NON-LINEAR DYNAMIC ANALYSIS OF ANISOTROPIC CYLINDRICAL SHELLS

by

A.A. LAKIS, A. SELMANE and A. TOLEDANO

gratuit

ÉCOLE POLYTECHNIQUE DE MONTRÉAL
Department of Mechanical Engineering
Campus de l'Université de Montréal
C.P. 6079, Succ. Centre-ville
Montréal (Québec)
H3C 3A7

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A.A. LAKIS, A. SELMANE, A. TOLEDANO (Génie Mécanique)

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ABSTRACT

A theory to predict the influence of geometric non-linearities on the natural frequencies of an empty anisotropic cylindrical shell is presented in this paper. It is a hybrid of finite element and classical thin shell theories. Sanders-Koiter non-linear and strain-displacement relations are used. Displacement functions are evaluated using linearized equations of motion. Modal coefficients are then obtained for these displacement functions. Expressions for the mass, linear and non-linear stiffness matrices are derived through the finite element method (in terms of the elements of the matrix of elasticity, p_{ij}). The uncoupled equations are solved with the help of elliptic functions. The period and frequency variations are first determined as a function of shell amplitudes and then compared with the results in the literature.

LIST OF SYMBOLS

$a_{rs}^{(1)}$, $a_{rs}^{(2)}$	:	coefficients determined by equation (51)
$b_{rs}^{(1)}$	:	coefficient determined by equation (A-2.2a)
$c_{rs}^{(1)}$, $c_{rs}^{(2)}$	:	coefficients determined by equation (A-2, 2b, 2c)
A_p	:	motion amplitude
A_p, B_p, C_p	:	constants in equations U, V and W, respectively
A_{pq}, B_{pq}, C_{pq}	:	modal coefficients determined by equations (37, 38 and 39)
A_{prsq}, B_{prsq}, C_{prsq}, D_{prsq}, E_{prsq}	:	modal coefficients determined by equation (40)
cn	:	Jacobi elliptic function
e	:	exponential function
E	:	Young's modulus of elasticity
f, g, h	:	spatial functions
f_p	:	function determined by equation (65)
G	:	shear modulus of elasticity
G (p, q)	:	coefficient determined by equation (56)
h_p (p = 0, 1, 4, 6, 8)	:	coefficients of the characteristic equation (19)
H_{pl}	:	matrix elements H
J	:	number of applied constraints
k_p^*	:	parameter of the integral elliptic function K
K (k_p^*)	:	elliptic complete integral of the first kind
l	:	length of element
L	:	total length of shell
m	:	axial mode
M_{xx}, $M_{\theta\theta}$, $M_{x\theta}$, $M_{\theta x}$, $\overline{M}_{x\theta}$	:	resultant moments of a cylindrical shell
n	:	circumferential mode

N_{xx}, $N_{\theta\theta}$, $N_{x\theta}$, $N_{\theta x}$, $\overline{N}_{x\theta}$	:	resultant constraints for a cylindrical shell
N	:	number of finite elements
NDF	:	number of degrees of freedom
p_x, p_θ, p_n	:	loads applied to a cylindrical shell
p_{ij}	:	elements of the matrix of elasticity (i, j = 1, ..., 6)
Q_x, Q_θ	:	shear constraint resultants for a cylindrical shell
R	:	radius of the shell
t	:	thickness of the shell
T_L	:	linear vibration period
T_{NL}	:	non-linear vibration period
U, V, W	:	axial, tangential and radial displacements, respectively
x	:	the coordinate generator of the shell
α_p, β_p	:	determined by equation (23) and given by equation (24)
ϵ_{xx}, $\epsilon_{\theta\theta}$, $\epsilon_{x\theta}$	:	deformations of the surface of reference
ϵ_{lk}	:	element of matrix E
κ_{xx}, $\kappa_{\theta\theta}$, $\kappa_{x\theta}$	:	rotations at the surface of reference
Λ_p	:	coefficient determined by equation (72)
λ_p	:	complex roots of characteristic equation (19)
ϕ, ϕ_x, ϕ_θ	:	determined by equation (10)
ν	:	Poisson's ratio
ρ	:	density of the shell
ω_p	:	linear frequency of free vibrations
ω_p^*	:	non-linear frequency of free vibrations
θ	:	circumferential coordinates
τ	:	time related coordinates

LIST OF MATRICES

$[A]$	:	determined by equation (27)
$[B]$, $[BB']$	:	determined by equations (30) and (34), respectively
$[A^{+}]$, $[B^{+}]$, $[C^{+}]$	:	matrices of modal coefficients determined by equation (37), (38) and (39)
$[A^{*}]$, $[B^{*}]$, $[C^{*}]$	:	determined by equations (42)
$[A']$, $[B']$, $[C']$	:	determined by equations (41)
$[A^{++}]$, $[B^{++}]$, $[C^{++}]$, $[D^{++}]$, $[E^{++}]$	:	matrices of modal coefficients determined by equation (40)
$[A^{**}]$, $[B^{**}]$, $[C^{**}]$, $[D^{**}]$, $[E^{**}]$	:	determined by equation (47)
$\{C\}$	:	vector of arbitrary constraints
$[E]$	:	matrix function of $[A]$
$\{F\}$	:	vector of external forces
$[H]$	:	determined by equation (18)
$[J]$	:	determined by equation (30)
$[k_L]$, $[k_{NL}]$	:	linear and non-linear stiffness matrices for a finite element, respectively
$[k^{*}_{NL}]$	:	determined by equation (54)
$[K_L]$, $[K_{NL}]$	:	linear and non-linear stiffness matrices, for the entire shell, respectively
$[K^{(r)}_{NL}]$, $[K^{(r)}_{NL}]$	:	reduced linear and non-linear stiffness matrices, for the entire shell, respectively
$[K^{(D)}_{L}]$	:	global diagonal linear stiffness matrix
$[L]$	:	determined by equation (26)
$[m]$	:	mass matrix of a finite element
$[M]$	:	mass matrix for total shell
$[M^{(r)}]$	:	reduced mass matrix for total shell

$[M^{(D)}]$	:	diagonal mass matrix for total shell
$[N]$	:	determined by equation (28)
$[P]$	:	matrix of elasticity
$\{q\}$	:	time-related vector coordinates
$[Q]$	:	determined by equation (29)
$[R]$, $[R']$	:	determined, respectively, by equations (25) and (34)
$[T]$	:	determined by equation (16)
$[X]$	:	diagonal matrix function of coordinate x
$\{\delta_i\}$	:	vector of degrees of freedom for node i
$\{\delta\}$	:	vector of degrees of freedom for total shell
$\{\delta^{(r)}\}$	:	reduced vector of degrees of freedom for total shell
$\{\epsilon\}$	:	deformation vector
$\{\epsilon_L\}$ $\{\epsilon_{NL}\}$	:	linear and non-linear components of the deformation vector, respectively
$[\phi]$	:	matrix of eigenvectors
$\{\sigma\}$	:	stress vector

NON-LINEAR DYNAMIC ANALYSIS OF ANISOTROPIC
CYLINDRICAL SHELLS

1. INTRODUCTION

Thin shells are widely used in a variety of fields. The diversity of their applications is extensive, from space vehicles to home appliances. Consequently, the analysis of thin shells under static or dynamic loads has been the focus of many investigations. Most of the research in this field has involved analysis of linear thin shells. The results have proven to be satisfactory in cases where deflections of the shell were very small, especially low-level bending, even when allowing for the thickness of the shell itself. In several practical experiments, however, the linear analysis was not sufficiently accurate for satisfactory design. In those cases, a non-linear analysis was required.

The first attempt to formulate a theory for thin shells was derived from Aron's general equation of elasticity in 1874. It was followed, in 1888, by an approximation theory known as the "First Approximation" of Love [1]. Since then, the linear theory of elastic shells has been re-examined repeatedly ([2] to [7]).

The non-linear theory of thin elastic shells has also been the focus of many studies. Thus, beginning with the tridimensional elasticity equations, there are now several articles available dealing with non-geometric linearities in shells of arbitrary shapes ([8] to [12]).

More specifically, several methods have been developed for the analysis of dynamic non-linear thin cylindrical shells. Among these were Galerkin's well-balanced method ([13] to [15]), the small perturbation method ([16] to [18]), the modal expansion method [19] and the finite element method [20]. All of these methods have their advantages and disadvantages. The best test of any method is probably its general content : i.e. the method should quantify the component displacements and provide for precise characterization of the high and low frequencies of the shell.

These criteria were not met in Galerkin's small perturbation method, and studies [13] to [18] applied only to the particular case where the shell was supported on both edges. The modal expansion and finite element methods, however, were satisfactory on both counts.

In references [13] to [15], only lateral displacement was applied. In [13], the restrictions of tangential displacement continuity were satisfied although to the detriment of actual bending at the edges of the shell. In order to meet the criteria of continuity by including bending at the edges of the shell, Evensen [14] modified the lateral displacement expression by using a symmetric mode to include the coupling. This modification, however, led to actual moments at the edge of the shell such that the boundary conditions lay somewhere between the simply supported and clamped cases. Boundary-condition effects on the other components of displacement were ignored, moreover, in [13] and [14].

Similarly, in [15] coupling with the symmetric mode led to the derivation of motion by assuming:

a) The condition of continuity for the tangential components of displacement.
b) A geometric boundary condition on the axial component.
c) A natural boundary condition.

These three conditions, however, were only satisfied in a general sense.

Alturi [16] also used these three conditions and suggested that a lateral displacement with three modes be included. The displacement and axial bending moment were zero at the edges of the shell. Contrary to Dowell and Ventres [15], Alturi [16] solved the problem by using the small perturbation method. The unknowns appearing in the modal equations were expressed by means of an asymptotic series and terms of small parameter.

The formulae in [13] to [16] have serious drawbacks:

a) Having only assumed the form of the lateral displacement, special attention must be given to the conditions of continuity for the other components. Should these not be satisfied automatically, it would be necessary to include other modes and these modes are obtained intuitively. This procedure can hardly be generalized to include other shell geometries.

b) It is extremely difficult to satisfy the geometric boundary conditions on tangential displacement, especially for a circular shell.

c) The analytical solution of the problem requires several manual calculations. These become increasingly difficult so that the inclusion of other means becomes necessary.

d) The formulation is not applicable when the shape of the modes are not simple analytic functions.

e) Generalizations from arbitrary shells are not valid.

Some of the disadvantages were eliminated in Chen and Babcock [17]. The small perturbation method was used to transform the non-linear equations to a linear system, by expanding the unknown variables in a power series with respect to a small parameter. Applying the boundary conditions of circumferential continuity for a simply supported shell, lateral displacement was then obtained. The major advantage of this technique, compared to other methods requiring an initial hypothesis regarding the form of the vibration mode, is that the results are not preconceived.

Other refinements were presented in Ginsberg's article [18]. The equations for a simply supported circular shell were obtained using an energy formulation. All three displacements, U, V, and W, were considered and a more exact theory was used. Due to algebraic difficulties encountered during derivation of the general equations, the perturbation technique had to be used. For this reason, therefore, limitations (d) and (e) still apply.

The above mentioned shortcomings restrict the use of the methods employed in [13] to [18] (Donnell's simplified non-linear theory), because the theory neglects the plane of inertia effect. By incorporating the nodal expansion technique, Radwan and Genin [19] improved upon the methods used in [13] to [18] and eliminated the weaknesses therein by using Sanders-Koiter [10,11] general non-linear theory.

The authors of the present paper derived and validated the general nodal equations for analysis of a static and dynamic arbitrary non-linear geometric shell. The three displacement components were considered in these cases.

There are two advantages to these formulations:

a) Greater simplicity in problem formulation and solution, compared with the other methods.

b) Whatever the shell structure might be, the formulation of the equations retains the same form once the corresponding non-linear nodal equations are derived.

However, this method too has serious disadvantages : the analytical forms for the displacement components apply only to those cases where a cylinder is supported at both ends.

References [13] to [19] adopted the analytical method as their numerical approach to solving the problem. The finite element method likewise suggests a numerical approach. This method offers many advantages, some of which are:

a) Arbitrary shell geometry : the method applies equally well to the cylinder, to the cone or to all other axisymmetric shells with positive or negative shaped curvatures.

b) Simple inclusion of thickness discontinuities, material property variations, differences in materials comprising the shell.

c) Arbitrary boundary conditions : the problem can be resolved for a supported, clamped-free or clamped-clamped shell without changing the displacement functions in each case.

d) High and low frequency characteristics are obtained immediately.

After adopting the finite element method, Raju and Rao [20] obtained, for various boundary conditions, frequency variation in conjunction with the maximum normal displacement of a point situated on the average surface of the shell.

The Sanders-Koiter relationship was derived from strain-displacement non-linear theory. A curved element with two nodes having six degrees of freedom each was used to restrain the shell. The displacement functions were not derived from thin shell theory but were instead described as a cubic polynomial in relation to the orthogonal coordinate. Their algorithm was iterative at each assumed normal displacement value, the approximate vector and frequencies were calculated until a prescribed convergence criterion was satisfied.

The research done in [14] to [20] was limited to studies of isotropic shells. Only Nowinski [13] made a generalization concerning orthotropic shells by incorporating Donnell's simplified theory. Ambartsumyan [21] produced an important work involving a number of cases for anisotropic shells.

The present paper presents a general approach to the analysis of non-linear thin cylindrical anisotropic shells. The finite element method was employed, but it is a hybrid, a combination of the finite element method and classical shell theory. The finite element chosen was a cylindrical one. This choice allowed us to use the complete equilibrium equations to determine the displacement functions and, further, the mass and stiffness matrices. This theory proved to be more accurate than the usual finite element methods.

1.1 Hypotheses

Non-linear elastic thin shell theory is derived by approximation from the tridimensional elasticity equation. Like linear theory, it is also based on Love's "First Approximation" but the assumption concerning the order of magnitude of the bending has been modified.

The non-linear theory is based on the following hypotheses:

a) Thickness (t) is infinitesimal in comparison with the minimum radius of curvature (R_{min});

b) The displacement gradients are small and the squares of the rotation do not exceed reference surface deformation in order of magnitude;

c) The normal constraints, normal to the surface of reference are negligible;

d) The normals to the surface of reference remain normal after deformation and are not subject to any elongation.

Hypothesis (a) represents the definition of thin shells ($R/t \geq 10$).

Hypothesis (b) corroborates the non-linearities of the equations. Explained by physical bending terminology, these elements have the same thickness as the shell itself.

Hypotheses (c) and (d) allow us to neglect the stresses normal to the surface and the transversal shear deformation.

The theory based on these four hypotheses is known as the "Sanders-Koiter non-linear theory" [10,11]; it has been used throughout this paper.

1.2 Method

The analysis in this paper was divided into two parts : the first deals with linear behaviour and the second with non-linearities and strain-displacement relationships.

The main steps in the method we propose are as follows:

a) The shell is subdivided into several cylindrical elements (Figure 1). Each shell element is defined by two nodal circles and two nodal points i and j (Figure 2). The displacement functions are defined by:

$$\begin{Bmatrix} U\ (x,\theta) \\ W\ (x,\theta) \\ V\ (x,\theta) \end{Bmatrix} = [N] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \quad (1)$$

where $\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}$ represent nodal displacements, and the elements of matrix [N] are in general a function of position. These displacement functions must, on the one hand, adequately express real displacements of the shell and, on the other hand, satisfy at least the geometric boundary conditions.

b) The linear component of the procedure is presented in reference [22] and [23], where the displacement functions are determined by solving the three differential equations of motion from Sanders' theory [5] (in terms of the elements p_{ij} of [P] where [P] is the matrix of elasticity).

c) For the non-linear component, the modal coefficients [19] are derived from the results obtained in the previous step.

d) The linear and non-linear natural vibration frequency ratio is then obtained for the cases of uncoupled modal equations.

1.3 Strain-displacement and stress-strain relations

The non-linear Sanders-Koiter theory for thin shells postulated differences in the first and second fundamental forms between the reference surfaces, deformed and non deformed, as deformation measures in elongation and bending respectively.

Generally, the deformation vector $\{\epsilon\}$ is written as:

$$\{\epsilon\} = \{\epsilon_L\} + \{\epsilon_{NL}\} = \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{\theta\theta} \\ 2\epsilon_{x\theta} \\ \kappa_{xx} \\ \kappa_{\theta\theta} \\ 2\kappa_{x\theta} \end{Bmatrix} \tag{2}$$

where subscripts "L" and "NL" mean "linear" and "non-linear", respectively.

For a cylindrical shell, the expressions for $\{\epsilon_L\}$ and $\{\epsilon_{NL}\}$ are given by:

$$\{\epsilon\} = \begin{Bmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{1}{R}\left[\dfrac{\partial V}{\partial \theta} + W\right] \\ \dfrac{\partial V}{\partial x} + \dfrac{1}{R}\dfrac{\partial U}{\partial \theta} \\ -\dfrac{\partial^2 W}{\partial x^2} \\ -\dfrac{1}{R^2}\left[\dfrac{\partial^2 W}{\partial \theta^2} - \dfrac{\partial V}{\partial \theta}\right] \\ -\dfrac{2}{R}\dfrac{\partial^2 W}{\partial x \partial \theta} + \dfrac{3}{2R}\dfrac{\partial V}{\partial x} - \dfrac{1}{2R^2}\dfrac{\partial U}{\partial \theta} \end{Bmatrix} \tag{3}$$

and

$$\{\epsilon_{NL}\} = \left\{ \begin{array}{c} \frac{1}{2}\left[\frac{\partial W}{\partial x}\right]^2 + \frac{1}{8}\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right]^2 \\ \frac{1}{2R^2}\left[V - \frac{\partial W}{\partial \theta}\right]^2 + \frac{1}{8}\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right]^2 \\ \frac{1}{2R}\left[\frac{\partial W}{\partial x}\frac{\partial W}{\partial \theta} - V\frac{\partial W}{\partial x}\right] \\ 0 \\ 0 \\ 0 \end{array} \right\} \tag{4}$$

where U, V and W are, respectively, the axial, tangential and radial displacements of the shell's surface of reference.

