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INFLUENCE OF GEOMETRIC NON-LINEARITIES ON THE FREE VIBRATIONS OF ANISOTROPIC OPEN CYLINDRICAL SHELLS

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ABSTRACT

This report presents a general approach to predict the influence of geometric non-linearities on the free vibration of elastic, thin, anisotropic and non-uniform open cylindrical shells. The open shells are assumed to be freely simply-supported along their curved edges and to have arbitrary straight edges boundary conditions. The method is a hybrid of finite element and classical thin shell theories. The solution is divided into two parts. In part one, the displacement functions are obtained from Sanders' linear shell theory and the mass and linear stiffness matrices are obtained by the finite element procedure. In part two, the modal coefficients derived from the Sanders-Koiter non-linear theory of thin shells are obtained for these displacement functions. Expressions for the second order and third order non-linear stiffness matrices are then determined through the finite element method. The non-linear equation of motion is solved by the fourth-order Runge-Kutta numerical method. The linear and non-linear natural frequency variations are determined as a function of shell amplitudes for different cases. The results obtained reveal that the frequencies calculated by this method are in good agreement with those obtained by other authors.

1. INTRODUCTION

The analysis of thin shells under static or dynamic load has been the focus of many investigations. Most of the research in this field has involved analysis of linear thin shells. The results have proven to be satisfactory in cases where deflections of the shell were very small compared to the thickness of the shell itself. In several practical experiments, however, the linear analysis was not sufficiently accurate for satisfactory design. In those cases, a non-linear analysis was required.

The first paper to deal with non-linear vibrations of shells was the pioneering work of Reissner [1]. There are now several theories available dealing with geometric non-linearities in shells ([2] to [5]) and many others.

More specifically, several methods have been developed for the analysis of dynamic non-linear thin cylindrical shells. Among these were Galerkin's method ([6] to [12]), the small perturbation method ([13] to [15]), the modal expansion method [16] and the finite element method ([17] to [20])

Most of the research done in [6] to [20] was limited to studies of isotropic shells. Only Nowinski [6], Raouf & Palazotto [11] and Jiang & Olsen [20] made a generalization concerning orthotropic shell theory. Ambartsumyan [21] produced an important work involving a number of cases for anisotropic shells.

All of these methods have their advantages and disadvantages. The best test of any method is probably its general content and the capacity to predict, with precision, both the high and the low frequencies of vibration. These criteria were not met in Galerkin's small perturbation method, and studies [6] to [15] applied only to the particular case where the shell was simply-supported on both edges. Further more the analytical forms for the displacement components in the modal expansion [16] apply only to those cases where a uniform cylinder is supported at both ends.

The finite element method appears to be ideally suited to the analysis of complex shell structures. Numerous general computer programmes are available for industrial use in the linear and non-linear analysis, where the displacement functions of the finite elements used are assumed to be polynomial. To be able to predict with precision, both the high and the low frequencies, requires the use of a great many elements in the classical finite element method. In order to acheive this, the present paper presents a new finite element for the static or dynamic analysis of non-linear, elastic, thin, anisotropic and circumferentially non-uniform open cylindrical shells (Figure 1). The shells are assumed to be freely simply-supported along their curved edges and to have arbitrary straight edge boundary conditions. The finite element method is employed, but it is a hybrid, a combination of the finite element method and shell theory. This choice

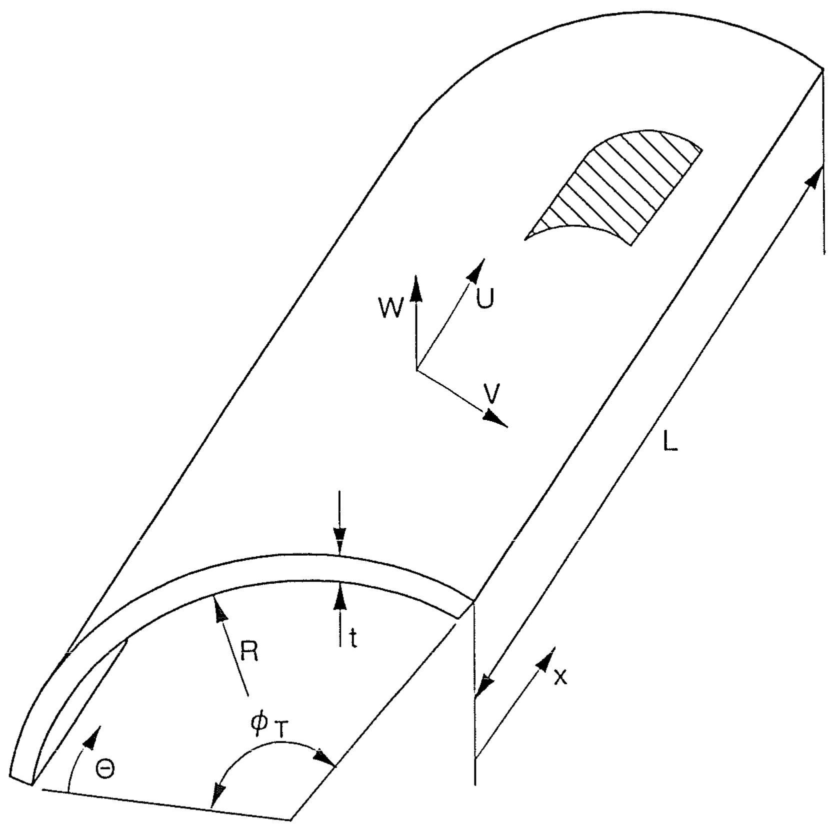

Figure 1 : Open cylindrical shell geometry.

allowes us to use the complete equilibrium equations to determine the displacement functions and, further, the mass and stiffnesses matrices. This method proves to be more accurate than the usual finite element methods.

The dynamic behaviour of an empty open or closed cylindrical shell, in the absence of external loads, can be represented by the following equation:

$$[M]\{\ddot{\delta}\} + [K_L]\{\delta\} + [K_{NL2}]\{\delta^2\} + [K_{NL3}]\{\delta^3\} = \{0\} \qquad (1)$$

where $\{\delta\}$ is the displacement vector; [M] the mass matrix, $[K_L]$ the linear stiffness matrix, $[K_{NL2}]$ the second order non-linear stiffess matrix and $[K_{NL3}]$ the third order non-linear stiffness matrix of the system.

The analytical solution involves two steps:

a) Using the linear strain-displacement and stress-strain relationships which are inserted into Sanders' equations of equilibrium [22], we determine the displacement functions by solving the linear equation system. We then determine the mass and linear stiffness matrices for a finite element and assemble the matrices for the complete shell [23].

b) Using strain-displacement relationships from the Sanders-Koiter non linear theory [3-4], the modal coefficients are obtained from the displacements functions. The second and third order non-linear stiffness matrices for a finite element are then calculated by precise analytical integration with respect to modal coefficients [16].

The linear and non-linear natural vibration frequency ratio is then obtained by solving equation (1).

2. EQUATIONS OF MOTION

2.1 Hypotheses

Non-linear elastic thin shell theory is derived by approximation from the tridimensional elasticity equation. As in the case of linear theory, it is based on Love's "First Approximation" but the assumption concerning the order of magnitude of the bending has been modified.

The non-linear theory is based on the following hypotheses:

a) Thickness (t) is infinitesimal in comparison with the minimum radius of curvature (R_{min}), ($R/t > 10$);

b) the displacement gradients are small and the squares of the rotation do not exceed reference surface deformation in order of magnitude;

c) the normal constraints, normal to the surface of reference, are negligible;

d) the normals to the surface of reference remain normal after deformation and are not subject to any elongation.

The theory based on these four hypotheses is known as the "Sanders-Koiter non-linear theory" [3,4]; it has been used throughout this paper.

2.2 Strain-displacement and stress-strain relations

The non-linear Sanders-Koiter theory for thin shells postulates differences in the first and second fundamental forms between the reference surfaces, deformed and non deformed, as deformation measures in elongation and bending respectively.

Generally, the deformation vector $\{\epsilon\}$ is written as:

$$\{\epsilon\} = \{\epsilon_L\} + \{\epsilon_{NL}\} = \left\{ \begin{array}{c} \epsilon_{xx} \\ \epsilon_{\theta\theta} \\ 2\epsilon_{x\theta} \\ \kappa_{xx} \\ \kappa_{\theta\theta} \\ 2\kappa_{x\theta} \end{array} \right\} \tag{2}$$

where subscripts "L" and "NL" mean "linear" and "non-linear", respectively.

For a cylindrical shell, the expressions for $\{\epsilon_L\}$ and $\{\epsilon_{NL}\}$ are given by :

$$\{\epsilon_L\} = \left\{ \begin{array}{c} \dfrac{\partial U}{\partial x} \\ \dfrac{1}{R}\left[\dfrac{\partial V}{\partial \theta} + W\right] \\ \dfrac{\partial V}{\partial x} + \dfrac{1}{R}\dfrac{\partial U}{\partial \theta} \\ -\dfrac{\partial^2 W}{\partial x^2} \\ -\dfrac{1}{R^2}\left[\dfrac{\partial^2 W}{\partial \theta^2} - \dfrac{\partial V}{\partial \theta}\right] \\ -\dfrac{2}{R}\dfrac{\partial^2 W}{\partial x \partial \theta} + \dfrac{3}{2R}\dfrac{\partial V}{\partial x} - \dfrac{1}{2R^2}\dfrac{\partial U}{\partial \theta} \end{array} \right\} \tag{3}$$

and

$$\{\epsilon_{NL}\} = \left\{ \begin{array}{c} \dfrac{1}{2}\left[\dfrac{\partial W}{\partial x}\right]^2 + \dfrac{1}{8}\left[\dfrac{\partial V}{\partial x} - \dfrac{1}{R}\dfrac{\partial U}{\partial \theta}\right]^2 \\ \dfrac{1}{2R^2}\left[V - \dfrac{\partial W}{\partial \theta}\right]^2 + \dfrac{1}{8}\left[\dfrac{\partial V}{\partial x} - \dfrac{1}{R}\dfrac{\partial U}{\partial \theta}\right]^2 \\ \dfrac{1}{2R}\left[\dfrac{\partial W}{\partial x}\dfrac{\partial W}{\partial \theta} - V\dfrac{\partial W}{\partial x}\right] \\ 0 \\ 0 \\ 0 \end{array} \right\} \tag{4}$$

Where U, V and W are, respectively, the axial, tangential and radial displacements of the shell's surface of reference.

It is evident that in equations (3) and (4) the expressions for components κ_{xx}, $\kappa_{\theta\theta}$, $2\kappa_{x\theta}$ are linear. This fits in with hypothesis (b) from paragraph 2.1.

The constituent relations between the stress and deformation vectors of the surface of reference for anisotropic shells are given as follows:

$$\{\sigma\} = \begin{Bmatrix} N_{xx} \\ N_{\theta\theta} \\ N_{x\theta} \\ M_{xx} \\ M_{\theta\theta} \\ M_{x\theta} \end{Bmatrix} = [P]\ \{\epsilon\} \tag{5}$$

where [P] is the matrix of elasticity. The elements p_{ij} in [P] determine the anisotropy of the shell, which depends on the mechanical characteristics of the structure's material.

In general, this implies that:

$$[P] = \begin{bmatrix} p_{11} & p_{12} & 0 & p_{14} & p_{15} & 0 \\ p_{21} & p_{22} & 0 & p_{24} & p_{25} & 0 \\ 0 & 0 & p_{33} & 0 & 0 & p_{36} \\ p_{41} & p_{42} & 0 & p_{44} & p_{45} & 0 \\ p_{51} & p_{52} & 0 & p_{54} & p_{55} & 0 \\ 0 & 0 & p_{63} & 0 & 0 & p_{66} \end{bmatrix} \tag{6}$$

2.3 Equations of equilibrium

By applying the virtual work principle to the infinitesimal element of the deformed surface of reference, the three equations of equilibrium, describing the non-linear behaviour of an arbitrarily formed shell, are obtained [3] (see Figure 2).

$$\frac{\partial N_{xx}}{\partial x} + \frac{1}{R}\frac{\partial \bar{N}_{x\theta}}{\partial \theta} - \frac{1}{2R^2}\frac{\partial \bar{M}_{x\theta}}{\partial \theta} - \frac{1}{2R}\frac{\partial}{\partial \theta}\left[\phi\,(N_{xx} + N_{\theta\theta})\right] = 0 \tag{7}$$

$$\frac{1}{R}\frac{\partial N_{\theta\theta}}{\partial \theta} + \frac{\partial \bar{N}_{x\theta}}{\partial x} + \frac{1}{R^2}\frac{\partial M_{\theta\theta}}{\partial \theta} + \frac{3}{2R}\frac{\partial \bar{M}_{x\theta}}{\partial x} - \frac{1}{R}(\phi_x \bar{N}_{x\theta} + \phi_\theta N_{\theta\theta}) + \frac{1}{2}\frac{\partial}{\partial x}\left[\phi\,(N_{xx} + N_{\theta\theta})\right] = 0 \tag{8}$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + \frac{2}{R}\frac{\partial^2 \bar{M}_{x\theta}}{\partial x \partial \theta} + \frac{1}{R^2}\frac{\partial^2 M_{\theta\theta}}{\partial \theta^2} - \frac{1}{R} N_{\theta\theta} - \frac{\partial}{\partial x}\left[\phi_x N_{xx} + \phi_\theta \bar{N}_{x\theta} \right] \tag{9}$$

$$- \frac{1}{R}\frac{\partial}{\partial \theta}\left[\phi_x \bar{N}_{x\theta} + \phi_\theta N_{\theta\theta} \right] = 0$$

where

$$\phi = \frac{1}{2}\left(\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta} \right) , \ \phi_x = - \frac{\partial W}{\partial x} \text{ et } \phi_\theta = - \frac{1}{R}\left(\frac{\partial W}{\partial \theta} - V \right) \tag{10}$$

Substituting equations (2) to (6) for the equilibrium equations (7) to (10), we obtain equation (11) as a function of elements p_{ij} in [P] and the axial, tangential and radial displacements U, V and W from a point of the shell surface of reference:

$$\begin{aligned} L_1 (U, V, W, p_{ij}) + N_1 (U, V, W, p_{ij}) &= 0 \\ L_2 (U, V, W, p_{ij}) + N_2 (U, V, W, p_{ij}) &= 0 \\ L_3 (U, V, W, p_{ij}) + N_3 (U, V, W, p_{ij}) &= 0 \end{aligned} \tag{11}$$

Functions L_i and N_i (i = 1 to 3) represent, respectively, the linear and non-linear equations of equilibrium. These equations are given in Appendix A-1.

