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GENERAL METHOD FOR ANALYSING CONTACT STRESSES ON CYLINDRICAL VESSELS

by

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ABSTRACT

A general theory is presented for the prediction of the stresses, displacements and interface pressure of cylindrical vessels partially filled with liquid, pressurized and having surfaces of contact with rigid supports. The supports are subdivided into a number of line and surface elements in the axial and circumferential directions. Each element is subjected to a load q_i, $i = 1, 2, \ldots, N$. The applied loads, expressed in double Fourier series and inserted into shells equations of motion, allow the determination of the corresponding displacements and stresses in terms of the q_i's. Prediction of the saddle-vessel interface pressure distribution is obtained by minimizing the potential energy of the system. Some calculations are conducted to illustrate the theory. The theory is compared with experiment, and agreement is found to be quite good.

INTRODUCTION

Knowledge of the pressure distribution of the surface of contact between fluid-filled cylindrical vessels and its supports is of considerable practical interest, considering that most cylindrical vessels are utilized in containing or conveying fluids. Although the problem of determining the stresses and displacements of saddle supported cylindrical vessels using beam's theory or semi-empirical approach has produced many papers (e.g. [1 - 4][1]), the litterature reveals a limited number of rigorous analysis which have been generally developed using cylindrical shells' theory.

For instance, Forbes and Tooth [5], and Wilson and Tooth [6] formulate an analysis capable of predicting the saddle/cylinder interface pressure and the stress resultant throughout the vessel. The most severe limitation of their theory is the assumption, in their investigation, a constant interface pressure distribution in the axial direction. The authors of this paper believe that the interface pressure is non-uniform in the axial as well as in the circumferential directions. The need is evident for a theory which can be used for the static or dynamic analysis of any kind of circular cylindrical shells partially filled with liquid, pressurized and having surfaces of contact with rigid supports. This theory may be used advantageously in the stress analysis of heat exchanger for nuclear power plants.

The work presented here based on a recently developed theory [7] by the authors, is an attempt to produce a general theory for the prediction of the saddle-vessel interface pressure distribution in the axial and circumferential directions as well as the stresses and displacements at any point

1 Number in brackets designates references at end of paper.

within the structure with wide range of applicability.

To this end, the support is subdivided into a sufficient number of line and surface elements each of which subjected to a load q_i, $i = 1, 2, \ldots, N$. Prediction of the saddle-vessel interface pressure distribution is obtained by minimizing the total Potential energy of the system which is derived in terms of the q_i's. A number of assumptions are made during the course of the investigation; a compendium of these assumptions and the limitations of the theory will be given in the text.

The theory of reference [7], and hence the theory of this paper, is capable of analysing geometrically axially non-symmetric, long or short, thin cylindrical shells subject to any set of boundary conditions (including supports other than at the two axial extremities of the shell).

The organization of the paper is as follows. First, the matrix formulation of the problem is presented and the field pressure distribution of the saddle/cylinder is transformed to a discrete set of forces. Secondly, the stresses and displacements of the shell are expressed in terms of these forces q_i, $i = 1, 2, \ldots, N$. Then, the potential energy of the system is presented, and its minimization is obtained. Finally, the method of calculation is developed and some results, conducted to illustrate the theory, are discussed.

2. EDGE LOADING AND SUPPORTS.

The basic equations which describe the static behaviour of cylindrical shells with bending resistance under arbitrary loads are determined from Sanders' equations of equilibrium of thin shells [8], [9]. This shell theory was preferred because all strains vanish for small rigid body motions. This matter is further elaborated is Appendix 1, where the equilibrium equations are given in terms of the axial, circumferential and radial displacements of the middle surface, U, V and W, respectively. The stress-strain and the strain-displacements relations are also listed in Appendix 1.

2.1 Shells under arbitrary edge loading.

In this development we take into account the effects of the boundary conditions. The solution of equation (50) for arbitrary edge loading is given in reference [9], and only an outline will be given here for the benefit of the reader.

We consider motions in the n^{th} circumferential wavenumber and write

$$\left\{\begin{matrix} U(x,\phi) \\ W\ (x,\phi) \\ V\ (x,\phi) \end{matrix}\right\}_c = \sum_{n=0}^{\infty} \left[T_n\right] \left\{\begin{matrix} u_n(x) \\ w_n(x) \\ v_n(x) \end{matrix}\right\} , \qquad (1)$$

where $\left[T_n\right]$ is given in Appendix 2. Substituting this into Sanders' equations of equilibrium (50) by assuming $p_x = p_\phi = p_r = 0$, and letting

$$u_n(x) = A\ e^{\lambda x/r} , \qquad v_n(x) = B\ e^{\lambda x/r} , \qquad w_n(x) = C\ e^{\lambda x/r} \qquad (2)$$

leads to three simultaneous equations in A, B, C and to a vanishing determinant, for non-trivial solution; this yields a characteristic equation which is an octic in λ. The complete solution is a summation of the eight independent solutions and involves the constants A_p, B_p, C_p $p = 1, 2, \ldots, 8$. As these constants are not independent, we express A_p and B_p in terms of C_p, we write $A_p = \alpha_p C_p$ and $B_p = \beta_p C_p$; in this way we can then express u_n, v_n w_n in terms of only eight constants $\bar{C}_p$ which are linear combination of C_p. This leads to the equation

$$\begin{Bmatrix} U(x,\phi) \\ W(x,\phi) \\ V(x,\phi) \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{bmatrix} T_n \end{bmatrix} \begin{bmatrix} R \end{bmatrix} \begin{Bmatrix} C_n \end{Bmatrix} , \tag{3}$$

where $\begin{bmatrix} R \end{bmatrix}$ is a 3x8 matrix given in [9], and $\begin{Bmatrix} C_n \end{Bmatrix}$ is the eight-order vector of the constants $\bar{C}_p$.

The strains are related to the displacements through equations (48) and (3); accordingly, we express $\begin{Bmatrix} \varepsilon \end{Bmatrix}$ in terms of $\begin{Bmatrix} C_n \end{Bmatrix}$ by

$$\begin{Bmatrix} \varepsilon \end{Bmatrix} = \sum_n \begin{bmatrix} T_n & 0 \\ 0 & T_n \end{bmatrix} \begin{bmatrix} Q \end{bmatrix} \begin{Bmatrix} C_n \end{Bmatrix} , \tag{4}$$

where the matrices $\begin{bmatrix} T_n \end{bmatrix}$ and $\begin{bmatrix} Q \end{bmatrix}$ are given in reference [9].

The corresponding stress-resultant vector may now be found from equations (49) and (48), i.e.,

$$\begin{Bmatrix} \sigma \end{Bmatrix}_c = \begin{bmatrix} P \end{bmatrix} \begin{Bmatrix} \varepsilon \end{Bmatrix} = \sum_n \begin{bmatrix} T_n & 0 \\ 0 & T_n \end{bmatrix} \begin{bmatrix} P \end{bmatrix} \begin{bmatrix} Q \end{bmatrix} \begin{Bmatrix} C_n \end{Bmatrix} , \tag{5}$$

where $\begin{bmatrix} P \end{bmatrix}$, the elasticity matrix, is given by equation (49) for isotropic shells.

2.2 Modeling of the support

A given cylindrical shell partially-filled with liquid, pressurized and having a surface of contact with a rectangular rigid support of dimensions (a, 2b) is shown in Figure 2. The locations of the support on the vessel is arbitrary, with coordinates δ_o and (x_o, x_f) in the circumferential and axial directions, respectively. The theory developed here for one support may be applied to shells having two or more supports. Also the assumption of a vertical plane of symmetry of the loads through the axis of the shell, permits the investigation of only half the support (a, b).

These dimensions "a" and "b" are first subdivided into N_1 and N_2 line elements, respectively; and then by assuming two point loads on the boundaries A and B of the support, Fig. 2, N finite elements are distributed over the area $(a \cdot b)$, as follows: 1) two concentrated loads of densities q_i (lb or kg), $i = 1, 2$, applied on the points A and B; 2) $(N_1 + 2N_2)$ line loads applied on the line elements of densities q_i (lb/in or kg/m), $i = 3, 4, \ldots, 2N_2 + N_1 + 2$; and 3) N_1N_2 surface loads of densities q_i (lb/in^2 or kg/m^2), $i = 2N_2 + N_1 + 3, \ldots, N$; where N is given by

$$N = N_1N_2 + 2N_2 + N_1 + 2 \quad . \tag{6}$$

The pressure distribution on the possible surface of contact between the rigid support and the shell is represented by the densities q_i's where $i = 1, 2, \ldots, N$.

3. STRESSES AND DISPLACEMENTS DUE TO ARBITRARY LOADING

In this section we shall develop a general procedure in order to obtain the displacements and the stresses induced by the q_i's, fluid surcharge pressure, weight of the vessel and its heads. To do so, the loads and the displacements are first represented by double Fourier series and then inserting these series into the equations of motion, the coefficients of the Fourier series are obtained for the displacement components in terms of the load factors p_{xmn}, $p_{\phi mn}$ and p_{rmn} for the axial, circumferential and radial directions, respectively. The second step is to determine these loads factors p_{xmn}, $p_{\phi mn}$ and p_{rmn} in terms of the unknowns q_i's for each particular case.

3.1 Arbitrary loadings.

Using the method of Fourier expansions and assuming that, for a cylinder of length ℓ,

$$\begin{Bmatrix} p_x \\ p_r \\ p_\phi \end{Bmatrix} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \begin{bmatrix} T_n \end{bmatrix} \begin{bmatrix} T_{mx} \end{bmatrix} \begin{Bmatrix} p_{xmn} \\ p_{rmn} \\ p_{\phi mn} \end{Bmatrix} , \tag{7}$$

and

$$\begin{Bmatrix} U_p \\ W_p \\ V_p \end{Bmatrix} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \begin{bmatrix} T_n \end{bmatrix} \begin{bmatrix} T_{mx} \end{bmatrix} \begin{Bmatrix} u_{mn} \\ w_{mn} \\ v_{mn} \end{Bmatrix} , \tag{8}$$

it is possible to obtaine, by introducing (7) and (8) into (50), the coefficients of the Fourier series for the displacement components as follows

$$\begin{Bmatrix} u_{mn} \\ w_{mn} \\ v_{mn} \end{Bmatrix} = \frac{r^2}{D} \begin{bmatrix} A_F \end{bmatrix}^{-1} \begin{Bmatrix} p_{xmn} \\ -p_{rmn} \\ p_{\phi mn} \end{Bmatrix} , \qquad (9)$$

where m and n are the axial and circumferential wave number, respectively, and the matrices $\begin{bmatrix} T_n \end{bmatrix}$, $\begin{bmatrix} T_{mn} \end{bmatrix}$ and $\begin{bmatrix} A_F \end{bmatrix}$ are shown in Appendix 2.

When the loading can be represented by Fourier series of the form

$$\begin{Bmatrix} p_x \\ p_r \\ p_\phi \end{Bmatrix} = \sum_{n=0,2,3,4,\dots,}^{\infty} \begin{bmatrix} T_n \end{bmatrix} \begin{Bmatrix} p_{xon} \\ p_{ron} \\ p_{\phi on} \end{Bmatrix} , \qquad (10)$$

The displacements can, in general, be assumed to have the form

$$\begin{Bmatrix} U_p \\ W_p \\ V_p \end{Bmatrix} = \sum_{n=0,2,3,4,\dots,}^{\infty} \begin{bmatrix} T_n \end{bmatrix} \begin{Bmatrix} u_{on} \\ w_{on} \\ v_{on} \end{Bmatrix} , \qquad (11)$$

where the coefficients u_{on}, w_{on} and v_{on} for $n \neq 1$, may be obtained from equation (9) by imposing $m = 0$. However, equation (9) and (11) cannot be applied to cases where the loadings are expressed in terms of the form ($a \cos\phi$) and ($b \sin\phi$); this is due to the fact that expressions in ($d \cos\phi$) and ($g \sin\phi$), for the radial displacement W, correspond to rigid body motion of the shell and therefore do not represent the true displacement caused by loadings expressed in terms of the same form. In order to avoid this difficulty, it was necessary to either expand the constants "a" and "b" in a Fourier sine series of the form $\Sigma D_n \sin (m\pi x/\ell)$ and thus to lengthen the numerical computations, or to obtain a particular solution using shells' membrane theory. It was decided to use the latter alternative, for these particular cases since such solution which is easily obtainable describes adequately the behavior of a cylindrical shell closed by stiff heads.

Such special cases occur when $m = 0$ and $n = 1$, e.g., for the weight of the shell and for the terms, $m = 0$ and $n = 1$, of the fluid pressure. In all other cases, where $(m = 0 \,,\, n \neq 1)$ and $m \neq 0$, the exact solution, developed in previous sections, was used to determine the corresponding displacements and stresses.

3.2 Load factors

Equations (7), (8) and (9) allow the loads and displacements to be expressed in terms of the load factors p_{xmn} , p_{rmn} and $p_{\phi mn}$. The object now is to determine the load factors corresponding to each particular case of loading.

a) Concentrated radial loads.

We assume two concentrated radial forces of intensity q_i (lb or kg) applied at the coordinate $(x,\phi) = (b_i \,,\, \pm\, \delta_i)$, Figure 3.

The load factors corresponding to such forces are as follows

$$p_{rmoi} = \frac{2q_i}{\pi r \ell} \sin \frac{m \pi b_i}{\ell} , \quad m = 1, 2, 3, \ldots, \tag{12a}$$

and

$$p_{rmni} = \frac{4q_i}{\pi r \ell} \cos n \delta_i \sin \frac{m \pi b_i}{\ell} , \quad n,m = 1,2,3,\ldots, \tag{12b}$$

where $i = 1$ and 2. By referring to figure 2 we obtain $\delta_1 = \delta_2 = \delta_0$, $b_1 = x_0$ and $b_2 = x_f$.

b) Line loads on two segments along the generator.

Let the segments, on which a constant inward line load q_i (lb/in or kg/m) is applied, be centered at $(x,\phi) = (b_i, \pm \delta_i)$ and of dimension $2 c_2$ along the axial direction, Figure 4.

The coefficients of the series expansion for this loading is given by

$$p_{rmoi} = \frac{4 q_i}{\pi^2 r m} \sin \frac{m \pi C_2}{\ell} \sin \frac{m \pi b_i}{\ell} , \quad m = 1,2,3,\ldots, \tag{13a}$$

and

$$p_{rmni} = \frac{8 q_i}{\pi^2 r m} \sin \frac{m \pi C_2}{\ell} \cos n \delta_i \sin \frac{m \pi b_i}{\ell} , \quad n,m = 1,2,3,\ldots, \tag{13b}$$

where $i = 3,4,\ldots, N_1 + 2$.

c) Line loads on two segements perpendicular to the generator.

For the cases where the shell is subjected to a line load q_i (lb/in or kg/m) perpendicular to the generator, figure 5, and of dimension $2r \phi_1$, the load factor p_{rmni} is given by

$$p_{rmoi} = \frac{4\ \phi_1\ q_i}{\pi\ \ell} \sin \frac{m\ \pi\ b_i}{\ell}, \qquad m = 1,2,3,..., \tag{14a}$$

and

$$p_{rmni} = \frac{8\ q_i}{\pi\ n\ \ell} \sin n\ \phi_1 \ \cos n\ \delta_i \ \sin \frac{m\ \pi\ b_i}{\ell}, \quad n,m = 1,2,3,..., \tag{14b}$$

where b_i is equal to x_o and x_f for $i = N_1 + 3$ to $N_1 + N_2 + 2$, and for $i = N_1 + N_2 + 3$ to $N_1 + 2N_2 + 2$, respectively; figure 2.

d) Constant pressure uniformly distributed over two rectangular areas.

Consider two rectangular areas subjected to a constant loading q_i, centered at the coordinates $(x,\phi) = (b_i, \pm\delta_i)$, and having the dimensions $2C_2$ and $2r\ \phi_1$ along the axial and circumferential directions, respectively (Figure 6).

The Fourier series expansion of the pressure, p_{ri}, due to the radial loading, q_i, being given by equation (7), one obtains the following expression for the load factor p_{rmni}

$$p_{rmoi} = \frac{8\ \phi_1\ q_i}{\pi^2\ m} \sin \frac{m\ \pi\ C_2}{\ell} \sin \frac{m\ \pi\ b_i}{\ell}, \quad \text{for} \quad m = 1,2,3,..., \tag{15a}$$

and

$$p_{rmni} = \frac{16\ q_i}{\pi^2\ m\ n} \sin n\ \phi_1 \ \sin \frac{m\ \pi\ C_2}{\ell} \cos n\ \delta_i \ \sin \frac{m\ \pi\ b_i}{\ell},$$

for $m,n = 1,2,3,...,$ (15b)

where $i = N_1 + 2N_2 + 3, ..., N$; and N is given by equation (6).

e) Pressure corresponding to a partially-filled vessel.

The pressure distribution on a shell partially or completely filled with stationary liquid is shown in figure 7. The series expansion for such loading is given by equation (10) and its corresponding load factor may be written in the form

$$p_{ro} = -(\gamma r/\pi)\left[\sin\phi_o - \phi_o\cos\phi_o\right],$$

$$p_{r1} = -(\gamma r/\pi)\left[\phi_o - (\sin 2\phi_o/2)\right],$$

$$p_{rn} = -\left[2\gamma r/(\pi n(n^2-1))\right]\left[\cos\phi_o \sin n\phi_o - n\sin\phi_o\cos n\phi_o\right],$$

$$n = 2,3,\ldots, \qquad (16)$$

where γ (lb/in^3 or kg/m^3) is the specific weight of the fluid and ϕ_o (radians) indicates the level of the liquid in the shell.

f) Surcharge pressure, weight of vessel and heads.

The loading corresponding to a surcharge pressure p_o is given by

$$p_r(x,\phi) = -p_o, \qquad (17)$$

and the loading due to the weight of the vessel may be written as

$$p_\phi(x,\phi) = -\gamma_s t\sin\phi, \qquad (18)$$

$$p_r(x,\phi) = -\gamma_s t\cos\phi, \qquad (19)$$

where γ_s is the specific weight of the shell's material and t the thickness of the vessel.

The heads of the vessel are assumed to be rigid and, consequently, their effects will be taken into account be prescribing the appropriate boundary conditions on the shell's edges.