It is evident that in equations (3) and (4) the expressions for components κ_{xx}, $\kappa_{\theta\theta}$, $2\kappa_{x\theta}$ are linear. This fits in with hypothesis (b) from paragraph 1.1.

The constituent relations between the stress and deformation vectors of the surface of reference for anisotropic shells are given as follows:

$$\{\sigma\} = \left\{ \begin{array}{c} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \\ M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \end{array} \right\} = [P]\ \{\epsilon\} \tag{5}$$

where [P] is the matrix of elasticity.

The elements P_{ij} in [P] determine the anisotropy of the shell, which depends on the mechanical characteristics of the structure's material.

In general, this implies that:

$$[P] = \begin{bmatrix} p_{11} & p_{12} & 0 & p_{14} & p_{15} & 0 \\ p_{21} & p_{22} & 0 & p_{24} & p_{25} & 0 \\ 0 & 0 & p_{33} & 0 & 0 & p_{36} \\ p_{41} & p_{42} & 0 & p_{44} & p_{45} & 0 \\ p_{51} & p_{52} & 0 & p_{54} & p_{55} & 0 \\ 0 & 0 & p_{63} & 0 & 0 & p_{66} \end{bmatrix} \tag{6}$$

1.4 Equations of equilibrium

By applying the virtual work principle to the infinitesimal element of the deformed surface of reference, the three equations of equilibrium, describing the non-linear behaviour of an arbitrarily formed shell, are then obtained [10] (see Figure 3).

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{R}\frac{\partial \bar{N}_{x\theta}}{\partial \theta} - \frac{1}{2R^2}\frac{\partial \bar{M}_{x\theta}}{\partial \theta} - \frac{1}{2R}\frac{\partial}{\partial \theta}\left[\phi\,(N_{xx} + N_{\theta\theta})\right] = 0 \tag{7}$$

$$\frac{1}{R}\frac{\partial N_{\theta\theta}}{\partial \theta} + \frac{\partial \bar{N}_{x\theta}}{\partial x} + \frac{1}{R^2}\frac{\partial M_{\theta\theta}}{\partial \theta} + \frac{3}{2R}\frac{\partial \bar{M}_{x\theta}}{\partial x} - \frac{1}{R}(\phi_x \bar{N}_{x\theta} + \phi_\theta N_{\theta\theta}) + \frac{1}{2}\frac{\partial}{\partial x}\left[\phi\,(N_{xx} + N_{\theta\theta})\right] = 0 \tag{8}$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{2}{R}\frac{\partial^2 \bar{M}_{x\theta}}{\partial x \partial \theta} + \frac{1}{R^2}\frac{\partial^2 M_{\theta\theta}}{\partial \theta^2} - \frac{1}{R}N_{\theta\theta} - \frac{\partial}{\partial x}\left[\phi_x N_{xx} + \phi_\theta \bar{N}_{x\theta}\right] - \frac{1}{R}\frac{\partial}{\partial \theta}\left[\phi_x \bar{N}_{x\theta} + \phi_\theta N_{\theta\theta}\right] = 0 \tag{9}$$

where

$$\phi = \frac{1}{2}\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right], \ \phi_x = -\frac{\partial W}{\partial x} \text{ et } \phi_\theta = -\frac{1}{R}\left[\frac{\partial W}{\partial \theta} - V\right] \tag{10}$$

Substituting equations (2) to (6) for the equilibrium equations (7) to (10), we obtain new equation (11) functions of elements p_{ij} in [P] and the axial, tangential and radial displacements U, V and W of the shell surface of reference:

$$\begin{aligned} L_1 (U, V, W, p_{ij}) + N_1 (U, V, W, p_{ij}) &= 0 \\ L_2 (U, V, W, p_{ij}) + N_2 (U, V, W, p_{ij}) &= 0 \\ L_3 (U, V, W, p_{ij}) + N_3 (U, V, W, p_{ij}) &= 0 \end{aligned} \tag{11}$$

Functions L_i and N_i ($i = 1$ to 3) represent, respectively, the linear and non-linear equations of equilibrium. These equations are given in Appendix A-1.

1.5 Matrix of elasticity

The matrix of elasticity [P] is generally given by equation (6); the present theory can therefore be applied to:

(i) Shells composed of only one layer or of an arbitrary number of isotropic of orthotropic layers;

(ii) Double-walled shells, with slabs or ribs;

(iii) Ring-stiffened shells with grooves of known characteristics;

(iv) Shells where [P] can be experimentally evaluated.

Here we will confirm ourselves to shells composed of only one layer or an arbitrary number of symmetric isotropic or orthotropic layers arranged relative to the surface coordinates.

For an arbitrary number of orthotropic layers [21], it is postulated that there is no slippage between the layers and that the principal directions of elasticity on every point of the shell coincide with the directions of the coordinate lines.

(i) For an even number of layers, equal to 2v, the elements p_{ij} of [P] can be written as:

$$P_{ij} = 2\sum_{s=1}^{v} B_{ij}^{s}\,(t_s - t_{s+1}) \qquad i = 1 \text{ to } 3 \text{ and } j = 1 \text{ to } 6$$
$$P_{ij} = \frac{2}{3}\sum_{s=1}^{v} B_{i-3,j-3}\,(t_s^3 - t_{s+1}^3) \qquad i = 4 \text{ to } 6 \text{ and } j = 4 \text{ to } 6 \tag{12}$$

(ii) For an odd number 2v + 1, we obtain:

$$p_{ij} = 2\left[B_{ij}^{v+1}\,t_{v+1} + \sum_{s=1}^{v} B_{ij}^{s}\,(t_s - t_{s+1})\right] \qquad i = 1 \text{ to } 3 \text{ and } j = 1 \text{ to } 6$$
$$P_{ij} = \frac{2}{3}\left[B_{i-3,j-3}^{v+1}\,t_{v+1}^3 + \sum_{s=1}^{v} B_{i-3,j-3}^{s}\,(t_s^3 - t_{s+1}^3)\right] \qquad i = 4 \text{ to } 6 \text{ and } j = 4 \text{ to } 6 \tag{13}$$

where

$$B_{11}^{s} = E_1^{s} / (1 - \nu_1^{s}\nu_2^{s}) \qquad B_{22}^{s} = E_2^{s} / (1 - \nu_1^{s}\nu_2^{s})$$
$$B_{12}^{s} = B_{21}^{s} = \nu_2^{s}E_1^{s} / (1 - \nu_1^{s}\nu_2^{s}) \qquad B_{33}^{s} = 0.5\,G_{12}^{s} \tag{14}$$
$$B_{ij}^{s} = 0 \quad \text{elsewhere}$$

t_s is the x^{th} layer coordinate having the surface of reference as shown in Figure 4; $(E_1^s\ ,\ \nu_1^s)$ and $(E_2^s\ ,\ \nu_2^s)$ are, respectively, Young's modulus and Poisson's ratio in direction x and θ and G_{12}^s is the shear modulus of elasticity.

2. LINEAR MATRIX CONSTRUCTION

The general equations of motion were derived in terms of the elements of the elasticity matrix (p_{ij}) in terms of the axial, tangential and radial displacements of the shell's surface of reference U, V and W, respectively. The solution of these non-linear differential equations was highly complicated.

To circumvent the difficulty, the problem was divided into two parts; the first dealing with the linear system, and the second with the non-linearities in the strain-displacement relations.

In order to obtain the stiffness and mass matrices, the displacement functions were derived from the shell's equations of motion.

2.1 Displacement functions

Following the procedure described in paragraph 1.2, the shell was subdivided into several finite elements defined by two nodes i and j and by components U, V and W, representing axial, tangential and radial displacements, respectively, from a point located on the shell's surface of reference.

The linear equations of motions are given by (see Appendix A-1):

$$\begin{aligned} L_1\ (U, V, W, p_{ij}) &= 0 \\ L_2\ (U, V, W, p_{ij}) &= 0 \\ L_3\ (U, V, W, p_{ij}) &= 0 \end{aligned} \tag{15}$$

The displacement functions are then assumed to be:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T] \begin{Bmatrix} u(x) \\ w(x) \\ v(x) \end{Bmatrix} \tag{16}$$

[T] is a (3 x 3) matrix in θ given in Appendix A-2 and u(x), w(x) and v(x) are functions of the x coordinate and the shell's characteristics.

Assuming:

$$u(x) = Ae^{\lambda x/R}, \; v(x) = Be^{\lambda x/R}, \; w(x) = Ce^{\lambda x/R} \tag{17}$$

Substituting (16) and (17) for the equations of motion (15), three homogeneous linear functions of constants A, B and C are obtained:

$$[H] \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \{0\} \tag{18}$$

For the solution to be non-trivial, the determinant of matrix [H] must be equal to zero. This brings us to the following polynomial equation [23]:

$$\text{Det}\,([H]) = h_8\lambda^8 - h_6\lambda^6 + h_4\lambda^4 - h_2\lambda^2 + h_0 = 0 \tag{19}$$

The values of coefficients h_p in this eighth-degree polynomial are given in Appendix A-2.

Each root of this equation yields a solution to the equations of motion (15). The complete solution is obtained by adding the eight solutions independently with the constants A_p, B_p and C_p ($p = 1, \ldots, 8$), so that:

$$u(x) = A_p e^{\lambda_p x/R} \tag{20}$$

$$v(x) = B_p e^{\lambda_p x/R} \tag{21}$$

$$w(x) = C_p e^{\lambda_p x/R} \tag{22}$$

The constants A_p, B_p and C_p are not independent. We can therefore express A_p and B_p as a function of C_p, for example:

$$A_p = \alpha_p C_p \quad \text{and} \quad B_p = \beta_p C_p \quad , \quad p = 1,\ldots,8 \tag{23}$$

The values of α_p and β_p can be obtained from the following relations:

$$\begin{Bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{Bmatrix} \begin{Bmatrix} \alpha_p \\ \beta_p \end{Bmatrix} = \begin{Bmatrix} -H_{13} \\ -H_{23} \end{Bmatrix} \tag{24}$$

where coefficients H_{ij} are as given in Appendix A-2.

Substituting expressions (20) to (23) into equations (16), the displacements $U(x,\theta)$, $V(x, \theta)$ and $W(x, \theta)$ can then be expressed in conjunction with the eight C_p constants only. We then have:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T]\,[R]\,\{C\} \tag{25}$$

where [R] is a (3 x 8) matrix given in Appendix A-2 and {C} is an 8^{th} order vector of the C_p constants:

$$\{C\} = \{C_1\, C_2 \ldots C_8\}^T$$

Setting [R] = [L] [X], equation (25) becomes:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T]\,[L]\,[X]\{C\} \tag{26}$$

where matrices [L] and [X] are given in Appendix A-2.

To determine the eight C_p constants, it is necessary to formulate eight boundary conditions for the finite elements. The axial, tangential and radial displacements, as well as rotation, will be specified for each node. The degrees of freedom at node i can be defined be the vector:

$$\{\delta_i\} = \left\{ u_i \; w_i \left[\frac{dw}{dx}\right]_i \; v_i \right\}^T$$

The elements which have two nodes and eight degrees of freedom will have i (x = 0) and j (x = l) as nodal displacements at the boundaries:

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \left\{ u_i \; w_i \left[\frac{dw}{dx}\right]_i \; v_i \; u_j \; w_j \left[\frac{dw}{dx}\right]_j \; v_j \right\}^T = [A]\{C\} \tag{27}$$

where the terms of matrix [A], given in Appendix A-2, are obtained from matrix [R] by successively setting x = 0 and x = l.

Multiplying equation (27) by $[A^{-1}]$ we obtain:

$$\{C\} = [A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}$$

Substituting for equations (26) we get:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T]\,[L]\,[X]\,[A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{28}$$

These equations determine the displacement functions.

2.2 Linear mass and stiffness matrices for an element

The deformation vector can be obtained from equations (3) and (28), therefore:

$$\{\epsilon_L\} = \begin{bmatrix} [T] & [O] \\ [O] & [T] \end{bmatrix} [Q]\,[A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{29}$$

Setting [Q] = [J] [X], equations (29) becomes:

$$\{\epsilon_L\} = \begin{bmatrix} [T] & [O] \\ [O] & [T] \end{bmatrix} [J]\,[X]\,[A^{-1}] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{30}$$

Matrix [J] is given in Appendix A-2.

Combining equations (5) and (30), the stress-strain relations can be written as:

$$\{\sigma_L\} = [P]\,[B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{31}$$

The mass and stiffness matrices can then be expressed as:

$$[m] = \rho t \int\int [N^T]\,[N]\,dA$$
$$[k_L] = \int\int [B^T]\,[P]\,[B]\,dA \tag{32}$$

where $dA = Rdxd\theta$. A quick reminder to the reader : "L" means "linear".

Using equations (28) and (30), equations (32), after integration with respect to θ over the interval, become:

$$[m] = \rho t\,[A^{-1}]^T \left\{ \pi R \int_0^l [X^T]\,[L^T]\,[L]\,[X]\,dx \right\} [A^{-1}]$$
$$[k_L] = [A^{-1}]^T \left\{ \pi R \int_0^l [X^T]\,[J^T]\,[P]\,[J]\,[X]\,dx \right\} [A^{-1}] \tag{33}$$

After working out the integration as a function of x, we obtain:

$$[m] = \pi R \rho t\,[A^{-1}]^T\,[R']\,[A^{-1}]$$
$$[k_L] = \pi R\,[A^{-1}]^T\,[BB']\,[A^{-1}] \tag{34}$$

where the (p,q) term from [R'] is:

$$R'(p,q) = \begin{cases} \dfrac{L'(p,q)}{(\lambda_p + \lambda_q)/R} \left[e^{(\lambda_p + \lambda_q)l/R} - 1 \right] & \text{si } \lambda_p + \lambda_q \neq 0 \\ L'(p,q)\,.\,l & \text{si } \lambda_p + \lambda_q = 0 \end{cases} \tag{35}$$

and where [BB'] is:

$$BB'(p,q) = \begin{cases} \dfrac{J'(p,q)}{(\lambda_p + \lambda_q)/R} \left[e^{(\lambda_p + \lambda_q)l/R} - 1 \right] & \text{si } \lambda_p + \lambda_q \neq 0 \\ J'(p,q) \, . \, l & \text{si } \lambda_p + \lambda_q = 0 \end{cases} \tag{36}$$

L'(p,q) and J'(p,q) are, respectively, the (p,q) terms of the products of matrices $[L^T]$ [L] and $[J^T]$ [P] [J].

3. NON-LINEAR MATRIX CONSTRUCTION

The following approach, developed in reference [19], was used with particular attention to geometric non-linearities. The coefficients of the modal equations were obtained through the Lagrange method. Thus, the non-linear stiffness matrix, once calculated, was overlaid onto the linear system. Before we embark on matrix formulation, however, a brief summary of the method is in order.

3.1 Method

This section will be limited to the relevant details of the method used to find the non-linear stiffness matrix.

The main steps of this method are as follows:

a) Shell displacements are expressed as generalized product coordinate sums and spatial functions;

b) the deformation vector is written as a function of the generalized coordinates by separating the linear portion from the non-linear;

c) these expressions are then introduced into the Lagrange equations up to and including the degree corresponding to the deformation energy;

d) substituting the expressions in a) into the strain-displacement relations in Sanders-Koiter [10,11] non-linear method, the generalized coordinate coefficients appearing in the equation derived under c) are determined in terms of spatial functions.

3.2 Coefficients of modal equations

If A_{pq}, B_{pq}, C_{pq}, A_{prsq}, B_{prsq}, C_{prsq}, D_{prsq} and E_{prsq} are designed as coefficients of the modal equations mentioned in step d) above for a cylindrical shell, the following expressions [19] are thus obtained:

$$A_{pq} = \frac{1}{8R^2}\left[R\frac{\partial g_p}{\partial x} - \frac{\partial f_p}{\partial \theta}\right]\cdot\left[R\frac{\partial g_q}{\partial x} - \frac{\partial f_q}{\partial \theta}\right] + \frac{1}{2}\frac{\partial h_p}{\partial x}\frac{\partial h_q}{\partial x} \tag{37}$$

where f, g, h are spatial functions determined by matrix [N] in equations (28) and:

$$B_{pq} = \frac{1}{8R^2}\left[R\frac{\partial g_p}{\partial x} - \frac{\partial f_p}{\partial \theta}\right]\cdot\left[R\frac{\partial g_q}{\partial x} - \frac{\partial f_q}{\partial \theta}\right] + \frac{1}{2R^2}\left[\frac{\partial h_p}{\partial \theta} - g_p\right]\cdot\left[\frac{\partial h_q}{\partial \theta} - g_q\right] \tag{38}$$

$$C_{pq} = \frac{1}{4R}\left[\frac{\partial h_p}{\partial x}\frac{\partial h_q}{\partial \theta} + \frac{\partial h_q}{\partial x}\frac{\partial h_p}{\partial \theta}\right] - \frac{1}{4R}\left[g_p\frac{\partial h_q}{\partial x} + g_q\frac{\partial h_p}{\partial x}\right] \tag{39}$$

$$
\begin{aligned}
A_{prsq} &= 2A_{pq}A_{rs} \\
B_{prsq} &= 2B_{pq}B_{rs} \\
C_{prsq} &= 2C_{pq}C_{rs} \\
D_{prsq} &= 2A_{pq}B_{rs} \\
E_{prsq} &= 2B_{pq}A_{rs}
\end{aligned}
\tag{40}
$$

In equations (39) and (40), the subscripts p,q and p, q, r, s represent the coupling between two modes. It is arranged in such a way that equations (40) is written, r = p and s = q.

