Effects of Transverse Shear Deformation and Rotary Inertia on the General Equations of Anisotropic Plates and Shells

gratuit

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NOMENCLATURE

A_1, A_2 :Lamé's parameters

A_{ij} : extensional stiffness eqn(20)

B_{ij} : bending-extensional coupling stiffness eqn(20)

D_{ij} :bending stiffness eqn(20)

I_i : inertia moment

L_i : motion equations eqn(36)

M_i ($i=1,2$): the moment resultants applied in α_i's direction

M_{ij} ($i,j=1,2$; $i \neq j$) :the moment resultants applied on the middle surface in α_j's direction (α_i =cte)

N_i ($i=1,2$): the in-plane force resultants applied in α_i's direction

N_{ij} ($i,j=1,2$; $i \neq j$) :the in-plane force resultants applied on the middle surface in α_j's direction (α_i=cte)

P_{ij} : terms of elasticity matrix($i=1,\ldots,10$; $j=1,\ldots,10$)

Q_{ij} ($i,j=1,2,3$) : the elastic stiffness in the material coordinates eqn(12)

$\overline{Q}_{ij}$ ($i,j=1,2,3$) : the elastic stiffness in the global coordinates eqn(16)

Q_i ($i=1,2$) : the transverse force resultants

q_1, q_2, q_n : the external force vector

R_i ($i=1,2$) : curvature radius

h : thickness of the shell

u_1, u_2, w : the displacement vector components

T_{ij} ($i,j=1,2,3$) :transformation matrix elements eqn(15)

α_1 and α_2 : curvilinear coordinates of the surface

β_1 and β_2 : the rotations of tangents to the reference surface

ε_i : deformation vector components

σ_i : stress vector component

ρ : density of the shell material

ζ : distance of the point from the corresponding point on the reference surface along the normal direction

$\varepsilon^0{}_1$ and $\varepsilon^0{}_2$: normal strains of the reference surface

$\gamma^0{}_1$ and $\gamma^0{}_2$: in-plane shearing strains of the reference surface

κ_1 and κ_2 : change in the curvature of the reference surface

τ_1 and τ_2 : torsion of the reference surface

$\mu^0{}_1$ and $\mu^0{}_2$: the shearing strains

Résumé

Ce travail présente l'analyse linéaire des coques anisotropes laminés et multicouches de forme générale basée sur la théorie des déformations de cisaillement avec le seul hypothèse de négliger la contrainte normale. Les résultats qui incluent les effets des déformations de cisaillement et l'inertie de rotation aussi bien que la courbure initiale, laquelle est incluse dans les résultantes de la contrainte et résultantes des contraintes de cisaillement transversales supposées, sont déduits par l'application de principe du travail virtuel, avec les déplacements et les rotations comme les variables indépendantes. Ces équations sont donc appliquées aux différentes géométries de coque comme les coques de révolution, cylindriques, sphériques et coniques aussi bien que des plaques rectangulaires et circulaires.

Abstract

The present work deals with a generalization of geometrically linear shear deformation theory for multilayered anisotropic shells of general shape. No assumptions are made other than to neglect the transverse normal strain. The results, which include the effects of shear deformations and rotary inertia as well as initial curvature(included in the stress resultants and assumed transverse shear stresses), are deduced by application of the virtual work principle, with displacements and transverse shear as independent variables. These equations are applied to different shell geometries, such as revolution, cylindrical, spherical and conical shells as well as rectangular and circular plates.

1. Introduction- Shells are widely used as structural elements in modern construction engineering, aircraft construction, ship building, rocket construction, the nuclear, aerospace and aeronautical industries, as well as the petroleum and petrochemical industries(pressure vessel, pipeline), etc. It is very important, therefore, that the static and dynamic behavior of these structure when subjected to different loads be clearly understood, in order that they may be used safely in industry.

The analysis of thin elastic shells under static or dynamic loads has been the focus of a great deal of research. These shells have been studied in the light of such different factors as the large displacements, thickness variation, residual stresses, rotary inertia, anisotropy, initial curvature and the effect of the surrounding medium(air, liquid), etc.

Many theories have been developed for thin elastic shells, in both linear and non-linear cases, and are based on the first approximation of Love-Kirchhoff theory which, because it does not take transverse shear deformations into account, can be grossly in error in predicting the transverse deflections, buckling loads and natural frequencies. In the case of plates and shells made of advanced laminated composite materials, the prediction errors are even more marked. The transverse shear effect on non-linear vibration and post-buckling behavior is significant especially for the laminates with moderately large thickness.

The present work presents the general equations of anisotropic shells(equilibrium, constitutive and kinematic relations) by considering the effects of shear deformation, rotary inertia and initial curvature. These relations are then applied to different shell geometries: shells of revolution, cylindrical, spherical and conical shells as well as the circular and rectangular plates.

2. Literature review- The literature review covers three broad areas. In the first, both linear and non-linear theories on the analysis of plates and shell structures are discussed. These theories were, in many instances, developed for isotropic materials before being extended to anisotropic material applications. The second part deals with the study of the effect of shear deformation on both the static and dynamic behavior of plates and shells, especially those made of advanced anisotropic materials. In the last part, we briefly discuss the effect of structure-fluid interaction on the vibrations of plates and shells. Special attention is given to cylindrical shells immersed in or filled with a liquid or subjected to a flowing fluid.

A shell structure may be defined as a body enclosed between two closely spaced and curved surface. In general, a shell has three fundamental identifying features; its reference surface, its thickness and its edges. Of these, the reference surface is the most significant because the behavior of the shell is governed by the behavior of its reference surface.

Many shell theories are derived from the equations of elasticity. The strain displacement relations of shells can be derived from kinematics and the *3-D* strain displacement relations written in terms of arbitrary curvilinear coordinates [1]. In reality ,the behavior of the top and bottom surfaces of a shell under load can vary widely.

The first attempt to formulate a bending theory of shells from the general equations of elasticity was made by Aron in 1874. A thin shell is one in which the thickness is small compared with the overall dimensions of the reference shell surface, and a two dimensional *(2-D)* theory is used to approximate three dimensional (3-D) phenomena. Many classical shell theories were developed originally for thin elastic shells, and are based on the Love-Kirchhoff assumptions which are: 1) the shell is thin ; 2) the displacements and rotations are small; 3) normals to the shell reference surface before deformation remain normal after deformation; and 4) transverse normal stresses is negligible.

These assumptions led to a thin shell theory that can be viewed as an extension to Kirchhoff flat plate theory and is often called Kirchhoff-Love shell theory. The effects of normal transverse strain are often neglected in the kinematics compared to the effects of the in-plane strains due to the thinness of the shell, and the shell is assumed to be an approximate state of plane stress. The in-plane stresses become dominant because the transverse normal stress is, in general, of order h/R times the bending stresses, whereas the transverse shear stresses, obtained from equilibrium conditions, are of order h/L times the bending stresses. Therefore, for L/R less than 10, the transverse normal stress is negligible compared to transverse shear stresses .

On the other hand, the normal transverse strain can generally be included in the analysis through the constitutive relations. In deriving the equilibrium equations, statically equivalent forces and moments acting on the reference surface can be defined by integrating stresses through the thickness. In this way, the *3-D* shell behavior can be fully described using a *2-D* approximation [1-4]. The third assumption of the Love-Kirchhoff theory is that transverse shear strains may not be written in terms of displacements, which leads to their being completely ignored although transverse shear stresses should be included in equilibrium equations.

Surveys of various classical shell theories can be found in the works of Bert [5], Reissner [6] and Naghdi [7]. The last truncate the Taylor's series expansion for tangential displacements after linear terms in the thickness coordinate, and many others followed him. An excellent collection of the research carried out on this topic has been produced by Leissa [8]. Elegant representations, both linear and non-linear, of Love's shell theory can be derived strictly via definitions from surface theory without reference to *3-D* relationships [3,9].

One of the best-known of these theories, Love's first approximation, yields sufficiently accurate results when (*i*) the lateral dimension to thickness ratio(L/h) is large; (*ii*) the dynamic excitations are within the low-frequency range; (*iii*) the material anisotropy is not severe. However, the application of such theories to layered anisotropic composites shells could lead to 20%-30%, or more, errors in prediction of natural frequencies, deflections, stresses and buckling loads .

There is an inconsistency in the original version of Love's theory since all strains do not vanish for rigid body motion. It was perhaps this inconsistency that encouraged many researchers to develop slightly different shell theories. Many shell theories based more or less on Love's assumptions have been developed, each different since each neglects or approximates small terms in its own way. Sanders [10] redefined the force and moment resultants in such a way that all strains vanish for any rigid body motion.

The thin shell assumptions in Love's theory have not been taken into account in the theories of Flügge, Lure and Byrne [3], which impose a less restrictive requirement on the thinness of the shell . Their theory also eliminates the rigid body strains anomaly. Koiter [11] discussed the significance of Love's first theory and, based on an order of magnitude study, states that refinements of the theory cannot consistently be made without including transverse deformation effects. Other prominent theories on this subject include those of Novozhilov [12].

Donnell applied Love's theory to shallow cylindrical shells [8]. Two types of basic equation, corresponding either to Flügge's or Donnell's equations for isotropic shells, have been formulated in the literature [2,3,13]. Donnell's derivation is not easy to follow, since it completely neglects a number of terms both in the relationships between the changes of curvature and twist and the displacement, and in the relations of stress resultants and moment resultants in terms of displacement.

A small displacement Love theory has been used by Dong et al. [14] for the bending analysis of thin anisotropic plates and shells. These are specialized to give linear Donnell equations for anisotropic cylindrical shells. Bogner et al.[15] developed a linear cylindrical isotropic shell finite element based on the classical shell theory. Morley [16] extended the limits of Donnell theory . Reissner [17] applied the Donnell's assumptions to a shallow spherical shell. The Donnell-Mushtari-Valsov equations [8] result when Donnell's assumptions are applied to a shallow shell of arbitrary geometry.

Cheng and He [18,19] have developed an exact linear theory for circular cylindrical shell based on Love's assumptions. By retaining all the small terms which are neglected, in varying degrees, by other theories, the usual eighth-order operator in the governing equilibrium equation of the transverse displacement can be separated into two fourth-order complex conjugate operators, thereby reducing the solution's complexity. A general theory for thin isotropic shells, which makes no simplifications for approximations beyond a fundamental hypothesis, was developed by Markov [20].

Padovan [21] used a complex multi-segment numerical integration procedure which can handle the static analysis of mechanically and thermally loaded laminated anisotropic shells of revolution with arbitrary meridional variation in thickness and material properties. The governing equations are based on the Love-Reissner theory. They did not considered the effects of shear deformation in their work.

Basar and Ding [22] used finite rotation elements for the non-linear analysis of thin shell structures. Their work is based on the Kirchhoff-Love hypothesis. In the development of a non-linear finite element using the Kirchhoff-Love hypothesis, the essential problem is the elimination of the rotation vector(the difference vector)without loss of accuracy. To do this, the Kirchhoff-Love hypothesis are expressed by two sets of equivalent conditions: one of them is used in the form of linear variational equations for elimination of the incremental rotational variables; the other, non-linear one, is needed for the exact calculation of the rotation vector of the fundamental state.

Most of the theories outlined above have been applied to a shell so thin that all transverse shear deformation effects, transverse stresses and strains can be neglected. These transverse effects become more pronounced as the shell becomes thicker relative to its in-plane dimensions and radius of curvature. This is particularly true of the transverse shear deformations [11] since classical theories can be grossly in error in error in predicting transverse deflections, buckling loads or natural frequencies. It is well known from experimental observations that the fact that classical plate theory neglects transverse shear strains leads to under-estimations of deflections and over-predictions of natural frequencies and buckling loads.

These errors are even higher in the case of plates and shells made up of advanced anisotropic laminated composites materials such as graphite-epoxy and boron-epoxy, where the ratio of elastic modulus to shear modulus is very great (i.e. of the order 25 to 40 instead of 2.6 for isotropic materials). As pointed out by Koiter [11], refinement to Love's approximation theory of thin elastics shells is meaningless unless the effects of transverse shear and normal stresses are taken into account. Transverse shear deformation plays a very important role in reducing the effective flexural stiffness of anisotropic laminated plates and shells because their in-plane elastic modulus to transverse shear modulus ratio is high.

The transverse shear effect on non-linear vibration and post-buckling behavior is significant, especially for laminates with moderately significant thickness, a high circumferential wave number and a greater number of layers. Study of shear deformation effects shows that these effects can become quite meaningful for some geometrical parameters, such as small radius-thickness or length-thickness ratios, as well as for shorter wavelengths or longer shells.

In addition to the transverse shear deformation, the initial curvature effect should be considered for the analysis of thick shells as indicated by Voyiadjis and Shi [23] for isotropic materials. The initial curvature effect is very important in making accurate predictions of stresses even in the central region. In the shell structures, the curvature of each parallel surface through the thickness of the shell is different. To consider the initial curvature effect, the term $1+z/R$ has to be included. The presence of curvature effectively increases the structural stiffness.

In refined shell theories that take the transverse shear deformation effect into account, the normals to the reference surface of shells are permitted to rotate such that plane sections originally perpendicular to the middle surface remain planar, but, as a result of the deformation, are no longer perpendicular. The transverse shear is represented by inclusion of an independent degree of freedom in the kinematics. The shell is still fully described by the behavior of the reference surface and therefore these approaches represent *2-D* theories [24].

Hilderbrand et al. [25] were the first to make significant contributions by dispensing with Love's assumption and assuming instead a three terms Taylor's series expansion for the displacement vector for orthotropic and homogeneous shells. Naghdi [26] has employed Reissner's [27] mixed variational principle to develop a complete shell formulation similar to that of Hilderbrand et al. [25], retaining two and three terms in the Taylor's series expansions for tangential and transverse displacement components, respectively.

The first analysis to incorporate bending and stretching coupling was carried out by Ambartsumyan [9]. He assumed that the individual orthotropic layers were oriented in such a way that the principal axes of material symmetry coincided with those of the principal coordinates of the shell reference surface. The effects of transverse shear deformations ,transverse normal stresses and transverse normal strain on the behavior of laminated shells can be incorporated, on the basis of a mathematical model, through the inclusion of higher order terms in the power series expansion of the assumed displacement field.

Dong and Tso [28] were perhaps the first to present a first order shear deformation theory , retaining one and two terms in the Taylor's series for transverse and tangential displacement components, respectively. The theory includes the effects of transverse shear deformation through the shell thickness, and thence they construct a laminated orthotropic shell theory. Hilderbrand et al. [25] found that the effects of the additional terms in the transverse displacement that resulted in non-zero transverse normal strains are negligible. Reissner used these kinematic relations to analyze first plates [29] and then sandwich shells . The rotatory inertia terms have been included in the dynamic analysis of plates by Mindlin [30].

The above-mentioned first order shear theories that result from the so-called Reissner-Mindlin (RM) kinematics do not satisfy the transverse shear boundary conditions on the top and bottom surfaces of the shell or plate, since a constant shear angle through the thickness is assumed, and plane sections remain plane. For this reason, the theories based on these kinematic relations usually require shear correction factors for equilibrium considerations. The shear correction factors are only functions of lamination parameters(number of layers, stacking sequence, degree of orthotropy and fiber orientation in each individual layer).

Levinson [31] and Reddy [32] have developed theories that include terms in the in-plane displacement kinematics. They used a parabolic shear strain distribution through the thickness for satisfying zero transverse shear stress on the top and bottom surfaces of the shell, thus producing closer agreement with linear elasticity. The parabolic shear strain distribution has been used to analyze the linear vibrational behavior of isotropic cylindrical shells by Bhimaraddi [33].

The effects of transverse shear deformation and transverse isotropy as well as thermal expansion through the thickness of cylindrical shells were considered by Gulati and Essenberg [34], Zukas and Vinson [35], Dong and his colleagues [14], Hsu and Wang [36], Chaudhuri and Abu-Arja [37] and Khedir et al. [38].

Whitney and Sun [39,40] developed a shear deformation theory for laminated cylindrical shells that includes both transverse shear deformation and transverse normal strain as well as expansional strains. The theory is based on a displacement field in which the displacements in the surface of the shell are expanded as linear functions of the thickness coordinate and the transverse displacement is expanded as a quadratic function of the thickness coordinate. They discussed some methods by which one can diagnose the mass matrix. They did not consider the some product of the first order derivatives of the tangential displacement component with respect to the x, y and z in the strain -displacement relations. These relations are based on the Von Karman's theory [12].

Reddy [41] extended Sanders' [42] theory for simply supported cross-ply laminated shells assuming five degrees of freedom per node. The theory is based on a displacement field in which the displacements of the middle surface are expanded as cubic functions of the thickness coordinate, and the transverse displacement is assumed to be constant through the thickness. The Navier-type exact solutions for bending and natural vibration are presented for cylindrical and spherical shells under simply supported boundary conditions.

A generalization of geometrically linear shear deformation theories for small elastic strains is presented for multilayered axisymmetric shells of general shape by Touratier [43]. He proposes a general shear deformation theory for multilayered, moderately thick, axisymmetric shells. The theory, which is geometrically linear, is developed for small elastic strain and is restricted to axisymmetric shells under axisymmetric loading and classical boundary conditions. The principal advantage of this work is that it does not need shear correction factors.

Static analysis of laminated shells using a refined shear deformation theory was done by Ji-Fan He [44]. According to this theory, the thickness of the shell must be small compared to the principal radii of curvature. It can be expected that the present theory would tend to be fairly accurate for laminated shells with many layers. Hsu and Wang [36] and Di Sciuva [45] proposed a specially designed displacement field with traction continuity at the layer interface; and Reissner [46] proposed another type of general shell theory for transversely isotropic materials based on the Reissner mixed variational principle with independently assumed transverse stresses.

More recently, Jing and Tzeng [47]derived a mixed shear deformation theory for thick laminated shells of general shape based on proposed method of Jing and Liao [48]. The displacement field uses a zig-zag function in addition to the Reissner -Mindlin type in-plane displacements and a constant transverse deflection. Kant and Ramesh [49] developed complete governing equations for a thick laminated composite shell. The theory is based on a three-term Taylor's series expansion of the displacement vector and generalized Hook's law, as is the displacement model of Hilderband et al.[25], and is applicable to orthotropic material lamina having planes of symmetry coincident with shell coordinates.

Advanced composites materials are being used more and more in a variety of industries due to their high strength and stiffness-to-weight ratios; this has led to a rapid increase in the use of these materials in structural applications during the past decade. Structural elements made up of advanced fiber-reinforced composite materials offer unique advantages over those made of isotropic materials. They are being extensively used in high and low technology areas, e.g., the aerospace industry, where complex shell configurations are common structural elements.

The filament-winding techniques for manufacturing composite shells of revolution has recently been expanded in aircraft, shipbuilding, petroleum and other industries. In general, these materials are fiber-reinforced laminates, symmetric or anti-symmetric cross-ply and angle-ply, which consist of numerous layers each with various fiber orientation. Although the total laminate may exhibit orthotropic-like properties, each layer of the laminate is usually anisotropic, thus the individual properties of each layer must be taken into account when attempting to gain insight into the actual stress and strain fields.

By optimizing the properties we can reduce the overall weight of a structure since stiffness and strength can be designed only where they are required. A lower weight structure translates into higher performance. Since optimized structural systems are often more sensitive to instabilities, it is necessary to exercise caution. The designer would be much better able to avoid any instabilities if, when predicting a maximum load capacity, he either knew the equilibrium paths of structural elements or had accurate modeling of the load-displacement behavior of structure.

Anisotropic laminated plates and shells have a further complication which must be considered during in design process: potentially large directional variations of stiffness properties in these structures due to tailoring mean that three-dimensional effects can become very important. The classical two-dimensional assumptions may lead to gross inaccuracies, although they may be valid for an identical shell structure made up of isotropic materials.

However, although they have properties that are superior to isotropic materials, advanced composite structures do present some technical problems in both manufacture and design. For computational reasons, the study of composite materials involves either their behaviors on the macroscopic level such as linear and non-linear loading responses, natural frequencies, buckling loads ,etc., or their micro-mechanical properties such as cracking, delamination ,fiber matrix debonding, etc.

A number of theories for layered anisotropic shells exist in the literature. Many of these theories were developed for thin shells and are based on the Kirchhoff-Love hypothesis. The first analysis that incorporated the bending -stretching coupling(due to asymmetric lamination in composites) was by Ambartsumayan [9]. In his analysis, he assumed that the individual orthotropic layers were oriented such that the principal axes of materials symmetry coincided with the principal coordinates of the shell reference surface. He [9] has written extensively on the matter, basing his work on Love's theory with some discussion of transverse stresses.

The simplifying assumption of laminated anisotropy is often used in applying a *2-D* theory to plates and shells consisting of layers of composite materials [24]. In this approach, the individual properties of the composite constituents, the fibers and the matrix, are "smeared" and thus each lamina is treated as an orthotropic material.

A survey of the analysis of multilayered composite shells using Reissner's mixed variational principle was done by Grigolyuk and Kulikov [50]. They maintain that laminated anisotropy assumes perfect bonding between layers, and that the interply adhesive has infinitesimal thickness but infinite stiffness. This approach leads to calculated laminated plate theory (*CLPT*) and the references by Jones [51] and Whitney and Pagano[52], to *CLPT* are based on the Kirchhoff-Love assumptions. However, both references point out that transverse shear deformation is more significant in laminated anisotropic than in similar isotropic constructions.

Bert [53] used Valsov shell theory to formulate a linear laminated shell theory similar to *CLPT*. Pagano and Wang [54-57] and Srinivas and Rao [58] have developed some exact solutions of *3-D* elasticity equations governing composite plates that have been used to validate the shear theory. They conclude that *CLPT* gives fairly good approximations for both the displacements and stresses if the plate is thin. Higher order shear theories do not give much better transverse stress results but displacements show a marked improvement over *CLPT* for the thicker plates. Transverse stresses are best calculated from equilibrium instead of from the constitutive relations [51]. Ren [59] similarly solved *3-D* elasticity equations for a laminated cylindrical shell in cylindrical bending .

His work dealt with what is now known as laminated orthotropic shells rather than with laminated anisotropic shells. In laminated anisotropic shells, the individual layers are, in general, anisotropic, and the principal axes of material symmetry of the individual layers coincide with only one of the principal coordinates of the shell (the thickness-normal coordinate). Whitney and Pagano [52] applied the Reissner-Mindlin theory to composite plate analysis. The buckling of laminated cylindrical shells was studied by Hirano [60]. Reddy and Chao [61] applied the closed form solution to thick composite plate.

Reddy [24,62] has extended the cubic kinematic approach to analysis laminated anisotropic plates and he has applied them to solving several linear static and buckling problems. Additionally, Soldatos applied the parabolic shear theory to examination of the stability of asymmetrically laminated cylindrical panels [63,64].

Cheng and Ho [65] presented an analysis of laminated anisotropic cylindrical shells using

Flügge's shell theory [2]. A first approximation theory for the asymmetric deformation of nonhomogeneous, anisotropic, elastic cylindrical shells was derived by Widera and his colleagues [66,67] by means of the asyomptic integration of the elasticity equations. For a homogeneous, isotropic material, the theory reduces to Donnell's equations.

Noor and Peters [68] presented the free vibration analysis of laminated anisotropic shells of revolution as well as the sensitivity of their response to anisotropic material coefficients. Their analytical formulation is based on a form of the Sanders-Budiansky shell theory, including the effects of both transverse shear deformation and the laminated anisotropic material response. Each of the shell variables is expressed in terms of trigonometric functions in the circumferential coordinate and a three-field mixed finite element model is used for the discretization in the meridional direction. They used a reduction method involving the successive use of the finite element method and the classical Bubnov-Galerkin technique to substantially reduce the size of the eigenvalue problem.

Zienkiewicz [69] introduced a finite element approach with independent transverse displacement and rotational degrees of freedoms such that a *RM* (Reissner-Mindlin) shear deformable shell element is obtained. A small rotation approach for anisotropic shells has been developed by Librescu and Schmidt [70]. Successive approximations, as steps towards an estimate of exact shell strain displacement relations where displacements, large strains and rotations are all initially allowed, are presented for isotropic shells by Sanders [10] and anisotropic shells by Librescu [70].