3.3 Stresses and displacements.

The displacements due to arbitrary loadings may be obtained by introducing the coefficients given by equations (12) - (19) into relations (9) and thence into equations (8) or (11) depending on the applied loadings. The stress-resultant vector, $\{\sigma\} = \{N_x , N_\phi , \bar{N}_{x\phi} , M_x , M_\phi , \bar{M}_{x\phi}\}^T$, for different loadings is obtained by substituting the corresponding displacement relations into equations (48) and (49).

a) Surcharge pressure

The surcharge pressure p_o induces displacements and stresses for the case n=0; accordingly, we may write

$$\left\{ \begin{array}{c} U \\ W \\ \partial W/\partial x \\ V \end{array} \right\}_{p_o} = \left\{ \begin{array}{l} p_o \; r \; x \; (1 - 2\nu)/2Et \\ p_o \; r^2 \; (1 - 0.5\nu)/Et \\ \quad 0 \\ \quad 0 \end{array} \right\}, \tag{20}$$

and

$$\{\sigma\}_{po} = \{p_o \; r/2, \;\; p_o \; r, \; 0, \; 0, \; 0, \; 0\}^T \quad . \tag{21}$$

b) Weight of the shell

By considering that the weight of the shell induces motions in the first circumferential wavenumber, $n = 1$, we obtain

$$\begin{Bmatrix} U \\ \\ W \\ \\ \\ \partial W/\partial x \\ \\ V \end{Bmatrix}_{p_1} = \begin{Bmatrix} -\frac{\gamma_s}{E}\left[\frac{\ell^3}{12r}\left(\frac{4x^3}{\ell^3} - \frac{6x^2}{\ell^2} + 1\right) + \frac{r\ell}{4}(1-4\nu)\left(\frac{1}{2} - \frac{x}{\ell}\right)\right]\cos\phi \\ \\ \frac{\gamma_s}{E}\left[\frac{\ell^4}{12r^2}\left(\frac{x^4}{\ell^4} - \frac{2x^3}{\ell^3} + \frac{x}{\ell}\right) + (x\ell - x^2)\left(2.125 + \frac{\nu}{2}\right) + \right. \\ \left. + \frac{r^2}{4}(4-\nu)\right]\cos\phi \\ \\ \frac{\gamma_s}{E}\left[\frac{\ell^3}{12r^2}\left(\frac{4x^3}{\ell^3} - \frac{6x^2}{\ell^2} + 1\right) + (\ell - 2x)\left(2.125 + \frac{\nu}{2}\right)\right]\cos\phi \\ \\ -\frac{\gamma_s}{E}\left[\frac{\ell^4}{12r^2}\left(\frac{x^4}{\ell^4} - \frac{2x^3}{\ell^3} + \frac{x}{\ell}\right) + (x\ell - x^2)(2.125 + 1.5\nu)\right]\sin\phi \end{Bmatrix}, \quad (22)$$

and

$$\{\sigma\}_{p_1} = \begin{Bmatrix} -\gamma_s t\left[(\ell^2/r)\left(\frac{x^2}{\ell^2} - \frac{x}{\ell}\right) - \frac{r}{4}\right]\cos\phi \\ \\ \gamma_s t \; r\cos\phi \\ \\ -2\gamma_s t\,(0.5\,\ell - x)\sin\phi \\ \\ 0 \\ \\ 0 \\ \\ 0 \end{Bmatrix}, \quad (23)$$

where ℓ, r, t and γ_s are, respectively, the length, mean radius, thickness and specific weight of the shell; ν is Poisson's ratio and E is Young's modulus.

c) Fluid pressure

Upon substituting relations (16) into equations ((9), (11)) and thence into equations (49) we obtain the following expressions for the displacements and the stress-resultants of the shell:

$$\left\{\begin{matrix} U \\ W \\ \partial W/\partial x \\ V \end{matrix}\right\}_{p_f} = \left\{\begin{matrix} U \\ W \\ \partial W/\partial x \\ V \end{matrix}\right\}_{p_f(n=0)} + \left\{\begin{matrix} U \\ W \\ \partial W/\partial x \\ V \end{matrix}\right\}_{p_f(n=1)} + \sum_{n=2}^{\infty} \left[\tau_n\right] \left\{\begin{matrix} u_{on} \\ w_{on} \\ 0 \\ v_{on} \end{matrix}\right\}_{p_f} , \qquad (24)$$

and

$$\{\sigma\}_{p_f} = \{\sigma\}_{p_f(n=0)} + \{\sigma\}_{p_f(n=1)} + \sum_{n=2}^{\infty} \begin{bmatrix} T_n & 0 \\ 0 & T_n \end{bmatrix} \left[E_{on}\right] \left\{\begin{matrix} u_{on} \\ w_{on} \\ v_{on} \end{matrix}\right\}, \qquad (25)$$

where

$$\left\{\begin{matrix} U \\ W \\ \partial W/\partial x \\ V \end{matrix}\right\}_{p_f(n=0)} = \left\{\begin{matrix} -\,(a_o\ r\ x/2Et)\ (1 - 2\nu) \\ -\,(a_o\ r^2/Et)\ (1 - 0.5\nu) \\ 0 \\ 0 \end{matrix}\right\} , \qquad (26)$$

$$\begin{Bmatrix} U \\ \\ W \\ \\ \partial W/\partial x \\ \\ V \end{Bmatrix}_{p_f(n=1)} = \begin{Bmatrix} (a_1/Et)\,\frac{\ell^3}{24r}\left[\left(\frac{4x^3}{\ell^3} - \frac{6x^2}{\ell^2} + 1\right) + \frac{r\ell}{4}\,(1-4\nu)\,\left(\frac{1}{2} - \frac{x}{\ell}\right)\right]\cos\phi \\ \\ -(a_1/Et)\left[\frac{\ell^4}{24r^2}\,\left(\frac{x^4}{\ell^4} - \frac{2x^3}{\ell^3} + \frac{x}{\ell}\right) + 1.125\,(x\ell - x^2)\,\frac{r^2}{4}\,(4-\nu)\right]\cos\phi \\ \\ -(a_1/Et)\left[\frac{\ell^3}{24r^2}\,\left(\frac{4x^3}{\ell^3} - \frac{6x^2}{\ell^2} + 1\right) + 1.125\,(\ell - 2x)\right]\cos\phi \\ \\ (a_1/Et)\left[\frac{\ell^4}{24r^2}\,\left(\frac{x^4}{\ell^4} - \frac{2x^3}{\ell^3} + \frac{x}{\ell}\right) + 1.125\,(x\ell - x^2)\right]\sin\phi \end{Bmatrix} \tag{27}$$

and

$$\{\sigma\}_{p_f(n=0)} = \{-a_o r/2,\ \ -a_o r,\ 0,\ 0,\ 0,\ 0\}^T \quad , \tag{28}$$

$$\{\sigma\}_{p_f(n=1)} = \{a_1\left[\frac{\ell^2}{2r}\,\left(\frac{x^2}{\ell^2} - \frac{x}{\ell}\right) - \frac{r}{4}\right]\cos\phi,\ -ra_1\cos\,\phi,\ a_1\left(\frac{\ell}{2} - x\right)\,\sin\,\phi, 0, 0, 0\}^T, \tag{29}$$

a_o and a_1 are, respectively, equal to p_{ro} and p_{r1} of equations (16); the matrices $\left[T_n\right]$, $\left[\tau_n\right]$ and $\left[E_{on}\right]$ are shown in Appendix 2; and the vector $\{u_{on},\ w_{on},\ v_{on}\}^T_{p_f(n\geq 2)}$ is determined by substituting relations (16) into equations (9).

d) Point, line and surface loads

Similarly, the stresses and displacements due to the applied contentrated loads, line loads and surface loads $(q_i,\ i = 1,\ N)$, Figure 2, may be written as follows

$$\begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_{q_i} = \sum_{n=0}^{\infty} \sum_{n=1}^{\infty} \begin{bmatrix} \tau_n \end{bmatrix} \begin{bmatrix} \tau_{mx} \end{bmatrix} \begin{Bmatrix} \sum\limits_{i=1}^{N} u_{mni} \\ \sum\limits_{i} w_{mni} \\ \sum\limits_{i} (m\pi/\ell) \; w_{mni} \\ \sum\limits_{i} v_{mni} \end{Bmatrix}_{q_i} , \qquad (30)$$

and

$$\{\sigma\}_{q_i} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \begin{bmatrix} T_n & 0 \\ 0 & T_n \end{bmatrix} \begin{bmatrix} X_m & 0 \\ 0 & X_m \end{bmatrix} \begin{bmatrix} P \end{bmatrix} \begin{bmatrix} C_{mn} \end{bmatrix} \begin{Bmatrix} \sum\limits_{i=1}^{N} u_{mni} \\ \Sigma_i \; w_{mni} \\ \Sigma_i \; v_{mni} \end{Bmatrix}_{q_i} , \qquad (31)$$

where P , the elasticity matrix, is determined by equation (49); the matrices $[\tau_n]$, $[\tau_{mx}]$, $[T_n]$, $[X_m]$ and $[C_{mn}]$ are given in Appendix 2; and the vector $\sum\limits_{i=1}^{N} \{u_{mni}, w_{mni}, v_{mni}\}^T_{q_i}$ is evaluated by substituting the relations (12), (13), (14) and (15) corresponding, respectively, to

$i = 1$ to 2, 3 to $N_1 + 2$, $N_1 + 3$ to $N_1 + 2N_2 + 2$, and $N_1 + 2N_2 + 3$ to N,

into equations (9); Figure 2.

4. BOUNDARY CONDITIONS

The displacements caused in the shells by an arbitrary edge loading are given by equation (3). Their final expressions may be written as

$$\begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_C = \sum_{n=0}^{\infty} \begin{bmatrix} \tau_n \end{bmatrix} \begin{bmatrix} AX \end{bmatrix} \{C_n\} \quad , \tag{32}$$

where the matrices $\begin{bmatrix} \tau_n \end{bmatrix}$ and $\begin{bmatrix} AX \end{bmatrix}$ are shown in Appendix 2, and $\{C_n\} = \{\bar{C}_1 , \ldots , \bar{C}_8\}^T$ is a set of eight constants. The corresponding stress-resultant vector $\{\sigma\}_C$ is given by equation (5) in terms of n, x, ϕ and $\{C_n\}$. The $\bar{C}_j$ where $j = 1,\ldots,8$, are the only free constants in our problem and must be determined from eight boundary conditions, four at each edge of constant x.

It was shown by Basset and Lamb that, for an edge of constant x, the boundary forces, $\{F_u , F_w , F_\beta , F_v\}$, are approximated by specified values of the following quantities

$$\{F_u , F_w , F_\beta , F_v\} = \{N_x , Q_x + \frac{1}{r} \frac{\partial \bar{M}_{x\phi}}{\partial \phi} , -M_x , \bar{N}_{x\phi} + \frac{3}{2r} \bar{M}_{x\phi}\} \quad , \tag{33}$$

where Q_x is equal to $\dfrac{\partial M_x}{\partial x} + \dfrac{1}{r} \dfrac{\partial \bar{M}_{x\phi}}{\partial \phi}$; and the appropriate boundary displacements are U, W, $\partial W/\partial x$ and V, respectively, figure 8.

For prescribed boundary conditions, the vector $\{C_n\}$ will be determined for a specific circumferential wavenumber, n. To do so, we express the appropriate forces and displacements at the edges $i(x=0)$ and $j(x=\ell)$ by their amplitudes associated with the n^{th} circumferential wavenumber. At both edges in Fig. 8, the axial, circumferential and radial displacements, as well as a rotation may be defined by the vector

$$\begin{Bmatrix} \delta_{ni} \\ \delta_{nj} \end{Bmatrix} = \{u_{ni}\ ,\ w_{ni},\ (dw_{ni}/dx),\ v_{ni},\ u_{nj},\ w_{nj},\ (dw_{nj}/dx),\ v_{nj}\}^{T}\ . \tag{34}$$

Similarly, the boundary forces may also be defined by

$$\begin{Bmatrix} F_{\delta ni} \\ F_{\delta nj} \end{Bmatrix} = \{F_{uni},\ F_{wni},\ F_{\beta ni},\ F_{vni},\ F_{unj},\ F_{wnj},\ F_{\beta nj},\ F_{vnj}\}^{T}\ ; \tag{35}$$

where all these components u_n, W_n, dW_n/dx, v_n, F_{un}, F_{Wn} , $F_{\beta n}$ and F_{vn} are, respectively, the amplitudes of U, W, $\partial W/\partial x$, V , F_u , F_w , F_β and F_v associated with the n^{th} circumferential wavenumber; the indices i and j are associated with the edges $x = 0$ and $x = \ell$, respectively.

Using the principle of superposition, it is now possible to determine the total boundary forces and displacements for each specific n.

a) for axisymmetric loads where $n = 0$, we have

$$\begin{Bmatrix} \delta_{oi} \\ \delta_{oj} \\ F_{\delta oi} \\ F_{\delta oj} \end{Bmatrix} = \begin{bmatrix} [A] \\ [AF_i] \\ [AF_j] \end{bmatrix} \{C_o\} + \{\tilde{p}_o\} + \{\tilde{p}_{fo}\} + \{\tilde{q}_o\} \quad , \tag{36}$$

b) for the case of the beam-like mode (n=1), we obtain

$$\begin{Bmatrix} \delta_{1i} \\ \delta_{1j} \\ F_{\delta 1i} \\ F_{\delta 1j} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} A \end{bmatrix} \\ \begin{bmatrix} AF_i \end{bmatrix} \\ \begin{bmatrix} AF_j \end{bmatrix} \end{bmatrix} \{C_1\} + \{\tilde{p}_1\} + \{\tilde{p}_{f1}\} + \{\tilde{q}_1\} \quad , \tag{37}$$

and finally the prescribed boundary forces and displacements for non-axisymmetric loads $(n \geqslant 2)$ are defined by

$$\begin{Bmatrix} \delta_{ni} \\ \delta_{nj} \\ F_{\delta ni} \\ F_{\delta nj} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} A \end{bmatrix} \\ \begin{bmatrix} AF_i \end{bmatrix} \\ \begin{bmatrix} AF_j \end{bmatrix} \end{bmatrix} \{C_n\} + \{\tilde{p}_{fn}\} + \{\tilde{q}_n\} \quad , \tag{38}$$

where the matrices $\begin{bmatrix} AF_i \end{bmatrix}$, $\begin{bmatrix} AF_j \end{bmatrix}$, $\{\tilde{p}_o\}$, $\{\tilde{p}_{fo}\}$, $\{\tilde{q}_o\}$, $\{\tilde{p}_1\}$, $\{\tilde{p}_{f1}\}$, $\{\tilde{q}_1\}$, $\{\tilde{p}_{fn}\}$ and $\{\tilde{q}_n\}$ are given in Appendix 2.

In equations (36) to (38), $\begin{bmatrix} A \end{bmatrix}$ is a 8x8 matrix obtained from $\begin{bmatrix} AX \end{bmatrix}$ which is a 4x8 matrix of equation (32): the upper part of $\begin{bmatrix} A \end{bmatrix}$ corresponds to $x = 0$ and the lower one refers to $x = \ell$. The (4x8) matrices $\begin{bmatrix} AF_i \end{bmatrix}_{x=0}$ and $\begin{bmatrix} AF_j \end{bmatrix}_{x=\ell}$ are calculated by introducing equations (5) into relations (33) and (35). The vectors $\{\tilde{p}_o\}$, $\{\tilde{p}_1\}$,$\{\tilde{p}_{fn}\}$ and $\{\tilde{q}_n\}$ where $n = 0$, 1, 2, ..., are the boundary displacements and forces corresponding, respectively, to the surcharge pressure, shell's weight, fluid loads, and point, line and surface loads. All these vectors are obtained by substituting equations ((20) - (31)) into relations (33) , (34) and (35). Eight of the sixteen components of the vector $\{\delta_{ni}, \delta_{nj}, F_{\delta ni}, F_{\delta nj}\}^T$, $n = 0,1,2,\ldots$, represent the total number of support conditions that are to be considered on the edges of the structure.

We now have the stresses and displacements of a shell subjected to external and arbitrary edge loadings given in terms of the unknowns q_i, $i = 1,2,\ldots, N$, by equations ((20) - (32)) with the edge constants $\{C_n\}$ given by equations ((36) - (38)) and the other terms involved given by equations ((12) - (19)).

5. FORMULATION OF THE CONTACT PROBLEM

Had the pressure distribution, q_i , $i = 1, \ldots, N$, been known, the response as expressed by equations ((20) - (32)) would have been the solution to the problem. In the case of unknown q_i's, however, we must proceed differently in order to obtain the pressure distribution and contact area when the applied loads and the supports' configurations are prescribed. In this section we shall establish the conditions for overall equilibrium of the system and express the total potential energy in terms of the q_i's, vessel self weight, surcharge pressure, fluid and edge loadings.

These q_i's may then be obtained by minimizing the quadratic form of the potential energy of the system with the necessary constraints to insure proper contact with the supports. Such minimisation is executed using SUMT method (Sequential Unconstrained Minimum Technique) [12]

5.1 Contact criteria

We consider rigid supports and a cylindrical shell governed by the laws of linear elasticity. As previously stated, the area between the supports and the shell is subdivided into N elements, each of which subjected to an unknown pressure distribution q_i where $i = 1,2,\ldots,N$. The corresponding forces F_{q_i} acting at the discrete points $(i = 1,\ldots, N)$ are as follows, (see Figs. 3,4,5 and 6).

$$\begin{aligned} F_{q_i} &= q_i \ , \quad i = 1, 2 \\ F_{q_i} &= 2C_2\, q_i \ , \quad i = 3, 4, \ldots, N_1 + 2 \ , \end{aligned} \tag{39}$$

$$F_{q_i} = 2\phi_1 r q_i , \quad i = N_1 + 3, \ldots, N_1 + 2N_2 + 2, \tag{39}$$

and

$$F_{q_i} = 4\phi_1 C_2 r q_i , \quad i = N_1 + 2N_2 + 3, \ldots, N ,$$

where N is given by equation (6).

The normal displacement w at any point i in the proposed zone of contact must be smaller or equal to zero (assuming no rigid body motion of the shell). On the other hand, the sum of all vertical components of the forces acting at the discrete points $(i = 1, \ldots, N)$ must balance the total weight of the system. This equilibrium condition can therefore be written as

$$\sum_{i=1}^{N} F_{q_i} \cos \delta_i = P , \tag{40}$$

where P is the total weight of the system and the δ_i's are shown in Figs 3-6. The criterion for contact may be given by the following constraints

$$\text{If } W_{q_i} < 0 , \quad \text{then } F_{q_i} = 0 , \tag{41}$$

and

$$\text{If } W_{q_i} = 0 , \quad \text{then } F_{q_i} \geqslant 0, \tag{42}$$

where W_{q_i} is radial elastic displacement at point i; and the equations (41) and (42) represent the "no contact" and "contact" regions, respectively.

5.2 Potential Energy and its minimization.

The total potential energy, $\pi\,(q_i)$, of the system may be expressed in terms of the strain energy, U_S , and the work of the external and edge forces W_{EX} and W_{BC} , respectively.

The determination of U_S , W_{EX} and W_{BC} is carried out in Appendix 3.