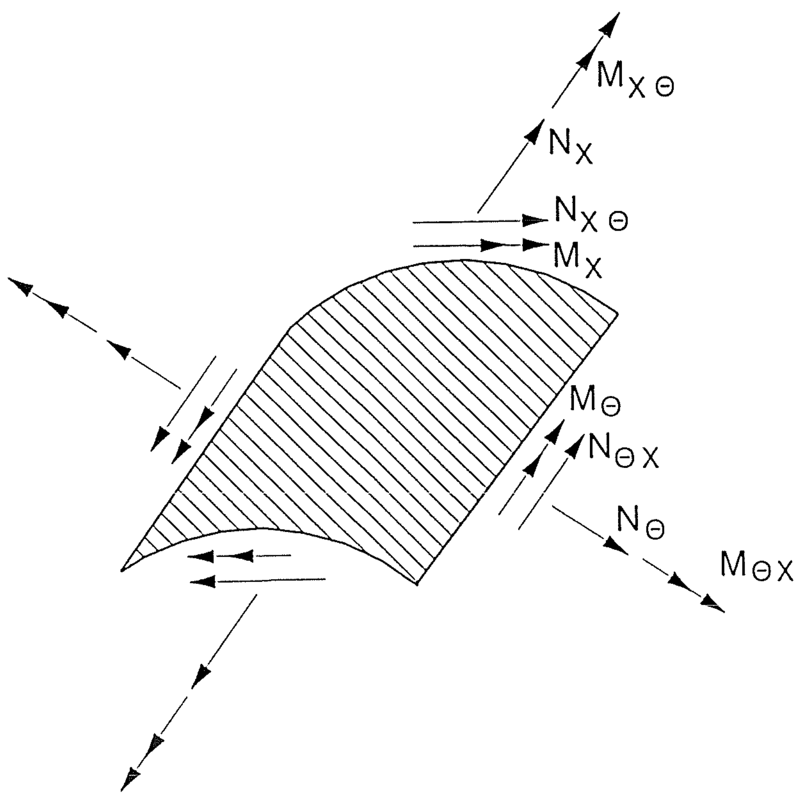

Figure 2 : Differential element for an open cylindrical shell.

3. DISPLACEMENT FUNCTIONS

The shell is subdivided into several finite elements defined by two nodes i and j and by components U, V, W and $dW/d\theta$, representing axial, tangential, radial displacements and the rotation, respectively, from a point located on the shell's surface of reference (Figure 3).

The linear equations of motion are given by (see Appendix A-1):

$$\begin{aligned} L_1\ (U,V,W,p_{ij}) &= 0 \\ L_2\ (U,V,W,p_{ij}) &= 0 \\ L_3\ (U,V,W,p_{ij}) &= 0 \end{aligned} \tag{12}$$

The displacement functions are then assumed to be:

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix} = [T_m] \begin{Bmatrix} U(\theta) \\ W(\theta) \\ V(\theta) \end{Bmatrix} \tag{13}$$

$[T_m]$ is a (3 x 3) matrix in x given in Appendix A-2 and $U(\theta)$, $W(\theta)$ and $V(\theta)$ are functions of the θ coordinate and the shell's characteristics.

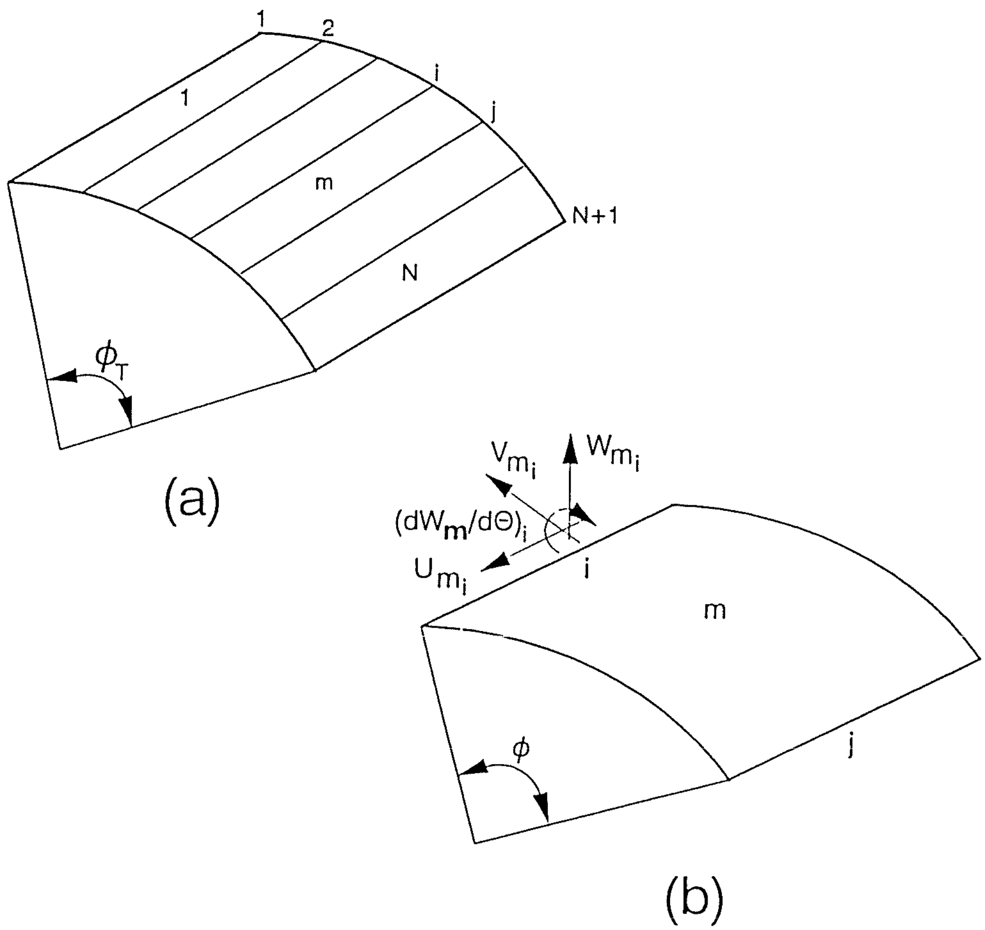

Figure 3 :

(a) Finite element idealization.

(b) Nodal displacements at node i.

Assuming:

$$U(\theta) = Ae^{\eta\theta}\ , \quad V(\theta) = Be^{\eta\theta}\ , \quad W(\theta) = Ce^{\eta\theta} \tag{14}$$

Substituting (13) and (14) into the equations of motion (12), three homogeneous linear functions of constants A, B and C are obtained:

$$[H]\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \{0\} \tag{15}$$

where the matrix [H] is given in reference [23]. For the solution to be non-trivial, the determinant of matrix [H] must be equal to zero. This brings us to the following characteristic equation [23]:

$$\text{Det}\ ([H]) = h_8\eta^8 + h_6\eta^6 + h_4\eta^4 + h_2\eta^2 + h_0 = 0 \tag{16}$$

Each root of this equation yields a solution to the equations of motion (12). The complete solution is obtained by adding the eight solutions independently with the constants A_p, B_p and C_p ($p = 1, \ldots, 8$), so that:

$$\begin{aligned} U(\theta) &= A_p e^{\eta_p\theta} \\ V(\theta) &= B_p e^{\eta_p\theta}, \quad p = 1,\ldots 8 \\ W(\theta) &= C_p e^{\eta_p\theta} \end{aligned} \tag{17}$$

The constants A_p, B_p and C_p are not independent. We can therefore express A_p and B_p as a function of C_p, for example:

$$A_p = \alpha_p C_p \quad \text{and} \quad B_p = \beta_p C_p \quad , \quad p = 1,\ldots,8 \tag{18}$$

The values of α_p and β_p can be obtained from system (15) by introducing relations (18).

Substituting expressions (17) and (18) into equations (13), the displacements $U(x,\theta)$, $V(x,\theta)$ and $W(x,\theta)$ can then be expressed in conjunction with the eight C_p constants only. We then have:

$$\left\{ \begin{array}{c} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{array} \right\} = [T]\ [R]\ \{C\} \tag{19}$$

where [R] is a (3 x 8) matrix given in Appendix A-2 and {C} is an 8^{th} order vector of the C_p constants:

$$\{C\} = \{C_1\, C_2 \ldots C_8\}^T \tag{20}$$

To determine the eight C_p constants, it is necessary to formulate eight boundary conditions for the finite elements. The axial, tangential and radial displacements, as well as rotation, will be specified for each node. The degrees of freedom at node i can be defined be the vector:

$$\{\delta_i\} = \left\{ U_i \, W_i \left[\frac{dW}{d\theta} \right]_i V_i \right\}^T \tag{21}$$

The elements which have two nodes and eight degrees of freedom will have i ($\theta = 0$) and j ($\theta = \phi$) as nodal displacements at the boundaries:

$$\left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} = \left\{ U_i \, W_i \left[\frac{dW}{d\theta} \right]_i V_i \, U_j \, W_j \left[\frac{dW}{d\theta} \right]_j V_j \right\}^T = [A]\{C\} \tag{22}$$

where the terms of matrix [A], given in Appendix A-2, are obtained from matrix [R] by successively setting $\theta = 0$ and $\theta = \phi$.

Multiplying equation (22) by $[A^{-1}]$ we obtain:

$$\{C\} = [A^{-1}] \left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} \tag{23}$$

Substituting equations (19) we get:

$$\left\{ \begin{matrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{matrix} \right\} = [T_m]\ [R]\ [A^{-1}] \left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} = [N] \left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} \tag{24}$$

These equations determine the displacement functions.

4. MASS AND LINEAR STIFFNESS MATRICES FOR AN ELEMENT

The deformation vector can be obtained from equations (3) and (24), therefore:

$$\{\epsilon_L\} = \begin{bmatrix} [T] & [O] \\ [O] & [T] \end{bmatrix} [Q]\ [A^{-1}] \left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} = [B] \left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} \tag{25}$$

where [Q] is a (6x8) matrix given in reference [23].

Combining equations (5) and (25), the stress-strain relations can be written as:

$$\{\sigma_L\} = [P]\ [B] \left\{ \begin{matrix} \delta_i \\ \delta_j \end{matrix} \right\} \tag{26}$$

The mass and linear stiffness matrices can then be expressed as:

$$[m] = \rho t \int \int [N^T] [N] \, dA$$
$$[k_L] = \int \int [B^T] [P] [B] \, dA \qquad (27)$$

where $dA = Rdxd\theta$, is the density of the shell, t its thickness, [P] the elasticity matrix and the matrices [N] and [B] are derived from equations (24) and (25) respectively. The matrices [m] and $[k_L]$ were obtained analytically by carring out the necessary matrix operations and integration over x and θ in equation (27). For more details on the linear analysis, see reference [23].

5. NON-LINEAR MATRIX CONSTRUCTION

The following approach, developed in reference [16], was used with particular attention to geometric non-linearities. The coefficients of the modal equations were obtained through the Lagrange method. Thus, the non-linear stiffness matrices, once calculated, was overlaid onto the linear system. Before we embark on matrix formulation, however, a brief summary of the method is in order.

5.1 Method

This section will cover the principal points of the method used to find the non-linear stiffness matrices.

The main steps of this method are as follows:

(a) Shell displacements are expressed as generalized product coordinate sums and spatial functions;

(b) the deformation vector is written as a function of the generalized coordinates by separating the linear portion from the non-linear;

(c) these expressions are then introduced into the Lagrange equations up to and including the degree corresponding to the deformation energy;

(d) by substituting the expressions in a) into the strain-displacement relations in the Sanders-Koiter [3-4] non-linear theory, the generalized coordinate coefficients appearing in the equation derived under c) are determined in terms of spatial functions.

5.2 Coefficients of modal equations

If a_p, b_p, c_p, A_{pq}, B_{pq}, C_{pq} and aA_{prs}, bB_{prs}, cC_{prs}, aB_{prs}, bA_{prs}, AA_{prsq}, BB_{prsq}, CC_{prsq}, AB_{prsq}, BA_{prsq} are designed as coefficients of the modal equations mentioned in step d) above for an open cylindrical shell, the following expressions [16] are thus obtained:

$$a_p = \frac{\partial f_p}{\partial x} \tag{28}$$

$$b_p = \frac{1}{R}\left[\frac{\partial g_p}{\partial \theta} + h_p\right] \tag{29}$$

$$c_p = \frac{1}{2}\left[\frac{\partial f_p}{R\partial \theta} + \frac{\partial g_p}{\partial x}\right] \tag{30}$$

$$A_{pq} = \frac{1}{8R^2}\left[R\frac{\partial g_p}{\partial x} - \frac{\partial f_p}{\partial \theta}\right] \cdot \left[R\frac{\partial g_q}{\partial x} - \frac{\partial f_q}{\partial \theta}\right] + \frac{1}{2}\frac{\partial h_p}{\partial x}\frac{\partial h_q}{\partial x} \tag{31}$$

$$B_{pq} = \frac{1}{8R^2}\left[R\,\frac{\partial g_p}{\partial x} - \frac{\partial f_p}{\partial \theta}\right] \cdot \left[R\,\frac{\partial g_q}{\partial x} - \frac{\partial f_q}{\partial \theta}\right] + \frac{1}{2R^2}\left[\frac{\partial h_p}{\partial \theta} - g_p\right] \cdot \left[\frac{\partial h_q}{\partial \theta} - g_q\right] \tag{32}$$

$$C_{pq} = \frac{1}{4R}\left[\frac{\partial h_p}{\partial x}\frac{\partial h_q}{\partial \theta} + \frac{\partial h_q}{\partial x}\frac{\partial h_p}{\partial \theta}\right] - \frac{1}{4R}\left[g_p\,\frac{\partial h_q}{\partial x} + g_q\,\frac{\partial h_p}{\partial x}\right] \tag{33}$$

where f, g, h are spatial functions determined by matrix [N] in equations (21) and:

$$\begin{aligned} aA_{prs} &= a_p A_{rs} + a_r A_{sp} + a_s A_{pr} \\ bB_{prs} &= b_p B_{rs} + b_r B_{sp} + b_s B_{pr} \\ cC_{prs} &= c_p C_{rs} + c_r C_{sp} + c_s C_{pr} \\ aB_{prs} &= a_r B_{sp} + a_s B_{pr} + b_p A_{rs} \\ bA_{prs} &= b_r A_{sp} + b_s A_{pr} + a_s B_{rs} \end{aligned} \tag{34}$$

$$
\begin{aligned}
AA_{prsq} &= 2A_{pq}A_{rs} \\
BB_{prsq} &= 2B_{pq}B_{rs} \\
CC_{prsq} &= 2C_{pq}C_{rs} \\
AB_{prsq} &= 2A_{pq}B_{rs} \\
BA_{prsq} &= 2B_{pq}A_{rs}
\end{aligned}
\tag{35}
$$

In equations (34) and (35), the subscripts 'p,q', 'p,q,r' and 'p,q,r,s' represent the coupling between two, three and four modes respectively.

For consistency, equations (28) to (35) are written in matrix format. The following notation is adopted : the matrices with the "+" superscript represent equations (28) to (33) and the ones with the "++" superscript represent the equations (34) and (35).

With equation (28), we obtain :

$$\{a^{+}\} = \{a^{*}\}[A^{-1}] \tag{36}$$

Where :

$$a_p^{*} = a_p' e^{\eta_p \theta}, \quad a_p' = a_p^{(1)} \sin \bar{m}x, \quad a_p^{(1)} = -\bar{m}\alpha_p, \quad p = 1,\ldots,8. \quad \text{and} \quad \bar{m} = m\pi/L \tag{37}$$

and with equation (31), we have :

$$[A^{+}] = [A^{-1}]^{T}[A^{*}][A^{-1}] \tag{38}$$

Where :

$$A^{*}_{pq} = a'_{pq}e^{(\eta_p+\eta_q)\theta}, \quad a'_{pq} = a^{(1)}_{pq}\cos^2 \bar{m}x, \quad p,q = 1,\ldots 8$$
$$a^{(1)}_{pq} = \frac{1}{8R^2}[R\bar{m}\beta_p - \alpha_p\eta_p][R\bar{m}\beta_q - \alpha_q\eta_q] + \frac{1}{2}\bar{m}^2 \tag{39}$$

Similarly, we obtain the expressions for $\{b^{+}\}$, $\{c^{+}\}$, $[B^{+}]$ and $[C^{+}]$. These matrices are given in Appendix A-2.

