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HYBRID FINITE ELEMENT ANALYSIS OF CIRCULAR AND ANNULAR PLATES

by

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gratuit

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ABSTRACT

The objective of this work is to present a new method for the dynamic and static analysis of thin, elastic, isotropic, non-uniform circular and annular plates. The method is a combination of plate theory and finite element analysis. The plate is divided into one circular and many annular finite elements. The displacement functions are derived from Sanders' classical plate theory, which is based on Love's first approximation and gives zero strain for small rigid-body motions. These displacement functions satisfy the convergence criteria of the finite element method. The matrices for mass and stiffness are determined by precise analytical integration. The free vibration problem becomes a problem of eigenvalues and eigenvectors. A computer programme has been developed, the convergence criteria have been established, and the natural frequencies and vibration modes have been computed for different cases. The results obtained reveal that the frequencies calculated by this method are in good agreement with those obtained by other authors.

LIST OF SYMBOLS

a	outside radius of an annular or circular plate
a_0	inside radius of an annular plate
B_j (j =1,4)	constant in the equation of U
C_j (j=1,4)	constant in the equation of V
$\bar{C}_j$ (j=1,8)	elements of vector $\{\bar{C}\}$
D	stiffness membrane = $Et/(1 - \nu^2)$
E	Young's modulus
J	number of boundary conditions
K	bending stiffness = $Et^3/12(1 - \nu^2)$
Ln	Napierian Logarithm
m	number of radial mode
M_r, M_θ, $\bar{M}_{r\theta}$	torque components for a circular plate
n	number of circumferential mode
N	number of finite elements
N_r, N_θ, $\bar{N}_{r\theta}$	stress components for a circular plate
t	thickness of the plate
U, V, W	radial, tangential, transversal displacement
u_n, v_n, w_n	amplitude of U, V, W associated with the n^{th} circumferential mode number
y	coordinate defined by y = r/a
y_0	coordinate defined by $y_0 = a_0/a$
α_j (j=1,4)	defined by equation 10
ε_r, ε_θ, $\bar{\varepsilon}_{r\theta}$	deformation of the mean surface of a circular plate
θ	circumferential coordinate
κ_r, κ_θ, $\bar{\kappa}_{r\theta}$	change of curvature and twist of the mean surface of a circular plate

λ_j (j=1,4)	roots of the characteristic equation (8)
ν	Poisson's ratio
ρ	density of the material of the plate
ω	natural angular frequency
Ω	non-dimensional natural frequency $= \omega a^2 \left(\frac{\rho t}{K} \right)^{\frac{1}{2}}$

LIST OF MATRICES

$[A_n]$	defined by equation 14
$[A_{nc}]$	defined by equation 17
$[B_n]$	defined by equation 33
$[B_{nc}]$	defined by equation 42
$[BB_n]$	defined by equation 19
$[BB_{nc}]$	defined by equation 22
$\{\bar{C}\}$	arbitrary constants vector
$[C_n]$	defined by equation 33
$[C_{nc}]$	defined by equation 42
$[D_n]$	defined by equation 34
$[D_{nc}]$	defined by equation 43
$[E_n]$	defined by equation 34
$[E_{nc}]$	defined by equation 43
$[F_n]$	defined by equation 39
$[F_{nc}]$	defined by equation 46
$[G_n]$	defined by equation 33
$[G_{nc}]$	defined by equation 42
$[H]$	defined by equation 7
$[H_n]$	defined by equation 39
$[H_{nc}]$	defined by equation 46
$[k]$	elementary stiffness matrix
$[K]$	global stiffness matrix
$[m]$	elementary mass matrix
$[M]$	global mass matrix
$[P]$	elasticity matrix
$[Q_n]$	defined by equation 18
$[Q_{nc}]$	defined by equation 21
$[R_n]$	defined by equation 11

$[R_{nc}]$	defined by equation 17
$[S_n]$	defined by equation 37
$[S_{nc}]$	defined by equation 46
$[ST_n]$	defined by equation 25
$[ST_{nc}]$	defined by equation 27
$[T_n]$	defined by equation 11
$[T_{nc}]$	defined by equation 17
$\{\delta_i\}$	vector of degrees of freedom at node i
$\{\varepsilon\}$	deformation vector
$\{\sigma\}$	stress vector

1. INTRODUCTION

Circular plates are widely used in engineering. They are used, for example, by the aerospace and aeronautical industry in aircraft fuselage, rockets and turbo-jets; by the nuclear industry in reactor walls; by the marine industry for ship and submarine parts; by the petroleum industry in holding tanks; and by civil engineers in domes and thin shells. Knowledge of the free vibration characteristics of these structures is important not only for the researcher who wishes to understand their behaviour, but also for the engineer whose duty it is to foresee and to prevent any failure which may occur in the course of the industrial use of such structures.

Research into the vibration of circular plates began in the 18th century [1, 2, 3]: in 1766, Euler formulated the first mathematical approximation of plate membrane theory. The German physician Chladin later found the vibration modes, and Lagrange developed a differential equation for free vibrations. Kirchoff (1877) is considered to be the founder of the plate theory which combines membrane effects with bending by analyzing plates with large deflections. He concluded that non-linear expressions could not be ignored and showed that the natural frequencies of plates can be determined by the virtual work method. Love applied Kirchoff's work to thick plates, whilst Timochenko made significant contributions to the work on circular plates with large deflections.

Foppl, Volmin and Panov worked on non-linear plate theory. The final form of the differential equation for plates with large deflections was developed by Von Karman. Chien Wien-Zang introduced the perturbation method to solve equations for plates with large deflections. Hodge extended elastic plate theory into the plastic domain.

Leissa collected the work of several researchers into one excellent book [4], which provides approximately 500 references. More recently, Leissa and Narita [5] have studied the influence of Poisson's ratio on the natural frequencies of a simply-supported circular plate.

The Japanese Irie, Yamada and Aomura [6] have determined the natural frequencies of clamped, simply-supported and free plates for thickness to plate radius ratios varying from 0.01

to 0.25. They used Mindlin's theory which takes rotational inertia into account. Itao and Crandall [7] have calculated the 701 first modes of vibration of a free circular plate.

The analysis of non-uniform plates of varying thickness has been carried out by several authors. Celep [8] used the initial functions method to determine the first and second vibration modes. Irie and Yamaha [9] used a spline technique in studying annular plates of which the thickness varied linearly, parabolically and exponentially. They also used Ritz' method to solve plate systems numerically. Sato and Shimzu [10] used the transfer matrix method to study the linear and non-linear behaviour of plates of varying thickness.

Narita [11] and Gorman [12] studied anisotropic plates. Bolotin formulated the problem of the dynamic stability of mechanical systems, and Lepore and Shah [13] applied this theory to circular plates, while Tani and Nakamura [14] studied the dynamic stability of annular plates.

In order to analyze complex plates, it has been necessary to employ new methods, the best known of which is the finite element method. Numerous general computer programmes, such as NASTRAN, SAP, ADINA and ABAQUS, are available for the industrial use of the finite element method, principally in the domain of the mechanics of solids. In general, triangular and square elements are used [15], where the displacement functions are polynomial, although curved elements [16] have been found to account more precisely for the geometry of the surface. The analytical formulation of these elements is complex.

One of the most important criteria in determining the versatility of a method is the capacity to predict, with precision, both the high and the low frequencies. This criterion demands the use of a great many elements in the finite element method, and, in order to meet it, our research group has developed a new type of finite element, a hybrid wherein the displacement functions in the finite element method are derived from Sanders' classical shell theory [17]. This method has been applied with satisfactory results to the dynamic linear and non-linear analysis of cylindrical [18-25], conical [26] and spherical [27], isotropic and anisotropic, uniform and non-uniform shells, both empty and liquid-filled. This method also has the advantage of giving good low frequencies, as well as high, with a small number of finite elements.

The method used here is a combination of circular plate theory and finite element analysis: We first determine the plate equations; second, we derive the displacement functions of plate theory and determine the stiffness and mass matrices by the finite element method. In this part of the study, we develop two new types of finite element, the first type being a circular plate and the second an annular plate, for circumferential modes $n \geq 2$.

2. FUNDAMENTAL EQUATIONS

2.1 Equilibrium equations

To study the equilibrium of the plate, taking into account membrane effects as well as bending effects, we use Sanders' [17] equations. It should be remembered that these equations are based on Love's [28] first approximation, and show zero deformation due to rigid-body motion. This is not the case with other theories.

The geometry of the mean surface of the plate studied and the coordinates used are shown in Figure 1. The unit vectors of the membrane force, the shear forces and the moment with reference to the mean surface are indicated in Figure 2.

The equilibrium equations of the plate are:

$$r \frac{\partial N_r}{\partial r} + N_r + \frac{\partial \bar{N}_{r\theta}}{\partial \theta} - N_\theta = 0 \quad \text{(a)}$$

$$r \frac{\partial \bar{N}_{r\theta}}{\partial r} + 2\bar{N}_{r\theta} + \frac{\partial \bar{N}_\theta}{\partial \theta} = 0 \quad \text{(b)} \qquad (1)$$

$$r^2 \frac{\partial^2 M_r}{\partial r^2} + 2 \frac{\partial M_r}{\partial r} + 2 \frac{\partial^2 \bar{M}_{r\theta}}{\partial r \partial \theta} + \frac{2}{r} \frac{\partial \bar{M}_{r\theta}}{\partial \theta} + \frac{1}{r} \frac{\partial^2 M_\theta}{\partial \theta^2} - \frac{\partial M_\theta}{\partial \theta} = 0 \quad \text{(c)}$$

where N_r, N_θ, $\bar{N}_{r\theta}$, M_r, M_θ and $\bar{M}_{r\theta}$ are the stress components, and r and θ are the coordinates of the plate.

2.2 Kinematic relationships

The relationship between the deformation and the displacements for a circular plate can be written as follows:

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ 2\epsilon_{r\theta} \\ \kappa_r \\ \kappa_\theta \\ 2\bar{\kappa}_{r\theta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial U}{\partial r} \\ \frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{U}{r} \\ \frac{1}{r}\frac{\partial U}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{V}{r}\right) \\ -\frac{\partial^2 W}{\partial r^2} \\ -\frac{1}{r}\frac{\partial W}{\partial r} - \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} \\ -2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W}{\partial \theta}\right) \end{Bmatrix} \tag{2}$$

where U is the radial displacement, V is the tangential displacement and W is the transversal displacement.

2.3 Constitutive equations

For an isotropic and elastic material, the constitutive equations which link the stress components to the deformations are:

$$\begin{Bmatrix} N_r \\ N_\theta \\ \bar{N}_{r\theta} \\ M_r \\ M_\theta \\ \bar{M}_{r\theta} \end{Bmatrix} = [P] \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ 2\bar{\epsilon}_{r\theta} \\ \kappa_r \\ \kappa_\theta \\ 2\bar{\kappa}_{r\theta} \end{Bmatrix} \tag{3}$$

where [P] is the elasticity matrix given in Appendix A.

By substituting (2) and (3) in the equilibrium equations (1), we obtain new equations (4) in terms of the radial, tangential and transversal displacements (U, V, W) of the mean surface of

the plate:

$$rU'' + U' + \left(\frac{1-\nu}{2}\right)\frac{\ddot{U}}{r} - \frac{U}{r} + \left(\frac{1+\nu}{2}\right)\dot{V}' - \left(\frac{3-\nu}{2}\right)\frac{\dot{V}}{r} = 0 \quad \text{(a)}$$

$$\left(\frac{1+\nu}{2}\right)\dot{U}' + \left(\frac{3-\nu}{2}\right)\frac{U}{r} + \left(\frac{1-\nu}{2}\right) r\,V'' + \left(\frac{1-\nu}{2}\right)V' + \frac{\ddot{V}}{r} - \left(\frac{1-\nu}{2}\right)\frac{V}{r} = 0 \quad \text{(b)} \qquad (4)$$

$$W'''' + 2\frac{\ddot{W}''}{r^2} + \frac{\dddot{W}}{r^4} + 2\frac{W'''}{r} - 2\frac{\ddot{W}'}{r^3} - \frac{W''}{r^2} + 4\frac{\dot{W}'}{r^4} + \frac{W'}{r^3} = 0 \quad \text{(c)}$$

The terms (') and (˙) represent $[\partial(\ )/\partial r]$ and $[\partial(\ )/\partial\theta]$ respectively.

By solving these equations it is possible to derive the displacement functions in terms of the nodal displacements.

3. DISPLACEMENT FUNCTIONS

3.1 Characteristic equations

Two types of finite element will be developed, the first being an element of the circular plate type and the second an element of the annular plate type (Figure 3). In this way, circular plate theory can be used to determine displacement functions.

