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VIBRATION ANALYSIS OF ANISOTROPIC OPEN CYLINDRICAL SHELLS CONTAINING FLOWING FLUID

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ABSTRACT

A theory is presented for the determination of the effects of a flowing fluid on the vibration characteristics of an open, anisotropic cylindrical shell. The case of an open shell partially or completely filled with liquid is also investigated. The structure may be uniform or non uniform in the circumferential direction. The formulation used is a combination of finite element method and classical shell theory. The displacement functions are derived from exact solutions of the Sanders' shell equations.

The velocity potential and Bernoulli's equation for a liquid finite element yield an expression for fluid pressure as a function of the nodal displacements of the element and three forces (inertial, centrifugal and Coriolis) of the moving fluid. An analytical integration of the fluid pressure over the liquid element leads to three components: mass, stiffness and damping matrices.

Calculations are given to illustrate the dynamic behaviour of open and closed cylindrical shells subjected to a flowing fluid and shells partially or completely filled with liquid. Reasonable agreement is found with others theories and experiments.

1. INTRODUCTION

Knowledge of the vibration characteristics of fluid-filled cylindrical shells and panels is of considerable practical interest, since cylindrical shells and panels are commonly used to contain or convey fluids. There are many ways in which the presence of the fluid may influence the dynamics of the structure. If the structure contains a stationary gas at low pressure, then the vibration of the shell differs only slightly from that of the same shell in vacuo. If the fluid is compressible, the compressibility of the fluid alters the effective stiffness of the system. Also, if the density of the fluid is relatively high, as in the case of a liquid, then the fluid exerts considerable inertial loading on the shell, and this results in a significant lowering of the resonant frequencies. Other effects of coupled fluid-shell motions occur when the fluid is flowing. Depending upon the boundary conditions, if the flow velocities are large, buckling or oscillatory flexural instabilities are possible.

The dynamics of coupled fluid-shells were reviewed extensively by Brown [1] and Yang [2] and others [3] to [10]. There have been few analysis of closed cylindrical shells having axially varying thickness. Again, While there is extensive literature relevant to the vibration of empty open cylindrical shells (cylindrical panels), no analyses have been found of open cylindrical shells, non-uniform in the circumferential direction and containing a flowing fluid.

The purpose of this study is to present a method for the dynamic and static analysis of open, thin, anisotropic cylindrical shells containing flowing fluid.

The structure may be uniform or non-uniform in the circumferential direction and we consider the problem of open cylindrical shells which are freely simply-supported along their curved edges and to have arbitrary straight edge boundary conditions.

The method is a hybrid of finite element method, classical shell theories and fluid theories. The structure is subdivided into a cylindrical panel segment finite elements. The displacement functions are derived from Sanders' equation of thin cylindrical shells [11]. In this approach, it is possible to determine the mass and stiffness matrices of the individual finite elements by exact analytical integration. Accordingly, this method is more accurate than the more usual finite element methods based on polynomial displacement functions.

To account for the fluid effect on the structure, a panel finite fluid element bounded by two nodal lines was considered. By solving the equations of motion for the fluid element, an expression for fluid pressure as a function of the displacements of the element was obtained. Analytical integration for the pressure distribution along the element yielded three components: the mass, stiffness and damping matrices for a fluid element.

Global matrices are, then, obtained by superimposing the individual matrices. The eigenvalue and eigenvector problem is solved by means of the equation reduction technique.

The hybrid approach (Finite element - Shell theory - Fluid theory) has been applied with satisfactory results to the dynamic linear and non-linear analysis of cylindrical [12-19], conical [20], spherical [21], isotropic and anisotropic, uniform and axially non-uniform shells both empty and liquid filled. This method has been applied also to the dynamic analysis of circular and annular plates [22], [23] and to an open anisotropic and circumferentially non-uniform cylindrical shell [24]. This study is an attempt to determine the vibration of a circumferentially non-uniform open cylindrical shell, containing flowing fluid. The case of an open cylindrical shell partially or completely filled with liquid is also studied.

2. DETERMINATION OF THE DISPLACEMENT FUNCTIONS:

Sanders' equations [11] for thin, cylindrical shells, in terms of axial, tangential and radial displacements (U, V, W) of the mean surface of the shell (figure 1) and in terms of element P_{ij} of the anisotropic matrix of elasticity [P] are:

$$\begin{aligned} L_1\,(U,V,W,P_{ij}) &= 0 \\ L_2\,(U,V,W,P_{ij}) &= 0 \\ L_3\,(U,V,W,P_{ij}) &= 0 \end{aligned} \tag{1}$$

where L_k (k = 1, 2, 3) are three linear differential operators, the form of which is fully explained in [24].

The strain-displacement relation is given by:

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_\theta \\ 2\bar{\epsilon}_{x\theta} \\ \kappa_x \\ \kappa_\theta \\ 2\bar{\kappa}_{x\theta} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{1}{R}\dfrac{\partial V}{\partial \theta} + \dfrac{W}{R} \\ \dfrac{\partial V}{\partial x} + \dfrac{1}{R}\dfrac{\partial U}{\partial \theta} \\ -\dfrac{\partial^2 W}{\partial x^2} \\ -\dfrac{1}{R^2}\dfrac{\partial^2 W}{\partial \theta^2} + \dfrac{1}{R^2}\dfrac{\partial V}{\partial \theta} \\ -\dfrac{2}{R}\dfrac{\partial^2 W}{\partial x \partial \theta} + \dfrac{3}{2R}\dfrac{\partial V}{\partial \theta} - \dfrac{1}{2R^2}\dfrac{\partial U}{\partial \theta} \end{Bmatrix} \tag{2}$$

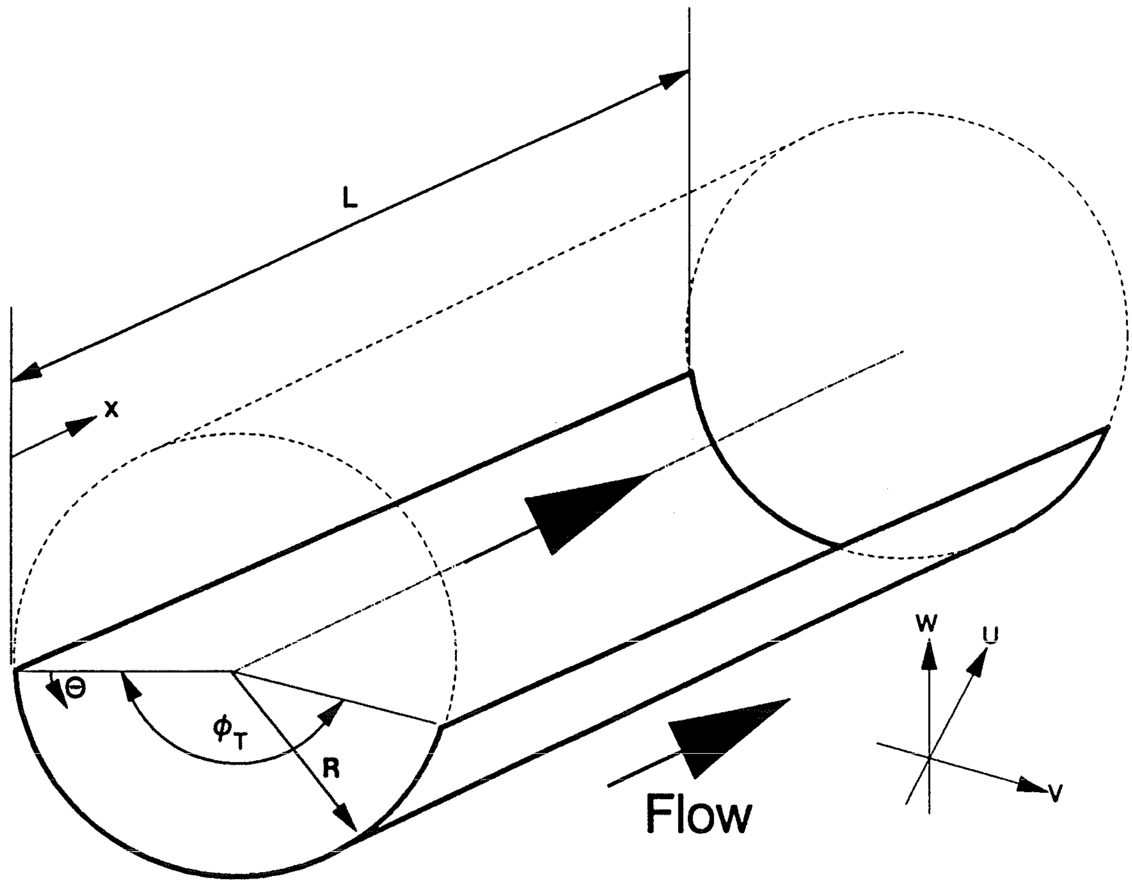

Figure 1: Open Cylindrical Shell Geometry

The finite element used is shown in Figure 2. It is a cylindrical panel segment defined by two line nodes i and j. Each node has four degrees of freedom: three displacements (axial, circumferential and radial) and one rotation. The panels are assumed to be freely simply-supported along their curved edges and to have arbitrary straight edge boundary conditions.

For motions associated with the m th axial wave number, we may write:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = \begin{bmatrix} \cos m\pi x/L & 0 & 0 \\ 0 & \sin m\pi x/L & 0 \\ 0 & 0 & \sin m\pi x/L \end{bmatrix} \begin{Bmatrix} U_m(\theta) \\ W_m(\theta) \\ V_m(\theta) \end{Bmatrix} = [T_m] \begin{Bmatrix} U_m(\theta) \\ W_m(\theta) \\ V_m(\theta) \end{Bmatrix} \tag{3}$$

By substituting equation (3) into equation (1) and letting

$$\begin{aligned} U_m(\theta) &= A\, e^{\eta\theta} \\ V_m(\theta) &= B\, e^{\eta\theta} \\ W_m(\theta) &= C\, e^{\eta\theta} \end{aligned} \tag{4}$$

