DYNAMIC ANALYSIS OF NON-UNIFORM CIRCULAR PLATES

by

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ABSTRACT

The objective of this work is to present a new method for the dynamic and static analysis of thin, elastic, isotropic, non-uniform circular plates. The method is a combination of plate theory and finite element analysis.

The plate is divided into one circular and many annular finite elements. The displacement functions are derived from Sanders' classical plate theory, which is based on Love's first approximation and gives zero strain for small rigid-body motions. These displacement functions satisfy the convergence criteria of the finite element method.

The matrices for mass and stiffness are determined by precise analytical integration, and the method for constructing the equivalent global matrices is given.

The free vibration problem becomes a problem of eigenvalues and eigenvectors. A computer programme has been developed, the convergence criteria have been established, and the frequencies and vibration modes have been computed for different cases.

The results obtained reveal that the frequencies calculated by this method are in good agreement with those obtained by other authors.

SOMMAIRE

Ce rapport présente une nouvelle méthode pour l'analyse dynamique et statique des plaques circulaires minces, élastiques, isotropes et non-uniformes. C'est une méthode hybride d'éléments finis et de la théorie des plaques.

La plaque est divisée en éléments finis de type circulaire et annulaire. Les fonctions de déplacement sont dérivées de la théorie des plaques de Sanders qui est basée sur la première approximation de Love et permet de trouver des déformations nulles pour un mouvement de corps rigide. Ces déplacements satisfont les critères de convergence de la méthode des éléments finis.

Les matrices de masse et de rigidité sont déterminées par intégration analytique exacte. Une méthode d'assemblage des matrices de masse et de rigidité est donnée.

Le problème des vibrations libres se ramène alors à un problème de valeurs et de vecteurs propres. Un programme informatique a été développé, la convergence de la méthode a été établie et on a déterminé les fréquences et les modes de vibration pour plusieurs cas.

Les résultats obtenus nous permettent de conclure qu'il y a une bonne concordance entre les fréquences calculées par la nouvelle méthode et celles obtenues par d'autres auteurs.

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LIST OF SYMBOLS

a	outside radius of an annular or circular plate finite element
a_0	inside radius of an annular plate finite element
B_j $(j=1,4)$	constant in the equation of U
C_j $(j=1,4)$	constant in the equation of V
$\bar{C}_j$ $(j=1,4)$	elements of vector $\{\bar{C}\}$
D	stiffness membrane = $Et/(1-\nu^2)$
E	Young's modulus
J	number of boundary conditions
K	bending stiffness = $Et^3/12(1-\nu^2)$
Ln	Neperian Logarithm
m	number of radial mode
M_x, M_θ, $\bar{M}_{x\theta}$	torque components for a conical shell
M_r, M_θ, $\bar{M}_{r\theta}$	torque components for a circular plate
n	number of circumferential mode
N	number of finite elements
N_x, N_θ, $\bar{N}_{x\theta}$	stress components for a conical shell
N_r, N_θ, $\bar{N}_{r\theta}$	stress components for a circular plate
P_{ij} $(i=1,4;\ j=1,4)$	elements of matrix $[P_0]$
Q_x, Q_r	shear stress components for a conical shell
t	thickness of the plate
U, V, W	radial, tangential, transversal displacement

u_n, v_n, w_n	amplitude of U, V, W associated with the n^{th} circumferential mode number ($n \geqq 2$)
u_1, w_1	amplitude of U, W associated with circumferential mode number $n = 1$
u_0, w_0	amplitude of U,W associated with circumferential mode number $n = 0$
x	coordinate along the generator of the cone
y	coordinate defined by $y = r/a$
y_0	coordinate defined by $y_0 = a_0/a$
α	half-angle at the top of the cone
α_j $(j=1,4)$	defined by equation (3.1.6)
ε_x, ε_θ, $\bar{\varepsilon}_{x\theta}$	deformation of the mean surface of a conical shell
ε_r, ε_θ, $\bar{\varepsilon}_{r\theta}$	deformation of the mean surface of a circular plate
θ	circumferential coordinate
χx, $\chi\theta$, $\bar{\chi}x\theta$	change of curvature and twist of the mean surface of a conical shell
χr, $\chi\theta$, $\bar{\chi}r\theta$	change of curvature and twist of the mean surface of a circular plate
$\lambda_j(j=1,4)$	roots of the characteristic equation
ν	Poisson's ratio
ρ	density of the material of the plate
ω	natural angular frequency
Ω	non-dimensional natural frequency = $\omega a^2 \left(\frac{\rho t}{K}\right)^{\frac{1}{2}}$