The nodal displacements are $(U, W, \frac{dW}{dr}, V)$ where U is the radial displacement, W is the transversal displacement, $\frac{dW}{dr}$ is the rotation and V is the tangential displacement.

As the plate is circular, the displacements are periodic as a function of θ and can be developed in a Fourier series:

$$\begin{aligned} U(r,\theta) &= u_n(r)\cos n\theta \\ W(r,\theta) &= w_n(r)\cos n\theta \\ V(r,\theta) &= v_n(r)\sin n\theta \end{aligned} \tag{5}$$

n is the number of the circumferential mode u_n, w_n, v_n are functions solely of r.

Examination of the equilibrium equations (1) reveals that (a) and (b) show the membrane effect while (c) shows the bending effect. These two groups of equations (a, b) and (c) are independent of each other and can be solved separately.

The generalised forms of the displacements u_n, v_n and w_n are:

$$\begin{aligned} & u_n = C\,y^{\frac{\lambda-1}{2}}, \quad v_n = B\,y^{\frac{\lambda-1}{2}} \\ & w_n = C_5 y^{n} + C_6 y^{-n} + C_7 y^{n+2} + C_8 y^{-n+2} \text{ (annular element)} \\ & w_n = C_3 y^{n} + C_4 y^{n+2} \text{ (circular element)} \end{aligned} \tag{6}$$

where λ is the solution to the characteristic equation.

$y = \frac{r}{a}$; r is the radius of the plate

a is the outside radius of the plate

By substituting (6) into (4a and 4b), we obtain a system of homogeneous equations in B and C of the form

$$[H]\begin{Bmatrix} C \\ B \end{Bmatrix} = \{0\} \tag{7}$$

where [H] is a second-order square matrix, the terms of which are functions of λ. This matrix is given in Appendix A. For a non-trivial solution, the determinant of [H] should be zero, giving the following characteristic equation:

$$\lambda^4 - 4\lambda^3 - 2(1+4n^2)\,\lambda^2 + 4(3+4n^2)\,\lambda + (9-40\,n^2+16n^4) \qquad (8)$$

This characteristic equation has 4 roots, all of which are real.

From the sum of the 4 values for λ_j, we can obtain the complete solution for U and V. Each value of λ_j involves two constants, B_j and C_j. As the values for B_j and C_j are not independent, one can express B_j as a function of C_j.

$$B_j = \alpha_j\, C_j \qquad (9)$$

By substituting (9) to (7), we may find α_j.

Thus we obtain

$$\alpha_j = \frac{-\frac{1}{4}(\lambda_j - 1)^2 + \left(\frac{1-\nu}{2}\right) n^2 + 1}{\left(\frac{1+\nu}{4}\right) n\,(\lambda_j - 1) - n\left(\frac{3-\nu}{2}\right)} \qquad (10)$$

3.2 Displacement functions for a finite element of annular plate type

In matrix form, displacements U, V and W can be written as follows:

$$\begin{Bmatrix} U \\ W \\ V \end{Bmatrix} = [T_n]\,[R_n]\,\{C\} \qquad (11)$$

where $[T_n]$ and $[R_n]$ are (3 x 3) matrices given in Appendix A and the vector $\{C\}$ is given by:

$$\{C\} = \begin{Bmatrix} \bar{C}_1 \\ \cdot \\ \cdot \\ \cdot \\ \bar{C}_8 \end{Bmatrix} \qquad (12)$$

Constants $\bar{C}_j$ are eliminated in favour of displacements at the nodes of the elements.

The displacement field at the node can be defined as:

$$\{\delta_i\} = \{u_{ni}, w_{ni}, \left(\frac{dw_n}{dr}\right)_i, v_{ni}\}^t \text{ at node i}$$

$$\{\delta_j\} = \{u_{nj}, w_{nj}, \left(\frac{dw_n}{dr}\right)_j, v_{nj}\}^t \text{ at node j} \tag{13}$$

$\{\delta_i\}$ and $\{\delta_j\}$ can be expressed as a function of constants $\{C_j\}$ in the following manner:

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [A_n] \; C \tag{14}$$

$[A_n]$ is the (8 x 8) matrix given in Appendix A.

Thus we have

$$\{C\} = [A_n]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{15}$$

Equation (11) becomes

$$\begin{Bmatrix} U \\ W \\ V \end{Bmatrix} = [T_n] \; [R_n] \; [A_n]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N_n] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{16}$$

where $[N_n]$ is the displacement matrix.

Since the circular plate element has a single node, the displacement field is defined by:

$$\{\delta_j\} = \{u_{nj}, w_{nj}, \left(\frac{dw_n}{dr}\right)_j, v_{nj}\}^t$$

The three displacements U, V and W can therefore be written as follows:

$$\begin{Bmatrix} U \\ W \\ V \end{Bmatrix} = [T_{nc}] \; [R_{nc}] \; [A_{nc}]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{17}$$

where $[T_{nc}]$ is the (3 x 3) matrix given in Appendix A.

$[R_{nc}]$ is the (3 x 4) matrix given in Appendix A.

$[A_{nc}]$ is the (4 x 4) matrix given in Appendix A.

4. DEFORMATION AND STRESS COMPONENT MATRICES

4.1 Deformation matrix for a finite element of the annular plate type

By substituting equation (16) in the kinematic equations (2), we obtain:

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ 2\epsilon_{r\theta} \\ \kappa_r \\ \kappa_\theta \\ 2\overline{\kappa}_{r\theta} \end{Bmatrix} = \begin{bmatrix} [T_n] & [0] \\ [0] & [T_n] \end{bmatrix} [Q_n]\ [A_n]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (18)$$

Matrices $[T_n]$, $[Q_n]$ and $[A_n]$ are given in Appendix A.

The deformation matrix is defined as follows:

$$[BB_n] = \begin{bmatrix} [T_n] & [0] \\ [0] & [T_n] \end{bmatrix} [Q_n]\ [A_n]^{-1} \qquad (19)$$

The deformation vector can be written:

$$\{\epsilon\} = [BB_n] = \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (20)$$

4.2 Deformation matrix for a finite element of the circular plate type

By substituting equations (17) in the kinematic equations (2), we obtain:

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ 2\epsilon_{r\theta} \\ \kappa_r \\ \kappa_\theta \\ 2\overline{\kappa}_{r\theta} \end{Bmatrix} = \begin{bmatrix} [T_{nc}] & [0] \\ [0] & [T_{nc}] \end{bmatrix} [Q_{nc}]\ [A_{nc}]^{-1}\ \{\delta_j\} \tag{21}$$

Matrices $[T_{nc}]$, $[Q_{nc}]$ and $[A_{nc}]$ are given in Appendix A.

The deformation matrix is defined as follows:

$$[BB_{nc}] = \begin{bmatrix} [T_{nc}] & [0] \\ [0] & [T_{nc}] \end{bmatrix} [Q_{nc}]\ [A_{nc}]^{-1} \tag{22}$$

4.3 Stress component matrix for a finite element of the annular plate type

By replacing the vector $\{\epsilon\}$ of equation (3) by the expression in (20), we obtain:

$$\{\sigma\} = [P]\ [BB_n] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{23}$$

The stress component matrix for the group of nodes i and j of the finite element is produced:

$$\begin{Bmatrix} \sigma_i \\ \sigma_j \end{Bmatrix} = \begin{bmatrix} [T_n] & & & [0] \\ & [T_n] & & \\ & & [T_n] & \\ [0] & & & [T_n] \end{bmatrix} \begin{bmatrix} [P]\ [Q_{ni}]\ [A_n]^{-1} \\ [P]\ [Q_{nj}]\ [A_n]^{-1} \end{bmatrix} \begin{Bmatrix} \delta \\ \delta \end{Bmatrix} \tag{24}$$

$$= [ST_n] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{25}$$

where the matrices $[Q_{ni}]$ and $[Q_{nj}]$ are obtained from matrix $[Q_n]$ (Appendix A), by substituting $r = a_0$ and $r = a$, respectively. Matrices $[T_n]$, $[P]$ and $[A_n]$ are given in Appendix A.

4.4 Stress component matrix for a finite element of the circular plate type

Since the finite element of the circular plate type has only one node, the stress component matrix is given by:

$$\{\sigma_j\} = [ST_{nc}]\,\{\delta_j\} \tag{26}$$

where

$$[ST_{nc}] = \begin{bmatrix} [T_{nc}] & [0] \\ [0] & [T_{nc}] \end{bmatrix} [P]\,[Q_{ncj}]\,[A_{nc}]^{-1} \tag{27}$$

$[Q_{ncj}]$ is the matrix $[Q_{nc}]$ (Appendix A) calculated for $r = a$. $[T_{nc}]$, $[P]$ and $[A_{nc}]$ are the matrices given in Appendix (A).

5. STIFFNESS AND MASS MATRICES

5.1 Annular plate

5.1.1 Stiffness matrix

In a system of local axes (r, θ), the stiffness matrix is obtained as follows [15]:

$$[k] = \int_s [BB_n]^t\,[P]\,[BB_n]\,dS \tag{28}$$

where dS is the surface element $= r\,dr\,d\theta$.

$[P]$ is the elasticity matrix given in Appendix A and $[BB_n]$ is the matrix defined by equation (19).

By replacing $[BB_n]$ by its expression in (19), the stiffness matrix can be written as follows:

$$[k] = \int_s [[A_n]^{-1}]^t\,[Q_n]^t\,[P]\,[Q_n]\,[A_n]^{-1}\,r\,rd\,d\theta \tag{29}$$

The integration with respect to θ gives:

$$[k] = [[A_n]^{-1}]^t \{ \pi \int_{a_0}^{a} [Q_n]^t [P] [Q_n] r \, dr \} [A_n]^{-1} \tag{30}$$

where a and a_0 are the outside and inside radii of the annular plate element. Expressing:

$$[G] = \pi \int_{a_0}^{a} [Q_n]^t [P] [Q_n] r \, dr \tag{31}$$

The stiffness matrix can be expressed as:

$$[k] = [[A_n]^{-1}]^t [G_n] [A_n]^{-1} \tag{32}$$

After integration of equation (31), we obtain:

$$\begin{cases} G_n(i,j) = \pi a^2 \sum_{k=1}^{6} \dfrac{D_n(i,k) \, B_n(k,j)}{E_n(i,k) + C_n(k,j) + 2} \left[1 - y_0^{E_n(i,k) + C_n(k,j) + 2} \right] \\ (i=1,8 \text{ and } j=1,8) \\ \text{if } E_n(i,k) + C_n(k,j) + 2 \neq 0 \end{cases}$$

$$\begin{cases} G_n(i,j) = -\pi a^2 \sum_{k=1}^{6} D_n(i,k) \, B_n(k,j) \, \mathrm{Ln} \, y_0 \\ (i=1,8 \text{ and } j=1,8) \\ \text{if } E_n(i,k) + C_n(k,j) + 2 = 0 \end{cases} \tag{33}$$

where

$$[D_n] = [B_n]^t\,[P]$$
$$[E_n] = [C_n]^t \tag{34}$$

$$y_0 = \frac{a}{a_0}$$

$[B_n]$ and $[C_n]$ are the two (6 x 8) matrices given in Appendix (A) and Ln is the Napierian Logarithm. [P] is the elasticity matrix given in Appendix A.

5.1.2 *Mass matrix*

In a system of local axes (r, θ), the mass matrix is defined as follows [15]:

$$[m] = \rho t \int_s [N_n]^t\,[N_n]\,dS \tag{35}$$

where

dS is the surface element = r dr dθ

ρ is the density of the finite element

t is the thickness of the finite element

$[N_n]$ is the matrix defined by (16)

By inserting (16) in (35) and integrating it with respect to θ, we obtain:

$$[m] = \rho t\,[[A_n]^{-1}]^t\,\{\pi \int_{a_0}^{a} [R_n]^t\,[R_n]\,r\,dr\}\,[A_n]^{-1} \tag{36}$$

where $[A_n]$ and $[R_n]$ are given in Appendix (A).