Also, we can write equation (34.1) in matrix format as :

$$[aA^{++}] = [A^{-1}]^{T}[aA^{**}][A^{-1}] \tag{40}$$

Where :

$$aA^{**}(p,q) = \sum_{k=1}^{8}[a'_p A^{-1}_{pq} a'_{pk} + a'_q A^{-1}_{qk} a'_{kp} + a'_k A^{-1}_{kp} a'_{pq}]e^{(\eta_p+\eta_q+\eta_k)\theta} \tag{41}$$

and equation (35.1) is written as :

$$[AA^{++}] = 2[A^{-1}]^T[AA^{**}][A^{-1}] \tag{42}$$

Where :

$$AA^{**}(p,q) = \sum_{k=1}^{8} a'_{kq} \left[\sum_{l=1}^{8} a'_{pl} E_{lk} e^{(\eta_p + \eta_q + \eta_k + \eta_l)\theta} \right] \tag{43}$$

E_{lk} is the term (l,k) of matrix [E], where [E] represents a matrix of constants defined by :

$$[E] = [A^{-1}]^T[A^{-1}] \tag{44}$$

Similarly, equations (34.2) to (34.5) and equations (35.2) to (35.5) can be written in matrix format, and give the above matrices respectively: $[bB^{++}]$, $[cC^{++}]$, $[aB^{++}]$, $[bA^{++}]$ and $[BB^{++}]$, $[CC^{++}]$, $[AB^{++}]$, $[BA^{++}]$. The terms (p,q) of these matrices are described in Appendix A-2.

5.3 Second order non-linear stiffness matrix for an element

The second order non-linear stiffness matrix for an anisotropic open cylindrical shell is as follows:

$$[k_{NL2}] = \left\{ \int \int \left[p_{11}[aA^{++}] + p_{22}[bB^{++}] + p_{12}([aB^{++}] + [bA^{++}]) + p_{33}[cC^{++}] \right] dA \right\} \tag{45}$$

where $dA = Rdxd\theta$, $[aA^{++}]$ is given by relation (40) and matrices $[bB^{++}],\ldots,[cC^{++}]$ are given in appendix A-2.

After integrating over x and θ, we obtain:

$$[k_{NL2}] = [A^{-1}]^T[k^*_{NL2}][A^{-1}] \tag{46}$$

The (p,q) term in matrix $[k^*_{NL2}]$ is written:

$$k^{*}_{NL2}(p,q) = \begin{cases} \displaystyle\sum_{k=1}^{8} \frac{R\ G(p,q)}{(\eta_p + \eta_q + \eta_k)} \left[e^{(\eta_p + \eta_q + \eta_k)} - 1 \right] & \\ \qquad\qquad \text{if } \eta_p + \eta_q + \eta_k \neq 0 & \\ \displaystyle\sum_{k=1}^{8} R\ \ G(p,q)\ \phi \qquad \text{if } \eta_p + \eta_q + \eta_k = 0 & \end{cases} \tag{47}$$

G(p,q) is a coefficient in conjunction with α, β, η and element p_{ij} in matrix [P]. The general expression of G(p,q) is:

$$\begin{aligned} G(p,q) = {} & p_{11}I_1[\ a_p^{(1)}\ A_{pq}^{-1}\ a_{qk}^{(1)} + a_q^{(1)}\ A_{qk}^{-1}\ a_{kp}^{(1)} + a_k^{(1)}\ A_{kp}^{-1}\ a_{pq}^{(1)}\ ] + \\ & p_{22}I_1[\ b_p^{(1)}\ A_{pq}^{-1}\ b_{qk}^{(1)} + b_q^{(1)}\ A_{qk}^{-1}\ b_{kp}^{(1)} + b_k^{(1)}\ A_{kp}^{-1}\ b_{pq}^{(1)}\ ] + \\ & p_{22}I_2[\ b_p^{(1)}\ A_{pq}^{-1}\ b_{qk}^{(2)} + b_q^{(1)}\ A_{qk}^{-1}\ b_{kp}^{(2)} + b_k^{(1)}\ A_{kp}^{-1}\ b_{pq}^{(2)}\ ] + \\ & p_{33}I_1[\ c_p^{(1)}\ A_{pq}^{-1}\ c_{qk}^{(1)} + c_q^{(1)}\ A_{qk}^{-1}\ c_{kq}^{(1)} + c_k^{(1)}\ A_{kp}^{-1}\ c_{pq}^{(1)}\ ] + \\ & p_{12}I_1[\ a_q^{(1)}\ A_{qk}^{-1}\ b_{kp}^{(1)} + a_k^{(1)}\ A_{kp}^{-1}\ b_{pq}^{(1)} + b_p^{(1)}\ A_{pq}^{-1}\ a_{qk}^{(1)} + \\ & \quad b_q^{(1)}\ A_{qk}^{-1}\ a_{kp}^{(1)} + b_k^{(1)}\ A_{kp}^{-1}\ a_{pq}^{(1)} + a_p^{(1)}\ A_{pq}^{-1}\ b_{qk}^{(1)}\ ] + \\ & p_{12}I_2[\ a_q^{(1)}\ A_{qk}^{-1}\ b_{kp}^{(2)} + a_k^{(1)}\ A_{kp}^{-1}\ b_{pq}^{(2)} + a_p^{(1)}\ A_{pq}^{-1}\ b_{qk}^{(2)}\ ] \end{aligned} \tag{48}$$

where:

$$I_1 = \frac{1}{3\bar{m}}[1-(-1)^m], \quad I_2 = 2I_1$$

The terms $a_p^{(1)}$ and $a_{pq}^{(1)}$ are given by equations (37) and (39). Terms $b_{..}^{(1)}$, $c_{..}^{(1)}$ and $b_{..}^{(2)}$ are coefficients appearing in expressions for the elements of matrices $\{b^*\}$, $\{c^*\}$, $[B^*]$ and $[C^*]$ defined in equations (29) to (33). These coefficients are given in Appendix A-2 and A_{pq}^{-1} is the term (p,q) of matrix $[A^{-1}]$, where [A] is the matrix defined by relation (22).

5.4 Third order non-linear stiffness matrix for an element

The third order non-linear stiffness matrix for an anisotropic open cylindrical shell is as follows:

$$[k_{NL3}] = \left\{ \int \int \Big[p_{11}[AA^{++}] + p_{22}[BB^{++}] + p_{12}([AB^{++}] + [BA^{++}]) + p_{33}[CC^{++}] \Big] dA \right\} \tag{49}$$

where $dA = Rdxd\theta$, $[AA^{++}]$ is given by relation (42) and matrices $[BB^{++}]$,...$[CC^{++}]$ are given in appendix A-2.

After integration, we obtain:

$$[k_{NL3}] = [A^{-1}]^T [k^*_{NL3}] [A^{-1}] \tag{50}$$

The (p,q) term in matrix $[k^*_{NL3}]$ is written:

$$k^*_{NL3}(p,q) = \begin{cases} \displaystyle\sum_{k=1}^{8} \sum_{l=1}^{8} \frac{R\ L\ E(l,k)\ S(p,q)}{8(\eta_p + \eta_q + \eta_k + \eta_l)} \left[e^{(\eta_p + \eta_q + \eta_k + \eta_l)} - 1 \right] \\ \qquad\qquad \text{if } \eta_p + \eta_q + \eta_k + \eta_l \neq 0 \\ \displaystyle\sum_{k=1}^{8} \sum_{l=1}^{8} \frac{1}{8} R\ L\ E(l,k)\ S(p,q)\ \phi \qquad \text{if } \eta_p + \eta_q + \eta_k + \eta_l = 0 \end{cases} \tag{51}$$

S(p,q) is a coefficient in conjunction with α, β, η and element p_{ij} in matrix [P]. The general expression of S(p,q) is:

$$\begin{aligned} S(p,q) = {} & 3p_{11}\, a^{(1)}_{pl}\, a^{(1)}_{kq} + p_{22}\, (3b^{(1)}_{pl}\, b^{(1)}_{kq} + 3b^{(2)}_{pl}\, b^{(2)}_{kq} + b^{(1)}_{pl}\, b^{(2)}_{kq} + b^{(2)}_{pl}\, b^{(1)}_{kq}) \\ & + p_{33}\, c^{(1)}_{pl}\, c^{(1)}_{kq} + p_{12}\, (3a^{(1)}_{pl}\, b^{(1)}_{kq} + a^{(1)}_{pl}\, b^{(2)}_{kq} + 3b^{(1)}_{pl}\, a^{(1)}_{kq} + b^{(2)}_{pl}\, a^{(1)}_{kq}) \end{aligned} \tag{52}$$

where the term $a_{pq}^{(1)}$ is given by equation (39). Terms $b_{pq}^{(1)}$, $c_{pq}^{(1)}$ and $b_{pq}^{(2)}$ are coefficients appearing in expressions for the elements of matrices $[B^*]$ and $[C^*]$ defined in equations (32) and (33). These coefficients are given in Appendix A-2.

6. THE INFLUENCE OF GEOMETRIC NON-LINEARITIES OF THE WALLS ON THE NATURAL FREQUENCIES OF AN OPEN CYLINDRICAL SHELL

The mass and stiffness matrices obtained apply to only one element. After the shell is subdivided into several open cylindrical elements (Figure 3), the global mass and stiffness matrices are determined by assembling the matrices for each element. Assembling is done in such a way that all the equations of motion and the continuity of displacements at each node are satisfied. These matrices are designated as [M], $[K_L]$, $[K_{NL2}]$ and $[K_{NL3}]$ respectively. They are square matrices of order NDF * (N + 1), where N represents the number of finite elements and NDF represents the number of degrees of freedom at each node. In practice, very specific conditions are applied to the shell boundaries. Thus, matrices [M], $[K_L]$, $[K_{NL2}]$ and $[K_{NL3}]$ are reduced to square matrices of order NREDUC = NDF * (N + 1) - J, where J represents the number of constraints applied. These reduced matrices are written as $[M^{(r)}]$, $[K_L^{(r)}]$, $[K_{NL2}^{(r)}]$ and $[K_{NL3}^{(r)}]$. The superscript "r" means "reduced".

The (1) system of equations then becomes:

$$[M^{(r)}]\ \{\ddot{\delta}^{(r)}\} + [K_L^{(r)}]\ \{\delta^{(r)}\} + [K_{NL2}^{(r)}]\ \{\delta^{(r)^2}\} + [K_{NL3}^{(r)}]\ \{\delta^{(r)^3}\} = \{0\} \qquad (53)$$

Setting:

$$\{\delta^{(r)}\} = [\Phi]\ \{q\} \tag{54}$$

where $[\Phi]$ represents the square matrix for the eigenvectors of the linear system and $\{q\}$ is a time-related vector.

Substituting equation (54) into system (53) and multiplying by $[\Phi^T]$,we obtain:

$$\begin{gathered}[\Phi^T][M^{(r)}][\Phi]\{\ddot{q}\} + [\Phi^T][K_L^{(r)}][\Phi]\{q\} + \\ [\Phi^T][K_{NL2}^{(r)}][\Phi^2]\{q^2\} + [\Phi^T][K_{NL3}^{(r)}][\Phi^3]\{q^3\} = \{0\}\end{gathered} \tag{55}$$

The products of matrix $[\Phi^T]\ [M^{(r)}]\ [\Phi]$ and $[\Phi^T]\ [K_L^{(r)}]\ [\Phi]$ represent diagonal matrices, written as $[M^{(D)}]$ and $[K_L^{(D)}]$, respectively.

The (55) system of equations is written:

$$\begin{gathered}[M^{(D)}]\ \{\ddot{q}\} + [K_L^{(D)}]\ \{q\} + \\ [\Phi^T]\ [K_{NL2}^{(r)}]\ [\Phi^2]\ \{q^2\} + [\Phi^T]\ [K_{NL3}^{(r)}]\ [\Phi^3]\ \{q^3\} = \{0\}\end{gathered} \tag{56}$$

We saw how matrices contained in the linear part of the system (53) could be reduced to diagonal matrices. On the other hand, the matrix products $[\Phi^T]\ [K^{(r)}_{NL2}]\ [\Phi^2]$ and $[\Phi^T]\ [K^{(r)}_{NL3}]\ [\Phi^3]$ are not generally described as diagonal matrices.

A typical equation of the (56) system would yield:

$$m_{pp}\ \ddot{q}_p + k^{(L)}_{pp} q_p + \sum_{s=1}^{NREDUC} k^{(NL2)}_{ps} q_s^2 + \sum_{s=1}^{NREDUC} k^{(NL3)}_{ps} q_s^3 = 0 \qquad (57)$$

where coefficients m_{pp} and $k^{(L)}_{pp}$, represent the p^{th} diagonal terms of matrices $[M^{(D)}]$ and $[k^{(D)}_L]$, respectively. $k^{(NL2)}_{ps}$ and $k^{(NL3)}_{ps}$ are the (p,s) terms of the products $[\Phi^T]\ [K^{(r)}_{NL2}]\ [\Phi^2]$ and $[\Phi^T]\ [K^{(r)}_{NL3}]\ [\Phi^3]$ thereby becoming diagonal.

Equation (57) would then be written:

$$m_{pp}\,\ddot{q}_p + k_{pp}^{(L)} q_p + k_{pp}^{(NL2)} q_p^2 + k_{pp}^{(NL3)} q_p^3 = 0 \tag{58}$$

Setting:

$$q_p(\tau) = A_p f_p(\tau) \tag{59}$$

which satisfies the conditions:

$$f_p(0) = 1 \text{ and } \dot{f}_p(0) = 0 \tag{60}$$

Equation (58) becomes, after the A_p simplification:

$$m_{pp}\,\ddot{f}_p + k_{pp}^{(L)} f_p + k_{pp}^{(NL2)}\, t\,(A_p/t)\, f_p^2 + k_{pp}^{(NL3)}\, t^2 (A_p/t)^2 f_p^3 = 0 \tag{61}$$

where t represents shell thickness.

Dividing this last equation by m_{pp}, it becomes:

$$\ddot{f}_p + \omega_p^2 f_p + \Lambda_p^{(NL2)} (A_p/t)\, f_p^2 + \Lambda_p^{(NL3)} (A_p/t)^2 f_p^3 = 0 \tag{62}$$

where

$$\omega_p^2 = \frac{k_{pp}^{(L)}}{m_{pp}} \tag{63}$$

The coefficient $[k_{pp}^{(L)}/m_{pp}]$ represents the p^{th} linear vibration frequency of the shell. and

$$\Lambda_p^{(NL2)} = \frac{k_{pp}^{(NL2)}}{m_{pp}}\, t \tag{64}$$

$$\Lambda_p^{(NL3)} = \frac{k_{pp}^{(NL3)}}{m_{pp}}\, t^2 \tag{65}$$

The solution $f_p(\tau)$ of the non-linear differential equation (62) which satisfies the conditions in (60) is calculated by a fourth order Runge-Kutta numerical method. The linear and non linear natural frequencies are evaluated by a systematic search for the $f_p(\tau)$ roots as a function of time. The ω_{NL}/ω_L ratio of linear and non-linear frequency is expressed as a function of non-dimensional ratio (A_p/t) where A_p is the vibration amplitude.