LIST OF MATRICES

$[A_n]$	defined by equation (3.2.4)
$[A_1]$	defined by equation (3.2.7)
$[A_0]$	defined by equation (3.2.8)
$[A_{nc}]$	defined by equation (3.3.1)
$[A_{1c}]$	defined by equation (3.3.2)
$[A_{0c}]$	defined by equation (3.3.3)
$[B_n]$	defined by equation (5.1.6)
$[B_1]$	defined by equation (5.1.15)
$[B_{nc}]$	defined by equation (5.2.3)
$[B_{1c}]$	defined by equation (5.2.10)
$[BB_n]$	defined by equation (4.1.2)
$[BB_1]$	defined by equation (4.1.5)
$[BB_0]$	defined by equation (4.1.7)
$[BB_{nc}]$	defined by equation (4.4.2)
$[BB_{1c}]$	defined by equation (4.2.4)
$[BB_{0c}]$	defined by equation (4.2.6)
$\{\underline{C}\}$	arbitrary constants vector
$[C_n]$	defined by equation (5.1.6)
$[C_1]$	defined by equation (5.1.15)
$[C_{nc}]$	defined by equation (5.2.3)
$[C_{1c}]$	defined by equation (5.2.10)
$[D_n]$	defined by equation (5.1.7)
$[D_1]$	defined by equation (5.1.16)
$[D_{nc}]$	defined by equation (5.2.4)

$[D_{1c}]$	defined by equation (5.2.11)
$[E_n]$	defined by equation (5.1.7)
$[E_1]$	defined by equation (5.1.16)
$[E_{nc}]$	defined by equation (5.2.4)
$[E_{1c}]$	defined by equation (5.2.11)
$[F_n]$	defined by equation (5.1.12)
$[F_{nc}]$	defined by equation (5.2.7)
$[F_{1c}]$	defined by equation (5.2.14)
$[G_n]$	defined by equation (5.1.6)
$[G_1]$	defined by equation (5.1.15)
$[G_0]$	defined by equation (5.1.20)
$[G_{nc}]$	defined by equation (5.2.3)
$[G_{1c}]$	defined by equation (5.2.10)
$[G_{0c}]$	defined by equation (5.2.16)
$[H]$	defined by equation (3.1.3)
$[H_n]$	defined by equation (5.1.12)
$[H_{nc}]$	defined by equation (5.2.7)
$[H_{1c}]$	defined by equation (5.2.14)
$[k]$	elementary stiffness matrix
$[K]$	global stiffness matrix
$[m]$	elementary mass matrix
$[M]$	global mass matrix
$[P]$	elasticity matrix ($n \geq 1$)
$[P_0]$	elasticity matrix ($n = 0$)
$[Q_n]$	defined by equation (4.1.1)
$[Q_1]$	defined by equation (4.1.4)

$[Q_0]$	defined by equation (4.1.6)
$[Q_{nc}]$	defined by equation (4.2.11)
$[Q_{1c}]$	defined by equation (4.2.3)
$[Q_{0c}]$	defined by equation (4.2.5)
$[R_n]$	defined by equation (3.2.1)
$[R_1]$	defined by equation (3.2.7)
$[R_0]$	defined by equation (3.2.8)
$[R_{nc}]$	defined by equation (3.3.1)
$[R_{1c}]$	defined by equation (3.3.2)
$[R_{0c}]$	defined by equation (3.3.3)
$[S_n]$	defined by equation (5.1.10)
$[S_1]$	defined by equation (5.1.18)
$[S_0]$	defined by equation (5.1.22)
$[S_{nc}]$	defined by equation (5.2.7)
$[S_{1c}]$	defined by equation (5.2.14)
$[S_{0c}]$	defined by equation (5.2.17)
$[ST_n]$	defined by equation (4.3.3)
$[ST_1]$	defined by equation (4.3.5)
$[ST_0]$	defined by equation (4.3.7)
$[ST_{nc}]$	defined by equation (4.4.2)
$[ST_{1c}]$	defined by equation (4.4.4)
$[ST_{0c}]$	defined by equation (4.4.6)
$[T_n]$	defined by equation (3.2.1)
$[T_1]$	defined by equation (3.2.7)
$[T_0]$	defined by equation (3.2.8)
$[T_{nc}]$	defined by equation (3.3.1)