Letting

$$[S_n] = \pi \int_{a_0}^{a} [R_n]^t\,[R_n]\,r\,dr \tag{37}$$

The mass matrix can then be written as follows:

$$[m] = \rho\, t\, [[A_n]^{-1}]^t\, [S_n]\, [A_n]^{-1} \tag{38}$$

After integration of equation (37), we obtain:

$$\begin{cases} S_n(i,j) = \pi\, a^2 \sum_{k=1}^{3} \dfrac{F_n(i,k)\; F_n(k,j)}{H_n(k,i) + H_n(k,j) + 2} \left[1 - y_0^{\,H_n(k,i)+H_n(k,j)+2} \right] \\ (i=1,8 \text{ and } j=1,8) \\ \text{if } H_n(k,i) + H_n(k,j) + 2 \neq 0 \end{cases}$$

$$\begin{cases} S_n(i,j) = -\pi\, a^2 \sum_{k=1}^{3} F_n(k,i)\; F_n(k,j)\; \mathrm{Ln}\; y_0 \\ (i=1,8 \text{ and } j=1,8) \\ \text{if } H_n(k,i) + H_n(k,j) + 2 = 0 \end{cases} \tag{39}$$

where $[F_n]$ and $[H_n]$ are the (3 x 8) matrices given in Appendix A.

5.2 Circular Plate

5.2.1 Stiffness matrix

As the circular plate element has only one node, the stiffness matrix will be of the order (4 x 4). It is given by:

$$[k] = [[A_{nc}]^{-1}]^t\, [G_{nc}]\, [A_{nc}]^{-1} \tag{40}$$

where $[A_{nc}]$ is the (4 x 4) matrix given in Table (2) of Appendix A.

$$[G_{nc}] = \pi \int_0^a [Q_{nc}]^t\, [P]\, [Q_{nc}]\; r\; dr \tag{41}$$

"a" being the radius of the circular plate element.

After integration we obtain:

$$\begin{cases} G_{nc}(i,j) = \pi\, a^2 \sum_{k=1}^{6} \dfrac{D_{nc}(i,k)\ B_{nc}(k,j)}{E_{nc}(i,k) + C_{nc}(k,j) + 2} \\ (i=1,4 \text{ and } j=1,4) \\ \text{if } E_{nc}(i,k) + C_{nc}(k,j) + 2 \neq 0 \end{cases}$$

$$\begin{cases} G_{nc}(i,j) = -\pi\, a^2 \sum_{k=1}^{6} F_{nc}(i,k)\ B_{nc}(k,j)\ \mathrm{Ln}\ \epsilon\ ;\ \epsilon = 10^{-9} \\ (i=1,4 \text{ and } j=1,4) \\ \text{if } E_{nc}(i,k) + C_{nc}(k,j) + 2 = 0 \end{cases} \tag{42}$$

where $[D_{nc}]\ [B_{nc}]^t\ [P]$

$$[E_{nc}] = [C_{nc}]^t \tag{43}$$

$[B_{nc}]$ and $[C_{nc}]$ are two (6 x 4) matrices given in Appendix A.

$[P]$ is the elasticity matrix given in Appendix A.

5.2.2 Mass matrix

The mass matrix is given by:

$$[m] = \rho\, t\ [[A_{nc}]^{-1}]^t\ [S_{nc}]\ [A_{nc}]^{-1} \tag{44}$$

where $[A_{nc}]$ is the matrix given in Appendix A.

$$[S_{nc}] = \pi \int_{a_0}^{a} [R_{nc}]^t\ [R_{nc}]\ r\ dr \tag{45}$$

After integration we obtain:

$$\begin{cases} S_{nc}(i,j) = \pi a^2 \sum_{k=1}^{3} \dfrac{F_{nc}(k,i)\ F_{nc}(k,j)}{H_{nc}(k,i) + H_{nc}(k,j) + 2} \\ (i=1,4 \text{ and } j=1,4) \\ \text{if } E_{nc}(i,k) + C_{nc}(k,j) + 2 = 0 \end{cases}$$

$$\begin{cases} S_{nc}(i,j) = -\pi a^2 \sum_{k=1}^{3} F_{nc}(k,i)\ F_{nc}(k,j)\ \mathrm{Log}\ \epsilon\ ;\ \epsilon = 10^{-9} \\ (i=1,4 \text{ and } j=1,4) \\ \text{if } H_{nc}(k,i) + H_{nc}(k,j) + 2 = 0 \end{cases} \tag{46}$$

where $[F_{nc}]$ and $[H_{nc}]$ are the (3 x 4) matrices given in Appendix A.

6. CALCULATIONS AND DISCUSSION

6.1 Assembly of the finite elements

As has already been mentioned, the complete plate is divided into a finite number of circular and annular elements, the positions of which can be selected arbitrarily. Each finite element of the circular plate type has a node at the circumference end, while the elements of the annular plate type have one node at each extremity (Figure 3).

Once the stiffness and mass matrices have been obtained, it is possible to construct the global matrices for the complete plate using finite element assembly technique. If N is the number of finite elements, [M] and [K] are two matrices of the order 4(N+1) for an annular plate and of the order 4N for a circular plate. These matrices are symmetrical and semi-defined; they are also band matrices of which the half-width of the band is equal to 8.

6.2 Static Forces

The study of the static equilibrium is carried out in the following manner:

When: {FA} is the vector of the forces applied to the nodes of the plate
{FB} is the vector of unknown reactions
{dA} is the vector of unknown nodal displacements
{dB} is the vector of displacements defined by the boundary conditions

The static equilibrium equation [K] {d} = {F} becomes:

$$\begin{bmatrix} KAA & KAB \\ KBA & KBB \end{bmatrix} \begin{Bmatrix} dA \\ dB \end{Bmatrix} = \begin{Bmatrix} FA \\ FB \end{Bmatrix} \tag{47}$$

We have therefore:

$$\begin{aligned} \{dA\} &= [KAA]^{-1} \{FA\} - [KAB] \{dB\} \\ \{FB\} &= [KBA] \{dA\} + [KBB] \{dB\} \end{aligned} \tag{48}$$

Finally, the stresses can be found from the displacement by the relation (24) and (26).

6.3 Free vibrations

In the free vibrations, the equations of motion are:

$$[M] \{\ddot{\delta}\}_T + [K] \{\delta\}_T = \{0\} \tag{49}$$

where [M], [K] are the global mass and stiffness matrices and $\{\delta_T\}$ is the vector for the global displacements of the whole plate.

$$\{\delta_T\} = \{\delta_1, \delta_2, ..., \delta_{N+1}\}^T$$

N being the number of finite elements.

The vibration is harmonic:

$$\{\delta_T\} = \{\delta_0\}_T \sin(\omega t + \psi) \tag{50}$$

where ω is the natural angular frequency

ψ is the phase angle.

By introducing equation (50) in (49), we obtain:

$$([K] - \omega^2 [M]) \{\delta_0\}_T = \{0\} \tag{51}$$

This relation holds only for certain values of ω where the determinant of the matrix in parentheses is zero. These values define the natural angular frequencies of the structure and give rise to a typical problem of eigenvalues and eigenvectors.

$$\det [[K] - \omega^2 [M]] = 0 \tag{52}$$

Corresponding to each natural frequency for which equation (51) has been verified is a vector $\{\delta_0\}$ of which the components are defined to one close arbitrary multiplying constant. Such vectors are called the natural modes (or eigenvectors) of the system.

6.4 Boundary Conditions

If the plate has boundary constraints such as being simply-supported, clamped, etc., the appropriate lines and columns in [K] and [M] are eliminated to satisfy these constraints. Consequently, matrices [K] and [M] reduce to square matrices of order 4(N+1)-J for an annular plate and 4N-J for a circular plate, where J is the number of constraints applied.

Thus, for a plate:

- free : J = 0
- with one edge simply-supported (u = v = w = 0) : J = 3
- with two edges clamped ($u = v = w = \frac{dw}{dr} = 0$) : J = 8

6.5 Convergence

The finite element method permits us to reach an approximate solution to the problem of elasticity. This solution is marred by errors which fall into two categories:

- The discretisation error which stems from the replacement of the initial physical problem by an approximate model.

- The truncation error stemming from the numerical calculation.

From the convergence of the finite element method, we see that the solution to the problem is a function of the number of finite elements used to model the structure under consideration, that is to say, it is a function of the fineness of the net.

The calculation was carried out with one particular plate for a number of radial modes "m" equal to 1, 2 and 3, with a number of finite elements N = 2, 4, 6, 8, 10, 12.

The results for n = 2 are given in Figure 4. We conclude that the convergence of the system demands 6 finite elements for the relatively low radial modes "m", but 20 for radial mode m = 10.

6.6 Free vibrations of circular plates

The natural frequencies of a circular plate can unquestionably be calculated by simpler methods than these. Our principal objective, however, has been to verify the accuracy of the mass and stiffness matrices in their general forms.

For this reason, we have compared the non-dimensional natural frequencies determined by this method with those obtained by other authors, both for different boundary conditions (plate clamped, simply-supported, free) and for different values of the number of circumferential mode "n" and on the number of radial mode "m".

We have obtained very good agreement both for high and low frequencies (Figures 5). Figure 6 shows the non-dimensional natural frequency curves as a function of the number of circumferential mode "n" for different boundary conditions and for different values of the number of radial mode "m".

Detailed discussion of the results obtained and their significance will not be undertaken here as this has already been done by others, notably in [1] and [4]. The evident success of this method in analysing circular plates is considered to have provided adequate proof of the soundness of the theory as a whole and the correctness of the expressions of the stiffness and mass matrices derived in this paper.

6.7 Free vibrations of annular plates

Our method is remarkable for the fact that it enables us to determine with high precision both low and high frequencies.

The results obtained in the literature are only for relatively low modes ($n = 0, 1, 2$ and $m = 1, 2, 3$). To extend the range of results, Figures 7-13 show part of the results obtained for $n = 2$ and $m = 1\text{-}10$ for different boundary conditions and various dimensions of the annular plate.

Figures 7 and 8 show the natural frequencies of an annular plate as a function of the number of radial mode for different boundary conditions and different ratios (a_0/a : inside radius / outside radius of the annular plate). We observe that for small a_0/a ($= 0.1$), the frequencies increase uniformly with the increasing of the radial mode, but for high ratio a_0/a ($= 0.9$), there are some parts of the radial mode where the frequencies increase very slightly.

We see in Figures 9 to 11 that the frequency increases slightly with increasing a_0/a ratio in the range $0 < a_0/a < 0.4$ approximately, and then increase rapidly for high a_0/a ratio.

Figures 12 and 13 show the natural frequency for one particular boundary condition (clamped-free) as a function of the a_0/a ratio for different radial mode m and circumferential mode $n = 2$.

7. CONCLUSION

The objective of this work was to present a new method for the static and dynamic analysis of thin, elastic, isotropic, non-uniform circular and annular plates. The method combined circular plate theory with finite element analysis. Two types of finite element were developed, the first was an element of the circular plate type and the second was of the annular plate type. The displacement functions, the mass and stiffness matrices were developed for circumferential modes $n \geq 2$.

The convergence of the method was established and the natural frequencies were obtained for various boundary conditions and for different circumferential and radial modes. These were compared with the results of other investigations and satisfactory agreement was found.

This method combines the advantages of finite element analysis which deals with complex plates (variable thickness, non-uniform materials, various boundary conditions, different types of load), and the precision of formulation obtained when we use displacement functions derived from circular plate theory.

The method enabled us to supplement the few results available on high natural frequencies associated with high circumferential and radial modes. It also enabled us to determine the natural frequencies and vibration modes of non-uniform annular and circular plates. We consider that we have, here, a method by which it is possible to predict the vibrationary characteristics of circular and annular plates.

The next step in this line of work should be the analysis of linear and non-linear anisotropic plates, and of the dynamic stability of circular and annular plates.

REFERENCES

[1] SZELARD, R., "Theory and Analysis of Plates", Prentice-Hall Inc., New Jersey, 1974.

[2] LLOYD, H.D., "Beams, Plates and Shells", McGraw Hill Book Company, 1976.

[3] KRAUS, H., "Thin Elastic Shells", John Wiley and Sons, New York, 1967.

[4] LEISSA, A.W., "Vibration of Plates", NASA SP-160, 1969.

[5] LEISSA, A.W. and NARITA, Y., "Natural Frequencies of Simply Supported Circular Plates", Journal of Sound and Vibration, Vol. 70, pp. 221-229, 1980.

[6] IRIE, T., YAMADA, G. and AOMURA, S., "Natural Frequencies of Mindlin Circular Plates", Journal of Applied Mechanics, Vol. 47, pp. 652-655, 1980.

[7] ITAO, K. and CRANDALL, S.H., "Natural Mode and Natural Frequencies of Uniform, Circular, Free-Edge Plates", Journal of Applied Mechanics, Vol. 46, pp. 448-453, 1979.

[8] CELEP, Z., "Free Vibration of Some Circular Plates of Arbitrary Thickness", Journal of Sound and Vibration, Vol. 70, pp. 379-388, 1980.

[9] IRIE, T. and YAMADA, G., "Analysis of Free Vibration of Annular Plate of Variable Thickness by Use a Spline Technique Method", Bulletin of the JSME, Vol. 23, no. 176, pp. 286-292, 1980.