Kant and Kommineni [71] presented higher order theories for general orthotropic as well as laminated shells. These theories are derived from the three-dimensional elasticity equations by expanding the displacement vector in Taylor's series in the thickness coordinate. Reference [72] presents some elements which can be applied successfully to analysis of both thin and thick plate and shells. Kui et al. [73] applied the finite element method, displacement type, to analyses of thin shells and to overcoming the shear locking phenomena.

Prayor and Barker [74] developed a linear plate element based on the *RM* theory. They used a rectangular element with 28 degrees of freedom (8,12,8 for extension, bending and shear effects, respectively) to have the continuity of transverse stress at any interface. Hinrichsen and Palazotto [75] applied a cubic spline function to non-linear analysis of thick composite plates. Their theory is based on the usual Kirchhoff hypothesis. The theory was developed by considering the Lagrangian strains in conjunction with the second Piola-Kirchhoff stress hypothesis. This formulation leads to a quasi-three dimensional element that combines large displacement with moderately large rotation but is restricted to small strains.

Schmit and Monforton [76] formulated an anisotropic cylindrical shell element which allows them to predict the geometrically nonlinear behavior of sandwich plate and cylindrical shell structures, based on accepted thin shell theory assumptions. Other recent papers by Meroueh [77] and Surana [78,79] can be mentioned. Cylindrical shells are in general use in the aerospace, shipbuilding, structural and petroleum industries. They are the simplest shell structure to analyse yet have many of the characteristics of more complex shell geometries. The linear problem of composite cylindrical shells has been widely investigated by a number of researchers using different shell theories. Based on the Kirchhoff hypothesis, for example, Dong [80] studied the free vibration of laminated orthotropic cylindrical shells with homogeneous boundary conditions.

The governing equations of orthotropic cylindrical shells were solved through a pair of complex conjugate fourth-order differential equations by Cheng and He [19]. Their work is based on the kirchhoff hypothesis. For the static problem, Flügge and Kelkar [81] and Yao [82] obtained an exact solution for closed isotropic long cylinders under general two-dimensional surface traction.

Using the Forbenius method, Srinivas [58] developed an exact three-dimensional solution for orthtropic finite cylinders with simply supported conditions. Varadan and Bhaskar [83] also performed the static stress analysis using the procedures proposed by Srinivas[58]. Pagano [84] obtained the stress field for a homogeneous, anisotropic closed cylinder under two-dimensional surface loads in which the problems are independent of the axial coordinate.

Ren [85] presented an exact solution for simply-supported laminated cross-ply circular cylindrical panels of infinite and finite length in the axial direction. Leissa et al. [86] analysed the vibration of cantilevered cylindrical panels by using the Ritz method, with algebraic polynomial functions for the displacements. Widera and Logan [67] studied the non-homogeneous, anisotropic, circular cylindrical elastic shell, using the method of asymptotic expansion in terms of a small parameter in conjunction with Reissner's variational principle.

In their work, the procedure used to derive the shell equation starts with substitution of non-dimensional shell coordinates in terms of characteristic length scale for changes of stress and displacement and Reissner functional direction. The employment of the formulation in terms of Reissner's principle allows one to obtain automatically all the equations necessary to formulate a complete boundary value problem for a first approximation shell analysis. Non-dimensional stresses, displacements and Reissner functional direction are introduced and considered to be representable by asymptotic expansions in a power series in terms of a small shell parameter.

Recently, Bert and his colleagues [87,88] and Hsu et al. [89] presented an exact solutions for bending and vibration of two layer cross-ply, thin cylindrical shells. These solutions are limited to cylindrical shells and sinusoidal distribution of the transverse load, and the procedure used is similar to that used by Whitney and Leissa [90], Whiteny and Pagano [52], Bert and Chen [91], and Reddy and Chao [61] for laminated composite plates.

Hung-Sying Jing and Kuan-Goang Tzeng [92] proposed a mixed shear deformation theory for the bending analysis of arbitrarily laminated, anisotropic panels and closed cylinders. The initial curvature effect is included in the strain -displacement relations, stress resultants and assumed transverse shear stresses. Two types of shell geometry, infinitely long cylindrical panels and closed cylinders of finite length, are employed in the numerical study. Suzuki and Leissa [93,94] analysed the free vibration of circular and non-circular cylindrical shells having circumferentially varying thickness.

The static response to the axisymmetric problem of arbitrarily laminated, anisotropic cylindrical shells of finite length using three-dimensional elasticity equations was made by Jing and Zeng [95]. The closed cylinder is simply supported at both ends. The highly-coupled partial differential equations are reduced to ordinary differential equations with variable coefficients by choosing the solution composed of trigonometric functional along the axial direction.

Kant et al.[49,71] presented various higher order theories for laminated composite cylindrical shells using C_0 finite elements. Kant and co-workers did extensive numerical investigations on laminated plates and shells, both static and dynamic analysis, using C_0 finite elements and different higher order theories. They proved that the imposition of shear free boundary conditions on the top and bottom bounding planes of the laminate gives stiffer solutions when compared to three-dimensional(*3-D*) elasticity solutions and various displacement models for flat laminates. The one having nine degrees of freedom per node produces results very close to *3-D* elasticity solution.

A higher order shear deformation theory of plates accounting for the Von Karman strains is presented by Reddy [96]. This theory contains the same dependent unknowns as those in the Hencky-Mindlin type first-order shear deformation theory. The displacements are expanded in powers of the thickness of the plate, and accounts for parabolic distribution of the transverse shear strains through the thickness of plate. Hamilton's principle was used to derive the equations of motions and the Navier solution procedure was used to solve the equations of the simply supported plates.

Jing and Liao [48] proposed a mixed function with displacements and transverse shear stresses as independent variables and established the corresponding partial hybrid stress element for the analysis of thick laminated plates. Some comparison between the results obtained for plates by these two functional are made by Jing and Tzeng [97].

A refined laminated plate theory, is developed by J.M.Whitney and C.T.Sun [39], is applicable to fiber-reinforced composite materials under impact loading. The theory includes the first symmetric thickness shear and thickness stretch motion, as well as the first anti-symmetric thickness shear mode, by including higher order terms in the displacement expansion about the mid-plane of the laminate in a manner similar to that of Mindlin and Medick [98] for homogeneous isotropic plates.

Reddy and Phan [62] used a higher order shear deformation theory to determine the natural frequencies and buckling loads of elastic plates. The theory accounts for the transverse shear strain and rotary inertia. This work dealt with the exact solutions of the theory as applied to the free vibration and buckling of isotropic, orthotropic and laminated rectangular plates with simply supported edge conditions. Reddy [32] developed a higher order shear deformation theory for the laminated composite plates. The displacement fields used in the present theory and that of Levinson [31] are the same, the equations of motion differ from those of both Levinson and Murthy. This theory uses a displacement approach similar to that in the Reissner-Mindlin type theories.

The form is dictated by satisfying the conditions that the transverse shear stresses vanish on the plate surfaces and be non-zero elsewhere. This requires the use of a displacement field in which the in-plane displacements are expanded as cubic functions of the thickness coordinate and the transverse deflection is constant through plate thickness. Ren and Hui [99] formulated a simple theory for non-linear bending of generally laminated composite rectangular plates which accounts for transverse shear strains by using the principle of virtual displacements. Moreover, because the total deflection of a plate is decomposed into a deflection due to bending and a deflection due to shear, solution of the governing equations of the present theory becomes simpler.

Jing and Liao's functional, modified from the Hellinger-Reissner principle by separating the stress field into a flexural part and a transverse shear part and leaving only displacements and transverse shear stresses as independent variable, has been used by Jing and Tzeng [47] to analyze laminated plates with satisfactory accuracy.

There are many situations in mechanics in which some simplifying assumptions have been considered to help the analyst in getting timely and accurate results. However, various air, water and land vehicles and structures as aircraft, rockets, pressure vessel, petroleum and petrochemical units and etc., may be subjected to impacts, collisions, blasts and /or other intensive transient loads which can cause large transient structural deformation and damage. Thin shell subjected to dynamic loads could encounter deflections of the order of the shell thickness or higher. Thin shells could also encounter a phenomenon of dynamic impacts or dynamic buckling and collapse, which are attributed to the change in the equilibrium state characterizing the load-response mode.

Response of this kinds cannot be correctly predicted by using the small or intermediate displacement theory. In the intermediate non-linearity approach, the non-linear terms which represent in-plane rotations of the shell are neglected [100,101]. This theory is often used in stability analysis. The structural elements made up of the advanced composite materials undergo large deformations before they become inelastic, because of the high modules and high strength properties of composites materials. Therefore, an accurate prediction of transient response is possible only when one accounts for the geometric non-linearity.

There are also cases where structural elements experience only small strains under load but may fail catastrophically due to their geometric configuration. It turns out that this class of structural system can be accurately analyzed on the basis of small strain, non-linear geometrical and linear elastic material behavior. The need for accurate and efficient methods of structural analysis and design, especially for this category of large-deflection(geometrically non-linear) and elastic -plastic (materially non-linear) dynamic response problems, has recently become increasingly apparent.

In the proposed non-linear analysis methods, e.g. [10,12,102], many of the non-linear displacement terms may be considered negligible depending, of course, on the specific situation. For example, an accurate load displacement characterization of a flat plate is based on the Von Karman equation where many non-linear rotational terms have been omitted. Similar assumption for shell elements result in equations of the type proposed by Donnell, Sanders and Novozhilov [102]. These formulations are typically valid for so-called intermediate non-linearity or theories that allow only moderate rotations.

The strain displacement relations that include non-linear displacement terms are used to represent large displacements and rotations of differential elements of the shell. Non-linear vibrations of generally laminated circular cylindrical shells are examined using the Timoshenko-Mindlin kinematics hypothesis and an extension of Donnell's shell theory. The effects of the transverse shear deformation, rotary inertia and initial geometrically imperfection are included in the analysis. The Galerkin procedure furnishes an infinite system of equations for time functions which are solved by the method of harmonic balance [103].

It has been recognised that the non-linear behavior of composite cylindrical shells plays an important role in determining the stability and dynamic response of these shells. Chu [104] first presented an analysis for circular isotropic cylindrical shells with the hardening type of non-linearity for the amplitude-frequency response. Nowinski [105] confirmed the results of Chu [104] by investigating the non-linear vibration of orthotropic cylindrical shells. Later, Evensen [106] pointed out that the mode shape assumed by Chu does not satisfy the condition of continuity of the circumferential in-plane displacement. A more rigorous study of non-linear free flexural vibrations of circular cylindrical shells was conducted by Atluri [107] who compared his results with the available data and concluded by accepting the possibility of the softening type of non-linearity.

Chen and Babcook [108] adopted a perturbation technique in considering the large-amplitude vibration of a thin-walled cylindrical shell. Ramachandran [109] studied the non-linear vibration of cylindrical shells of varying thickness. Khot [110] studied the post-buckling behavior of a laminated cylindrical shell subjected to axial load and torsion using the Von Karman-Donnell equations. The results obtained by Khot [110] show that, in general, composite shells are less imperfection sensitive than isotropic shells.

Recently, Iu and Chia [111] discussed the non-linear vibration and post-buckling of antisymmetric cross-ply circular cylindrical shells on the basis of Von Karman-Donnell kinematic assumptions, and the effects of transverse shear on the non-linear behavior of these shells using the Timoshenko-Mindlin kinematic hypothesis. They neglected the some terms (e.g cross-product of displacement derivatives)in non-linear strain-displacement relations.

Neglecting the transverse rotational non-linear terms as well will result in a linear Love-type shell the theory. These successive approximations to the shell strain displacement relations are discussed in the paper by Librescu [112] and Sanders [10]. In the last work, the deformations are restricted by the Kirchhoff hypothesis(the transverse shear and normal strains were neglected), the middle surface strains are assumed small and the rotations are assumed to be moderately small. Most of the above approaches can include various degrees of non-linearity in the strain displacement relations representing the displacements and rotations. Considerable simplification was achieved in the Donnell equations by use of the assumption that the non-linear membrane strains derived only from out-of-plane rotations.

For example, Donnell's theory is not suitable for the analysis of shells in which the buckling mode involves fewer than three full waves around the circumference [102]. More accurate non-linear shell equations are given by Sanders and by Novozhilov, but these are somewhat more complex than the Donnell equations. More terms are retained because fewer assumptions are made about the relative magnitude of various terms in the non-linear strain-displacement. Reddy and Chandrashekhara [113] solve laminated shell problems, both cylindrical and spherical, assuming *RM* theory and an intermediate non-linearity. There are few such analytical closed-form solutions for shell geometries, especially those that govern non-linear behavior.

The formulation and computational procedure are presented for the geometrically non-linear analysis of laminated orthotropic and anisotropic composite shells based upon a modified incremental Hellinger-Reissner principal and the total Lagrangian description by Rothert and Di [114]. In this investigation a computational model for a geometrically non-linear analysis has been studied on the basis of a rational approach for a hybrid stress model.

The through-thickness assumption used in the total Lagrangian formulation is introduced, incorporating the nonlinear formulation for a large rotation assumption. Noor and Peters [115] analyzed the non-linear response of anisotropic cylindrical panel that included transverse shear deformation. Their formulations are based on the Rayleigh-Ritz technique and the Hu-Washizu mixed shallow shell finite element approach.

Stein [116] used truncated series expansions of an exact non-linear strain displacement relations in a shell approach that also included transverse shear deformation. The non-linear strain-displacement relations are expanded into a series that contains all first- and second-degree terms; only the first few terms have been retained for the displacements. Geometrically non-linear quasi-three-dimensional approaches for laminated composite plates and shells have been developed by Palazotto and Witt [117], Hinrichsen and Palazotto [75] and Dennis and Palazotto [118]. Their work is restricted to small strains; the exact Green's strain displacements and linear strain displacement relations are assumed for the in-plane strains and the transverse strains, respectively, so the accuracy in rotation is limited by linear assumption on the transverse shear strains.

Tsai and Palazotto [119] have developed a finite element formulation for the geometric non-linear vibration analysis of cylindrical shells, based upon a curved quadrilateral, 36 degree of freedoms, thin shell element. The equations of motion are based on a total Lagrangian frame of reference. A β method, which is a generalization of Newmark's time marching integration scheme and the Newton-Raphson iterative method, are both applied in order to solve the set of non-linear equations of motion numerically.

The solution of a set of non-linear, second order ,differential equations which describe an anisotropic shell of revolution was presented by Martin and Drew [120]. Their analysis is based upon Sanders' non-linear shell theory without considering the shear deformation effects. The method for solving these equations follows the procedure used by Budiansky and Radkowski.

Kant and Kommineni [121] presented the geometrically non-linear transient analysis of laminated composite(transversely isotropic) and sandwich shells, based on Von Karmman's theory. In the time domain, the explicit central difference integrator is used in conjunction with the special mass matrix diagonalization scheme which conserves the total mass of the element and includes effects due to rotary inertia terms.

Rotter and Jumikis [102] have presented a set of non-linear strain-displacement relations for axisymmetric thin shells subject to large displacements with moderate rotations, by retaining more terms. Their works is based on Kirchhoff's assumptions. They have shown that non-linear strains arising from products of in-plane strain terms, which were omitted in previous theories, may be important in certain buckling problems. The new relations are particularly important when branched shells are being studied and when the buckling mode may involve a translation of the branching joint. Their work dose not include any numerical results.

A modal approximations in deriving the equations of motion for the non-linear flexural vibrations of a cylindrical shell by using the Donnell's shallow shell theory was presented by Dowell and Ventres

[122]. The purpose of their work was to satisfy more accurately the boundary and the continuity conditions and investigation their effects on the form of the modal equations.

Horrigmoe and Bergan [123] presented classical variational principles for non-linear problems by considering incremental deformations of a continuum. Wunderlich [124] and Stricklen et al. [125] have reviewed various principles of incremental analysis and solution procedures for geometrical non-linear problems respectively. Noor and Hartely [126] employed the shallow shell theory with transverse shear strains and geometric non-linearities to develop triangular and quadrilateral finite elements.

Chao and Reddy [127], Reddy and Chandrasekhara [113] have presented a first order shear deformation theory based on kinematic and geometric assumption of Sander's thin shell theory for the geometrically non-linear analysis of doubly curved composite shells. An analysis of the dynamic responses of cylindrical shells including geometric and material non-linearities was made by Wu and Witmer [128]. The methods of finite element analysis are applied to the problem of large deflection, elastic-plastic dynamic response of cylindrical shells to transient loading. The formulation is based upon the virtual work principle and D'Alambert's principle. Wu and Witmer used a bilinear polynomial for the axial displacement, and bicubic polynomials for both the circumferential displacement and the transverse displacement, and explicitly excluded rigid body modes.

The analytical solution of shell motion equations is generally considered to be difficult. Approximation methods can be used (e.g. the finite difference, Galerkin, Rayleigh-Ritz, Transfer matrix and finite element methods). All of these methods have both the advantages and the disadvantages. One of the most important criteria in determining the versatility of the solution is the capacity to predict, with precision, both high and low frequencies.

In the finite difference method, the initial values are given and this method requires a great deal of calculation times. The Galerkin approach loses precision in the higher frequencies of shells. The Rayleigh-Ritz method presents several draw-backs, among which are the displacement function choice, which has to take the boundary conditions into account, and the necessity to use a large number of terms to express displacement functions. On the other hand, the finite element method [69,129-132] is satisfactory from these view points.

The accuracy of solutions reached by the finite element displacement formulation depends on whether the assumed functions accurately model the deformations modes of structures. To satisfy this criterion, Lakis and his group have developed a hybrid type of finite element, whereby the displacement functions in the finite element method are derived from Sander's classical shell theory [42]. This method has been applied with satisfactory results to the dynamic linear and non-linear analysis cylindrical shells, both closed and open [133-143], spherical [144], conical [145], isotropic and anisotropic, uniform and axially non-uniform shells, both empty and liquid-filled. This method has also been applied to the dynamic of circular and annular plates by Lakis and Selmane [146-148].

The effect of surrounding medium (air, liquid and etc.) upon the vibration of plates and shells is of primary interest to scientists and engineers working in aerospace, marine and reactor technology. The effect of the fluid on the structural response is usually significant except in the case of extremely thick shells. The dynamic response of the shells when subjected to a flowing fluid, as well as the influence of fluid speed on the shell free vibrations, were studied by several researcher: Lakis and Païdoussis [133-135], Païdoussis and Denis[149], Weaver and Unny [150], Cheng [151] and Jain [152]. Païdoussis and Li made an elaborate review in this field [153].

The fluid effect on the dynamic behavior of the structure can be taken into account by considering the hydrodynamic mass which is added to the mass matrix of the structure. The effective mass is a function of the mode shape being considered, the shell and liquid geometrical parameters, plus the physical parameters. In addition, the forces exerted by free surface motion have to be considered; but the pressure distribution due to surface motion during vibration could be neglected, however, since resonant sloshing frequencies of thin shells are considerably below the natural frequencies of the combined fluid-structure system.

The dynamics of coupled fluid-shells were reviewed extensively by Yang [154] and Brown [155]. Dynamic analysis of the structure-fluid systems was studied by Brenneman and Yang [156], using the modal and hybrid methods. They obtained the structure and fluid modes by applying the stiffness and flexibility methods, following MacNeal's approach. Crouzet-Pascal and Garnet [157] studied a ring-reinforced cylindrical shell immersed in a fluid medium, and its dynamic response to an axisymmetric step pulse. MacNeal [158] presented another approach which is based on a hybrid finite element formulation in which the structure is modeled with displacements as the unknown variables, and a fluid is modeled with pressure as the variables. To utilize existing mainframe structural analysis programs, MacNeal showed how to recover symmetry by manipulating the equations and adding auxiliary variables to the problem.

The free vibration of simply supported vertical cylindrical shells partially filled with or submerged in a fluid has been analyzed by Gonçalves and Batista [159]. The Rayleigh-Ritz method is used to obtain an approximate solution which coincides with the exact solution for the cases of an empty shell or a shell completely in contact with fluid. Their work is based upon the consistent shell theory of Sanders. The fluid is taken as non-viscous and compressible and the coupling between the deformable shell and this acoustic medium is taken into account.

Since the lowest natural frequency of bending vibration of shells, immersed in or filled with a fluid, is much less than the corresponding natural frequency of the shell in air, they investigated the effects of variable height of fluid on the vibration response of vertical cylinders filled with or submerged in an acoustic fluid medium. In general, the lowest frequency is dependent on liquid level, mode shapes and shell and liquid geometrical and physical parameters.

The free vibration analysis of cylindrical storage tanks with axial thickness variation and partially filled with liquid was studied by Han and Liu [160]. The tank is modeled using Flügge's thin shell theory (in the isotropic case)and the fluid in the tank, according to potential flow theory, is assumed to be inviscid and incompressible. In their work, the shear deformation effects have not been considered. They solved the partial differential equations by using the transfer matrix technique.

An analysis of the non-linear vibration of cylindrical shells of varying thickness in an incompressible fluid was made by Ramachandran [109]. The Rayleigh-Ritz procedure was used to analyze of non-linear transverse vibrations of elastic, orthotropic cylindrical shells of linearly varying thickness, embedded in an incompressible fluid (there is no shear deformation effect in his work).

3.THEORETICAL DEVELOPMENT

This work is based on the following assumptions:

1) Linear elastic behavior of laminated anisotropic materials;
2) Use the strain-displacement relations expressed in arbitrary orthogonal curvilinear coordinate system;
3) The shell is thin and therefore we assume that the normal stress is negligible compared with stress tangential to the shell surface;
4) The transverse shear deformation, rotary inertia and initial curvature are considered to influence the governing equations.