The general expression of the potential energy is given by

$$\pi\,(q_i) = \frac{1}{2}\,U_S - W_{EX} - W_{BC} \,, \qquad (43)$$

where

$$U_S = \frac{\pi r \ell}{2} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left\{ \begin{matrix} \sum\limits_{i=1}^{N} u_{mni} \\ \sum\limits_{i} w_{mni} \\ \sum\limits_{i} v_{mni} \end{matrix} \right\}_{q_i}^{T} \begin{bmatrix} C_{mn} \end{bmatrix}^T \begin{bmatrix} P \end{bmatrix} \begin{bmatrix} C_{mn} \end{bmatrix} \left\{ \begin{matrix} \sum\limits_{i=1}^{N} u_{mni} \\ \sum\limits_{i} w_{mni} \\ \sum\limits_{i} v_{mni} \end{matrix} \right\}_{q_i} +$$

$$+ 4\pi r \ell \sum_{m=1,3,5,\dots}^{\infty} (1/m) \left\{ \begin{matrix} u_{01} \\ w_{01} \\ v_{01} \end{matrix} \right\}_{P_1}^{T} \begin{bmatrix} C_{01} \end{bmatrix}^T \begin{bmatrix} P' \end{bmatrix} \begin{bmatrix} C_{m1} \end{bmatrix} \left\{ \begin{matrix} \sum\limits_{i} u_{m1i} \\ \sum\limits_{i} w_{m1i} \\ \sum\limits_{i} v_{m1i} \end{matrix} \right\}_{q_i} + \qquad (44)$$

$$+ 4\pi r \ell \sum_{n=0}^{\infty} \sum_{m=1,3,5,}^{\infty} (1/m) \begin{Bmatrix} u_{on} \\ w_{on} \\ v_{on} \end{Bmatrix}_{p_f}^{T} \begin{bmatrix} C_{on} \end{bmatrix}^{T} \begin{bmatrix} P' \end{bmatrix} \begin{bmatrix} C_{mn} \end{bmatrix} \begin{Bmatrix} \sum_i u_{mni} \\ \sum_i w_{mni} \\ \sum_i v_{mni} \end{Bmatrix}_{q_i} +$$

$$+ 4\pi r \sum_{m=1,3,5,}^{\infty} \frac{\ell}{m\pi} \left[1 - (-1)^m \right] \begin{Bmatrix} \sum_{i=1}^{N} u_{moi} \\ \sum_i w_{moi} \\ \sum_i v_{moi} \end{Bmatrix}_{q_i}^{T} \begin{bmatrix} C_{mo} \end{bmatrix}^{T} \begin{bmatrix} P' \end{bmatrix} \begin{Bmatrix} p_o \; r/D \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{p_o} + \qquad (44)$$

+ constant,

$$W_{BC} = \pi r \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \begin{Bmatrix} \sum_{i=1}^{N} u_{mni} \\ 0 \\ \sum_i (m\pi/\ell) \; w_{mni} \\ 0 \\ \sum_i (-1)^m \; u_{mni} \\ 0 \\ \sum_i (-1)^m (m\pi/\ell) \; w_{mni} \\ 0 \end{Bmatrix}_{q_i}^{T} \begin{bmatrix} - D_{on} \\ D_{on} \end{bmatrix} \begin{Bmatrix} u_{on} \\ w_{on} \\ v_{on} \end{Bmatrix}_{p_f} + \qquad (45)$$

$$+ \pi r \sum_{m=1}^{\infty} \begin{Bmatrix} \sum_{i=1}^{N} u_{m1i} \\ 0 \\ \sum_i (m\pi/\ell)\ w_{m1i} \\ 0 \\ \sum_i (-1)^m\ u_{m1i} \\ 0 \\ \sum_i (-1)^m (m\pi/\ell)\ w_{m1i} \\ 0 \end{Bmatrix}_{q_i}^{T} \begin{bmatrix} -D_{01} \\ D_{01} \end{bmatrix} \begin{Bmatrix} u_{01} \\ w_{01} \\ v_{01} \end{Bmatrix}_{p_1} +$$

$$+ 2\pi r \sum_{m=1}^{\infty} \begin{Bmatrix} \sum_{i=1}^{N} u_{moi} \\ 0 \\ \sum_i (m\pi/\ell)\ w_{moi} \\ 0 \\ \sum_i (-1)^m\ u_{moi} \\ 0 \\ \sum_i (-1)^m (m\pi/\ell)\ w_{moi} \\ 0 \end{Bmatrix}_{q_i}^{T} \begin{Bmatrix} -\nu\ p_o\ r \\ 0 \\ 0 \\ 0 \\ \nu\ p_o\ r \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{p_o} + \qquad (45\ \text{cont})$$

$$+ \pi r \sum_{m=1}^{\infty} \begin{Bmatrix} u_{01} \\ w_{01} \\ 0 \\ v_{01} \\ u_{01} \\ w_{01} \\ 0 \\ v_{01} \end{Bmatrix}_{p_1}^{T} \begin{bmatrix} S_i \\ S_j \end{bmatrix}_{n=1} \begin{Bmatrix} \sum_i u_{m1i} \\ \sum_i w_{m1i} \\ \sum_i v_{m1i} \end{Bmatrix}_{q_i} + 2\pi r \sum_{m=1}^{\infty} \begin{Bmatrix} 0 \\ p_o\, r^2/D \\ 0 \\ 0 \\ p_o\, r\, \ell(1-2\nu)/2Et \cdot \\ p_o\, r^2/D \\ 0 \\ 0 \end{Bmatrix}_{p_o}^{T}$$

(45 cont)

$$\cdot \begin{bmatrix} S_i \\ S_j \end{bmatrix}_{n=0} \begin{Bmatrix} \sum_i u_{moi} \\ \sum_i w_{moi} \\ \sum_i v_{moi} \end{Bmatrix}_{q_i} + \pi r \sum_n \sum_m \begin{Bmatrix} u_{on} \\ w_{on} \\ 0 \\ v_{on} \\ \omega_{on} \\ w_{on} \\ 0 \\ v_{on} \end{Bmatrix}_{p_f}^{T} \begin{bmatrix} S_i \\ S_j \end{bmatrix} \begin{Bmatrix} \sum_i u_{mni} \\ \sum_i w_{mni} \\ \sum_i v_{mni} \end{Bmatrix}_{q_i} + \text{constant},$$

and

$$W_{EXT} = 4rp_o\ell \sum_{m=1,3,5}^{\infty} (1/m) \left(\sum_{i=1}^{N} w_{moi} \right) q_i + \qquad (46)$$

$$+ 2r\gamma_s t\ell \sum_{m=1,3,5}^{\infty} (1/m) \sum_i (w_{mli} - v_{mli})_{q_i} +$$

$$+ 2r\ell \sum_{n=0}^{\infty} p_{fn} \sum_{m=1,3,5}^{\infty} (1/m) \left(\sum_i w_{mni}\right)_{q_i} + \text{constant}; \tag{46}$$

In equations (44) - (46) the words "constant" represent all the terms of the potential energy which are not function of the interface pressure distribution q_i, $i = 1,\ldots,N$; the vectors $\left[\sum_i u_{mni}, \sum_i w_{mni}, \sum_i v_{mni}\right]^T_{q_i}$ and $\left[u_{01}, w_{01}, v_{01}\right]^T_{P_1}$ are given by equations (30) and (22), respectively; the displacement factors $\left[u_{on}, w_{on}, v_{on}\right]^T_{P_f}$ due to the fluid pressure are determined by substituting relations (16) into equation (9); the matrices $\left[C_{mn}\right]$ $\left[D_{on}\right]$ and $\left[P\right]$ are given in tables 1, 7 and equation (49), respectively; $\left[P'\right]$ is equivalent to matrix $\left[P\right]$ with $p'(3,j) = p'(6,j) = 0$, $j = 1,\ldots,6$; and finally the general terms of the matrices $\left[S_i\right]$ and $\left[S_j\right]$ are given by $S_i(p,q) = S_{mn}(p,q)$, $S_j(p,q) = S_{mn}(p+4,q)$, $p = 1, \ldots, 4$, $q = 1,\ldots,3$ and $\left[S_{mn}\right]$ is listed in table 7.

The original contact problem may then be stated as follows:

$$\left.\begin{array}{l} \text{Minimize } \pi(q_i), \\[2ex] \text{such that } \displaystyle\sum_{i=1}^{N} F_{q_i} \cos\delta_i = P, \end{array}\right\} \tag{47}$$

and subject to the condition that either $W_{qi} = 0$ or $F_{q_i} = 0$;

$F_{q_i} \geqslant 0$, $W_{q_i} \geqslant 0$; where π, F_{q_i} and P are given by equations (43),

(39) and (40) , respectively.

6. CALCULATIONS AND DISCUSSION

To determine the interface pressure distribution q_i's, the displacements and the stresses of a given cylindrical vessel completely or partially filled with liquid, pressurized and having a surface of contact with rigid supports, we first specify the imposed boundary conditions, their number J, the location and dimensions of the supports. The surfaces of these supports are then subdivided into a sufficient number, N, of line and surface elements each of which subjected to a load q_i where i = 1, 2, ..., N (sufficiency in this context is related to the required degree of precision). A computer programme, written in Fortran V language for the CDC CYBER 74 computer, determines all the q_i's by minimizing, for given input data, the potential energy of the system, calculates the displacements and stresses for each particular loading, obtains the values of the constants $\{C_n\}$ corresponding to the appropriate boundary conditions and finally determines the total response at any point of the structure.

The additional necessary input data are the mean radius r, wall thickness t, length of the vessel ℓ, specific weights of material of the shell and of the fluid γ_S and γ, respectively; Poisson's ratio ν, modulus of elasticity E, internal pressure p_o and the angle ϕ_o (rad.) which represents the level of the liquid in the shell.

The computer programme proceeds as follows for each circumferential wavenumber ($n \geqslant 0$).

(1) The eight complex roots, λj, of the characteristic equation given in Appendix 2, are calculated by the Newton-Raphson iterative technique, and hence, we obtain the parameters κ_1, κ_2, μ_1, μ_2, α_p, β_p and $\bar{\alpha}_p$. $\bar{\beta}_p$

(p = 1, 2,...,8) shown in the matrices $\begin{bmatrix} Q \end{bmatrix}$, $\begin{bmatrix} AX \end{bmatrix}$, $\begin{bmatrix} AF_i \end{bmatrix}$ and $\begin{bmatrix} AF_j \end{bmatrix}$.

(2) The displacements and stress-resultant matrices corresponding to the surcharge pressure, shell's weight and fluid pressure are computed, respectively, by the relations given by equations (20) - (21), (22) - (23), and (24) - (29).

(3) The total potential energy of the system, $\pi\ (q_i)$, given by equations (43) - (46) in terms of the unknown q_i's , i = 1, ..., N, and subjected to the conditions given by relation (47) is minimized by using SUMT method (Sequential Unconstrained Minimum Technique) [12] in order to determine the pressure distribution. Then, the stresses and displacements due to the q_i's are computed by the relationships given by equations (30) - (31).

(4) If the boundary conditions of the shell are under consideration, then appropriate rows of relations (36) - (38) are deleted to satisfy these conditions, reducing the matrix equations to one of order J where J is the number of boundary conditions imposed. Thus for non-axisymmetrical loads, four boundary conditions will have to be prescribed at each edge and J = 8; and for axisymmetric loads (n = 0) only two boundary conditions will be required at each edge and J = 4. With the reduced equations ((36) - (37)) and its intermediate matrices determined, the computer program proceeds to find the eight constants of the vector $\{C_n\}$; and consequently, the stress-resultants and displacements caused by the imposed boundary conditions and given, respectively, by equations (5) and (32) may be determined at any point of the shell.

This analysis proceeds separately for each circumferential wavenumber n, and the total response may then be found by summing over n. The total number of n required for the computation is reached when the relative error of each displacement components approaches 10^{-9}.

The necessary time for the minimization of the potential energy in order to obtain the q_i's seems to be high. However if only a few elements are used in the calculation, the response may be computed to an acceptable degree of accuracy, but with saving in computational cost.

Of course, as the answer to any particular problem has to be obtained numerically, the proof of usefulness of this theory, as compared to other theories, will also depend on its efficiency (in terms of cost and effort), as well as its precision. A paper currently under preparation will deal with the numerical minimization of the potential energy, its limitations and usefulness as well as the necessary simplifications to avoid high computational cost. Two categories of contact problem will be dealt with: (a) the evaluation of the contact area and the pressure distribution when the applied loads are known and (b) the design of systems capable of giving the best possible distribution of pressure over the contacting regions.

Here only one typical case has been calculated in order to check the correctness of the theory. This calculation was undertaken to determine the displacements and stresses of a particular twin saddle supported vessel partially filled with liquid. The shell analysed is one already studied by Forbes and Tooth [5], with whose results those of this theory will be compared. The vessel, manufactured from an aluminium alloy sheet, is subjected at various water levels to self weight and interface pressure. The resulting interface pressure distribution between the saddle and the vessel is shown in Figure 9 as given by reference [5]. The data of the simply supported vessel are as follows: $r = 5.625$ in (14.287 cm), $t = 0.028$ in (0.07112 cm), $\ell = 53$ in (1.3462 m), $E = 10^7$ lbf/in^2 (0.703103×10^{10} kg/m^2), $\nu = 0.3$, $\gamma = 0.03611$ lbf/in^3 (999.52 kg/m^3), $\gamma_S = 0.09754$ lbf/in^3 (2.7×10^3 kg/m^3). The saddles are maintained at a constant distance ($11\,\ell/60$) from the vessel's ends.

In the experiments of [5] the liquid depth was varied such that the angle which represent the level of the liquid in the shell took the values $\phi_o = 0.4644\,\pi$, $0.60833\,\pi$, $0.78333\,\pi$ and π radians. The effects of the closed ends were taken into account. For each ϕ_o, the radial and circumferential displacements were measured by Forbes and Tooth at the saddle outer rim profile and at the vessel center profile for a number of values of the circumferential coordinate, ϕ. Forbes and Tooth also developed a method based on Flügge's theory which, however, only applies when the shell is empty ($\phi_o = 0$) or completely full ($\phi_o = \pi$). In their investigation, the distribution and magnitude of the interface pressure is assumed to be (1) the same for both saddles, (2) symmetric with respect to the generator passing through the center of the saddle arc and (3) constant accross the saddle width. Finally the saddle arc length is subdivided into a series of equal angular parts each of axial length equal to the width of the saddle and loaded by a uniform pressure.

The stresses and displacements were calculated by our theory using 20 and 12 elements in the axial and circumferential directions, respectively, for a total number of 286 elements (see equation 6) in the case of $\phi_o = 0.4644\,\pi$, $0.60833\,\pi$, $0.78333\,\pi$ and π radians. In all cases the finite elements were of equal length. Figures 10-15 show our computed results compared with the experimental data of [5]. Agreement between theory and experiment is quite good in most cases.

One noteworthy observation is that, generally, the theory somewhat overestimates the radial displacements, w, at all fractional fillings, while it estimates them almost accurately when the shell is completely filled. A possible, reasonable explanation is that the shell is not ideally simply-supported assuming of course that the experimental values are correct.

CONCLUSION

As developed previously, the present theory and the computer program based upon it, is capable of determining the stresses, displacements and the interface pressure distribution of the general case of a thin cylindrical shell with arbitrary boundary conditions, pressurized, partially or completely filled with liquid and having two or more supports. To this end the supports are subdivided, in the circumferential and axial directions, into a sufficient number, N, of line and surface elements each of which subjected to a load q_i, $i = 1, \ldots, N$. SUMT method (Sequential Unconstrained Minimum Technique) is used to minimize the potential energy in order to determine those q_i's.

This theory was computarized so that if the dimensions and material properties of the vessel, and the properties of the saddle, are given as inputs, the program gives as output the displacements and stresses at any point of the structure. The analysis proceeds separately for each circumferential wavenumber, n, and the total stresses and displacements may than be found by summing over n.

As stated previously, further computations are under way to test the limitations and the usefulness of the minimization of the potential energy and to design a system capable of giving the best possible distribution of pressure over the contacting regions. Such development is the subject of another paper currently under preparation.

A number of cases, the authors believe, could have been takled to illustrate the capabilities of the theory but were not because of the computational cost. Thus, shells with several discontinuities in loading could be analysed with the same ease as those presented here. Similarly, cases of anisotropic shells may be analysed equally easily [13].

APPENDIX 1

Shells' equations of motion.

In this development we use a shell's theory based on Love's first approximation which is quite adequate for thin shells. Most forms of the equations of motion based on this approximation contain an inconsistency, namely that, except for the special case of axisymmetric loading, the strains do not all vanish for small rigid body rotations (e.g., the theories of Love, Reissner [10] and Timoshenko [11]). Here we shall use a theory developed by Sanders [8] , which removes this inconsistency, remarkably, without complicating the equations. Consequently, as is shown in [8] , all strains obtained by Sanders vanish for small rigid-body motions. These modified strain-displacmeent relations are given by

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\phi \\ 2\varepsilon_{x\phi} \\ \kappa_x \\ \kappa_\phi \\ 2\bar{\kappa}_{x\phi} \end{Bmatrix} = \begin{Bmatrix} \partial U/\partial x \\ (1/r)\ (\partial V/\partial\phi) + (W/r) \\ \partial V/\partial x + (1/r)\ (\partial U/\partial\phi) \\ -\ \partial^2 W/\partial x^2 \\ -\ (1/r^2)\ \left[(\partial^2 W/\partial\phi^2) - (\partial V/\partial\phi) \right] \\ -\ (2/r)\ (\partial^2 W/\partial x\partial\phi) + (3/2r)\ (\partial V/\partial x) - (1/2r^2)\ (\partial U/\partial\phi). \end{Bmatrix} \tag{48}$$

where U, V and W are, respectively, the axial, circumferential and radial displacements of the middle surface of the shell and r its mean radius.

The appropriate set of stress-strain relations (see Fig. 1) is given by

$$\{\sigma\} = \{N_x , N_\phi , \bar{N}_{x\phi} , M_x , M_\phi , \bar{M}_{x\phi}\}^T = \Big[P\Big]\{\varepsilon\} \quad , \tag{49}$$

where $\Big[P\Big]$, the elasticity matrix for an isotropic elastic material, is given by

$$\Big[P\Big] = \begin{bmatrix} D & \nu D & 0 & 0 & 0 & 0 \\ \nu D & D & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{D(1-\nu)}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & K & \nu K & 0 \\ 0 & 0 & 0 & \nu K & K & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{D(1-\nu)}{2} \end{bmatrix}$$

and the stiffness parameters K and D are given by

$$K = Et^3/12(1-\nu^2) \ ,$$

$$D = Et/(1-\nu^2)$$

where t is the thickness of the shell and ν is Poisson's ratio.

Upon substituting equations (48) and (49) into Sanders' equations of motion, one obtains the equations of equilibrium of a circular cylindrical shells in terms of U, V and W, namely

$$r^2 \frac{\partial^2 U}{\partial x^2} + \frac{(1-\nu)}{2} \frac{\partial^2 U}{\partial \phi^2} + \frac{r(1+\nu)}{2} \frac{\partial^2 V}{\partial x\, \partial \phi} + r\nu \frac{\partial W}{\partial x} + k\Big[\frac{(1-\nu)}{8} \frac{\partial^2 U}{\partial \phi^2} - $$

$$- \frac{3(1-\nu)}{8} r \frac{\partial^2 V}{\partial x\, \partial \phi} + \frac{(1-\nu)}{2} r \frac{\partial^3 W}{\partial x\, \partial \phi^2}\Big] = - p_x \frac{r^2}{D} \quad ,$$

$$\frac{(1+\nu)r}{2} \frac{\partial^2 U}{\partial x\, \partial \phi} + \frac{\partial^2 V}{\partial \phi^2} + \frac{(1-\nu)r^2}{2} \frac{\partial^2 V}{\partial x^2} + \frac{\partial W}{\partial \phi} + k\Big[- \frac{3(1-\nu)r}{8} \frac{\partial^2 U}{\partial x\, \partial \phi} + \tag{50}$$

$$+ \frac{9}{8}(1-\nu)r^2 \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial \phi^2} - \frac{(3-\nu)r^2}{2} \frac{\partial^3 W}{\partial x^2\, \partial \phi} - \frac{\partial^3 W}{\partial \phi^3}\Big] = - p_\phi \frac{r^2}{D} \quad ,$$

$$- \nu r \frac{\partial U}{\partial x} - \frac{\partial V}{\partial \phi} - W + k\Big[\frac{(\nu-1)r}{2} \frac{\partial^3 U}{\partial x\, \partial \phi^2} + \frac{(3-\nu)r^2}{2} \frac{\partial^3 V}{\partial x^2\, \partial \phi} +$$

$$+ \frac{\partial^3 V}{\partial \phi^3} - r^4 \frac{\partial^4 W}{\partial x^4} - 2r^2 \frac{\partial^4 W}{\partial x^2 \partial \phi^2} - \frac{\partial^4 W}{\partial \phi^4} \Bigg] = p_r \frac{r^2}{D} \quad ,$$

where k = (1/12) $(t/r)^2$.