7. CALCULATIONS AND DISCUSSION

The influence of the wall's geometric non-linearity on the open or closed cylindrical shell's free vibrations is expressed by equation (62). For a shell having the particular physical characteristics given, the ratio ω_{NL}/ω_L of linear and non-linear frequency have been graphically represented in Figures 4 to 9 with respect to the non-dimensional ratio, A_p/t. The straight horizontal line represents the linear vibration cases, where the frequency is independent of the motion's amplitude.

7.1 Non-linear free vibration of closed cylindrical shell

The first example of calculations to determine the influence of non-linearities in strain-displacement relations on the free vibrations of a simply-supported cylindrical shell is shown in the analyses in references [6] and [17]. The shell has the following properties:

$$E = 200\,\mathrm{GPa},\ \nu = 0.3,\ \rho = 7800\,\mathrm{Kg/m^3},$$

$$R = 2.54\,\mathrm{cm},\ L = 40\,\mathrm{cm},\ t = 0.0254\,\mathrm{cm},\ \phi_T = 360°.$$

The variation in natural frequencies of this shell was calculated using the method we propose, and compared to the results Nowinski [6] and Raju and Rao [17] obtained for the case of n = 4 and m = 1 (Figure 4).

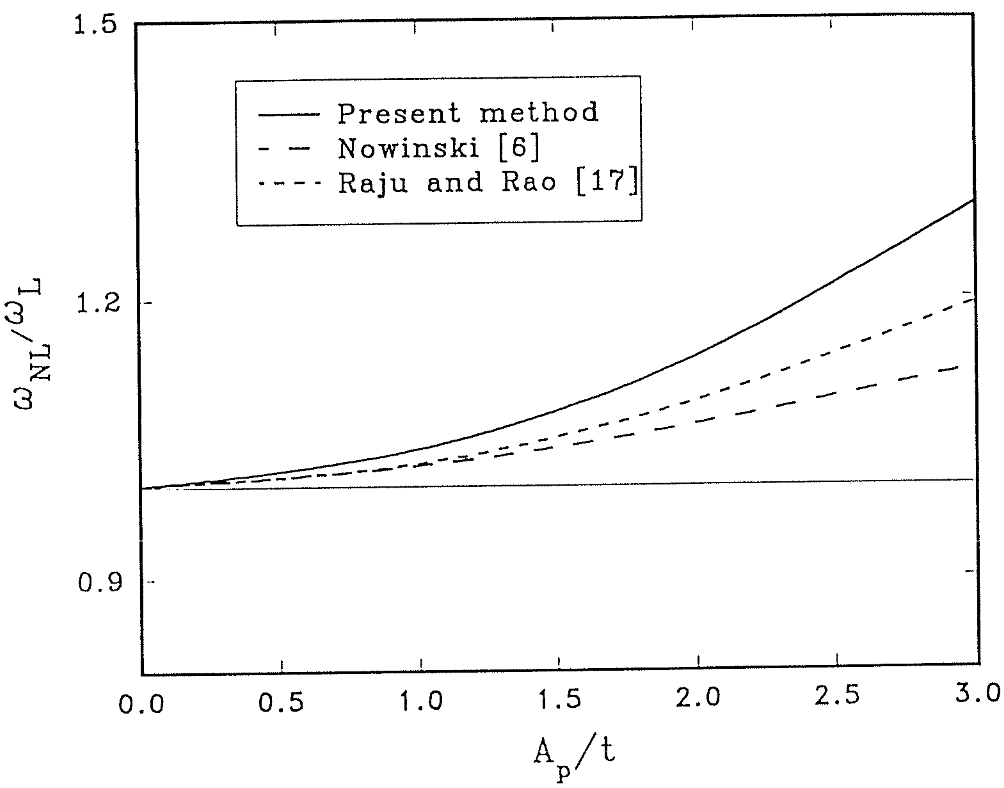

Figure 4 : Comparison of the effect of amplitude upon frequency for a simply–supported cylindrical shell, n=4, m=1.

Nowinski [6] based his analytical development on Donnell's simplified non-linear method. Only lateral displacement was considered. For their part, Raju and Rao [17], beginning with an energy formulation, used the finite element method.

In Figure 4, we observe that the variation ratio between the linear and non-linear frequency increases as ratio A/t increases. Non-linearity has a hardening effect. These variations are small for values A/t below 1.0. For values above 1.0, the variation is more pronounced than that which Nowinski [6] and Raju and Rao [17] obtained.

Its appears that these differences might be due to the fact that Nowinski [6] neglected plane inertia and took into account only lateral displacement. As for Raju and Rao [17], who used the Sanders-Koiter [3,4] non-linear theory, they expressed the displacements of components along the shell generator in polynomial form.

The second example of comparison is shown in Figure 5. The shell is simply-supported at both ends and the pertinent data are as follows:

$$\zeta = \pi Rm/nL = 2, \quad \epsilon = (n^2 t/R)^2 = 1 \quad \text{and} \quad \nu = 0.3$$

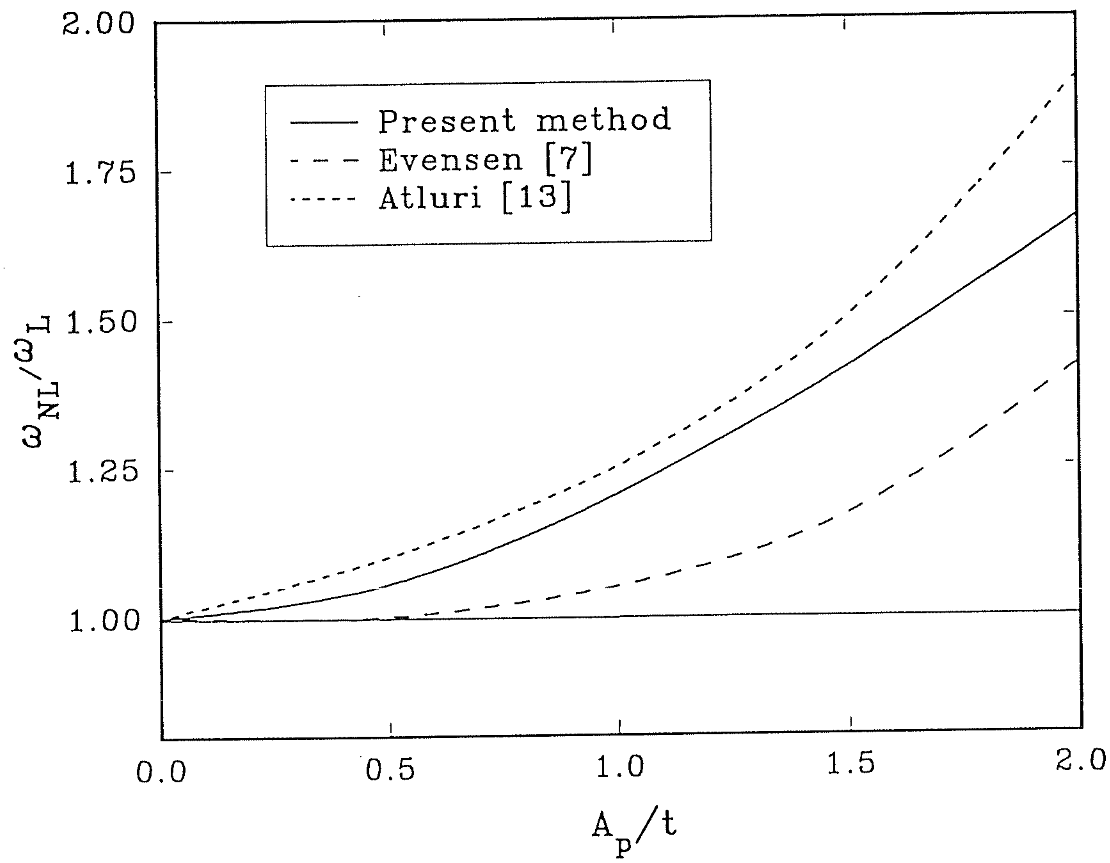

Figure 5 : Comparison of the effect of amplitude upon frequency for a simply-supported cylindrical shell,

$$\zeta = \pi Rm/nL = 2,\ \epsilon = (n^2t/R)^2 = 1,\ \nu = 0.3$$

The variations in frequency ratio as a function of A/t for this shell were calculated using the present method, and compared to the results of Evensen [7] and Atluri [13].

Evensen's [7] analysis involved a two modes approximation and his equation was obtained from Galerkin procedure. The work of Atluri [13] is based on Donnell's equations, a modal expansion was used for displacements and Galerkin technique was used to reduce the problem to a non-linear ordinary differential equation for the modal amplitudes.

As may be seen, the results obtained by the present method are in satisfactory agreement with those of other authors.

7.2 Non-linear free vibration of open cylindrical shell

One of the great advantages of the finite element method is the ease with which it can be applied to any geometry and boundary condition. Thus, the second step of calculation is to study the non-linear dynamic characteristics of open cylindrical shells as a function of circumferential and axial modes for various boundary conditions.

In Figure 6, we present the effect of large amplitude on the frequency of vibration for axial mode m = 1 and various circumferential mode n (1 to 12). The open shell is simply-supported at the four edges and the data are as follows :

$$E = 200\,\mathrm{GPa},\ \nu = 0.3,\ \rho = 7800\,\mathrm{Kg/m^3},$$

$$R = 2.54\,\mathrm{cm},\ L = 40\,\mathrm{cm},\ t = 0.0254\,\mathrm{cm},\ \phi_T = 135°.$$

The Figure shows that the non-linearity is of the hardening type for circumferential mode n = 1 and n > 9 and is of softening type for n between 2 and 9. We see also that the non-linear effect is more pronounced for the mode n = 5 and the variation is small for the case of n = 1.

Figure 7 shows the variation in frequency ratio as a function of A/t for axial mode m = 2 and circumferential mode n = 1 to 12. As in Figure 6, the same phenomena can be observed for this axial mode, the non-linearity is of the hardening type for circumferential mode n = 1, 2, 3 and n > 9 and is of softening type for n between 4 and 9. We see also that the non-linear effect is more pronounced for the mode n = 7 and the variation is small for the case of n = 1.

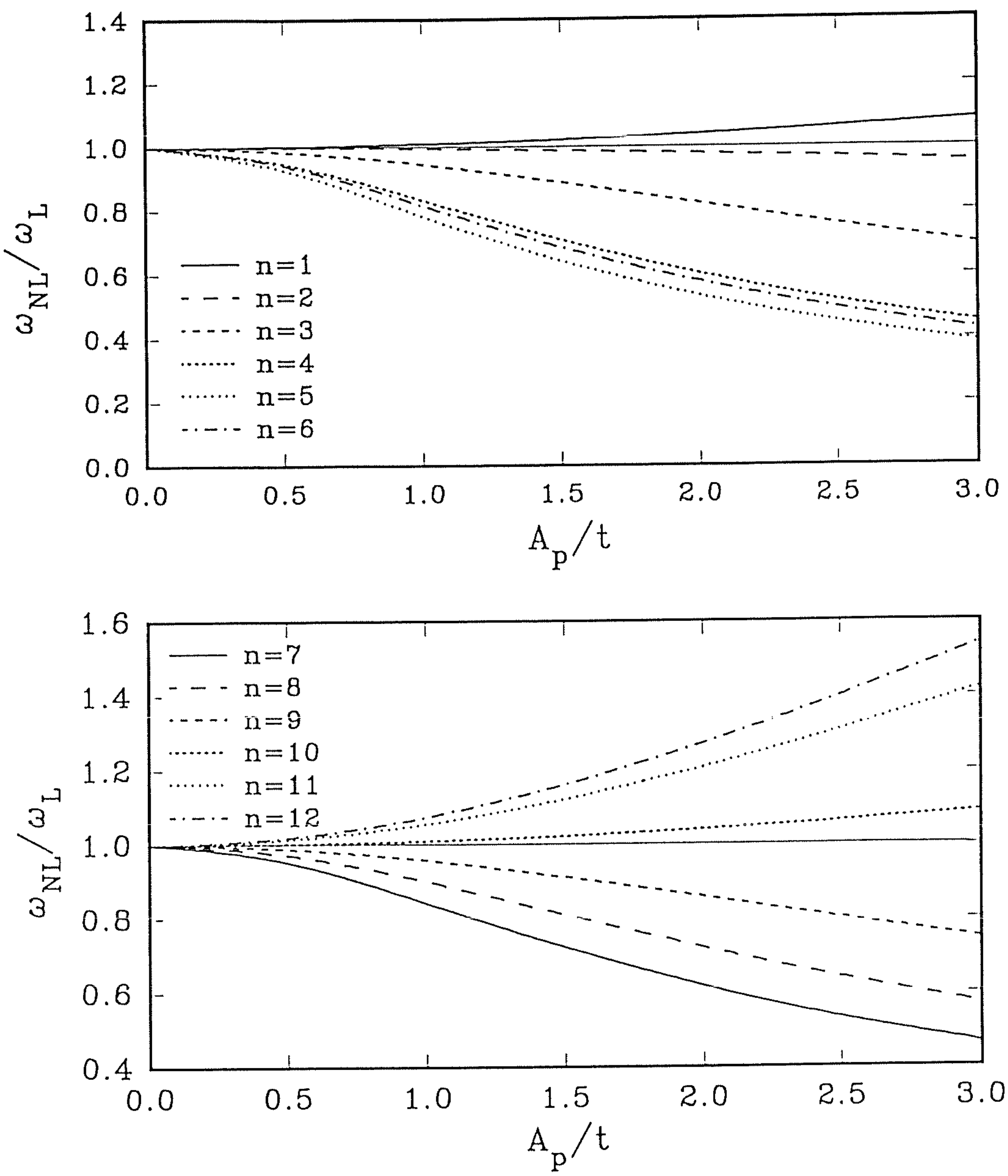

Figure 6 : Influence of large amplitude on natural frequency of simply-supported open cylindrical shell for various circumferential mode n and axial mode m=1.