3.1 ***Strain-Displacement Relations-*** The normal and shear strain components are related to the components of the displacement vector by [3]:

$$\varepsilon_i=\frac{\partial}{\partial\alpha_i}\left(\frac{U_i}{\sqrt{g_i}}\right)+\frac{1}{2g_i}\sum_{k=1}^{3}\frac{\partial g_i}{\partial\alpha_k}\frac{U_k}{\sqrt{g_k}} \qquad i=1,2,3$$

$$\gamma_{ij}=\frac{1}{\sqrt{g_i g_j}}\left[g_i\frac{\partial}{\partial\alpha_j}\left(\frac{U_i}{\sqrt{g_i}}\right)+g_j\frac{\partial}{\partial\alpha_i}\left(\frac{U_j}{\sqrt{g_i}}\right)\right] \qquad i=1,2,3;\ i\neq j \tag{1}$$

where α_i ; u_i and g_i are, respectively, the curvilinear coordinates of the surface, components of the displacement vector and geometrical scale factor quantities, and are defined below for application to shells (Figure1):

$$\alpha_1=\alpha_1 \quad \alpha_2=\alpha_2 \quad \alpha_3=\zeta$$
$$u_1=U_1 \quad u_2=U_2 \quad u_3=W \tag{2}$$
$$g_1=A_1^{\,2}(1+\zeta/R_1)^2 \quad g_2=A_2^{\,2}(1+\zeta/R_2)^2 \quad g_3=1$$

where U_1, U_2, W, A_i, R_i and ζ are, respectively, the displacement vector components, Lamé's parameters, the curvature radius and the thickness coordinate. If we substitute equations (2) into equations (1), we obtain the following strain displacements equations in the shell space:

$$\varepsilon_1=\frac{1}{A_1(1+\zeta/R_1)}\left(\frac{\partial U_1}{\partial\alpha_1}+\frac{U_2}{A_2}\frac{\partial A_1}{\partial\alpha_2}+\frac{A_1W}{R_1}\right)$$

$$\varepsilon_2=\frac{1}{A_2(1+\zeta/R_2)}\left(\frac{\partial U_2}{\partial\alpha_2}+\frac{U_1}{A_1}\frac{\partial A_2}{\partial\alpha_1}+\frac{A_2W}{R_2}\right) \tag{3}$$

$$\varepsilon_n=\frac{\partial W}{\partial\zeta}$$

$$\gamma_{1n}=\frac{1}{A_1(1+\zeta/R_1)}\frac{\partial W}{\partial\alpha_1}+A_1(1+\zeta/R_1)\frac{\partial}{\partial\zeta}[\frac{U_1}{A_1(1+\zeta/R_1)}]$$

$$\gamma_{2n}=\frac{1}{A_2(1+\zeta/R_2)}\frac{\partial W}{\partial\alpha_2}+A_2(1+\zeta/R_2)\frac{\partial}{\partial\zeta}[\frac{U_2}{A_2(1+\zeta/R_2)}] \qquad \text{(3-Cont.)}$$

$$\gamma_{12}=\frac{A_2(1+\zeta/R_2)}{A_1(1+\zeta/R_1)}\frac{\partial}{\partial\alpha_1}[\frac{U_2}{A_2(1+\zeta/R_2)}]+\frac{A_1(1+\zeta/R_1)}{A_2(1+\zeta/R_2)}\frac{\partial}{\partial\alpha_2}[\frac{U_1}{A_1(1+\zeta/R_1)}]$$

Where ε_i and γ_i are, respectively, the normal and shearing strain components. We can assume that the displacement's components are presented by the following relationships:

$$\begin{aligned} U_1(\alpha_1,\alpha_2,\zeta)&=u_1(\alpha_1,\alpha_2)+\zeta\beta_1(\alpha_1,\alpha_2)\\ U_2(\alpha_1,\alpha_2,\zeta)&=u_2(\alpha_1,\alpha_2)+\zeta\beta_2(\alpha_1,\alpha_2)\\ W(\alpha_1,\alpha_2,\zeta)&=w(\alpha_1,\alpha_2)\end{aligned} \qquad (4)$$

The β_1 and β_2 present the rotation of tangents to the reference surface oriented along the parametric lines α_1 and α_2 respectively (Figure 1). We substitute equations (4) into equations (3):

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\\ \gamma_{1n}\\ \gamma_{2n}\end{Bmatrix}=\begin{bmatrix}\frac{1}{(1+\zeta/R_1)} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{(1+\zeta/R_2)} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{(1+\zeta/R_1)} & \frac{1}{(1+\zeta/R_2)} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{(1+\zeta/R_1)} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{(1+\zeta/R_2)}\end{bmatrix}\left[\begin{Bmatrix}\varepsilon^o{}_1\\ \varepsilon^o{}_2\\ \gamma^o{}_1\\ \gamma^o{}_2\\ \mu^o{}_1\\ \mu^o{}_2\end{Bmatrix}+\zeta\begin{Bmatrix}\kappa_1\\ \kappa_2\\ \tau_1\\ \tau_2\\ 0\\ 0\end{Bmatrix}\right] \qquad (5)$$

Where:

$$\varepsilon^o{}_1=\frac{1}{A_1}\frac{\partial U_1}{\partial \alpha_1}+\frac{U_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}+\frac{W}{R_1} \quad ; \quad \kappa_1=\frac{1}{A_1}\frac{\partial \beta_1}{\partial \alpha_1}+\frac{\beta_2}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}$$

$$\varepsilon^o{}_2=\frac{1}{A_2}\frac{\partial U_2}{\partial \alpha_2}+\frac{U_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}+\frac{W}{R_2} \quad ; \quad \kappa_2=\frac{1}{A_2}\frac{\partial \beta_2}{\partial \alpha_2}+\frac{\beta_1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}$$

$$\gamma^o{}_1=\frac{1}{A_1}\frac{\partial U_2}{\partial \alpha_1}-\frac{U_1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2} \quad ; \quad \tau_1=\frac{1}{A_1}\frac{\partial \beta_2}{\partial \alpha_1}-\frac{\beta_1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}$$

$$\gamma^o{}_2=\frac{1}{A_2}\frac{\partial U_1}{\partial \alpha_2}-\frac{U_2}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} \quad ; \quad \tau_2=\frac{1}{A_2}\frac{\partial \beta_1}{\partial \alpha_2}-\frac{\beta_2}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1} \qquad (6)$$

$$\mu^o{}_1=\frac{1}{A_1}\frac{\partial W}{\partial \alpha_1}-\frac{U_1}{R_1}+\beta_1$$

$$\mu^o{}_2=\frac{1}{A_2}\frac{\partial W}{\partial \alpha_2}-\frac{U_2}{R_2}+\beta_2$$

Where $\varepsilon^o{}_i$; $\gamma^o{}_i$; κ_i; τ_i and $\mu^o{}_i$ are, respectively, the normal and in-plane shearing strain, the change in the curvature and torsion of the reference surface and the shearing strain components. The Coddazi conditions which were used for the above equations are:

$$\frac{\partial}{\partial \alpha_2}\left[A_1\left(1+\zeta/R_1\right)\right]=\frac{\partial A_1}{\partial A_2}\left(1+\zeta/R_2\right) \qquad (7)$$

$$\frac{\partial}{\partial \alpha_1}\left[A_2\left(1+\zeta/R_2\right)\right]=\frac{\partial A_2}{\partial \alpha_1}\left(1+\zeta/R_1\right) \qquad (8)$$

Where R_i, ζ, A_i and α_i were earlier defined by equation (1,2).

3.2 <u>***The relationship between the stress and strain vectors (Hook's law)-***</u>The relationship between the stress and strain vectors (Hook's law):

$$\{\sigma\} = [Q]\{\varepsilon\} \tag{9}$$

The constitutive equation of the *Kth* lamina (for a lamina of fibre reinforced composite material) in the lamina reference axes (α , β ,γ) can be written as follows (for only one lamina) (Figure 2):

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \\ \tau_{\beta\gamma} \\ \tau_{\alpha\gamma} \\ \tau_{\alpha\beta} \end{Bmatrix} = \begin{bmatrix} Q_{\alpha\alpha} & Q_{\alpha\beta} & Q_{\alpha\gamma} & 0 & 0 & 0 \\ Q_{\beta\alpha} & Q_{\beta\beta} & Q_{\beta\gamma} & 0 & 0 & 0 \\ Q_{\gamma\alpha} & Q_{\gamma\beta} & Q_{\gamma\gamma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2Q_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2Q_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \\ \gamma_{\beta\gamma} \\ \gamma_{\alpha\gamma} \\ \gamma_{\alpha\beta} \end{Bmatrix} \tag{10}$$

$$\tau_{\beta\gamma}=\frac{1}{2}G_{\beta\gamma}\gamma_{\beta\gamma} \text{ , } \tau_{\alpha\gamma}=\frac{1}{2}G_{\alpha\gamma}\gamma_{\alpha\gamma} \text{ , } \tau_{\alpha\beta}=\frac{1}{2}G_{\alpha\beta}\gamma_{\alpha\beta} \tag{11}$$

Note : The [Q] matrix denotes the elastic stiffness in the material coordinates (local axes).

Q_{ij}'s elements are defined as follows:

$$\begin{gathered} Q_{\alpha\alpha}=E_{\alpha\alpha}(1-\nu_{\beta\gamma}\nu_{\gamma\beta})/\Delta \quad ; \quad Q_{44}=G_{\beta\gamma} \\ Q_{\beta\beta}=E_{\beta\beta}(1-\nu_{\gamma\alpha}\nu_{\alpha\gamma})/\Delta \quad ; \quad Q_{55}=G_{\alpha\gamma} \\ Q_{\gamma\gamma}=E_{\alpha\alpha}(1-\nu_{\alpha\beta}\nu_{\beta\alpha})/\Delta \quad ; \quad Q_{66}=G_{\alpha\beta} \\ Q_{\alpha\beta}=(\nu_{\beta\alpha}+\nu_{\gamma\alpha}\nu_{\beta\gamma})E_{\alpha\alpha}/\Delta=(\nu_{\alpha\beta}+\nu_{\gamma\beta}\nu_{\alpha\gamma})E_{\beta\beta}/\Delta \\ Q_{\alpha\gamma}=(\nu_{\gamma\alpha}+\nu_{\beta\alpha}\nu_{\gamma\beta})E_{\alpha\alpha}/\Delta=(\nu_{\alpha\gamma}+\nu_{\alpha\beta}\nu_{\beta\gamma})E_{\gamma\gamma}/\Delta \\ Q_{\beta\gamma}=(\nu_{\gamma\beta}+\nu_{\alpha\beta}\nu_{\gamma\alpha})E_{\beta\beta}/\Delta=(\nu_{\beta\gamma}+\nu_{\beta\alpha}\nu_{\alpha\gamma})E_{\gamma\gamma}/\Delta \\ \Delta=1-\nu_{\alpha\beta}\nu_{\beta\alpha}-\nu_{\beta\gamma}\nu_{\gamma\beta}-\nu_{\gamma\alpha}\nu_{\alpha\gamma}-2\nu_{\beta\alpha}\nu_{\gamma\beta}\nu_{\alpha\gamma} \end{gathered} \tag{12}$$

Where $E_{\alpha\beta}$, $G_{\alpha\beta}$ and $\nu_{\alpha\beta}$ are, respectively, Young's moduli of elasticity in the principal directions, rigidity moduli characterizing the change of angles between the principal directions, and the Poisson ratios characterizing the transverse contraction (expansion) under tension (compression) in the directions of the coordinate axes.

The stress-strain relations of the *Kth* lamina in the laminate coordinate axes (1,2,3 global coordinates) can be written as (Figure 3):

$$\{\bar{\sigma}\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_n \\ \tau_{2n} \\ \tau_{1n} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \overline{Q_{11}} & \overline{Q_{12}} & \overline{Q_{13}} & 0 & 0 & 2\overline{Q_{16}} \\ \overline{Q_{21}} & \overline{Q_{22}} & \overline{Q_{23}} & 0 & 0 & 2\overline{Q_{26}} \\ \overline{Q_{31}} & \overline{Q_{32}} & \overline{Q_{33}} & 0 & 0 & 2\overline{Q_{36}} \\ 0 & 0 & 0 & 2\overline{Q_{44}} & 2\overline{Q_{45}} & 0 \\ 0 & 0 & 0 & 2\overline{Q_{54}} & 2\overline{Q_{55}} & 0 \\ \overline{Q_{16}} & \overline{Q_{26}} & \overline{Q_{36}} & 0 & 0 & 2\overline{Q_{66}} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_n \\ \gamma_{2n} \\ \gamma_{1n} \\ \gamma_{12} \end{Bmatrix} \qquad (13)$$

Where :

$$[\overline{Q}] = [T]^{-1}[Q][T] \qquad (14)$$

The transformation matrix [T] is defined by:

$$[T] = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & (m^2-n^2) \end{bmatrix} \qquad (15)$$

où: $m \equiv \cos\alpha$, $n \equiv \sin\alpha$

The orientation angle θ is measured counter-clockwise from the *1*-axis to the α-axis.

$[\overline{Q}]$'s elements are defined as follows:

$$
\begin{aligned}
\overline{Q_{11}}&=Q_{\alpha\alpha}m^{4}+2(Q_{\alpha\beta}+2Q_{66})m^{2}n^{2}+Q_{\beta\beta}n^{4}\\
\overline{Q_{12}}&=(Q_{\alpha\alpha}+Q_{\beta\beta}-4Q_{66})m^{2}n^{2}+Q_{\alpha\beta}(m^{4}+n^{4})\\
\overline{Q_{13}}&=Q_{\alpha\gamma}m^{2}+Q_{\beta\gamma}n^{2}\\
\overline{Q_{16}}&=-mn^{3}Q_{\beta\beta}+m^{3}nQ_{\alpha\alpha}-mn(m^{2}-n^{2})(Q_{\alpha\beta}+2Q_{66})\\
\overline{Q_{22}}&=Q_{\alpha\alpha}n^{4}+2(Q_{\alpha\beta}+2Q_{66})m^{2}n^{2}+Q_{\beta\beta}m^{4}\\
\overline{Q_{23}}&=Q_{\alpha\gamma}n^{2}+Q_{\beta\gamma}m^{2}\\
\overline{Q_{26}}&=-m^{3}nQ_{\beta\beta}+mn^{3}Q_{\alpha\alpha}+mn(m^{2}-n^{2})(Q_{\alpha\beta}+2Q_{66})\\
\overline{Q_{33}}&=Q_{\gamma\gamma}\\
\overline{Q_{36}}&=(Q_{\alpha\gamma}-Q_{\beta\gamma})mn\\
\overline{Q_{44}}&=Q_{44}m^{2}+Q_{55}n^{2}\\
\overline{Q_{45}}&=(Q_{55}-Q_{44})mn\\
\overline{Q_{55}}&=Q_{55}m^{2}+Q_{44}n^{2}\\
\overline{Q_{66}}&=(Q_{\alpha\alpha}+Q_{\beta\beta}-2Q_{\alpha\beta})m^{2}n^{2}+Q_{66}(m^{2}-n^{2})^{2}
\end{aligned}
\tag{16}
$$

3.3 ***<u>The equations of motion:-</u>***Using the virtual work principle for the present case yields:

$$
\begin{aligned}
&\frac{\partial A_2N_1}{\partial\alpha_1}+\frac{\partial A_1N_{21}}{\partial\alpha_2}+N_{12}\frac{\partial A_1}{\partial\alpha_2}-N_2\frac{\partial A_2}{\partial\alpha_1}+\frac{Q_1A_1A_2}{R_1}+A_1A_2q_1=I_1\ddot{u}_1+I_2\ddot{\beta}_1\\
&\frac{\partial A_1N_2}{\partial\alpha_2}+\frac{\partial A_2N_{12}}{\partial\alpha_1}+N_{21}\frac{\partial A_2}{\partial\alpha_1}-N_1\frac{\partial A_1}{\partial\alpha_2}+\frac{Q_2A_1A_2}{R_2}+A_1A_2q_2=I_1\ddot{u}_2+I_2\ddot{\beta}_2\\
&\frac{\partial A_2Q_1}{\partial\alpha_1}+\frac{\partial A_1Q_2}{\partial\alpha_2}-A_1A_2\left(\frac{N_1}{R_1}+\frac{N_2}{R_2}\right)-A_1A_2q_n=I_1\ddot{w}\\
&\frac{\partial A_2M_1}{\partial\alpha_1}+\frac{\partial A_1M_{21}}{\partial\alpha_2}+M_{12}\frac{\partial A_1}{\partial\alpha_2}-M_2\frac{\partial A_2}{\partial\alpha_1}-Q_1A_1A_2=I_2\ddot{u}_1+I_3\ddot{\beta}_1\\
&\frac{\partial A_1M_2}{\partial\alpha_2}+\frac{\partial A_2M_{12}}{\partial\alpha_1}+M_{21}\frac{\partial A_2}{\partial\alpha_1}-M_1\frac{\partial A_1}{\partial\alpha_2}-Q_2A_1A_2=I_2\ddot{u}_2+I_3\ddot{\beta}_2
\end{aligned}
\tag{17}
$$

Where :

$$I_1,I_2,I_3=\sum_{k=1}^{N}\int_{h_k}^{h_{k-1}}\rho^{(k)}(1,\zeta,\zeta^2)d\zeta \tag{18}$$

Where I_i , $\rho^{(k)}$ and ζ are, respectively, inertia moments, density of the K^{th}'s lamina material and the thickness coordinate.

Now, we see that there are five independent boundary conditions to be applied at given edges. The transverse shear deformations do not vanish in the present theory and, therefore, the β_i cannot be expressed in terms of U_i and W. The transverse shear theory recommended here leads to no strains during rigid body motion.

3.4 <u>The stress resultants and stress couples-</u> The stress resultants and stress couples are given by [3] :

$$\begin{Bmatrix} N_1 \\ N_{12} \\ Q_1 \\ M_1 \\ M_{12} \end{Bmatrix} = \int_\zeta \begin{Bmatrix} \sigma_1 \\ \tau_{12} \\ \tau_{1n} \\ \sigma_1 \\ \tau_{12} \end{Bmatrix} (1+\zeta/R_2)\,d\zeta \quad ; \quad \begin{Bmatrix} N_2 \\ N_{21} \\ Q_2 \\ M_2 \\ M_{21} \end{Bmatrix} = \int_\zeta \begin{Bmatrix} \sigma_2 \\ \tau_{21} \\ \tau_{2n} \\ \sigma_2 \\ \tau_{21} \end{Bmatrix} (1+\zeta/R_1)\,d\zeta \tag{19}$$

The quantities (N_{11}, N_{22}, N_{12}, N_{21}) are called the *in-plane force resultants*, and (M_{11}, M_{22}, M_{12}, M_{21}) are called the *moment resultants*; (Q_{11}, Q_{22}) denote the *transverse force resultants*. We notice, in equations (11), that the symmetry of the stress tensor ($\tau_{12}=\tau_{21}$) does not necessarily imply that N_{12} and N_{21} are equal or that M_{12} and M_{21} are equal except in the case of a spherical shell, a flat plate or a thin shell of any shape.

3.5 The Constitutive Equations-The stress resultants and stress couples that correspond to the remaining stress are given by equations (19); therefore, by using equations (5), (13) and (19) we have:

$$\begin{Bmatrix} N_1 \\ N_{12} \\ N_2 \\ N_{21} \end{Bmatrix} = \underset{(4\times4)}{\begin{bmatrix} G_{ij} & A_{ij} \\ A_{ij} & GG_{ij} \end{bmatrix}} \begin{Bmatrix} \varepsilon^o{}_1 \\ \gamma^o{}_1 \\ \varepsilon^o{}_2 \\ \gamma^o{}_2 \end{Bmatrix} + \underset{(4\times4)}{\begin{bmatrix} H_{ij} & B_{ij} \\ B_{ij} & HH_{ij} \end{bmatrix}} \begin{Bmatrix} \kappa_1 \\ \tau_1 \\ \kappa_2 \\ \tau_2 \end{Bmatrix} \qquad i,j=1,6,2,6$$

$$\begin{Bmatrix} M_1 \\ M_{12} \\ M_{22} \\ M_{21} \end{Bmatrix} = \underset{(4\times4)}{\begin{bmatrix} I_{ij} & B_{ij} \\ B_{ij} & II_{ij} \end{bmatrix}} \begin{Bmatrix} \varepsilon^o{}_1 \\ \gamma^o{}_1 \\ \varepsilon^o{}_2 \\ \gamma^o{}_2 \end{Bmatrix} + \underset{(4\times4)}{\begin{bmatrix} J_{ij} & D_{ij} \\ D_{ij} & JJ_{ij} \end{bmatrix}} \begin{Bmatrix} \kappa_1 \\ \tau_1 \\ \kappa_2 \\ \tau_2 \end{Bmatrix} \qquad i,j=1,6,2,6 \tag{20}$$

Where

$$\begin{aligned} G_{ij}&=A_{ij}+a_1B_{ij}+a_2D_{ij}+a_3E_{ij} \quad ; \quad H_{ij}=B_{ij}+a_1D_{ij}+a_2E_{ij}+a_3F_{ij} \\ GG_{ij}&=A_{ij}+b_1B_{ij}+b_2D_{ij}+b_3E_{ij} \quad ; \quad HH_{ij}=B_{ij}+b_1D_{ij}+b_2E_{ij}+b_3F_{ij} \\ I_{ij}&=B_{ij}+a_1D_{ij}+a_2E_{ij}+a_3F_{ij} \quad ; \quad J_{ij}=D_{ij}+a_1E_{ij}+a_2F_{ij}+a_3C_{ij} \\ II_{ij}&=B_{ij}+b_1D_{ij}+b_2E_{ij}+b_3F_{ij} \quad ; \quad JJ_{ij}=D_{ij}+b_1E_{ij}+b_2F_{ij}+b_3C_{ij} \end{aligned} \tag{21}$$

and

$$a_1=\frac{1}{R_2}-\frac{1}{R_1} \quad ; \quad b_1=\frac{1}{R_1}-\frac{1}{R_2}$$
$$a_2=\frac{1}{R_1}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \quad ; \quad b_2=\frac{1}{R_2}\left(\frac{1}{R_2}-\frac{1}{R_1}\right) \tag{22}$$
$$a_3=\frac{1}{R^2{}_1R_2} \quad ; \quad b_3=\frac{1}{R_2^2R_1}$$

$$A_{ij}=\sum_{k=1}^{N}(\overline{Q_{ij}})_k(h_k-h_{k-1})$$
$$B_{ij}=\frac{1}{2}\sum_{k=1}^{N}(\overline{Q_{ij}})_k(h^2{}_k-h^2{}_{k-1})$$
$$D_{ij}=\frac{1}{3}\sum_{k=1}^{N}(\overline{Q_{ij}})_k(h^3{}_k-h^3{}_{k-1})$$
$$E_{ij}=\frac{1}{4}\sum_{k=1}^{N}(\overline{Q_{ij}})_k(h^4{}_k-h^4{}_{k-1}) \qquad i,j=1,6,2,6 \tag{23}$$
$$F_{ij}=\frac{1}{5}\sum_{k=1}^{N}(\overline{Q_{ij}})_k(h^5{}_k-h^5{}_{k-1})$$
$$C_{ij}=\frac{1}{6}\sum_{k=1}^{N}(\overline{Q_{ij}})_k(h^6{}_k-h^6{}_{k-1})$$

N: *Number of lamina*

Note: $N_{12} \neq N_{21}$ and $M_{12} \neq M_{21}$ and

$$\frac{1}{(1+\zeta/R)} \cong 1-\zeta/R+(\zeta/R)^2-+\ldots \tag{24}$$

This expansion requires only that $(\zeta / R)^2 < 1$. So:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1+\frac{\zeta}{R_2}}{1+\frac{\zeta}{R_1}}\, d\zeta = h\left[1+\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{h^2}{12R_1}\right]$$
$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1+\frac{\zeta}{R_2}}{1+\frac{\zeta}{R_1}}\, \zeta\, d\zeta = -\frac{h^3}{12}\left(\frac{1}{R_1}-\frac{1}{R_2}\right) \tag{25}$$

We also have:

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} \int \tau_{1n}(1+\zeta/R_2)d\zeta \\ \int \tau_{2n}(1+\zeta/R_2)d\zeta \end{Bmatrix} = \begin{bmatrix} AA_{55} & A_{45} \\ A_{45} & BB_{44} \end{bmatrix} \begin{Bmatrix} \mu^o{}_1 \\ \mu^o{}_2 \end{Bmatrix} \tag{26}$$

Where:

$$\begin{aligned} &AA_{55}=A_{55}+a_1B_{55}+a_2D_{55}+a_3E_{55} \quad ; \quad BB_{44}=A_{44}+b_1B_{44}+b_2D_{44}+b_3E_{44} \\ &A_{\alpha\beta}=\sum_{k=1}^{N}(\overline{Q_{\alpha\beta}})_k(h_k-h_{k-1}) \quad ; \quad B_{\alpha\beta}=\frac{1}{2}\sum_{k=1}^{N}(\overline{Q_{\alpha\beta}})_k(h^2{}_k-h^2{}_{k-1}) \qquad \alpha,\beta=4,5 \\ &D_{\alpha\beta}=\frac{1}{3}\sum_{k=1}^{N}(\overline{Q_{\alpha\beta}})_k(h^3{}_k-h^3{}_{k-1}) \quad ; \quad E_{\alpha\beta}=\frac{1}{4}\sum_{k=1}^{N}(\overline{Q_{\alpha\beta}})_k(h^4{}_k-h^4{}_{k-1}) \end{aligned} \tag{27}$$

finally:

$$\{ N_{11}N_{12}Q_{11}N_{22}N_{21}Q_{22}M_{11}M_{12}M_{22}M_{21} \}^T = [P]_{(10\times10)} \{ \varepsilon^o{}_1\gamma^o{}_1\mu^o{}_1\varepsilon^o{}_2\gamma^o{}_2\mu^o{}_2\kappa_1\tau_1\kappa_2\tau_2 \}^T \tag{28}$$

The ε_1^o ; γ_1^o ; ... and τ_2 were given earlier in equations (5). Matrix [P] is given as follows:

$$\begin{Bmatrix} N_{11} \\ N_{12} \\ Q_{11} \\ N_{22} \\ N_{21} \\ Q_{22} \\ M_{11} \\ M_{12} \\ M_{22} \\ M_{21} \end{Bmatrix} = \begin{bmatrix} P_{11} & P_{12} & 0 & P_{14} & P_{15} & 0 & P_{17} & P_{18} & P_{19} & P_{1,10} \\ P_{21} & P_{22} & 0 & P_{24} & P_{25} & 0 & P_{27} & P_{28} & P_{29} & P_{2,10} \\ 0 & 0 & P_{33} & 0 & 0 & P_{36} & 0 & 0 & 0 & 0 \\ P_{41} & P_{42} & 0 & P_{44} & P_{45} & 0 & P_{47} & P_{48} & P_{49} & P_{4,10} \\ P_{51} & P_{52} & 0 & P_{54} & P_{55} & 0 & P_{57} & P_{58} & P_{59} & P_{5,10} \\ 0 & 0 & P_{63} & 0 & 0 & P_{66} & 0 & 0 & 0 & 0 \\ P_{71} & P_{72} & 0 & P_{74} & P_{75} & 0 & P_{77} & P_{78} & P_{79} & P_{7,10} \\ P_{81} & P_{82} & 0 & P_{84} & P_{85} & 0 & P_{87} & P_{88} & P_{89} & P_{8,10} \\ P_{91} & P_{92} & 0 & P_{94} & P_{95} & 0 & P_{97} & P_{98} & P_{99} & P_{9,10} \\ P_{10,1} & P_{10,2} & 0 & P_{10,4} & P_{10,5} & 0 & P_{10,7} & P_{10,8} & P_{10,9} & P_{10,10} \end{bmatrix}_{(10\times10)} \begin{Bmatrix} \varepsilon^o{}_1 \\ \gamma^o{}_1 \\ \mu^o{}_1 \\ \varepsilon^o{}_2 \\ \gamma^o{}_2 \\ \mu^o{}_2 \\ \kappa_1 \\ \tau_1 \\ \kappa_2 \\ \tau_2 \end{Bmatrix}_{(10\times1)} \tag{29}$$

Where P_{ij}'s elements are given in Appendix A and defined by equations (21-23) and (27).