For consistency, equations (37) to (39) and (40) are written in matrix format.

Hence, these different matrices can be expressed in conjunction with matrices [T], [L], [X] and $[A^{-1}]$.

The following notation is adopted : the matrices with the "+" superscript represent equations (37) to (39) and the ones with the "++" superscript represent the equations in (40).

With equations (37) to (39), we obtain:

$$
\left\{ \begin{matrix} [A^{+}] \\ [B^{+}] \\ [C^{+}] \end{matrix} \right\} = [A^{-1}]^{T} \, [X^{T}] \left\{ \begin{matrix} [A'] \\ [B'] \\ [C'] \end{matrix} \right\} [X] \, [A^{-1}] \tag{41}
$$

where matrices [A'], [B'] and [C'] are a function of n, θ and α_p, β_p from roots λ_p of the specific equation in (19) and of constants defined in equations (23).

Setting:

$$\begin{Bmatrix} [A^{*}] \\ [B^{*}] \\ [C^{*}] \end{Bmatrix} = [X^{T}] \begin{Bmatrix} [A'] \\ [B'] \\ [C'] \end{Bmatrix} [X] \tag{42}$$

equations (41) become:

$$\begin{Bmatrix} [A^{+}] \\ [B^{+}] \\ [C^{+}] \end{Bmatrix} = [A^{-1}]^{T} \begin{Bmatrix} [A^{*}] \\ [B^{*}] \\ [C^{*}] \end{Bmatrix} [A^{-1}] \tag{43}$$

Matrices $[A^{+}]$, $[B^{+}]$ and $[C^{+}]$ are square (8 x 8) matrices.

When $r = p$ and $s = q$, the equations are written:

$$\begin{Bmatrix} [A^{++}] \\ [B^{++}] \\ [C^{++}] \\ [D^{++}] \\ [E^{++}] \end{Bmatrix} = 2 \begin{Bmatrix} [A^{+}]\,[A^{+}] \\ [B^{+}]\,[B^{+}] \\ [C^{+}]\,[C^{+}] \\ [A^{+}]\,[B^{+}] \\ [B^{+}]\,[A^{+}] \end{Bmatrix} \tag{44}$$

Using equations (41) we then get:

$$\begin{Bmatrix} [A^{++}] \\ [B^{++}] \\ [C^{++}] \\ [D^{++}] \\ [E^{++}] \end{Bmatrix} = 2\, [A^{-1}]^{T}\, [X] \begin{Bmatrix} [A'] \\ [B'] \\ [C'] \\ [A'] \\ [B'] \end{Bmatrix} [X]\, [A^{-1}]\, [A^{-1}]^{T}\, [X] \begin{Bmatrix} [A'] \\ [B'] \\ [C'] \\ [B'] \\ [A'] \end{Bmatrix} [X]\, [A^{-1}] \tag{45}$$

Using the symmetrical properties of matrices [X], [A'], [B'] and [C'].

The product $[A^{-1}][A^{-1}]^T$ represents a matrix of a constant, written as [E]. Substituting equations (42) in equations (45), we obtain:

$$\begin{Bmatrix} [A^{++}] \\ [B^{++}] \\ [C^{++}] \\ [D^{++}] \\ [E^{++}] \end{Bmatrix} = 2\,[A^{-1}]^T \begin{Bmatrix} [A^{*}] \\ [B^{*}] \\ [C^{*}] \\ [A^{*}] \\ [B^{*}] \end{Bmatrix} [E] \begin{Bmatrix} [A^{*}] \\ [B^{*}] \\ [C^{*}] \\ [B^{*}] \\ [A^{*}] \end{Bmatrix} [A^{-1}] \qquad \textbf{(46)}$$

Setting:

$$\begin{Bmatrix} [A^{**}] \\ [B^{**}] \\ [C^{**}] \\ [D^{**}] \\ [E^{**}] \end{Bmatrix} = \begin{Bmatrix} [A^{*}] \\ [B^{*}] \\ [C^{*}] \\ [A^{*}] \\ [B^{*}] \end{Bmatrix} [E] \begin{Bmatrix} [A^{*}] \\ [B^{*}] \\ [C^{*}] \\ [B^{*}] \\ [A^{*}] \end{Bmatrix} \qquad \textbf{(47)}$$

Equations (46) are then written:

$$\begin{Bmatrix} [A^{++}] \\ [B^{++}] \\ [C^{++}] \\ [D^{++}] \\ [E^{++}] \end{Bmatrix} = 2\,[A^{-1}]^T \begin{Bmatrix} [A^{**}] \\ [B^{**}] \\ [C^{**}] \\ [D^{**}] \\ [E^{**}] \end{Bmatrix} [A^{-1}] \qquad \textbf{(48)}$$

Let us now illustrate the development of the expressions for the (p,q) term of matrices $[A^*]$ and $[A^{**}]$.

For $[A^*]$ there is:

$$A^*(p,q) = a_{pq} e^{(\lambda_p + \lambda_q)x/R} \tag{49}$$

and for $[A^{**}]$ there is:

$$A^{**}(p,q) = \sum_{k=1}^{8} a_{kq} \left[\sum_{l=1}^{8} a_{pl} \epsilon_{lk} e^{(\lambda_p + \lambda_q + \lambda_k + \lambda_l)x/R} \right] \tag{50}$$

ϵ_{lk} is the term (l,k) of matrix [E], and

$$a_{rs} = \frac{1}{2R^2} \left[\frac{1}{4} a_{rs}^{(1)} \sin^2 n\theta + a_{rs}^{(2)} \cos^2 n\theta \right]$$

with

$$\begin{aligned} a_{rs}^{(1)} &= (\beta_r \lambda_r + n\alpha_r)(\beta_s \lambda_s + n\alpha_s) \\ a_{rs}^{(2)} &= \lambda_r \lambda_s \end{aligned} \qquad r,s = 1,\dots,8 \tag{51}$$

Similarly, matrices $[B^{++}]\dots[E^{++}]$ can be written as a function of α, β, λ, x and θ. The (p,q) terms of these matrices are described in Appendix A-2.

3.3 Non-linear stiffness matrix for an element

The non-linear stiffness matrix for an orthotropic cylindrical shell is as follows:

$$[k_{NL}] = \int\int \left\{ p_{11}[A^{++}] + p_{22}[B^{++}] + p_{12}([D^{++}] + [E^{++}]) + p_{33}[C^{++}] \right\} dA \tag{52}$$

where $dA = Rdxd\theta$.

Using equations (37) to (39), equation (52) is written:

$$[k_{NL}] = 2[A^{-1}]^T \left\{ \int \int \left[p_{11}[A^{**}] + p_{22}[B^{**}] + p_{12}([D^{**}] + [E^{**}]) + p_{33}[C^{**}] \right] dA \right\} [A^{-1}] \tag{53}$$

After integration, we obtain:

$$[k_{NL}] = \frac{1}{8R^2} [A^{-1}]^T [k^*_{NL}] [A^{-1}] \tag{54}$$

The (p,q) term in matrix $[k^*_{NL}]$ is written:

$$k^*_{NL}(p,q) = \begin{cases} \displaystyle\sum_{k=1}^{8} \sum_{l=1}^{8} \frac{\epsilon_{lk}}{(\lambda_p + \lambda_q + \lambda_k + \lambda_l)} G(p,q) \left[e^{(\lambda_p + \lambda_q + \lambda_k + \lambda_l)} - 1 \right] & \text{if } \lambda_p + \lambda_q + \lambda_k + \lambda_l \neq 0 \\ \displaystyle\sum_{k=1}^{8} \sum_{l=1}^{8} \epsilon_{lk} \, G(p,q) l/R & \text{if } \lambda_p + \lambda_q + \lambda_k + \lambda_l = 0 \end{cases} \tag{55}$$

G(p,q) is a coefficient in conjunction with α, β, λ and element p_{ij} in matrix [P]. The general expression of G(p,q) is:

$$\begin{aligned} G(p,q) = {} & \frac{3}{16} (p_{11} + p_{22} + 2p_{12}) \, a^{(1)}_{pl} \, a^{(1)}_{kq} + 3 \, (p_{11} \, a^{(2)}_{pl} \, a^{(2)}_{kq} + p_{22} \, b^{(1)}_{pl} \, b^{(1)}_{kq}) \\ & + p_{12} \, (a^{(2)}_{pl} \, b^{(1)}_{kq} + b^{(1)}_{pl} \, a^{(2)}_{kq}) + \frac{1}{4}(p_{11} + p_{12}) \, (a^{(2)}_{pl} \, a^{(1)}_{kq} + a^{(1)}_{pl} \, a^{(2)}_{kq}) \\ & + \frac{3}{4} (p_{12} + p_{22}) \, (a^{(1)}_{pl} \, b^{(1)}_{kq} + b^{(1)}_{pl} \, a^{(1)}_{kq}) \\ & + \frac{1}{4} p_{33} \, (c^{(1)}_{pl} \, c^{(1)}_{kq} + c^{(2)}_{pl} \, c^{(2)}_{kq} + c^{(2)}_{pl} \, c^{(1)}_{kq} + c^{(1)}_{pl} \, c^{(2)}_{kq}) \end{aligned} \tag{56}$$

where the terms $a^{(1)}$ and $a^{(2)}$ are given by equations (51). Terms $b_{...}^{(1)}$,$c_{...}^{(1)}$ and $c_{...}^{(2)}$ are coefficients appearing in expressions for the elements of matrices $[B^*]$ and $[C^*]$ defined in equations (42). These coefficients are given in Appendix A-2.

4. THE INFLUENCE OF GEOMETRIC NON-LINEARITIES OF THE WALLS ON THE NATURAL FREQUENCIES OF A CYLINDRICAL SHELL

The mass and stiffness matrices obtained apply to only one element. After the shell is subdivided into several cylindrical elements, the global mass and stiffness matrices are determined by assembling the matrices for each element. Assembling is done such that all the equations of motion and the continuity of displacements at each node are satisfied.

Vectors $\{F_i\}$ and $\{F_j\}$ represent the internal forces at each i,j node and $\{\delta_i\}$ and $\{\delta_j\}$ are the displacements associated with $\{F_i\}$ and $\{F_j\}$. The sums of the forces and moments at each node must be equal to the sum of the external forces and the moments applied to the node:

$$\{F\}^e = F_j + F_{i+1}$$

$$\text{and} \quad \delta_j = \delta_{i+1}$$

Using these relations we can overlay the mass and stiffness matrices for the individual elements in order to obtain the mass and stiffness matrices for the whole shell. These matrices are designated as $[M]$, $[K_L]$ and $[K_{NL}]$, respectively. They are square matrices of order NDF * (N + 1), where N represents the number of finite elements and NDF represents the number of degrees of freedom at each node.

4.1 Equations of motion

The dynamic behaviour of an empty cylindrical shell, in the absence of external loads, can be represented by the following system:

$$[M]\ \{\ddot{\delta}\} + [K_L]\ \{\delta\} + [K_{NL}]\ \{\delta^3\} = \{0\} \tag{57}$$

where $\{\delta\}$ is the displacement vector; $[M]$, $[K_L]$ and $[K_{NL}]$ are, respectively, the linear and non-linear mass stiffness matrices of the system.

In practice, very specific conditions are applied to the shell boundaries. Thus, matrices [M], $[K_L]$ and $[K_{NL}]$ are reduced to square matrices of order NREDUC = NDF * (N + 1) - J, where J represents the number of essential constraints. These reduced matrices are written as $[M^{(r)}]$, $[K_L^{(r)}]$ and $[K_{NL}^{(r)}]$. As noted previously and to apply hereafter, the superscript "r" means "reduced".

The (57) system of equations then becomes:

$$[M^{(r)}]\ \{\ddot{\delta}^{(r)}\} + [K_L^{(r)}]\ \{\delta^{(r)}\} + [K_{NL}^{(r)}]\ \{\delta^{(r)^3}\} = \{0\} \tag{58}$$

Setting:

$$\{\delta^{(r)}\} = [\Phi]\ \{q\} \tag{59}$$

Where $[\Phi]$ represents the square matrix for the eigenvectors of the linear system and $\{q\}$ is a time-related vector.

Substituting equation (59) into system (58), it becomes:

$$[M^{(r)}]\ [\Phi]\ \{\ddot{q}\} + [K_L^{(r)}]\ [\Phi]\ \{q\} + [K_{NL}^{(r)}]\ [\Phi^3]\ \{q^3\} = \{0\} \tag{60}$$

Multiplying equation (60) by $[\Phi^T]$, we obtain:

$$[\Phi^T][M^{(r)}][\Phi]\{\ddot{q}\} + [\Phi^T][K_L^{(r)}][\Phi]\{q\} + [\Phi^T][K_{NL}^{(r)}][\Phi^3]\{q^3\} = \{0\} \tag{61}$$

The products of matrix $[\Phi^T]$ $[M^{(r)}]$ $[\Phi]$ and $[\Phi^T]$ $[K_L^{(r)}]$ $[\Phi]$ represent diagonal matrices, written as $[M^{(D)}]$ and $[K_L^{(D)}]$, respectively.

Finally, the (57) system of equations is written:

$$[M^{(D)}]\ \{\ddot{q}\} + [K_L^{(D)}]\ \{q\} + [\Phi^T]\ [K_{NL}^{(r)}]\ [\Phi^3]\ \{q^3\} = \{0\} \tag{62}$$

4.2 Solution of uncoupled equations

We saw in the preceding paragraph how matrices contained in the linear part of the system (57) could be reduced to diagonal matrices. On the other hand, the matrix product $[\Phi^T]$ $[K_{NL}^{(r)}]$ $[\Phi^3]$ is not generally described as a diagonal matrix.

A typical equation of the (62) system would yield:

$$m_{pp}\ \ddot{q}_p + k_{pp}^{(L)} q_p + \sum_{s=1}^{NREDUC} k_{ps}^{(NL)} q_s^3 = 0 \tag{63}$$

where coefficients m_{pp} and $[k_{pp}^{(L)}]$, represent the pth diagonal terms of matrices $[M^{(D)}]$ and $[k_L^{(D)}]$, respectively, and $[k_{ps}^{(NL)}]$ is the (p,s) term of the product $[\Phi^T]$ $[K_{NL}^{(r)}]$ $[\Phi^3]$ thereby becoming diagonal.

Equation (63) would then be written:

$$m_{pp}\ \ddot{q}_p + k_{pp}^{(L)} q_p + k_{pp}^{(NL)} q_p^3 = 0 \tag{64}$$

Setting:

$$q_p(\tau) = A_p f_p(\tau) \tag{65}$$

which satisfies the conditions:

$$f_p(0) = 1 \text{ and } \dot{f}_p(0) = 0 \tag{66}$$

Equation (64) becomes, after the A_p simplification:

$$m_{pp} \ddot{f}_p + k_{pp}^{(L)} f_p + k_{pp}^{(NL)} A_p^2 f_p^3 = 0 \tag{67}$$

which is equivalent to:

$$m_{pp} \ddot{f}_p + k_{pp}^{(L)} f_p + k_{pp}^{(NL)} t^2 (A_p/t)^2 f_p^3 = 0 \tag{68}$$

where t represents shell thickness.

Dividing this last equation by m_{pp}, it becomes:

$$\ddot{f}_p + \frac{k_{pp}^{(L)}}{m_{pp}} f_p + \frac{k_{pp}^{(NL)}}{m_{pp}} t^2 (A_p/t)^2 f_p^3 = 0 \tag{69}$$

The coefficient $[k_{pp}^{(L)}/m_{pp}]$ represents the p^{th} linear vibration frequency of the shell. We then obtain:

$$\ddot{f}_p + \omega_p^2 f_p + \Lambda_p (A_p/t)^2 f_p^3 = 0 \tag{70}$$

where

$$\omega_p^2 = \frac{k_{pp}^{(L)}}{m_{pp}} \tag{71}$$

and

$$\Lambda_p = \frac{k_{pp}^{(NL)}}{m_{pp}} t^2 \tag{72}$$

The solution $f_p(\tau)$ of this non-linear differential equation which satisfies the conditions in (66) is the Jacobi elliptic function $cn(\omega^* t, k^*)$, given by:

$$cn(\omega_p^* t, k_p^*) = \cos(am\ u) = \cos(\psi) \tag{73}$$

where

$$u = \int_0^{\psi} \frac{d\theta}{\sqrt{1 - k_p^{*2} \sin^2\theta}} \tag{74}$$

(ψ = am u is called the amplitude of u).

In this case:

$$\omega_p^{*2} = \left[\omega_p^2 + \Lambda_p (A_p/t)^2\right]^{1/2} \tag{75}$$

$$k_p^{*2} = \frac{\Lambda_p (A_p/t)^2}{2\left[\omega_p^2 + \Lambda_p (A_p/t)^2\right]} \tag{76}$$

The ratio of the non-linear to linear period is then determined by:

$$T_{NL}/T_L = 2K/\left(\pi[1 + (\Lambda_p/\omega_p^2)(A_p/t)^2]^{1/2}\right) \tag{77}$$

where $K = K(k_p^*)$ represents the complete integral elliptic of the first kind given by the infinite sum:

$$K(k_p^*) = \frac{1}{2}\pi\,[1 + \left[\frac{1}{2}\right]^2 k_p^{*2} + \left[\frac{1.3}{2.4}\right]^2 k_p^{*4} + \ldots + \left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k_p^{*2n} + \ldots] \tag{78}$$

The equation in (77) represents the influence of the geometric non-linearity of the walls on the natural frequencies of an empty shell when the equations are uncoupled. The ratio T_{NL}/T_L is expressed in conjunction with non-dimensional ratio (A_p/t) where A_p is the vibration amplitude.