we obtain

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T_m]\,[R]\,\{C\} \tag{5}$$

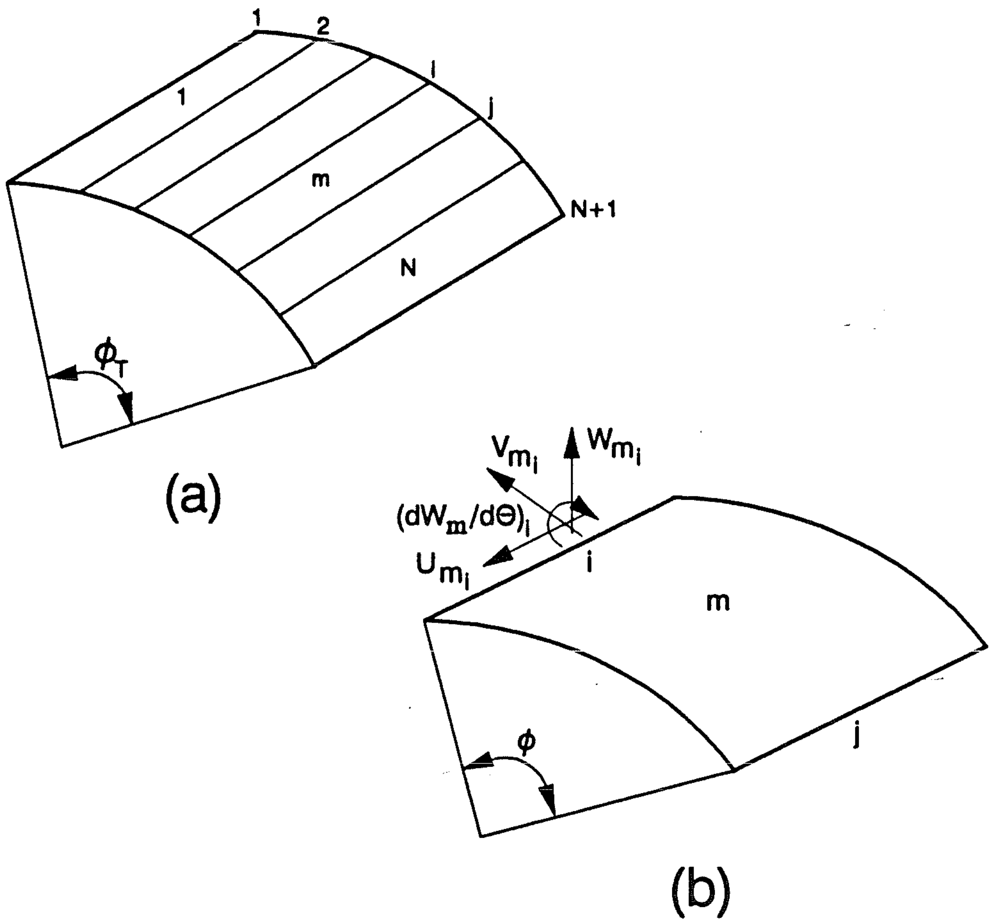

Figure 2: **(a)** **Finite element idealization**

(b) **Nodal displacements at node i for the finite element m**

N : number of finite elements

where [R] is a (3 × 8) matrix given by:

$$R(1,j) = \alpha_j \, e^{\eta_j \theta} \qquad j = 1,\ldots,8$$
$$R(2,j) = e^{\eta_j \theta} \qquad j = 1,\ldots,8 \qquad (6)$$
$$R(3,j) = \beta_j \, e^{\eta_j \theta} \qquad j = 1,\ldots,8$$

η_j (j = 1,..., 8) are the roots of the characteristic equation of the empty panel. As A, B and C are not independent, we may write A = αC and B = βC, which determine α_j and β_j. {C} is a vector of eight constants which are linear combinations of the C_j. The eight C_j are the only free constants, which must be determined from eight boundary conditions, four at each straight edge of the finite element.

We now express the nodal displacement vectors as follows

$$\{\delta_i\} = \left\{ U_{mi},\ W_{mi},\ \left(\frac{dW_m}{d\theta}\right)_i,\ V_{mi} \right\}^T \qquad (7)$$

Each $\{\delta_i\}$ may be determined from equation (5), where θ in [R] now has a definite value, $\theta = 0$ or $\theta = \phi$, as the case may be; hence we obtain

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [A]\,\{C\} \qquad (8)$$

where the elements of matrix [A] are determined from those of matrix [R].

Finally, combining equation (5) and (8), we obtain:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T_m]\,[R]\,[A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{9}$$

which defines the displacements functions.

3. MASS AND STIFFNESS MATRICES FOR EMPTY FINITE ELEMENTS

The strains are related to the displacements through equations (2); accordingly, we may now express $\{\epsilon\}$ in terms of δ_i and δ_j, and after lengthy manipulations we obtain:

$$\{\epsilon\} = \begin{bmatrix} [T_m] & 0 \\ 0 & [T_m] \end{bmatrix} [Q]\,[A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{10}$$

where [Q] is a (6×8) matrix given in Ref. [24].

The corresponding stresses may be related to the strains by the elasticity matrix [P].

$$\{\sigma\} = [P]\,\{\epsilon\} = [P]\,[B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{11}$$

The general term P_{ij} of anisotropic matrix [P] is found from [25].

The mass and stiffness matrices, $[m_s]$ and $[k_s]$ respectively, for one finite element may be written as follows:

$$[m_s] = \rho_s t \int_0^L \int_0^\phi [N]^T [N] \, dA \quad \text{and} \quad [k_s] = \int_0^L \int_0^\phi [B]^T [P] [B] \, dA \,, \tag{12}$$

where ρ_s is the density of the shell, t its thickness, dA a surface element, [P] the elasticity matrix and the matrices [N] and [B] are derived from equations (9) and (10), respectively.

The matrices $[m_s]$ and $[k_s]$ were obtained analytically by carrying out the necessary matrix operations and integration over x and θ in equation (12). The global matrices $[M_s]$ and $[K_s]$ may be obtained, respectively, by superimposing the mass $[m_s]$ and stiffness $[k_s]$ matrices for each individual panel finite element. See reference [24] for more details.

4. BEHAVIOUR OF THE FLUID-SHELL INTERACTION

4.1 Equations of motion

The dynamic behaviour of an open shell subjected to a pressure field can be represented by the following system:

$$\big[[M_s] - [M_f]\big] \{\ddot{\delta}\} - [C_f]\{\dot{\delta}\} + \big[[K_s] - [K_f]\big] \{\delta\} = \{F\} \tag{13}$$

where $\{\delta\}$ is the displacement vector, $[M_s]$ and $[K_s]$ are, respectively, the mass and stiffness matrices of the system in vacuo; $[M_f]$ and $[C_f]$ and $[K_f]$ represent the inertial, Coriolis and centrifugal forces of the liquid flow and $\{F\}$ represents the external forces.

4.2 Assumptions

We assume here that the structure is subjected only to potential flow which induces inertial, Coriolis and centrifugal forces to participate in the vibration pattern. These forces are coupled with the elastic deformation of the shell.

The mathematical model which is developed is based on the following hypothesis:

(i) the fluid flow is potential ;

(ii) vibration is linear (small deformation) ;

(iii) pressure on the wall is purely lateral ;

(iv) the fluid mean velocity distribution is assumed to be constant across a shell section ;

and (v) the fluid is incompressible.

4.3 Mass, stiffness and damping matrices of the moving fluid

The potential flow may be governed by the equation:

$$\nabla^2 \Phi = \frac{1}{c^2} \left\{ \frac{\partial^2 \Phi}{\partial t^2} + 2U_x \frac{\partial^2 \Phi}{\partial x \partial t} + U_x^2 \frac{\partial^2 \Phi}{\partial x^2} \right\} \tag{14}$$

where c is the speed of sound in the fluid; U_x is the velocity of the liquid through the shell section and Φ is the potential function that represents the velocity potential.

Therefore:

$$V_x = U_x + \frac{\partial \Phi}{\partial x} \; ; \; V_\theta = \frac{1}{R} \frac{\partial \Phi}{\partial \theta} \; ; \; V_r = \frac{\partial \Phi}{\partial r} \tag{15}$$

where V_x, V_θ and V_r are respectively the axial, tangential and radial components of the fluid velocity.

The Bernouilli equation is given by:

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} V^2 + \frac{P}{\rho_f} \Big|_{r=\xi} = 0 \tag{16}$$

Introducing equation (15) into equation (16) and taking into account only the linear terms, we find the dynamic pressure P:

$$P_u = \rho_{fu} \left\{ \frac{\partial \Phi_u}{\partial t} + U_{xu} \frac{\partial \Phi_u}{\partial x} \right\} \Big|_{r=\xi} \tag{17}$$

where u subscript represents "inside" or "outside" fluid as the case may be:

$$\begin{aligned} &\text{if} \quad u = i \qquad \text{then} \quad \xi = R_i = R - \frac{t}{2} \\ &\text{if} \quad u = o \qquad \text{then} \quad \xi = R_o = R - \frac{t}{2} \end{aligned} \tag{18}$$

For steady flow, the velocity potential must satisfy the Laplace equation. This relation is expressed in the cylindrical coordinate system by:

$$\nabla^2 \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial x^2} \tag{19}$$

A full definition of the flow requires that a condition be applied to the structure-fluid interface. The impermeability condition ensures contact between the shell and the fluid. This should be:

$$V_r \big|_{r=R} = \frac{\partial \Phi}{\partial r} \Big|_{r=R} = \left(\frac{\partial W}{\partial t} + U_x \frac{\partial W}{\partial x} + \frac{U_x^2}{2} \frac{\partial^2 W}{\partial x^2} \right) \Big|_{r=R} \tag{20}$$

From the theory of shells (equation 5), we have:

$$W(x,\theta,t) = \sum_{j=1}^{8} C_j \, e^{\eta_j \theta} \sin \frac{m\pi x}{L} \, e^{i\omega t} \tag{21}$$

Assuming then,

$$\Phi(x,\theta,r,t) = \sum_{j=1}^{8} R_j(r) \, S_j(x,\theta,t) \tag{22}$$

and applying the impermeability condition (equation 20) with the radial displacement given by relation (21), we determine the function S_j (x, θ, t). Introducing this explicit term S_j (x, θ, t) into equation (22) and then in equation (17), we find a relation for the dynamic pressure as a function of the displacement W_j and the function $R_j(r)$:

$$P_u = -\rho_f \sum_{j=1}^{8} \frac{R_j(r)}{R_j'(R)} \left[\ddot{W}_j + 2U_{xu}\dot{W}_j' + \frac{U_{xu}^2}{2}\dot{W}_j'' + U_{xu}^2 W_j'' + \frac{U_{xu}^3}{2} W_j''' \right] \tag{23}$$

where ('),(·) and (')represent $\frac{\partial(\ )}{\partial r}$, $\frac{\partial(\ )}{\partial t}$ and $\frac{\partial(\ )}{\partial x}$ respectively., By using the relation

(14) and (19), we obtain the following differential Bessel equation:

$$r^2 \frac{d^2R_j(r)}{dr^2} + r\frac{dR_j(r)}{dr} + R_j(r)\left[\left(\frac{im\pi}{L}\right)^2 r^2 - (i\eta_j)^2\right] = 0 \tag{24}$$

where i is the complex number, $i^2 = -1$ and η_j is the complex solution of the characteristic equation.

The general solution of equation (24) is given by:

$$R_j(r) = A\,J_{i\eta_j}\left(\frac{im\pi}{L}r\right) + B\,Y_{i\eta_j}\left(\frac{im\pi}{L}r\right) \tag{25}$$

where $J_{i\eta_j}$ and $Y_{i\eta_j}$ are, respectively, the Bessel functions of the first and second kind of order "$i\eta_j$".

For inside flow, the solution (25) must be finite on the axis of the shell ($r = 0$); this means we have to set the constant 'B' equal to zero. For outside flow ($r \rightarrow \infty$); this means that the constant 'A' is equal to zero. When the shell is simultaneously subjected to internal and external flow, we have to take the complete solution (25).