[10] SATO, N. and SHIMIZU, C., "Transfert Matrix Analysis of Non-Linear Free Vibrations of Circular Plates with Variable Thickness", Journal of Sound and Vibration, Vol. 97, pp. 587-595, 1984.

[11] NARITA, Y., "Natural Frequencies of Completely Free Annular and Circular Plates Having Polar Orthotropy", Journal of Sound and Vibration, Vol. 82, pp. 33-38, 1984.

[12] GORMAN, D.G., "Natural Frequencies of Transverse Vibration of Polar Orthotropic Variable Thickness Annular Plates", Journal of Sound and Vibration, Vol. 86, pp. 47-60, 1983.

[13] LERORE, J.A. and SHAH, H.C., "Dynamic Stability of Circular Plates under Stochastic Excitations", Journal of Spacecraft and Rockets, Vol. 7, pp. 582-587, 1970.

[14] TANI, J. and NARAMURA, T., "Dynamic Stability of Annular Plates under Periodic Radial Loads", Journal of the Acoustical Society of America, Vol. 64, pp. 827-831, 1978.

[15] ZIENKIEWICZ, O.C., "The Finite Element Method", 3th Edition, McGraw Hill Book Company, 1977.

[16] DHATT, G. et TOUZOT, G., "Une présentation de la méthode des éléments finis", Moloine, S.A. Editeur Paris, 1984.

[17] SANDERS, J.L., "An Improved First Approximation Theory for Thin Shells", NASA, TR-24, 1959.

[18] LAKIS, A.A. and PAIDOUSSIS, M.P., "Dynamic Analysis of Axially Non-Uniform Thin Cylindrical Shells", J. Mech. Eng. Science, Vol. 14, No. 1, pp. 49-71, 1972.

[19] LAKIS, A.A. and PAIDOUSSIS, M.P., "Free Vibration of Cylindrical Shells Partially Filled with Liquid", Journal of Sound and Vibration, Vol. 19, pp. 1-15, 1971.

[20] LAKIS, A.A. and PAIDOUSSIS, M.P., "Prediction of the Response of a Cylindrical Shell to Arbitrary of Boundary-Layer-Induced Random Pressure Field", Journal of Sound and Vibration, Vol. 25, pp. 1-27, 1972.

[21] LAKIS, A.A., "Theoretical Model of Cylindrical Structures Containing Turbulent Flowing Fluids", 2nd Int. Symposium on Finite Element Methods in Flow Problems, Santa Margherita Ligure (Italy), June 1976.

[22] LAKIS, A.A., SAMI, S.M. and ROUSSELET, J., "Turbulent Two Phases Flow Loop Faicility for Predicting Wall-Pressure Fluctuation and Shell Response", 24th Int. Instrumentation Symposium Albuquerque (New-Mexico), May 1978.

[23] LAKIS, A.A. and PAIDOUSSIS, M.P., "Shell Natural Frequencies of the Pickering Steam Generator", Atomic Energy of Canada Ltd, AECL Report No. 4362, 1973.

[24] LAKIS, A.A. and SINNO, M., "Free Vibration of Axisymmetric and Beam-Like Cylindrical Shells partially Filled with Liquid", Int. Jl. for Numerical Methods in Eng., Vol. 33, pp. 235-268, 1992.

[25] LAKIS, A. and LAVEAU, A., "Non-Linear Dynamic Analysis of Anisotropic Cylindrical Shells Containing a Flowing Fluid", Int. Journal Solids Structures, Vol. 28, No. 9, pp. 1079-1094, 1991.

[26] LAKIS, A.A., VAN DYKE, P. and OURICHE, H., "Dynamic Analysis of Anisotropic Fluid-Filled Conical Shells", Journal of Fluids and Structures, Vol. 6, pp. 135-162, 1992.

[27] LAKIS, A., TUY, N, LAVEAU, A. and SELMANE, A., "Analysis of Axially Non-Uniform Thin Spherical Shells", International Symposium, STRUCOPT-COMPUMAT, Paris, France, pp. 80-85, 1989.

[28] LOVE, A.E.H., "A Treatise on the Mathematical Theory of Elasticity", Dover Publication, New York, 1944.

APPENDIX A

List of Matrices

MATRIX [H]

$$H(1,1) = \frac{\lambda^2}{4} - \frac{\lambda}{2} - \left(\frac{1-\nu}{2}\right) n^2 - \frac{3}{4}$$

$$H(1,2) = \left(\frac{1+\nu}{4}\right) n\lambda - \frac{n}{4}(7-\nu)$$

$$H(2,1) = \left(\frac{1+\nu}{4}\right) n\lambda - \frac{n}{4}(5-3\nu)$$

$$H(2,2) = \left(\frac{1-\nu}{4}\right) \lambda^2 - \left(\frac{1-\nu}{4}\right) \lambda - 3\left(\frac{1-\nu}{8}\right) - n^2$$

MATRICES $[T_n]$ and $[T_{nc}]$

$$[T_n] = [T_{nc}] = \text{diag}\ [\cos n\theta\ ,\ \cos n\theta\ ,\ \sin n\theta]$$

MATRIX $[A_n]$

$$A_n\ (1,j) = y_o^{\frac{\lambda_j - 1}{2}}\ ,\ j = 1,4$$

$$A_n\ (2,5) = y_o^{n}$$

$$A_n\ (2,6) = y_o^{-n}$$

$$A_n\ (2,7) = y_o^{n+2}$$

$$A_n\ (2,8) = y_o^{-n+2}$$

$$A_n\ (3,5) = \frac{n}{a}\ y_o^{n-1}$$

$$A_n\ (3,6) = -\frac{n}{a}\, y_o^{-n-1}$$

$$A_n\ (3,7) = \frac{n+2}{a}\, y_o^{n+1}$$

$$A_n\ (3,8) = \frac{-n+2}{2}\, y_o^{-n+1}$$

$$A_n\ (4,j) = \alpha_j\, y_o^{\frac{\lambda_j - 1}{2}} \quad , \ j = 1,4$$

$$A_n\ (5,j) = 1 \ , \ j = 1,4$$

$$A_n\ (6,j) = 1 \ , \ j = 5,8$$

$$A_n\ (7,5) = \frac{n}{a}$$

$$A_n\ (7,6) = -\frac{n}{a}$$

$$A_n\ (7,7) = \frac{n+2}{a}$$

$$A_n\ (7,8) = \frac{-n+2}{a}$$

$$A_n\ (8,j) = \alpha_j \ , \ j = 1,4$$

$A_n\ (i,j) = 0$ for all other i and j

MATRIX $[A_{nc}]$

$$A(1,1) = A(1,2) = A(2,3) = A(2,4) = 1$$

$$A(3,3) = \frac{n}{a}$$

$$A(3,4) = \frac{n+2}{a}$$

$$A(4,1) = \alpha_1$$

$$A(4,2) = \alpha_2$$

$A(i,j) = 0$ for all other i and j

MATRICES $[R_n]$, $[F_n]$ et $[H_n]$

$$R_n(i,j) = F_n(i,j)\ y^{H_n(i,j)} \qquad i = 1,3; \quad j = 1,8$$

The matrix $[F_n]$ is defined by:

$F_n(1,j) = 1$ for $j = 1,4$
$F_n(2,1) = 1$ for $j = 5,8$
$F_n(3,j) = \alpha_j$ for $j = 1,4$
$F_n(i,j) = 0$ for all other i and j

The matrix $[H_n]$ is defined by:

$$H_n\ (1,j) = \frac{\lambda_j - 1}{2}\ ,\ j = 1,4$$

$$H_n\ (2,5) = n$$

$$H_n\ (2,6) = -n$$

$$H_n\ (2,7) = n+2$$

$$H_n\ (2,8) = -n+2$$

$$H_n\ (3,j) = \frac{\lambda_j - 1}{2} \quad \text{for } j=1,4$$

$H_n\ (i,j) = 0$ for all other i and j

Matrices $[R_{nc}]$, $[F_{nc}]$ et $[H_{nc}]$

$$R_{nc}(i,j) = F_{nc}(i,j)\ y^{H_{nc}(i,j)} \qquad i = 1,3; \quad j = 4$$

The matrix $[F_{nc}]$ is defined by:

$$F_{nc}(1,1) = F_{nc}(1,2) = F_{nc}(2,3) = F_{nc}(2,4) = 1$$

$$F_{nc}(3,1) = \alpha_1$$

$$F_{nc}(3,2) = \alpha_2$$

$F_{nc}(i,j) = 0$ for all other i and j

The matrix $[H_{nc}]$ is defined by

$$H_{nc}(1,j) = \frac{\lambda_j - 1}{2}, \quad j = 1,2$$

$$H_{nc}(2,3) = n$$

$$H_{nc}(2,4) = n+2$$

$$H_{nc}(3,j) = \frac{\lambda_j - 1}{2} \quad \text{for } j=1,2$$

$H_{nc}(i,j) = 0$ for all other i and j

Matrices $[Q_n]$, $[B_n]$ et $[C_n]$

$$Q_n(i,j) = B_n(i,j)\, y^{C_n(i,j)} \qquad i = 1,6 \;; \quad j = 1,8$$

The matrix $[B_n]$ is defined by:

$$B_n(1,j) = \frac{\lambda_j - 1}{2a} \qquad \text{for } j=1,4$$

$$B_n(2,j) = \frac{1 + \alpha_j n}{a} \qquad \text{for } j=1,4$$

$$B_n(3,j) = \frac{1}{a}\left[-n + \alpha_j \left(\frac{\lambda_j - 3}{2} \right) \right] \qquad \text{for } j=1,4$$

$$B_n(4,5) = \frac{-n(n-1)}{a^2}$$

$$B_n(4,6) = \frac{-n(n+1)}{a^2}$$

$$B_n(4,7) = \frac{-n(n+2)(n+1)}{a^2}$$

$$B_n(4,8) = \frac{-n(1-n)(2-n)}{a^2}$$

$$B_n(5,5) = \frac{n(n-1)}{a^2}$$

$$B_n(5,6) = \frac{n(n+1)}{a^2}$$

$$B_n(5,7) = \frac{n^2-n-2}{a^2}$$

$$B_n(5,8) = \frac{n^2 + n - 2}{a^2}$$

$$B_n(6,5) = \frac{2n(n-1)}{a^2}$$

$$B_n(6,6) = \frac{-2n(n+1)}{a^2}$$

$$B_n(6,7) = \frac{2n(n+1)}{a^2}$$

$$B_n(6,8) = \frac{-2n(n-1)}{a^2}$$

$B_n(i,j) = 0$ for all other i and j

The matrix $[C_n]$ is defined by

$C_n(i,j) = \frac{\lambda_j - 3}{2}$ for $i=1,2,3$ and $j=1,...4$

$C_n(i,5) = n-2$ for $i=4,5,6$

$C_n(i,6) = -n-2$ for $i=4,5,6$

$C_n(i,7) = n$ for $i=4,5,6$

$C_n(i,8) = -n$ for $i=4,5,6$

$C_n(i,j) = 0$ for all other i and j

$$Q_{nc}(i,j) = B_{nc}(i,j)\ y^{C_{nc}(i,j)} \qquad i = 1,6\ ;\ j = 1,4$$

The matric $[B_{nc}]$ is defined by:

$$B_{nc}(1,j) = \frac{\lambda_j - 1}{2a} \qquad \text{for } j=1,2$$

$$B_{nc}(2,j) = \frac{1+\alpha_j n}{a} \qquad \text{for } j=1,2$$

$$B_{nc}(3,j) = \frac{1}{a}\left[-n + \alpha_j \left(\frac{\lambda_j - 3}{2} \right) \right]$$

$$B_{nc}(4,3) = \frac{-n(n-1)}{a^2}$$

$$B_{nc}(4,4) = \frac{-n(n+2)(n+1)}{a^2}$$

$$B_{nc}(5,3) = \frac{n(n-1)}{a^2}$$

$$B_{nc}(5,4) = \frac{n^2-n-2}{a^2}$$

$$B_{nc}(6,3) = \frac{2n(n-1)}{a^2}$$

$$B_{nc}(6,4) = \frac{2n(n+1)}{a^2}$$

$B_{nc}(i,j) = 0$ for all other i and j

The matrix $[C_{nc}]$ is defined by:

$$C_{nc}(i,j) = \frac{\lambda_j - 3}{2} \quad \text{for } i=1,2,3 \text{ and } j=1,2$$

$$C_n(i,3) = n-2 \quad \text{for } i=4,5,6$$

$$C_{nc}(i,4) = n \quad \text{for } i=4,5,6$$

$$C_{nc}(i,j) = 0 \quad \text{for all other } i \text{ and } j$$

Matrix [P]

$$P(1,1) = P(2,2) = D, \qquad P(4,4) = P(5,5) = K$$

$$P(1,2) = P(2,1) = \nu D, \qquad P(4,5) = P(5,4) = \nu K$$

$$P(3,3) = \left(\frac{1-\nu}{2}\right) = D, \qquad P(6,6) = \left(\frac{1-\nu}{2}\right) K$$

where $D = \left(\frac{Et}{1-\nu^2}\right)$ and $K = \left(\frac{Et}{12(1-\nu^2)}\right)$