APPENDIX 2

In this Appendix are given the matrices referred to in the text which were too large to be included therein.

The matrices are listed as follows.

$\left[T_n\right]$, $\left[T_{mx}\right]$, $\left[\tau_n\right]$, $\left[X_m\right]$, $\left[\tau_{mx}\right]$ (see table 1)

$\left[E_{on}\right]$, $\left[C_{mn}\right]$, $\left[A_F\right]$ (see table 2)

$\left[AX\right]$ (see table 3)

$\left[AF_i\right]$ (see table 4)

$\left[AF_j\right]$ (see table 5)

$\{\tilde{p}_o\}$, $\{\tilde{p}_{fo}\}$, $\{\tilde{p}_1\}$, $\{\tilde{p}_{f1}\}$ (see table 6)

$\{\tilde{p}_{fn}\}_{n \geqslant 2}$, $\{\tilde{q}_n\}_{n \geqslant 0}$ (see table 7)

The quantities ω_1 , ω_2 , η_1 , η_2 , ψ_1 , ψ_2 , ζ_1 and ζ_2 which appear in the matrices $\left[AX\right]$, $\left[AF_i\right]$ and $\left[AF_j\right]$ are given by $\omega_j = \kappa_j \ell/r$, $\eta_j = \mu_j \ell/r$, $\psi_j = \kappa_j x/r$, $\zeta_j = \mu_j x/r$; $j = 1$, 2 , where κ_1 , κ_2 , μ_1 , μ_2 are the real and imaginary components of the eight characteristic values λ_p , which may be written as [9] ,

$$\lambda_{1,2} = -\kappa_1 \pm \mu_1 i \quad , \quad \lambda_{3,4} = -\kappa_2 \pm \mu_2 i \ ,$$

$$\lambda_{5,6} = \kappa_1 \pm \mu_1 i \quad , \quad \lambda_{7,8} = \kappa_2 \pm \mu_2 i \ .$$

The quantities $\bar{\alpha}_p$, $\bar{\beta}_p$, $p = 1 , 2 , \ldots , 8$, are real or imaginary parts of the α_p and β_p defined in the paragraph following equation (2), such that $\alpha_{1,2} = \bar{\alpha}_1 \pm \bar{\alpha}_2 i$, $\alpha_{3,4} = \bar{\alpha}_3 \pm \bar{\alpha}_4 i$, ... , $\bar{\alpha}_{7,8} = \bar{\alpha}_7 \pm \bar{\alpha}_8 i$; and similarly for the $\bar{\beta}_p$. The method for determining α_p , β_p is given in ref. [9] .

Table 1. Matrices $\left[T_n\right]$, $\left[T_{mx}\right]$, $\left[\tau_n\right]$, $\left[X_m\right]$ and $\left[\tau_{mx}\right]$.

$$\left[T_n\right] = \begin{bmatrix} \cos n\phi & 0 & 0 \\ 0 & \cos n\phi & 0 \\ 0 & 0 & \sin n\phi \end{bmatrix}$$

$$\left[T_{mx}\right] = \begin{bmatrix} \cos(m\pi x/\ell) & 0 & 0 \\ 0 & \sin(m\pi x/\ell) & 0 \\ 0 & 0 & \sin(m\pi x/\ell) \end{bmatrix}$$

$$\left[\tau_n\right] = \begin{bmatrix} \cos n\phi & 0 & 0 & 0 \\ 0 & \cos n\phi & 0 & 0 \\ 0 & 0 & \cos n\phi & 0 \\ 0 & 0 & 0 & \sin n\phi \end{bmatrix}$$

$$\left[X_m\right] = \begin{bmatrix} \sin(m\pi x/\ell) & 0 & 0 \\ 0 & \sin(m\pi x/\ell) & 0 \\ 0 & 0 & \cos(m\pi x/\ell) \end{bmatrix}$$

$$\left[\tau_{mx}\right] = \begin{bmatrix} \cos(m\pi x/\ell) & 0 & 0 & 0 \\ 0 & \sin(m\pi x/\ell) & 0 & 0 \\ 0 & 0 & \cos(m\pi x/\ell) & 0 \\ 0 & 0 & 0 & \sin(m\pi x/\ell) \end{bmatrix}$$

Table 2. Matrices $\left[E_{on}\right]$, $\left[C_{mn}\right]$ and $\left[A_F\right]$.

$$\left[E_{on}\right] = \begin{bmatrix} 0 & \nu D/r & \nu nD/r \\ 0 & D/r & nD/r \\ -nD(1-\nu)/2r & 0 & 0 \\ 0 & \nu n^2K/r^2 & \nu nK/r^2 \\ 0 & n^2K/r^2 & nK/r^2 \\ nK(1-\nu)/4r^2 & 0 & 0 \end{bmatrix}$$

$$\left[C_{mn}\right] = \begin{bmatrix} -m\pi/\ell & 0 & 0 \\ 0 & 1/r & n/r \\ -n/r & 0 & m\pi/\ell \\ 0 & (m\pi/\ell)^2 & 0 \\ 0 & n^2/r^2 & n/r^2 \\ n/2r^2 & (2n/r)\cdot(m\pi/\ell) & (3/2r)\cdot(m\pi/\ell) \end{bmatrix}$$

$$A_F\ (1,1) = r^2(m\pi/\ell)^2 + n^2\frac{(1-\nu)}{2}\left(1+\frac{k}{4}\right)$$

$$A_F\ (1,2) = -r\frac{(m\pi)}{\ell}\left[\nu-\frac{(1-\nu)}{2}\ k\ n^2\right]$$

$$A_F\ (1,3) = \frac{-rn}{2}\ \frac{(m\pi)}{\ell}\left[\left(1-\frac{3k}{4}\right)+\nu\left(1+\frac{3k}{4}\right)\right]$$

$$A_F(2,1) = A_F\ (1,2)$$

$$A_F\ (2,2) = 1+kr^4\left[\frac{n^2}{r^2}+(m\pi/\ell)^2\right]^2$$

$$A_F\ (2,3) = n(1+n^2k)+\frac{(3-\nu)}{2}\ kr^2n\ (m\pi/\ell)^2$$

$$A_F\ (3,1) = A_F\ (1,3)$$

$$A_F\ (3,2) = A_F\ (2,3)$$

$$A_F\ (3,3) = \frac{(1-\nu)}{2}r^2\ (m\pi/\ell)^2\left(1+\frac{9k}{4}\right)+n^2(1+k)$$

Table 3. Matrix $\left[\underset{4\times 8}{AX}\right]$

$$\begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_c = \sum_n \underset{4\times 4}{\left[\tau_n\right]} \underset{4\times 8}{\left[AX\right]} \underset{8\times 1}{\{C\}}$$

$$AX\ (1,1) = e^{-\Psi_1}\left[\bar{\alpha}_1 \cos\zeta_1 - \bar{\alpha}_2 \sin\zeta_1\right] \quad ; \quad AX\ (2,1) = e^{-\Psi_1} \cos\zeta_1$$

$$AX\ (1,2) = e^{-\Psi_1}\left[\bar{\alpha}_2 \cos\zeta_1 + \bar{\alpha}_1 \sin\zeta_1\right] \quad ; \quad AX\ (2,2) = e^{-\Psi_1} \sin\zeta_1$$

$$AX\ (1,3) = e^{-\Psi_2}\left[\bar{\alpha}_3 \cos\zeta_2 - \bar{\alpha}_4 \sin\zeta_2\right] \quad ; \quad AX\ (2,3) = e^{-\Psi_2} \cos\zeta_2$$

$$AX\ (1,4) = e^{-\Psi_2}\left[\bar{\alpha}_4 \cos\zeta_2 + \bar{\alpha}_3 \sin\zeta_2\right] \quad ; \quad AX\ (2,4) = e^{-\Psi_2} \sin\zeta_2$$

$$AX\ (1,5) = e^{\Psi_1}\left[\bar{\alpha}_5 \cos\zeta_1 - \bar{\alpha}_6 \sin\zeta_1\right] \quad ; \quad AX\ (2,5) = e^{\Psi_1} \cos\zeta_1$$

$$AX\ (1,6) = e^{\Psi_1}\left[\bar{\alpha}_6 \cos\zeta_1 + \bar{\alpha}_5 \sin\zeta_1\right] \quad ; \quad AX\ (2,6) = e^{\Psi_1} \sin\zeta_1$$

$$AX\ (1,7) = e^{\Psi_2}\left[\bar{\alpha}_7 \cos\zeta_2 - \bar{\alpha}_8 \sin\zeta_2\right] \quad ; \quad AX\ (2,7) = e^{\Psi_2} \cos\zeta_2$$

$$AX\ (1,8) = e^{\Psi_2}\left[\bar{\alpha}_8 \cos\zeta_2 + \bar{\alpha}_7 \sin\zeta_2\right] \quad ; \quad AX\ (2,8) = e^{\Psi_2} \sin\zeta_2$$

$$AX\ (3,1) = \frac{e^{-\Psi_1}}{r}\left[-\kappa_1 \cos\zeta_1 - \mu_1 \sin\zeta_1\right] \quad ; \quad AX\ (3,5) = \frac{e^{\Psi_1}}{r}\left[\kappa_1 \cos\zeta_1 - \mu_1 \sin\zeta_1\right]$$

$$AX\ (3,2) = \frac{e^{-\Psi_1}}{r}\left[\mu_1 \cos\zeta_1 - \kappa_1 \sin\zeta_1\right] \quad ; \quad AX\ (3,6) = \frac{e^{\Psi_1}}{r}\left[\mu_2 \cos\zeta_2 + \kappa_2 \sin\zeta_2\right]$$

$$AX\ (3,3) = \frac{e^{-\Psi_2}}{r}\left[-\kappa_2 \cos \zeta_2 - \mu_2 \sin \zeta_2\right] \quad ; \quad AX\ (3,7) = \frac{e^{\Psi_2}}{r}\left[\kappa_2 \cos \zeta_2 - \mu_2 \sin \zeta_2\right]$$

$$AX\ (3,4) = \frac{e^{-\Psi_2}}{r}\left[\mu_2 \cos \zeta_2 - \kappa_2 \sin \zeta_2\right] \quad ; \quad AX\ (3,8) = \frac{e^{\Psi_2}}{r}\left[\mu_2 \cos \zeta_2 + \kappa_2 \sin \zeta_2\right]$$

$$AX\ (4,1) = e^{-\Psi_1}\left[\bar{\beta}_1 \cos \zeta_1 - \bar{\bar{\beta}}_2 \sin \zeta_1\right] \quad ; \quad AX\ (4,5) = e^{\Psi_1}\left[\bar{\beta}_5 \cos \zeta_1 - \bar{\beta}_6 \sin \zeta_1\right]$$

$$AX\ (4,2) = e^{-\Psi_1}\left[\bar{\beta}_2 \cos \zeta_1 + \bar{\beta}_1 \sin \zeta_1\right] \quad ; \quad AX\ (4,6) = e^{\Psi_1}\left[\bar{\beta}_6 \cos \zeta_1 + \bar{\beta}_5 \sin \zeta_1\right]$$

$$AX\ (4,3) = e^{-\Psi_2}\left[\bar{\beta}_3 \cos \zeta_2 - \bar{\beta}_4 \sin \zeta_2\right] \quad ; \quad AX\ (4,7) = e^{\Psi_2}\left[\bar{\beta}_7 \cos \zeta_2 - \bar{\beta}_8 \sin \zeta_2\right]$$

$$AX\ (4,4) = e^{-\Psi_2}\left[\bar{\beta}_4 \cos \zeta_2 + \bar{\beta}_3 \sin \zeta_2\right] \quad ; \quad AX\ (4,8) = e^{\Psi_2}\left[\bar{\beta}_8 \cos \zeta_2 + \bar{\beta}_7 \sin \zeta_2\right]$$

Table 4. Matrix $\left[AF_i \right]_{4x8}$

$$\begin{Bmatrix} F_{u_i} \\ F_{w_i} \\ F_{\beta_i} \\ F_{v_i} \end{Bmatrix} = \begin{Bmatrix} N_x \\ Q_x + \frac{1}{r} \frac{\partial M_{x\phi}}{\partial \phi} \\ -M_x \\ \bar{N}_{x\phi} + \frac{3}{2r} \bar{M}_{x\phi} \end{Bmatrix}_{x = 0} = \underset{4x4}{\left[\tau_n \right]} \underset{4x8}{\left[AF_i \right]} \underset{8x1}{\begin{Bmatrix} \bar{C}_1 \\ \vdots \\ \bar{C}_8 \end{Bmatrix}}$$

$$AF_i\ (1,1) = \frac{D}{r} \left[-\kappa_1 \bar{\alpha}_1 - \mu_1 \bar{\alpha}_2 + \nu(n\bar{\beta}_1 + 1) \right]$$

$$AF_i\ (1,2) = \frac{D}{r} \left[-\kappa_1 \bar{\alpha}_2 + \mu_1 \bar{\alpha}_1 + \nu n\bar{\beta}_2 \right]$$

$$AF_i\ (1,3) = \frac{D}{r} \left[-\kappa_2 \bar{\alpha}_3 - \mu_2 \bar{\alpha}_4 + \nu(n\bar{\beta}_3 + 1) \right]$$

$$AF_i\ (1,4) = \frac{D}{r} \left[-\kappa_2 \bar{\alpha}_4 + \mu_2 \bar{\alpha}_3 + \nu n\bar{\beta}_4 \right]$$

$$AF_i\ (1,5) = \frac{D}{r} \left[\kappa_1 \bar{\alpha}_5 - \mu_1 \bar{\alpha}_6 + \nu(n\bar{\beta}_5 + 1) \right]$$

$$AF_i\ (1,6) = \frac{D}{r} \left[\kappa_1 \bar{\alpha}_6 + \mu_1 \bar{\alpha}_5 + \nu n\bar{\beta}_6 \right]$$

$$AF_i\ (1,7) = \frac{D}{r} \left[\kappa_2 \bar{\alpha}_7 - \mu_2 \bar{\alpha}_8 + \nu(n\bar{\beta}_7 + 1) \right]$$

$$AF_i\ (1,8) = \frac{D}{r} \left[\kappa_2 \bar{\alpha}_8 + \mu_2 \bar{\alpha}_7 + \nu n\bar{\beta}_8 \right]$$

$$AF_i\ (2,1) = \frac{K}{r^3}\Big[n(1-\nu)(-2n\kappa_1 - \frac{3}{2}\kappa_1\bar{\beta}_1 - \frac{3}{2}\mu_1\bar{\beta}_2 + \frac{n\bar{\alpha}_1}{2}) + \kappa_1(\kappa_1^2 - \mu_1^2) - \nu\kappa_1(n^2 + n\bar{\beta}_1) -$$

$$- 2\kappa_1\mu_1^2 - \nu n\mu_1\bar{\beta}_2\Big]$$

$$AF_i\ (2,2) = \frac{K}{r^3}\Big[n(1-\nu)(2n\mu_1 - \frac{3}{2}\kappa_1\bar{\beta}_2 + \frac{3}{2}\mu_1\bar{\beta}_1 + \frac{n\bar{\alpha}_2}{2}) - 2\kappa_1^2\mu_1 - \nu n\kappa_1\bar{\beta}_2 -$$

$$- \mu_1(\kappa_1^2 - \mu_1^2) + \nu\mu_1(n^2 + n\bar{\beta}_1)\Big]$$

$$AF_i\ (2,3) = \frac{K}{r^3}\Big[n(1-\nu)(-2n\kappa_2 - \frac{3}{2}\kappa_2\bar{\beta}_3 - \frac{3}{2}\mu_2\bar{\beta}_4 + \frac{n\bar{\alpha}_3}{2}) + \kappa_2(\kappa_2^2 - \mu_2^2) - \nu\kappa_2(n^2 + n\bar{\beta}_3) -$$

$$- 2\kappa_2\mu_2^2 - \nu n\mu_2\bar{\beta}_4\Big]$$

$$AF_i\ (2,4) = \frac{K}{r^3}\Big[n(1-\nu)(2n\mu_2 - \frac{3}{2}\kappa_2\bar{\beta}_4 + \frac{3}{2}\mu_2\bar{\beta}_3 + \frac{n\bar{\alpha}_4}{2}) - 2\kappa_2^2\mu_2 - \nu n\kappa_2\bar{\beta}_4 - \mu_2(\kappa_2^2 - \mu_2^2) +$$

$$+ \nu\mu_2(n^2 + n\bar{\beta}_3)\Big]$$

$$AF_i\ (2,5) = \frac{K}{r^3}\Big[n(1-\nu)(2n\kappa_1 + \frac{3}{2}\kappa_1\bar{\beta}_5 - \frac{3}{2}\mu_1\bar{\beta}_6 + \frac{n\bar{\alpha}_5}{2}) - \kappa_1(\kappa_1^2 - \mu_1^2) + \nu\kappa_1(n^2 + n\bar{\beta}_5) +$$

$$+ 2\kappa_1\mu_1^2 - \nu n\mu_1\bar{\beta}_6\Big]$$

$$AF_i\ (2,6) = \frac{K}{r^3}\Big[n(1-\nu)(2n\mu_1 + \frac{3}{2}\kappa_1\bar{\beta}_6 + \frac{3}{2}\mu_1\bar{\beta}_5 + \frac{n\bar{\alpha}_6}{2}) - 2\kappa_1^2\mu_1 + \nu n\kappa_1\bar{\beta}_6 - \mu_1(\kappa_1^2 - \mu_1^2) +$$

$$+ \nu\mu_1(n^2 + n\bar{\beta}_5)\Big]$$

$$AF_i\ (2,7) = \frac{K}{r^3}\Big[n(1-\nu)(2n\kappa_2 + \frac{3}{2}\kappa_2\bar{\beta}_7 - \frac{3}{2}\mu_2\bar{\beta}_8 + \frac{n\bar{\alpha}_7}{2}) - \kappa_2(\kappa_2^2 - \mu_2^2) + \nu\kappa_2(n^2 + n\bar{\beta}_7) +$$

$$+ 2\kappa_2\mu_2^2 - \nu n\mu_2\bar{\beta}_8\Big]$$

$$AF_i\ (2,8) = \frac{K}{r^3}\left[n(1-\nu)(2n\mu_2 + \frac{3}{2}\kappa_2\bar{\beta}_8 + \frac{3}{2}\mu_2\bar{\beta}_7 + \frac{n\bar{\alpha}_8}{2}) - 2\kappa_2^2\mu_2 + \nu n\kappa_2\bar{\beta}_8 - \mu_2(\kappa_2^2 - \mu_2^2) + \right.$$

$$\left. + \nu\mu_2\ (n^2 + n\bar{\beta}_7)\right]$$

$$AF_i\ (3,1) = \frac{-K}{r^2}\left[-(\kappa_1^2 - \mu_1^2) + \nu(n^2 + n\bar{\beta}_1)\right]; \quad AF_i\ (3,5) = \frac{-K}{r^2}\left[-(\kappa_1^2 - \mu_1^2) + \nu(n^2 + n\bar{\beta}_5)\right]$$

$$AF_i\ (3,2) = \frac{-K}{r^2}\left[2\kappa_1\mu_1 + \nu n\bar{\beta}_2\right]; \quad AF_i\ (3,6) = \frac{-K}{r^2}\left[-2\kappa_1\mu_1 + \nu n\bar{\beta}_6\right]$$