Finally, we obtain the equation for the pressure on the wall as follows:

$$P_u = -\rho_u \sum_{j=1}^{8} Z_{uj}\left(\frac{im\pi R_u}{L}\right)\left[\ddot{W}_j + 2U_{xu}\dot{W}_j' + \frac{U_{xu}^2}{2}\dot{W}_j'' + U_{xu}^2 W_j'' + \frac{U_{xu}^3}{2}W_j'''\right] \tag{26}$$

where $(\cdot)$ and $(')$ represent $\frac{\partial(\,)}{\partial t}$ and $\frac{\partial(\,)}{\partial x}$ respectively, and

$$Z_{uj}\left(\frac{im\pi R_u}{L}\right) = \frac{R_u}{i\eta_j - \dfrac{im\pi R_u}{L}\,\dfrac{J_{i\eta_j+1}(im\pi R_u/L)}{J_{i\eta_j}(im\pi R_u/L)}} \quad \text{if } u = i \tag{27}$$

$$Z_{uj}\left(\frac{im\pi R_u}{L}\right) = \frac{R_u}{i\eta_j - \dfrac{im\pi R_u}{L}\,\dfrac{Y_{i\eta_j+1}(im\pi R_u/L)}{Y_{i\eta_j}(im\pi R_u/L)}} \quad \text{if } u = o \tag{28}$$

By introducing the displacement function (9), into the dynamic pressure expression (26) and performing the matrix operation required by the finite element method, the mass, damping and stiffness matrices for fluid are obtained by evaluating the following integral:

$$\int_A [N]^T \{P_u\}\ dA \tag{29}$$

we obtain

$$[m_f] = [A^{-1}]^T\ [S_f]\ [A^{-1}] \tag{30}$$

$$[c_f] = [A^{-1}]^T\ [D_f]\ [A^{-1}] \tag{31}$$

$$[k_f] = [A^{-1}]^T\ [G_f]\ [A^{-1}] \tag{32}$$

The matrix [A] is given by equation (8) and the elements of [S_f], [D_f] and [G_f] are given, as follows.

$$S_f(r,s) = -\frac{RL}{2} I_{rs} \left(\rho_i Z_{is} - \rho_o Z_{os}\right) \qquad (33)$$

$$D_f(r,s) = \frac{R m^2 \pi^2}{4L} I_{rs} \left(\rho_i U_{xi}^2 Z_{is} - \rho_o U_{xo}^2 Z_{os}\right) \qquad (34)$$

$$G_f(r,s) = \frac{R m^2 \pi^2}{2L} I_{rs} \left(\rho_i U_{xi}^2 Z_{is} - \rho_o U_{xo}^2 Z_{os}\right) \qquad (35)$$

where r, s = 1, ..., 8

and

$$\begin{cases} I_{rs} = \dfrac{1}{(\eta_r + \eta_s)} \left[e^{(\eta_r+\eta_s)\phi} - 1 \right] & \text{for } \eta_r + \eta_s \neq 0 \\ I_{rs} = \phi & \text{for } \eta_r + \eta_s = 0 \end{cases} \qquad (36)$$

Finally, the global matrices [M_f], [C_f] and [K_f] may be obtained, respectively, by superimposing the mass [m_f], damping [c_f] and stiffness [k_f] matrices for each individual fluid finite element.

5. EIGENVALUE AND EIGENVECTOR PROBLEM

The eigenvalue and eigenvector problem is solved by means of the equation reduction technique. Equation (13) may be rewritten as follows:

$$\begin{bmatrix} [0] & \frac{1}{\omega_o}[M] \\ \frac{1}{\omega_o^2}[M] & \frac{1}{\omega_o}[C] \end{bmatrix} \begin{Bmatrix} \ddot{\delta} \\ \dot{\delta} \end{Bmatrix} + \begin{bmatrix} -\frac{1}{\omega_o}[M] & [0] \\ [0] & [K] \end{bmatrix} \begin{Bmatrix} \dot{\delta} \\ \delta \end{Bmatrix} = \{0\} \qquad (37)$$

where

$$[M] = [M_s] - [M_f]\,, \quad [K] = [K_s] - [K_f]\,, \quad [C] = [C_f] \qquad (38)$$

$[M_s]$ and $[K_s]$ are the global mass and stiffness matrices for the empty shell, $[M_f]$, $[C_f]$ and $[K_f]$ are the global mass, damping and stiffness for the fluid.

$\omega_o = P_{11}$: the first element of elasticity matrix.

The problem for eigenvalues is given by:

$$|\,[DD] - \Lambda\,[I]\,| = 0 \qquad (39)$$

where

$$[DD] = \begin{bmatrix} [0] & [I] \\ -\frac{1}{\omega_o^2}[K]^{-1}[M] & -\frac{1}{\omega_o}[K]^{-1}[C] \end{bmatrix} \qquad (40)$$

and $$\Lambda = \frac{1}{\omega_o^2 \omega^2},$$

ω is the natural frequency of the system, and $\omega_o = P_{11}$: the first element of the elasticity matrix.

Particular case: If the velocity of the fluid ($U_x = 0$), the eigenvalue problem may be reduced to:

$$\left| \frac{1}{\omega_o^2} [K]^{-1} [M] - \Lambda [I] \right| = 0 \qquad (41)$$

and ω (rad/s) $= \dfrac{1}{\omega_o \Lambda}$.

Matrices [K], [M] and [C] are square matrices of order NDF (N+1)-J, where NDF is the number of degrees of freedom at each node, N is the number of finite elements in the structure and J is the number of constraints applied.

6. CALCULATIONS AND DISCUSSION

Calculations have already been conducted to test the theory in the case of EMPTY open and closed shells. The free vibrations of uniform and circumferentially non-uniform open and closed shells were obtained for a variety of boundary conditions [24]. The computed natural frequencies were compared with those obtained by other theories and from experiments; agreement was found to be good. Here we present some calculations to test the theory in the case of liquid-filled open and closed cylindrical shells. In the case when the shell is subjected to flowing fluid, the dynamic stability of this type of problem is analysed.

6.1 Free vibration of closed cylindrical shells partially or completely filled with liquid

a) For the first set of calculations, we determine the frequency parameters (Ω) for different values of R/t and L/R for shells completely filled with liquid (inside). The results obtained (10 elements) for $n = 1$ are given in Table 1 in the case of free simply-supported shells. We concluded that, as a result of the lateral pressure exerted by the liquid on the structure, the frequency parameters (Ω) depended both on L/R and R/t, in contrast to the case of the empty shell, where R/t ratio had only a slight effect upon the results.

TABLE 1: Vibration parameter (Ω) of cylindrical shells simply-supported at both ends and filled with liquid

($n = 1$, $m = 1$, $\nu = 0.3$, $\rho_i = 1000$ kg/m^3 and

$\Omega = \omega R \sqrt{\rho (1 - \nu^2)/E}$).

L/r \ R/t		20	50	100	200	Baron and Bleich [26] all values of R/t
Empty	2.0	0.5775	0.5900	0.6067	0.5711	0.5728
Full	2.0	0.4196	0.3288	0.2629	0.1810	----
Empty	4.0	0.2572	0.2581	0.2594	0.2603	0.2569
Full	4.0	0.1809	0.1372	0.1065	0.07998	----
Empty	8.0	0.08744	0.08747	0.08752	0.08756	0.0874
Full	8.0	0.06020	0.04489	0.03424	0.02269	----
Empty	10.0	0.05911	0.05911	0.05913	0.05914	0.0592
Full	10.0	0.04044	0.03005	0.02283	0.01684	----

b) Next calculations were made for a cylindrical shell simply supported at both ends in which the liquid level was varied from zero to full in the circumferential direction. The pertinent data are as follows: $R = 37.7$ mm, $t = 0.229$ mm, $L = 234$ mm, $\nu = 0.29$, $\rho_i / \rho_s = 0.128$. The effects of the inertial force were calculated by this theory assuming $U_x = 0$ in equations (33) to (35).

Table 2 shows some frequencies computed by the present method and compared with experimental results [27] in the case of closed cylindrical shell both empty and completely filled with liquid.

As may be seen the results obtained by the present method are in good agreement with experimental results.

Figures 3 and 4 show some frequencies computed by the present method in which the liquid level was varied from zero to full in the circumferential direction.

We see for some modes that the frequency decreases rapidly with increasing d_1/d in the range $0 < d_1/d < 1/4$ approximately and then decreases only slighly for higher fractional fillings. For other modes, however, the frequencies decrease appreciably with increasing d_1/d over the whole range of d_1/d, as might be expected.

TABLE 2: Natural Frequencies (Hz) of a simply-supported closed cylindrical shell, both when empty and when completely filled with liquid.

(m, n)	Empty		Full (inside fluid)	
	Present Method	Experimental [27]	Present Method	Experimental [27]
(1,2)	1133	1150	376	375
(1,3)	629	640	234	250
(1,4)	655	688	270	300
(1,5)	942	995	422	430
(1,6)	1353	1430	651	680
(1,7)	1853	1938	940	970
(2,3)	2067	2070	784	813
(2,4)	1368	1430	568	600
(2,5)	1248	1313	561	625
(2,6)	1489	1570	714	755
(2.7)	1927	2050	978	1000

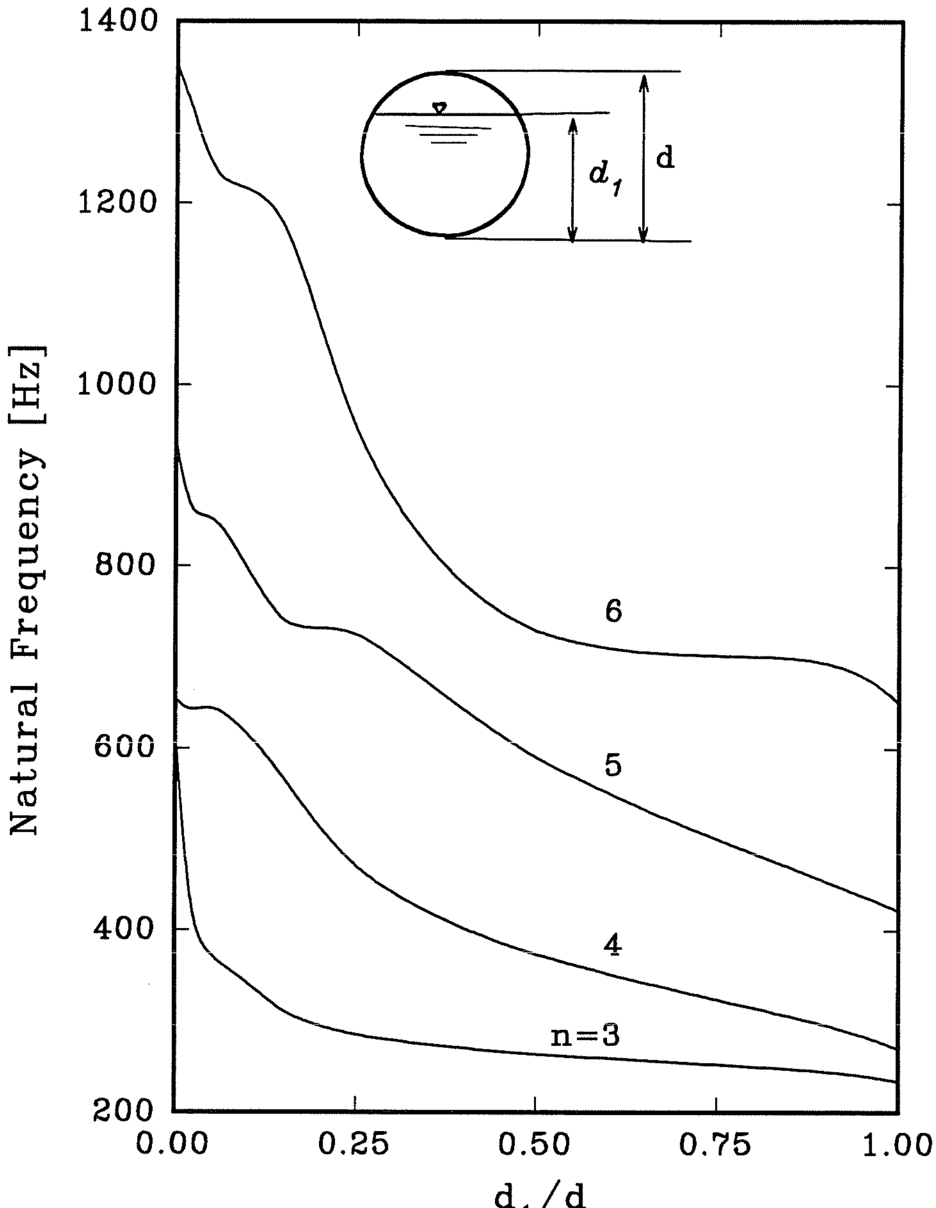

Figure 3 : Natural frequencies of a partially filled closed cylindrical shell supported at both ends as a function of liquid level, m=1.

