Renew. Energy 165 (2021) 359–368, https://doi.org/10.1016/j.renene.2020.10.074.
[17] I.M. Tanash, T. Riihonen, Global Minimax approximations and bounds for the Gaussian Q-function by sums of exponentials, IEEE Trans. Commun. 68 (2020) 6514–6524, https://doi.org/10.1109/TCOMM.2020.3006902.
[18] P. Virtanen, R. Gommers, T.E. Oliphant, M. Haberland, T. Reddy, D. Cournapeau, E. Burovski, P. Peterson, W. Weckesser, J. Bright, S.J. van der Walt, M. Brett, J. Wilson, K.J. Millman, N. Mayorov, A.R.J. Nelson, E. Jones, R. Kern, E. Larson, C. J. Carey, i. Polat, Y. Feng, E.W. Moore, J. VanderPlas, D. Laxalde, J. Perktold, R. Cimrman, I. Henriksen, E.A. Quintero, C.R. Harris, A.M. Archibald, A.H. Ribeiro, F. Pedregosa, P. van Mulbregt, SciPy 1.0: fundamental algorithms for scientific computing in Python, Nat. Methods 17 (2020) 261–272, https://doi.org/10.1038/s41592-019-0686-2.