$$AF_i\ (3,3) = \frac{-K}{r^2}\left[-(\kappa_2^2 - \mu_2^2) + \nu(n^2 + n\bar{\beta}_3)\right]; \quad AF_i\ (3,7) = \frac{-K}{r^2}\left[-(\kappa_2^2 - \mu_2^2) + \nu(n^2 + n\bar{\beta}_7)\right]$$

$$AF_i\ (3,4) = \frac{-K}{r^2}\left[2\kappa_2\mu_2 + \nu n\bar{\beta}_4\right]; \quad AF_i\ (3,8) = \frac{-K}{r^2}\left[-2\kappa_2\mu_2 + \nu n\bar{\beta}_8\right]$$

$$AF_i\ (4,1) = \frac{D(1-\nu)}{2}\left[\frac{1}{r}(-\kappa_1\bar{\beta}_1 - \mu_1\bar{\beta}_2 - n\bar{\alpha}_1) + \frac{3t^2}{24r^3}(-2n\kappa_1 - \frac{3}{2}\kappa_1\bar{\beta}_1 - \frac{3}{2}\mu_1\bar{\beta}_2 + \frac{n\bar{\alpha}_1}{2})\right]$$

$$AF_i\ (4,2) = \frac{D(1-\nu)}{2}\left[\frac{1}{r}(-\kappa_1\bar{\beta}_2 + \mu_1\bar{\beta}_1 - n\bar{\alpha}_2) + \frac{3t^2}{24r^3}(2n\mu_1 - \frac{3}{2}\kappa_1\bar{\beta}_2 + \frac{3}{2}\mu_1\bar{\beta}_1 + \frac{n\bar{\alpha}_2}{2})\right]$$

$$AF_i\ (4,3) = \frac{D(1-\nu)}{2}\left[\frac{1}{r}(-\kappa_2\bar{\beta}_3 - \mu_2\bar{\beta}_4 - n\bar{\alpha}_3) + \frac{3t^2}{24r^3}(-2n\kappa_2 - \frac{3}{2}\kappa_2\bar{\beta}_3 - \frac{3}{2}\mu_2\bar{\beta}_4 + \frac{n\bar{\alpha}_3}{2})\right]$$

$$AF_i\ (4,4) = \frac{D(1-\nu)}{2}\left[\frac{1}{r}(-\kappa_2\bar{\beta}_4 + \mu_2\bar{\beta}_3 - n\bar{\alpha}_4) + \frac{3t^2}{24r^3}(2n\mu_2 - \frac{3}{2}\kappa_2\bar{\beta}_4 + \frac{3}{2}\mu_2\bar{\beta}_3 + \frac{n\bar{\alpha}_4}{2})\right]$$

$$AF_i\ (4,5) = \frac{D(1-\nu)}{2}\left[\frac{1}{r}(\kappa_1\bar{\beta}_5 - \mu_1\bar{\beta}_6 - n\bar{\alpha}_5) + \frac{3t^2}{24r^3}(2n\kappa_1 + \frac{3}{2}\kappa_1\bar{\beta}_5 - \frac{3}{2}\mu_1\bar{\beta}_6 + \frac{n\bar{\alpha}_5}{2})\right]$$

$$AF_i\ (4,6) = \frac{D(1-\nu)}{2}\left[\frac{1}{r}(\kappa_1\bar{\beta}_6 + \mu_1\bar{\beta}_5 - n\bar{\alpha}_6) + \frac{3t^2}{24r^3}(2n\mu_1 + \frac{3}{2}\kappa_1\bar{\beta}_6 + \frac{3}{2}\mu_1\bar{\beta}_5 + \frac{n\bar{\alpha}_6}{2})\right]$$

$$AF_i\ (4,7) = \frac{D(1-\nu)}{2}\left[\frac{1}{r}(\kappa_2\bar{\beta}_7 - \mu_2\bar{\beta}_8 - n\bar{\alpha}_7) + \frac{3t^2}{24r^3}\left(2n\kappa_2 + \frac{3}{2}\kappa_2\bar{\beta}_7 - \frac{3}{2}\mu_2\bar{\beta}_8 + \frac{n\bar{\alpha}_7}{2}\right)\right]$$

$$AF_i\ (4,8) = \frac{D(1-\nu)}{2}\left[\frac{1}{r}(\kappa_2\bar{\beta}_8 + \mu_2\bar{\beta}_7 - n\bar{\alpha}_8) + \frac{3t^2}{24r^3}\left(2n\mu_2 + \frac{3}{2}\kappa_2\bar{\beta}_8 + \frac{3}{2}\mu_2\bar{\beta}_7 + \frac{n\bar{\alpha}_8}{2}\right)\right]$$

Table 5. Matrix $\left[AF_j\right]_{4x8}$

$$\begin{Bmatrix} F_{u_j} \\ F_{w_j} \\ F_{\beta_j} \\ F_{v_j} \end{Bmatrix} = \begin{Bmatrix} N_x \\ Q_x + \frac{1}{r} \frac{\partial \bar{M}x\phi}{\partial \phi} \\ - M_x \\ \bar{N}_{x\phi} + \frac{3}{2r} \bar{M}_{x\phi} \end{Bmatrix}_{x = \ell} = \left[\tau_n\right]_{4x4} \left[AF_j\right]_{4x8} \begin{Bmatrix} \bar{C}1 \\ \vdots \\ \bar{C}8 \end{Bmatrix}$$

$$AF_j\ (1,1) = \frac{De^{-\omega_1}}{r}\left\{ \left[-\kappa_1\bar{\alpha}_1 - \mu_1\bar{\alpha}_2 + \nu(n\bar{\beta}_1 + 1)\right] \cos\eta_1 + \left[\kappa_1\bar{\alpha}_2 - \mu_1\bar{\alpha}_1 - \nu n\bar{\beta}_2\right] \sin\eta_1 \right\}$$

$$AF_j\ (1,2) = \frac{De^{-\omega_1}}{r}\left\{ \left[-\kappa_1\bar{\alpha}_2 + \mu_1\bar{\alpha}_1 + \nu n\bar{\beta}_2\right] \cos\eta_1 + \left[-\kappa_1\bar{\alpha}_1 - \mu_1\bar{\alpha}_2 + \nu(n\bar{\beta}_1 + 1)\right] \sin\eta_1 \right\}$$

$$AF_j\ (1,3) = \frac{De^{-\omega_2}}{r}\left\{ \left[-\kappa_2\bar{\alpha}_3 - \mu_2\bar{\alpha}_4 + \nu(n\bar{\beta}_3 + 1)\right] \cos\eta_2 + \left[\kappa_2\bar{\alpha}_4 - \mu_2\bar{\alpha}_3 - \nu n\bar{\beta}_4\right] \sin\eta_2 \right\}$$

$$AF_j\ (1,4) = \frac{De^{-\omega_2}}{r}\left\{ \left[-\kappa_2\bar{\alpha}_4 + \mu_2\bar{\alpha}_3 + \nu n\bar{\beta}_4\right] \cos\eta_2 + \left[-\kappa_2\bar{\alpha}_3 - \mu_2\bar{\alpha}_4 + \nu(n\bar{\beta}_3 + 1)\right] \sin\eta_2 \right\}$$

$$AF_j\ (1,5) = \frac{De^{\omega_1}}{r}\left\{ \left[\kappa_1\bar{\alpha}_5 - \mu_1\bar{\alpha}_6 + \nu(n\bar{\beta}_5 + 1)\right] \cos\eta_1 + \left[-\kappa_1\bar{\alpha}_6 - \mu_1\bar{\alpha}_5 - \nu n\bar{\beta}_6\right] \sin\eta_1 \right\}$$

$$AF_j\ (1,6) = \frac{De^{\omega_1}}{r}\left\{ \left[\kappa_1\bar{\alpha}_6 + \mu_1\bar{\alpha}_5 + \nu n\bar{\beta}_6\right] \cos\eta_1 + \left[\kappa_1\bar{\alpha}_5 - \mu_1\bar{\alpha}_6 + \nu(n\bar{\beta}_5 + 1)\right] \sin\eta_1 \right\}$$

$$AF_j\ (1,7) = \frac{De^{\omega_2}}{r}\left\{ \left[\kappa_2\bar{\alpha}_7 - \mu_2\bar{\alpha}_8 + \nu(n\bar{\beta}_7 + 1)\right] \cos\eta_2 + \left[-\kappa_2\bar{\alpha}_8 - \mu_2\bar{\alpha}_7 - \nu n\bar{\beta}_8\right] \sin\eta_2 \right\}$$

$$AF_j\ (1,8) = \frac{De^{\omega_2}}{r}\left\{ \left[\kappa_2\bar{\alpha}_8 + \mu_2\bar{\alpha}_7 + \nu n\bar{\beta}_8\right] \cos\eta_2 + \left[\kappa_2\bar{\alpha}_7 - \mu_2\bar{\alpha}_8 + \nu(n\bar{\beta}_7 + 1)\right] \sin\eta_2 \right\}$$

$$AF_j\ (2,1) = AF_i\ (2,1)\ e^{-\omega_1} \cos\eta_1 + \frac{Ke^{-\omega_1}}{r^3} \Big[n(1-\nu)\big(-2n\mu_1 + \tfrac{3}{2}\kappa_1\bar{\beta}_2 - \tfrac{3}{2}\mu_1\bar{\beta}_1 - \frac{n\bar{\alpha}_2}{2}\big) +$$

$$+ 2\kappa_1^2\mu_1 + \nu n\kappa_1\bar{\beta}_2 + \mu_1(\kappa_1^2 - \mu_1^2) - \nu\mu_1(n^2 + n\bar{\beta}_1) \Big] \sin\eta_1$$

$$AF_j\ (2,2) = AF_i\ (2,2)\ e^{-\omega_1} \cos\eta_1 + \frac{Ke^{-\omega_1}}{r^3} \Big[n(1-\nu)\big(-2n\kappa_1 - \tfrac{3}{2}\kappa_1\bar{\beta}_1 - \tfrac{3}{2}\mu_1\bar{\beta}_2 + \frac{n\bar{\alpha}_1}{2}\big) +$$

$$+\kappa_1(\kappa_1^2 - \mu_1^2) - \nu\kappa_1(n^2 + n\bar{\beta}_1) - 2\kappa_1\mu_1^2 - \nu n\mu_1\bar{\beta}_2 \Big] \sin\eta_1$$

$$AF_j\ (2,3) = AF_i\ (2,3)\ e^{-\omega_2} \cos\eta_2 + \frac{Ke^{-\omega_2}}{r^3} \Big[n(1-\nu)\big(-2n\mu_2 + \tfrac{3}{2}\kappa_2\bar{\beta}_4 - \tfrac{3}{2}\mu_2\bar{\beta}_3 - \frac{n\bar{\alpha}_4}{2}\big) +$$

$$+ 2\kappa_2^2\mu_2 + \nu n\kappa_2\bar{\beta}_4 + \mu_2(\kappa_2^2 - \mu_2^2) - \nu\mu_2(n^2 + n\bar{\beta}_3) \Big] \sin\eta_2$$

$$AF_j\ (2,4) = AF_i\ (2,4)\ e^{-\omega_2} \cos\eta_2 + \frac{Ke^{-\omega_2}}{r^3} \Big[n(1-\nu)\big(-2n\kappa_2 - \tfrac{3}{2}\kappa_2\bar{\beta}_3 - \tfrac{3}{2}\mu_2\bar{\beta}_4 + \frac{n\bar{\alpha}_3}{2}\big) +$$

$$+ \kappa_2(\kappa_2^2 - \mu_2^2) - \nu\kappa_2(n^2 + n\bar{\beta}_3) - 2\kappa_2\mu_2^2 - \nu n\mu_2\bar{\beta}_4 \Big] \sin\eta_2$$

$$AF_j\ (2,5) = AF_i\ (2,5)\ e^{\omega_1} \cos\eta_1 + \frac{Ke^{\omega_1}}{r^3} \Big[n(1-\nu)\big(-2n\mu_1 - \tfrac{3}{2}\kappa_1\bar{\beta}_6 - \tfrac{3}{2}\mu_1\bar{\beta}_5 - \frac{n\bar{\alpha}_6}{2}\big) +$$

$$+ 2\kappa_1^2\mu_1 - \nu n\kappa_1\bar{\beta}_6 + \mu_1(\kappa_1^2 - \mu_1^2) - \nu\mu_1(n^2 + n\bar{\beta}_5) \Big] \sin\eta_1$$

$$AF_j\ (2,6) = AF_i\ (2,6)\ e^{\omega_1} \cos\eta_1 + \frac{Ke^{\omega_1}}{r^3} \Big[n(1-\nu)\big(2n\kappa_1 + \tfrac{3}{2}\kappa_1\bar{\beta}_5 - \tfrac{3}{2}\mu_1\bar{\beta}_6 + \frac{n\bar{\alpha}_5}{2}\big) -$$

$$- \kappa_1(\kappa_1^2 - \mu_1^2) + \nu\kappa_1(n^2 + n\bar{\beta}_5) + 2\kappa_1\mu_1^2 - \nu n\mu_1\bar{\beta}_6 \Big] \sin\eta_1$$

$$AF_j\ (2,7) = AF_i\ (2,7)\ e^{\omega_2} \cos\eta_2 + \frac{Ke^{\omega_2}}{r^3} \Big[n(1-\nu)\big(-2n\mu_2 - \tfrac{3}{2}\kappa_2\bar{\beta}_8 - \tfrac{3}{2}\mu_2\bar{\beta}_7 - \frac{n\bar{\alpha}_8}{2}\big) +$$

$$+ 2\kappa_2^2\mu_2 - \nu n\kappa_2\bar{\beta}_8 + \mu_2(\kappa_2^2 - \mu_2^2) - \nu\mu_2(n^2 + n\bar{\beta}_7) \Big] \sin\eta_2$$

$$AF_j\ (2,8) = AF_i\ (2,8)\ e^{\omega_2} \cos\eta_2 + \frac{Ke^{\omega_2}}{r^3} \Big[n(1-\nu)(2n\kappa_2 + \frac{3}{2}\kappa_2\bar{\beta}_7 - \frac{3}{2}\mu_2\bar{\beta}_8 + \frac{n\bar{\alpha}_7}{2}) -$$

$$- \kappa_2(\kappa_2^2 - \mu_2^2) + \nu\kappa_2(n^2 + n\bar{\beta}_7) + 2\kappa_2\mu_2^2 - \nu n\mu_2\bar{\beta}_8 \Big] \sin\eta_2$$

$$AF_j\ (3,1) = -\frac{Ke^{-\omega_1}}{r^2} \Big\{ \Big[-(\kappa_1^2 - \mu_1^2) + \nu(n^2 + n\bar{\beta}_1) \Big] \cos\eta_1 - \Big[2\kappa_1\mu_1 + \nu n\bar{\beta}_2 \Big] \sin\eta_1 \Big\}$$

$$AF_j\ (3,2) = -\frac{Ke^{-\omega_1}}{r^2} \Big\{ \Big[2\kappa_1\mu_1 + \nu n\bar{\beta}_2 \Big] \cos\eta_1 - \Big[\kappa_1^2 - \mu_1^2 - \nu(n^2 + n\bar{\beta}_1) \Big] \sin\eta_1 \Big\}$$

$$AF_j\ (3,3) = -\frac{Ke^{-\omega_2}}{r^2} \Big\{ \Big[-(\kappa_2^2 - \mu_2^2) + \nu(n^2 + n\bar{\beta}_3) \Big] \cos\eta_2 - \Big[2\kappa_2\mu_2 + \nu n\bar{\beta}_4 \Big] \sin\eta_2 \Big\}$$

$$AF_j\ (3,4) = -\frac{Ke^{-\omega_2}}{r^2} \Big\{ \Big[2\kappa_2\mu_2 + \nu n\bar{\beta}_4 \Big] \cos\eta_2 - \Big[\kappa_2^2 - \mu_2^2 - \nu(n^2 + n\bar{\beta}_3) \Big] \sin\eta_2 \Big\}$$

$$AF_j\ (3,5) = -\frac{Ke^{\omega_1}}{r^2} \Big\{ \Big[-(\kappa_1^2 - \mu_1^2) + \nu(n^2 + n\bar{\beta}_5) \Big] \cos\eta_1 - \Big[-2\kappa_1\mu_1 + \nu n\bar{\beta}_6 \Big] \sin\eta_1 \Big\}$$

$$AF_j\ (3,6) = -\frac{Ke^{\omega_1}}{r^2} \Big\{ \Big[-2\kappa_1\mu_1 + \nu n\bar{\beta}_6 \Big] \cos\eta_1 - \Big[\kappa_1^2 - \mu_1^2 - \nu(n^2 + n\bar{\beta}_5) \Big] \sin\eta_1 \Big\}$$

$$AF_j\ (3,7) = -\frac{Ke^{\omega_2}}{r^2} \Big\{ \Big[-(\kappa_2^2 - \mu_2^2) + \nu(n^2 + n\bar{\beta}_7) \Big] \cos\eta_2 - \Big[-2\kappa_2\mu_2 + \nu n\bar{\beta}_8 \Big] \sin\eta_2 \Big\}$$

$$AF_j\ (3,8) = -\frac{Ke^{\omega_2}}{r^2} \Big\{ \Big[-2\kappa_2\mu_2 + \nu n\bar{\beta}_8 \Big] \cos\eta_2 - \Big[\kappa_2^2 - \mu_2^2 - \nu(n^2 + n\bar{\beta}_7) \Big] \sin\eta_2 \Big\}$$

$$AF_j\ (4,1) = AF_i\ (4,1)\ e^{-\omega_1} \cos\eta_1 + \frac{D(1-\nu)}{2r} e^{-\omega_1} \Big[\kappa_1\bar{\beta}_2 - \mu_1\bar{\beta}_1 + n\bar{\alpha}_2 + \frac{3t^2}{24r^2} \cdot$$

$$\cdot(-2n\mu_1 + \frac{3}{2}\kappa_1\bar{\beta}_2 - \frac{3}{2}\mu_1\bar{\beta}_1 - \frac{n\bar{\alpha}_2}{2}) \Big] \sin\eta_1$$

$$AF_j\ (4,2) = AF_i\ (4,2)\ \bar{e}^{\omega_1} \cos\eta_1 + \frac{D(1-\nu)}{2r}\ \bar{e}^{\omega_1} \Big[-\kappa_1\bar{\beta}_1 - \mu_1\bar{\beta}_2 - n\bar{\alpha}_1 + \frac{3t^2}{24r^2} \cdot$$

$$\cdot(-2n\kappa_1 - \frac{3}{2}\kappa_1\bar{\beta}_1 - \frac{3}{2}\mu_1\bar{\beta}_2 + \frac{n\bar{\alpha}_1}{2}) \Big] \sin\eta_1$$

$$AF_j\ (4,3) = AF_i\ (4,3)\ \bar{e}^{\omega_2} \cos\eta_2 + \frac{D(1-\nu)}{2r}\ \bar{e}^{\omega_2} \Big[\kappa_2\bar{\beta}_4 - \mu_2\bar{\beta}_3 + n\bar{\alpha}_4 + \frac{3t^2}{24r^2} \cdot$$

$$\cdot(-2n\mu_2 + \frac{3}{2}\kappa_2\bar{\beta}_4 - \frac{3}{2}\mu_2\bar{\beta}_3 - \frac{n\bar{\alpha}_4}{2}) \Big] \sin\eta_2$$