5. CALCULATIONS AND DISCUSSION

The influence of the wall's geometric non-linearity on the cylindrical shell's free vibrations is expressed by equations (75) and (77). For a cylindrical shell having the particular physical characteristics given, equations (75) and (77) have been graphically represented in Figures 5 to 11 with respect to the non-dimensional ratio, A_p/t. The straight horizontal line separating the two types of curvature represents the linear vibration cases, where the frequency is independent of the motion's amplitude. Two types of boundary conditions were studied. The circumferential mode was kept constant, at $n = 4$.

5.1 Non-linear free vibration of an empty cylindrical shell

The first example of calculations to determine the influence of non-linearities in strain-displacement relations on the free vibrations of a cylindrical shell is shown in the analyses in references [13] and [20]. The shell has the following properties:

$E = 2.96 \times 10^7 \, lb/in.^2$, $\nu = 0.3$, $R = 1$ in., $t = 0.01$ in., $L = \pi/2$ in. and $\rho = 7.33 \times 10^{-4} \, lb.s^2/in^4$.

The boundary conditions were for a shell simply supported at both ends, such that $U = V = W = 0$.

The variation in natural frequencies of this shell was calculated using the method we propose, and compare to the results Nowinski [13] and Raju and Rao [20] obtained for the case of $m = 1$ (Figure 5).

Nowinski [13] based his analytical development upon Donnell's simplified non-linear method. Only lateral displacement was considered. For their part, Raju and Rao [20], beginning with an energy formulation, used the finite element method.

The shell was subdivided into four equal finite elements and our findings matched results obtained by others, in particular Raju and Rao [20].

In the case where $n = 4$ and $m = 1$ (Figure 5), we observed that the variation ratio between the linear and non-linear periods decreased as ratio A/t increased. The frequency ratios demonstrated inverse behaviour. A non-linear trend of the hardening type resulted from the Λ/ω ratio being positive. These variations are small for values A/t below 1.0. For values above 1.0, the identified variation was more pronounced than that which Nowinski [13] and Raju and Rao [20] obtained.

We were able to ascertain that these differences might be due to the fact that Nowinski [13] neglected plane inertia. Furthermore, the authors noted a radial displacement that was not cancelled out at the ends of the shell. As for Raju and Rao [20], who used Sanders-Koiter's

[10,11] non-linear theory, they expressed the displacements of components along the shell generator in polynomial form.

The present method also accounts for the high frequency characteristics found for a given value of circumferential mode n. Typical curvatures are shown in Figures 6, 7 and 8. Here too, non-linearity had a hardening effect.

Figure 6 shows the variations in the period and frequency ratios as a function of A/t for m = 2 and 3 on the one hand, and for m = 4 and 5 on the other, with the more accurate form being closer to the second. The same phenomena can be observed in Figure 7 for m = 6 and 9, and for m = 7 and 8. However in this Figure, the gaps between each pair of curves are approximately the same.

Finally, for high frequencies, the variation is small in the case of m = 11 and m = 12 and more pronounced for m = 10. The variations in ratios T_{NL}/T_L and ω^*/ω corresponding to the last two modes, m = 13 and m = 14, are left out. With reference to T_{NL}/T_L, these variations are less than 1.5 % and 0.004 % for m = 13 and m = 14, respectively.

One of the great advantages of the finite element method is the ease with which it can be applied to any boundary condition. Thus, the second calculation example, the one in the Raju and Rao [20] analysis, dealt with a cylindrical shell with circumferential constraints at both ends. We then have V = 0 as the boundary condition. The shell has the same physical properties as the preceding one e.g.:

$E=2.96\times 10^7 \text{lb/in.}^2$, $\nu=0.3$, $R=1$ in., $t=0.01$ in., $L=\pi/2$ in. and $\rho=7.33\times 10^{-4}\text{lb.s}^2/\text{in}^4$.

The shell was divided into four equal finite elements and the results obtained by the present methods are the same as in reference [20] and are shown in Figure 5. As in the first calculation example, the same differences were observed between the two methods. Again, the trends in non-linearities are of the hardening type, the ω^*/ω ratio increasing as A/t increases.

More and more, we found that ratio T_{NL}/T_L decreased more rapidly when V = 0 at both ends. So, for A/t = 3.0, for example, the present method showed that T_{NL}/T_L goes from 0.64 to 0.56, whereas with Raju and Rao's method [20], the decrease was from 0.84 to 0.76. This is probably due to the greater flexibility of the shell, where only the circumferential displacement being constraint. On the other hand, for both types of boundary condition considered, the gap between the ω^*/ω vs A/t curve is greater than between T_{NL}/T_L vs A/t.

As in the first example, the characteristics of frequencies were obtained. A few typical curves are shown in Figure 9, 10 and 11. The non-linearity trends are again of the hardening type.

Thus, in Figure 9, the period and frequency variations in conjunction with the ratio corresponding to modes m = 2, 3 and 4 were plotted. The Λ/ω ratio for m = 5 scarcely differs from ratio Λ/ω for m = 4 (less than 0.02 %), and is the reason why the curves for m = 5 were not drawn. It should be noted that the gaps between the T_{NL}/T_L and ω^*/ω ratios are almost identical when going from m = 2 to m = 3, and from m = 3 to m = 4. This statement is equally valid for Figures 10 and 11, which correspond to modes 6, 7, 8, 9 and 10, 11, 13, 15, respectively. However, in these last two cases, the curves are much closer to each other than in the preceding example. The exception is the case where m = 15, when the behaviour of the cylinder is closer to linear.

The variations in modes m = 12, 14, 16, 17 and 18 are negligible ($\leq$ 3 %). It was noted that mode m = 18 is ; of the softening type, ratio Λ/ω being negative. In this case, the maximum deviation in the linear behaviour is of the order of 0.005 %.

On the whole, by comparing the high frequency curves for both types of boundary condition studied, it can be concluded that these curves are closer to each other where V = 0 and more spread out where U = V = W = 0.

Finally, two points that are common to the two types of boundary condition should be emphasized:

a) For m = 1 and for all other modes, the variation in the ratio of periods T_{NL}/T_L seems to possess an asymptotic limit when ratio A/t rises above 2.0.

b) The influence of the geometric non-linearity of the walls is left out in the last frequencies (m $\geq$ 15 for V = 0).

5.2 Coupling of modes

The coupling between different modes was ignored in our study. Nevertheless, the present theory constitutes a general approach to the dynamic study of non-linear cylindrical shells.

The dynamic behaviour of the shell however is not adequately described by equation (64). When we keep the non-linearities in mind the coupling between different modes can no longer be left out. It then becomes necessary to develop a method for solving the system of uncoupled equations (63).

6. CONCLUSION

The method discussed in this report demonstrates the influence of geometric non-linearities of the walls on the free vibration of empty cylindrical shells. It is a hybrid method, based on a combination of thin shell theory and the finite element method.

A cylindrical finite element was used, so that the displacement functions could be derived directly from classical thin shell theory.

The solution was divided into two parts. In part one, the displacement functions were obtained from linear shell theory [22,23] and the mass and linear stiffness matrices were determined by the finite element procedure. In part two, the modal coefficients corresponding to non-linearities in strain-displacement relations were obtained for the displacement functions by the method developed in reference [19]. The non-linear stiffness matrix was then calculated using the finite element method.

With the help of a computer program, variations in the free vibration frequencies and periods were determined in conjunction with motion amplitude for a cylindrical shell. Deviations in terms of linear vibrations were observed. The results obtained with this numerical method for the two types of boundary condition were in agreement with other analytical and numerical methods.

The methods developed in the present research may be applied to the study of forced vibrations of a cylindrical shell under dynamic loads. This theory may also be applicable to problems of normal cones with circular sections.

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APPENDIX A-1
EQUATIONS OF MOTION

This appendix contains the equations of motion for a thin cylindrical anisotropic shell which were referred to this paper. The appendix is divided into two parts : part one deals with the linear system operators and part two, with the non-linear.

a) <u>Equations of motion for a cylindrical shell : linear system</u>

$$L_1(U,V,W,P_{ij}) = p_{11}\frac{\partial^2 U}{\partial x^2} + \frac{p_{12}}{R}\left(\frac{\partial^2 V}{\partial x\partial\theta} + \frac{\partial W}{\partial x}\right) - p_{14}\frac{\partial^3 W}{\partial x^3} + \frac{p_{15}}{R^2}\left(-\frac{\partial^3 W}{\partial x\partial\theta^2} + \frac{\partial^2 V}{\partial x\partial\theta}\right) +$$

$$\left(\frac{p_{33}}{R} - \frac{p_{63}}{2R^2}\right)\left(\frac{\partial^2 V}{\partial x\partial\theta} + \frac{1}{R}\frac{\partial^2 U}{\partial\theta^2}\right) + \left(\frac{p_{36}}{R^2} - \frac{p_{66}}{2R^3}\right)\left(-\frac{2\partial^3 W}{\partial x\partial\theta^2} + \frac{3}{2}\frac{\partial^2 V}{\partial x\partial\theta} - \frac{1}{2R}\frac{\partial^2 U}{\partial\theta^2}\right)$$

$$L_2(U,V,W,P_{ij}) = \left(\frac{p_{21}}{R} + \frac{p_{51}}{R^2}\right)\frac{\partial^2 U}{\partial x\partial\theta} + \frac{1}{R}\left(\frac{p_{22}}{R} + \frac{p_{52}}{R^2}\right)\left(\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial W}{\partial\theta}\right) - \left(\frac{p_{24}}{R} + \frac{p_{54}}{R^2}\right)$$

$$\left(\frac{\partial^3 W}{\partial x^2\partial\theta}\right) + \frac{1}{R^2}\left(\frac{p_{25}}{R} + \frac{p_{55}}{R^2}\right)\left(-\frac{\partial^3 W}{\partial\theta^3} + \frac{\partial^2 V}{\partial\theta^2}\right) + \left(p_{33} + \frac{3p_{63}}{2R}\right)\left(\frac{\partial^2 V}{\partial x^2} + \frac{1}{R}\frac{\partial^2 U}{\partial x\partial\theta}\right) +$$

$$\frac{1}{R}\left(p_{36} + \frac{3p_{66}}{2R}\right)\left(-2\frac{\partial^3 W}{\partial x^2\partial\theta} + \frac{3}{2}\frac{\partial^2 V}{\partial x^2} - \frac{1}{2R}\frac{\partial^2 U}{\partial x\partial\theta}\right)$$

$$L_3(U,V,W,P_{ij}) = p_{41}\frac{\partial^3 U}{\partial x^3} + \frac{p_{42}}{R}\left(\frac{\partial^3 V}{\partial x^2\partial\theta} + \frac{\partial^2 W}{\partial x^2}\right) - p_{44}\frac{\partial^4 W}{\partial x^4} + \frac{p_{45}}{R^2}$$

$$\left(-\frac{\partial^4 W}{\partial x^2\partial\theta^2} + \frac{\partial^3 V}{\partial x^2\partial\theta}\right) + \frac{2p_{63}}{R}\left\{\frac{\partial^3 V}{\partial x^2\partial\theta} + \frac{1}{R}\frac{\partial^3 U}{\partial x\partial\theta^2}\right) + \frac{2p_{66}}{R^2}\left(-\frac{2\partial^4 W}{\partial x^2\partial\theta^2} + \right.$$

$$\left.\frac{3}{2}\frac{\partial^3 V}{\partial x^2\partial\theta} - \frac{1}{2R}\frac{\partial^3 U}{\partial x\partial\theta^2}\right) + \frac{p_{51}}{R^2}\frac{\partial^3 U}{\partial x\partial\theta^2} + \frac{p_{52}}{R^3}\left(\frac{\partial^3 V}{\partial\theta^3} + \frac{\partial^2 W}{\partial\theta^2}\right) + \frac{p_{55}}{R^4}\left(-\frac{\partial^4 W}{\partial\theta^4} + \frac{\partial^3 V}{\partial\theta^3}\right)$$

$$- \frac{p_{54}}{R^2}\frac{\partial^4 W}{\partial x^2\partial\theta^2}$$

b) <u>Equations of motion for a cylindrical shell : non-linear system</u>

$$N_1(U,V,W,P_{ij}) = p_{11}\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2} + \frac{p_{12}}{R^2}\left(\frac{\partial W}{\partial\theta}\frac{\partial^2 W}{\partial x\partial\theta} - V\frac{\partial^2 W}{\partial x\partial\theta} - \frac{\partial W}{\partial\theta}\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial x}\right) +$$

$$\frac{1}{R}\left(\frac{p_{33}}{R} - \frac{p_{63}}{2R^2}\right)\cdot\left(\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial\theta^2} + \frac{\partial W}{\partial\theta}\frac{\partial^2 W}{\partial x\partial\theta} - V\frac{\partial^2 W}{\partial x\partial\theta} - \frac{\partial W}{\partial x}\frac{\partial V}{\partial\theta}\right) + (p_{11} + p_{12})\cdot$$

$$\left[\frac{1}{4}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 V}{\partial x^2} + \frac{1}{4R^2}\frac{\partial U}{\partial\theta}\cdot\frac{\partial^2 U}{\partial x\partial\theta} - \frac{1}{4R}\frac{\partial U}{\partial\theta}\cdot\frac{\partial^2 V}{\partial x^2} - \frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 U}{\partial x\partial\theta}\right] -$$

$$\left(\frac{p_{11} + p_{21}}{4R}\right)\cdot\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial\theta}\right]\cdot\left[\frac{\partial^2 U}{\partial x\partial\theta} + \frac{\partial W}{\partial x}\cdot\frac{\partial^2 W}{\partial x\partial\theta}\right] - \left(\frac{p_{12} + p_{22}}{4R}\right)\cdot$$

$$\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial\theta}\right]\cdot\left[\frac{1}{R}\frac{\partial^2 V}{\partial\theta^2} + \frac{1}{R}\frac{\partial W}{\partial\theta} + \frac{1}{R^2}\frac{\partial W}{\partial\theta}\cdot\frac{\partial^2 W}{\partial\theta^2} - \frac{V}{R^2}\frac{\partial^2 W}{\partial\theta^2} - \frac{1}{R^2}\frac{\partial W}{\partial\theta}\cdot\frac{\partial V}{\partial\theta} + \right.$$

$$\frac{V}{R^2} \cdot \frac{\partial V}{\partial \theta}\Bigg] + \left(\frac{p_{14} + p_{24}}{4R}\right) \cdot \left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot \left[\frac{\partial^3 W}{\partial x^2 \partial \theta}\right] - \left(\frac{p_{15} + p_{25}}{4R}\right) \cdot$$

$$\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot \left[-\frac{1}{R^2}\frac{\partial^3 W}{\partial \theta^3} + \frac{1}{R^2}\frac{\partial^2 V}{\partial \theta^2}\right] - \left(\frac{p_{11} + p_{21} + p_{12} + p_{22}}{4R}\right) \cdot$$

$$\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot \Bigg[\frac{1}{4}\frac{\partial V}{\partial x} \cdot \frac{\partial^2 V}{\partial x \partial \theta} + \frac{1}{4R^2}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 U}{\partial \theta^2} - \frac{1}{4R}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 V}{\partial x \partial \theta} -$$

$$\frac{1}{4R}\frac{\partial V}{\partial x}\frac{\partial^2 U}{\partial \theta^2}\Bigg] - \left(\frac{p_{11} + p_{21}}{4R}\right) \cdot \left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \left[\frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right] -$$

$$\left(\frac{p_{12} + p_{22}}{4R}\right) \cdot \left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \Bigg[\frac{1}{R}\frac{\partial V}{\partial \theta} + \frac{W}{R} + \frac{1}{2R^2}\left(\frac{\partial W}{\partial \theta}\right)^2 - \frac{V}{R^2}\frac{\partial W}{\partial \theta} +$$

$$\frac{V^2}{2R^2}\Bigg] + \left(\frac{p_{14} + p_{24}}{4R}\right) \cdot \left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \left[\frac{\partial^2 W}{\partial x^2}\right] - \left(\frac{p_{15} + p_{25}}{4R}\right) \cdot$$

$$\left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \left[-\frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{1}{R^2}\frac{\partial V}{\partial \theta}\right] - \left(\frac{p_{11} + p_{21} + p_{12} + p_{22}}{4R}\right) \cdot$$

$$\left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \left[\frac{1}{8}\left(\frac{\partial V}{\partial x}\right)^2 + \frac{1}{8R^2}\left(\frac{\partial U}{\partial \theta}\right)^2 - \frac{1}{4R}\frac{\partial V}{\partial x} \cdot \frac{\partial U}{\partial \theta}\right]$$

$$N_2(U,V,W,P_{ij}) = (\frac{P_{21}}{R} + \frac{P_{51}}{R^2}) \frac{\partial W}{\partial x} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{1}{R^2} (\frac{P_{22}}{R} + \frac{P_{52}}{R^2}) (\frac{\partial W}{\partial \theta} \frac{\partial^2 W}{\partial \theta^2} - V \frac{\partial^2 W}{\partial \theta^2} -$$

$$\frac{\partial W}{\partial \theta} \frac{\partial V}{\partial \theta} + V \frac{\partial V}{\partial \theta}) + \frac{1}{R} (P_{33} + \frac{3P_{63}}{2R}) (\frac{\partial W}{\partial x} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{\partial W}{\partial \theta} \frac{\partial^2 W}{\partial x^2} - V \frac{\partial^2 W}{\partial x^2} - \frac{\partial W}{\partial x} \frac{\partial V}{\partial x}) +$$