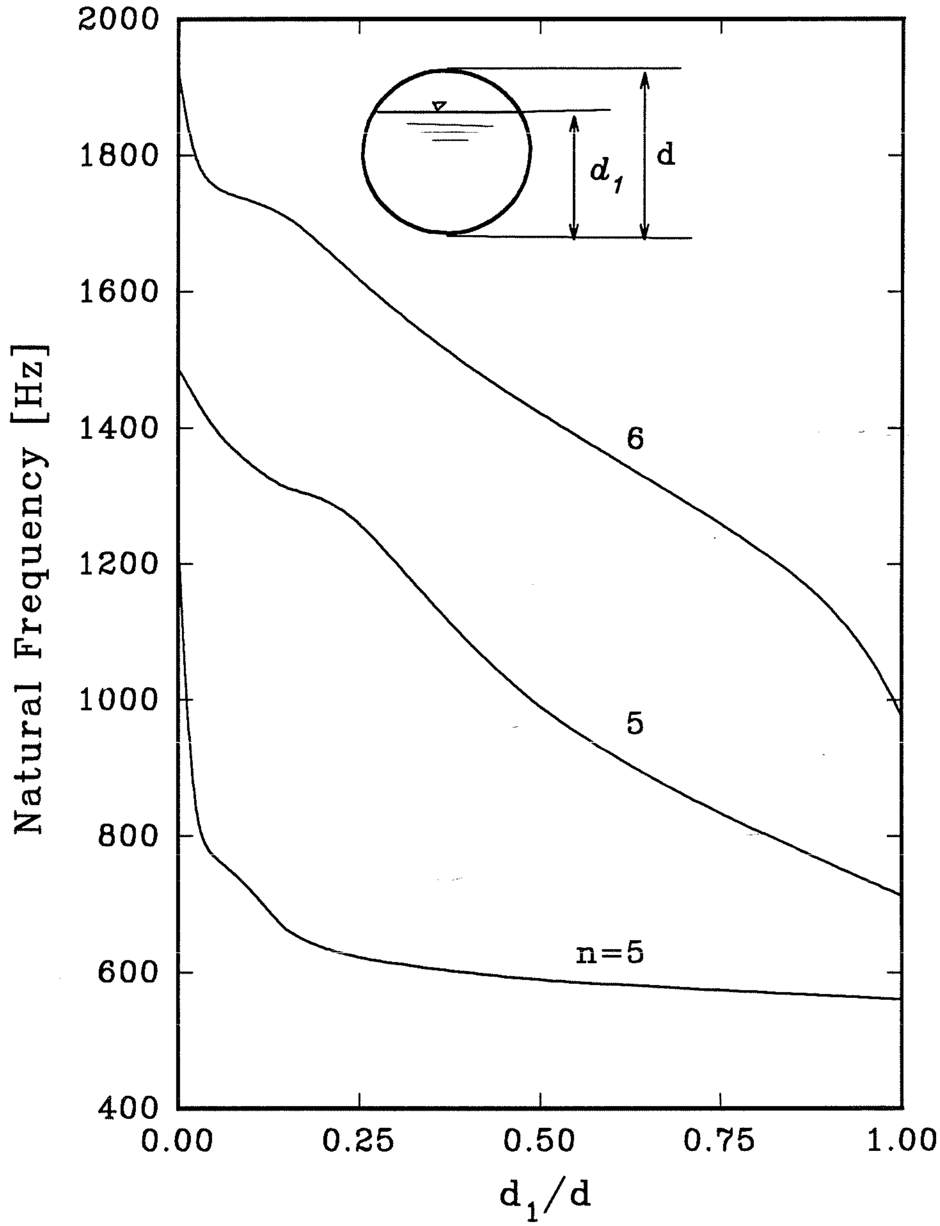

Figure 4 : Natural frequencies of a partially filled closed cylindrical shell supported at both ends as a function of liquid level, m=2.

c) The third calculation is to analyse the transverse vibration of isotropic and orthotropic cylindrical shells embedded in an incompressible fluid, simply-supported at both ends. This case was analysed by Ramachandran [28] who use the Rayleigh-Ritz procedure.

In Table 3, the values of the material properties used in the calculations are shown.

TABLE 3:Material and physical properties of the shell

	E_x (x 10^{11}N/m^2)	E_θ (x 10^{11} N/m^2)	G (x 10^{11}N/m^2)	ν_x	ν_θ
Isotropy	21.981	21.981	0.8454	0.3	0.3
Orthotropy	1.0	0.5	0.1	0.05	0.025

R = 0.235 m, t = 0.00235 m. ρ_s = 7850 N/m^3, ρ_f = 1000 N/m^3

The natural frequencies of this shell-liquid system for n = 4 , 8 ; m = 1 ; L/R = 2, 4 and different material properties of the shell are given in Table 4. Four cases were studied, when the shell is empty; when the fluid is inside or outside of the shell; and when the shell is embedded in a fluid.

TABLE 4: Frequency values (Hz) for simply-supported cylindrical shells, empty and filled with liquid.

Mat.	L/R	(n, m)	Theories	Empty	Inside and outside fluid (full)	Inside fluid	Outside fluid
Isotropy	4	(4,1)	Present Method	659	251.4	333.2	331.4
			Ramachandran [28]	700	294.2	----	----
			Lakis [12]*	659	251.7	333.8	331.7
		(8,1)	Present Method	2187	1064	1361	1361
			[28]	2200	944.1	----	----
			[12]*	2177	1073	1362	1360
Orthotropy	2	(4,1)	Present Method	240.1	92.2	121.9	121.6
			[28]	----	183.1	----	----
			[12]*	238.8	92.4	121.7	121.9
		(8,1)	Present Method	327.3	158.5	203.3	200.2
			[28]	----	248.5	----	----
			[12]*	324.1	160.7	203.2	203.9

* These results are computed from a computer program developed by A.A. Lakis & his co-workers and based on the theory presented in [12].

6.2 Dynamic stability of closed cylindrical shell containing flowing fluid

When the fluid is flowing, the shell will be subjected to centrifugal, Coriolis and intertia forces. A simply-supported shell with the following characteristics: $L/R = 2$, $t/R = 0.01$, $\rho_i\ \rho_s = 0.128$, $n = 5$ has been analysed, to see the influence of the speed of the flow U_{xi} on the frequencies (internal flow).

The dimensionless parameters of frequency and velocity are $\bar{\omega} = \omega/\omega_o$ and $\bar{U} = U/U_o$ where:

$$\omega_o = \frac{\pi^2}{L^2}(K/\rho_s t)^{1/2} \quad , \quad K = \frac{Et^3}{12(1-\nu^2)}$$

$$U_o = \frac{\pi^2}{L}(K/\rho_s t)^{1/2}$$

ω and U are respectively the natural frequency and the velocity of the flowing fluid.

The results appear in Figure 5. In a previous analysis of this case by Weaver and Unny [29], we observe that the natural frequencies decrease with flow velocity. At zero flow velocity, the two methods give the same results but, as the flow velocity increases the two term Galerkin method used in reference [29] generates significantly different results from those of the present hybrid finite element method. Our results predict that the first mode frequency becomes negative imaginary at $\bar{U} = 3.1$, indicating static divergence instability in this mode. If the velocity is increased further, the first mode reappears and coalesces at $\bar{U} = 3.95$ with that of the second mode to produce coupled mode flutter.

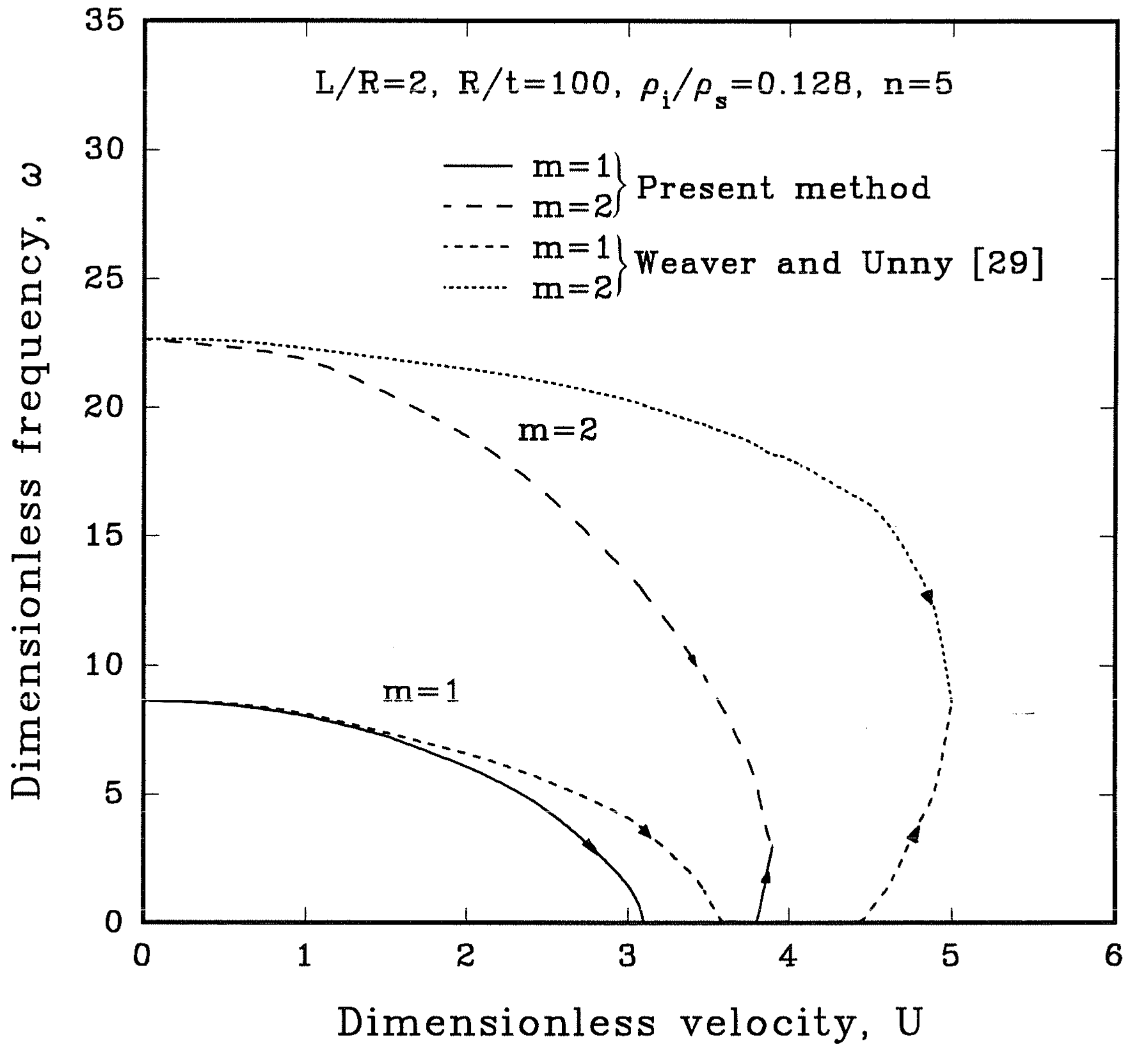

Figure 5 : Stability of a simply–supported closed cylindrical shell as a function of flow velecity.