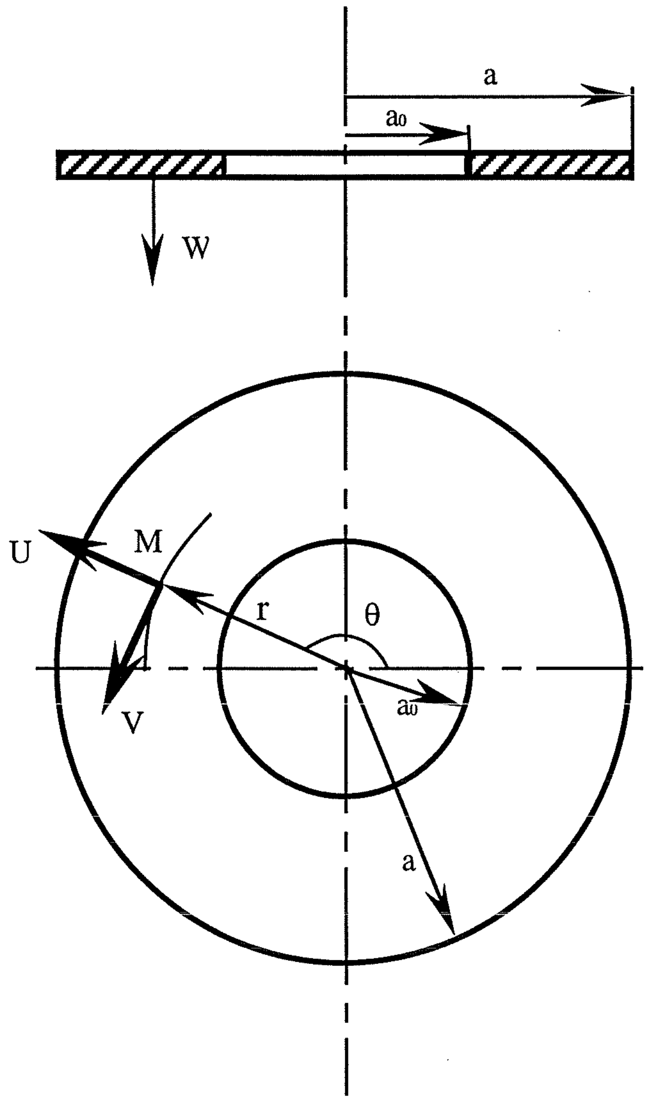

Figure 1 : Geometry of the mean surface of the plate

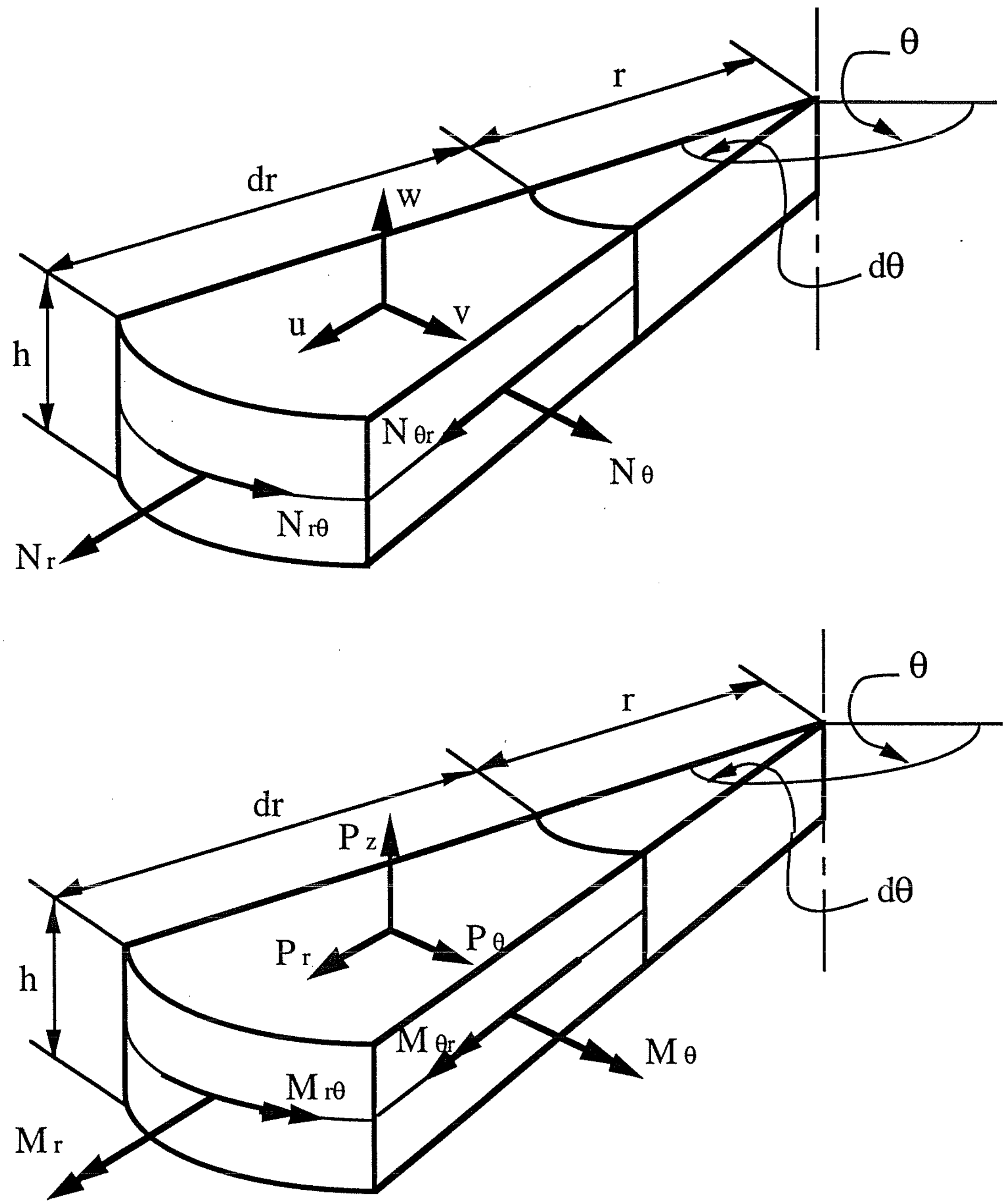

Figure 2 : Differential element for a plate

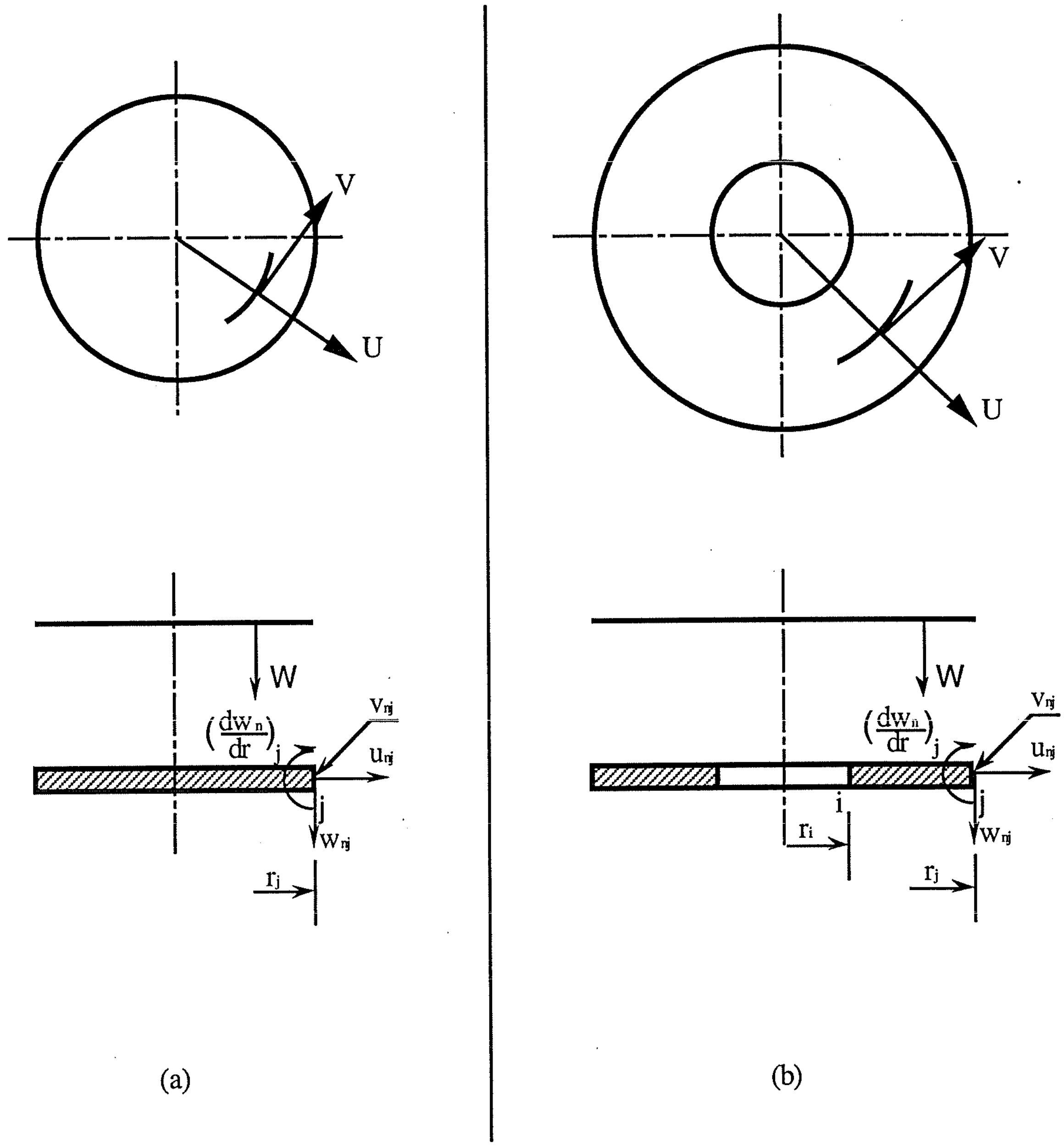

Figure 3 : Displacements and degrees of freedom
(a) Finite element of the 'circular plate' type
(b) Finite element of the 'annular plate' type