$$(\frac{P_{21} + P_{22}}{R} + \frac{P_{51} + P_{52}}{R^2}) \cdot \left[\frac{1}{4} \frac{\partial V}{\partial x} \cdot \frac{\partial^2 V}{\partial x \partial \theta} + \frac{1}{4R^2} \frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 U}{\partial \theta^2} - \frac{1}{4R} \frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{4R} \frac{\partial V}{\partial x}\right.$$

$$\left.\frac{\partial^2 U}{\partial \theta^2}\right] + \frac{P_{33}}{R} \cdot \left[\frac{\partial W}{\partial x}\right] \cdot \left[\frac{\partial V}{\partial x} + \frac{1}{R} \frac{\partial U}{\partial \theta} + \frac{1}{R} \frac{\partial W}{\partial x} \cdot \frac{\partial W}{\partial \theta} - \frac{V}{R} \cdot \frac{\partial W}{\partial x}\right] + \frac{P_{36}}{R} \cdot$$

$$\left[\frac{\partial W}{\partial x}\right] \cdot \left[- \frac{2}{R} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{3}{2R} \frac{\partial V}{\partial x} - \frac{1}{2R^2} \frac{\partial U}{\partial \theta}\right] + \left[(\frac{P_{11} + P_{21}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) -\right.$$

$$\left.\frac{P_{21}}{R} \cdot (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R})\right] \cdot \left[\frac{\partial U}{\partial x} + \frac{1}{2} (\frac{\partial W}{\partial x})^2\right] + \left[(\frac{P_{12} + P_{22}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) -\right.$$

$$\left.\frac{P_{22}}{R} \cdot (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R})\right] \cdot \left[\frac{1}{R} \cdot \frac{\partial V}{\partial \theta} + \frac{W}{R} + \frac{1}{2R^2} (\frac{\partial W}{\partial \theta})^2 - \frac{V}{R^2} \frac{\partial W}{\partial \theta} + \frac{V^2}{2R^2}\right] +$$

$$\left[\frac{P_{24}}{R} \cdot (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R}) - (\frac{P_{14} + P_{24}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta})\right] \cdot \left[\frac{\partial^2 W}{\partial x^2}\right] +$$

$$\left[(\frac{P_{15} + P_{25}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - \frac{P_{25}}{R} (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R})\right] \cdot \left[\frac{-1}{R^2} \frac{\partial^2 W}{\partial \theta^2} +\right.$$

$$\left.\frac{1}{R^2} \frac{\partial V}{\partial \theta}\right] + \left[(\frac{P_{11} + P_{21} + P_{12} + P_{22}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - (\frac{P_{21} + P_{22}}{R}) \cdot\right.$$

$$\left(-\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{V}{R}\right]\cdot\left[\frac{1}{8}\left(\frac{\partial V}{\partial x}\right)^2 + \frac{1}{8R^2}\left(\frac{\partial U}{\partial \theta}\right)^2 - \frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial U}{\partial \theta}\right] + \left(\frac{p_{11} + p_{21}}{4}\right)\cdot$$

$$\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right]\cdot\left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial W}{\partial x}\cdot\frac{\partial^2 W}{\partial x^2}\right] + \left(\frac{p_{12} + p_{22}}{4}\right)\cdot\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right]\cdot$$

$$\left[\frac{1}{R}\frac{\partial^2 V}{\partial x\partial\theta} + \frac{1}{R}\frac{\partial W}{\partial x} + \frac{1}{R^2}\frac{\partial W}{\partial \theta}\cdot\frac{\partial^2 W}{\partial x\partial\theta} - \frac{V}{R^2}\frac{\partial^2 W}{\partial x\partial\theta} - \frac{1}{R^2}\cdot\frac{\partial W}{\partial \theta}\cdot\frac{\partial V}{\partial x} + \frac{V}{R^2}\cdot\frac{\partial V}{\partial x}\right] -$$

$$\left(\frac{p_{24} + p_{14}}{4}\right)\cdot\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right]\cdot\left[\frac{\partial^3 W}{\partial x^3}\right] + \left(\frac{p_{15} + p_{25}}{4}\right)\cdot\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right]\cdot$$

$$\left[-\frac{1}{R^2}\frac{\partial^3 W}{\partial x\partial\theta^2} + \frac{1}{R^2}\frac{\partial^2 V}{\partial x\partial\theta}\right] + \left(\frac{p_{11} + p_{21} + p_{12} + p_{22}}{4}\right)\cdot\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right]\cdot$$

$$\left[\frac{1}{4}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 V}{\partial x^2} + \frac{1}{4R^2}\frac{\partial U}{\partial \theta}\cdot\frac{\partial^2 U}{\partial x\partial\theta} - \frac{1}{4R}\frac{\partial U}{\partial \theta}\cdot\frac{\partial^2 V}{\partial x^2} - \frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 U}{\partial x\partial\theta}\right]$$

$$N_3(U,V,W,P_{ij}) = p_{41}\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \frac{\partial W}{\partial x}\frac{\partial^3 W}{\partial x^3}\right] + \frac{p_{42}}{R^2}\left[\left(\frac{\partial^2 W}{\partial x\partial\theta}\right)^2 + \frac{\partial W}{\partial \theta}\frac{\partial^3 W}{\partial x^2\partial\theta} -\right.$$

$$\left.\frac{\partial V}{\partial x}\frac{\partial^2 W}{\partial x\partial\theta} - V\frac{\partial^3 W}{\partial x^2\partial\theta} - \frac{\partial^2 W}{\partial x\partial\theta}\frac{\partial V}{\partial x} - \frac{\partial W}{\partial \theta}\frac{\partial^2 V}{\partial x^2} + \left(\frac{\partial V}{\partial x}\right)^2 + V\frac{\partial^2 V}{\partial x^2}\right] + \frac{2p_{63}}{R^2}$$

$$\left[\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{\partial W}{\partial x}\frac{\partial^3 W}{\partial x\partial\theta^2} + \left(\frac{\partial^2 W}{\partial x\partial\theta}\right)^2 + \frac{\partial W}{\partial \theta}\frac{\partial^3 W}{\partial x^2\partial\theta} - \frac{\partial V}{\partial x}\frac{\partial^2 W}{\partial x\partial\theta} - V\frac{\partial^3 W}{\partial x^2\partial\theta} -\right.$$

$$\frac{\partial^2 W}{\partial x^2}\frac{\partial V}{\partial\theta} - \frac{\partial W}{\partial x}\frac{\partial^2 V}{\partial x\partial\theta}\Bigg] + \frac{p_{51}}{R^2}\left[\left(\frac{\partial^2 W}{\partial x\partial\theta}\right)^2 + \frac{\partial W}{\partial x}\frac{\partial^3 W}{\partial x\partial\theta^2}\right] + (p_{41} + p_{42})\cdot$$

$$\Bigg[\frac{1}{4}\left(\frac{\partial^2 V}{\partial x^2}\right)^2 + \frac{1}{4}\frac{\partial V}{\partial x}\cdot\frac{\partial^3 V}{\partial x^3} + \frac{1}{4R^2}\left(\frac{\partial^2 U}{\partial x\partial\theta}\right)^2 + \frac{1}{4R^2}\frac{\partial U}{\partial\theta}\cdot\frac{\partial^3 U}{\partial x^2\partial\theta} - \frac{1}{2R}\frac{\partial^2 U}{\partial x\partial\theta}\cdot$$

$$\frac{\partial^2 V}{\partial x^2} - \frac{1}{4R}\cdot\frac{\partial U}{\partial\theta}\cdot\frac{\partial^3 V}{\partial x^3} - \frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial^3 U}{\partial x^2\partial\theta}\Bigg] + \frac{p_{52}}{R^2}\Bigg[\frac{1}{R^2}\left(\frac{\partial^2 W}{\partial\theta^2}\right)^2 + \frac{1}{R^2}\cdot\frac{\partial W}{\partial\theta}\cdot$$

$$\frac{\partial^3 W}{\partial\theta^3} - \frac{1}{R^2}\frac{\partial V}{\partial\theta}\cdot\frac{\partial^2 W}{\partial\theta^2} - \frac{V}{R^2}\frac{\partial^3 W}{\partial\theta^3} - \frac{1}{R^2}\cdot\frac{\partial^2 W}{\partial\theta^2}\cdot\frac{\partial V}{\partial\theta} - \frac{1}{R^2}\frac{\partial W}{\partial\theta}\cdot\frac{\partial^2 V}{\partial\theta^2} + \frac{1}{R^2}\left(\frac{\partial V}{\partial\theta}\right)^2 +$$

$$\frac{V}{R^2}\frac{\partial^2 V}{\partial\theta^2}\Bigg] + \left(\frac{p_{51} + p_{52}}{R^2}\right)\cdot\Bigg[\frac{1}{4}\left(\frac{\partial^2 V}{\partial x\partial\theta}\right)^2 + \frac{1}{4}\frac{\partial V}{\partial x}\cdot\frac{\partial^3 V}{\partial x\partial\theta^2} + \frac{1}{4R^2}\left(\frac{\partial^2 U}{\partial\theta^2}\right)^2 +$$

$$\frac{1}{4R^2}\cdot\frac{\partial U}{\partial\theta}\cdot\frac{\partial^3 U}{\partial\theta^3} - \frac{1}{2R}\frac{\partial^2 U}{\partial\theta^2}\cdot\frac{\partial^2 V}{\partial x\partial\theta} - \frac{1}{4R}\frac{\partial U}{\partial\theta}\cdot\frac{\partial^3 V}{\partial x\partial\theta^2} - \frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial^3 U}{\partial\theta^3}\Bigg] +$$

$$p_{11}\left[\frac{\partial W}{\partial x}\right]\cdot\left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial W}{\partial x}\cdot\frac{\partial^2 W}{\partial x^2}\right] + p_{12}\cdot\left[\frac{\partial W}{\partial x}\right]\cdot\Bigg[\frac{1}{R}\frac{\partial^2 V}{\partial x\partial\theta} + \frac{1}{R}\frac{\partial W}{\partial x} +$$

$$\frac{1}{R^2}\frac{\partial W}{\partial\theta}\cdot\frac{\partial^2 W}{\partial x\partial\theta} - \frac{V}{R^2}\cdot\frac{\partial^2 W}{\partial x\partial\theta} - \frac{1}{R^2}\frac{\partial W}{\partial\theta}\cdot\frac{\partial V}{\partial x} + \frac{V}{R^2}\cdot\frac{\partial V}{\partial x}\Bigg] - p_{14}\left[\frac{\partial W}{\partial x}\right]\cdot$$

$$\left[\frac{\partial^3 W}{\partial x^3}\right] + p_{15}\left[\frac{\partial W}{\partial x}\right]\cdot\left[-\frac{1}{R^2}\frac{\partial^3 W}{\partial x\partial\theta^2} + \frac{1}{R^2}\frac{\partial^2 V}{\partial x\partial\theta}\right] + (p_{11} + p_{12})\cdot\left[\frac{\partial W}{\partial x}\right]\cdot$$

$$\left[\frac{1}{4}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 V}{\partial x^2}+\frac{1}{4R^2}\frac{\partial U}{\partial \theta}\cdot\frac{\partial^2 U}{\partial x\partial\theta}-\frac{1}{4R}\frac{\partial U}{\partial \theta}\cdot\frac{\partial^2 V}{\partial x^2}-\frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 U}{\partial x\partial\theta}\right]$$

$$\left[\frac{\partial U}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right]\cdot\left[-p_{11}\frac{\partial^2 W}{\partial x^2}+\frac{p_{21}}{R}\left(-\frac{1}{R}\frac{\partial^2 W}{\partial \theta^2}+\frac{1}{R}\frac{\partial V}{\partial \theta}+1\right)\right]-\left[\frac{1}{R}\frac{\partial V}{\partial \theta}+\right.$$

$$\left.\frac{W}{R}+\frac{1}{2R^2}\left(\frac{\partial W}{\partial \theta}\right)^2-\frac{V}{R^2}\frac{\partial W}{\partial \theta}+\frac{V^2}{2R^2}\right]\cdot\left[-p_{12}\frac{\partial^2 W}{\partial x^2}+\frac{p_{22}}{R}\left(-\frac{1}{R}\cdot\frac{\partial^2 W}{\partial \theta^2}+\frac{1}{R}\frac{\partial V}{\partial \theta}+\right.\right.$$

$$\left.\left.1\right)\right]-\left[\frac{1}{8}\left(\frac{\partial V}{\partial x}\right)^2+\frac{1}{8R^2}\left(\frac{\partial U}{\partial \theta}\right)^2-\frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial U}{\partial \theta}\right]\cdot\left[-(p_{11}+p_{12})\cdot\right.$$

$$\left.\left(\frac{\partial^2 W}{\partial x^2}\right)+\left(\frac{p_{21}+p_{22}}{R}\right)\cdot\left(-\frac{1}{R}\frac{\partial^2 W}{\partial \theta^2}+\frac{1}{R}\frac{\partial V}{\partial \theta}+1\right)\right]+\frac{\partial^2 W}{\partial x^2}\cdot\left[-p_{14}\frac{\partial^2 W}{\partial x^2}+\right.$$

$$\left.\frac{p_{24}}{R}\left(-\frac{1}{R}\frac{\partial^2 W}{\partial \theta^2}+\frac{1}{R}\frac{\partial V}{\partial \theta}\right)\right]-\left[-\frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2}+\frac{1}{R^2}\frac{\partial V}{\partial \theta}\right]\cdot\left[-p_{15}\frac{\partial^2 W}{\partial x^2}+\right.$$

$$\left.\frac{p_{25}}{R}\left(-\frac{1}{R}\frac{\partial^2 W}{\partial \theta^2}+\frac{1}{R}\frac{\partial V}{\partial \theta}\right)\right]-p_{33}\cdot\left[-\frac{1}{R}\frac{\partial W}{\partial \theta}+\frac{V}{R}\right]\cdot\left[\frac{\partial^2 V}{\partial x^2}+\frac{1}{R}\frac{\partial^2 U}{\partial x\partial\theta}+\right.$$

$$\left.\frac{1}{R}\frac{\partial W}{\partial x}\cdot\frac{\partial^2 W}{\partial x\partial\theta}+\frac{1}{R}\frac{\partial W}{\partial \theta}\cdot\frac{\partial^2 W}{\partial x^2}-\frac{V}{R}\cdot\frac{\partial^2 W}{\partial x^2}-\frac{1}{R}\frac{\partial W}{\partial x}\cdot\frac{\partial V}{\partial x}\right]-p_{36}\left[-\frac{1}{R}\frac{\partial W}{\partial \theta}+\right.$$

$$\left.\frac{V}{R}\right]\cdot\left[-\frac{2}{R}\frac{\partial^3 W}{\partial x^2\partial\theta}+\frac{3}{2R}\frac{\partial^2 V}{\partial x^2}-\frac{1}{2R^2}\frac{\partial^2 U}{\partial x\partial\theta}\right]-\left[-\frac{2}{R}\frac{\partial^2 W}{\partial x\partial\theta}+\frac{1}{R}\frac{\partial V}{\partial x}\right]\cdot$$

$$\left[p_{33}\left(\frac{\partial V}{\partial x}+\frac{1}{R}\frac{\partial U}{\partial \theta}+\frac{1}{R}\frac{\partial W}{\partial x}\cdot\frac{\partial W}{\partial \theta}-\frac{V}{R}\cdot\frac{\partial W}{\partial x}\right)+p_{36}\left(-\frac{2}{R}\frac{\partial^2 W}{\partial x\partial\theta}+\frac{3}{2R}\frac{\partial V}{\partial x}-\right.\right.$$

$$\frac{1}{2R^2}\frac{\partial U}{\partial\theta})\Bigg] + \frac{P_{33}}{R}\left[\frac{\partial W}{\partial x}\right] \cdot \left[\frac{\partial^2 V}{\partial x\partial\theta} + \frac{1}{R}\frac{\partial^2 U}{\partial\theta^2} + \frac{1}{R}\frac{\partial W}{\partial x} \cdot \frac{\partial^2 W}{\partial\theta^2} + \frac{1}{R}\frac{\partial W}{\partial\theta} \cdot \frac{\partial^2 W}{\partial x\partial\theta} -\right.$$

$$\left.\frac{V}{R} \cdot \frac{\partial^2 W}{\partial x\partial\theta} - \frac{1}{R}\frac{\partial W}{\partial x} \cdot \frac{\partial V}{\partial\theta}\right] + \frac{P_{36}}{R}\left[\frac{\partial W}{\partial x}\right] \cdot \left[-\frac{2}{R}\frac{\partial^3 W}{\partial x\partial\theta^2} + \frac{3}{2R}\frac{\partial^2 V}{\partial x\partial\theta} - \frac{1}{2R^2}\frac{\partial^2 U}{\partial\theta^2}\right] -$$

$$\frac{P_{21}}{R}\left[-\frac{1}{R}\frac{\partial W}{\partial\theta} + \frac{V}{R}\right] \cdot \left[\frac{\partial^2 U}{\partial x\partial\theta} + \frac{\partial W}{\partial x} \cdot \frac{\partial^2 W}{\partial x\partial\theta}\right] - \frac{P_{22}}{R} \cdot \left[-\frac{1}{R}\frac{\partial W}{\partial\theta} + \frac{V}{R}\right] \cdot$$

$$\left[\frac{1}{R}\frac{\partial^2 V}{\partial\theta^2} + \frac{1}{R}\frac{\partial W}{\partial\theta} + \frac{1}{R^2}\frac{\partial W}{\partial\theta} \cdot \frac{\partial^2 W}{\partial\theta^2} - \frac{V}{R^2} \cdot \frac{\partial^2 W}{\partial\theta^2} - \frac{1}{R^2}\frac{\partial W}{\partial\theta} \cdot \frac{\partial V}{\partial\theta} + \frac{V}{R^2}\frac{\partial V}{\partial\theta}\right] +$$

$$\frac{P_{24}}{R}\left[-\frac{1}{R}\frac{\partial W}{\partial\theta} + \frac{V}{R}\right] \cdot \left[\frac{\partial^3 W}{\partial x^2\partial\theta}\right] - \frac{P_{25}}{R}\left[-\frac{1}{R}\frac{\partial W}{\partial\theta} + \frac{V}{R}\right] \cdot \left[-\frac{1}{R^2}\frac{\partial^3 W}{\partial\theta^3} +\right.$$

$$\left.\frac{1}{R^2}\frac{\partial^2 V}{\partial\theta^2}\right] - \left(\frac{P_{21} + P_{22}}{R}\right)\left[-\frac{1}{R}\frac{\partial W}{\partial\theta} + \frac{V}{R}\right] \cdot \left[\frac{1}{4}\frac{\partial V}{\partial x} \cdot \frac{\partial^2 V}{\partial x\partial\theta} + \frac{1}{4R^2}\frac{\partial U}{\partial\theta} \cdot \frac{\partial^2 U}{\partial\theta^2} -\right.$$

$$\left.\frac{1}{4R}\frac{\partial U}{\partial\theta} \cdot \frac{\partial^2 V}{\partial x\partial\theta} - \frac{1}{4R}\frac{\partial V}{\partial x}\frac{\partial^2 U}{\partial\theta^2}\right]$$

APPENDIX A-2

The matrices referred to the course of our analytical developments are given in this appendix.