6.3 Free vibration of an open cylindrical shell partially or completely filled with liquid

The articles in literature which deal with open shells interacting with a fluid are not available. Here, we present some results for an open cylindrical shell partially or completely filled with liquid. The open cylindrical shell is constructed of steel, is filled with water and is simply-supported at its four edges.

The pertinent data are as follows (see Figure 1):

$\phi_T = 180°$, R = 37.7 mm, t = 0.229 mm , L = 234 mm , $\nu = 0.29$, $\rho_i / \rho_s = 0.128$

a) In Figure 6, we see the behaviour of an open cylindrical shell empty and filled with liquid as a function of the number of circumferential modes. For a given *m*, the frequencies decrease to a minimum before they increase as the number of circumferential waves (*n*) is increased. This behaviour was first observed for a shell in vacuo by Arnold and Warburton [30] , who were able to explain it by a consideration of the strain energy associated with bending and stretching of the reference surface. It may be concluded from their work, that at low *n* the bending strain energy is low and the stretching strain energy is high; while at the higher *n*, the relative contributions from the two types of strain energy are reversed. The interchange in the relative contributions of the bending and stretching strain energy as the circumferencial wave number *n* is increased explains the decrease and subsequent increase in the natural frequencies indicated in Figure 6. This behaviour is also true for an open cylindrical shell partially or completely filled with liquid.

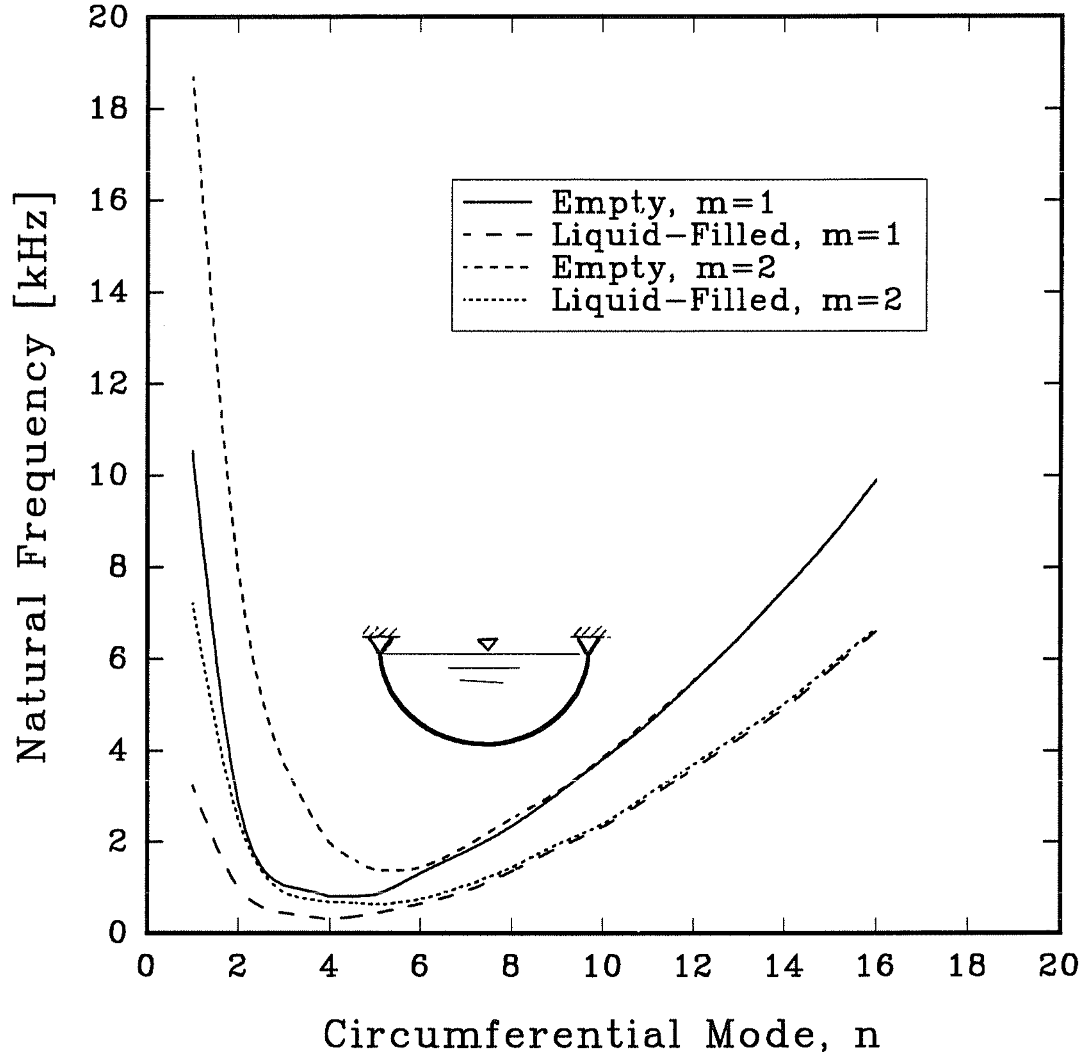

Figure 6 : Natural frequency of an empty and liquid–filled open cylindrical shell with W=V=0 at the four edges.

b) Figure 7 shows that when an open cylindrical shell is partially filled with liquid, the curves show a rapid decrease of the natural frequencies as a_1 / a_2 increases from 0 to 3/4 approximately, and then decreases only slightly for higher fractional fillings.

c) To see the influence of the orientation of the shell, we present in Figure 8, the natural frequency as a function of the orientation of the shell and the free surface of the liquid, the liquid level a_1 / a_2 = 0.46 (see Figure 7). We observe that the natural frequencies of the system decrease between the two extreme positions. The reduction is about 11% for the two modes (m = 1, n = 7) and (m = 2, n = 7).

We can use this type of analysis to study the dynamic response of a case such as a tanker on an inclined road when the free surface of the liquid oscillates.

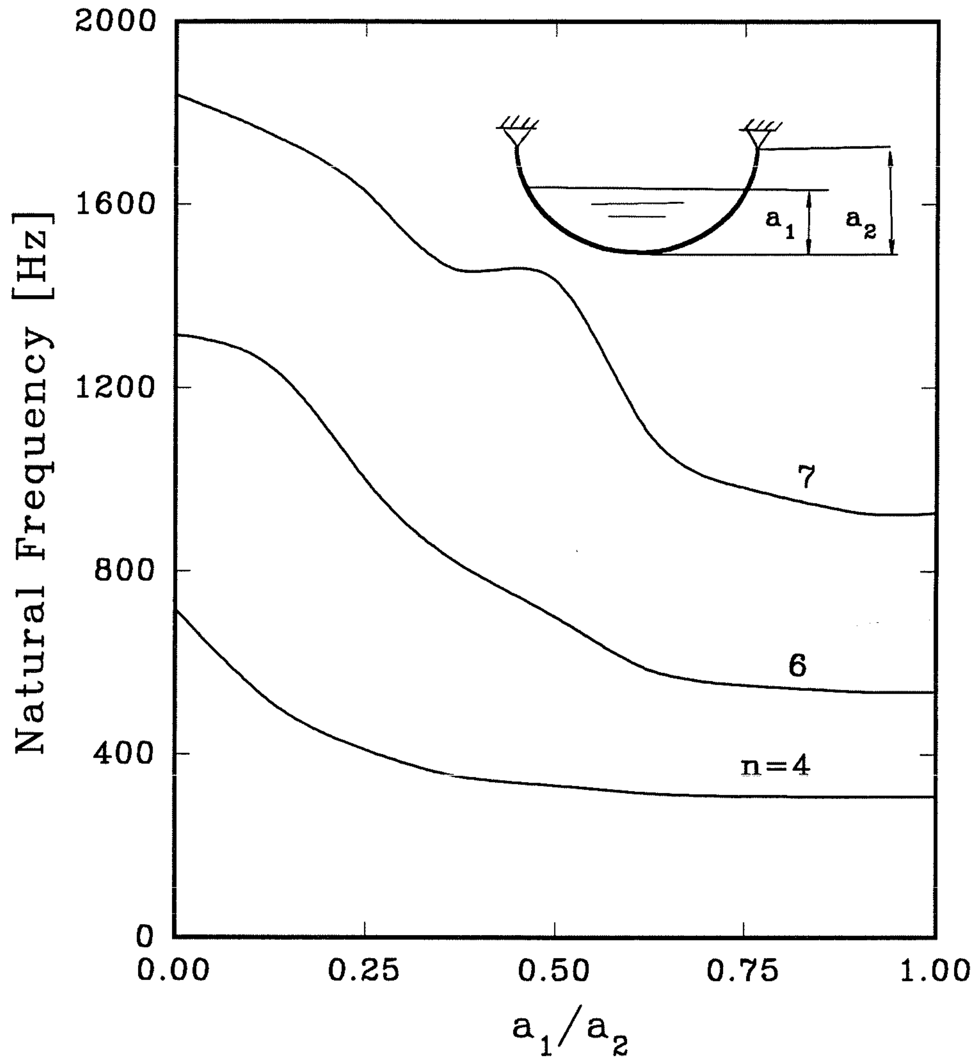

Figure 7 : Natural frequencies of an open cylindrical shell with W=V=0 at the four edges as a function of liquid level, m=1.

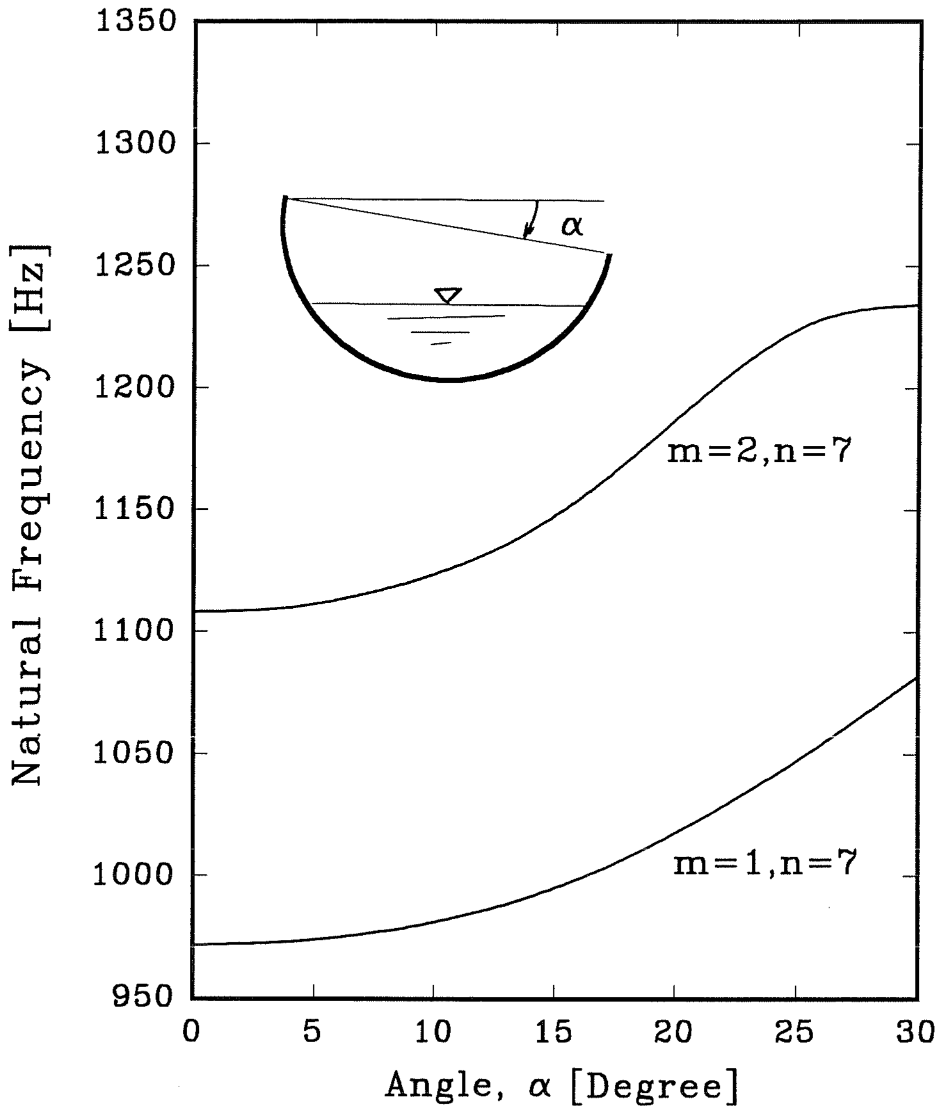

Figure 8 : Natural frequencies of an open cylindrical shell with W=V=0 at the four edges as a function of the orientation of the liquid level and the shell.