$[T_{1c}]$	defined by equation (3.3.2)
$[T_{0c}]$	defined by equation (3.3.3)
$\{\delta_i\}$	vector of degrees of freedom at node i
$\{\varepsilon\}$	deformation vector
$\{\sigma\}$	stress vector

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CHAPTER I

INTRODUCTION

1.1 General

Circular plates are widely used in engineering. They are used, for example, by the aerospace and aeronautical industry in aircraft fuselage, rockets and turbo-jets; by the nuclear industry in reactor walls; by the marine industry for ship and submarine parts; by the petroleum industry in holding tanks; and by civil engineers in domes and thin shells. Knowledge of the free vibration characteristics of these structures is important not only for the researcher who wishes to understand their behaviour, but also for the engineer whose duty it is to foresee and to prevent any failure which may occur in the course of the industrial use of such structures.

Research into the vibration of circular plates began in the 18th century [1, 2, 3]: in 1766, Euler formulated the first mathematical approximation of plate membrane theory. The German physician Chladin later found the vibration modes, and Lagrange developed a differential equation for free vibrations.

Kirchoff (1877) is considered to be the founder of the plate theory which combines membrane effects with bending by analysing plates with large deflections. He concluded that non-linear expressions could not be ignored and showed that the natural frequencies of plates can be determined by the virtual work method.

Love applied Kirchoff's work to thick plates, whilst Timochenko made significant contributions to the work on circular plates with large deflections.

Foppl, Volmin and Panov worked on non-linear plate theory. The final form of the differential equation for plates with large deflections was developed by von Karman.

Chien Wien-Zang introduced the perturbation method to solve equations for plates with large deflections. Hodge extended elastic plate theory into the plastic domain.

Leissa collected the work of several researchers into one excellent book [4], which provides approximately 500 references. More recently, Leissa and Narita [5] have studied the influence of Poisson's ratio on the natural frequencies of a simply-supported circular plate.

The Japanese Irie, Yamada and Aomura [6] have determined the natural frequencies of clamped, simply-supported and free plates for thickness-to-plate radius ratios varying from 0.01 to 0.25. They used Mindlin's theory which takes rotational inertia into account. Itao and Crandall [7] have calculated the 701 first modes of vibration of a free circular plate.

The analysis of non-uniform plates of varying thickness has been carried out by several authors. Celep [8] used the initial functions method to determine the first and second vibration modes.

Irie and Yamada [9] used a spline technique in studying annular plates of which the thickness varied linearly, parabolically and exponentially. They also used Ritz' method to solve plate systems numerically.

Sato and Shimzu [10] used the transfer matrix method to study the linear and non-linear behaviour of plates of varying thickness.

Narita [11] and Gorman [12] studied anisotropic plates. Bolotin formulated the problem of the dynamic stability of mechanical systems, and Lepore and Shah [13] applied this theory to circular plates, while

Tani and Nakamura [14] studied the dynamic stability of annular plates.

In order to analyse complex plates, it has been necessary to employ new methods, the best known of which is the finite element method.

Numerous general computer programmes, such as NASTRAN, SAP, ADINA and ABAQUS, are available for the industrial use of the finite element method, principally in the domain of the mechanics of solids.

In general, triangular and square elements are used [15], where the displacement functions are polynomial, although curved elements [16] have been found to account more precisely for the geometry of the surface. The analytical formulation of these elements is complex.