The matrices are classified as follows:

[H]	(Table 1)
[A]	(Table 2)
[T], [L], [X]	(Table 3)
[J]	(Table 4)
$[B^*]$, $[C^*]$	(Table 5)
$[B^{**}]$, $[C^{**}]$, $[D^{**}]$, $[E^{**}]$	(Table 6)

The eight roots of the characteristic equation (19) are represented by λ_p (p = 1,...8). The values for α_p and β_p are defined by equations (24).

Quantities ℓ and R are the length and radius, respectively, of each finite element.

TABLE 1

MATRIX $\underset{(3x3)}{[H]}$

$$\underset{(3x3)}{[H]} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \{0\} \qquad [H] = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}$$

Here we shall only be presenting the coefficients appearing in equation (24).

$$H_{11} = n^2 h_1 - \lambda^2 p_{11}$$

$$H_{12} = -n\lambda h_3$$

$$H_{21} = H_{12}$$

$$H_{22} = -n^2 h_7 + \lambda^2 h_9$$

$$H_{13} = -\lambda(n^2 h_5 + p_{12}) + \lambda^3 p_{14}/R$$

$$H_{23} = -n(1 + n^2)\, p_{25}/R - np_{22} - n^3 p_{55}/R^2 + n\lambda^2 h_{11}$$

with $h_1 = p_{33} - p_{36}/R + p_{66}/4R^2$

$$h_3 = p_{12} + p_{33} + (p_{15} + p_{36})/R - 3p_{66}/4R^2$$

$$h_5 = (p_{15} + 2p_{36} - p_{66}/R)/R$$

$$h_7 = p_{22} + p_{55}/R^2 + 2p_{25}/R$$

$$h_9 = p_{33} + 3p_{36}/R + 9p_{66}/4R^2$$

$$h_{11} = (2p_{36} + p_{24} + 3p_{66}/R + p_{54}/R)/R$$

The characteristic equation (19) is:

$$h_8\lambda^8 - h_6\lambda^6 + h_4\lambda^4 - h_2\lambda^2 + h_0 = 0$$

where $$h_8 = (h_9/r^2)(p_{11}p_{44} - p_{14}^2)$$

$$h_6 = (n^2/r^2)\,[\,h_9(h_1p_{44} + 2p_{11}p_{45} + 4p_{11}p_{66} - 2h_5rp_{14}) + h_7(p_{11}p_{44} - p_{14}^2) - r^2h_{11}^2p_{11} - h_3^2p_{44} + 2rh_3h_{11}p_{14}\,] + (2/r)\,h_9(p_{11}p_{24} - p_{14}p_{12})$$

$$h_4 = (n^4/r^2)\,[\,h_1h_7p_{44} + h_9p_{11}p_{55} + (2p_{45} + 4p_{66})(h_1h_9 + h_7p_{11} - h_3^2) + (p_{25} + (1/r)\,p_{55}) \cdot (2h_3p_{14} - 2h_{11}p_{11}r) + h_{11}r^2\,(2h_3h_5 - h_1h_{11}) - rh_5\,(2h_7p_{14} + rh_5h_9)\,] +$$

$$(n^2/r) \cdot [2\ (p_{25} + rp_{22})((h_3/r)\ p_{14} - h_{11}p_{11}) -$$

$$2p_{12}\ (h_5h_9r + h_7p_{14} - h_3h_{11}r) - 2p_{24}\ (h_3^2 - h_1h_9 - h_7p_{11}) +$$

$$2h_9p_{11}p_{25}] + h_9\ (p_{11}p_{22} - p_{12}^2)$$

$$h_2 = (n^6/r^2)\ [h_1h_7\ (2p_{45} + 4p_{66}) + p_{55}\ (h_1h_9 + h_7p_{11} - h_3^2) -$$

$$r^2h_5^2h_7 + (p_{25} + (1/r)\ p_{55}) \cdot (-2rh_1h_{11} + 2rh_3h_5 - p_{11}p_{25} -$$

$$(1/r)\ p_{11}p_{55})] + (n^4/r)\ [2h_1h_7p_{24} + 2p_{25}\ (h_1h_9 + h_7p_{11} -$$

$$h_3^2) - 2p_{12}\ (rh_5h_7 - h_3p_{25} - (h_3/r)\ p_{55}) - 2\ (p_{25} + rp_{22})$$

$$(h_1h_{11} + (1/r)\ p_{11}p_{25} + (1/r^2)\ p_{11}p_{55} - h_3h_5)] +$$

$$n^2\ [p_{22}\ (h_1h_9 + h_7p_{11} - h_3^2) - (1/r)(p_{25} + rp_{22})((1/r)$$

$$p_{11}p_{25} + p_{11}p_{22} - 2h_3p_{12}) - h_7p_{12}^2]$$

$$h_0 = n^4h_1h_7\ [p_{22} + (2/r)\ n^2p_{25} + (n^4/r^2)\ p_{55}] - n^2h_1$$

$$[(n^3/r)(p_{25} + (1/r)\ p_{55}) + (n/r)(p_{25} + rp_{22})]^2$$

TABLE 2

MATRIX $\underset{(8x8)}{[A]}$

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \underset{(8,8)}{[A]} \underset{(8,1)}{\{C\}}$$

with $\{C\} = \{C_1, C_2, \ldots, C_8\}^T$

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \left\{ u_i \ w_i \ (\frac{dw}{dx})_i \ v_i \ u_j \ w_j \ (\frac{dw}{dx})_j \ v_j \right\}^T$$

$$A(1,q) = \alpha_q \qquad \text{(A-2.1)}$$

$$A(2,q) = 1$$

$$A(3,q) = \frac{\lambda_q}{R}$$

$$A(4,q) = \beta_q$$

$$A(5,q) = A(1,q)\ a_q$$

$$A(6,q) = a_q$$

$$A(7,q) = A(3,q)\ a_q$$

$$A(8,q) = A(4,q)\ a_q$$

$$a_q = e^{\lambda_q \ell/R} \text{ and } q = 1, \ldots, 8$$

TABLE 3

MATRICES $\underset{(8x8)}{[T]}$ and $\underset{(3x8)}{[R]}$

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = \underset{(3,3)}{[T]}\ \underset{(3,8)}{[R]}\ \underset{(8,1)}{\{C\}}$$

with $\{C\} = \{C_1, C_2, \ldots, C_8\}^T$

$$[T] = \begin{bmatrix} \cos n\theta & 0 & 0 \\ 0 & \cos n\theta & 0 \\ 0 & 0 & \sin n\theta \end{bmatrix}$$

$$\underset{(3,8)}{[R]} = \underset{(3,8)}{[L]}\ \underset{(8,8)}{[X]}$$

$$L(1,q) = \alpha_q$$

$$L(2,q) = 1 \qquad q = 1, \ldots, 8$$

$$L(3,q) = \beta_q$$

$$X(p,q) = e^{\lambda_p x/R} \quad \text{si } p = q$$

$$X(p,q) = 0 \quad \text{si } p \neq q \qquad p,q = 1, \ldots, 8$$

TABLE 4

MATRIX $\underset{(6x8)}{[Q]}$

$$\{\varepsilon_L\} = \underset{(6,6)}{\begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix}} \underset{(6,8)}{[Q]} \underset{(8,8)}{[A^{-1}]} \underset{(8,1)}{\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}}$$

with $\underset{(6,8)}{[Q]} = \underset{(6,8)}{[J]} \underset{(8,8)}{[X]}$

$$J(1,q) = \alpha_q \frac{\lambda_q}{R}$$

$$J(2,q) = \frac{1}{R} (n\beta_q + 1)$$

$$J(3,q) = \frac{1}{R} (\beta_q\lambda_q - n\alpha_q)$$

$$J(4,q) = -(\frac{\lambda_q}{R})^2 \qquad q = 1, \ldots, 8$$

$$J(5,q) = \frac{1}{R^2} (n^2 + \beta_q n)$$

$$J(6,q) = \frac{1}{R^2} (2n\lambda_q + \frac{3}{2} \beta_q\lambda_q + \frac{1}{2} n\alpha_q)$$

TABLE 5

MATRICES $\underset{(8x8)}{[B^*]}$ and $\underset{(8x8)}{[C^*]}$

$$\underset{(8,8)}{\begin{Bmatrix} [B^+] \\ [C^+] \end{Bmatrix}} = \underset{(8,8)}{[A^{-1}]}^T \underset{(8,8)}{\begin{Bmatrix} [B^*] \\ [C^*] \end{Bmatrix}} \underset{(8,8)}{[A^{-1}]}$$

$$B^*(p,q) = b_{pq}\, e^{(\lambda_p + \lambda_q)\, x/R}$$

$$p,q = 1, \ldots, 8$$

$$C^*(p,q) = c_{pq}\, e^{(\lambda_p + \lambda_q)\, x/R}$$

where $$b_{pq} = \frac{1}{2R^2} \left[\frac{1}{4} a_{pq}^{(1)} + b_{pq}^{(1)}\right] \sin^2 n\theta$$

$$c_{pq} = -\frac{1}{4R^2} \left[c_{pq}^{(1)} + c_{pq}^{(2)}\right] \sin n\theta \cos n\theta$$

with $$a_{pq}^{(1)} = (\beta_p\lambda_p + n\alpha_p)(\beta_q\lambda_q + n\alpha_q)$$

$$b_{pq}^{(1)} = (n + \beta_p)(n + \beta_q)$$

$$p,q = 1, \ldots, 8$$

$$c_{pq}^{(1)} = n\,(\lambda_p + \lambda_q)$$

$$c_{pq}^{(2)} = (\beta_p\lambda_q + \beta_q\lambda_p)$$

TABLE 6

MATRICES $\underset{(8x8)}{[B^{**}]}$, $\underset{(8x8)}{[C^{**}]}$, $\underset{(8x8)}{[D^{**}]}$ and $\underset{(8x8)}{[E^{**}]}$

$$\underset{(8,8)}{\begin{Bmatrix} [B^{++}] \\ [C^{++}] \\ [D^{++}] \\ [E^{++}] \end{Bmatrix}} = 2 \underset{(8,8)}{[A^{-1}]^T} \underset{(8,8)}{\begin{Bmatrix} [B^{**}] \\ [C^{**}] \\ [D^{**}] \\ [E^{**}] \end{Bmatrix}} \underset{(8,8)}{[A^{-1}]}$$

$$B^{**}(p,q) = \sum_{k=1}^{8} b_{kq} \left[\sum_{l=1}^{8} b_{pl}\ \varepsilon_{lk}\ e^{(\lambda_p+\lambda_q+\lambda_k+\lambda_l)\ x/R} \right]$$

$$C^{**}(p,q) = \sum_{k=1}^{8} c_{kq} \left[\sum_{l=1}^{8} c_{pl}\ \varepsilon_{lk}\ e^{(\lambda_p+\lambda_q+\lambda_k+\lambda_l)\ x/R} \right]$$

$$D^{**}(p,q) = \sum_{k=1}^{8} b_{kq} \left[\sum_{l=1}^{8} a_{pl}\ \varepsilon_{lk}\ e^{(\lambda_p+\lambda_q+\lambda_k+\lambda_l)\ x/R} \right]$$

$$E^{**}(p,q) = \sum_{k=1}^{8} a_{kq} \left[\sum_{l=1}^{8} b_{pl}\ \varepsilon_{lk}\ e^{(\lambda_p+\lambda_q+\lambda_k+\lambda_l)\ x/R} \right]$$