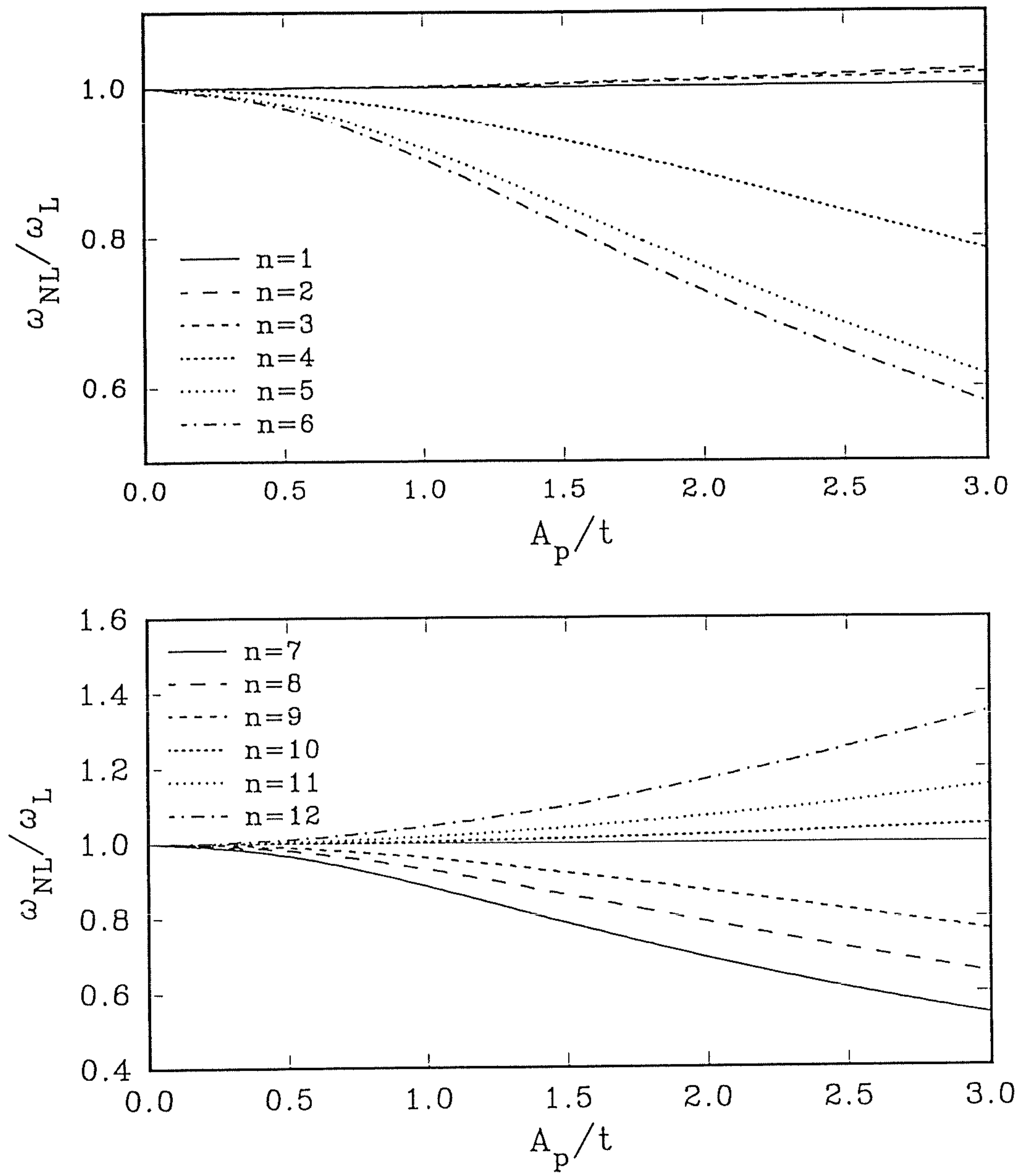

Figure 7 : Influence of large amplitude on natural frequency of simply-supported open cylindrical shell for various circumferential mode n and axial mode m=2.

In order to establish the effect of boundary condition on non-linear free vibration, we turn to Figure 8. We observe for the mode (n = 1, m = 2) and the same open shell with different boundary conditions, that the shell with the clamped - simply supported boundary conditions in its straight edges is the one on which the effect of non-linearity is the more pronounced, The effect is small for a panel with clamped - free boundary conditions. The steel panel analysis in Figure 8 has the following data:

$$R = 37.7 \text{ mm}, L = 234 \text{ mm}, t = 0.229 \text{ and } \phi_T = 180 \text{ deg.}$$

With the same data, Figure 9 shows the effect of the opening angle ϕ_T on the non-linear free vibration of open cylindrical shell. It shows that the panel with opening angle $\phi_T = 135$ deg. is the one which has the smaller effect on the non-linearity and the more pronounced effect is for $\phi_T = 10$ deg.

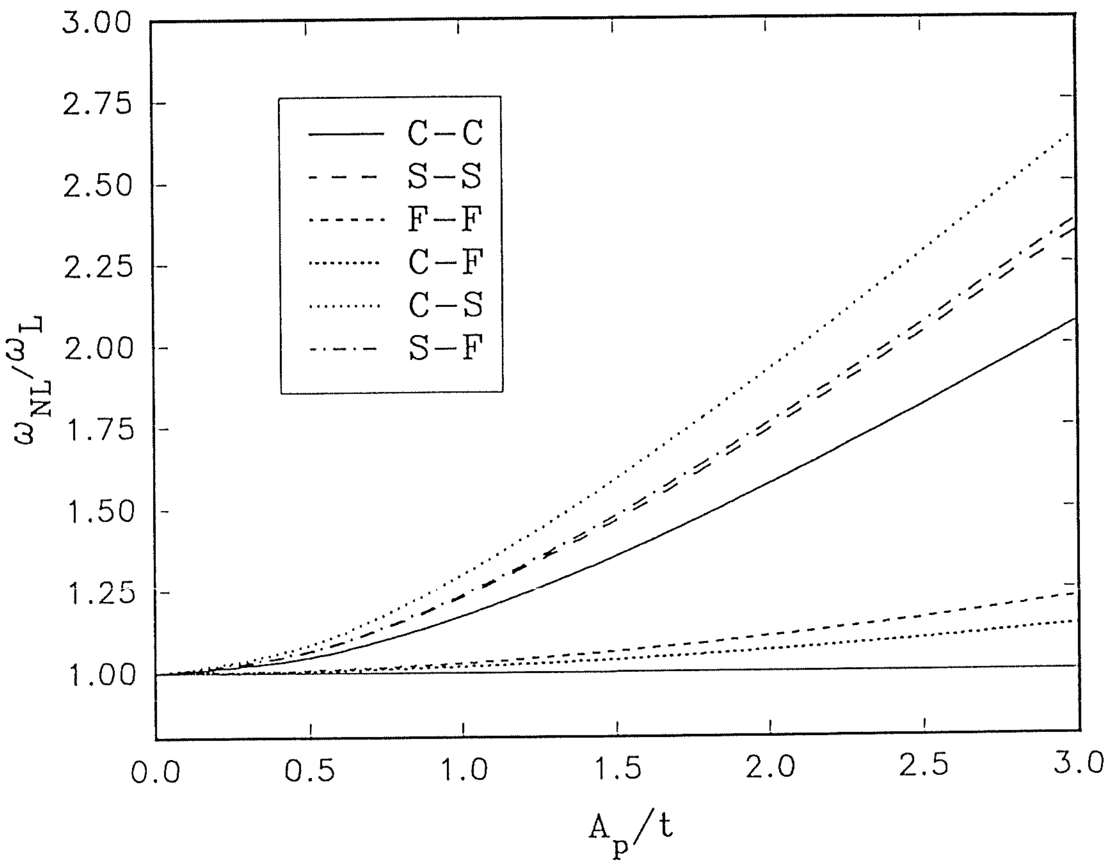

Figure 8 : Influence of large amplitude on the natural frequency of an open cylindrical shell for different boundary conditions, n=1, m=2.
(F : Free, S : Simply−supported, C : Clamped)

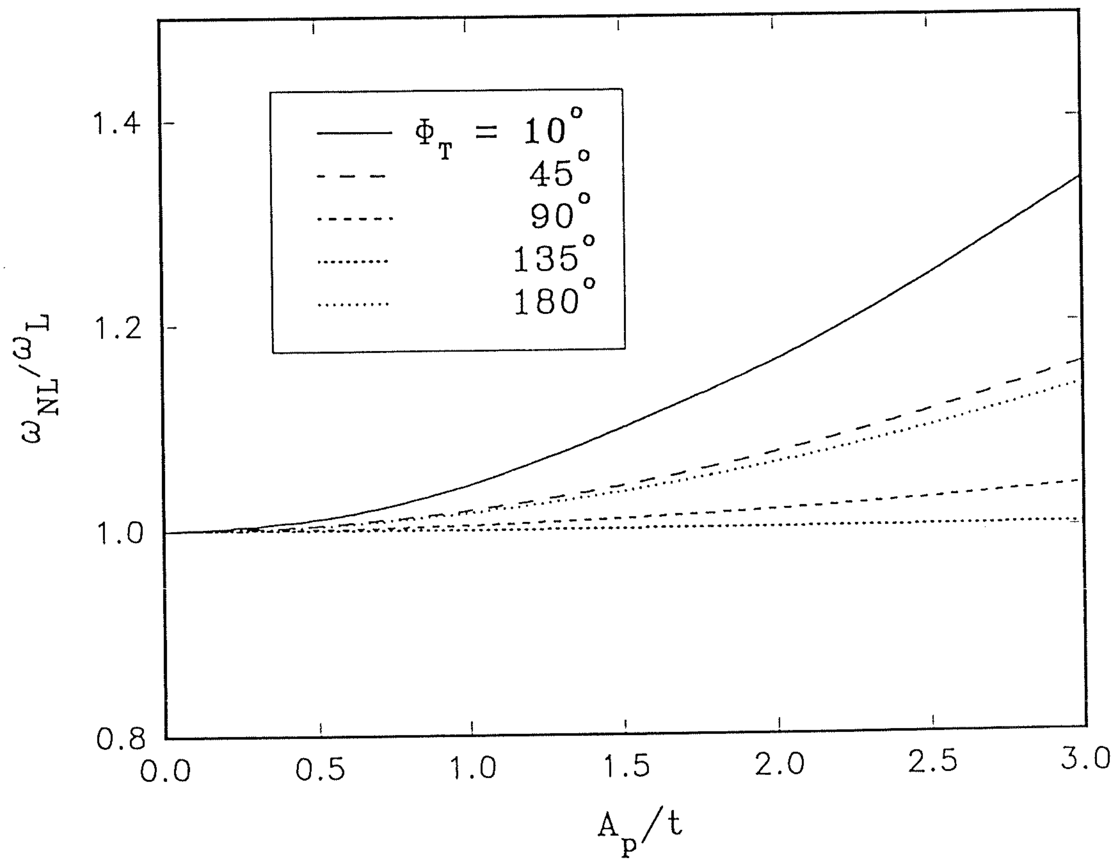

Figure 9 : Influence of large amplitude on the natural frequency of clamped–free open cylindrical shell for different opening angle Φ_T, (n=1, m=2).

8. CONCLUSION

The method discussed in this report demonstrates the influence of geometric non-linearities of the walls on the free vibrations of empty open or closed cylindrical shells. It is a hybrid method, based on a combination of thin shell theory and the finite element method.

An open cylindrical finite element was developed, so that the displacement functions could be derived directly from classical thin shell theory.

The solution was divided into two parts. In part one, the displacement functions were obtained from linear shell theory and the mass and linear stiffness matrices were determined by the finite element procedure. In part two, the modal coefficients corresponding to non-linearities in strain-displacement relations were obtained for the displacement functions. The second and third order non-linear stiffness matrices were then calculated using the finite element method.

With the help of a computer program, variations in the free vibration frequencies were determined in conjunction with motion amplitude for a closed or open cylindrical shell. Deviations in terms of linear vibrations were observed. The results obtained with this method were in agreement with other analytical and numerical methods.

A paper currently under preparation will deal with non-linear free vibration analysis of liquid-filled open or closed cylindrical shells. The non-linear dynamic stability of shells containing flowing fluid will also be investigated.

9. REFERENCES

[1] REISSNER, E., "Non-Linear effects in Vibrations of Cylindrical Shells", AIAA, J. 1, 617-620, 1963.

[2] NAGHDI, P.M. and NORDGREN, R.P., " On the Nonlinear Theory of Elastic Shells Under the Kirchhoff Hypothesis", Quart. Appl. Math. 21, 49-59, 1963.

[3] SANDERS, J.L., "Nonlinear Theories for Thin Shells", Quart, Appl. Math. 21, 21-36, 1963.

[4] KOITER, W.T., "On the Nonlinear Theory; of Thin Elastic Shells I, II, III", Proc. K. ned. Akad. Wet. B69, 1-54, 1966.

[5] YOKOO, Y. and MATSUNAGA, H., "A General Nonlinear Theory; of Elastic Shells", Int. J. Solids Struct. 10, 261-274, 1974.

[6] NOWINSKI, J.L., "Nonlinear Transverse Vibrations of Orthotropic Cylindrical Shells", AIAA J. 1, 617-620, 1963.

[7] EVENSEN, D.A., "Nonlinear Flexural Vibrations of Thin-Walled Circular Cylinders", NASA-TN-D4090, 1967.

[8] DOWELL, E.H. and VENTRES, C.S., "Modal Equations for the Nonlinear Flexural Vibrations of a Cylindrical Shell", Int. J. Solids Struct. 4, 975-991, 1968.

[9] LEISSA, A.W. and KADI, A.S., "Curvature Effects on Shallow Shell Vibrations", J. Sound Vib. 16, 173- 187,1971.

[10] BIRMAN, V. and BERT, C.W., "Non-Linear Beam-Type Vibrations of Long Cylindrical Shells", Int. J. Non-linear Mech. 22, 327-334, 1987.

[11] RAOUF, R.A. and PALAZOTTO, A.N., "On the Non-Linear Free Vibrations of Curved Orthotropic Panels", Int. J. Non-linear Mech. 29, 507-514, 1994.

[12] KOBAYASHI, Y. and LEISSA, A.W., "Large Amplitude Free Vibration of Thick Shallow Shells Supported by Shear Diaphragms", Int. J. Non-linear Mech. 30, 57-66, 1995.

[13] ATLURI, S., "A Perturbation Analysis of Non-linear Free Flexural Vibrations of Circular Cylindrical Shell", Int. J. Solids Struct. 8, 549-569, 1972.

[14] CHEN, J.C. and BABCOCK, C.D., "Nonlinear Vibrations of Cylindrical Shells", AIAA J. 13, 868-876, 1975.

[15] GINSBERG, J., "Nonlinear Resonant Vibrations of Infinitely Long Cylindrical Shells", AIAA J. 10, 979-980, 1972.

[16] RADWAN, H. and GENIN, J., "Non-linear modal Equations for Thin Elastic Shells", Int. J. Non-linear Mech. 10, 15-29, 1975.

[17] RAJU, K.K. and RAO, G.V., "Large Amplitude Asymmetric Vibrations of Some Thin Shells of Revolution", J. Sound Vib. 44, 327-333, 1976.

[18] BASAR, Y. and DING, Y., "Finite-Rotation Element For The Non-Linear Analysis of Thin Shell Structures", Int. J. Solids Struct. 26, 83-97, 1990.

[19] TSAI, C.T. and PALAZOTTO, A.N., "On the Finite Element Analysis of Non-Linear Vibration for Cylindrical Shells With High-Order Shear Deformation Theory", Int. J. Non-linear Mech. 26, 379-388, 1991.

[20] JIANG, J. and OLSEN, M.D, " Nonlinear Dynamic Analysis of Blast Loaded Cylindrical Shell Structures", Computer and Struc. 41, 41-52, 1991.

[21] AMBARTSUMYAN, S.A., "Theory of Anisotropic Shells", NASA-TT-F-118, 1961.

[22] SANDERS, J.L., "An Improved First Approximation Theory for Thin Shells", NASA-TR-R24, 1959.

[23] SELMANE, A. and LAKIS, A.A., "Dynamic Analysis of Anisotropic Open Cylindrical Shells", Tech. Report, No. EPM/RT-95/10, Ecole Polytechnique de Montréal, Canada, 1995.