One of the most important criteria in determining the versatility of a method is the capacity to predict, with precision, both the high and the low frequencies. This criterion demands the use of a great many elements in the finite element method, and, in order to meet it, the research group working under Dr. Lakis has developed a new type of finite element, a hybrid wherein the displacement functions in the finite element method are derived from Sanders' classical shell theory [17]. This method has been applied with satisfactory results to the dynamic linear and non-linear analysis of cylindrical [18-26], conical [27, 28] and spherical [29], isotropic and anisotropic, uniform and non-uniform shells, both empty and liquid-filled. This method also has the advantage of giving good low frequencies, as well as high, with a small number of finite elements.

1.2 Research Objectives

We shall outline a new method for determining the natural frequencies of a non-uniform circular or annular plate under different boundary conditions. Furthermore, by this method we shall be able to establish the stresses at each point on the plate.

The method used here is a combination of circular plate theory and finite element analysis. It is an extension of the hybrid finite element theory developed by the research group directed by Dr. Lakis for different types of shell of revolution.

We first determine the plate equations using conical shell theory; second, we derive the displacement functions of plate theory and determine the stiffness and mass matrices by the finite element method. In this part of the study, we develop two new types of finite element, the first type being a circular plate and the second an annular plate, for circumferential modes $n = 0$, $n = 1$ and $n \geqq 2$.

This work is a sequel to that which was begun by Dr. Lakis. The main objectives are:

- In terms of theory: To verify and modify (where necessary)
  a) the characteristic equation
  b) the displacement functions
  c) the stiffness and mass matrices

- In terms of programming: To write a computer programme to do the dynamic and static analysis of any circular or annular plate for different circumferential modes and a variety of boundary conditions.

1.3 Report Outline

The following section contains a summary of the eight chapters comprising this report.

I Introduction

II Fundamental equations for thin, elastic, isotropic, circular plates

III Displacement functions stemming from the solution of the characteristic equation

IV Deformation and stress resultant matrices

V, VI Determination of the stiffness and mass matrices and formulation of the problem in eigenvalues

VII Numerical calculations and the different tests to which the method has been submitted in order to establish its validity

VIII Conclusion and recommendations

CHAPTER II

FUNDAMENTAL EQUATIONS FOR CIRCULAR PLATES

2.1 Hypothesis

The problem is formulated on the basis of Love's first approximation [30].

The assumptions are:

- The plate is thin.
- The displacements of the plate are sufficiently small to conserve the linearity of the equations used.
- The normal stresses acting along an axis perpendicular to the mean surface are negligible.
- The normals to the mean surface remain perpendicular after deformation and do not undergo any lengthening.
- Rotational inertia terms are ignored.

2.2 Equilibrium equations

To study the equilibrium of the circular plate, taking into account membrane effects as well as bending effects, we use Sanders' [17] equation for conical cones and assume a half-angle at the top of the cone equal to 90°.

It should be remembered that these equations are based on Love's [30] first approximation, and show zero deformation due to rigid-body motion. This is not the case with other theories.

The equations for conical shells are summarised in references [27] and [28].

The geometry of the mean surface of the plate studied and the coordinates used are shown in Figure 2.1.

The unit vectors of the membrane force, the shear forces and the moment with reference to the mean surface are indicated in Figure 2.2.

Taking the half-angle at the top of the cone to be equal to 90° with $x = r$, equations (A-1.2) give the equilibrium equations of a circular plate:

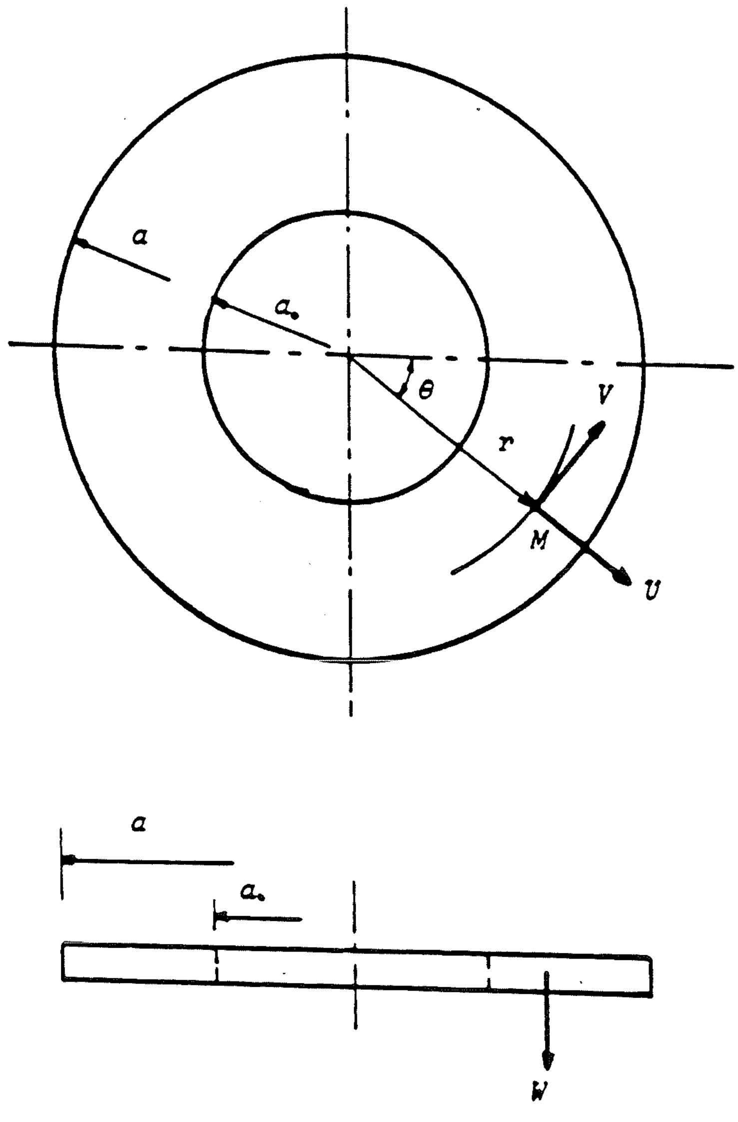
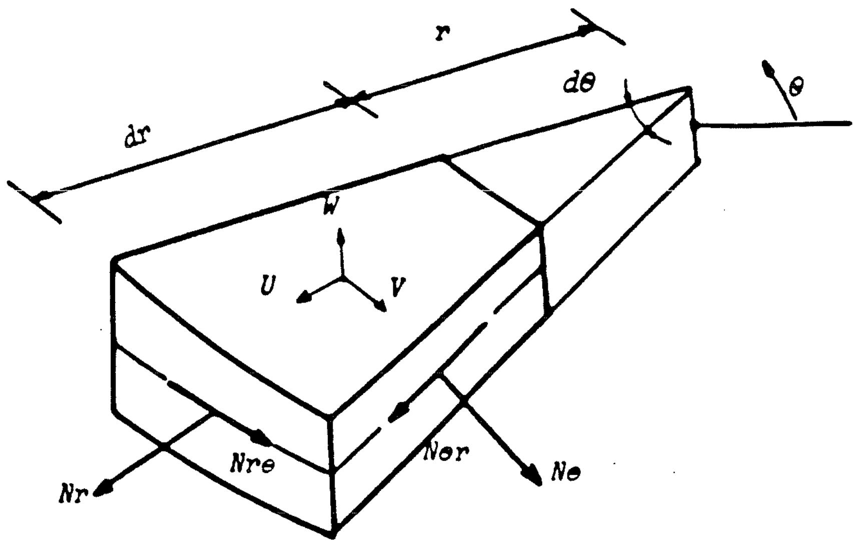


Figure 2.1: Geometry of the mean surface of a circular plate.