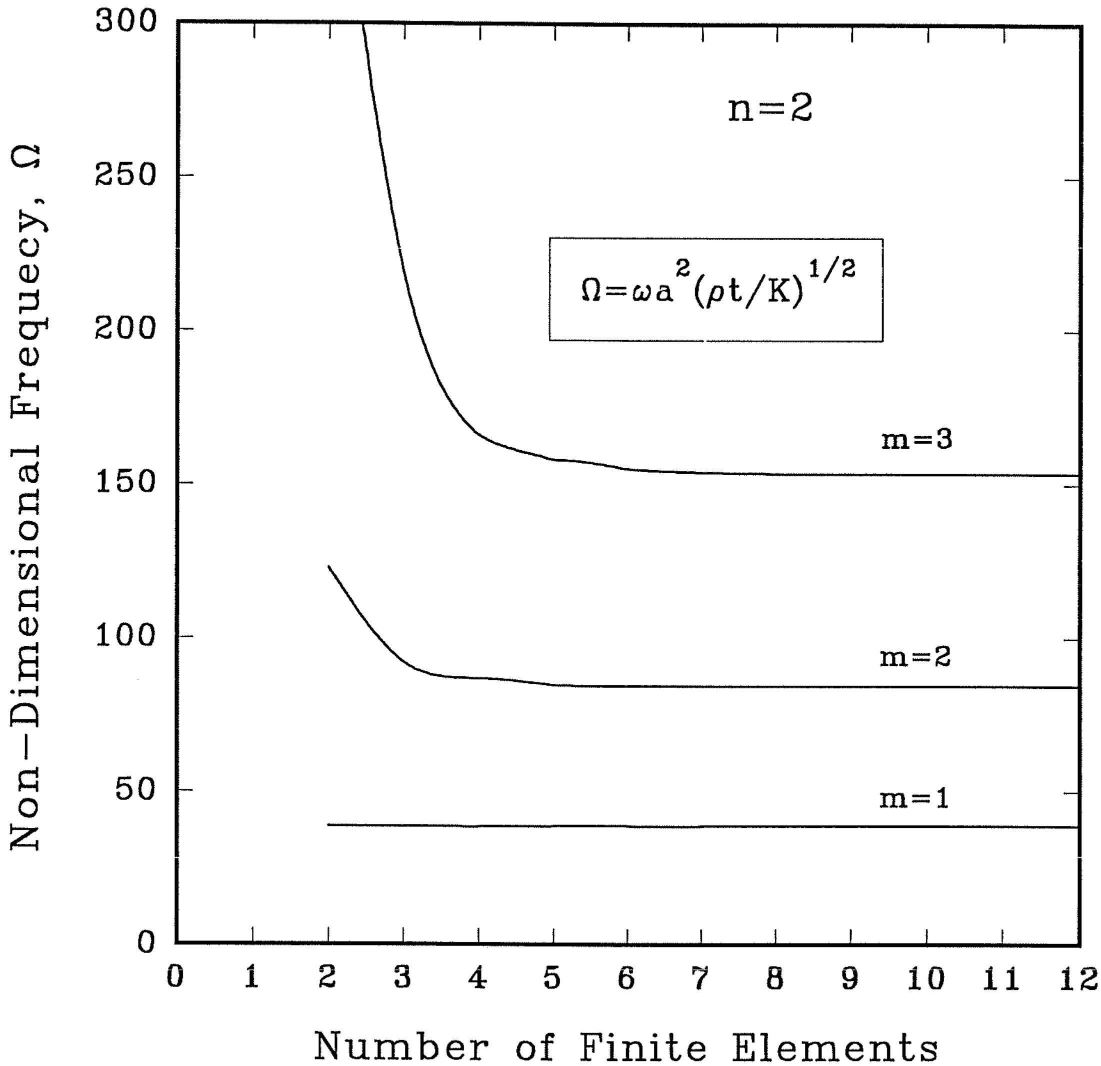

Figure 4 : Non-dimensional natural frequency Ω of a clamped circular plate as a function of the number of finite elements; where n is the number of circumferential mode, m is the number of radial mode, a is the plate's radius, t its thickness, ρ is the material density, K is the bending stiffness and ω is the natural angular frequency.

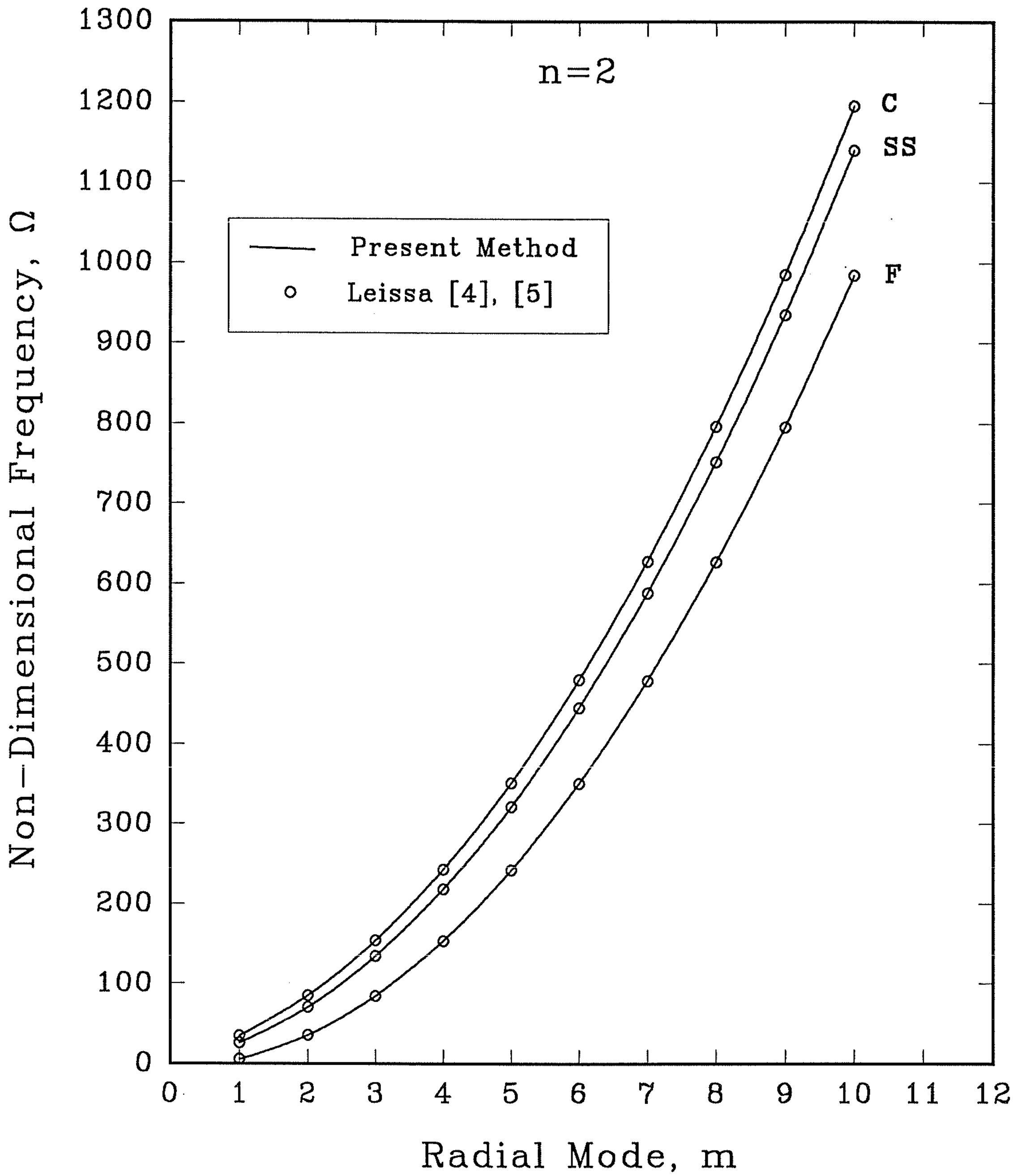

Figure 5 : Non-dimensional natural frequency of circular plate as a function of the number of radial mode, m
C : Clamped, SS : Simply-Supported, F : Free

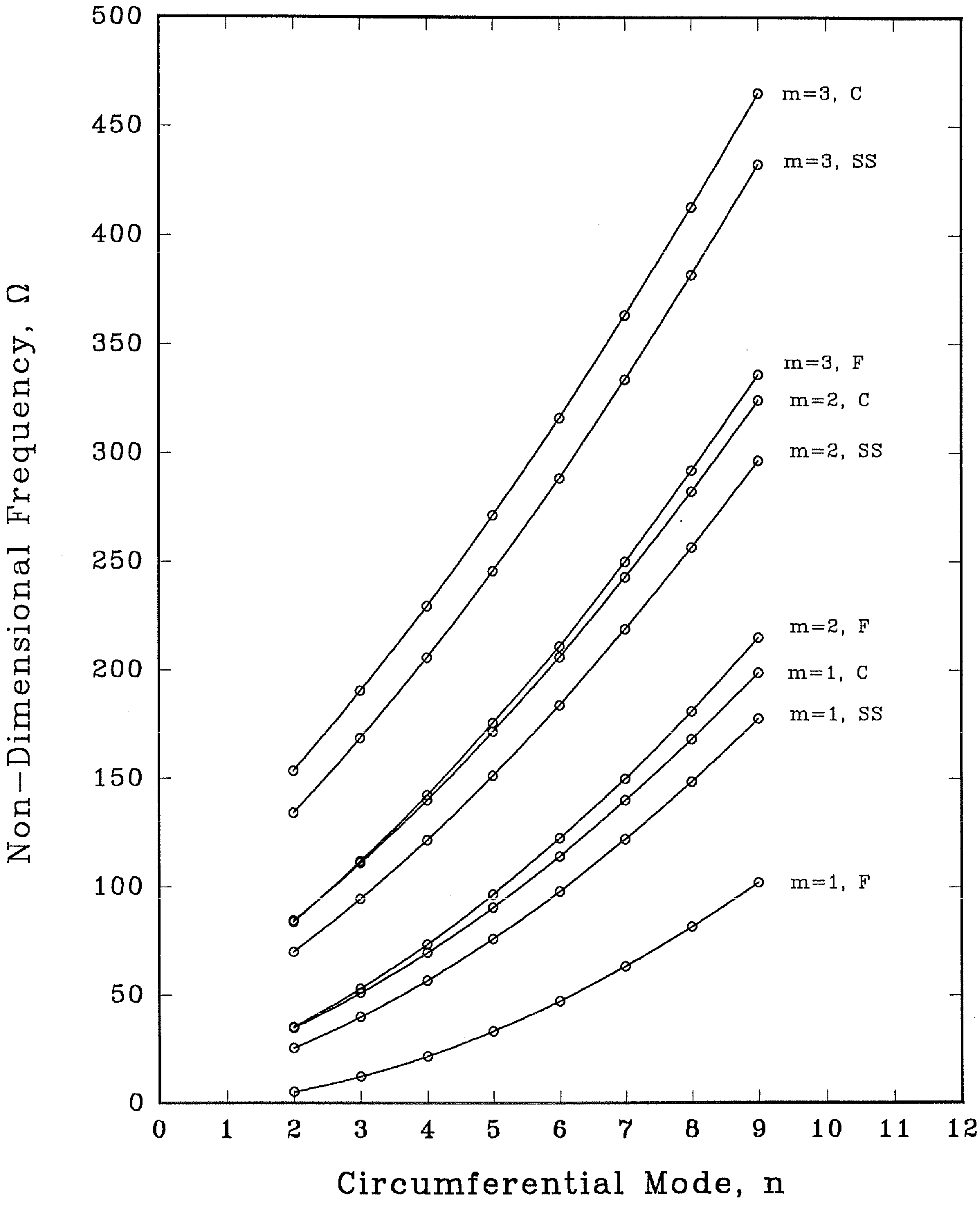

Figure 6 : Non−dimensional natural frequency Ω of circular plate as a function of the number of circumferential mode, n.
C : Clamped, SS : Simply−Supported, F : Free
m : number of radial mode

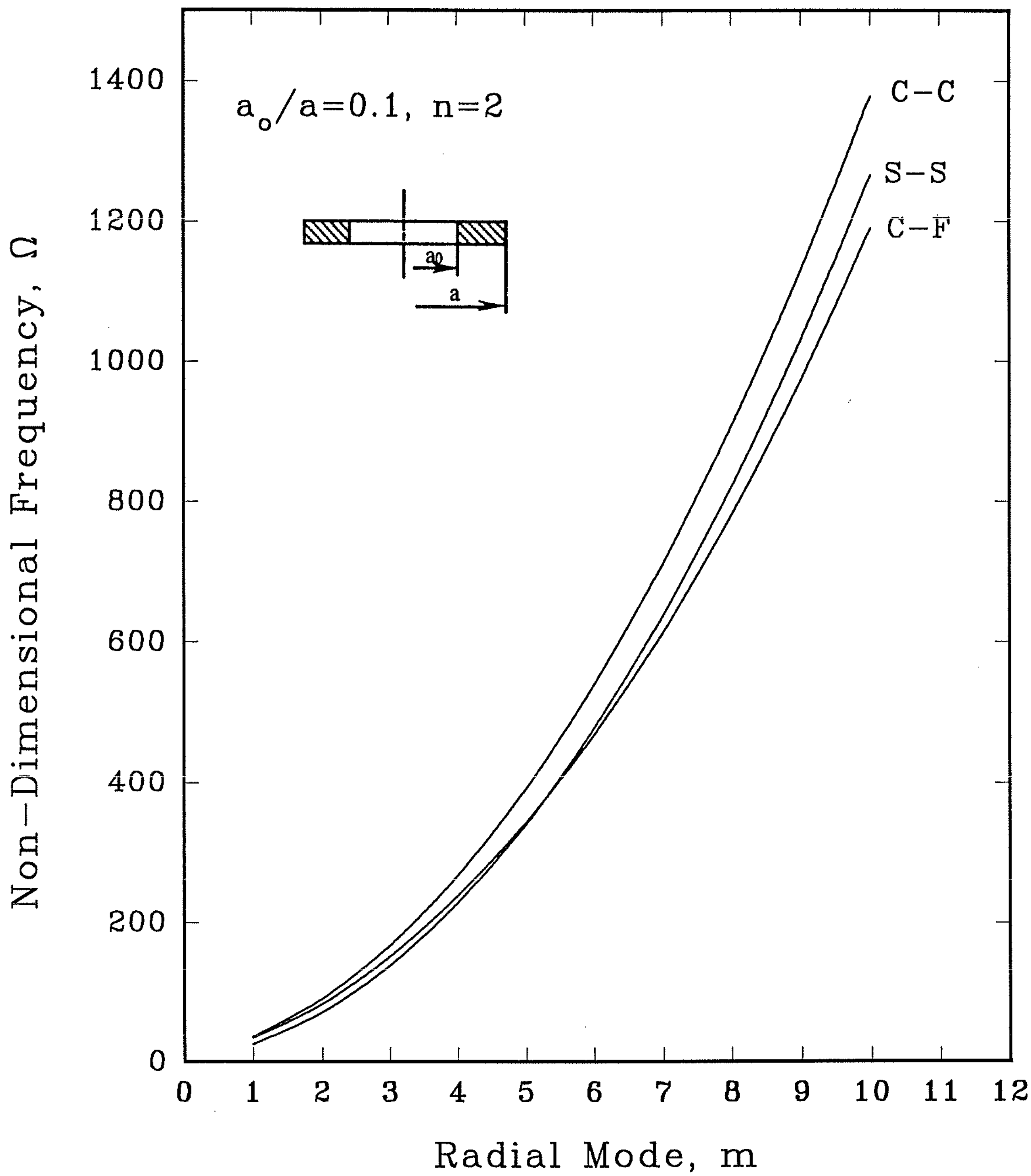

Figure 7 : Non−dimensional natural frequency Ω of annular plate as a function of of the radial mode, m.
C : Clamped, S : Simply−Supported, F : Free