$$AF_j\ (4,4) = AF_i\ (4,4)\ \bar{e}^{\omega_2} \cos\eta_2 + \frac{D(1-\nu)}{2r}\ \bar{e}^{\omega_2} \Big[-\kappa_2\bar{\beta}_3 - \mu_2\bar{\beta}_4 - n\bar{\alpha}_3 + \frac{3t^2}{24r^2} \cdot$$

$$\cdot(-2n\kappa_2 - \frac{3}{2}\kappa_2\bar{\beta}_3 - \frac{3}{2}\mu_2\bar{\beta}_4 + \frac{n\bar{\alpha}_3}{2}) \Big] \sin\eta_2$$

$$AF_j\ (4,5) = AF_i\ (4,5)\ e^{\omega_1} \cos\eta_1 + \frac{D(1-\nu)}{2r}\ e^{\omega_1} \Big[-\kappa_1\bar{\beta}_6 - \mu_1\bar{\beta}_5 + n\bar{\alpha}_6 + \frac{3t^2}{24r^2} \cdot$$

$$\cdot(-2n\mu_1 - \frac{3}{2}\kappa_1\bar{\beta}_6 - \frac{3}{2}\mu_1\bar{\beta}_5 - \frac{n\bar{\alpha}_6}{2}) \Big] \sin\eta_1$$

$$AF_j\ (4,6) = AF_i\ (4,6)\ e^{\omega_1} \cos\eta_1 + \frac{D(1-\nu)}{2r}\ e^{\omega_1} \Big[\kappa_1\bar{\beta}_5 - \mu_1\bar{\beta}_6 - n\bar{\alpha}_5 + \frac{3t^2}{24r^2} \cdot$$

$$\cdot(2n\kappa_1 + \frac{3}{2}\kappa_1\bar{\beta}_5 - \frac{3}{2}\mu_1\bar{\beta}_6 + \frac{n\bar{\alpha}_5}{2}) \Big] \sin\eta_1$$

$$AF_j\ (4,7) = AF_i\ (4,7)\ e^{\omega_2} \cos\eta_2 + \frac{D(1-\nu)}{2r}\ e^{\omega_2} \Big[-\kappa_2\bar{\beta}_8 - \mu_2\bar{\beta}_7 + n\bar{\alpha}_8 + \frac{3t^2}{24r^2} \cdot$$

$$\cdot(-2n\mu_2 - \frac{3}{2}\kappa_2\bar{\beta}_8 - \frac{3}{2}\mu_2\bar{\beta}_7 - \frac{n\bar{\alpha}_8}{2}) \Big] \sin\eta_2$$

$$AF_j\ (4,8) = AF_i\ (4,8)\ e^{\omega_2} \cos\eta_2 + \frac{D(1-\nu)}{2r}\ e^{\omega_2} \Big[\kappa_2\bar{\beta}_7 - \mu_2\bar{\beta}_8 - n\bar{\alpha}_7 + \frac{3t^2}{24r^2} \cdot$$

$$\cdot(2n\kappa_2 + \frac{3}{2}\kappa_2\bar{\beta}_7 - \frac{3}{2}\mu_2\bar{\beta}_8 + \frac{n\bar{\alpha}_7}{2}) \Big] \sin\eta_2$$

Table 6. Vectors $\{\tilde{p}_o\}$,$\{\tilde{p}_{fo}\}$,$\{\tilde{p}_1\}$and $\{\tilde{p}_{f1}\}$

$$
\{\tilde{p}_o\} = \left\{ \begin{array}{c}
0 \\
p_o r^2 \ (1-0.5\nu)/Et \\
0 \\
0 \\
\hline
p_o r\ell \ (1-2\nu)/Et \\
p_o r^2 \ (1-0.5\nu)/Et \\
0 \\
0 \\
\hline
\nu p_o r \\
0 \\
0 \\
0 \\
\hline
\nu p_o r \\
0 \\
0 \\
0
\end{array} \right\}
, \quad \{\tilde{p}_{fo}\} = \left\{ \begin{array}{c}
0 \\
-a_o r^2 \ (1-0.5\nu)/Et \\
0 \\
0 \\
\hline
0 \\
-a_o r^2 \ (1-0.5\nu)/Et \\
0 \\
0 \\
\hline
-a_o r/2 \\
0 \\
0 \\
0 \\
\hline
-a_o r/2 \\
0 \\
0 \\
0
\end{array} \right\}
$$

$$\{\tilde{p}_1\} = \left\{ \begin{array}{l} -(\gamma_s/E)\left[(\ell^3/12r) + (r\ell/8)(1-4\nu)\right] \\ (\gamma_s/4E)r^2\ (4-\nu) \\ (\gamma_s/E)\left[(\ell^3/12r^2) + \ell(2.125 + 0.5\nu)\right] \\ 0 \\ \hline (\gamma_s/E)\left[(\ell^3/12r) + (r\ell/8)(1-4\nu)\right] \\ (\gamma_s/4E)r^2\ (4-\nu) \\ -(\gamma_s/E)\left[(\ell^3/12r^2) + \ell(2.125 + 0.5\nu)\right] \\ 0 \\ \hline \gamma_s tr/4 \\ 0 \\ 0 \\ -\gamma_s t\ell - w_B \\ \hline \gamma_s tr/4 \\ 0 \\ 0 \\ +\gamma_s t\ell + w_B \end{array} \right\} \quad \{\tilde{p}_{f1}\} = \left\{ \begin{array}{l} (a_1/Et)\left[(\ell^3/24r) + (r\ell/8)(1-4\nu)\right] \\ -(a_1/4Et)r^2\ (4-\nu) \\ -(a_1/Et)\left[(\ell^3/24r^2) + 1.125\ell\right] \\ 0 \\ \hline -(a_1/Et)\left[(\ell^3/24r) + (r\ell/8)(1-4\nu)\right] \\ -(a_1/4Et)r^2\ (4-\nu) \\ (a_1/Et)\left[(\ell^3/24r^2) + 1.125\ell\right] \\ 0 \\ \hline -a_1 r/4 \\ 0 \\ 0 \\ a_1\ell/2 \\ \hline -a_1 r/4 \\ 0 \\ 0 \\ -a_1\ell/2 \end{array} \right\}$$

where $a_o = -\gamma r\ (\sin\phi_o - \phi_o \cos\phi_o)/\pi$, $a_1 = -\gamma r\ (\phi_o - 0.5\sin 2\phi_o)/\pi$, w_B is the weight of bulkheads and ϕ_o indicates the level of the liquid.

Table 7. Vector $\{\tilde{p}_{fn}\}_{n \geq 2}$ and $\{\tilde{q}_n\}_{n \geq 0}$.

$$\{\tilde{p}_{fn}\}_{n \geqslant 2} = \left\{ \begin{array}{c} u_{on} \\ w_{on} \\ 0 \\ v_{on} \\ \hline u_{on} \\ w_{on} \\ 0 \\ v_{on} \\ \hline \underset{8x3}{\begin{bmatrix} [D_{on}] \\ [D_{on}] \end{bmatrix}} \underset{3x1}{\begin{Bmatrix} u_{on} \\ w_{on} \\ v_{on} \end{Bmatrix}} \end{array} \right\} ,$$

$$\{\tilde{q}_n\}_{n \geqslant 0} = \sum_{m=0}^{\infty} \left\{ \begin{array}{c} \sum_{i=1}^{N} u_{mni} \\ 0 \\ \sum_i (m\pi/\ell)\, w_{mni} \\ 0 \\ \hline \sum_i (-1)^m\, u_{mni} \\ 0 \\ \sum\sum_i (-1)^m\, (m\pi/\ell)\, w_{mni} \\ 0 \\ \hline \underset{8x3}{\begin{bmatrix} S_{mn} \end{bmatrix}} \underset{3x1}{\begin{Bmatrix} \sum_i u_{mni} \\ \sum_i w_{mni} \\ \sum_i v_{mni} \end{Bmatrix}} \end{array} \right\} ,$$

ıere

$$\left[\begin{matrix}\text{on}\end{matrix}\right]_{\geqslant 2} = \begin{bmatrix} 0 & \nu D/r & \nu nD/r \\ n^2K(1-\nu)/2r^3 & 0 & 0 \\ 0 & -\nu n^2K/r^2 & -\nu nK/r^2 \\ (nD(1-\nu)/2r)\cdot\left[(3t^2/48r^2)-1\right] & 0 & 0 \end{bmatrix}$$

$$\left[S_{mn}\right]_{\substack{m\geqslant 0\\ n\geqslant 0}} = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{K(1-\nu)n^2}{2r^3} & K\left[\left(\dfrac{m\pi}{\ell}\right)^3 + (2-\nu)\cdot\dfrac{n^2}{r^2}\cdot\left(\dfrac{m\pi}{\ell}\right)\right] & \dfrac{nK}{r^2}\cdot\dfrac{m\pi}{\ell}\cdot\left(\dfrac{3-\nu}{2}\right) \\ 0 & 0 & 0 \\ \dfrac{nD}{2r}(1-\nu)\cdot\left(\dfrac{3t^2}{48r^2}-1\right) & \dfrac{3}{2r^2}\cdot K(1-\nu)\cdot n\,\dfrac{m\pi}{\ell} & D\left(\dfrac{1-\nu}{2}\right)\cdot\dfrac{m\pi}{\ell}\cdot\left(1+\dfrac{9}{48}\,\dfrac{t^2}{r^2}\right) \\ 0 & 0 & 0 \\ \dfrac{(-1)^m\, n^2\,(1-\nu)K}{2r^3} & (-1)^m K\cdot\left[\left(\dfrac{m\pi}{\ell}\right)^3 + (2-\nu)\,\dfrac{n^2}{r^2}\,\left(\dfrac{m\pi}{\ell}\right)\right] & (-1)^m\,\dfrac{nK}{r^2}\,\dfrac{m\pi}{\ell}\,\dfrac{(3-\nu)}{2} \\ 0 & 0 & 0 \\ (-1)^m\,\dfrac{nD(1-\nu)}{2r}\cdot\left(\dfrac{3t^2}{48r^2}-1\right) & (-1)^m\,\dfrac{3K}{2r^2}\,(1-\nu)n\,\dfrac{m\pi}{\ell} & D\dfrac{(1-\nu)}{2}\cdot(-1)^m\cdot\dfrac{m\pi}{\ell}\,\left(1+\dfrac{9}{48}\,\dfrac{t^2}{r^2}\right) \end{bmatrix}$$

APPENDIX 3

Total potential energy.

The potential energy of the system is given by equation (48) as follows.

$$\pi = \frac{1}{2} U_S - W_{EX} - W_{BC} \quad ,$$

where the strain energy, U_s, may be expressed by

$$U_S = \int_o^{\ell} \int_o^{2\pi} \{\varepsilon\}^T \{\sigma\} \; r \; dx \; d\phi \quad , \tag{51}$$

$\{\varepsilon\}$ and $\{\sigma\}$ are the strain and stress-resultant vectors, respectively; r is the mean radius of the shell and ℓ its length.

Substituting equation (49) into (51), we obtain

$$U_S = \int_o^{\ell} \int_o^{2\pi} \{\varepsilon\}^T \begin{bmatrix} P \end{bmatrix} \{\varepsilon\} \; r \; dx \; d\phi \quad , \tag{52}$$

where $\begin{bmatrix} P \end{bmatrix}$, the elasticity matrix, is listed in appendix 1 and $\{\varepsilon\}$ is given by relation (48).

Introducing equations (20), (22), (24), (30) and (32) into relation (48) and thence into equation (52) we obtain expression for the strain energy in terms of the edge and applied loads.

$$U_S = \int_o^{\ell} \int_o^{2\pi} \Bigg[\sum_n \underset{1x8}{\{C_n\}^T} \underset{8x6}{\begin{bmatrix} Q \end{bmatrix}}^T \underset{6x6}{\begin{bmatrix} T_n & 0 \\ 0 & T_n \end{bmatrix}} + \underset{1x3}{\begin{Bmatrix} u_{01} \\ w_{01} \\ v_{01} \end{Bmatrix}_{P1}^T} \underset{3x6}{\begin{bmatrix} C_{01} \end{bmatrix}}^T \underset{6x6}{\begin{bmatrix} T_1 & 0 \\ 0 & T_1 \end{bmatrix}} +$$

$$+ \sum_n \begin{Bmatrix} u_{on} \\ w_{on} \\ v_{on} \end{Bmatrix}_{pf}^T \underset{3x6}{\begin{bmatrix} C_{on} \end{bmatrix}}^T \underset{6x6}{\begin{bmatrix} T_n & 0 \\ 0 & T_n \end{bmatrix}} + \sum_n \sum_m \underset{1x3}{\begin{Bmatrix} \sum_i u_{mni} \\ \sum_i w_{mni} \\ \sum_i v_{mni} \end{Bmatrix}_{q_i}^T} \underset{3x6}{\begin{bmatrix} C_{mn} \end{bmatrix}}^T \underset{6x6}{\begin{bmatrix} X_m & 0 \\ 0 & X_m \end{bmatrix}} \underset{6x6}{\begin{bmatrix} T_n & 0 \\ 0 & T_n \end{bmatrix}} +$$

$$+\begin{bmatrix}0, & p_o r/D, & 0, & 0, & 0, & 0\end{bmatrix}\Bigg]\cdot\begin{bmatrix}P\end{bmatrix}_{6x6}\cdot\Bigg[\sum_n \begin{bmatrix}T_n & 0\\ 0 & T_n\end{bmatrix}_{6x6}\begin{bmatrix}Q\end{bmatrix}_{6x8}\begin{Bmatrix}C_n\end{Bmatrix}_{8x1}+$$

$$+\begin{bmatrix}T_1 & 0\\ 0 & T_1\end{bmatrix}_{6x6}\begin{bmatrix}C_{01}\end{bmatrix}_{6x3}\begin{Bmatrix}u_{01}\\ w_{01}\\ v_{01}\end{Bmatrix}_{p_1}+\sum_n\begin{bmatrix}T_n & 0\\ 0 & T_n\end{bmatrix}_{6x6}\begin{bmatrix}C_{on}\end{bmatrix}_{6x3}\begin{Bmatrix}u_{on}\\ w_{on}\\ v_{on}\end{Bmatrix}_{p_f}+\sum_n\sum_m\begin{bmatrix}T_n & 0\\ 0 & T_n\end{bmatrix}_{6x6}\cdot \tag{53}$$

$$\cdot\begin{bmatrix}X_m & 0\\ 0 & X_m\end{bmatrix}\begin{bmatrix}C_{mn}\end{bmatrix}_{6x3}\begin{Bmatrix}\sum_i u_{mni}\\ \sum_i w_{mni}\\ \sum_i v_{mni}\end{Bmatrix}_{q_i}+\begin{Bmatrix}0\\ p_o r/D\\ 0\\ 0\\ 0\\ 0\end{Bmatrix}\Bigg]\, r\, dx\, d\phi \quad ,$$

where the matrix $\begin{bmatrix}Q\end{bmatrix}$ is given by equation (4) and all other matrices are listed in appendix 2.

W_{BC} represents the work of the edge forces and may be defined as follows.

$$W_{BC} = \int_o^{2\pi}\{F_{\delta nj}\}^T\begin{Bmatrix}U\\ W\\ -\partial W/\partial x\\ V\end{Bmatrix}_{x=\ell} rd\phi - \int_o^{2\pi}\{F_{\delta ni}\}^T\begin{Bmatrix}U\\ W\\ -\partial W/\partial x\\ V\end{Bmatrix}_{x=0} rd\phi \quad , \tag{54}$$

where $\{F_{\delta nj}\}$ and $\{F_{\delta ni}\}$ are given by equation (35).

And finally the work of external forces, W_{EX}, is given by

$$W_{EX} = \int_o^{\ell}\int_o^{2\pi}\Big[W_{qi}\, p_o + W_{qi}\,\gamma_s t\cos\phi + W_{qi}\sum_n p_{fn}\cos n\phi - V_{qi}\,\gamma_s t\sin\phi\Big]\cdot \tag{55}$$

$$r\, dx\, d\phi\ + cte.$$

Now, integrating equations (53) - (55) over $\dot{x}$ and ϕ, we obtain relations (44), (45) and (46), given in the main text, where the terms which are not function of the q_i's, have been omitted and replaced by a constant, for simplicity.

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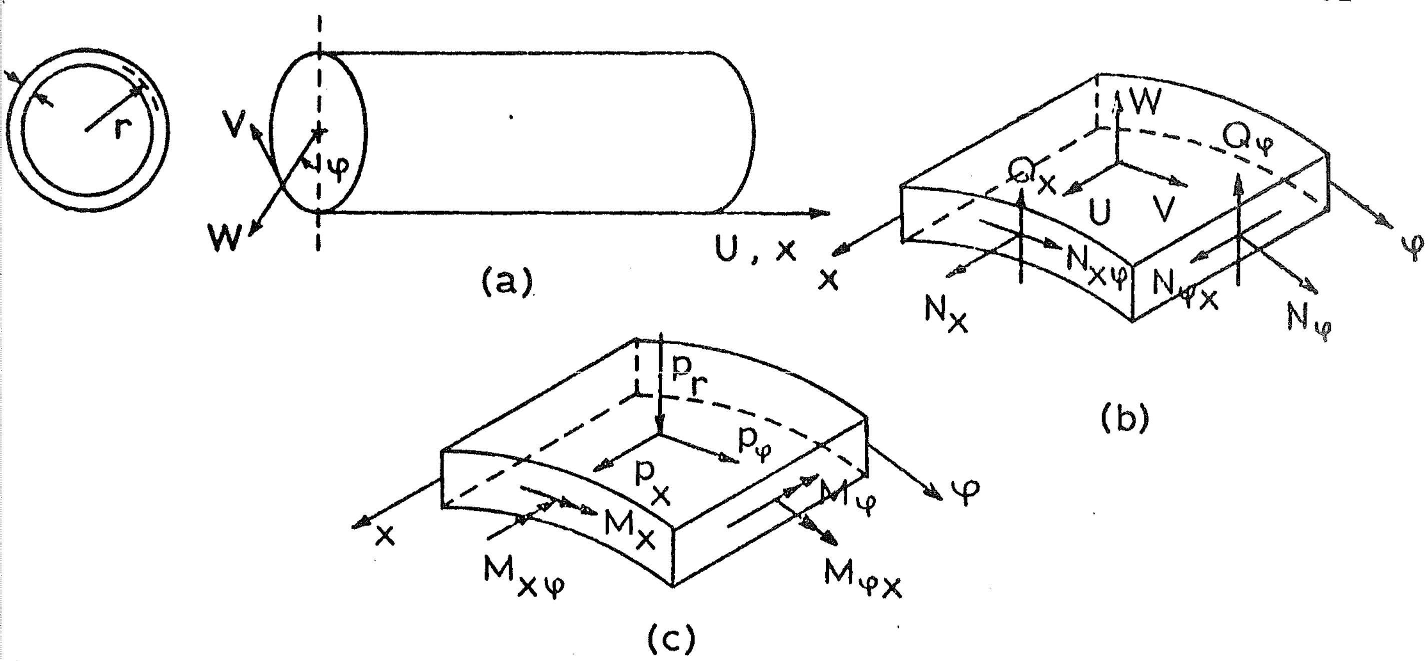

Figure 1 (a) Definition of the displacements U, V and W.
(b) Stress-resultants and displacements acting upon a differential elements.
(c) Stress couples and surface loads acting upon a differential elements.