APPENDIX A-1

EQUATIONS OF MOTION

This appendix contains the equations of motion for a thin cylindrical anisotropic shell which were referred to in this paper. The appendix is divided into two parts : part one deals with the linear system operators and part two, with the non-linear.

a) Equations of motion for a cylindrical shell : linear system

$$L_1(U,V,W,P_{ij}) = p_{11}\frac{\partial^2 U}{\partial x^2} + \frac{p_{12}}{R}\left(\frac{\partial^2 V}{\partial x\partial\theta} + \frac{\partial W}{\partial x}\right) - p_{14}\frac{\partial^3 W}{\partial x^3} + \frac{p_{15}}{R^2}\left(-\frac{\partial^3 W}{\partial x\partial\theta^2} + \frac{\partial^2 V}{\partial x\partial\theta}\right) +$$

$$\left(\frac{p_{33}}{R} - \frac{p_{63}}{2R^2}\right)\left(\frac{\partial^2 V}{\partial x\partial\theta} + \frac{1}{R}\frac{\partial^2 U}{\partial\theta^2}\right) + \left(\frac{p_{36}}{R^2} - \frac{p_{66}}{2R^3}\right)\left(-\frac{2\partial^3 W}{\partial x\partial\theta^2} + \frac{3}{2}\frac{\partial^2 V}{\partial x\partial\theta} - \frac{1}{2R}\frac{\partial^2 U}{\partial\theta^2}\right)$$

$$L_2(U,V,W,P_{ij}) = \left(\frac{p_{21}}{R} + \frac{p_{51}}{R^2}\right)\frac{\partial^2 U}{\partial x\partial\theta} + \frac{1}{R}\left(\frac{p_{22}}{R} + \frac{p_{52}}{R^2}\right)\left(\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial W}{\partial\theta}\right) - \left(\frac{p_{24}}{R} + \frac{p_{54}}{R^2}\right)$$

$$\left(\frac{\partial^3 W}{\partial x^2\partial\theta}\right) + \frac{1}{R^2}\left(\frac{p_{25}}{R} + \frac{p_{55}}{R^2}\right)\left(-\frac{\partial^3 W}{\partial\theta^3} + \frac{\partial^2 V}{\partial\theta^2}\right) + \left(p_{33} + \frac{3p_{63}}{2R}\right)\left(\frac{\partial^2 V}{\partial x^2} + \frac{1}{R}\frac{\partial^2 U}{\partial x\partial\theta}\right) +$$

$$\frac{1}{R}\left(p_{36} + \frac{3p_{66}}{2R}\right)\left(-2\frac{\partial^3 W}{\partial x^2\partial\theta} + \frac{3}{2}\frac{\partial^2 V}{\partial x^2} - \frac{1}{2R}\frac{\partial^2 U}{\partial x\partial\theta}\right)$$

$$L_3(U,V,W,P_{ij}) = p_{41}\frac{\partial^3 U}{\partial x^3} + \frac{p_{42}}{R}\left(\frac{\partial^3 V}{\partial x^2\partial\theta} + \frac{\partial^2 W}{\partial x^2}\right) - p_{44}\frac{\partial^4 W}{\partial x^4} + \frac{p_{45}}{R^2}$$

$$\left(-\frac{\partial^4 W}{\partial x^2\partial\theta^2} + \frac{\partial^3 V}{\partial x^2\partial\theta}\right) + \frac{2p_{63}}{R}\left(\frac{\partial^3 V}{\partial x^2\partial\theta} + \frac{1}{R}\frac{\partial^3 U}{\partial x\partial\theta^2}\right) + \frac{2p_{66}}{R^2}\left(-\frac{2\partial^4 W}{\partial x^2\partial\theta^2} + \right.$$

$$\left.\frac{3}{2}\frac{\partial^3 V}{\partial x^2\partial\theta} - \frac{1}{2R}\frac{\partial^3 U}{\partial x\partial\theta^2}\right) + \frac{p_{51}}{R^2}\frac{\partial^3 U}{\partial x\partial\theta^2} + \frac{p_{52}}{R^3}\left(\frac{\partial^3 V}{\partial\theta^3} + \frac{\partial^2 W}{\partial\theta^2}\right) + \frac{p_{55}}{R^4}\left(-\frac{\partial^4 W}{\partial\theta^4} + \frac{\partial^3 V}{\partial\theta^3}\right)$$

$$-\frac{p_{54}}{R^2}\frac{\partial^4 W}{\partial x^2\partial\theta^2}$$

b) <u>Equations of motion for a cylindrical shell : non-linear system</u>

$$N_1(U,V,W,P_{ij}) = p_{11}\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x^2} + \frac{p_{12}}{R^2}\left(\frac{\partial W}{\partial\theta}\frac{\partial^2 W}{\partial x\partial\theta} - V\frac{\partial^2 W}{\partial x\partial\theta} - \frac{\partial W}{\partial\theta}\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial x}\right) +$$

$$\frac{1}{R}\left(\frac{p_{33}}{R} - \frac{p_{63}}{2R^2}\right)\cdot\left(\frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial\theta^2} + \frac{\partial W}{\partial\theta}\frac{\partial^2 W}{\partial x\partial\theta} - V\frac{\partial^2 W}{\partial x\partial\theta} - \frac{\partial W}{\partial x}\frac{\partial V}{\partial\theta}\right) + (p_{11} + p_{12})\cdot$$

$$\left[\frac{1}{4}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 V}{\partial x^2} + \frac{1}{4R^2}\frac{\partial U}{\partial\theta}\cdot\frac{\partial^2 U}{\partial x\partial\theta} - \frac{1}{4R}\frac{\partial U}{\partial\theta}\cdot\frac{\partial^2 V}{\partial x^2} - \frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 U}{\partial x\partial\theta}\right] -$$

$$\left(\frac{p_{11} + p_{21}}{4R}\right)\cdot\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial\theta}\right]\cdot\left[\frac{\partial^2 U}{\partial x\partial\theta} + \frac{\partial W}{\partial x}\cdot\frac{\partial^2 W}{\partial x\partial\theta}\right] - \left(\frac{p_{12} + p_{22}}{4R}\right)\cdot$$

$$\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial\theta}\right]\cdot\left[\frac{1}{R}\frac{\partial^2 V}{\partial\theta^2} + \frac{1}{R}\frac{\partial W}{\partial\theta} + \frac{1}{R^2}\frac{\partial W}{\partial\theta}\cdot\frac{\partial^2 W}{\partial\theta^2} - \frac{V}{R^2}\frac{\partial^2 W}{\partial\theta^2} - \frac{1}{R^2}\frac{\partial W}{\partial\theta}\cdot\frac{\partial V}{\partial\theta} +\right.$$

$$\frac{V}{R^2} \cdot \frac{\partial V}{\partial \theta}\Bigg] + \left(\frac{p_{14} + p_{24}}{4R}\right) \cdot \left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot \left[\frac{\partial^3 W}{\partial x^2 \partial \theta}\right] - \left(\frac{p_{15} + p_{25}}{4R}\right) \cdot$$

$$\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot \left[- \frac{1}{R^2}\frac{\partial^3 W}{\partial \theta^3} + \frac{1}{R^2}\frac{\partial^2 V}{\partial \theta^2}\right] - \left(\frac{p_{11} + p_{21} + p_{12} + p_{22}}{4R}\right) \cdot$$

$$\left[\frac{\partial V}{\partial x} - \frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot \Bigg[\frac{1}{4}\frac{\partial V}{\partial x} \cdot \frac{\partial^2 V}{\partial x \partial \theta} + \frac{1}{4R^2}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 U}{\partial \theta^2} - \frac{1}{4R}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 V}{\partial x \partial \theta} -$$

$$\frac{1}{4R}\frac{\partial V}{\partial x}\frac{\partial^2 U}{\partial \theta^2}\Bigg] - \left(\frac{p_{11} + p_{21}}{4R}\right) \cdot \left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \left[\frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right] -$$

$$\left(\frac{p_{12} + p_{22}}{4R}\right) \cdot \left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \Bigg[\frac{1}{R}\frac{\partial V}{\partial \theta} + \frac{W}{R} + \frac{1}{2R^2}\left(\frac{\partial W}{\partial \theta}\right)^2 - \frac{V}{R^2}\frac{\partial W}{\partial \theta} +$$

$$\frac{V^2}{2R^2}\Bigg] + \left(\frac{p_{14} + p_{24}}{4R}\right) \cdot \left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \left[\frac{\partial^2 W}{\partial x^2}\right] - \left(\frac{p_{15} + p_{25}}{4R}\right) \cdot$$

$$\left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \left[- \frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2} + \frac{1}{R^2}\frac{\partial V}{\partial \theta}\right] - \left(\frac{p_{11} + p_{21} + p_{12} + p_{22}}{4R}\right) \cdot$$

$$\left[\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2}\right] \cdot \left[\frac{1}{8}\left(\frac{\partial V}{\partial x}\right)^2 + \frac{1}{8R^2}\left(\frac{\partial U}{\partial \theta}\right)^2 - \frac{1}{4R}\frac{\partial V}{\partial x} \cdot \frac{\partial U}{\partial \theta}\right]$$

$$N_2(U,V,W,P_{ij}) = (\frac{p_{21}}{R} + \frac{p_{51}}{R^2}) \frac{\partial W}{\partial x} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{1}{R^2} (\frac{p_{22}}{R} + \frac{p_{52}}{R^2}) (\frac{\partial W}{\partial \theta} \frac{\partial^2 W}{\partial \theta^2} - V \frac{\partial^2 W}{\partial \theta^2} -$$

$$\frac{\partial W}{\partial \theta} \frac{\partial V}{\partial \theta} + V \frac{\partial V}{\partial \theta}) + \frac{1}{R} (^{p_{33}} + \frac{3p_{63}}{2R}) (\frac{\partial W}{\partial x} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{\partial W}{\partial \theta} \frac{\partial^2 W}{\partial x^2} - V \frac{\partial^2 W}{\partial x^2} - \frac{\partial W}{\partial x} \frac{\partial V}{\partial x}) +$$

$$(\frac{p_{21} + p_{22}}{R} + \frac{p_{51} + p_{52}}{R^2}) \cdot \left[\frac{1}{4} \frac{\partial V}{\partial x} \cdot \frac{\partial^2 V}{\partial x \partial \theta} + \frac{1}{4R^2} \frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 U}{\partial \theta^2} - \frac{1}{4R} \frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{4R} \frac{\partial V}{\partial x} \cdot \right.$$

$$\left. \frac{\partial^2 U}{\partial \theta^2} \right] + \frac{p_{33}}{R} \cdot \left[\frac{\partial W}{\partial x}\right] \cdot \left[\frac{\partial V}{\partial x} + \frac{1}{R} \frac{\partial U}{\partial \theta} + \frac{1}{R} \frac{\partial W}{\partial x} \cdot \frac{\partial W}{\partial \theta} - \frac{V}{R} \cdot \frac{\partial W}{\partial x}\right] + \frac{p_{36}}{R} \cdot$$

$$\left[\frac{\partial W}{\partial x}\right] \cdot \left[- \frac{2}{R} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{3}{2R} \frac{\partial V}{\partial x} - \frac{1}{2R^2} \frac{\partial U}{\partial \theta}\right] + \left[(\frac{p_{11} + p_{21}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - \right.$$

$$\left. \frac{p_{21}}{R} \cdot (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R}) \right] \cdot \left[\frac{\partial U}{\partial x} + \frac{1}{2} (\frac{\partial W}{\partial x})^2\right] + \left[(\frac{p_{12} + p_{22}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - \right.$$

$$\left. \frac{p_{22}}{R} \cdot (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R}) \right] \cdot \left[\frac{1}{R} \cdot \frac{\partial V}{\partial \theta} + \frac{W}{R} + \frac{1}{2R^2} (\frac{\partial W}{\partial \theta})^2 - \frac{V}{R^2} \frac{\partial W}{\partial \theta} + \frac{V^2}{2R^2}\right] +$$

$$\left[\frac{p_{24}}{R} \cdot (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R}) - (\frac{p_{14} + p_{24}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta})\right] \cdot \left[\frac{\partial^2 W}{\partial x^2}\right] +$$

$$\left[(\frac{p_{15} + p_{25}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - \frac{p_{25}}{R} (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R})\right] \cdot \left[\frac{-1}{R^2} \frac{\partial^2 W}{\partial \theta^2} + \right.$$

$$\left. \frac{1}{R^2} \frac{\partial V}{\partial \theta} \right] + \left[(\frac{p_{11} + p_{21} + p_{12} + p_{22}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - (\frac{p_{21} + p_{22}}{R}) \cdot \right.$$

$$N_2(U,V,W,P_{ij}) = (\frac{P_{21}}{R} + \frac{P_{51}}{R^2}) \frac{\partial W}{\partial x} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{1}{R^2} (\frac{P_{22}}{R} + \frac{P_{52}}{R^2}) (\frac{\partial W}{\partial \theta} \frac{\partial^2 W}{\partial \theta^2} - V \frac{\partial^2 W}{\partial \theta^2} -$$

$$\frac{\partial W}{\partial \theta} \frac{\partial V}{\partial \theta} + V \frac{\partial V}{\partial \theta}) + \frac{1}{R} (P_{33} + \frac{3P_{63}}{2R}) (\frac{\partial W}{\partial x} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{\partial W}{\partial \theta} \frac{\partial^2 W}{\partial x^2} - V \frac{\partial^2 W}{\partial x^2} - \frac{\partial W}{\partial x} \frac{\partial V}{\partial x}) +$$

$$(\frac{P_{21} + P_{22}}{R} + \frac{P_{51} + P_{52}}{R^2}) \cdot \left[\frac{1}{4} \frac{\partial V}{\partial x} \cdot \frac{\partial^2 V}{\partial x \partial \theta} + \frac{1}{4R^2} \frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 U}{\partial \theta^2} - \frac{1}{4R} \frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{4R} \frac{\partial V}{\partial x} \cdot \right.$$

$$\left. \frac{\partial^2 U}{\partial \theta^2}\right] + \frac{P_{33}}{R} \cdot \left[\frac{\partial W}{\partial x}\right] \cdot \left[\frac{\partial V}{\partial x} + \frac{1}{R} \frac{\partial U}{\partial \theta} + \frac{1}{R} \frac{\partial W}{\partial x} \cdot \frac{\partial W}{\partial \theta} - \frac{V}{R} \cdot \frac{\partial W}{\partial x}\right] + \frac{P_{36}}{R} \cdot$$

$$\left[\frac{\partial W}{\partial x}\right] \cdot \left[- \frac{2}{R} \frac{\partial^2 W}{\partial x \partial \theta} + \frac{3}{2R} \frac{\partial V}{\partial x} - \frac{1}{2R^2} \frac{\partial U}{\partial \theta}\right] + \left[(\frac{P_{11} + P_{21}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - \right.$$

$$\left. \frac{P_{21}}{R} \cdot (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R})\right] \cdot \left[\frac{\partial U}{\partial x} + \frac{1}{2} (\frac{\partial W}{\partial x})^2\right] + \left[(\frac{P_{12} + P_{22}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - \right.$$