6.4 Dynamic stability of an open cylindrical shell containing flowing fluid

An open cylindrical shell containing flowing fluid has been analysed. The data for the shell are as follows: $R/t = 165$, $L/R = 6.2$, $\phi_T = 180°$, $\rho_f / \rho_s = 0.128$, $\nu = 0.29$, and the dimensionless parameters of frequency and velocity are $\bar{\omega} = \omega / \omega_o$ and $\bar{U} = U / U_o$ where

$$\omega_0 = \frac{\pi^2}{L^2} (K / \rho_s t)^{1/2} \quad , \quad K = \frac{Et^3}{12(1-\nu^2)}$$

$$U_0 = \frac{\pi^2}{L} (K / \rho_s t)^{1/2}$$

ω and U are respectively the natural frequency and the velocity of the flowing fluid.

We present here an examination of the natural frequencies of the system as functions of the flow velocity, and thereby a determination of the effect of flow on the dynamic behaviour of the system.

Different sets of results are presented to illustrate the method as well as the effect of various parameters, in particular the effect of internal or internal and external flow and the effect of boundary conditions.

a) Simply-supported - simply-supported shell

A simply-supported open cylindrical shell containing flowing fluid (internal and external) has been analysed. Figure 9 shows the frequencies of the system as a function of the flow velocity. As the velocity increases from zero, the frequencies associated with all modes decrease, they remain real (the system being conservative) , until at sufficiently high velocities, they vanish, indicating the existence of buckling-type (divergence) instability. At higher flow velocity the frequencies become purely imaginary.

We predict the first loss of stability at a flow velocity equal to $\bar{U} = 7.75$ for the mode ($m = 1$, $n = 4$).

b) Free-Free Shell

The case of an open cylindrical shell having its straight edges free and the curved edges freely simply-supported has been studied by the present theory. Figure 10 shows that natural frequencies associated with all modes decrease with increasing flow velocity until at a value of $\bar{U} = 8.5$ ($m = 1$, $n = 6$) the system buckles.

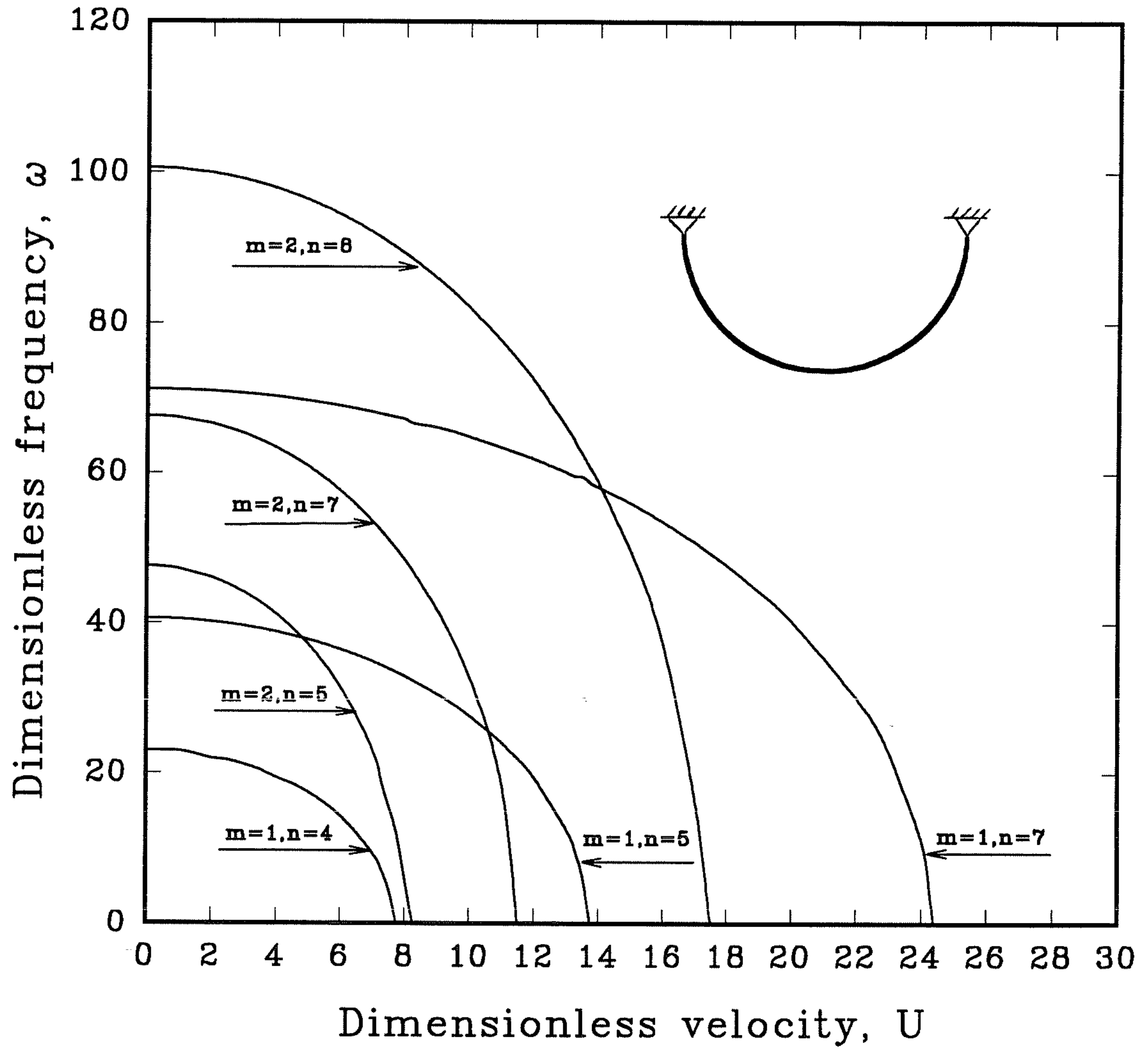

Figure 9 : Stability of a simply supported open cylindrical shell as a function of flow velocity. (inside and outside flow)

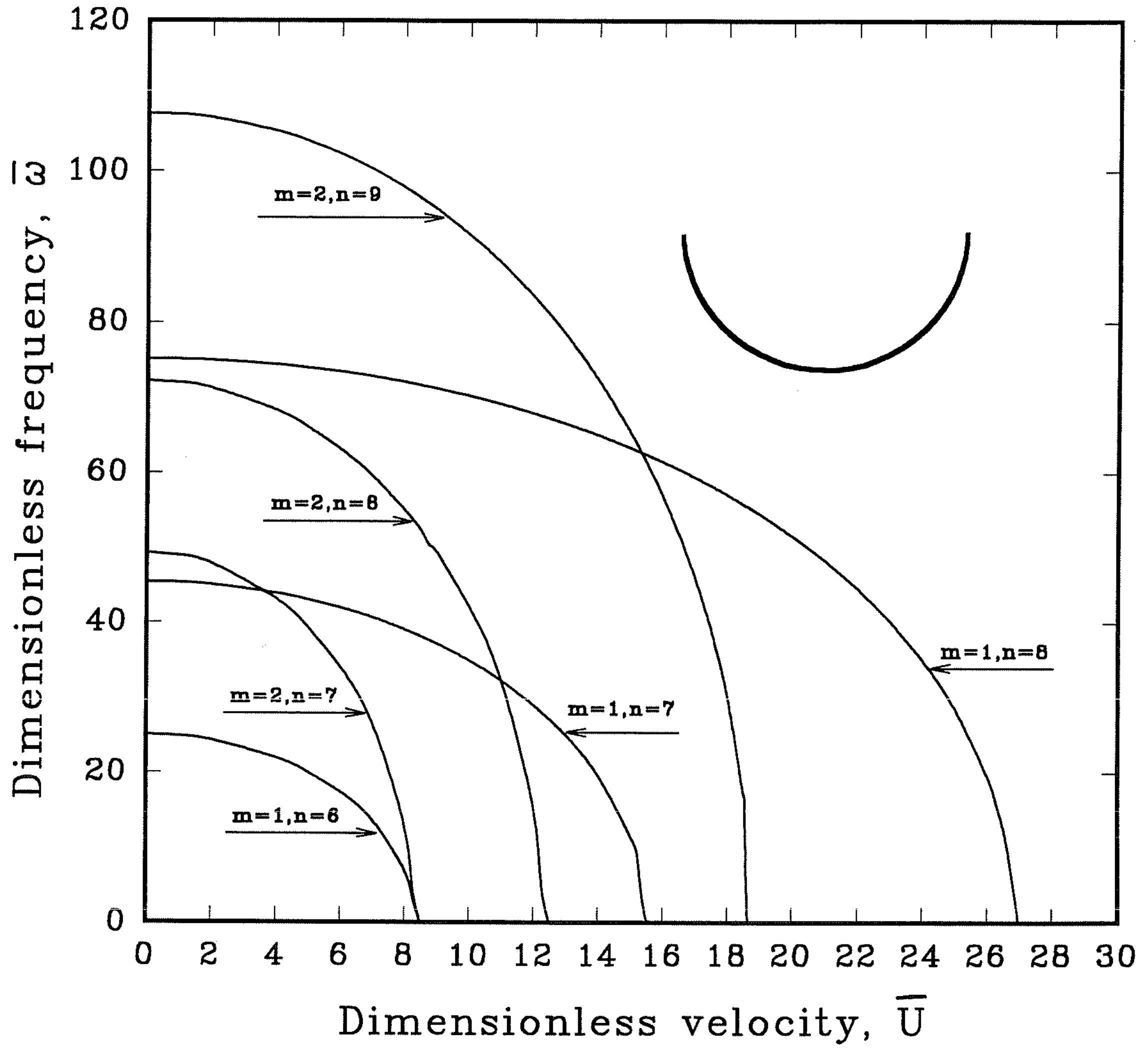

Figure 10 : Stability of a free–free open cylindrical shell as a function of flow velocity. (inside and outside flow)

c) Clamped-Clamped Shell:

The calculations were performed for one open cylindrical shell having its straight edge clamped and the curved edges freely simply-supported. Two sets of calculations were made:

(i) In the first set of calculations, we study the influence of the flow velocity on the dynamic stability of the open shell containing internal and external flow. We observe in Figure 11 that the frequencies associated with all modes decrease with increasing flow velocity, and similarly to the case of simply supported-simply supported and free-free open shells, the frequencies remain real until at a sufficiently high velocities, they vanish, indicating the instability. For the stipulated boundary conditions, we predict the first loss of instability at $\bar{U} = 8.25$ for the mode (m = 1, n = 4).