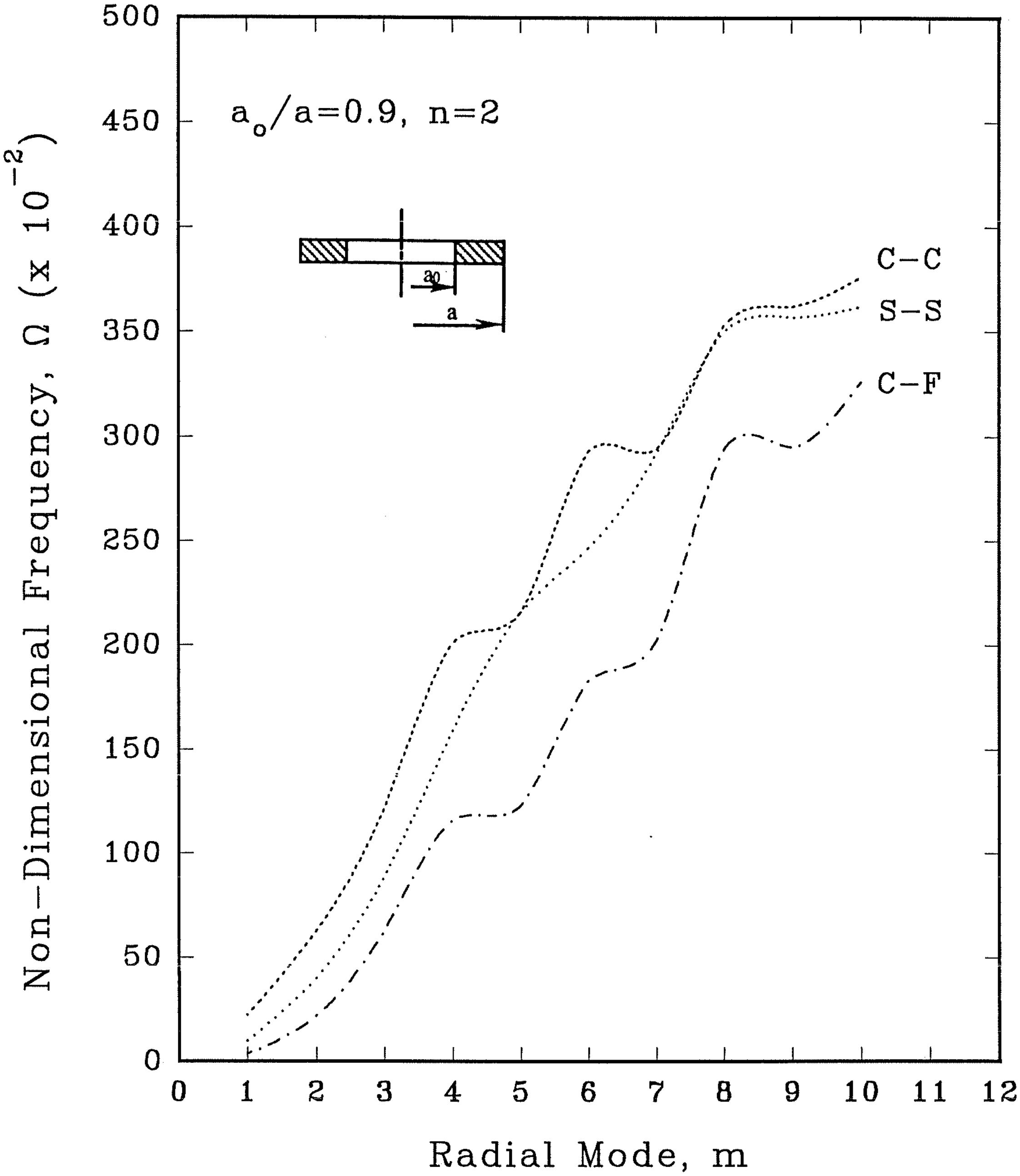

Figure 8 : Non−dimensional natural frequency of annular plate as a function of of the radial mode, m.
C : Clamped, S : Simply−Supported, F : Free

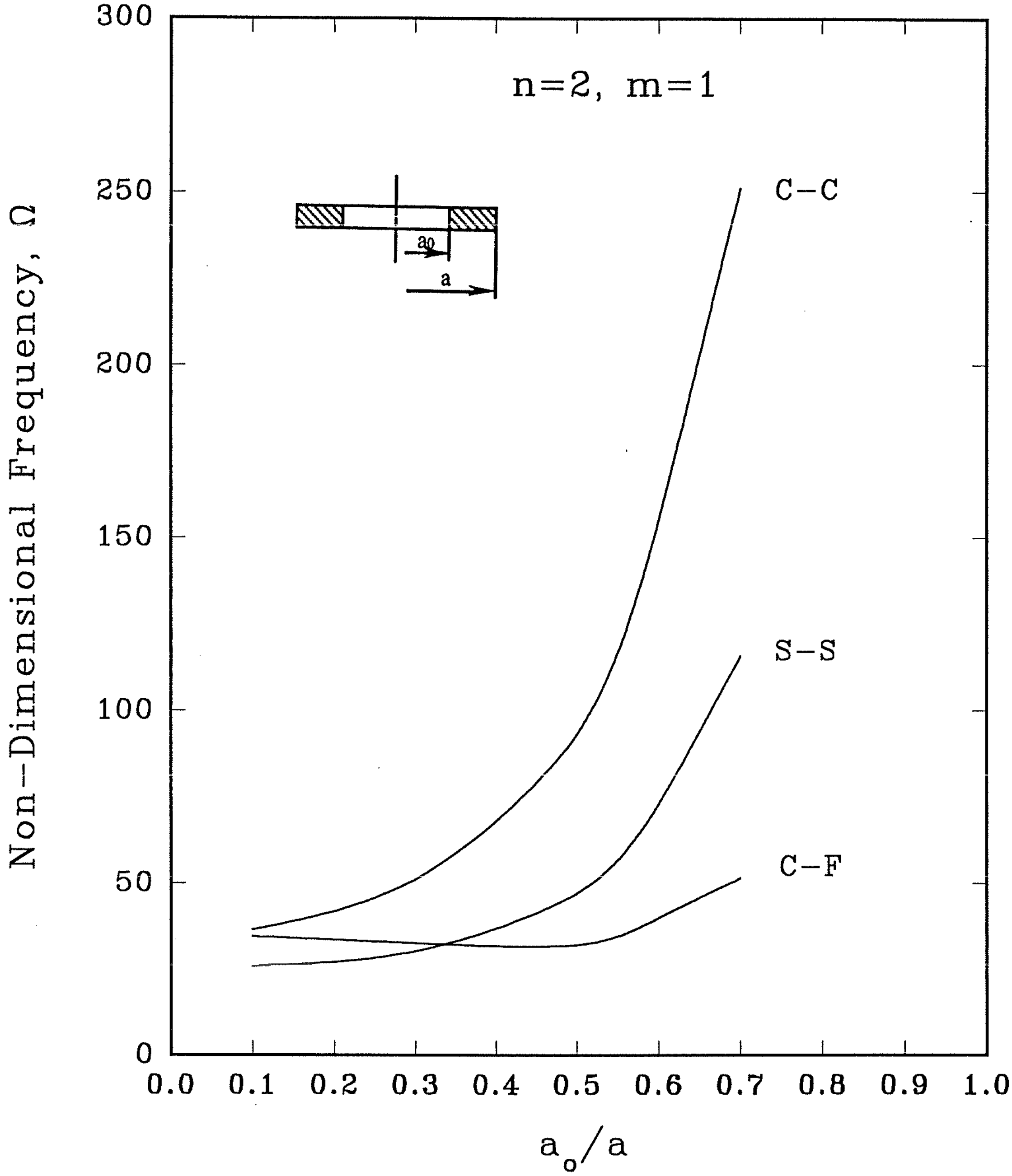

Figure 9 : Non−dimensional natural frequency of annular plate as a function of ratio a_o/a.

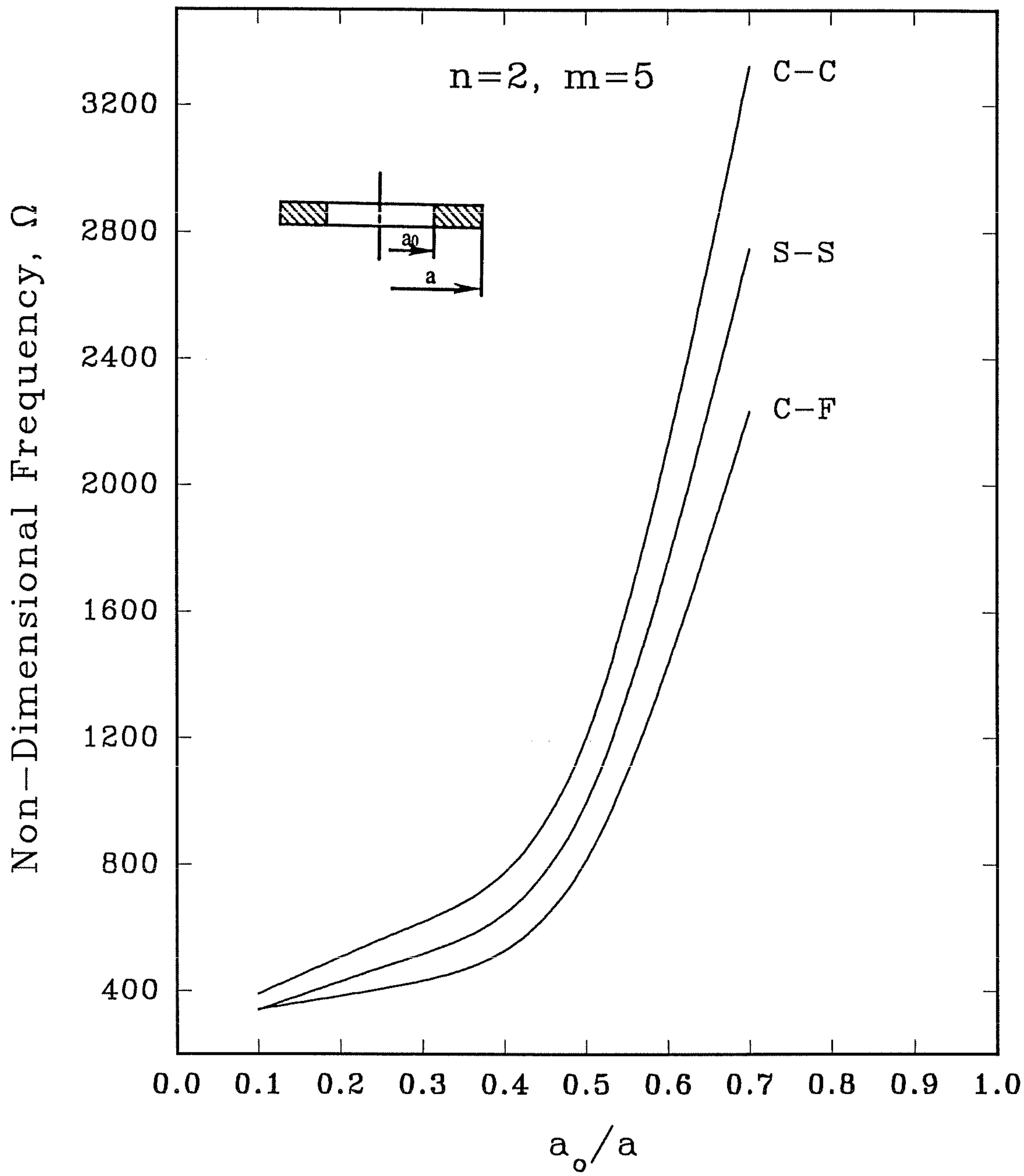

Figure 10 : Non−dimensional natural frequency of annular plate as a function of ratio a_o/a.