$p,q = 1, \ldots, 8$

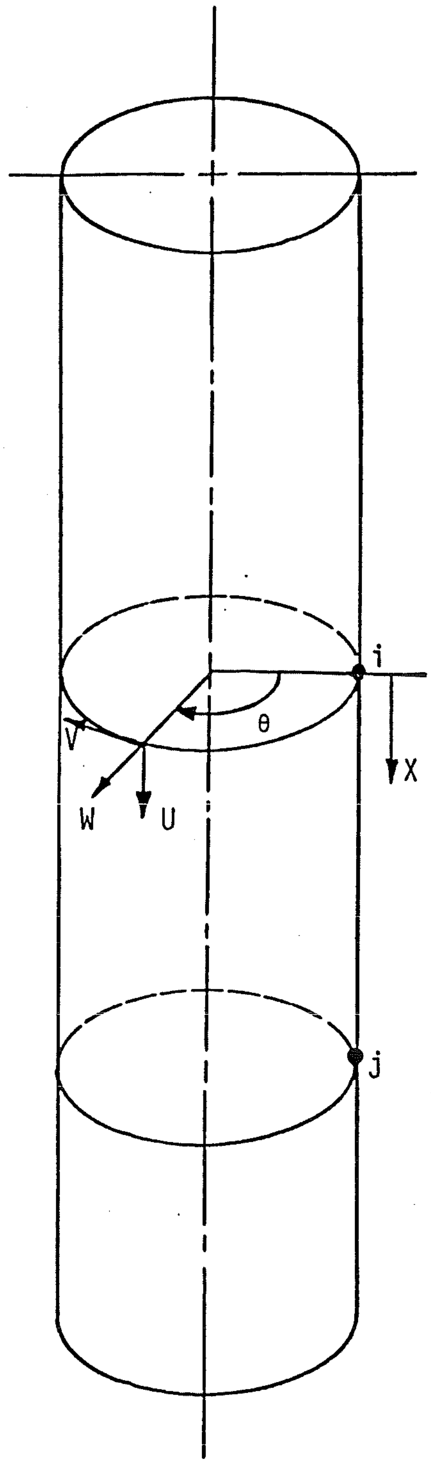


Figure 1 Geometry of the surface of a cylindrical shell

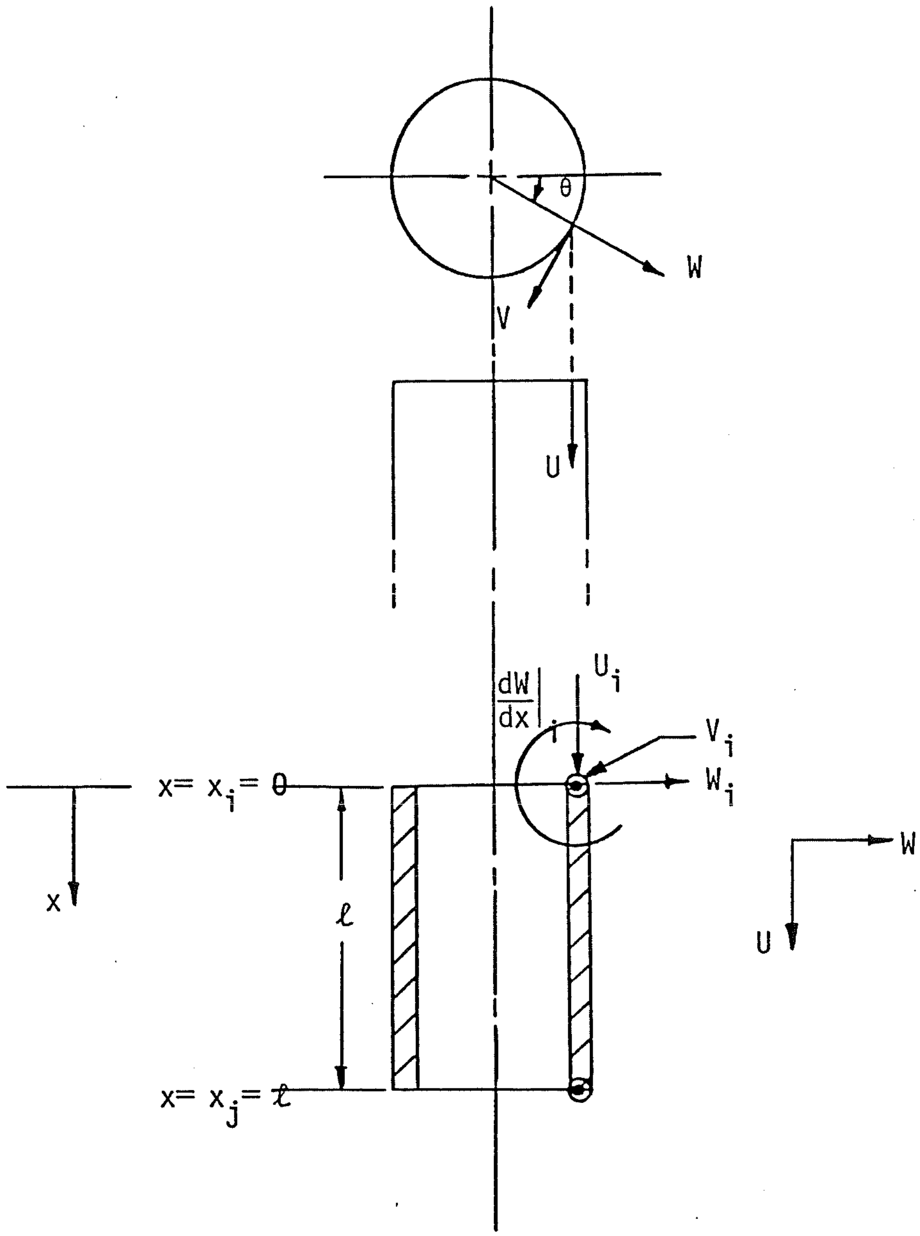
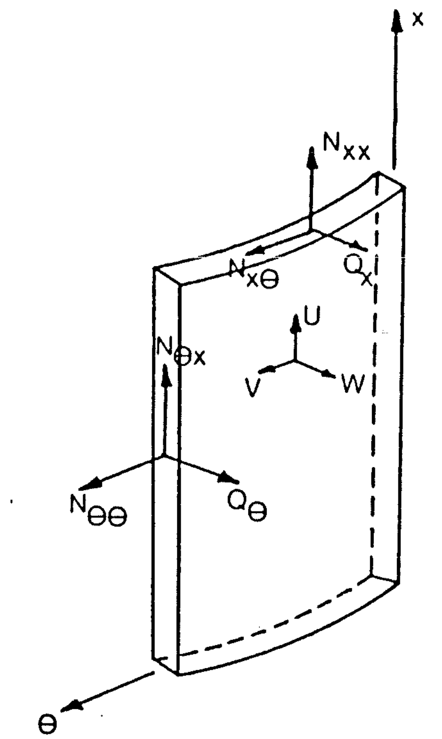
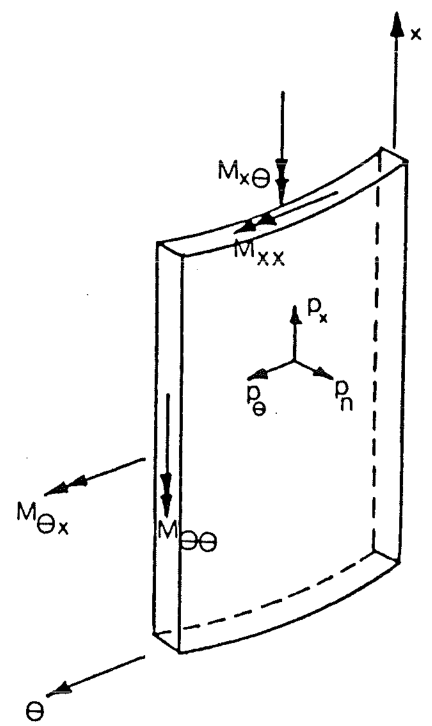


Figure 2 Nodal displacements at nodes i and j of a cylindrical element



(a) Resultant constraints and displacements



(b) Resultant couples and external loads

Figure 3 Differential elements for cylindrical shells

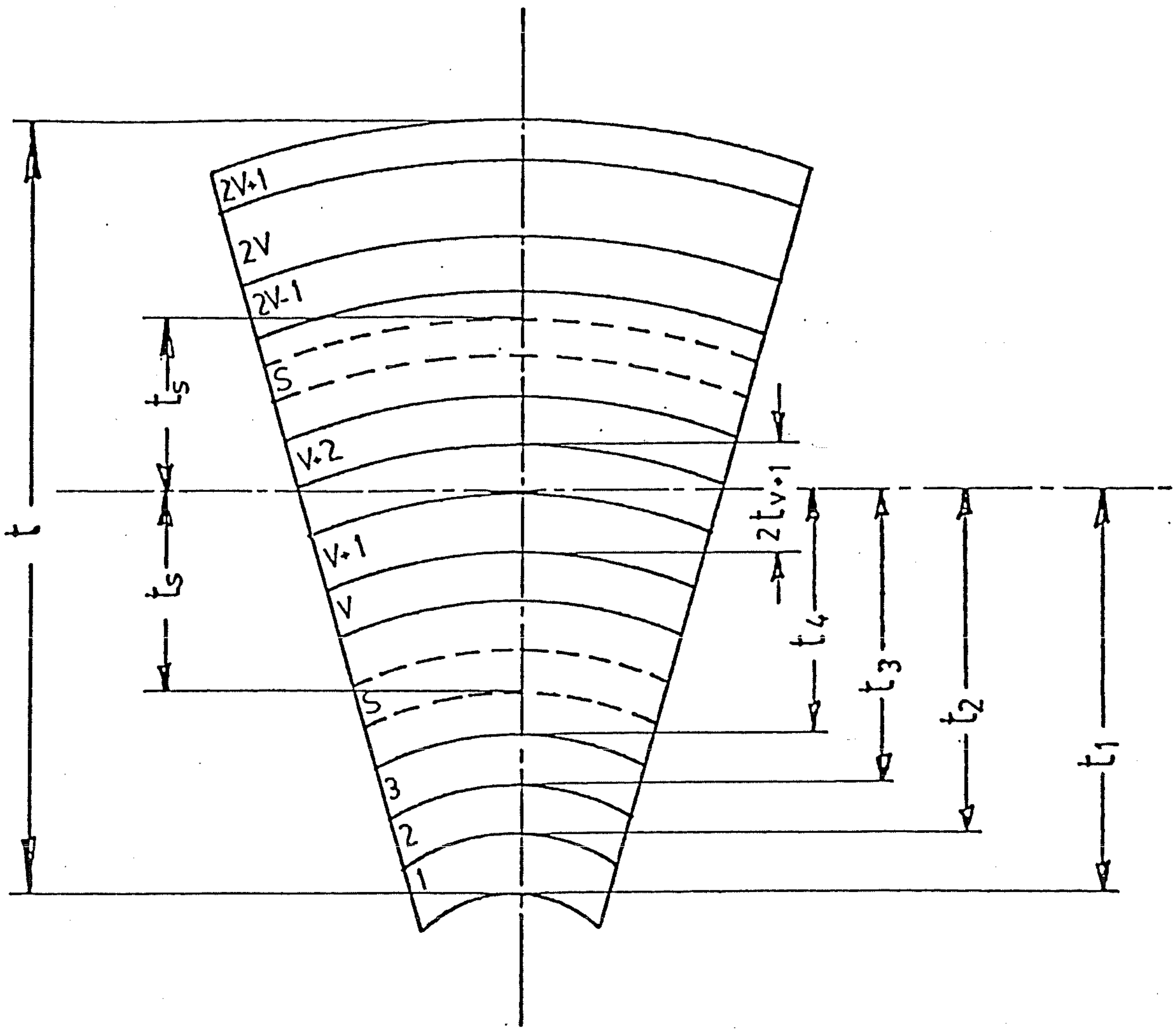


Figure 4 Shell composed of an odd number of anisotropic layers

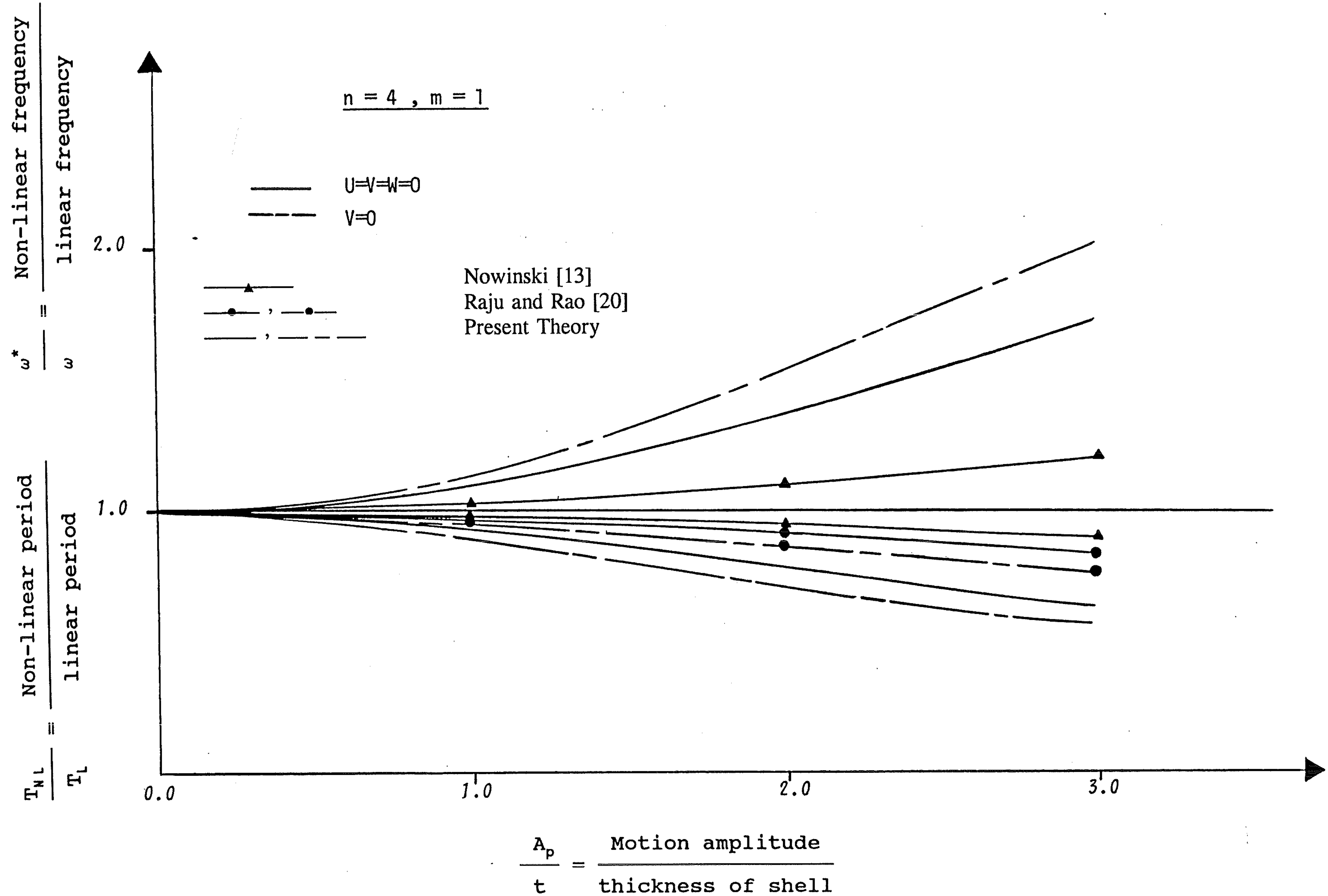


Figure 5 Free vibration variations in period and frequency as a function of motion amplitude. n = 4; m = 1

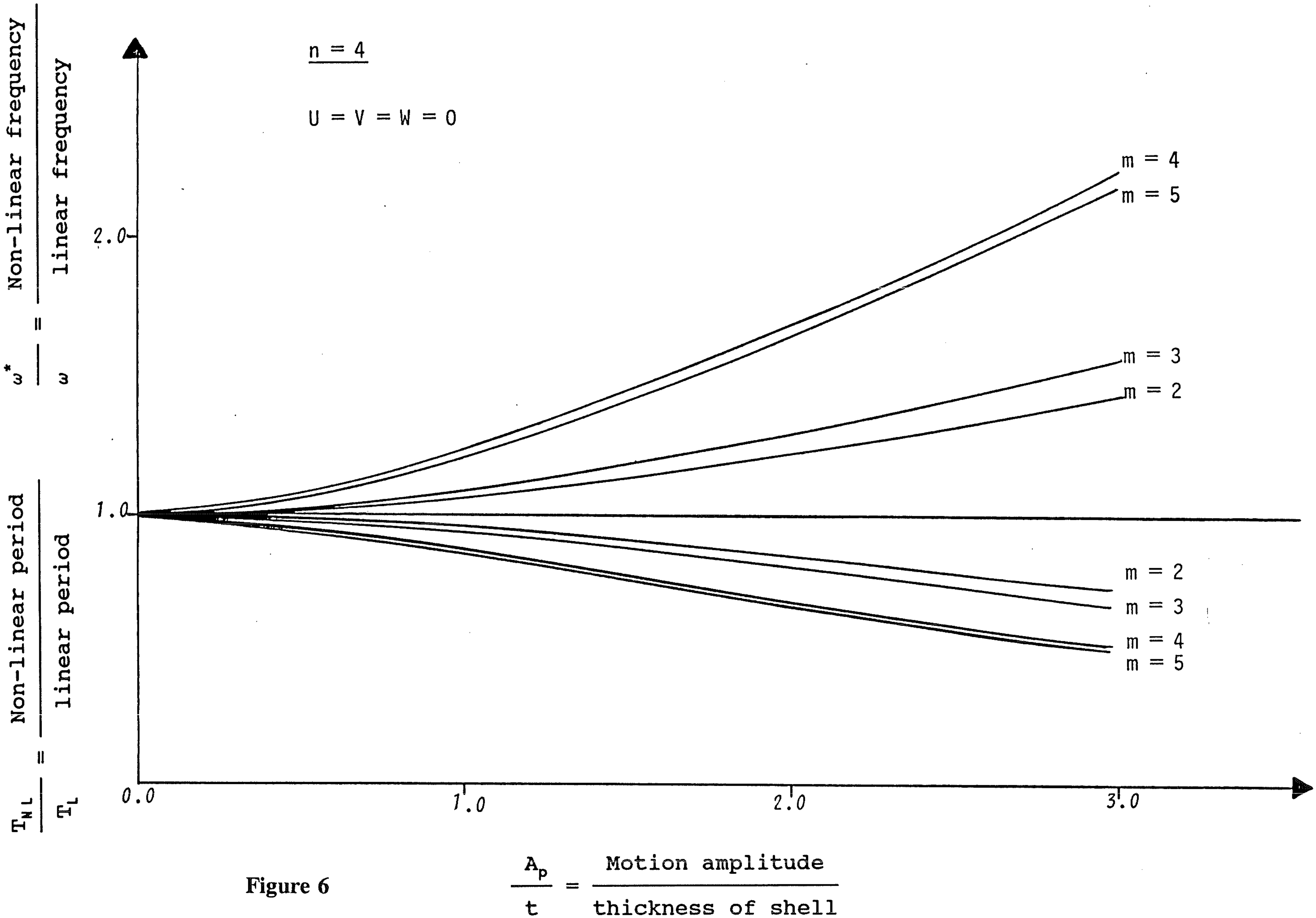

n = 4
U = V = W = 0
m = 4
m = 5
m = 3
m = 2
m = 2
m = 3
m = 4
m = 5
2.0
1.0
0.0
1.0
2.0
3.0
$\frac{\omega^*}{\omega}$ = $\frac{\text{Non-linear frequency}}{\text{linear frequency}}$
$\frac{T_{NL}}{T_L}$ = $\frac{\text{Non-linear period}}{\text{linear period}}$
$\frac{A_p}{t}$ = $\frac{\text{Motion amplitude}}{\text{thickness of shell}}$

Figure 6

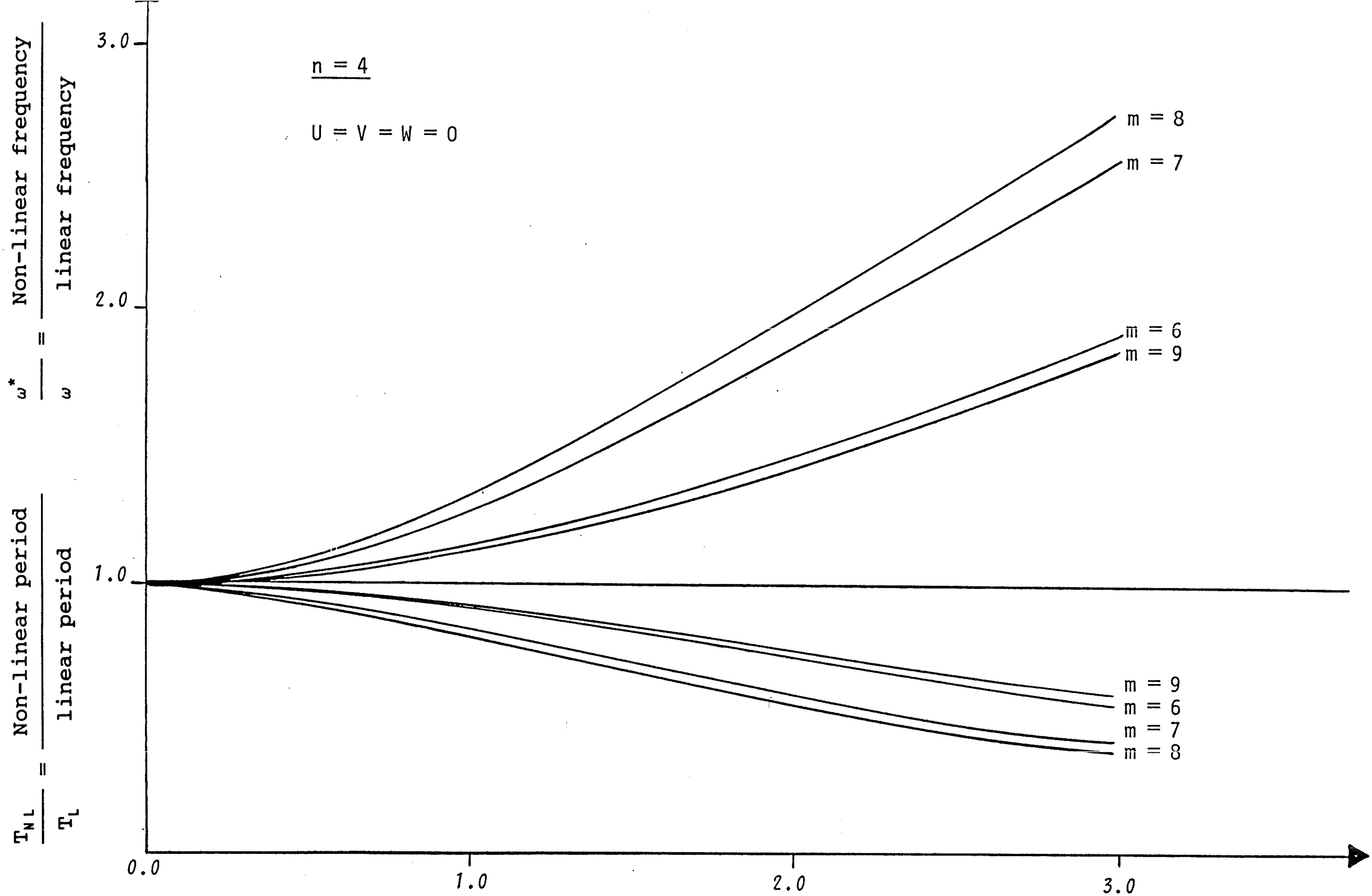


Figure 7 $\frac{A_p}{t} = \frac{\text{Motion amplitude}}{\text{thickness of shell}}$