Now, we develop **1)Equilibrium Equations, 2)Constitutive Equations,3)Kinematic Relations (Strain-Displacement Relations)** for the following cases: *a) Shells of Revolution; b) Cylindrical Shells; c) Rectangular Plates; d) Spherical Shells; e) Conical Shells* and *f) Circular Plates.*

4. ***Shells of Revolution***

4.1 The equilibrium equations- We substitute the geometry definitions of shells of revolution (Figure 4) into equations (17).

$$\frac{1}{R_\varphi R_\theta \sin\varphi}[R_\theta\cos\varphi N_\varphi + R_\theta\sin\varphi N_{\varphi,\varphi} + R_\varphi N_{\theta\varphi,\theta} - R_\theta\cos\varphi N_\theta] + \frac{Q_\varphi}{R_\varphi} + q_\varphi = I_1\ddot{u}_\varphi + I_2\ddot{\beta}_\varphi$$

$$\frac{1}{R_\varphi R_\theta \sin\varphi}[R_\varphi N_{\theta,\theta} + R_\theta\cos\varphi N_{\theta\varphi} + R_\theta\sin\varphi N_{\varphi\theta,\varphi} + R_\theta\cos\varphi N_{\varphi\theta}] + \frac{Q_\theta}{R_\theta} + q_\theta = I_1\ddot{u}_\theta + I_2\ddot{\beta}_\theta$$

$$\frac{1}{R_\varphi R_\theta \sin\varphi}[R_\theta\cos\varphi Q_\varphi + R_\theta\sin\varphi Q_{\varphi,\varphi} + R_\varphi Q_{\theta,\theta}] - \frac{N_\varphi}{R_\varphi} - \frac{N_\theta}{R_\theta} + q_n = I_1\ddot{w} \tag{30}$$

$$\frac{1}{R_\varphi R_\theta \sin\varphi}[R_\theta\sin\varphi M_{\varphi,\varphi} + R_\theta\cos\varphi M_\varphi + R_\varphi M_{\theta\varphi,\theta} - R_\theta\cos\varphi M_\theta] - Q_\varphi = I_2\ddot{u}_\varphi + I_3\ddot{\beta}_\varphi$$

$$\frac{1}{R_\varphi R_\theta \sin\varphi}[R_\varphi M_{\theta,\theta} + R_\theta\sin\varphi M_{\varphi\theta,\varphi} + R_\theta M_{\varphi\theta} + R_\theta\cos\varphi M_{\theta\varphi}] - Q_\theta = I_2\ddot{u}_\theta + I_3\ddot{\beta}_\theta$$

4.2 Constitutive equations- We have the same equations as those of (29), but the definitions given in equations (22) must be changed.

$$a_1 = \frac{1}{R_\theta} - \frac{1}{R_\varphi} \ ; \ a_2 = \frac{1}{R_\varphi}\left(\frac{1}{R_\varphi} - \frac{1}{R_\theta}\right) \ ; \ a_3 = \frac{1}{R^2_\varphi R_\theta}$$

$$b_1 = \frac{1}{R_\varphi} - \frac{1}{R_\theta} \ ; \ b_2 = \frac{1}{R_\theta}\left(\frac{1}{R_\theta} - \frac{1}{R_\varphi}\right) \ ; \ b_3 = \frac{1}{R_\varphi R^2_\theta} \tag{31}$$

The constitutive equation is given in Appendix A.

4.3 ***Kinematic relations (Linear Strain-Displacement Relations)-*** By using equations (30), equations (5) can be defined as shown below:

$$\begin{Bmatrix}\varepsilon_\varphi\\ \varepsilon_\theta\\ \gamma_{\varphi\theta}\\ \gamma_{\varphi n}\\ \gamma_{\theta n}\end{Bmatrix}=\begin{bmatrix}\frac{1}{(1+\zeta/R_1)} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{(1+\zeta/R_2)} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{(1+\zeta/R_1)} & \frac{1}{(1+\zeta/R_2)} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{(1+\zeta/R_1)} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{(1+\zeta/R_2)}\end{bmatrix}\left[\begin{Bmatrix}\varepsilon^o_\varphi\\ \varepsilon^o_\theta\\ \gamma^o_\varphi\\ \gamma^o_\theta\\ \mu^o_\varphi\\ \mu^o_\theta\end{Bmatrix}+\zeta\begin{Bmatrix}\kappa_\varphi\\ \kappa_\theta\\ \tau_\varphi\\ \tau_\theta\\ 0\\ 0\end{Bmatrix}\right]$$

where

$$\varepsilon^o_\varphi=\frac{1}{R_\varphi}\left(W+\frac{\partial U_\varphi}{\partial\varphi}\right) \quad ; \quad \kappa_\varphi=\frac{1}{R_\varphi}\frac{\partial\beta_\varphi}{\partial\varphi} \tag{32}$$

$$\varepsilon^o_\theta=\frac{1}{R_\theta\sin\varphi}\frac{\partial U_\theta}{\partial\theta}+\frac{1}{R_\varphi}cotg\varphi U_\varphi+\frac{W}{R_\theta} \quad ; \quad \kappa_\theta=\frac{1}{R_\theta\sin\varphi}\frac{\partial\beta_\theta}{\partial\theta}+\frac{\beta_\varphi}{R_\varphi}cotg\varphi$$

$$\gamma^o_\varphi=\frac{1}{R_\varphi}\frac{\partial U_\theta}{\partial\varphi} \quad ; \quad \tau_\varphi=\frac{1}{R_\varphi}\frac{\partial\beta_\theta}{\partial\varphi}$$

$$\gamma^o_\theta=\frac{1}{R_\theta\sin\varphi}\frac{\partial U_\varphi}{\partial\theta}-\frac{U_\theta}{R_\varphi}cotg\varphi \quad ; \quad \tau_\theta=\frac{1}{R_\theta\sin\varphi}\frac{\partial\beta_\varphi}{\partial\theta}-\frac{\beta_\theta}{R_\varphi}cotg\varphi$$

$$\mu^o_\varphi=\frac{1}{R_\varphi}\frac{\partial W}{\partial\varphi}-\frac{U_\varphi}{R_\varphi}+\beta_\varphi$$

$$\mu^o_\theta=\frac{1}{R_\theta\sin\varphi}\frac{\partial W}{\partial\theta}-\frac{U_\theta}{R_\theta}+\beta_\theta$$

5. *Cylindrical Shells*

5.1 ***The Equilibrium Equations-*** Using the geometry definitions of circular cylindrical shells given in Figure (5), equations (30) will become:

$$\frac{\partial N_{xx}}{\partial x}+\frac{1}{R}\frac{\partial N_{\theta x}}{\partial \theta}+q_x=I_1\ddot{u}_x+I_2\ddot{\beta}_x$$
$$\frac{\partial N_{x\theta}}{\partial x}+\frac{1}{R}\frac{\partial N_{\theta\theta}}{\partial \theta}+\frac{Q_{\theta\theta}}{R}+q_\theta=I_1\ddot{u}_\theta+I_2\ddot{\beta}_\theta$$
$$\frac{\partial Q_{xx}}{\partial x}+\frac{1}{R}\frac{\partial Q_{\theta\theta}}{\partial \theta}-\frac{N_{\theta\theta}}{R}+q_n=I_1\ddot{w} \tag{33}$$
$$\frac{\partial M_{xx}}{\partial x}+\frac{1}{R}\frac{\partial M_{\theta x}}{\partial \theta}-Q_{xx}=I_2\ddot{u}_x+I_3\ddot{\beta}_x$$
$$\frac{1}{R}\frac{\partial M_{\theta\theta}}{\partial \theta}+\frac{\partial M_{x\theta}}{\partial x}-Q_{\theta\theta}=I_2\ddot{u}_\theta+I_3\ddot{\beta}_\theta$$

5.2 ***Constitutive equation-*** Equation (29) can be used by changing the definitions given in Figure (5). This equation is given in Appendix A.

$$a_1=\frac{1}{R}\ ;\ a_2=0.\ ;\ a_3=0.\ ;\ b_1=-\frac{1}{R}\ ;\ b_2=\frac{1}{R^2}\ ;\ b_3=0. \tag{34}$$

5.3 ***Kinematic Relations(Linear Strain Displacement Relations)-*** The kinematic relations are obtained by using equation (32) and shell geometry definitions.

$$\begin{aligned}
\varepsilon^o{}_x&=\frac{\partial U_x}{\partial x} &;\quad \kappa_x&=\frac{\partial \beta_x}{\partial x}\\
\varepsilon^o{}_\theta&=\frac{1}{R}\frac{\partial U_\theta}{\partial \theta}+\frac{W}{R} &;\quad \kappa_\theta&=\frac{1}{R}\frac{\partial \beta_\theta}{\partial \theta}\\
\gamma^o{}_x&=\frac{\partial U_\theta}{\partial x} &;\quad \tau_x&=\frac{\partial \beta_\theta}{\partial x}\\
\gamma^o{}_\theta&=\frac{1}{R}\frac{\partial U_x}{\partial \theta} &;\quad \tau_\theta&=\frac{1}{R}\frac{\partial \beta_x}{\partial \theta}\\
\mu^o{}_x&=\frac{\partial W}{\partial x}+\beta_x &;\quad \mu^o{}_\theta&=\frac{1}{R}\frac{\partial W}{\partial \theta}-\frac{U_\theta}{R}+\beta_\theta
\end{aligned} \tag{35}$$

Substituting the above equations into the constitutive equations (taking into account the coefficients which were given in equations (34)) and then into equations (33), we will obtain :

$$
\begin{aligned}
L_1&=(U,V,W,\beta_x,\beta_\theta,\overline{P}_{ij})=0.\\
L_2&=(U,V,W,\beta_x,\beta_\theta,\overline{P}_{ij})=0.\\
L_3&=(U,V,W,\beta_x,\beta_\theta,\overline{P}_{ij})=0.\\
L_4&=(U,V,W,\beta_x,\beta_\theta,\overline{P}_{ij})=0.\\
L_5&=(U,V,W,\beta_x,\beta_\theta,\overline{P}_{ij})=0.
\end{aligned}
\tag{36}
$$

These relations are defined fully by the equations given in Appendix A. In order to compare them with *classical shell theory,* the three equations of motion for cylindrical shells are also given in Appendix A [143].

6. *Rectangular Plates*

6.1 Equilibrium equations - The same cylindrical shell equations are used, taking into account the rectangular plate geometry definitions (Figure 6), so equations (33) become:

$$
\begin{gathered}
\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{yx}}{\partial y}+q_x=I_1\ddot{u}_x+I_2\ddot{\beta}_x\\
\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{yy}}{\partial y}+q_y=I_1\ddot{u}_y+I_2\ddot{\beta}_y\\
\frac{\partial Q_{xx}}{\partial x}+\frac{\partial Q_y}{\partial y}+q_n=I_1\ddot{w}\\
\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{yx}}{\partial y}-Q_{xx}=I_2\ddot{u}_x+I_3\ddot{\beta}_x\\
\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}-Q_{yy}=I_2\ddot{u}_y+I_3\ddot{\beta}_y
\end{gathered}
\tag{37}
$$

6.2 Constitutive equations- We have the same equations as those of (29), but the definitions (22) must be changed. This equation is defined in Appendix A.

$$
a_1=a_2=a_3=b_1=b_2=b_3=0. \tag{38}
$$

6.3 Kinematic Relations(Linear Strain-Displacement Relations)- These relations can be obtained by substituting the structural geometry definitions into the kinematic relations of cylindrical shells (35).

$$
\begin{aligned}
\varepsilon^o{}_x &= \frac{\partial U_x}{\partial x} \quad ; \quad \kappa_x = \frac{\partial \beta_x}{\partial x} \\
\varepsilon^o{}_y &= \frac{\partial U_y}{\partial y} \quad ; \quad \kappa_y = \frac{\partial \beta_y}{\partial y} \\
\gamma^o{}_x &= \frac{\partial U_y}{\partial x} \quad ; \quad \tau_x = \frac{\partial \beta_y}{\partial x} \\
\gamma^o{}_y &= \frac{\partial U_x}{\partial y} \quad ; \quad \tau_y = \frac{\partial \beta_x}{\partial y} \\
\mu^o{}_x &= \frac{\partial W}{\partial x} + \beta_x \quad ; \quad \mu^o{}_y = \frac{\partial W}{\partial y} + \beta_y
\end{aligned}
\tag{39}
$$

Now, we can substitute the constitutive equations into equations (37) in the same way that we obtained the five differential equations for the case of cylindrical shells, and can obtain the implicit equations as (36). These equations are given fully in Appendix B.

7. *Spherical Shells*:

*7.1 **Equilibrium equations*** -The equilibrium equations for the spherical shells can be derived by using the equations (30) and following definitions (Figure 7).

$$
\begin{aligned}
& r_\theta = r_\varphi = r \\
& \frac{\mathrm{cosec}\varphi}{r}[\cos\varphi N_\varphi + \sin\varphi N_{\varphi,\varphi} + N_{\theta\varphi,\theta} - \cos\varphi N_\theta] + \frac{Q_\varphi}{r} + q_\varphi = I_1\ddot{u}_\varphi + I_2\ddot{\beta}_\varphi \\
& \frac{1}{r}[\mathrm{cosec}\varphi N_{\theta,\theta} + cotg\varphi N_{\theta\varphi} + N_{\varphi\theta,\varphi} + cotg\varphi N_{\varphi\theta}] + \frac{Q_\theta}{r} + q_\theta = I_1\ddot{u}_\theta + I_2\ddot{\beta}_\theta \\
& \frac{1}{r}[cotg\varphi Q_\varphi + Q_{\varphi,\varphi} + \mathrm{cosec}\varphi Q_{\theta,\theta}] - \frac{N_\varphi}{r} - \frac{N_\theta}{r} + q_n = I_1\ddot{w} \\
& \frac{1}{r}[M_{\varphi,\varphi} + cotg\varphi M_\varphi + \mathrm{cosec} M_{\theta\varphi,\theta} - cotg\varphi M_\theta] - Q_\varphi = I_2\ddot{u}_\varphi + I_3\ddot{\beta}_\varphi \\
& \frac{1}{r}[\mathrm{cosec}\varphi M_{\theta,\theta} + M_{\varphi\theta,\varphi} + cotg\varphi M_{\varphi\theta} + cotg\varphi M_{\theta\varphi}] - Q_\theta = I_2\ddot{u}_\theta + I_3\ddot{\beta}_\theta
\end{aligned}
\tag{40}
$$

7.2 ***Constitutive Equation***-We have the same equations as in (29), but the definitions given in (22) must be changed. These relations are given fully in Appendix A.

$$a_1=a_2=b_1=b_2=0. \quad ; \quad a_3=b_3=\frac{1}{R^3} \tag{41}$$

7.3 ***Kinematic Equations***- Substituting $r_\theta=r_\varphi=r$ into the definitions of (32), equations (5) are defined as below:

$$\begin{aligned}
\varepsilon^o_\varphi&=\frac{1}{r}\left(\frac{\partial U_\varphi}{\partial\varphi}+W\right) &;\quad \kappa_\varphi&=\frac{1}{r}\frac{\partial\beta_\varphi}{\partial\varphi}\\
\varepsilon^o_\theta&=\frac{1}{r\sin\varphi}\frac{\partial U_\theta}{\partial\theta}+\frac{1}{r}cotg\varphi U_\varphi+\frac{W}{r} &;\quad \kappa_\theta&=\frac{1}{r\sin\varphi}\frac{\partial\beta_\theta}{\partial\theta}+\frac{1}{r}cotg\varphi\beta_\varphi\\
\gamma^o_\varphi&=\frac{1}{r}\frac{\partial U_\theta}{\partial\varphi} &;\quad \tau_\varphi&=\frac{1}{r}\frac{\partial\beta_\theta}{\partial\varphi}\\
\gamma^o_\theta&=\frac{1}{r\sin\varphi}\frac{\partial U_\varphi}{\partial\theta}-\frac{1}{r}cotg\varphi U_\theta &;\quad \tau_\theta&=\frac{1}{r\sin\varphi}\frac{\partial\beta_\varphi}{\partial\theta}-\frac{1}{r}cotg\varphi\beta_\theta\\
\mu^o_\varphi&=\frac{1}{r}\frac{\partial W}{\partial\varphi}-\frac{U_\varphi}{r}+\beta_\varphi &;\quad \mu^o_\theta&=\frac{1}{r\sin\varphi}\frac{\partial W}{\partial\theta}-\frac{U_\theta}{r}+\beta_\theta
\end{aligned} \tag{42}$$

Now, we substitute relations (42) into the constitutive equations and then into equations (40), giving five differential equations which describe the equations of motion in terms of the displacement field and mechanical properties of the shell, so that we have the same implicit equations as in (36). ***Li***'s equations are given in Appendix C.

8. ***Conical Shells***:

8.1 ***Equilibrium Equations***- We substitute the geometry definitions of conical shells (Figure 8) into equations (30):

$$
\begin{aligned}
&\frac{\operatorname{coseca}}{x}N_{\theta x,\theta}+N_{xx}+q_x=I_1\ddot{u}_x+I_2\ddot{\beta}_x\\
&\frac{\operatorname{coseca}}{x}N_{\theta,\theta}+N_{x\theta,x}+\frac{1}{x\tan\alpha}Q_\theta+q_\theta=I_1\ddot{u}_\theta+I_2\ddot{\beta}_\theta\\
&\frac{\operatorname{coseca}}{x}Q_{\theta,\theta}+Q_{x,x}-\frac{1}{x\tan\alpha}N_\theta+q_n=I_1\ddot{w}\\
&\frac{\operatorname{coseca}}{x}M_{\theta x,\theta}+M_{x,x}-Q_x=I_2\ddot{u}_x+I_3\ddot{\beta}_x\\
&\frac{\operatorname{coseca}}{x}M_{\theta,\theta}+M_{x\theta,x}-Q_\theta=I_2\ddot{u}_\theta+I_3\ddot{\beta}_\theta
\end{aligned}
\tag{43}
$$

8.2 ***Constitutive Equation-*** Equation (29) has to be modified by changing the definitions given in (22) to obtain the constitutive equation of the conical shells. This equation is defined in Appendix A.

$$
\begin{aligned}
&a_1=\frac{1}{x\tan\alpha} \qquad a_2=0. \qquad a_3=0.\\
&b_1=-\frac{1}{x\tan\alpha} \qquad b_2=\frac{1}{x^2\tan^2\alpha} \qquad b_3=0.
\end{aligned}
\tag{44}
$$

8.3 ***Kinematic Relations (Linear Strain-Displacement Relations)-*** These relations can be obtained by using the strain-displacement relations of shells of revolution (32) and conical shell geometry definitions given in (Figure 8).

$$
\begin{aligned}
&\varepsilon^o{}_x=\frac{\partial U_x}{\partial x} && ; \quad \kappa_x=\frac{\partial\beta_x}{\partial x}\\
&\varepsilon^o{}_\theta=\frac{1}{x\sin\alpha}\frac{\partial U_\theta}{\partial\theta}+\frac{W}{x\tan\alpha} && ; \quad \kappa_\theta=\frac{1}{x\sin\alpha}\frac{\partial\beta_\theta}{\partial\theta}\\
&\gamma^o{}_x=\frac{\partial U_\theta}{\partial x} && ; \quad \tau_x=\frac{\partial\beta_\theta}{\partial x}\\
&\gamma^o{}_\theta=\frac{1}{x\sin\alpha}\frac{\partial U_x}{\partial\theta} && ; \quad \tau_\theta=\frac{1}{x\sin\alpha}\frac{\partial\beta_x}{\partial\theta}\\
&\mu^o{}_x=\frac{\partial W}{\partial x}+\beta_x && ; \quad \mu^o{}_\theta=\frac{1}{x\sin\alpha}\frac{\partial W}{\partial\theta}-\frac{U_\theta}{xtaan\alpha}+\beta_\theta
\end{aligned}
\tag{45}
$$

The five differential equations of motion for conical shells, in terms of the displacement field and mechanical properties of shells, can be obtained by substituting the kinematic relations first into the constitutive equations, and then into the equilibrium equations. These implicit equations L_i's are given fully in Appendix D.

9. *Circular Plates*:

9.1 ***The equilibrium equations -*** These equations are obtained by using circular plate geometry definitions (Figure 9) and the same equations as we used for conical shells (43).

$$\begin{aligned}
&\frac{1}{r}\frac{\partial N_{\theta r}}{\partial\theta}+\frac{\partial N_{rr}}{\partial r}+q_r=I_1\ddot{u}_r+I_2\ddot{\beta}_r\\
&\frac{1}{r}\frac{\partial N_{\theta\theta}}{\partial\theta}+\frac{\partial N_{r\theta}}{\partial r}+q_\theta=I_1\ddot{u}_\theta+I_2\ddot{\beta}_\theta\\
&\frac{1}{r}\frac{\partial Q_{\theta\theta}}{\partial\theta}+\frac{\partial Q_{rr}}{\partial r}+q_n=I_1\ddot{w}\\
&\frac{1}{r}\frac{\partial M_{\theta r}}{\partial\theta}+\frac{\partial M_{rr}}{\partial r}-Q_{rr}=I_2\ddot{u}_r+I_3\ddot{\beta}_r\\
&\frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial\theta}+\frac{\partial M_{r\theta}}{\partial r}-Q_{\theta\theta}=I_2\ddot{u}_\theta+I_3\ddot{\beta}_\theta
\end{aligned}\qquad(46)$$

9.2 ***Constitutive equation-*** Changing the relations defined in (22) and substituting in Equations (29), the constitutive equation for a circular plate can be obtained and is given in Appendix A.

$$a_1=a_2=a_3=b_1=b_2=b_3=0.\qquad(47)$$

9.3 ***Kinematic relations (Linear Strain-Displacement Relations)-*** These equations are obtained by substituting the geometry definitions of circular plates into the conical shell kinematic relations:

$$\begin{aligned}
\varepsilon^o_r &= \frac{\partial U_r}{\partial r} \quad ; \quad \kappa_r = \frac{\partial \beta_r}{\partial r} \\
\varepsilon^o_\theta &= \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} \quad ; \quad \kappa_\theta = \frac{1}{r}\frac{\partial \beta_\theta}{\partial \theta} \\
\gamma^o_r &= \frac{\partial U_\theta}{\partial r} \quad ; \quad \tau_r = \frac{\partial \beta_\theta}{\partial r} \\
\gamma^o_\theta &= \frac{1}{r}\frac{\partial U_r}{\partial \theta} \quad ; \quad \tau_\theta = \frac{1}{r}\frac{\partial \beta_r}{\partial \theta} \\
\mu^o_r &= \frac{\partial W}{\partial r} + \beta_r \quad ; \quad \mu^o_\theta = \frac{1}{r}\frac{\partial W}{\partial \theta} + \beta_\theta
\end{aligned} \tag{48}$$

We substitute relations (48) first into the constitutive equations and then into equations (46), and obtain five differential equations which are defined in Appendix E.