$$\left. \frac{P_{22}}{R} \cdot (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R})\right] \cdot \left[\frac{1}{R} \cdot \frac{\partial V}{\partial \theta} + \frac{W}{R} + \frac{1}{2R^2} (\frac{\partial W}{\partial \theta})^2 - \frac{V}{R^2} \frac{\partial W}{\partial \theta} + \frac{V^2}{2R^2}\right] +$$

$$\left[\frac{P_{24}}{R} \cdot (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R}) - (\frac{P_{14} + P_{24}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta})\right] \cdot \left[\frac{\partial^2 W}{\partial x^2}\right] +$$

$$\left[(\frac{P_{15} + P_{25}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - \frac{P_{25}}{R} (- \frac{1}{R} \frac{\partial W}{\partial \theta} + \frac{V}{R})\right] \cdot \left[\frac{-1}{R^2} \frac{\partial^2 W}{\partial \theta^2} + \right.$$

$$\left. \frac{1}{R^2} \frac{\partial V}{\partial \theta}\right] + \left[(\frac{P_{11} + P_{21} + P_{12} + P_{22}}{4}) \cdot (\frac{\partial^2 V}{\partial x^2} - \frac{1}{R} \frac{\partial^2 U}{\partial x \partial \theta}) - (\frac{P_{21} + P_{22}}{R}) \cdot \right.$$

$$\left(-\frac{1}{R}\frac{\partial W}{\partial \theta}+\frac{V}{R}\right)\Bigg] \cdot \left[\frac{1}{8}\left(\frac{\partial V}{\partial x}\right)^2+\frac{1}{8R^2}\left(\frac{\partial U}{\partial \theta}\right)^2-\frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial U}{\partial \theta}\right]+\left(\frac{p_{11}+p_{21}}{4}\right) \cdot$$

$$\left[\frac{\partial V}{\partial x}-\frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot \left[\frac{\partial^2 U}{\partial x^2}+\frac{\partial W}{\partial x}\cdot\frac{\partial^2 W}{\partial x^2}\right]+\left(\frac{p_{12}+p_{22}}{4}\right) \cdot \left[\frac{\partial V}{\partial x}-\frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot$$

$$\left[\frac{1}{R}\frac{\partial^2 V}{\partial x\partial \theta}+\frac{1}{R}\frac{\partial W}{\partial x}+\frac{1}{R^2}\frac{\partial W}{\partial \theta}\cdot\frac{\partial^2 W}{\partial x\partial \theta}-\frac{V}{R^2}\frac{\partial^2 W}{\partial x\partial \theta}-\frac{1}{R^2}\cdot\frac{\partial W}{\partial \theta}\cdot\frac{\partial V}{\partial x}+\frac{V}{R^2}\cdot\frac{\partial V}{\partial x}\right] -$$

$$\left(\frac{p_{24}+p_{14}}{4}\right) \cdot \left[\frac{\partial V}{\partial x}-\frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot \left[\frac{\partial^3 W}{\partial x^3}\right]+\left(\frac{p_{15}+p_{25}}{4}\right) \cdot \left[\frac{\partial V}{\partial x}-\frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot$$

$$\left[-\frac{1}{R^2}\frac{\partial^3 W}{\partial x\partial \theta^2}+\frac{1}{R^2}\frac{\partial^2 V}{\partial x\partial \theta}\right]+\left(\frac{p_{11}+p_{21}+p_{12}+p_{22}}{4}\right) \cdot \left[\frac{\partial V}{\partial x}-\frac{1}{R}\frac{\partial U}{\partial \theta}\right] \cdot$$

$$\left[\frac{1}{4}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 V}{\partial x^2}+\frac{1}{4R^2}\frac{\partial U}{\partial \theta}\cdot\frac{\partial^2 U}{\partial x\partial \theta}-\frac{1}{4R}\frac{\partial U}{\partial \theta}\cdot\frac{\partial^2 V}{\partial x^2}-\frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 U}{\partial x\partial \theta}\right]$$

$$N_3(U,V,W,P_{ij}) = p_{41}\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2+\frac{\partial W}{\partial x}\frac{\partial^3 W}{\partial x^3}\right]+\frac{p_{42}}{R^2}\Bigg[\left(\frac{\partial^2 W}{\partial x\partial \theta}\right)^2+\frac{\partial W}{\partial \theta}\frac{\partial^3 W}{\partial x^2\partial \theta}-$$

$$\frac{\partial V}{\partial x}\frac{\partial^2 W}{\partial x\partial \theta}-V\frac{\partial^3 W}{\partial x^2\partial \theta}-\frac{\partial^2 W}{\partial x\partial \theta}\frac{\partial V}{\partial x}-\frac{\partial W}{\partial \theta}\frac{\partial^2 V}{\partial x^2}+\left(\frac{\partial V}{\partial x}\right)^2+V\frac{\partial^2 V}{\partial x^2}\Bigg]+\frac{2p_{63}}{R^2}$$

$$\Bigg[\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial \theta^2}+\frac{\partial W}{\partial x}\frac{\partial^3 W}{\partial x\partial \theta^2}+\left(\frac{\partial^2 W}{\partial x\partial \theta}\right)^2+\frac{\partial W}{\partial \theta}\frac{\partial^3 W}{\partial x^2\partial \theta}-\frac{\partial V}{\partial x}\frac{\partial^2 W}{\partial x\partial \theta}-V\frac{\partial^3 W}{\partial x^2\partial \theta}-$$

$$\frac{\partial^2 W}{\partial x^2}\frac{\partial V}{\partial \theta} - \frac{\partial W}{\partial x}\frac{\partial^2 V}{\partial x \partial \theta}\Bigg] + \frac{p_{51}}{R^2}\left[\left(\frac{\partial^2 W}{\partial x \partial \theta}\right)^2 + \frac{\partial W}{\partial x}\frac{\partial^3 W}{\partial x \partial \theta^2}\right] + (p_{41} + p_{42}) \cdot$$

$$\Bigg[\frac{1}{4}\left(\frac{\partial^2 V}{\partial x^2}\right)^2 + \frac{1}{4}\frac{\partial V}{\partial x} \cdot \frac{\partial^3 V}{\partial x^3} + \frac{1}{4R^2}\left(\frac{\partial^2 U}{\partial x \partial \theta}\right)^2 + \frac{1}{4R^2}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^3 U}{\partial x^2 \partial \theta} - \frac{1}{2R}\frac{\partial^2 U}{\partial x \partial \theta} \cdot$$

$$\frac{\partial^2 V}{\partial x^2} - \frac{1}{4R} \cdot \frac{\partial U}{\partial \theta} \cdot \frac{\partial^3 V}{\partial x^3} - \frac{1}{4R}\frac{\partial V}{\partial x} \cdot \frac{\partial^3 U}{\partial x^2 \partial \theta}\Bigg] + \frac{p_{52}}{R^2}\Bigg[\frac{1}{R^2}\left(\frac{\partial^2 W}{\partial \theta^2}\right)^2 + \frac{1}{R^2} \cdot \frac{\partial W}{\partial \theta} \cdot$$

$$\frac{\partial^3 W}{\partial \theta^3} - \frac{1}{R^2}\frac{\partial V}{\partial \theta} \cdot \frac{\partial^2 W}{\partial \theta^2} - \frac{V}{R^2}\frac{\partial^3 W}{\partial \theta^3} - \frac{1}{R^2} \cdot \frac{\partial^2 W}{\partial \theta^2} \cdot \frac{\partial V}{\partial \theta} - \frac{1}{R^2}\frac{\partial W}{\partial \theta} \cdot \frac{\partial^2 V}{\partial \theta^2} + \frac{1}{R^2}\left(\frac{\partial V}{\partial \theta}\right)^2 +$$

$$\frac{V}{R^2}\frac{\partial^2 V}{\partial \theta^2}\Bigg] + \left(\frac{p_{51} + p_{52}}{R^2}\right) \cdot \Bigg[\frac{1}{4}\left(\frac{\partial^2 V}{\partial x \partial \theta}\right)^2 + \frac{1}{4}\frac{\partial V}{\partial x} \cdot \frac{\partial^3 V}{\partial x \partial \theta^2} + \frac{1}{4R^2}\left(\frac{\partial^2 U}{\partial \theta^2}\right)^2 +$$

$$\frac{1}{4R^2} \cdot \frac{\partial U}{\partial \theta} \cdot \frac{\partial^3 U}{\partial \theta^3} - \frac{1}{2R}\frac{\partial^2 U}{\partial \theta^2} \cdot \frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{4R}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^3 V}{\partial x \partial \theta^2} - \frac{1}{4R}\frac{\partial V}{\partial x} \cdot \frac{\partial^3 U}{\partial \theta^3}\Bigg] +$$

$$p_{11}\left[\frac{\partial W}{\partial x}\right] \cdot \left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial W}{\partial x} \cdot \frac{\partial^2 W}{\partial x^2}\right] + p_{12} \cdot \left[\frac{\partial W}{\partial x}\right] \cdot \Bigg[\frac{1}{R}\frac{\partial^2 V}{\partial x \partial \theta} + \frac{1}{R}\frac{\partial W}{\partial x} +$$

$$\frac{1}{R^2}\frac{\partial W}{\partial \theta} \cdot \frac{\partial^2 W}{\partial x \partial \theta} - \frac{V}{R^2} \cdot \frac{\partial^2 W}{\partial x \partial \theta} - \frac{1}{R^2}\frac{\partial W}{\partial \theta} \cdot \frac{\partial V}{\partial x} + \frac{V}{R^2} \cdot \frac{\partial V}{\partial x}\Bigg] - p_{14}\left[\frac{\partial W}{\partial x}\right] \cdot$$

$$\left[\frac{\partial^3 W}{\partial x^3}\right] + p_{15}\left[\frac{\partial W}{\partial x}\right] \cdot \left[-\frac{1}{R^2}\frac{\partial^3 W}{\partial x \partial \theta^2} + \frac{1}{R^2}\frac{\partial^2 V}{\partial x \partial \theta}\right] + (p_{11} + p_{12}) \cdot \left[\frac{\partial W}{\partial x}\right] \cdot$$

$$\left[\frac{1}{4}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 V}{\partial x^2}+\frac{1}{4R^2}\frac{\partial U}{\partial\theta}\cdot\frac{\partial^2 U}{\partial x\partial\theta}-\frac{1}{4R}\frac{\partial U}{\partial\theta}\cdot\frac{\partial^2 V}{\partial x^2}-\frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial^2 U}{\partial x\partial\theta}\right]$$

$$\left[\frac{\partial U}{\partial x}+\frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right]\cdot\left[-p_{11}\frac{\partial^2 W}{\partial x^2}+\frac{p_{21}}{R}\left(-\frac{1}{R}\frac{\partial^2 W}{\partial\theta^2}+\frac{1}{R}\frac{\partial V}{\partial\theta}+1\right)\right]-\left[\frac{1}{R}\frac{\partial V}{\partial\theta}+\right.$$

$$\left.\frac{W}{R}+\frac{1}{2R^2}\left(\frac{\partial W}{\partial\theta}\right)^2-\frac{V}{R^2}\frac{\partial W}{\partial\theta}+\frac{V^2}{2R^2}\right]\cdot\left[-p_{12}\frac{\partial^2 W}{\partial x^2}+\frac{p_{22}}{R}\left(-\frac{1}{R}\cdot\frac{\partial^2 W}{\partial\theta^2}+\frac{1}{R}\frac{\partial V}{\partial\theta}+\right.\right.$$

$$\left.\left.1\right)\right]-\left[\frac{1}{8}\left(\frac{\partial V}{\partial x}\right)^2+\frac{1}{8R^2}\left(\frac{\partial U}{\partial\theta}\right)^2-\frac{1}{4R}\frac{\partial V}{\partial x}\cdot\frac{\partial U}{\partial\theta}\right]\cdot\left[-(p_{11}+p_{12})\cdot\right.$$

$$\left.\left(\frac{\partial^2 W}{\partial x^2}\right)+\left(\frac{p_{21}+p_{22}}{R}\right)\cdot\left(-\frac{1}{R}\frac{\partial^2 W}{\partial\theta^2}+\frac{1}{R}\frac{\partial V}{\partial\theta}+1\right)\right]+\frac{\partial^2 W}{\partial x^2}\cdot\left[-p_{14}\frac{\partial^2 W}{\partial x^2}+\right.$$

$$\left.\frac{p_{24}}{R}\left(-\frac{1}{R}\frac{\partial^2 W}{\partial\theta^2}+\frac{1}{R}\frac{\partial V}{\partial\theta}\right)\right]-\left[-\frac{1}{R^2}\frac{\partial^2 W}{\partial\theta^2}+\frac{1}{R^2}\frac{\partial V}{\partial\theta}\right]\cdot\left[-p_{15}\frac{\partial^2 W}{\partial x^2}+\right.$$

$$\left.\frac{p_{25}}{R}\left(-\frac{1}{R}\frac{\partial^2 W}{\partial\theta^2}+\frac{1}{R}\frac{\partial V}{\partial\theta}\right)\right]-p_{33}\cdot\left[-\frac{1}{R}\frac{\partial W}{\partial\theta}+\frac{V}{R}\right]\cdot\left[\frac{\partial^2 V}{\partial x^2}+\frac{1}{R}\frac{\partial^2 U}{\partial x\partial\theta}+\right.$$

$$\left.\frac{1}{R}\frac{\partial W}{\partial x}\cdot\frac{\partial^2 W}{\partial x\partial\theta}+\frac{1}{R}\frac{\partial W}{\partial\theta}\cdot\frac{\partial^2 W}{\partial x^2}-\frac{V}{R}\cdot\frac{\partial^2 W}{\partial x^2}-\frac{1}{R}\frac{\partial W}{\partial x}\cdot\frac{\partial V}{\partial x}\right]-p_{36}\left[-\frac{1}{R}\frac{\partial W}{\partial\theta}+\right.$$

$$\left.\frac{V}{R}\right]\cdot\left[-\frac{2}{R}\frac{\partial^3 W}{\partial x^2\partial\theta}+\frac{3}{2R}\frac{\partial^2 V}{\partial x^2}-\frac{1}{2R^2}\frac{\partial^2 U}{\partial x\partial\theta}\right]-\left[-\frac{2}{R}\frac{\partial^2 W}{\partial x\partial\theta}+\frac{1}{R}\frac{\partial V}{\partial x}\right]\cdot$$

$$\left[p_{33}\left(\frac{\partial V}{\partial x}+\frac{1}{R}\frac{\partial U}{\partial\theta}+\frac{1}{R}\frac{\partial W}{\partial x}\cdot\frac{\partial W}{\partial\theta}-\frac{V}{R}\cdot\frac{\partial W}{\partial x}\right)+p_{36}\left(-\frac{2}{R}\frac{\partial^2 W}{\partial x\partial\theta}+\frac{3}{2R}\frac{\partial V}{\partial x}-\right.\right.$$

$$\frac{1}{2R^2}\frac{\partial U}{\partial \theta})\Bigg] + \frac{P_{33}}{R}\left[\frac{\partial W}{\partial x}\right] \cdot \left[\frac{\partial^2 V}{\partial x \partial \theta} + \frac{1}{R}\frac{\partial^2 U}{\partial \theta^2} + \frac{1}{R}\frac{\partial W}{\partial x} \cdot \frac{\partial^2 W}{\partial \theta^2} + \frac{1}{R}\frac{\partial W}{\partial \theta} \cdot \frac{\partial^2 W}{\partial x \partial \theta} -\right.$$