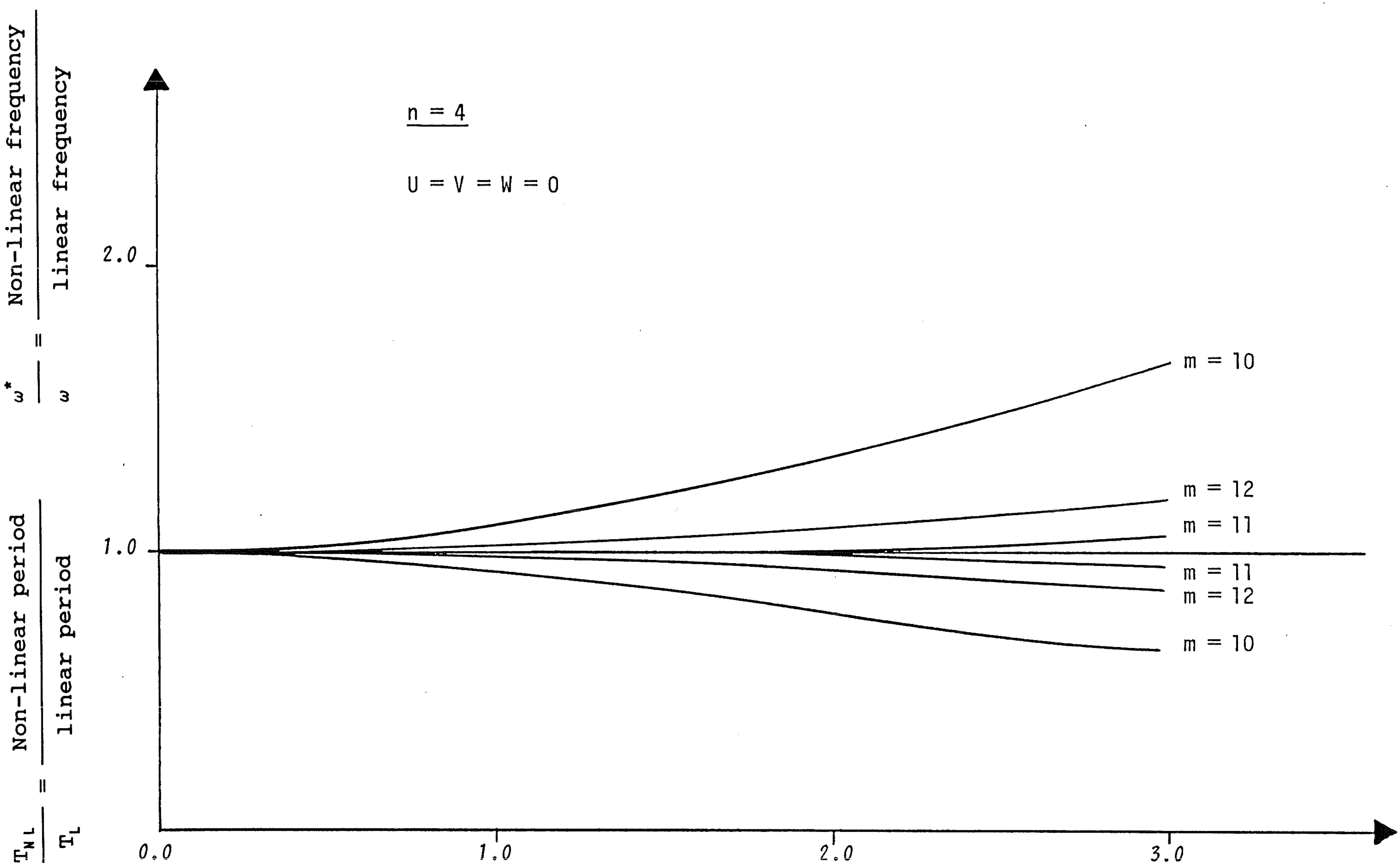
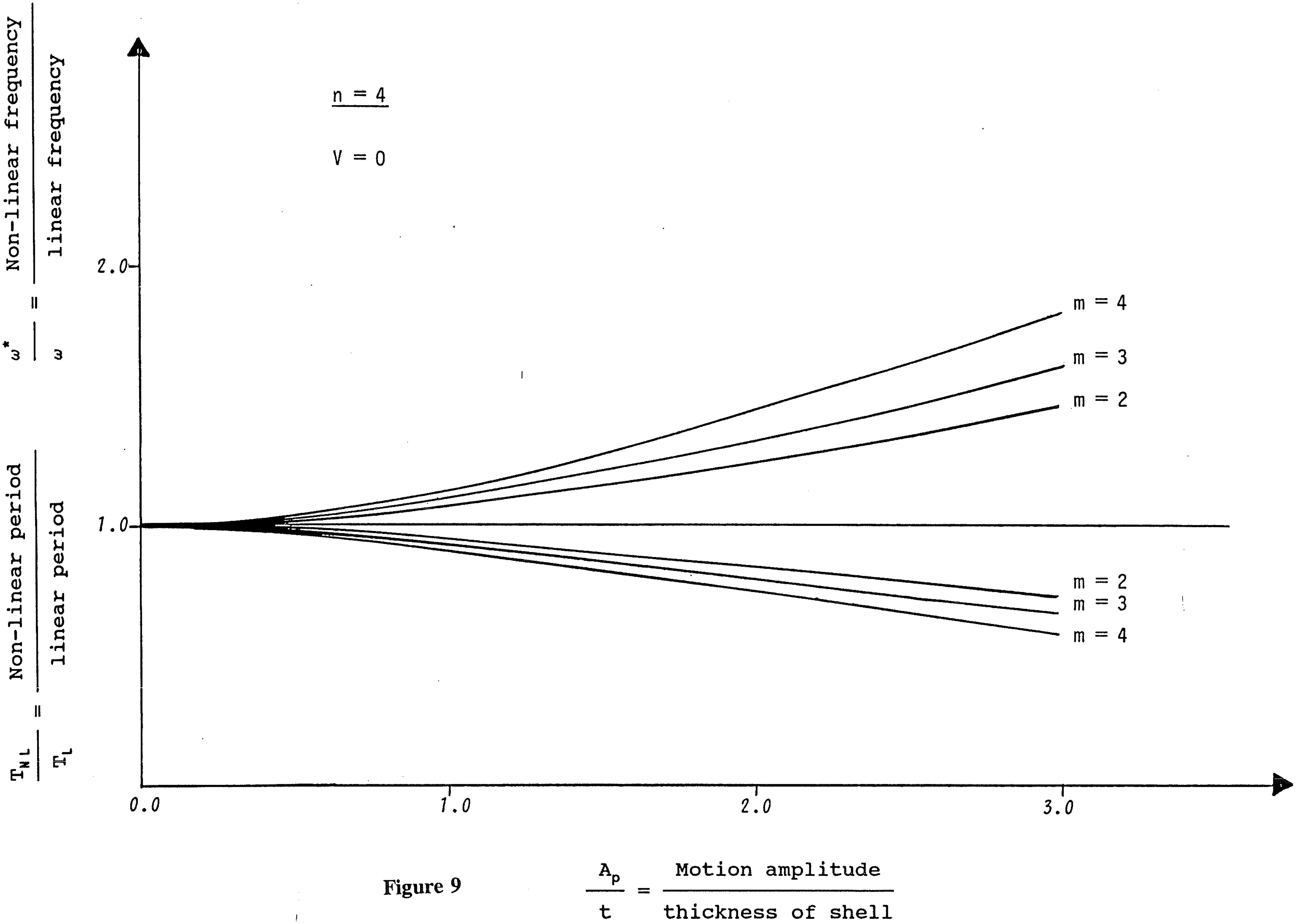


Figure 8

$$\frac{A_p}{t} = \frac{\text{Motion amplitude}}{\text{thickness of shell}}$$



n = 4
V = 0
$\frac{\omega^*}{\omega} = \frac{\text{Non-linear frequency}}{\text{linear frequency}}$
$\frac{T_{NL}}{T_L} = \frac{\text{Non-linear period}}{\text{linear period}}$
2.0
1.0
m = 4
m = 3
m = 2
m = 2
m = 3
m = 4
0.0
1.0
2.0
3.0
$\frac{A_p}{t} = \frac{\text{Motion amplitude}}{\text{thickness of shell}}$

Figure 9

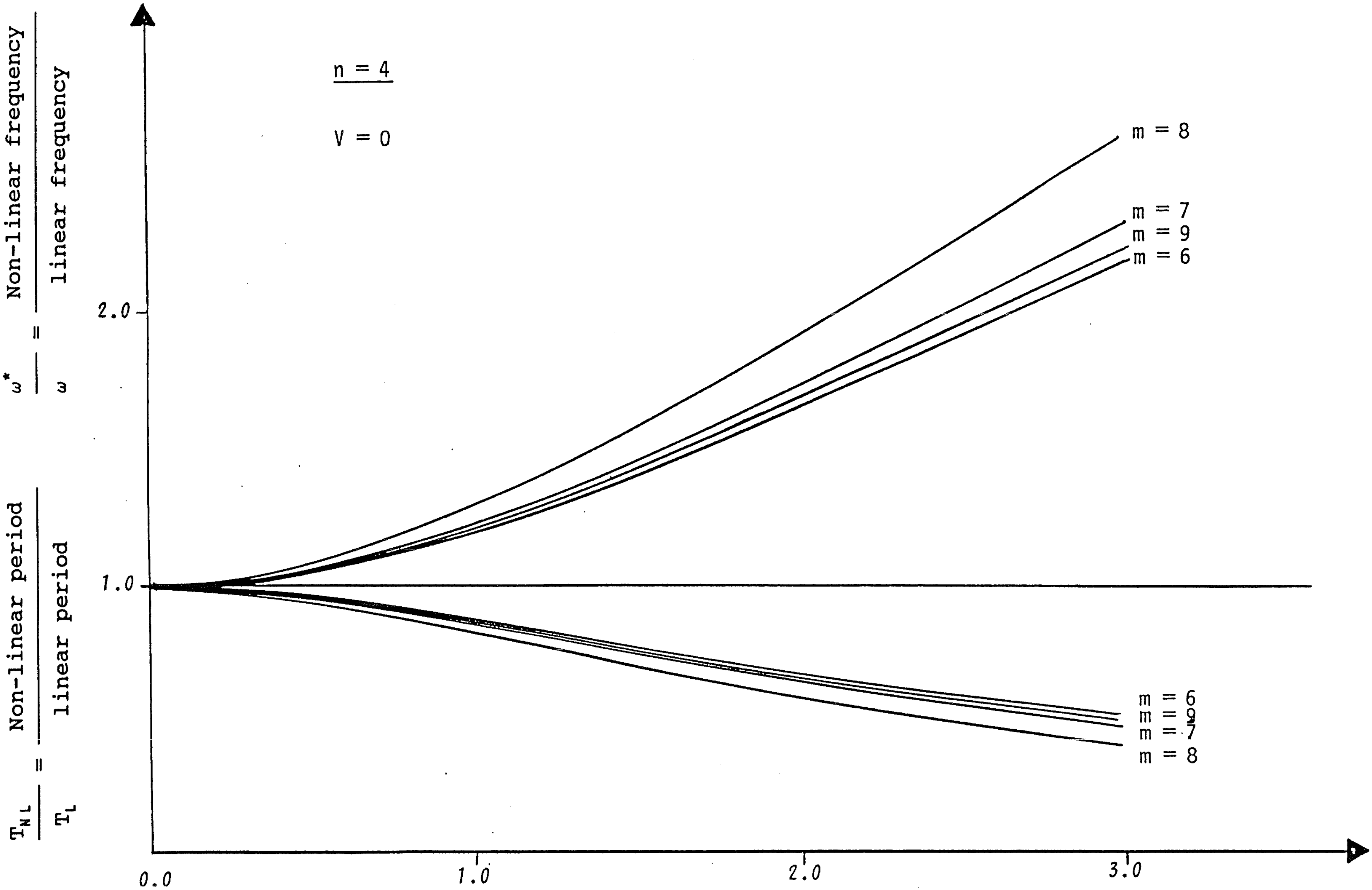


Figure 10 $\frac{A_p}{t} = \frac{\text{Motion amplitude}}{\text{thickness of shell}}$

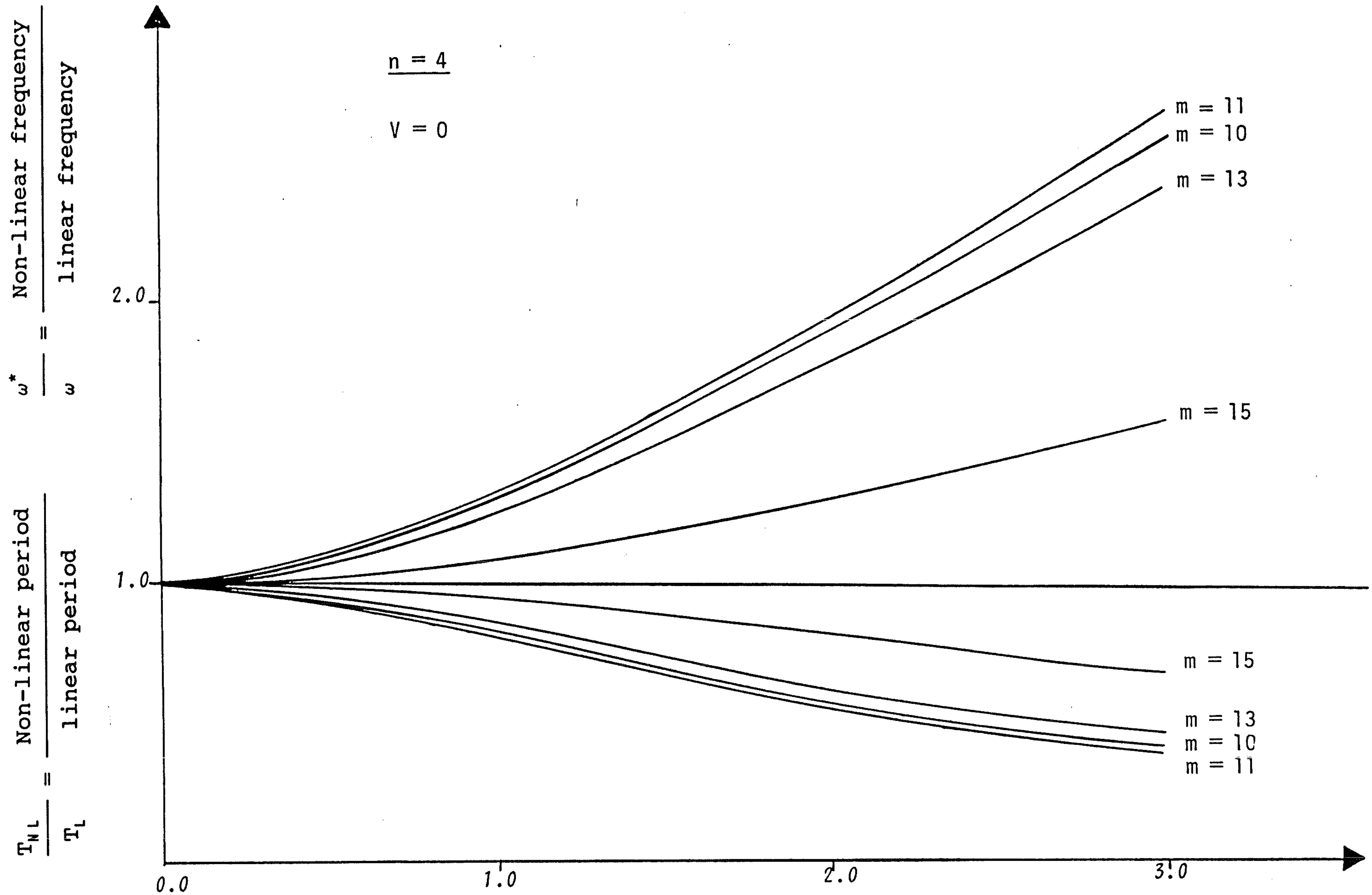


Figure 11 $\frac{A_p}{t} = \frac{\text{Motion amplitude}}{\text{thickness of shell}}$

École Polytechnique de Montréal
C.P. 6079, Succ. Centre-ville
Montréal (Québec)
H3C 3A7