10. ***<u>Discussion-</u>*** Linear equations of equilibrium, constitutive and kinematic relations that include the effects of transverse shear, rotary inertia and initial curvature are derived for different geometries of anisotropic plates (circular and rectangular plates) and shells (shells of revolution, cylindrical, spherical and conical shells). The stress-strain relationships are given for multilayered anisotropic laminated materials in order to derive the constitutive equations of each of the above-mentioned geometries. These are given in Appendix A. The work of several researchers on this particular subject has been reviewed and summarized.

In the present theory, β_1 and β_2 which represent the rotation of tangents to the reference surface oriented along parametric lines α_1 and α_2, cannot be expressed in terms of U_i and W. Therefore, the five differential equations of motion cannot be reduced to 3 as in classical shell theory. In the case of cylindrical shells, we obtain five differential equations of motion as shown in A2-A6 in Appendix A. Also listed in Appendix A are the three differential equations (A7-A9) of Sanders' cylindrical shell theory.
The characteristic equations of vibration analysis of anisotropic laminated open circular cylindrical shells,

formulated on the basis of the present theory, have been compared to that of Sanders' shell theory [Ref. 143]. Assuming the displacement functions for the dynamic analysis of anisotropic open circular cylindrical shells to be as follows :

$$\begin{Bmatrix} U(x,\theta) \\ V(x,\theta) \\ W(x,\theta) \\ \beta_x(x,\theta) \\ \beta_\theta(x,\theta) \end{Bmatrix} = \sum_{i=1}^{10} \begin{bmatrix} Cos\bar{m}x & 0 & 0 & 0 & 0 \\ 0 & Sin\bar{m}x & 0 & 0 & 0 \\ 0 & 0 & Sin\bar{m}x & 0 & 0 \\ 0 & 0 & 0 & Cos\bar{m}x & 0 \\ 0 & 0 & 0 & 0 & Sin\bar{m}x \end{bmatrix} \begin{Bmatrix} u_i(\theta) \\ v_i(\theta) \\ w_i(\theta) \\ \beta_{x_i}(\theta) \\ \beta_{\theta_i}(\theta) \end{Bmatrix} = \sum_{i=1}^{10} [T_1] \begin{Bmatrix} A_i e^{\eta_i \theta} \\ B_i e^{\eta_i \theta} \\ C_i e^{\eta_i \theta} \\ D_i e^{\eta_i \theta} \\ E_i e^{\eta_i \theta} \end{Bmatrix} \tag{49}$$

where:

$$\bar{m} = \frac{m\pi}{L}$$

we substitute these definitions into the equations of motion for cylindrical shells (36). We then take into account that the non-trivial solution leads to a tenth order polynomial equation (50) (characteristic equation) due to five degrees of freedom per node, instead of an 8^{th} order equation (51) [Ref. 143, equation 10]:

$$f_{10}\eta^{10} + f_8\eta^8 + f_6\eta^6 + f_4\eta^4 + f_2\eta^2 + f_0 = 0. \tag{50}$$

Where f_i (i = 0 to 10) are the coefficients of the determinant of the matrix *[H]* given in Appendix A. For the case of isotropic cylindrical shells based on classical shell theory, we obtain:

$$h_8\eta^8 + h_6\eta^6 + h_4\eta^4 + h_2\eta^2 + h_0 = 0. \tag{51}$$

where h_i (i= 0 to 8). The coefficients characteristic equation of cylindrical shells based on Sanders' shell theory, are given in [Ref. 143]. Each root of the characteristic equation (50) yields a solution to the equations of motion (36). The complete solution is obtained by finding the sum of all ten solutions independently with the constants A_i, B_i, C_i, D_i and E_i. The fundamental unknowns consist of the ten strain components, ten stress resultants and the five generalized displacements of plates or shells.

11. Conclusion- General equations of multi-layered laminated anisotropic shells were developed by taking into account the shear deformation and rotary inertia effects as well as the initial curvature, the only assumption being that the transverse normal strain be neglected. The derivation was from geometrically linear theory for small elastic strains and from strains expressed in orthogonal curvilinear coordinates for general shells. The virtual work principle was applied in order to derive the equilibrium equations.

The theory used yields five coupled linear second-order differential equations with constant coefficients, instead of 3 equations, as in the case of Sanders' theory. The reason for this is that transverse shear strains do not vanish in the present theory and, therefore, the β_i cannot be expressed in terms of displacement components. This theory leads to no strain during rigid body motions.

A paper currently under preparation will deal with the static and dynamic analysis of open and closed non-uniform anisotropic laminated circular cylindrical shells. The method used is a combination of hybrid finite element analysis and the displacement functions are obtained using the shell equations based on this theory.

The effects on the vibration characteristics of cylindrical shells of variation of shell geometrical (*R/t*, *L/R* and *L/t*) and material (isotropic, symmetric and anti-symmetric cross-ply laminated shells) parameters, as well as axial and circumferential wave number (m, n), are handled through several numerical examples with reasonable agreement with other theories. Further work is underway to apply this theory to the dynamic analysis of liquid-filled open and closed anisotropic cylindrical shells.

Appendix A

This appendix contains the constitutive equations and equations of motion for thin anisotropic plates and shells which were referred to this paper. The Appendix is divided into five parts, covering respectively cylindrical shells, rectangular plates, spherical and conical shells, and circular plates.

$$\begin{Bmatrix} N_{11} \\ N_{12} \\ Q_{11} \\ N_{22} \\ N_{21} \\ Q_{22} \\ M_{11} \\ M_{12} \\ M_{22} \\ M_{21} \end{Bmatrix} = \begin{bmatrix} G_{11} & G_{16} & 0 & A_{12} & A_{16} & 0 & H_{11} & H_{16} & B_{12} & B_{16} \\ G_{61} & G_{66} & 0 & A_{62} & A_{66} & 0 & H_{61} & H_{66} & B_{62} & B_{66} \\ 0 & 0 & AA_{55} & 0 & 0 & A_{45} & 0 & 0 & 0 & 0 \\ A_{21} & A_{26} & 0 & GG_{22} & GG_{26} & 0 & B_{21} & B_{26} & HH_{22} & HH_{26} \\ A_{61} & A_{66} & 0 & GG_{62} & GG_{66} & 0 & B_{61} & B_{66} & HH_{62} & HH_{66} \\ 0 & 0 & A_{45} & 0 & 0 & BB_{44} & 0 & 0 & 0 & 0 \\ I_{11} & I_{16} & 0 & B_{12} & B_{16} & 0 & J_{11} & J_{16} & D_{12} & D_{16} \\ I_{61} & I_{66} & 0 & B_{62} & B_{66} & 0 & J_{61} & J_{66} & D_{62} & D_{66} \\ B_{21} & B_{26} & 0 & II_{22} & II_{26} & 0 & D_{21} & D_{26} & JJ_{22} & JJ_{26} \\ B_{61} & B_{66} & 0 & II_{62} & II_{66} & 0 & D_{61} & D_{66} & JJ_{62} & JJ_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon^o{}_1 \\ \gamma^o{}_1 \\ \mu^o{}_1 \\ \varepsilon^o{}_2 \\ \gamma^o{}_2 \\ \mu^o{}_2 \\ \kappa_1 \\ \tau_1 \\ \kappa_2 \\ \tau_2 \end{Bmatrix} \tag{A-1}$$

The P_{ij}'s elements (A_{ij}, B_{ij}, D_{ij}, G_{ij}, GG_{ij}, H_{ij}, HH_{ij}, I_{ij}, II_{ij}, J_{ij} and JJ_{ij}) have been defined by equations (21-23) and (27).

Cylindrical Shells- The equations of motion are defined by the following equations:

$$\begin{gathered} L_1(U,V,W,\beta_x,\beta_\theta,\overline{P_{ij}}) = \\ P_{11}\frac{\partial^2 U_x}{\partial x^2} + \frac{1}{R}(P_{15}+P_{51})\frac{\partial^2 U_x}{\partial x \partial\theta} + \frac{P_{55}}{R^2}\frac{\partial^2 U_x}{\partial\theta^2} - I_1\frac{\partial^2 U_x}{\partial t^2} + \\ +P_{12}\frac{\partial^2 U_\theta}{\partial x^2} + \frac{1}{R}(P_{14}+P_{52})\frac{\partial^2 U_\theta}{\partial x\partial\theta} + \frac{P_{54}}{R^2}\frac{\partial^2 U_\theta}{\partial\theta^2} + \\ \frac{P_{14}}{R}\frac{\partial W}{\partial x} + \frac{P_{54}}{R^2}\frac{\partial W}{\partial\theta} + \\ P_{17}\frac{\partial^2\beta_x}{\partial x^2} + \frac{1}{R}(P_{1,10}+P_{57})\frac{\partial^2\beta_x}{\partial x\partial\theta} + P_{5,10}\frac{\partial^2\beta_x}{\partial\theta^2} - I_2\frac{\partial^2\beta_x}{\partial t^2} + \\ P_{18}\frac{\partial^2\beta_\theta}{\partial x^2} + \frac{1}{R}(P_{19}+P_{58})\frac{\partial^2\beta_\theta}{\partial x\partial\theta} + \frac{P_{59}}{R^2}\frac{\partial^2\beta_\theta}{\partial\theta^2} \end{gathered} \tag{A-2}$$

$$
\begin{aligned}
&L_2(U,V,W,\beta_x,\beta_\theta,\overline{P_{ij}})= \\
&P_{21}\frac{\partial^2 U_x}{\partial x^2}+\frac{1}{R}(P_{25}+P_{41})\frac{\partial^2 U_x}{\partial x\partial\theta}+\frac{P_{45}}{R^2}\frac{\partial^2 U_x}{\partial\theta^2}+ \\
&P_{22}\frac{\partial^2 U_\theta}{\partial x^2}+\frac{1}{R}(P_{24}+P_{42})\frac{\partial^2 U_\theta}{\partial x\partial\theta}+\frac{P_{44}}{R^2}\frac{\partial^2 U_\theta}{\partial\theta^2}-\frac{P_{66}}{R^2}U_\theta-I_1\frac{\partial^2 U_\theta}{\partial t^2}+ \\
&+\frac{1}{R}(P_{24}+P_{63})\frac{\partial W}{\partial x}+\frac{1}{R^2}(P_{44}+P_{66})\frac{\partial W}{\partial\theta}+ \\
&+P_{27}\frac{\partial^2\beta_x}{\partial x^2}+\frac{1}{R}(P_{2,10}+P_{47})\frac{\partial^2\beta_x}{\partial x\partial\theta}+\frac{P_{4,10}}{R^2}\frac{\partial^2\beta_x}{\partial\theta^2}+\frac{P_{36}}{R}\beta_x+ \\
&+P_{28}\frac{\partial^2\beta_\theta}{\partial x^2}+\frac{1}{R}(P_{29}+P_{48})\frac{\partial^2\beta_\theta}{\partial x\partial\theta}+\frac{P_{49}}{R^2}\frac{\partial^2\beta_\theta}{\partial\theta^2}+\frac{P_{66}}{R}\beta_\theta-I_2\frac{\partial^2 U_\theta}{\partial t^2}
\end{aligned}
\tag{A-3}
$$

$$
\begin{aligned}
&L_3(U,V,W,\beta_x,\beta_\theta,\overline{P_{ij}})= \\
&-\frac{P_{41}}{R}\frac{\partial U_x}{\partial x}-\frac{P_{45}}{R^2}\frac{\partial U_x}{\partial\theta}- \\
&-\frac{1}{R}(P_{36}+P_{42})\frac{\partial U_\theta}{\partial x}-\frac{1}{R^2}(P_{66}+P_{44})\frac{\partial U_\theta}{\partial\theta}+ \\
&+P_{33}\frac{\partial^2 W}{\partial x^2}+\frac{1}{R}(P_{36}+P_{63})\frac{\partial^2 W}{\partial x\partial\theta}+\frac{P_{66}}{R^2}\frac{\partial^2 W}{\partial\theta^2}-\frac{P_{44}}{R^2}W-I_1\frac{\partial^2 W}{\partial t^2}+ \\
&+(P_{33}-\frac{P_{47}}{R})\frac{\partial\beta_x}{\partial x}+\frac{1}{R}(P_{63}-\frac{P_{4,10}}{R})\frac{\partial\beta_x}{\partial\theta}+ \\
&+(P_{36}-\frac{P_{48}}{R})\frac{\partial\beta_\theta}{\partial x}+\frac{1}{R}(P_{66}-\frac{P_{49}}{R})\frac{\partial\beta_\theta}{\partial\theta}
\end{aligned}
\tag{A-4}
$$

$$
\begin{aligned}
&L_4(U,V,W,\beta_x,\beta_\theta,\overline{P_{ij}})= \\
&P_{71}\frac{\partial^2 U_x}{\partial x^2}+\frac{1}{R}(P_{75}+P_{10,1})\frac{\partial^2 U_x}{\partial x\partial\theta}-I_2\frac{\partial^2 U_x}{\partial t^2}+\frac{P_{10,5}}{R^2}\frac{\partial^2 U_x}{\partial\theta^2}-I_2\frac{\partial^2 U_x}{\partial t^2}+ \\
&P_{72}\frac{\partial^2 U_\theta}{\partial x^2}+\frac{1}{R}(P_{74}+P_{10,2})\frac{\partial^2 U_\theta}{\partial x\partial\theta}+\frac{P_{10,4}}{R^2}\frac{\partial^2 U_\theta}{\partial\theta^2}+\frac{P_{36}}{R}U_\theta+ \\
&+(\frac{P_{74}}{R}-P_{33})\frac{\partial W}{\partial x}+\frac{1}{R}(\frac{P_{10,4}}{R}-P_{36})\frac{\partial W}{\partial\theta}+ \\
&+P_{77}\frac{\partial^2\beta_x}{\partial x^2}+\frac{1}{R}(P_{7,10}+P_{10,7})\frac{\partial^2\beta_x}{\partial x\partial\theta}+\frac{P_{10,10}}{R^2}\frac{\partial^2\beta_x}{\partial\theta^2}-P_{33}\beta_x-I_3\frac{\partial^2\beta_x}{\partial t^2}+ \\
&+P_{78}\frac{\partial^2\beta_\theta}{\partial x^2}+\frac{1}{R}(P_{79}+P_{10,8})\frac{\partial^2\beta_\theta}{\partial x\partial\theta}+\frac{P_{10,9}}{R^2}\frac{\partial^2\beta_\theta}{\partial\theta^2}-P_{36}\beta_\theta
\end{aligned}
\tag{A-5}
$$

$$
\begin{aligned}
&L_5(U,V,W,\beta_x,\beta_\theta,\overline{P}_\psi)= \\
&P_{81}\frac{\partial^2 U_x}{\partial x^2}+\frac{1}{R}(P_{91}+P_{85})\frac{\partial^2 U_x}{\partial x\partial\theta}+\frac{P_{95}}{R^2}\frac{\partial^2 U_x}{\partial\theta^2}+ \\
&P_{82}\frac{\partial^2 U_\theta}{\partial x^2}+\frac{1}{R}(P_{84}+P_{92})\frac{\partial^2 U_\theta}{\partial x\partial\theta}+\frac{P_{94}}{R^2}\frac{\partial^2 U_\theta}{\partial\theta^2}+\frac{P_{66}}{R}U_\theta-I_2\frac{\partial^2 U_\theta}{\partial t^2}+ \\
&+(\frac{P_{84}}{R}-P_{63})\frac{\partial W}{\partial x}+\frac{1}{R}(\frac{P_{94}}{R}-P_{66})\frac{\partial W}{\partial\theta}+ \\
&+P_{87}\frac{\partial^2\beta_x}{\partial x^2}+\frac{1}{R}(P_{8,10}+P_{97})\frac{\partial^2\beta_x}{\partial x\partial\theta}+\frac{P_{9,10}}{R^2}\frac{\partial^2\beta_x}{\partial\theta^2}-P_{63}\beta_x+ \\
&+P_{88}\frac{\partial^2\beta_\theta}{\partial x^2}+\frac{1}{R}(P_{89}+P_{98})\frac{\partial^2\beta_\theta}{\partial x\partial\theta}+\frac{P_{99}}{R^2}\frac{\partial^2\beta_\theta}{\partial\theta^2}-P_{66}\beta_\theta-I_3\frac{\partial^2\beta_\theta}{\partial t^2}
\end{aligned}
\tag{A-6}
$$

The equations of motion for a thin cylindrical shell (Hybrid finite element method based on Sanders' shell theory) are defined as below [143]:

$$
\begin{aligned}
&L_1(U,V,W,P_\psi)=P_{11}\frac{\partial^2 U}{\partial x^2}+\frac{P_{12}}{R}(\frac{\partial^2 V}{\partial x\partial\theta}+\frac{\partial W}{\partial x})-P_{14}\frac{\partial^3 W}{\partial x^3}+ \\
&\frac{P_{15}}{R^2}(\frac{\partial^3 W}{\partial x\partial\theta^2}+\frac{\partial^2 V}{\partial x\partial\theta})+(\frac{P_{33}}{R}-\frac{P_{63}}{2R^2})(\frac{\partial^2 V}{\partial x\partial\theta}+\frac{1}{R}\frac{\partial^2 U}{\partial\theta^2})+ \\
&(\frac{P_{36}}{R^2}-\frac{P_{66}}{2R^3})(-\frac{2\partial^3 W}{\partial x\partial\theta^2}+\frac{3}{2}\frac{\partial^2 V}{\partial x\partial\theta}-\frac{1}{2R}\frac{\partial^2 U}{\partial\theta^2})
\end{aligned}
\tag{A-7}
$$

$$
\begin{aligned}
&L_2(U,V,W,P_\psi)=(\frac{P_{21}}{R}+\frac{P_{51}}{R^2})(\frac{\partial^2 U}{\partial x\partial\theta})+\frac{1}{R}(\frac{P_{22}}{R}+\frac{P_{52}}{R^2}) \\
&(\frac{\partial^2 V}{\partial\theta^2}+\frac{\partial W}{\partial\theta})-(\frac{P_{24}}{R}+\frac{P_{54}}{R^2})(\frac{\partial^3 W}{\partial x^2\partial\theta})+\frac{1}{R^2}(\frac{P_{25}}{R}+\frac{P_{55}}{R^2}) \\
&(-\frac{\partial^3 w}{\partial\theta^3}+\frac{\partial^2 V}{\partial\theta^2})+(P_{33}+\frac{3P_{63}}{2R})(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 U}{R\partial x\partial\theta})+ \\
&\frac{1}{R}(P_{36}+\frac{3P_{66}}{2R})(-2\frac{\partial^3 W}{\partial x^2\partial\theta}+\frac{3}{2}\frac{\partial^2 V}{\partial x^2}-\frac{\partial^2 U}{2R\partial x\partial\theta})
\end{aligned}
\tag{A-8}
$$

$$
\begin{gathered}
L_5(U,V,W,P_{ij})=P_{41}\frac{\partial^3 U}{\partial x^3}+\frac{P_{42}}{R}\left(\frac{\partial^3 V}{\partial x^2\partial\theta}+\frac{\partial^2 W}{\partial x^2}\right)-P_{44}\frac{\partial^4 W}{\partial x^4}+ \\
\frac{P_{45}}{R^2}\left(-\frac{\partial^4 W}{\partial x^2\partial\theta^2}+\frac{\partial^3}{\partial x^2\partial\theta}\right)+\frac{2P_{63}}{R}\left(\frac{\partial^3 U}{R\partial x\partial\theta^2}+\frac{\partial^3 V}{\partial x^2\partial\theta}\right)+\left(\frac{2P_{66}}{R^2}\right) \\
\left(-2\frac{\partial^4 W}{\partial x^2\partial\theta^2}+\frac{3}{2}\frac{\partial^3 V}{\partial x^2\partial\theta}-\frac{\partial^3 U}{2R\partial x\partial\theta^2}\right)+\frac{P_{51}}{R^2}\frac{\partial^3 U}{\partial x\partial\theta^2}+\frac{P_{52}}{R^3}\left(\frac{\partial^3 V}{\partial\theta^3}\frac{\partial^2 W}{\partial\theta^2}\right)+ \\
\frac{P_{55}}{R^4}\left(-\frac{\partial^4 W}{\partial\theta^4}+\frac{\partial^3 V}{\partial\theta^3}\right)-\frac{P_{21}}{R}\frac{\partial U}{\partial x}-\frac{P_{54}}{R^2}\frac{\partial^4 W}{\partial x^2\partial\theta^2} \\
-\frac{P_{22}}{R^2}\left(\frac{\partial V}{\partial\theta}+W\right)+\frac{P_{24}}{R}\frac{\partial^2 W}{\partial\theta^2}-\frac{P_{25}}{R^3}\left(-\frac{\partial^2 W}{\partial\theta^2}+\frac{\partial V}{\partial\theta}\right)
\end{gathered}
\qquad \text{(A-9)}
$$

The P_{ij}'s elements are defined (only for one lamina) [143]:

$$
\begin{gathered}
P_{11}=C_{11} \quad P_{44}=D_{11} \\
P_{12}=C_{12} \quad P_{45}=D_{12} \\
P_{21}=P_{12} \quad P_{54}=P_{45} \\
P_{22}=C_{22} \quad P_{55}=D_{22} \\
P_{33}=C_{33} \quad P_{66}=D_{33} \\
\textit{where} \\
C_{11}=E_x t/\Delta \quad D_{11}=E_x t^3/12\Delta \\
C_{22}=E_\theta t/\Delta \quad D_{22}=E_\theta t^3/12\Delta \\
C_{12}=\nu_x E_\theta t/\Delta \quad D_{12}=\nu_x E_\theta t^3/12\Delta \\
C_{33}=G_{x\theta} t \quad D_{33}=G_{x\theta} t^3/12 \\
\textit{where} \\
\Delta=(1-\nu_x\nu_\theta)
\end{gathered}
\qquad \text{(A-10)}
$$

Matrix [H]:

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} & H_{15} \\ H_{21} & H_{22} & H_{23} & H_{24} & H_{25} \\ H_{31} & H_{32} & H_{33} & H_{34} & H_{35} \\ H_{41} & H_{42} & H_{43} & H_{44} & H_{45} \\ H_{51} & H_{52} & H_{53} & H_{54} & H_{55} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \\ D \\ E \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

where:

$$\begin{aligned}
H_{11} &= P_{11}(-\bar{m}^2)-[\frac{P_{5,10}+P_{10,5}}{2R^3}-\frac{P_{10,10}}{4R^4}-\frac{P_{55}}{R^2}]\eta^2-[\frac{P_{15}+P_{51}}{R}-\frac{P_{10,1}+P_{1,10}}{2R^2}](\bar{m}\eta) \\
H_{12} &= (P_{12}+\frac{P_{18}}{2R})(-\bar{m}^2)+[\frac{P_{14}+P_{52}}{R}+\frac{P_{58}-P_{10,2}}{2R^2}-\frac{P_{10,8}}{4R^3}]\bar{m}\eta+(\frac{P_{54}}{R^2}-\frac{P_{10,4}}{2R^3})\eta^2 \\
H_{13} &= \frac{P_{14}}{R}\bar{m}+(\frac{P_{54}}{R^2}-\frac{P_{10,4}}{2R^3})\eta \\
H_{14} &= P_{17}(-\bar{m}^2)-[\frac{P_{1,10}+P_{57}}{R}-\frac{P_{10,7}}{2R^2}]\bar{m}\eta-(\frac{P_{10,10}}{2R^3}-\frac{P_{5,10}}{R^2})\eta^2 \\
H_{15} &= P_{18}(-\bar{m}^2)+[\frac{P_{19}+P_{58}}{R}-\frac{P_{10,8}}{2R^2}]\bar{m}\eta-(\frac{P_{10,9}}{2R^3}-\frac{P_{59}}{R^2})\eta^2 \\
H_{22} &= [P_{22}+\frac{P_{28}+P_{82}}{2R}+\frac{P_{88}}{4R^2}](-\bar{m}^2)+[\frac{P_{24}+P_{42}}{R}+\frac{P_{48}+P_{84}}{2R^2}]\bar{m}\eta-\frac{1}{R^2}(P_{66}-P_{44}\eta^2) \\
H_{23} &= (\frac{P_{24}+P_{63}}{R}+\frac{P_{84}}{2R^2})\bar{m}+(P_{44}+P_{66})\frac{\eta}{R^2} \\
H_{24} &= (P_{27}+\frac{P_{87}}{2R})(-\bar{m}^2)-[\frac{P_{2,10}+P_{47}}{R}+\frac{P_{8,10}}{2R^2}]\bar{m}\eta+\frac{\eta^2}{R^2}P_{4,10}+\frac{P_{63}}{R} \\
H_{25} &= (P_{28}+\frac{P_{88}}{2R})(-\bar{m}^2)+(\frac{P_{29}+P_{48}}{R}+\frac{P_{89}}{2R^2})\bar{m}\eta+\frac{P_{49}}{R^2}\eta^2+\frac{P_{66}}{R} \\
H_{33} &= P_{33}(-\bar{m}^2)+(\frac{P_{36}+P_{63}}{R})\bar{m}\eta+\frac{P_{66}}{R^2}\eta^2-\frac{P_{44}}{R^2} \\
H_{34} &= (\frac{P_{47}}{R}-P_{33})(\bar{m})+(\frac{P_{63}}{R}-\frac{P_{4,10}}{R^2})\eta \\
H_{35} &= (P_{36}-\frac{P_{48}}{R})(\bar{m})+(P_{66}-\frac{P_{49}}{R})\frac{\eta}{R} \\
H_{44} &= P_{77}(-\bar{m}^2)-(P_{7,10}+P_{10,7})\frac{\bar{m}\eta}{R}+\frac{P_{10,10}}{R^2}\eta^2-P_{33} \\
H_{45} &= P_{78}(-\bar{m}^2)+(P_{79}+P_{10,8})\frac{\bar{m}\eta}{R}+P_{10,9}\frac{\eta^2}{R^2}-P_{36} \\
H_{55} &= P_{88}(-\bar{m}^2)+(P_{98}+P_{89})\frac{\bar{m}\eta}{R}+\frac{P_{99}}{R^2}\eta^2-P_{66}
\end{aligned} \tag{A-11}$$

where

$$\bar{m} = \frac{m\pi}{L}$$

Appendix B

<u>*Rectangular Plates*</u>-The L_i's equations are given below:

$$
\begin{aligned}
&L_1(U,V,W,\beta_x,\beta_y,\overline{P_{ij}})= \\
&P_{11}\frac{\partial^2 U_x}{\partial x^2}+(P_{15}+P_{51})\frac{\partial^2 U_x}{\partial x\partial y}+P_{55}\frac{\partial^2 U_x}{\partial y^2}-I_1\frac{\partial^2 U_x}{\partial t^2}+ \\
&+P_{12}\frac{\partial^2 U_y}{\partial x^2}+(P_{14}+P_{52})\frac{\partial^2 U_y}{\partial x\partial y}+P_{54}\frac{\partial^2 U_y}{\partial y^2}+ \\
&P_{17}\frac{\partial^2 \beta_x}{\partial x^2}+(P_{1,10}+P_{57})\frac{\partial^2 \beta_x}{\partial x\partial y}+P_{5,10}\frac{\partial^2 \beta_x}{\partial y^2}-I_2\frac{\partial^2 \beta_x}{\partial t^2}+ \\
&P_{18}\frac{\partial^2 \beta_y}{\partial x^2}+(P_{19}+P_{58})\frac{\partial^2 \beta_y}{\partial x\partial y}+P_{59}\frac{\partial^2 \beta_y}{\partial y^2}
\end{aligned}
\tag{B-1}
$$

$$
\begin{aligned}
&L_2(U,V,W,\beta_x,\beta_y,\overline{P_{ij}})= \\
&P_{21}\frac{\partial^2 U_x}{\partial x^2}+(P_{25}+P_{41})\frac{\partial^2 U_x}{\partial x\partial y}+P_{45}\frac{\partial^2 U_x}{\partial y^2}+ \\
&+P_{22}\frac{\partial^2 U_y}{\partial x^2}+(P_{24}+P_{42})\frac{\partial^2 U_y}{\partial x\partial y}+P_{44}\frac{\partial^2 U_y}{\partial y^2}-I_1\frac{\partial^2 U_y}{\partial t^2}+ \\
&+P_{27}\frac{\partial^2 \beta_x}{\partial x^2}+(P_{2,10}+P_{47})\frac{\partial^2 \beta_x}{\partial x\partial y}+P_{4,10}\frac{\partial^2 \beta_x}{\partial y^2}+ \\
&+P_{28}\frac{\partial^2 \beta_y}{\partial x^2}+(P_{29}+P_{48})\frac{\partial^2 \beta_y}{\partial x\partial y}+P_{49}\frac{\partial^2 \beta_y}{\partial y^2}-I_2\frac{\partial^2 \beta_y}{\partial t^2}
\end{aligned}
\tag{B-2}
$$

$$
\begin{aligned}
&L_3(U,V,W,\beta_x,\beta_y,\overline{P_{ij}})= \\
&P_{33}\frac{\partial^2 W}{\partial x^2}+(P_{36}+P_{63})\frac{\partial^2 W}{\partial x\partial y}+P_{66}\frac{\partial^2 W}{\partial y^2}-I_1\frac{\partial^2 W}{\partial t^2}+ \\
&+P_{33}\frac{\partial \beta_x}{\partial x}+P_{63}\frac{\partial \beta_x}{\partial y}+P_{36}\frac{\partial \beta_y}{\partial x}+P_{66}\frac{\partial \beta_y}{\partial y}
\end{aligned}
\tag{B-3}
$$

$$L_4(U,V,W,\beta_x,\beta_y,\overline{P_{ij}})=$$
$$P_{71}\frac{\partial^2U_x}{\partial x^2}+(P_{75}+P_{10,1})\frac{\partial^2U_x}{\partial x\partial y}+P_{10,5}\frac{\partial^2U_x}{\partial y^2}-I_2\frac{\partial^2U_x}{\partial t^2}+$$
$$+P_{72}\frac{\partial^2U_y}{\partial x^2}+(P_{74}+P_{10,2})\frac{\partial^2U_y}{\partial x\partial y}+P_{10,4}\frac{\partial^2U_y}{\partial y^2}+$$
$$(-)P_{33}\frac{\partial W}{\partial x}-P_{36}\frac{\partial w}{\partial y}+ \tag{B-4}$$
$$P_{77}\frac{\partial^2\beta_x}{\partial x^2}+(P_{7,10}+P_{10,7})\frac{\partial^2\beta_x}{\partial x\partial y}+P_{10,10}\frac{\partial^2\beta_x}{\partial y^2}-P_{33}\beta_x-I_3\frac{\partial^2\beta_x}{\partial t^2}+$$
$$P_{78}\frac{\partial^2\beta_y}{\partial x^2}+(P_{79}+P_{10,8})\frac{\partial^2\beta_y}{\partial x\partial y}+P_{10,9}\frac{\partial^2\beta_y}{\partial y^2}-P_{36}\beta_y$$

$$L_5(U,V,W,\beta_x,\beta_y,\overline{P_{ij}})=$$
$$P_{81}\frac{\partial^2U_x}{\partial x^2}+(P_{85}+P_{91})\frac{\partial^2U_x}{\partial x\partial y}+P_{95}\frac{\partial^2U_x}{\partial y^2}+$$
$$+P_{82}\frac{\partial^2U_y}{\partial x^2}+(P_{84}+P_{92})\frac{\partial^2U_y}{\partial x\partial y}+P_{94}\frac{\partial^2U_y}{\partial y^2}-I_2\frac{\partial^2U_y}{\partial t^2}+$$
$$(-)P_{63}\frac{\partial W}{\partial x}-P_{66}\frac{\partial W}{\partial y}+ \tag{B-5}$$
$$P_{87}\frac{\partial^2\beta_x}{\partial x^2}+(P_{8,10}+P_{97})\frac{\partial^2\beta_x}{\partial x\partial y}+P_{9,10}\frac{\partial^2\beta_x}{\partial y^2}-P_{63}\beta_x+$$
$$P_{88}\frac{\partial^2\beta_y}{\partial x^2}+(P_{89}+P_{98})\frac{\partial^2\beta_y}{\partial x\partial y}+P_{99}\frac{\partial^2\beta_y}{\partial y^2}-P_{66}\beta_y-I_3\frac{\partial^2\beta_y}{\partial t^2}$$

Appendix C

Spherical Shells- **The L_i's equations (equations of motion) are given below:**

$$L_1(U_\varphi,U_\theta,W,\beta_\varphi,\beta_\theta,\overline{P_{ij}})=$$

$$\frac{P_{11}}{r^2}\frac{\partial^2 U_\varphi}{\partial\varphi^2}+\frac{(P_{15}+P_{51})}{r^2\sin\varphi}\frac{\partial^2 U_\varphi}{\partial\varphi\partial\theta}+\frac{P_{55}}{r^2\sin^2\varphi}\frac{\partial^2 U_\varphi}{\partial\theta^2}+$$

$$+\frac{P_{11}}{r^2}cotg\varphi\frac{\partial U_\varphi}{\partial\varphi}+\frac{P_{54}}{r^2}\frac{\cos\varphi}{\sin^2\varphi}\frac{\partial U_\varphi}{\partial\theta}-\frac{1}{r^2}(P_{14}+P_{33}+P_{44}cotg^2\varphi)U_\varphi-I_1\frac{\partial^2 U_\varphi}{\partial t^2}+$$

$$+\frac{P_{12}}{r^2}\frac{\partial^2 U_\theta}{\partial\varphi^2}+\frac{(P_{52}+P_{14})}{r^2\sin\varphi}\frac{\partial^2 U_\theta}{\partial\varphi\partial\theta}+\frac{P_{54}}{r^2\sin^2\varphi}\frac{\partial^2 U_\theta}{\partial\theta^2}+$$

$$+\frac{(P_{12}-P_{15}-P_{42})}{r^2}cotg\varphi\frac{\partial U_\theta}{\partial\varphi}-\frac{(P_{55}+P_{44})}{r^2}\frac{\cos\varphi}{\sin^2\varphi}\frac{\partial U_\theta}{\partial\theta}+$$

$$+\frac{1}{r^2}((P_{15}+P_{45})cotg^2\varphi-P_{36})U_\theta+\frac{(P_{11}+P_{14}+P_{33})}{r^2}\frac{\partial W}{\partial\varphi}+\frac{(P_{51}+P_{54}+P_{36})}{r^2\sin\varphi}\frac{\partial W}{\partial\theta}+ \qquad \text{(C-1)}$$

$$+(\frac{P_{11}}{\sin\varphi}-P_{44})\frac{cotg\varphi}{r^2}W+\frac{P_{17}}{r^2}\frac{\partial^2\beta\varphi}{\partial\varphi^2}+\frac{1}{r^2\sin\varphi}(P_{1,10}+P_{57})\frac{\partial^2\beta_\varphi}{\partial\varphi\partial\theta}+$$

$$+\frac{P_{5,10}}{r^2\sin^2\varphi}\frac{\partial^2\beta_\varphi}{\partial\theta^2}+\frac{(P_{17}+P_{19}-P_{47})}{r^2}cotg\varphi\frac{\partial\beta_\varphi}{\partial\varphi}+$$

$$+\frac{(P_{59}-P_{4,10})}{r^2}\frac{\cos\varphi}{\sin^2\varphi}\frac{\partial\beta_\varphi}{\partial\theta}-(\frac{1}{r^2}(P_{19}+P_{49})cotg^2\varphi-\frac{P_{33}}{r})\beta_\varphi-I_2\frac{\partial^2\beta_\varphi}{\partial t^2}+$$

$$+\frac{P_{18}}{r^2}\frac{\partial^2\beta_\theta}{\partial\varphi^2}+\frac{(P_{19}+P_{58})}{r^2\sin\varphi}\frac{\partial^2\beta_\theta}{\partial\varphi\partial\theta}+\frac{P_{59}}{r^2\sin^2\varphi}\frac{\partial^2\beta_\theta}{\partial\theta^2}+$$

$$+\frac{(P_{18}-P_{48}-P_{1,10})}{r^2}cotg\varphi\frac{\partial\beta_\theta}{\partial\varphi}-\frac{(P_{49}+P_{5,10})}{r^2}\frac{\cos\varphi}{\sin^2\varphi}\frac{\partial\beta_\theta}{\partial\theta}+\frac{1}{r^2}(rP_{36}+(P_{1,10}+P_{4,10}cotg^2\varphi))\beta_\theta$$

$$L_2(U_\varphi,U_\theta,W,\beta_\varphi,\beta_\theta,\overline{P_{ij}})=$$

$$\frac{P_{21}}{r^2}\frac{\partial^2 U_\varphi}{\partial\varphi^2}+\frac{(P_{41}+P_{25})}{r^2\sin\varphi}\frac{\partial^2 U_\varphi}{\partial\varphi\partial\theta}+\frac{p_{45}}{r^2\sin^2\varphi}\frac{\partial^2 U_\varphi}{\partial\theta^2}+\frac{(P_{51}+P_{24}+P_{21})}{r^2}cotg\varphi\frac{\partial U_\varphi}{\partial\varphi}+$$

$$+\frac{(P_{44}+P_{55})\cos\varphi}{r^2\sin^2\varphi}\frac{\partial U_\varphi}{\partial\theta}+\frac{1}{r^2}(P_{54}cotg^2\varphi-P_{24}-P_{63})U_\varphi+$$

$$\frac{P_{22}}{r^2}\frac{\partial^2 U_\theta}{\partial\varphi^2}+\frac{(P_{42}+P_{24})}{r^2\sin\varphi}\frac{\partial^2 U_\theta}{\partial\varphi\partial\theta}+\frac{p_{44}}{r^2\sin^2\varphi}\frac{\partial^2 U_\theta}{\partial\theta^2}+\frac{P_{22}}{r^2}cotg\varphi\frac{\partial U_\theta}{\partial\varphi}-$$

$$-\frac{P_{54}\cos\varphi}{r^2\sin^2\varphi}\frac{\partial U_\theta}{\partial\theta}+\frac{1}{r^2}(P_{25}-P_{55}cotg^2\varphi-P_{66})U_\theta-I_1\frac{\partial^2 U_\theta}{\partial t^2}+$$

$$\frac{(P_{21}+P_{24}+P_{63})}{r^2}\frac{\partial W}{\partial\varphi}+\frac{(P_{41}+P_{44}+P_{66})}{r^2\sin\varphi}\frac{\partial W}{\partial\theta}+\frac{1}{r^2}(p_{51}+P_{54}+P_{21}+P_{24})cotg\varphi W+ \tag{C-2}$$

$$\frac{P_{27}}{r^2}\frac{\partial^2\beta_\varphi}{\partial\varphi^2}+\frac{(P_{47}+P_{2,10})}{r^2\sin\varphi}\frac{\partial^2\beta_\varphi}{\partial\varphi\partial\theta}+\frac{p_{4,10}}{r^2\sin^2\varphi}\frac{\partial^2\beta_\varphi}{\partial\theta^2}+\frac{(P_{57}+P_{29}+P_{27})}{r^2}cotg\varphi\frac{\partial\beta_\varphi}{\partial\varphi}+$$

$$+\frac{(P_{49}+P_{5,10})\cos\varphi}{r^2\sin^2\varphi}\frac{\partial\beta_\varphi}{\partial\theta}+[\frac{1}{r^2}(P_{59}cotg^2\varphi-P_{29})+\frac{P_{63}}{r}]\beta_\varphi+$$

$$\frac{P_{28}}{r^2}\frac{\partial^2\beta_\theta}{\partial\varphi^2}+\frac{(P_{48}+P_{29})}{r^2\sin\varphi}\frac{\partial^2\beta_\theta}{\partial\varphi\partial\theta}+\frac{p_{49}}{r^2\sin^2\varphi}\frac{\partial^2\beta_\theta}{\partial\theta^2}+\frac{(P_{58}+P_{28}-P_{2,10})}{r^2}cotg\varphi\frac{\partial\beta_\theta}{\partial\varphi}+$$

$$+\frac{(P_{59}-P_{4,10})\cos\varphi}{r^2\sin^2\varphi}\frac{\partial\beta_\theta}{\partial\theta}+\frac{1}{r^2}(rP_{66}+P_{2,10}-P_{5,10}cotg^2\varphi)\beta_\theta-I_2\frac{\partial^2 U_\theta}{\partial t^2}$$

$$L_3(U_\varphi, U_\theta, W, \beta_\varphi, \beta_\theta, \overline{P_{ij}})=$$

$$-\frac{1}{r^2}(P_{33}+P_{11}+P_{41})\frac{\partial U_\varphi}{\partial\varphi}-\frac{1}{r^2\sin\varphi}(P_{63}+P_{15}+P_{45})\frac{\partial U_\varphi}{\partial\theta}-$$

$$-\frac{1}{r^2}(P_{33}+P_{14}+P_{44})cotg\varphi U_\varphi-\frac{1}{r^2}(P_{36}+P_{12}+P_{42})\frac{\partial U_\theta}{\partial\varphi}-$$

$$-\frac{1}{r^2\sin\varphi}(P_{66}+P_{14}+P_{44})\frac{\partial U_\theta}{\partial\theta}+\frac{1}{r^2}(P_{15}+P_{45}-P_{36})cotg\varphi U_\theta+$$

$$+\frac{P_{33}}{r^2}\frac{\partial^2 W}{\partial\varphi^2}+\frac{(P_{36}+P_{63})}{r^2\sin\varphi}\frac{\partial^2 W}{\partial\varphi\partial\theta}+\frac{P_{66}}{r^2\sin^2\varphi}\frac{\partial^2 W}{\partial\theta^2}+ \tag{C-3}$$

$$+\frac{P_{33}}{r^2}cotg\varphi\frac{\partial W}{\partial\varphi}-\frac{1}{r^2}(P_{11}+P_{14}+P_{41}+P_{44})W-I_1\frac{\partial^2 W}{\partial t^2}+$$

$$+\frac{1}{r}(P_{33}-\frac{1}{r}(P_{17}+P_{47}))\frac{\partial\beta_\varphi}{\partial\varphi}+\frac{1}{r\sin\varphi}(P_{63}-\frac{1}{r}(P_{1,10}+P_{4,10}))\frac{\partial\beta_\varphi}{\partial\theta}+$$

$$+\frac{1}{r^2}(rP_{33}-P_{19}-P_{49})cotg\varphi\beta_\varphi+$$

$$\frac{1}{r^2}(rP_{36}-P_{18}-P_{48})\frac{\partial\beta_\theta}{\partial\varphi}+\frac{1}{r^2\sin\varphi}(rP_{66}-P_{19}-P_{49})\frac{\partial\beta_\theta}{\partial\theta}+\frac{1}{r}[P_{36}+\frac{1}{r}(P_{1,10}+P_{4,10})]cotg\varphi\beta_\theta$$

$$L_4(U_\varphi,U_\theta,W,\beta_\varphi,\beta_\theta,\overline{P_{ij}})=$$

$$\frac{P_{71}}{r^2}\frac{\partial^2 U_\varphi}{\partial\varphi^2}+\frac{1}{r^2\sin\varphi}(P_{75}+P_{10,1})\frac{\partial^2 U_\varphi}{\partial\varphi\partial\theta}+\frac{1}{r^2\sin^2\varphi}P_{10,5}\frac{\partial^2 U_\varphi}{\partial\theta^2}+$$

$$+\frac{1}{r^2}(P_{74}+P_{71}-P_{91})cotg\varphi\frac{\partial U_\varphi}{\partial\varphi}+\frac{(P_{10,4}-P_{95})}{r^2\sin^2\varphi}\cos\varphi\frac{\partial U_\varphi}{\partial\theta}+$$

$$+\frac{1}{r}(P_{33}-\frac{P_{74}}{r\sin^2\varphi}-\frac{P_{94}}{r}cotg^2\varphi)U_\varphi-I_2\frac{\partial^2 U_\varphi}{\partial t^2}+$$

$$\frac{P_{72}}{r^2}\frac{\partial^2 U_\theta}{\partial\varphi^2}+\frac{(P_{74}+P_{10,2})}{r^2\sin\varphi}\frac{\partial^2 U_\theta}{\partial\varphi\partial\theta}+\frac{P_{10,4}}{r^2\sin^2\varphi}\frac{\partial^2 U_\theta}{\partial\theta^2}+$$

$$+\frac{(P_{72}-P_{75}-P_{92})}{r^2}cotg\varphi\frac{\partial U_\theta}{\partial\varphi}-\frac{1}{r^2\sin^2\varphi}(P_{10,5}+P_{94})\cos\varphi\frac{\partial U_\theta}{\partial\theta}+$$

$$+\frac{1}{r^2}(rP_{36}+P_{75}++P_{95}cotg^2\varphi)U_\theta+$$

$$\frac{(P_{71}+P_{74}-rP_{33})}{r^2}\frac{\partial W}{\partial\varphi}+\frac{1}{r^2\sin\varphi}(P_{10,4}+P_{10,1}-rP_{36})\frac{\partial W}{\partial\theta}+\frac{1}{r^2}(P_{71}+P_{74}-P_{91}-P_{94})cotg\varphi W+ \qquad \text{(C-4)}$$

$$\frac{P_{77}}{r^2}\frac{\partial^2\beta_\varphi}{\partial\varphi^2}+\frac{1}{r^2\sin\varphi}(P_{10,7}+P_{7,10})\frac{\partial^2\beta_\varphi}{\partial\varphi\partial\theta}+\frac{1}{r^2\sin^2\varphi}P_{10,10}\frac{\partial^2\beta_\varphi}{\partial\theta^2}+$$

$$+\frac{1}{r^2}(P_{79}+P_{77}-P_{97})cotg\varphi\frac{\partial\beta_\varphi}{\partial\varphi}+\frac{1}{r^2\sin^2\varphi}(P_{10,9}-P_{9,10})\cos\varphi\frac{\partial\beta_\varphi}{\partial\theta}+$$

$$-\frac{1}{r^2}(P_{79}+r^2P_{33}+P_{99}cotg^2\varphi)\beta_\varphi-I_3\frac{\partial^2\beta_\varphi}{\partial t^2}+$$

$$\frac{P_{78}}{r^2}\frac{\partial^2\beta_\theta}{\partial\varphi^2}+\frac{1}{r^2\sin\varphi}(P_{79}+P_{10,8})\frac{\partial^2\beta_\theta}{\partial\varphi\partial\theta}+\frac{1}{r^2\sin^2\varphi}P_{10,9}\frac{\partial^2\beta_\theta}{\partial\theta^2}+$$

$$+\frac{1}{r^2}[P_{78}-(P_{98}+P_{7,10})]cotg\varphi\frac{\partial\beta_\theta}{\partial\varphi}-\frac{1}{r^2\sin^2\varphi}(P_{99}+P_{10,10})\cos\varphi\frac{\partial\beta_\theta}{\partial\theta}-$$

$$-\frac{1}{r^2}(r^2P_{36}-P_{7,10}-P_{9,10}cotg^2\varphi)\beta_\theta$$

$$L_5(U_\varphi, U_\theta, W, \beta_\varphi, \beta_\theta, \overline{P_{ij}}) =$$

$$\frac{P_{81}}{r^2}\frac{\partial^2 U_\varphi}{\partial\varphi^2} + \frac{1}{r^2\sin\varphi}(P_{85}+P_{91})\frac{\partial^2 U_\varphi}{\partial\varphi\partial\theta} + \frac{1}{r^2\sin^2\varphi}P_{95}\frac{\partial^2 U_\varphi}{\partial\theta^2} +$$

$$+\frac{1}{r^2}(P_{84}+P_{81}+P_{10,1})cotg\varphi\frac{\partial U_\varphi}{\partial\varphi} + \frac{1}{r^2\sin^2\varphi}(P_{94}+P_{10,5})\cos\varphi\frac{\partial U_\varphi}{\partial\theta} +$$

$$+\frac{1}{r^2}(rP_{63}-P_{84}+P_{10,4}cotg^2\varphi)U_\varphi +$$

$$\frac{P_{82}}{r^2}\frac{\partial^2 U_\theta}{\partial\varphi^2} + \frac{1}{r^2\sin\varphi}(P_{84}+P_{92})\frac{\partial^2 U_\theta}{\partial\varphi\partial\theta} + \frac{1}{r^2\sin^2\varphi}P_{94}\frac{\partial^2 U_\theta}{\partial\theta^2} +$$

$$+\frac{1}{r^2}(P_{82}-P_{85}+P_{10,2})cotg\varphi\frac{\partial U_\theta}{\partial\varphi} + \frac{1}{r^2\sin^2\varphi}(P_{10,4}-P_{95})\cos\varphi\frac{\partial U_\theta}{\partial\theta} +$$

$$+\frac{1}{r^2}(rP_{66}+P_{85}-P_{10,5}cotg^2\varphi)U_\theta - I_2\frac{\partial^2 U_\theta}{\partial t^2} +$$

$$\frac{1}{r^2}(P_{81}+P_{84}-rP_{63})\frac{\partial W}{\partial\varphi} + \frac{1}{r^2\sin\varphi}(P_{91}+P_{94}-rP_{66})\frac{\partial W}{\partial\theta} + \frac{1}{r^2}(P_{81}+P_{84}+P_{10,1}+P_{10,4})cotg\varphi\, W + \tag{C-5}$$

$$\frac{P_{87}}{r^2}\frac{\partial^2 \beta_\varphi}{\partial\varphi^2} + \frac{1}{r^2\sin\varphi}(P_{8,10}+P_{97})\frac{\partial^2 \beta_\varphi}{\partial\varphi\partial\theta} + \frac{1}{r^2\sin^2\varphi}P_{9,10}\frac{\partial^2 \beta_\varphi}{\partial\theta^2} +$$

$$+\frac{1}{r^2}(P_{89}+P_{87}+P_{10,7})cotg\varphi\frac{\partial \beta_\varphi}{\partial\varphi} + \frac{1}{r^2\sin^2\varphi}(P_{99}+P_{10,10})\cos\varphi\frac{\partial \beta_\varphi}{\partial\theta} +$$

$$-\frac{1}{r^2}[P_{89}-P_{10,9}cotg^2\varphi + r^2P_{63}]\beta_\varphi +$$

$$\frac{P_{88}}{r^2}\frac{\partial^2 \beta_\theta}{\partial\varphi^2} + \frac{1}{r^2\sin\varphi}(P_{89}+P_{98})\frac{\partial^2 \beta_\theta}{\partial\varphi\partial\theta} + \frac{1}{r^2\sin^2\varphi}P_{99}\frac{\partial^2 \beta_\theta}{\partial\theta^2} +$$

$$+\frac{1}{r^2}(P_{88}+P_{10,8}+P_{8,10})cotg\varphi\frac{\partial \beta_\theta}{\partial\varphi} + \frac{1}{r^2\sin^2\varphi}P_{10,9}\cos\varphi\frac{\partial \beta_\theta}{\partial\theta} +$$

$$+\frac{1}{r^2}(P_{8,10}-P_{10,10}cotg^2\varphi - r^2P_{66})\beta_\theta - I_3\frac{\partial^2 U_\theta}{\partial t^2}$$