$$\left.\frac{V}{R} \cdot \frac{\partial^2 W}{\partial x \partial \theta} - \frac{1}{R}\frac{\partial W}{\partial x} \cdot \frac{\partial V}{\partial \theta}\right] + \frac{P_{36}}{R}\left[\frac{\partial W}{\partial x}\right] \cdot \left[-\frac{2}{R}\frac{\partial^3 W}{\partial x \partial \theta^2} + \frac{3}{2R}\frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{2R^2}\frac{\partial^2 U}{\partial \theta^2}\right] -$$

$$\frac{P_{21}}{R}\left[-\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{V}{R}\right] \cdot \left[\frac{\partial^2 U}{\partial x \partial \theta} + \frac{\partial W}{\partial x} \cdot \frac{\partial^2 W}{\partial x \partial \theta}\right] - \frac{P_{22}}{R} \cdot \left[-\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{V}{R}\right] \cdot$$

$$\left[\frac{1}{R}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{1}{R^2}\frac{\partial W}{\partial \theta} \cdot \frac{\partial^2 W}{\partial \theta^2} - \frac{V}{R^2} \cdot \frac{\partial^2 W}{\partial \theta^2} - \frac{1}{R^2}\frac{\partial W}{\partial \theta} \cdot \frac{\partial V}{\partial \theta} + \frac{V}{R^2}\frac{\partial V}{\partial \theta}\right] +$$

$$\frac{P_{24}}{R}\left[-\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{V}{R}\right] \cdot \left[\frac{\partial^3 W}{\partial x^2 \partial \theta}\right] - \frac{P_{25}}{R}\left[-\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{V}{R}\right] \cdot \left[-\frac{1}{R^2}\frac{\partial^3 W}{\partial \theta^3} +\right.$$

$$\left.\frac{1}{R^2}\frac{\partial^2 V}{\partial \theta^2}\right] - \left(\frac{P_{21} + P_{22}}{R}\right)\left[-\frac{1}{R}\frac{\partial W}{\partial \theta} + \frac{V}{R}\right] \cdot \left[\frac{1}{4}\frac{\partial V}{\partial x} \cdot \frac{\partial^2 V}{\partial x \partial \theta} + \frac{1}{4R^2}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 U}{\partial \theta^2} -\right.$$

$$\left.\frac{1}{4R}\frac{\partial U}{\partial \theta} \cdot \frac{\partial^2 V}{\partial x \partial \theta} - \frac{1}{4R}\frac{\partial V}{\partial x}\frac{\partial^2 U}{\partial \theta^2}\right]$$

APPENDIX A-2

The matrices referred to in the course of our analytical developments are given in this appendix.

The matrices are classified as follows:

$[T_m]$, $[R]$, $[A]$	(Table 1)
$[b^+]$, $[c^+]$	(Table 2)
$[B^+]$, $[C^+]$	(Table 3)
$[BB^{++}]$, $[CC^{++}]$, $[AB^{++}]$, $[BA^{++}]$	(Table 4)
$[bB^{++}]$, $[cC^{++}]$, $[aB^{++}]$, $[bA^{++}]$	(Table 5)

TABLE 1

MATRICES $[T_m]$, [R] and [A]

MATRIX $[T_m]_{(3x3)}$

$$[T_m] = \mathrm{Diag}[\ \cos\bar{m}x,\ \sin\bar{m}x,\ \sin\bar{m}x\ ]$$
$$\bar{m} = m\pi/L$$

MATRIX $[R]_{(3x8)}$

$$\begin{aligned} R(1,j) &= \alpha_j e^{\eta_j\theta} \\ R(2,j) &= e^{\eta_j\theta}, \qquad j = 1,\ldots 8 \\ R(3,j) &= \beta_j e^{\eta_j\theta} \end{aligned}$$

MATRIX $[A]_{(8x8)}$

$$\begin{aligned} A(1,j) &= \alpha_j \\ A(2,j) &= 1 \\ A(3,j) &= \eta_j \\ A(4,j) &= \beta_j \\ A(5,j) &= \alpha_j e^{\eta_j\phi} \qquad j = 1,\ldots,8 \\ A(6,j) &= e^{\eta_j\phi} \\ A(7,j) &= \eta_j e^{\eta_j\phi} \\ A(8,j) &= \beta_j e^{\eta_j\phi} \end{aligned}$$

TABLE 2

MATRICES $[b^+]_{(1x8)}$ and $[c^+]_{(1x8)}$

$$\begin{Bmatrix} [b^+] \\ [c^+] \end{Bmatrix} = \begin{Bmatrix} [b^*] \\ [c^*] \end{Bmatrix} [A^{-1}]$$

Where:

$$b_p^* = b_p' e^{\eta_p \theta}, \quad b_p' = b_p^{(1)} \sin \bar{m}x, \quad b_p^{(1)} = \frac{\eta_p B_p + 1}{R}$$

$$c_p^* = c_p' e^{\eta_p \theta}, \quad c_p' = c_p^{(1)} \cos \bar{m}x, \quad c_p^{(1)} = \frac{\eta_p \alpha_p}{2R} + \frac{\bar{m}\beta_p}{2}$$

TABLE 3

MATRICES $[B^{+}]_{(8x8)}$ and $[C^{+}]_{(8x8)}$

$$\left\{ \begin{matrix} [B^{+}] \\ [C^{+}] \end{matrix} \right\} = [A^{-1}]^{T} \left\{ \begin{matrix} [B^{*}] \\ [C^{*}] \end{matrix} \right\} [A^{-1}]$$

Where:

$$B^{*}_{pq} = b^{\prime}_{pq} e^{(\eta_p + \eta_q)\theta}, \quad b^{\prime}_{pq} = b^{(1)}_{pq} \cos^2 \bar{m}x + b^{(2)}_{pq} \sin^2 \bar{m}x \ ,$$

$$b^{(1)}_{pq} = \frac{1}{8R^2} [R\bar{m}\beta_p - \alpha_p \eta_p][R\bar{m}\beta_q - \alpha_q \eta_q]$$

$$b^{(2)}_{pq} = \frac{1}{2R^2} [\eta_p - \beta_p][\eta_q - \beta_q]$$

$$C^{*}_{pq} = c^{\prime}_{pq} e^{(\eta_p + \eta_q)\theta}, \quad c^{\prime}_{pq} = c^{(1)}_{pq} \cos \bar{m}x \sin \bar{m}x ,$$

$$c^{(1)}_{pq} = \frac{\bar{m}}{4R} [\eta_p + \eta_q - \beta_p - \beta_q]$$

TABLE 4

MATRICES $[bB^{++}]$, $[cC^{++}]$, $[aB^{++}]$ and $[bA^{++}]$

$$\begin{Bmatrix} [bB^{++}] \\ [cC^{++}] \\ [aB^{++}] \\ [bA^{++}] \end{Bmatrix} = [A^{-1}]^T \begin{Bmatrix} [bB^{**}] \\ [cC^{**}] \\ [aB^{**}] \\ [bA^{**}] \end{Bmatrix} [A^{-1}]$$

where

$$bB^{**}(p,q) = \sum_{k=1}^{8} [b_p^{\prime} A_{pq}^{-1} b_{pk}^{\prime} + b_q^{\prime} A_{qk}^{-1} b_{kp}^{\prime} + b_k^{\prime} A_{kp}^{-1} b_{pq}^{\prime}] e^{(\eta_p + \eta_q + \eta_k)\theta}$$

$$cC^{**}(p,q) = \sum_{k=1}^{8} [c_p^{\prime} A_{pq}^{-1} c_{pk}^{\prime} + c_q^{\prime} A_{qk}^{-1} c_{kp}^{\prime} + c_k^{\prime} A_{kp}^{-1} c_{pq}^{\prime}] e^{(\eta_p + \eta_q + \eta_k)\theta}$$

$$aB^{**}(p,q) = \sum_{k=1}^{8} [a_q^{\prime} A_{qk}^{-1} b_{kp}^{\prime} + a_k^{\prime} A_{kp}^{-1} b_{pq}^{\prime} + b_p^{\prime} A_{pq}^{-1} a_{qk}^{\prime}] e^{(\eta_p + \eta_q + \eta_k)\theta}$$

$$bA^{**}(p,q) = \sum_{k=1}^{8} [b_q^{\prime} A_{qk}^{-1} a_{kp}^{\prime} + b_k^{\prime} A_{kp}^{-1} a_{pq}^{\prime} + a_p^{\prime} A_{pq}^{-1} b_{qk}^{\prime}] e^{(\eta_p + \eta_q + \eta_k)\theta}$$

TABLE 5

MATRICES [BB^{++}], [CC^{++}], [AB^{++}] and [BA^{++}]

$$\begin{Bmatrix} [BB^{++}] \\ [CC^{++}] \\ [AB^{++}] \\ [BA^{++}] \end{Bmatrix} = 2\,[A^{-1}]^{T} \begin{Bmatrix} [BB^{**}] \\ [CC^{**}] \\ [AB^{**}] \\ [BA^{**}] \end{Bmatrix} [A^{-1}]$$

Where:

$$BB^{**}(p,q) = \sum_{k=1}^{8} b'_{kq} \left[\sum_{l=1}^{8} b'_{pl} E_{lk} e^{(\eta_p + \eta_q + \eta_k + \eta_l)\theta} \right]$$

$$CC^{**}(p,q) = \sum_{k=1}^{8} c'_{kq} \left[\sum_{l=1}^{8} c'_{pl} E_{lk} e^{(\eta_p + \eta_q + \eta_k + \eta_l)\theta} \right]$$

$$AB^{**}(p,q) = \sum_{k=1}^{8} b'_{kq} \left[\sum_{l=1}^{8} a'_{pl} E_{lk} e^{(\eta_p + \eta_q + \eta_k + \eta_l)\theta} \right]$$

$$BA^{**}(p,q) = \sum_{k=1}^{8} a'_{kq} \left[\sum_{l=1}^{8} b'_{pl} E_{lk} e^{(\eta_p + \eta_q + \eta_k + \eta_l)\theta} \right]$$

APPENDIX A-3

LIST OF SYMBOLS

$a_{rs}^{(1)}$	:	coefficient determined by equation (37)
$b_{rs}^{(1)}$, $b_{rs}^{(2)}$	:	coefficients determined by equation (A-2.3)
$c_{rs}^{(1)}$	:	coefficient determined by equation (A-2.3)
A_p	:	motion amplitude
a_p, b_p, c_p	:	modal coefficients determined by equations (28, 29 and 30)
aA_{prs}, bB_{prs}, cC_{prs}, aB_{prs}, bA_{prs}	:	modal coefficients determined by equation (34)
A_{pq}, B_{pq}, C_{pq}	:	modal coefficients determined by equations (31, 32 and 33)
A_{prsq}, B_{prsq}, C_{prsq}, AB_{prsq}, BA_{prsq}	:	modal coefficients determined by equation (35)
E	:	Young's modulus of elasticity
f, g, h	:	spatial functions
f_p	:	function determined by equation (59)
G (p, q)	:	coefficient determined by equation (48)
L	:	total length of shell
m	:	axial mode
M_{xx}, $M_{\theta\theta}$, $M_{x\theta}$, $M_{\theta x}$, $\overline{M}_{x\theta}$	:	resultant moments of a cylindrical shell
n	:	circumferential mode
N_{xx}, $N_{\theta\theta}$, $N_{x\theta}$, $N_{\theta x}$, $\overline{N}_{x\theta}$	:	resultant constraints for a cylindrical shell
N	:	number of finite elements
p_{ij}	:	elements of the matrix of elasticity (i, j = 1, ..., 6)
R	:	radius of the shell
S(p,q)	:	coefficient determined by equation (52)
t	:	thickness of the shell
U, V, W	:	axial, tangential and radial displacements, respectively
x	:	the coordinate generator of the shell

α_p, β_p	:	determined by equation (18)
ϵ_{xx}, $\epsilon_{\theta\theta}$, $\epsilon_{x\theta}$	:	deformations of the surface of reference
E_{lk}	:	element of matrix E determined by equation (44)
κ_{xx}, $\kappa_{\theta\theta}$, $\kappa_{x\theta}$	:	rotations at the surface of reference
Λ_p^{NL2}	:	coefficient determined by equation (64)
Λ_p^{NL3}	:	coefficient determined by equation (65)
η_p	:	complex roots of characteristic equation (16)
$\bar{\phi}$, $\bar{\phi}_x$, $\bar{\phi}_\theta$	:	determined by equation (10)
ϕ	:	opening angle for one finite element
ϕ_T	:	opening angle for total shell
ν	:	Poisson's ratio
ρ	:	density of the shell
ω_L	:	linear frequency of free vibrations
ω_{NL}	:	non-linear frequency of free vibrations
θ	:	circumferential coordinates
τ	:	time related coordinates

LIST OF MATRICES

$[A]$	:	determined by equation (22)
$[B]$	:	determined by equation (33)
$\{a^+\}$	:	determined by equation (36)
$\{b^+\}$, $\{c^+\}$	:	given in Appendix A-2, Table 2
$[aA^{++}]$	:	determined by equation (40)
$[bB^{++}]$, $[cC^{++}]$, $[aB^+]$, $[bA^{++}]$	:	given in Appendix A-2, Table 5
$[AA^{++}]$	:	determined by equation (42)
$[BB^{++}]$, $[CC^{++}]$, $[AB^{++}]$, $[BA^{++}]$	:	given in Appendix A-2, Table 4
$[A^+]$	:	determined by equation (38)
$[B^+]$, $[C^+]$	:	given in Appendix A-2, Table 3
$\{C\}$	:	vector of arbitrary constraints
$[E]$	:	matrix function of [A] determined by equation (44)
$[k_L]$, $[k_{NL2}]$, $[k_{NL3}]$	:	linear and non-linear stiffness matrices for a finite element, respectively
$[k^*_{NL2}]$, $[k^*_{NL3}]$	:	determined by equation (47) and (51), respectively
$[K_L]$, $[K_{NL2}]$, $[K_{NL3}]$	:	linear and non-linear stiffness matrices, for the entire shell, respectively
$[K_L^{(r)}]$, $[K_{NL2}^{(r)}]$, $[K_{NL3}^{(r)}]$	:	reduced linear and non-linear stiffness matrices, for the entire shell, respectively
$[K_L^{(D)}]$	:	global diagonal linear stiffness matrix
$[m]$	:	mass matrix of a finite element
$[M]$	:	mass matrix for total shell
$[M^{(r)}]$	:	reduced mass matrix for total shell
$[M^{(D)}]$	:	diagonal mass matrix for total shell
$[N]$	:	determined by equation (24)
$[P]$	:	matrix of elasticity
$\{q\}$	:	time-related vector coordinates

[R]	:	determined, by equation (19)
[T]	:	determined by equation (13)
$\{\delta_i\}$	:	vector of degrees of freedom for node i
$\{\delta\}$	:	vector of degrees of freedom for total shell
$\{\delta^{(r)}\}$	:	reduced vector of degrees of freedom for total shell
$\{\epsilon\}$	:	deformation vector
$\{\epsilon_L\}$ $\{\epsilon_{NL}\}$	:	linear and non-linear components of the deformation vector, respectively
$[\Phi]$	:	matrix of eigenvectors
$\{\sigma\}$	:	stress vector

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