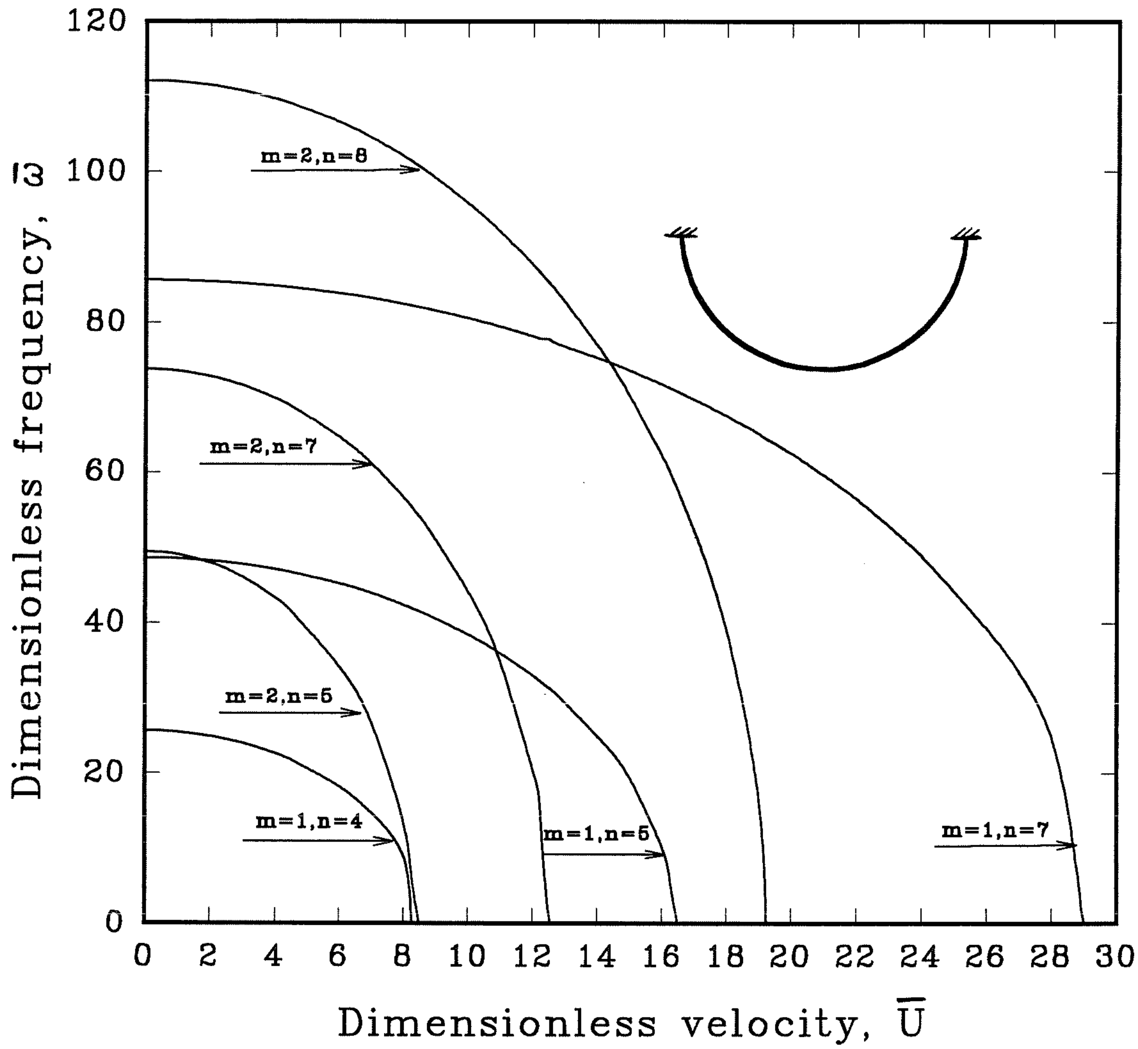

Figure 11 : Stability of a clamped–clamped open cylindrical shell as a function of flow velocity. (inside and outside flow)

(ii) In order to assess the effect of the external flow on the dynamic behaviour of the system, a set of calculations was performed in which the dynamic forces arising from the external fluid were neglected. As may be seen from Figure 12, the role of the external flow is to reduce the natural frequencies of the system and the critical velocities at which the system becomes unstable. We observe that the critical velocities are reduced from $\bar{U} = 23$ (m = 1, n = 6) and $\bar{U} = 17.75$ m = 2, n = 7) for the system with internal flow to $\bar{U} = 16.5$ (m = 1, n = 6) and $\bar{U} = 12.5$ (m = 2, n = 7) for the system with internal and external flow respectively. The discrepancy is about 40%. We conclude that the role of external flow is not negligible and tends to reduce the natural frequencies and the critical velocities of the system.

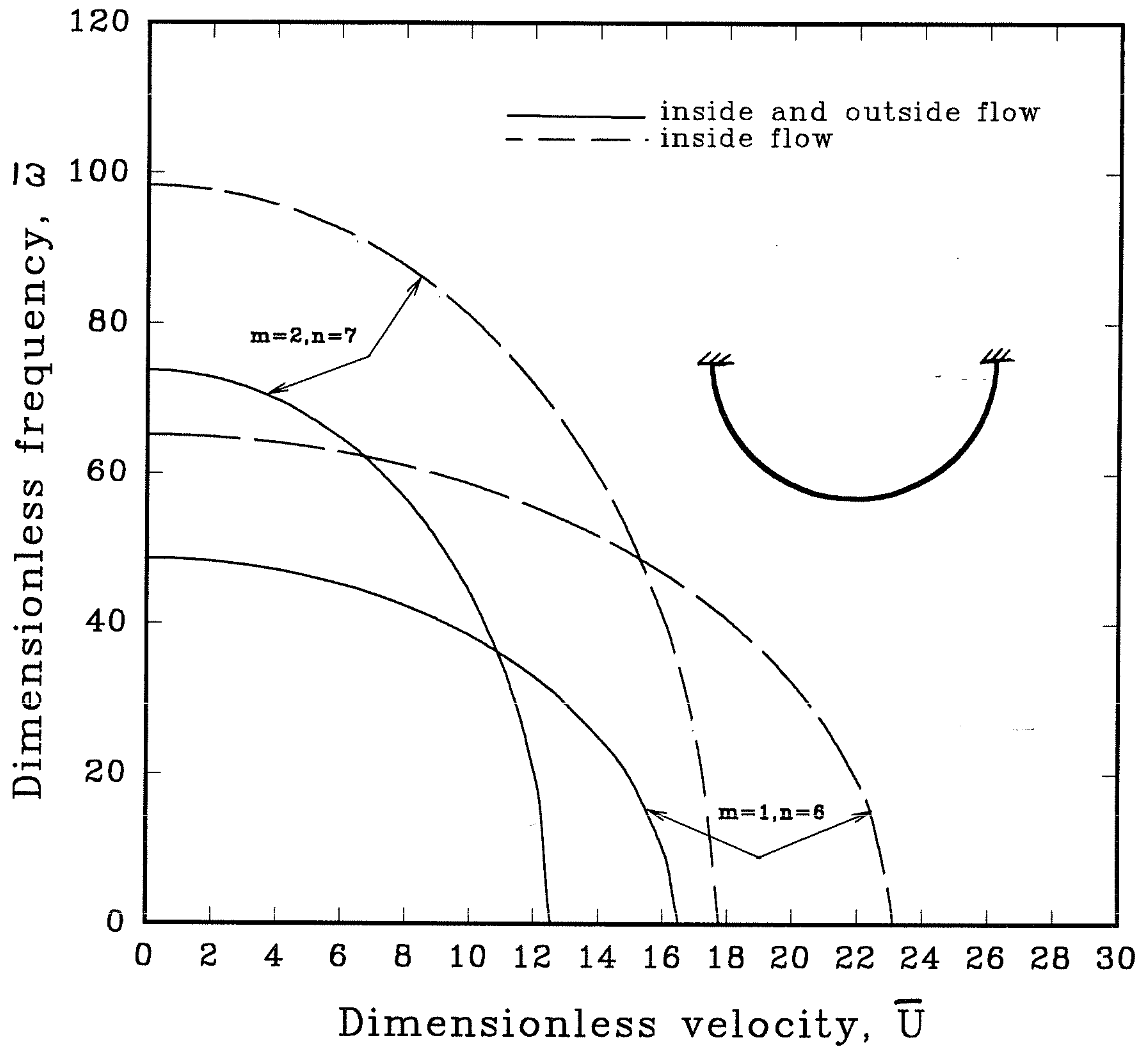

Figure 12 : Effect of inside and outside flow on the stability of a clamped–clamped cylindrical shell.

d) Comparaison between the boundary conditions:

In order to establish the effects of boundary conditions on the critical flow velocities which render the system dynamically unstable, we turn to Figure 13. We observe for the same mode and the same open shell with different boundary conditions, that the shell with free-free boundary conditions in its straight edges is the one which loses dynamic stability first.

For the mode (m = 1, n = 7) we have critical velocities as follows: Free-Free shell ($\bar{U} = 15.5$), simply supported - simply supported shell ($\bar{U} = 24.4$) and clamped-clamped shell ($\bar{U} = 29$). For the mode (m = 2, n = 7), we have respectively $\bar{U} = 8.5$; 11,5 and 12.5.