Appendix D

Conical Shells-**The L_i's equations are given as follows:**

$$L_1(U_x,U_\theta,W,\beta_x,\beta_\theta,\overline{P_{ij}})=$$

$$P_{11}\frac{\partial^2 U_x}{\partial x^2}+\frac{(P_{15}+P_{51})}{x\sin\alpha}\frac{\partial^2 U_x}{\partial x\partial\theta}+\frac{P_{55}}{x^2\sin^2\alpha}\frac{\partial^2 U_x}{\partial\theta^2}-\frac{P_{15}}{x^2\sin\alpha}\frac{\partial U_x}{\partial\theta}-I_1\frac{\partial^2 U_x}{\partial t^2}+$$

$$+P_{12}\frac{\partial^2 U_\theta}{\partial x^2}+\frac{(P_{14}+P_{52})}{x\sin\alpha}\frac{\partial^2 U_\theta}{\partial x\partial\theta}+\frac{P_{54}}{x^2\sin^2\alpha}\frac{\partial^2 U_\theta}{\partial\theta^2}-\frac{P_{14}}{x^2\sin\alpha}\frac{\partial U_\theta}{\partial\theta}+$$

$$+\frac{P_{14}}{x\tan\alpha}\frac{\partial W}{\partial x}+\frac{P_{54}}{x^2\sin^2\alpha}\cos\alpha\frac{\partial W}{\partial\theta}-\frac{P_{14}}{x^2\tan\alpha}W+$$

$$+P_{17}\frac{\partial^2\beta_x}{\partial x^2}+\frac{(P_{1,10}+P_{57})}{x\sin\alpha}\frac{\partial^2\beta_x}{\partial x\partial\theta}+\frac{P_{5,10}}{x^2\sin^2\alpha}\frac{\partial^2\beta_x}{\partial\theta^2}-\frac{P_{1,10}}{x^2\sin\alpha}\frac{\partial\beta_x}{\partial\theta}-I_2\frac{\partial^2\beta_x}{\partial t^2}+$$

$$+P_{18}\frac{\partial^2\beta_\theta}{\partial x^2}+\frac{(P_{19}+P_{58})}{x\sin\alpha}\frac{\partial^2\beta_\theta}{\partial x\partial\theta}+\frac{P_{59}}{x^2\sin^2\alpha}\frac{\partial^2\beta_\theta}{\partial\theta^2}-\frac{P_{19}}{x^2\sin\alpha}\frac{\partial\beta_\theta}{\partial\theta}$$

(D-1,2)

$$L_2(U_x,U_\theta,W,\beta_x,\beta_\theta,\overline{P_{ij}})=$$

$$P_{21}\frac{\partial^2 U_x}{\partial x^2}+\frac{(P_{25}+P_{41})}{x\sin\alpha}\frac{\partial^2 U_x}{\partial x\partial\theta}+\frac{P_{45}}{x^2\sin^2\alpha}\frac{\partial^2 U_x}{\partial\theta^2}-\frac{P_{25}}{x^2\sin\alpha}\frac{\partial U_x}{\partial\theta}+$$

$$+P_{22}\frac{\partial^2 U_\theta}{\partial x^2}+\frac{(P_{24}+P_{42})}{x\sin\alpha}\frac{\partial^2 U_\theta}{\partial x\partial\theta}+\frac{P_{44}}{x^2\sin^2\alpha}\frac{\partial^2 U_\theta}{\partial\theta^2}-\frac{P_{24}}{x^2\sin\alpha}\frac{\partial U_\theta}{\partial\theta}-\frac{P_{66}}{x^2\tan^2\alpha}U_\theta-I_1\frac{\partial^2 U_\theta}{\partial t^2}+$$

$$+\frac{P_{24}+P_{63}}{x\tan\alpha}\frac{\partial W}{\partial x}+\frac{P_{44}+P_{66}}{x^2\sin^2\alpha}\cos\alpha\frac{\partial W}{\partial\theta}-\frac{P_{24}}{x^2\tan\alpha}W+$$

$$+P_{27}\frac{\partial^2\beta_x}{\partial x^2}+\frac{(P_{2,10}+P_{47})}{x\sin\alpha}\frac{\partial^2\beta_x}{\partial x\partial\theta}+\frac{P_{4,10}}{x^2\sin^2\alpha}\frac{\partial^2\beta_x}{\partial\theta^2}-\frac{P_{2,10}}{x^2\sin\alpha}\frac{\partial\beta_x}{\partial\theta}+\frac{P_{63}}{x\tan\alpha}\beta_x+$$

$$+P_{28}\frac{\partial^2\beta_\theta}{\partial x^2}+\frac{(P_{29}+P_{48})}{x\sin\alpha}\frac{\partial^2\beta_\theta}{\partial x\partial\theta}+\frac{P_{49}}{x^2\sin^2\alpha}\frac{\partial^2\beta_\theta}{\partial\theta^2}-\frac{P_{29}}{x^2\sin\alpha}\frac{\partial\beta_\theta}{\partial\theta}+\frac{P_{66}}{x\tan\alpha}\beta_\theta-I_2\frac{\partial^2\beta_\theta}{\partial t^2}$$

$$L_3(U_x,U_\theta,W,\beta_x,\beta_\theta,\overline{P_{ij}})=$$

$$-\frac{P_{41}}{x\tan\alpha}\frac{\partial U_x}{\partial x}-\frac{P_{45}}{x^2\sin^2\alpha}\cos\alpha\frac{\partial U_x}{\partial\theta}+$$

$$(-)\frac{(P_{36}+P_{42})}{x\tan\alpha}\frac{\partial U_\theta}{\partial x}-\frac{(P_{44}+P_{66})}{x^2\sin^2\alpha}\cos\alpha\frac{\partial U_\theta}{\partial\theta}+\frac{P_{36}}{x^2\tan\alpha}U_\theta+$$

$$+P_{33}\frac{\partial^2 W}{\partial x^2}+\frac{(P_{36}+P_{63})}{x\sin\alpha}\frac{\partial^2 W}{\partial x\partial\theta}+\frac{P_{66}}{x^2\sin^2\alpha}\frac{\partial^2 W}{\partial\theta^2}-\frac{P_{36}}{x^2\sin\alpha}\frac{\partial W}{\partial\theta}-\frac{P_{44}}{x^2\tan^2\alpha}W-I_1\frac{\partial^2 W}{\partial t^2}+$$

$$+(P_{33}-\frac{P_{47}}{x\tan\alpha})\frac{\partial\beta_x}{\partial x}+(\frac{P_{63}}{x\sin\alpha}-\frac{P_{4,10}\cos\alpha}{x^2\sin^2\alpha})\frac{\partial\beta_x}{\partial\theta}+$$

$$+(P_{36}-\frac{P_{48}}{x\tan\alpha})\frac{\partial\beta_\theta}{\partial x}+\frac{(P_{66}-P_{49}cotg\alpha)}{x\sin\alpha}\frac{\partial\beta_\theta}{\partial\theta}$$

$$L_4(U_x,U_\theta,W,\beta_x,\beta_\theta,\overline{P_{ij}})= \tag{D-3,4}$$

$$P_{71}\frac{\partial^2 U_x}{\partial x^2}+\frac{(P_{75}+P_{10,1})}{x\sin\alpha}\frac{\partial^2 U_x}{\partial x\partial\theta}+\frac{P_{10,5}}{x^2\sin^2\alpha}\frac{\partial^2 U_x}{\partial\theta^2}-\frac{P_{75}}{x^2\sin\alpha}\frac{\partial U_x}{\partial\theta}-I_2\frac{\partial^2 U_x}{\partial t^2}+$$

$$+P_{72}\frac{\partial^2 U_\theta}{\partial x^2}+\frac{(P_{74}+P_{10,2})}{x\sin\alpha}\frac{\partial^2 U_\theta}{\partial x\partial\theta}+\frac{P_{10,4}}{x^2\sin^2\alpha}\frac{\partial^2 U_\theta}{\partial\theta^2}-\frac{P_{74}}{x^2\sin\alpha}\frac{\partial U_\theta}{\partial\theta}+\frac{P_{36}}{x\tan\alpha}U_\theta+$$

$$+(\frac{P_{74}}{x\tan\alpha}-P_{33})\frac{\partial W}{\partial x}+\frac{(\frac{P_{10,4}}{x}cotg\alpha-P_{36})}{x\sin\alpha}\frac{\partial W}{\partial\theta}-\frac{P_{74}}{x^2\tan\alpha}W+$$

$$+P_{77}\frac{\partial^2\beta_x}{\partial x^2}+\frac{(P_{10,7}+P_{7,10})}{x\sin\alpha}\frac{\partial^2\beta_x}{\partial x\partial\theta}+\frac{P_{10,10}}{x^2\sin^2\alpha}\frac{\partial^2\beta_x}{\partial\theta^2}-\frac{P_{7,10}}{x^2\sin\alpha}\frac{\partial\beta_x}{\partial\theta}-P_{33}\beta_x-I_3\frac{\partial^2\beta_x}{\partial t^2}+$$

$$+P_{78}\frac{\partial^2\beta_\theta}{\partial x^2}+\frac{(P_{10,8}+P_{79})}{x\sin\alpha}\frac{\partial^2\beta_\theta}{\partial x\partial\theta}+\frac{P_{10,9}}{x^2\sin^2\alpha}\frac{\partial^2\beta_\theta}{\partial\theta^2}-\frac{P_{79}}{x^2\sin\alpha}\frac{\partial\beta_\theta}{\partial\theta}-P_{36}\beta_\theta$$

$$
\begin{aligned}
&L_5(U_x,U_\theta,W,\beta_x,\beta_\theta,\overline{P_{ij}})=\\
&P_{81}\frac{\partial^2 U_x}{\partial x^2}+\frac{(P_{85}+P_{91})}{x\sin\alpha}\frac{\partial^2 U_x}{\partial x\partial\theta}+\frac{P_{95}}{x^2\sin^2\alpha}\frac{\partial^2 U_x}{\partial\theta^2}-\frac{P_{85}}{x^2\sin\alpha}\frac{\partial U_x}{\partial\theta}+\\
&+P_{82}\frac{\partial^2 U_\theta}{\partial x^2}+\frac{(P_{84}+P_{92})}{x\sin\alpha}\frac{\partial^2 U_\theta}{\partial x\partial\theta}+\frac{P_{94}}{x^2\sin^2\alpha}\frac{\partial^2 U_\theta}{\partial\theta^2}-\frac{P_{84}}{x^2\sin\alpha}\frac{\partial U_\theta}{\partial\theta}+\frac{P_{66}}{x\tan\alpha}U_\theta-I_2\frac{\partial^2 U_\theta}{\partial t^2}+\\
&+\left(\frac{P_{84}}{x\tan\alpha}-P_{63}\right)\frac{\partial W}{\partial x}+\frac{\left(\frac{P_{94}}{x}cotg\alpha-P_{66}\right)}{x\sin\alpha}\frac{\partial W}{\partial\theta}-\frac{P_{84}}{x^2\tan\alpha}W+\\
&+P_{87}\frac{\partial^2\beta_x}{\partial x^2}+\frac{(P_{97}+P_{8,10})}{x\sin\alpha}\frac{\partial^2\beta_x}{\partial x\partial\theta}+\frac{P_{9,10}}{x^2\sin^2\alpha}\frac{\partial^2\beta_x}{\partial\theta^2}-\frac{P_{8,10}}{x^2\sin\alpha}\frac{\partial\beta_x}{\partial\theta}-P_{63}\beta_x+\\
&+P_{88}\frac{\partial^2\beta_\theta}{\partial x^2}+\frac{(P_{98}+P_{89})}{x\sin\alpha}\frac{\partial^2\beta_\theta}{\partial x\partial\theta}+\frac{P_{99}}{x^2\sin^2\alpha}\frac{\partial^2\beta_\theta}{\partial\theta^2}-\frac{P_{89}}{x^2\sin\alpha}\frac{\partial\beta_\theta}{\partial\theta}-P_{66}\beta_\theta-I_3\frac{\partial^2\beta_\theta}{\partial t^2}
\end{aligned}
\tag{D-5}
$$

Appendix E

***Circular Plates*-The five differential equations of motion are defined as follows:**

$$
\begin{aligned}
&L_1(U_r,U_\theta,W,\beta_r,\beta_\theta,\overline{P_{ij}})=\\
&P_{11}\frac{\partial^2 U_r}{\partial r^2}+\frac{1}{r}(P_{51}+P_{15})\frac{\partial^2 U_r}{\partial r\partial\theta}+\frac{P_{55}}{r^2}\frac{\partial^2 U_r}{\partial\theta^2}-\frac{P_{15}}{r^2}\frac{\partial U_r}{\partial\theta}-I_1\frac{\partial^2}{\partial t^2}+\\
&P_{12}\frac{\partial^2 U_\theta}{\partial r^2}+\frac{1}{r}(P_{52}+P_{14})\frac{\partial^2 U_\theta}{\partial r\partial\theta}+\frac{P_{54}}{r^2}\frac{\partial^2 U_\theta}{\partial\theta^2}-\frac{P_{14}}{r^2}\frac{\partial U_\theta}{\partial\theta}+\\
&P_{17}\frac{\partial^2\beta_r}{\partial r^2}+\frac{1}{r}(P_{1,10}+P_{57})\frac{\partial^2\beta_r}{\partial r\partial\theta}+\frac{P_{5,10}}{r^2}\frac{\partial^2\beta_r}{\partial\theta^2}-\frac{P_{1,10}}{r^2}\frac{\partial\beta_r}{\partial\theta}-I_2\frac{\partial^2}{\partial t^2}+\\
&P_{18}\frac{\partial^2\beta_\theta}{\partial r^2}+\frac{1}{r}(P_{19}+P_{58})\frac{\partial^2\beta_\theta}{\partial r\partial\theta}+\frac{P_{59}}{r^2}\frac{\partial^2\beta_\theta}{\partial\theta^2}-\frac{P_{19}}{r^2}\frac{\partial\beta_\theta}{\partial\theta}
\end{aligned}
\tag{E-1}
$$

$$L_2(U_r,U_\theta,W,\beta_r,\beta_\theta,\overline{P_{ij}})=$$

$$P_{21}\frac{\partial^2 U_r}{\partial r^2}+\frac{1}{r}(P_{41}+P_{25})\frac{\partial^2 U_r}{\partial r\partial\theta}+\frac{P_{45}}{r^2}\frac{\partial^2 U_r}{\partial\theta^2}-\frac{P_{25}}{r^2}\frac{\partial U_r}{\partial\theta}+$$
$$P_{22}\frac{\partial^2 U_\theta}{\partial r^2}+\frac{1}{r}(P_{42}+P_{24})\frac{\partial^2 U_\theta}{\partial r\partial\theta}+\frac{P_{44}}{r^2}\frac{\partial^2 U_\theta}{\partial\theta^2}-\frac{P_{24}}{r^2}\frac{\partial U_\theta}{\partial\theta}-I_1\frac{\partial^2}{\partial t^2}+ \tag{E-2}$$
$$P_{27}\frac{\partial^2 \beta_r}{\partial r^2}+\frac{1}{r}(P_{2,10}+P_{47})\frac{\partial^2 \beta_r}{\partial r\partial\theta}+\frac{P_{4,10}}{r^2}\frac{\partial^2 \beta_r}{\partial\theta^2}-\frac{P_{2,10}}{r^2}\frac{\partial \beta_r}{\partial\theta}+$$
$$P_{28}\frac{\partial^2 \beta_\theta}{\partial r^2}+\frac{1}{r}(P_{29}+P_{48})\frac{\partial^2 \beta_\theta}{\partial r\partial\theta}+\frac{P_{49}}{r^2}\frac{\partial^2 \beta_\theta}{\partial\theta^2}-\frac{P_{29}}{r^2}\frac{\partial \beta_\theta}{\partial\theta}-I_2\frac{\partial^2}{\partial t^2}$$

$$L_3(U_r,U_\theta,W,\beta_r,\beta_\theta,\overline{P_{ij}})=$$
$$P_{33}\frac{\partial^2 W}{\partial r^2}+\frac{1}{r}(P_{36}+P_{63})\frac{\partial^2 W}{\partial r\partial\theta}+\frac{P_{66}}{r^2}\frac{\partial^2 W}{\partial\theta^2}-\frac{P_{36}}{r^2}\frac{\partial W}{\partial\theta}-I_1\frac{\partial^2}{\partial t^2}+$$
$$P_{33}\frac{\partial \beta_r}{\partial r}+\frac{P_{63}}{r}\frac{\partial \beta_r}{\partial\theta}+ \tag{E-3}$$
$$P_{36}\frac{\partial \beta_\theta}{\partial r}+\frac{P_{66}}{r}\frac{\partial \beta_\theta}{\partial\theta}$$

$$L_4(U_r,U_\theta,W,\beta_r,\beta_\theta,\overline{P_{ij}})=$$
$$P_{71}\frac{\partial^2 U_r}{\partial r^2}+\frac{1}{r}(P_{10,1}+P_{75})\frac{\partial^2 U_r}{\partial r\partial\theta}+\frac{P_{10,5}}{r^2}\frac{\partial^2 U_r}{\partial\theta^2}-\frac{P_{75}}{r^2}\frac{\partial U_r}{\partial\theta}-I_2\frac{\partial^2}{\partial t^2}+$$
$$P_{72}\frac{\partial^2 U_\theta}{\partial r^2}+\frac{1}{r}(P_{74}+P_{10,2})\frac{\partial^2 U_\theta}{\partial r\partial\theta}+\frac{P_{10,4}}{r^2}\frac{\partial^2 U_\theta}{\partial\theta^2}-\frac{P_{74}}{r^2}\frac{\partial U_\theta}{\partial\theta}+$$
$$(-)P_{33}\frac{\partial W}{\partial r}-\frac{P_{36}}{r}\frac{\partial W}{\partial\theta}+ \tag{E-4}$$
$$P_{77}\frac{\partial^2 \beta_r}{\partial r^2}+\frac{1}{r}(P_{7,10}+P_{10,7})\frac{\partial^2 \beta_r}{\partial r\partial\theta}+\frac{P_{10,10}}{r^2}\frac{\partial^2 \beta_r}{\partial\theta^2}-\frac{P_{7,10}}{r^2}\frac{\partial \beta_r}{\partial\theta}-P_{33}\beta_r-I_3\frac{\partial^2}{\partial t^2}+$$
$$P_{78}\frac{\partial^2 \beta_\theta}{\partial r^2}+\frac{1}{r}(P_{79}+P_{10,8})\frac{\partial^2 \beta_\theta}{\partial r\partial\theta}+\frac{P_{10,9}}{r^2}\frac{\partial^2 \beta_\theta}{\partial\theta^2}-\frac{P_{79}}{r^2}\frac{\partial \beta_\theta}{\partial\theta}-P_{36}\beta_\theta$$

$$L_5(U_r,U_\theta,W,\beta_r,\beta_\theta,\overline{P_{ij}})=$$
$$P_{81}\frac{\partial^2 U_r}{\partial r^2}+\frac{1}{r}(P_{91}+P_{85})\frac{\partial^2 U_r}{\partial r\partial\theta}+\frac{P_{95}}{r^2}\frac{\partial^2 U_r}{\partial\theta^2}-\frac{P_{85}}{r^2}\frac{\partial U_r}{\partial\theta}+$$
$$P_{82}\frac{\partial^2 U_\theta}{\partial r^2}+\frac{1}{r}(P_{84}+P_{92})\frac{\partial^2 U_\theta}{\partial r\partial\theta}+\frac{P_{94}}{r^2}\frac{\partial^2 U_\theta}{\partial\theta^2}-\frac{P_{84}}{r^2}\frac{\partial U_\theta}{\partial\theta}-I_2\frac{\partial^2}{\partial t^2}+$$
$$(-)P_{63}\frac{\partial W}{\partial r}-\frac{P_{66}}{r}\frac{\partial W}{\partial\theta}+ \tag{E-5}$$
$$P_{87}\frac{\partial^2\beta_r}{\partial r^2}+\frac{1}{r}(P_{8,10}+P_{97})\frac{\partial^2\beta_r}{\partial r\partial\theta}+\frac{P_{9,10}}{r^2}\frac{\partial^2\beta_r}{\partial\theta^2}-\frac{P_{8,10}}{r^2}\frac{\partial\beta_r}{\partial\theta}-P_{63}\beta_r+$$
$$P_{88}\frac{\partial^2\beta_\theta}{\partial r^2}+\frac{1}{r}(P_{89}+P_{98})\frac{\partial^2\beta_\theta}{\partial r\partial\theta}+\frac{P_{99}}{r^2}\frac{\partial^2\beta_\theta}{\partial\theta^2}-\frac{P_{89}}{r^2}\frac{\partial\beta_\theta}{\partial\theta}-P_{66}\beta_\theta-I_3\frac{\partial^2}{\partial t^2}$$

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Legends of Figures

FIG.1 a) Differential element of a shell.

b) Definition of shell coordinate system.

FIG.2 Unidirectional lamina and principal coordinate axes.