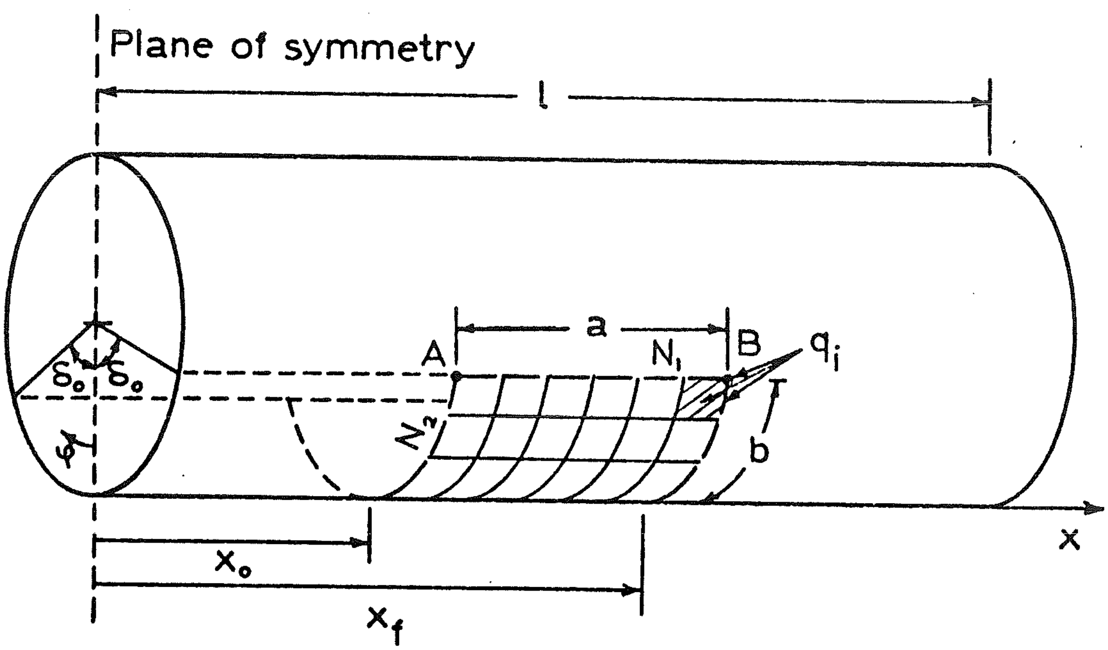

Figure 2 Location of the support on the shell.
(N_1 and N_2 are the number of line elements at the boundaries of the support in x and φ directions, respectively; and, $N = N_1N_2 + 2N_2 + N_1 + 2$, is the total number of elements).

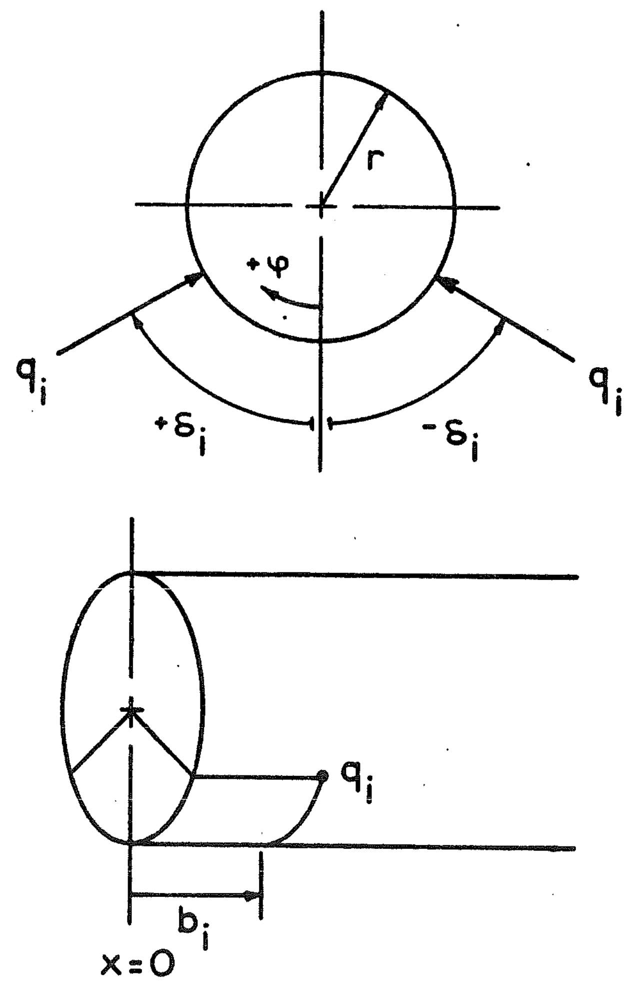

Figure 3 Concentrated radial loads, q_i (lb or kg).

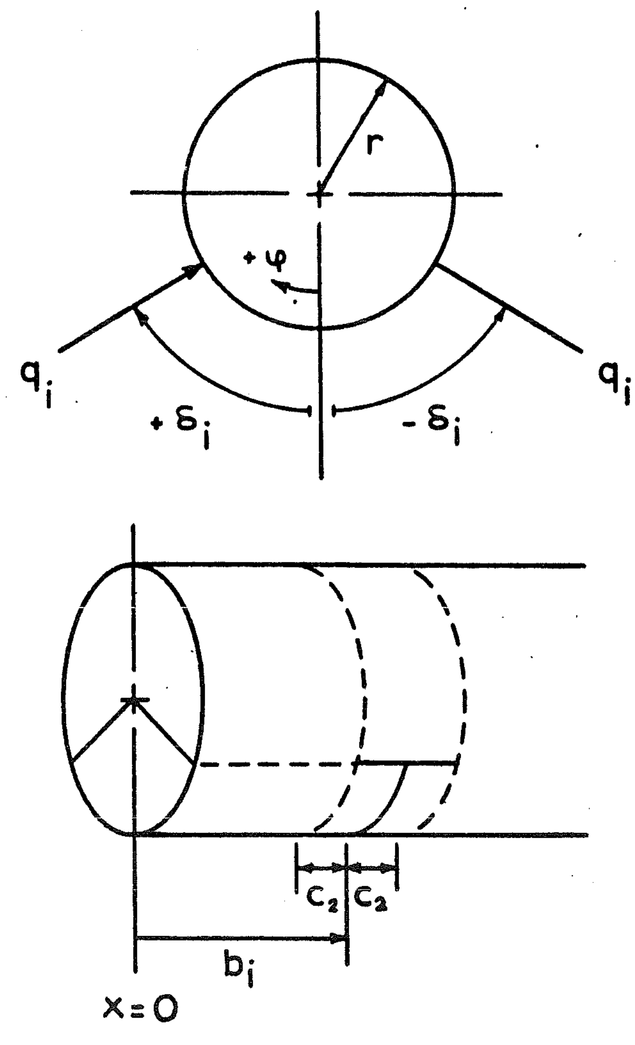

Figure 4 Line load along a generator, q_i (lb/in or kg/m).

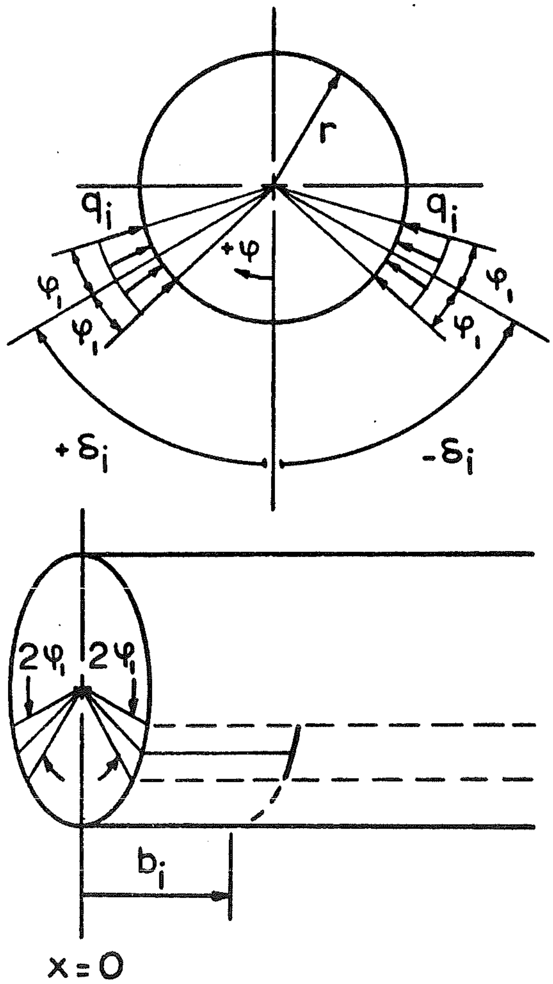

Figure 5 Line load perpendicular to the generator, q_i (lb/in or kg/m).

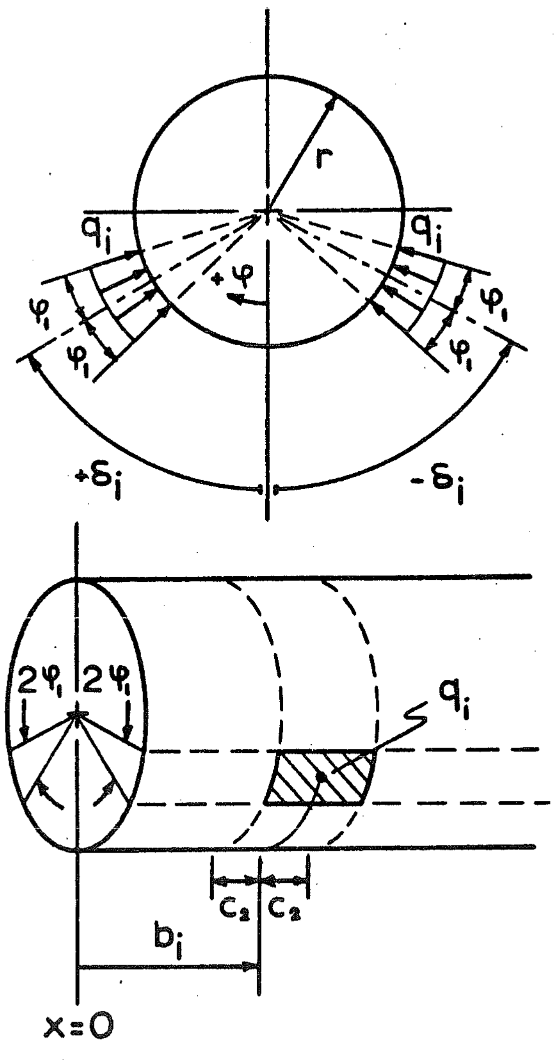

Figure 6 Distributed loads, q_i (lb/in^2 or kg/m^2).

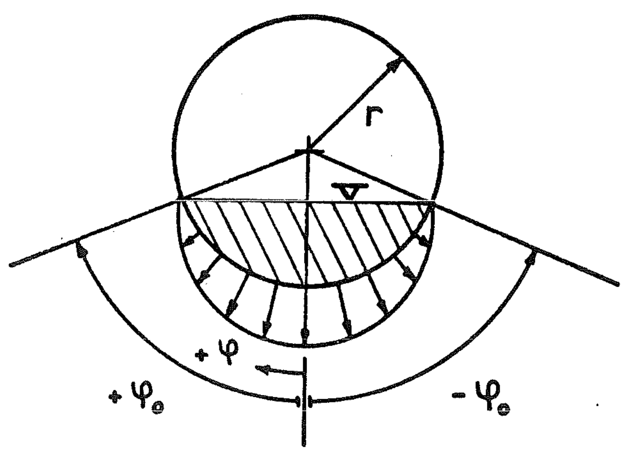

Figure 7 Pressure distribution for a partially-filled shell.

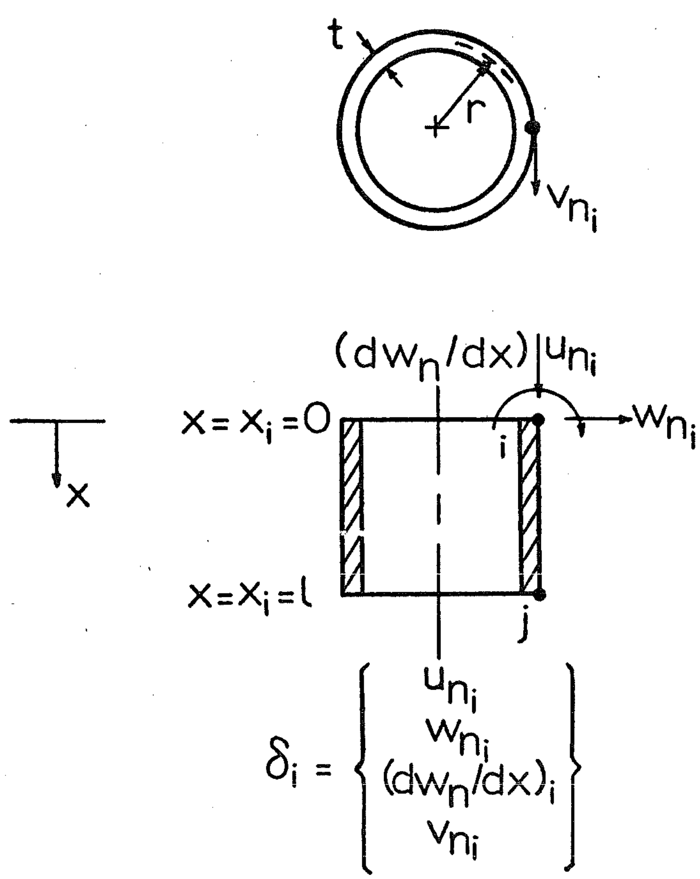

Figure 8. Displacements at the edges i and j.

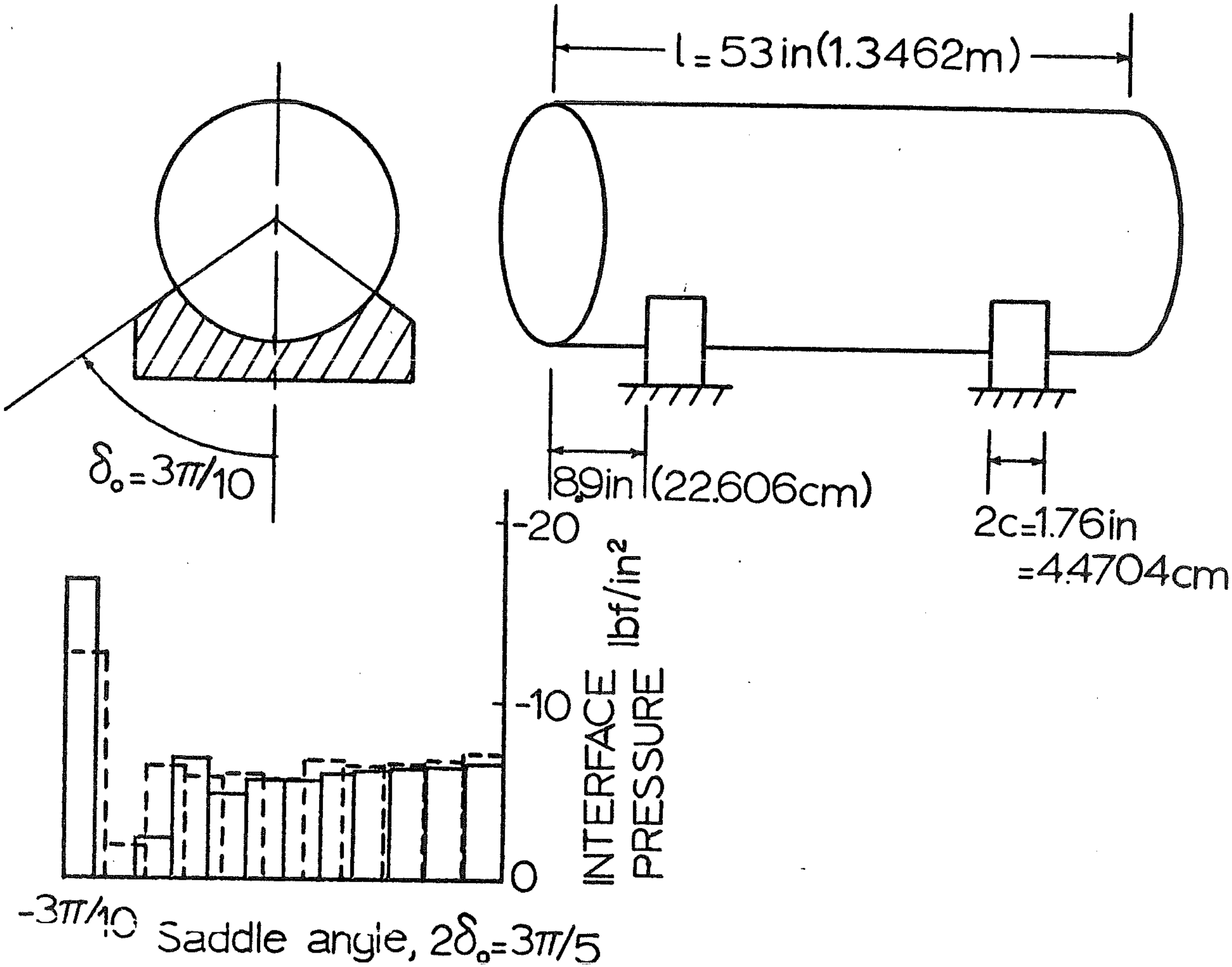

Figure 9. Analytical and experimental saddle/cylinder interface pressure of [5].

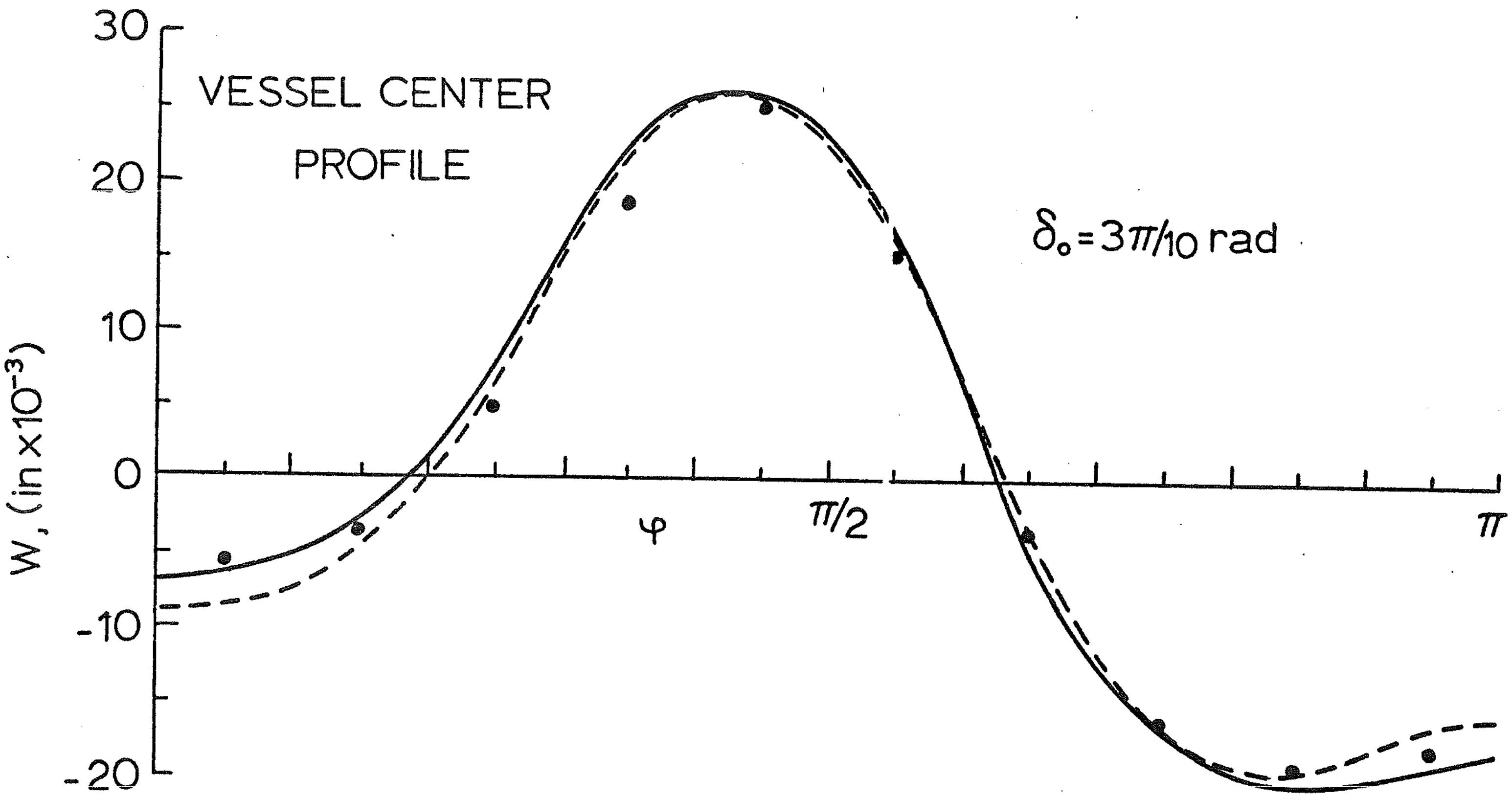

Figure 10. Radial displacements of a twin saddle supported cylindrical vessel full of liquid. – , this theory; --- theory of Forbes and Tooth [5]; •, experimental results of [5].