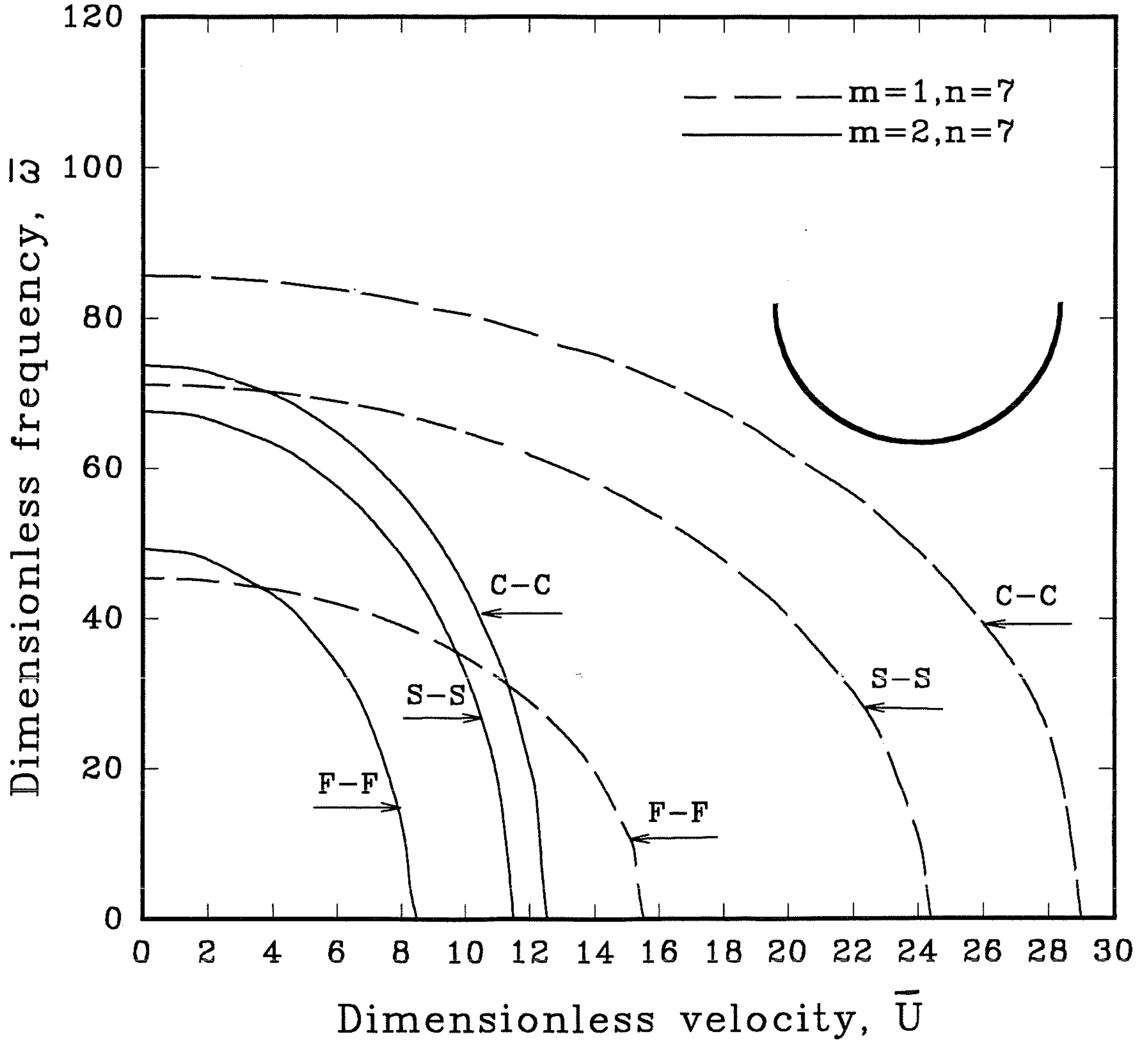

Figure 13 : Effect of boundary conditions on the stability of an open cylindrical shell.
(inside and outside flow)
F:Free, S:Simply−supported, C:Clamped

e) Clamped-Free Shell:

For this set of boundary conditions, the fluid flow renders the system non-conservative. The shell frequency becomes complex in contrast that of a system which is supported, free or clamped at both straight edges, when real frequencies are generated until the system buckles.

The evolution of the eigenfrequencies with the increasing dimensionless flow velocity is shows in the Argand diagram (Figure 14) [Note that Re (ω) is the oscillations frequency, while Im (ω) is related to the damping], the dimensionless flow velocity $\bar{U}$ being the parameter. For $\bar{U} \neq 0$ the frequencies are complex. It is noted that the effect of flow is to damp the system in all modes, the frequencies having positive imaginary parts.

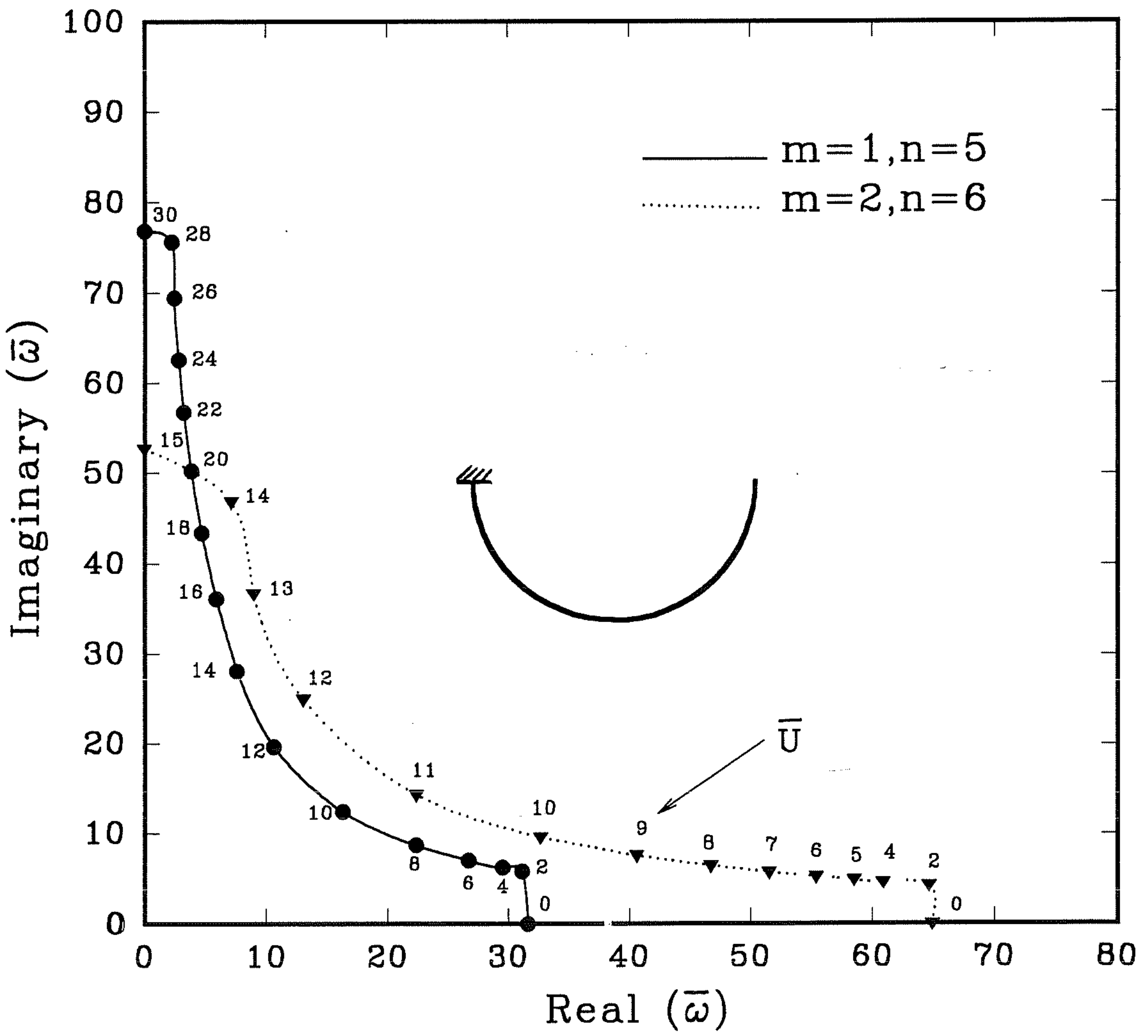

Figure 14 : Argand diagram of the dimensionless frequency, $\bar{\omega}$, as function of the dimensionless flow velocity, $\bar{U}$, for a clamped-free open cylindrical shell. (inside flow)

7. CONCLUSIONS

The theory developed in this paper is used to obtain the effects of inertia, Coriolis and centrifugal forces of a moving fluid on the vibration characteristics of anisotropic open and closed cylindrical shells.

A cylindrical panel finite element was developed, making possible the derivation of the displacement functions from the equations of motion of the shell. Mass and stiffness of each element were obtained by exact analytical integration.

The fluid pressure was derived from the velocity potential and from the linear impermeability and dynamic conditions applied to the shell-fluid interface. The finite element method was used to obtain the mass, stiffness and damping of fluid element. The results obtained by this method were compared with other investigations and satisfactory agreement was obtained. This method combines the advantages of finite element analysis which deals with complex shells, and the precision of formulation which the use of displacement functions derived from shell and fluid theories contributes.

The method enables us to predict the vibrationary characteristics of open cylindrical shells partially or completely filled with liquid and the dynamic stability of open shells subjected to flowing fluid.

The next step in this line of work should be the study of the non-linear dynamic analysis of open cylindrical shells containing a flowing fluid.

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NOMENCLATURE

LIST OF SYMBOLS

A, B, C	Constants in equations defining U, V, W respectively
a_1/a_2	Liquid level ratio for an open cylindrical shell
c	Velocity of sound in fluid
d_1/d	Liquid level ratio for closed cylindrical shell
E	Young's modulus
e	exponential
i	$i^2 = -1$
$J_{i\eta j}$	Bessel function of the first kind and of order $i\eta j$
K	Bending stiffness, $Et^3/12(1 - \nu^2)$
L	Length of the shell
m	Axial mode number
n	Circumferential mode number
P_u	Lateral pressure exerted on the shell, u = i for inside pressure and u = o for outside pressure
P_{ij}	Terms of elasticity matrix (i= 1,...,6 ; j= 1, ..., 6)
R	Mean radius of the shell
R_j	Solution of Bessel equation (25)

S_j	Defined by equation (22)
t	Thickness of the shell
U, V, W	Axial, tangential and radial displacements
U_{xu}	Velocity of the liquid
U_o	Defined by $(\pi^2/L)\ (K/\rho_s t)^{1/2}$
$\bar{U}$	Nondimensional velocity, U_{xu}/U_o
V_x , V_θ , V_r	Axial, tangential and radial fluid velocity (15)
x	Axial coordinate
$Y_{i\eta j}$	Bessel function of the second kind and of order $i\eta j$
Z_{uj}	Defined by equation (27) for $u = i$ and equation (28) for $u = o$
η_i	Complex roots of the characteristic equation
ϵ_x, ϵ_θ, $\bar{\epsilon}_{x\theta}$	Deformation of reference surface
κ_x, κ_θ, $\bar{\kappa}_{x\theta}$	Changes in curvature and torsion of reference surface
θ	Circumferential coordinate
ν	Poisson's ratio
ϕ	Angle for one finite element
ϕ_T	Angle for the whole open shell
Φ	Velocity potential
ρ_s	Density of the shell material

ρ_f Density of fluid, f = i for inside fluid and f = o for outside fluid

ω Natural frequency (rad/s)

ω_0 Defined by $(\pi^2/L^2)\ (K/\rho_s t)^{1/2}$

$\bar{\omega}$ Nondimensional frequency, ω/ω_0

LIST OF MATRICES

[A] Defined by equation (8)

[B] Defined by equation (10)

$[c_f]$ Damping matrix for a fluid finite element

$[C_f]$ Damping matrix for the whole fluid

{C} Vector of arbitrary constants

$[D_f]$ Defined by equation (34)

$[G_f]$ Defined by equation (35)

$[k_f]$ Stiffness matrix for a fluid finite lement

$[k_s]$ Stiffness matrix for a shell finite element

$[K_f]$ Stiffness matrix for the whole fluid

$[K_s]$ Stiffness matrix for the whole shell

$[m_f]$ Mass matrix for a fluid finite element

$[m_s]$ Mass matrix for a shell finite element

$[M_f]$ Mass matrix for the whole fluid

$[M_s]$	Mass matrix for the whole shell
[N]	Displacement function defined by equation (9)
[P]	Elasticity matrix
[Q]	Defined by equation (10)
[R]	Defined by equation (6)
$[S_f]$	Defined by equation (33)
$[T_m]$	Defined by equation (3)
$\{\delta_i\}$	Degree of freedom at node i
$\{\epsilon\}$	Deformation vector
$\{\sigma\}$	Stress vector

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