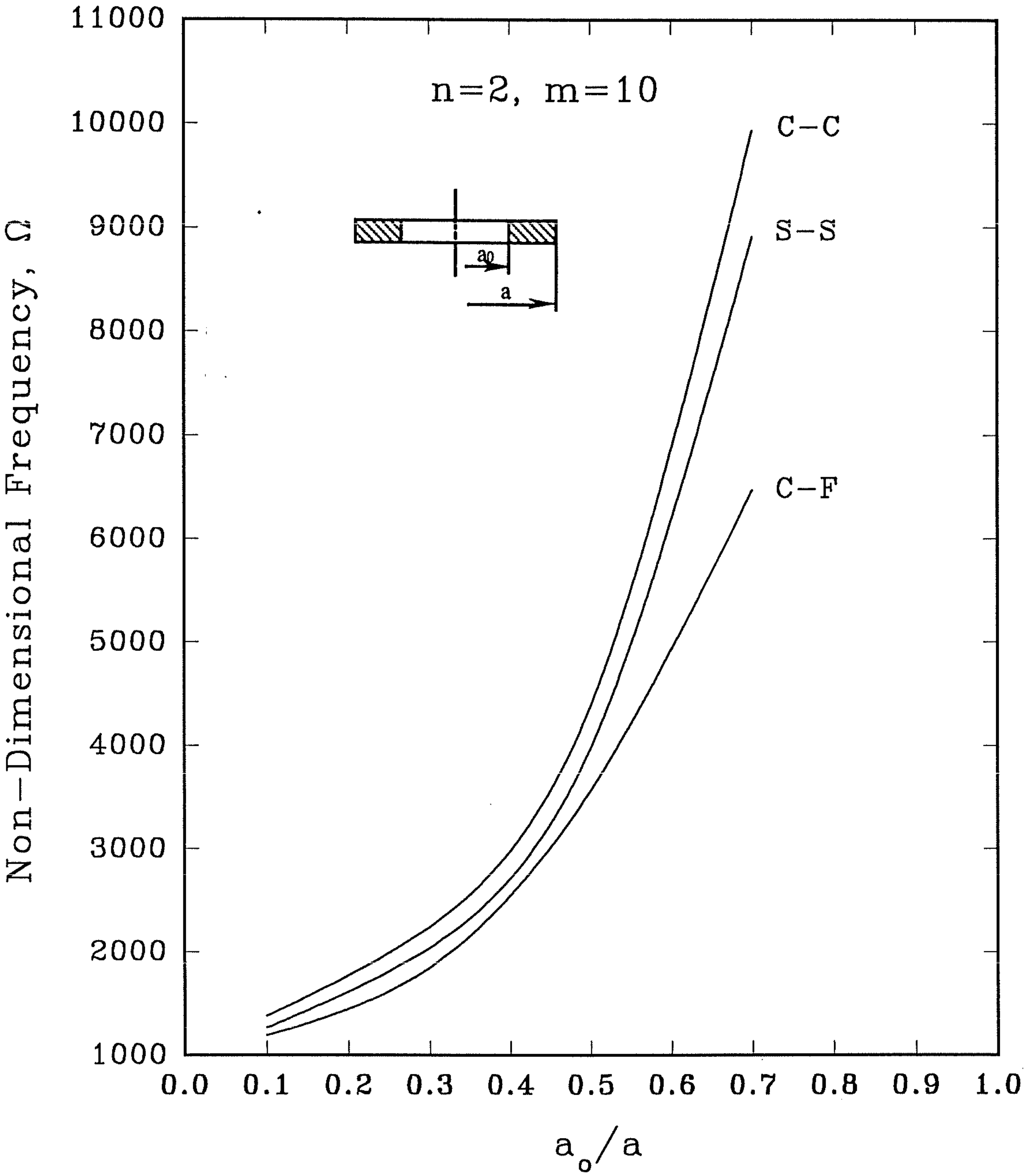

Figure 11 : Non−dimensional natural frequency of annular plate as a function of ratio a_o/a.

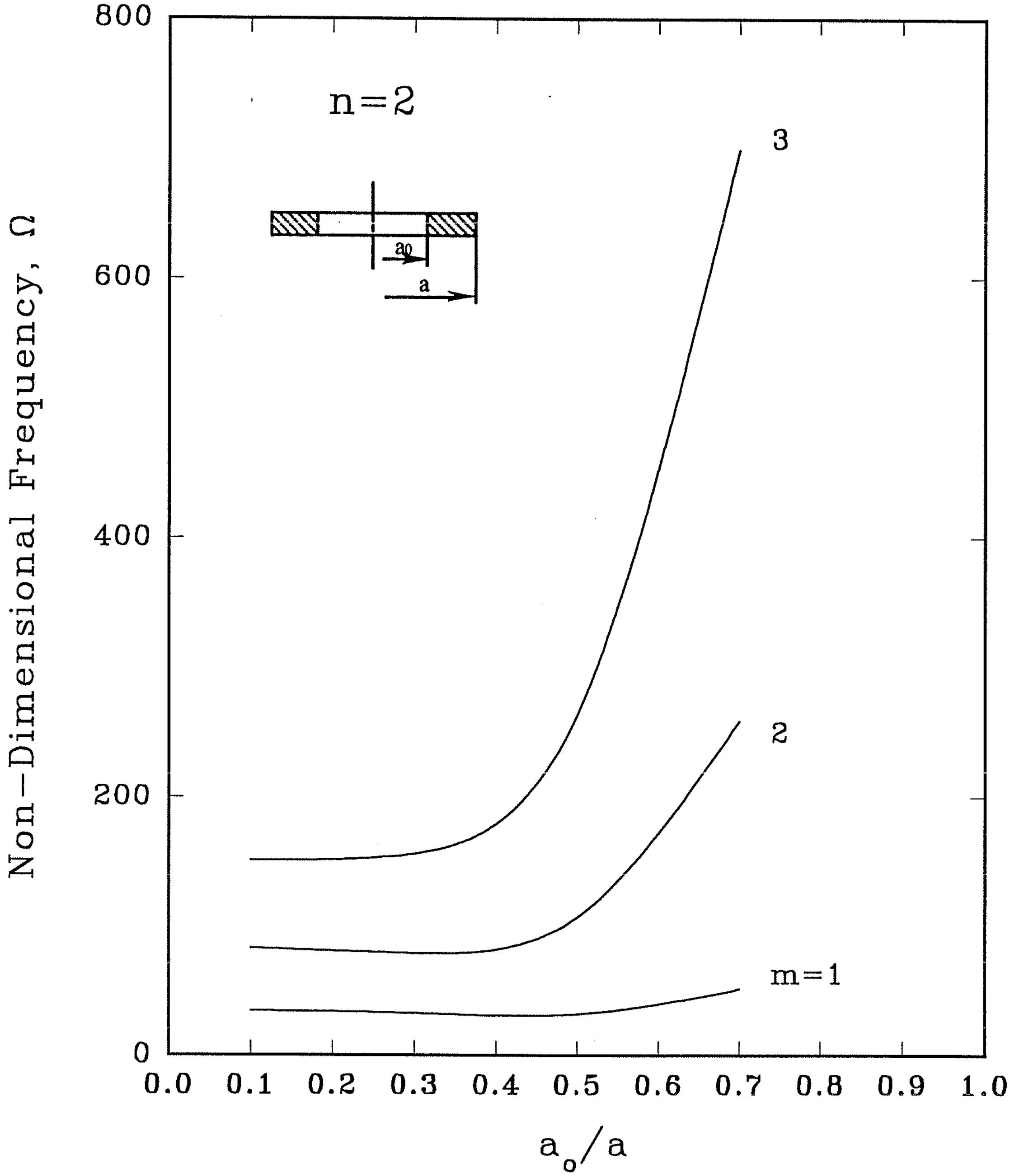

Figure 12 : Non-dimensional natural frequency of a Clamped-Free annular plate as a function of ratio a_o/a.

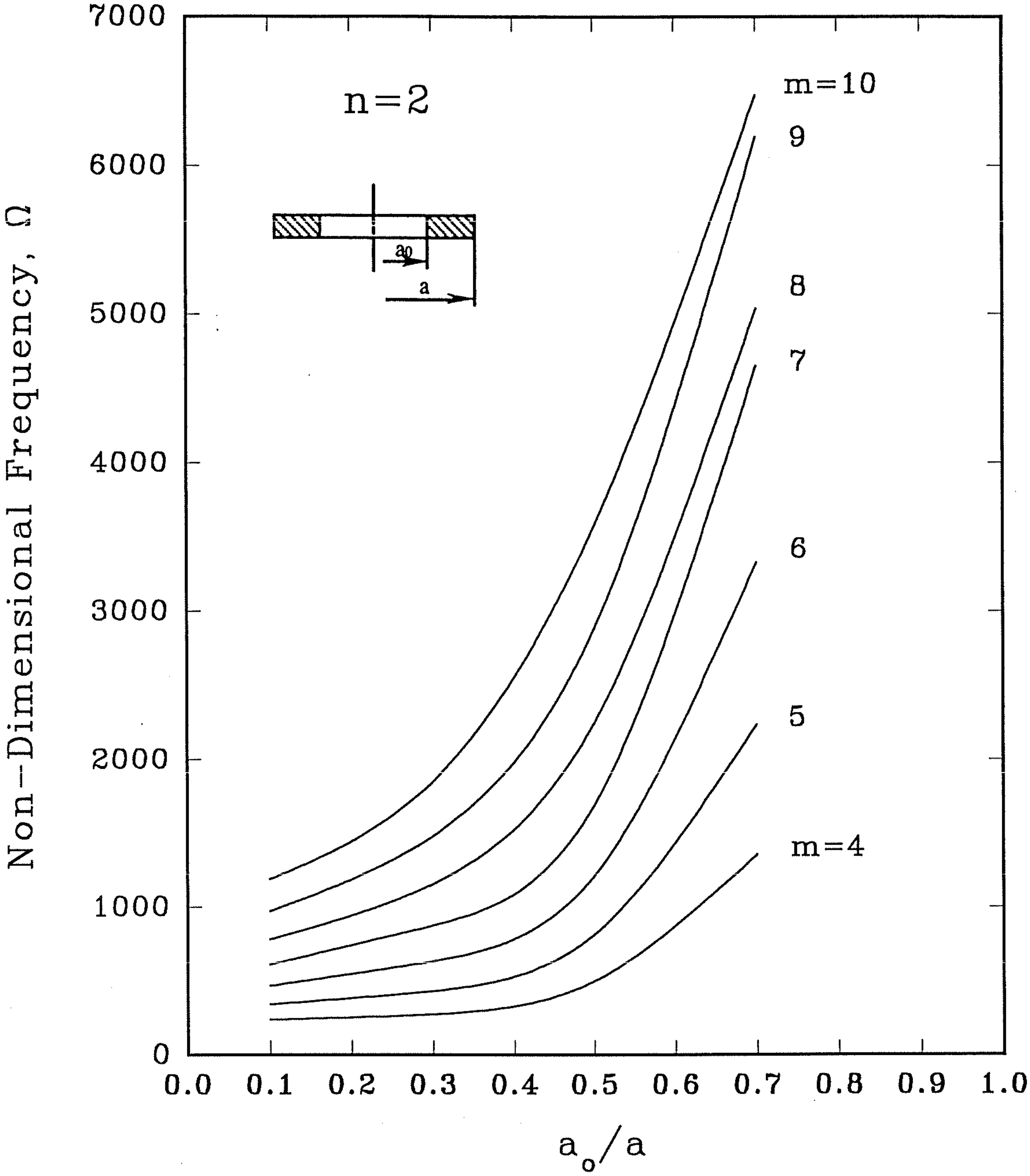

Figure 13 : Non-dimensional natural frequency of a Clamped-Free annular plate as a function of ratio a_o/a.

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