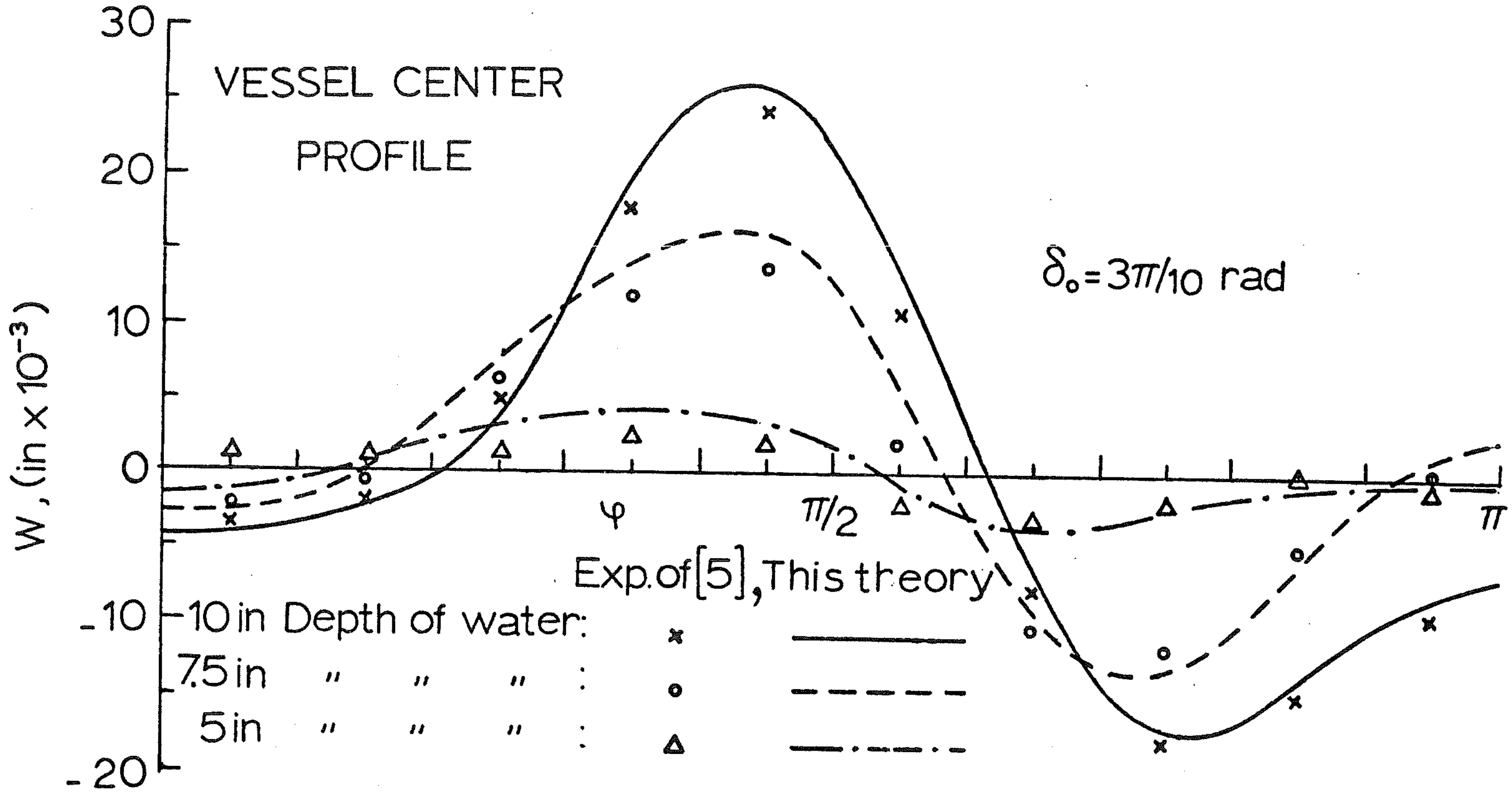

Figure 11. Radial displacements of a twin saddle supported cylindrical vessel at various water levels. ——, ---, —·—·—, this theory; x, o, Δ, experimental results of [5].

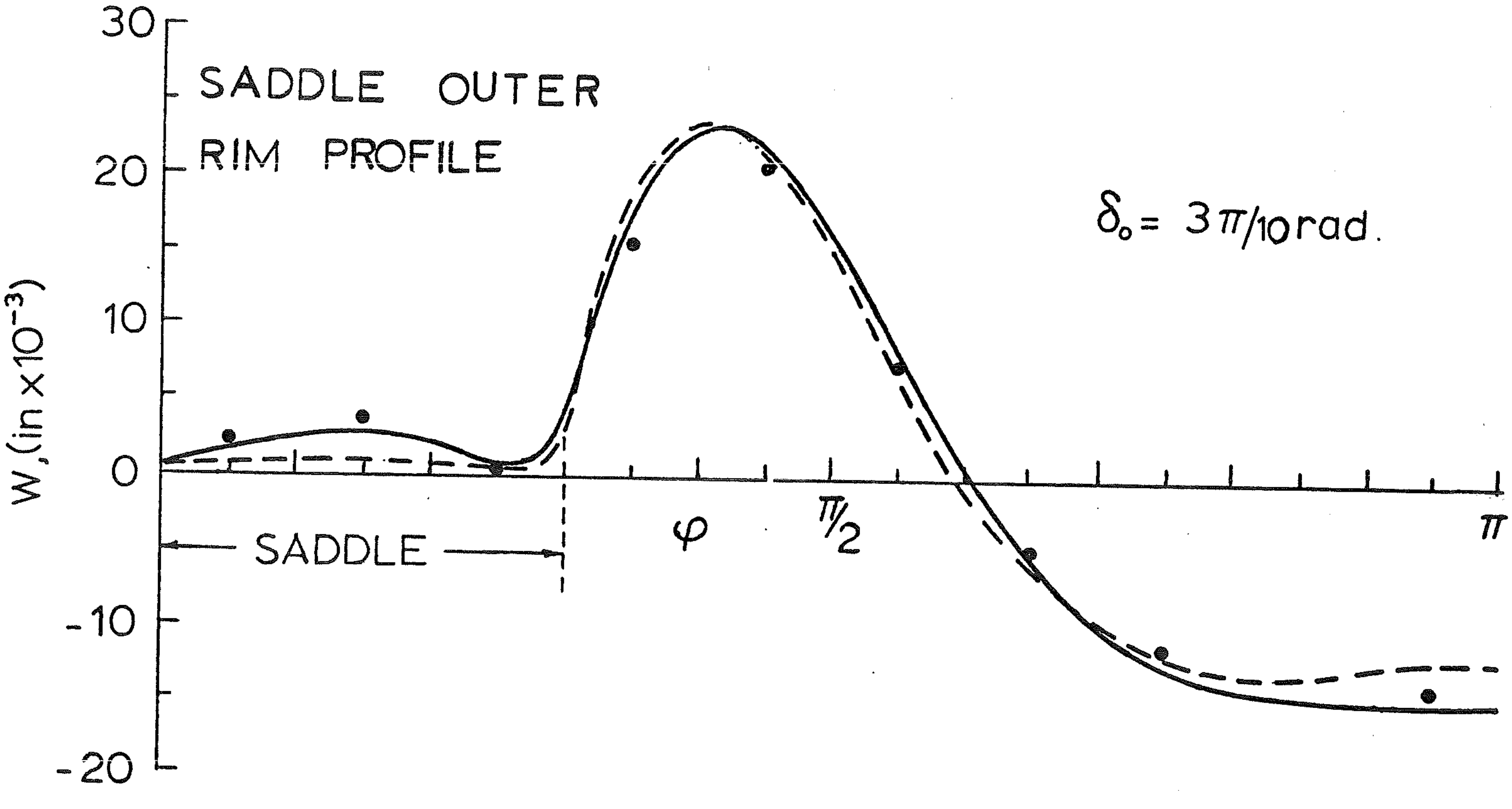

Figure 12. Radial displacements at saddle outer rim profile of saddle supported cyl. vessel full of liquid; ——, this theory; --- theory of [5]; ●, experimental results of [5].

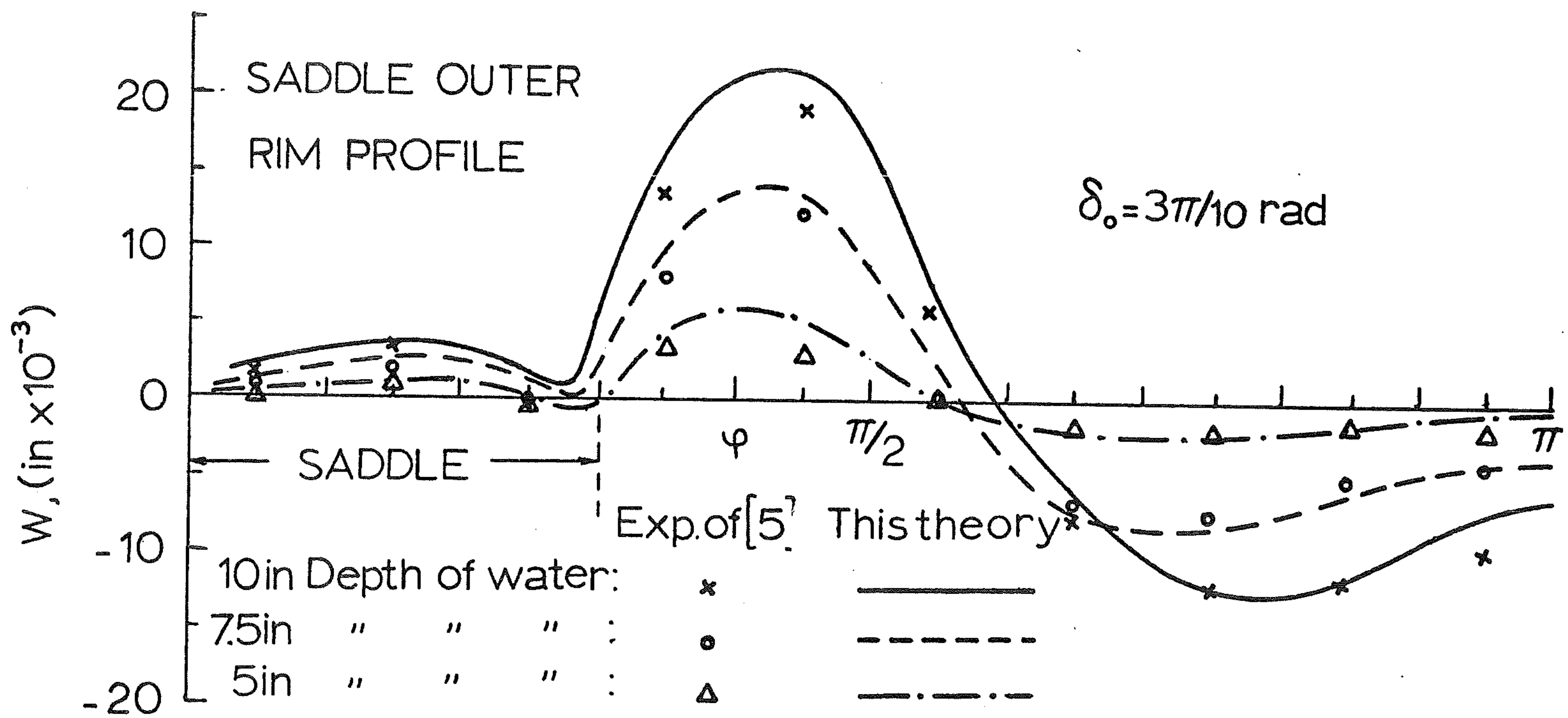

Figure 13. Radial displacements at various water levels. —— , ---, — . —, this theory; ×, o, △, experimental results of [5].

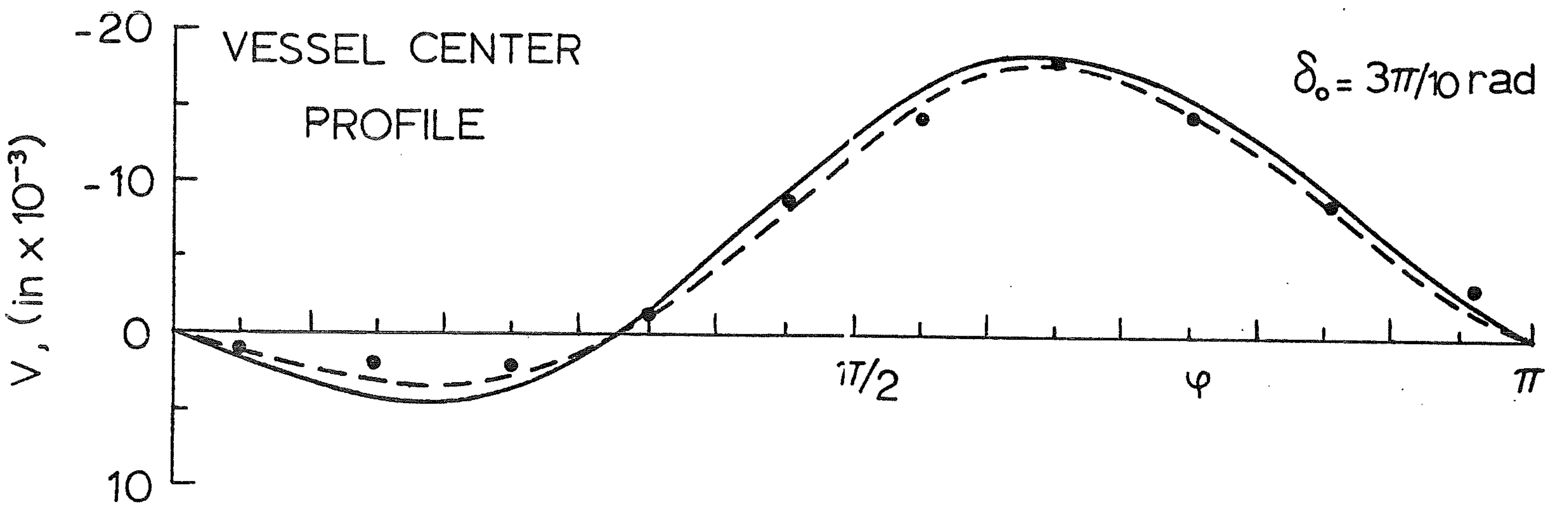

Figure 14. Circumferential displacements of saddle-supported cylindrical vessel full of liquid. ——— , this theory; ---, theory of [5]; ●, experimental results of [5].

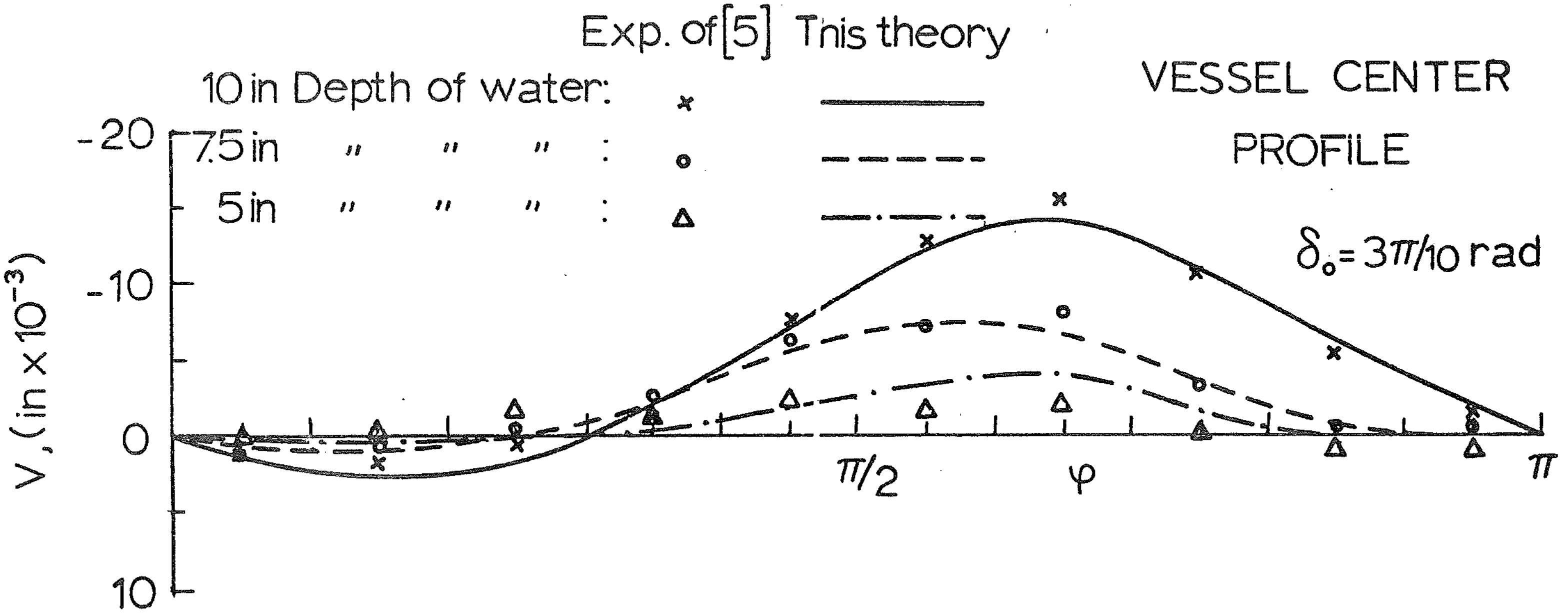

Figure 15. Circumferential displacements at various water levels. ——, ---, — . —, this theory; ×, o, Δ, experimental results of [5].