FIG.3 a) Multidirectional laminate with coordinate notation of individual plies.
b) A fiber reinforced lamina with global and material coordinate systems.

FIG.4) Surface of revolution.

FIG.5 a) Circular cylindrical shell geometry.
b) Positive direction of integrated stress quantities.

FIG.6 Force and moment resultant on a plate element.

FIG.7 Geometry of spherical shell.

FIG.8 Geometry of conical shell.

FIG.9 Circular plate element.

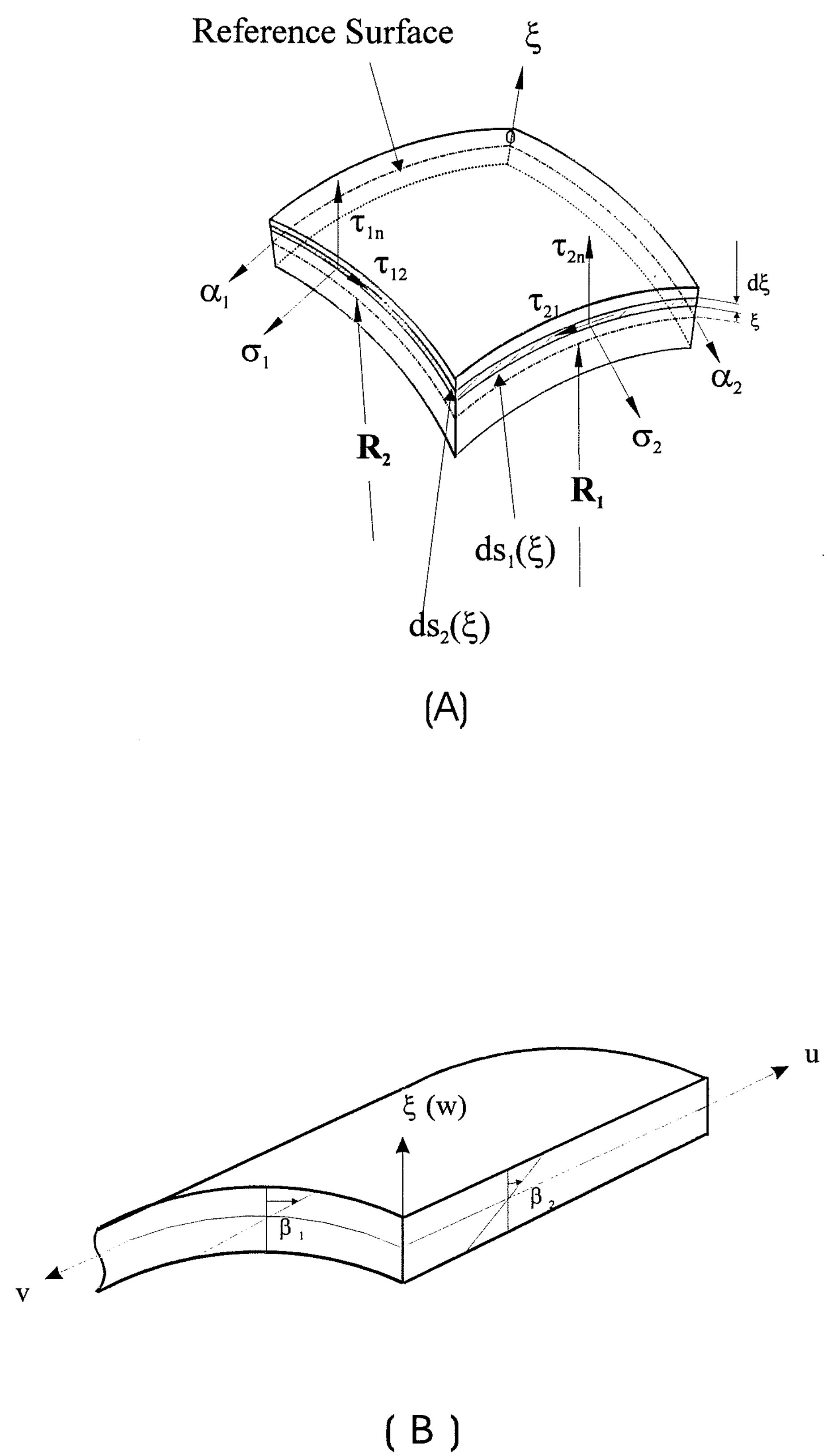


Figure 1: A)Differential element of a shell.
B)Definition of shell coordinate system.

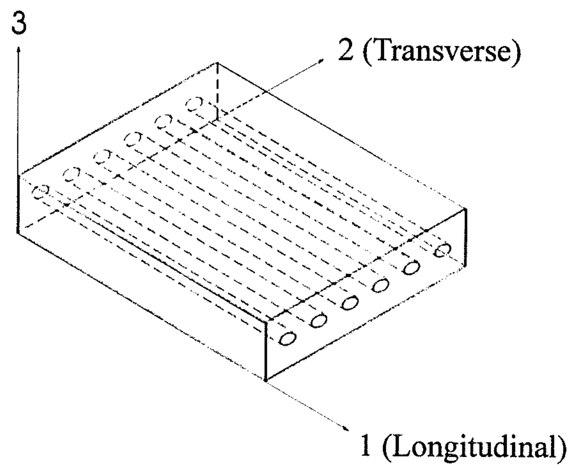
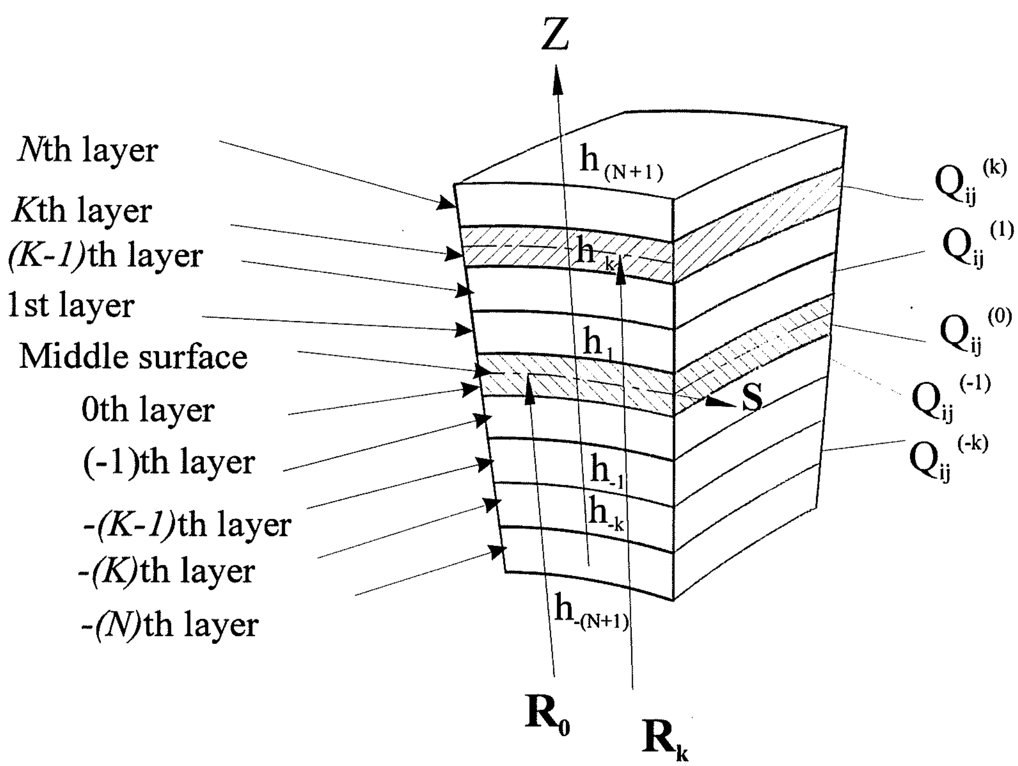
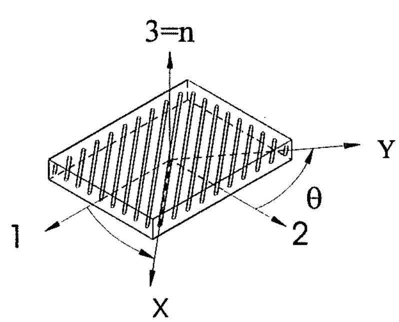


Figure 2: Unidirectional lamina and principal coordinate axes.

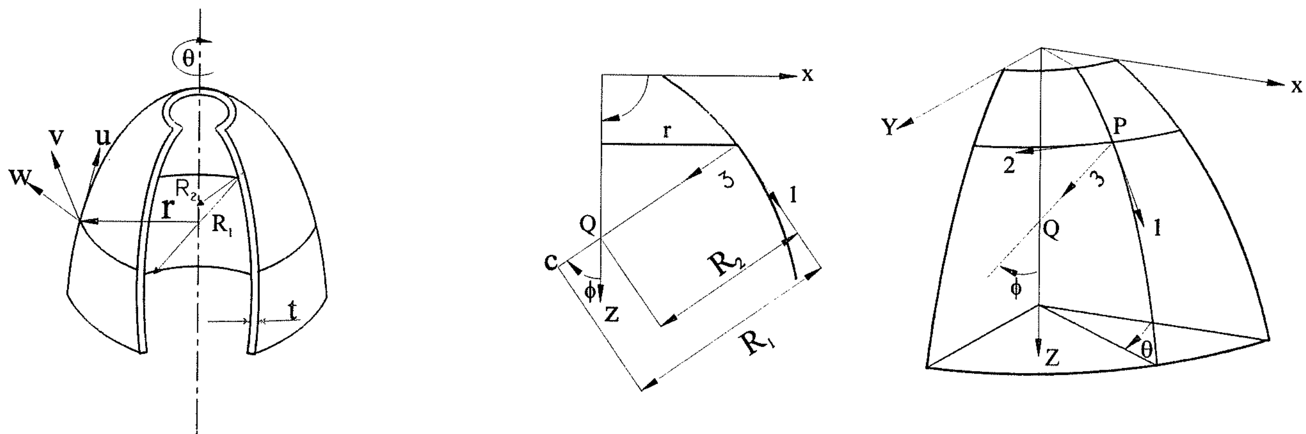


(A)



(B)

Figure 3: A)Multidirectional laminate with coordinate notation of individual plies.
B) A fiber reinforced lamina with global and material coordinate systems.

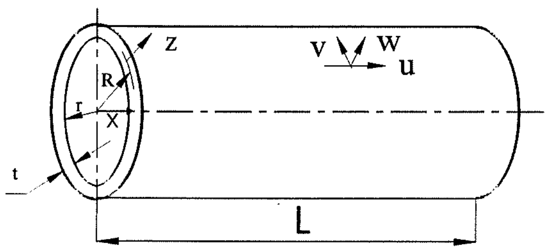


θ=cte : $ds = r_\phi d\phi$ $\quad A_1 = R_\phi \quad A_2 = R_\theta \sin\phi$

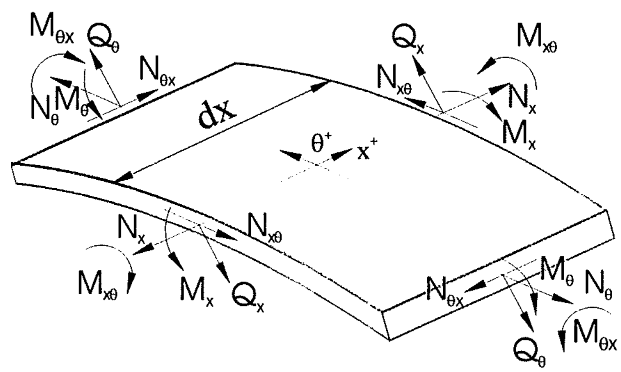
ϕ=cte : $ds = r d\theta$ $\quad R_1 = R_\phi \quad R_2 = R_\theta$

$\alpha_1 = \phi \quad \alpha_2 = \theta$ $\quad \partial A_1/\partial\alpha_2 = 0. \quad \partial A_2/\partial\alpha_1 = R_\theta \cos\phi$

Figure 4: Surface of revolution



(A)



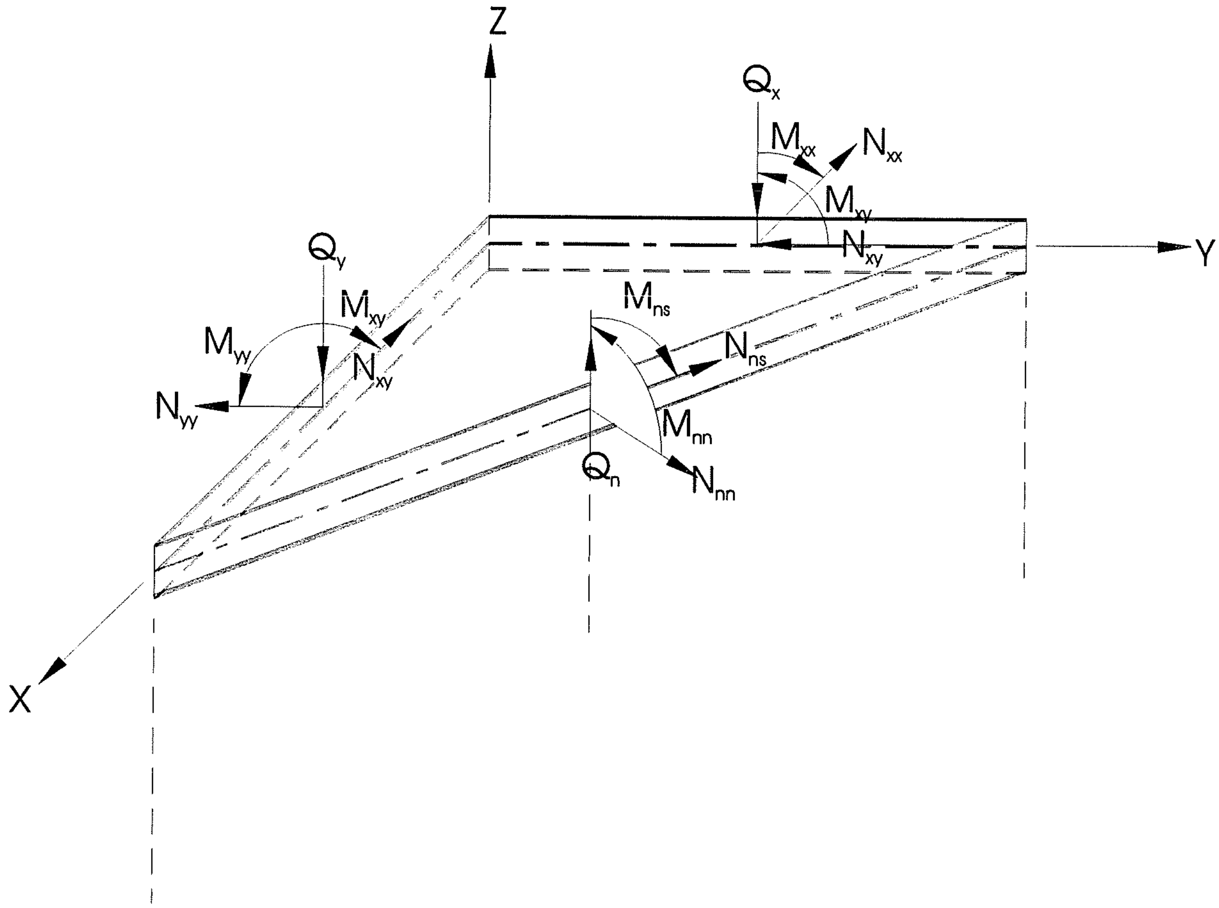
(B)

$\theta=\theta \qquad R_\phi d\phi=dx$

$\phi=x \qquad R\phi=\infty \quad R_\theta=R$

$\phi=\pi/2 \quad \cos\phi=0. \quad \mathrm{Sin}\ \phi=1.$

Figure 5: A)Circular cylindrical shell geometry.
B)Positive direction of integrated stress quantities.



$r \rightarrow \infty \quad \theta \rightarrow \infty \quad rd\theta \rightarrow dy$

Figure 6: Force and moment resultant on a plate element

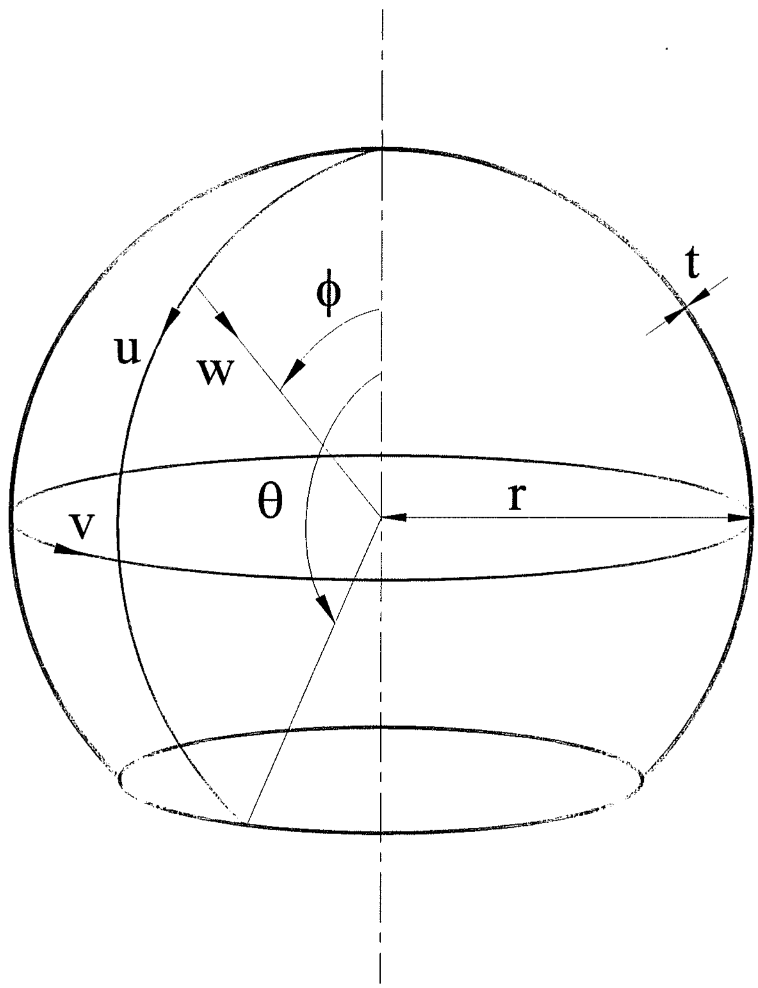
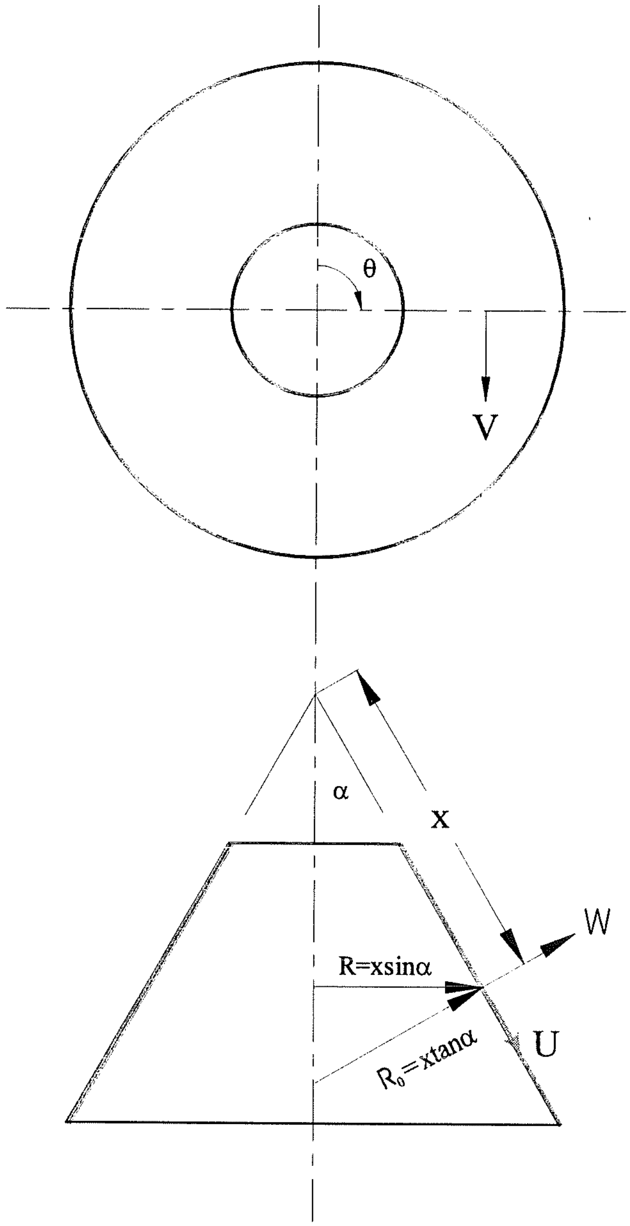


Figure 7: Geometry of spherical shell.

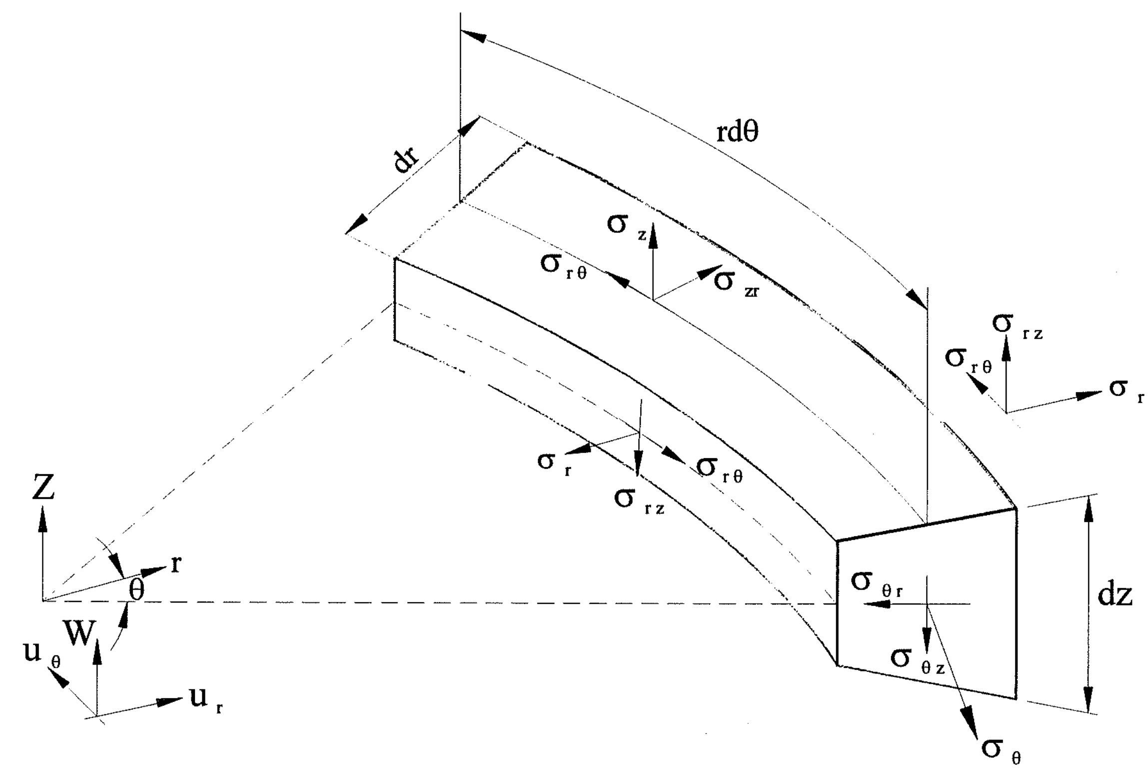


$R_\phi = \infty \qquad \sin\phi = \cos\alpha$

$R_\theta = x\tan\alpha \qquad \cos\phi = \sin\alpha$

$\phi = \pi/2 - \alpha \qquad r_\phi d_\phi \rightarrow dx$

Figure 8: Geometry of conical shell.



$\alpha=\pi/2$
$x=r$

Figure 9: Circular plate element.