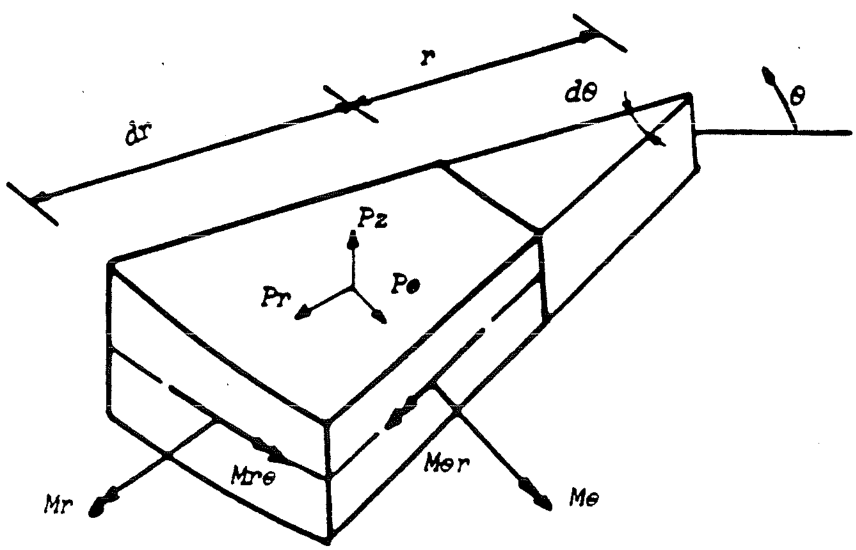


Figure 2.2: Differential element for a circular plate
(a) Displacements and stress components
(b) Torque components and load factors.

$$r \frac{\partial N_r}{\partial_r} + N_r + \frac{\partial \bar{N}_{r\theta}}{\partial \theta} - N_\theta = 0 \qquad \text{(a)}$$

$$r \frac{\partial \bar{N}_{r\theta}}{\partial_r} + 2\bar{N}_{r\theta} + \frac{\partial \bar{N}_\theta}{\partial \theta} = 0 \qquad \text{(b)} \qquad (2.2.1)$$

$$r^2 \frac{\partial^2 M_r}{\partial r^2} + 2 \frac{\partial M_r}{\partial_r} + 2 \frac{\partial^2 \bar{M}_{r\theta}}{\partial_r \partial \theta} + \frac{2}{r} \frac{\partial \bar{M}_{r\theta}}{\partial \theta} + \frac{1}{r} \frac{\partial^2 M_\theta}{\partial \theta^2} - \frac{\partial M_\theta}{\partial \theta} = 0 \qquad \text{(c)}$$

where N_r, N_θ, $\bar{N}_{r\theta}$, M_r, $\bar{M}_\theta$ and $M_{r\theta}$ are the stress components, and r and θ are the coordinates of the plate.

2.3 Kinematic relationships

In considering $\alpha = 90°$ in equations (A-1.3), the relationship between the deformation and the displacements for a circular plate can be written as follows:

$$\begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ 2\overline{\epsilon_{r\theta}} \\ \chi_r \\ \chi_\theta \\ 2\overline{\chi_{r\theta}} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial U}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial V}{\partial \theta} + \dfrac{U}{r} \\ \dfrac{1}{r}\dfrac{\partial U}{\partial \theta} + r\dfrac{\partial}{\partial r}\left(\dfrac{V}{r}\right) \\ -\dfrac{\partial^2 W}{\partial r^2} \\ -\dfrac{1}{r}\dfrac{\partial W}{\partial r} - \dfrac{1}{r^2}\dfrac{\partial^2 W}{\partial \theta^2} \\ -2\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial W}{\partial \theta}\right) \end{Bmatrix} \qquad (2.3.1)$$

where U is the radial displacement, V is the tangential displacement and W is the transversal displacement.

2.4 Constitutive equations

For an isotropic and elastic material, the constitutive equations which link the stress components to the deformations are:

$$\begin{Bmatrix} N_r \\ N_\theta \\ \overline{N}_{r\theta} \\ M_r \\ M_\theta \\ \overline{M}_{r\theta} \end{Bmatrix} = [P] \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ 2\overline{\epsilon_{r\theta}} \\ \chi_r \\ \chi_\theta \\ 2\overline{\chi_{r\theta}} \end{Bmatrix} \qquad (2.4.1)$$

where [P] is the elasticity matrix given in Table (6) of Appendix (A-2).

By substituting (2.3.1) and (2.4.1) in the equilibrium equations (2.2.1), we obtain new equations (2.4.2) in terms of the radial, tangential and transversal displacements (U, V, W) of the mean surface of the plate:

$$rU'' + U' + \left(\frac{1-\upsilon}{2}\right)\frac{\ddot{U}}{r} - \frac{U}{r} + \left(\frac{1+\upsilon}{2}\right)\dot{V}' - \left(\frac{3-\upsilon}{2}\right)\frac{\dot{V}}{r} = 0 \qquad \text{(a)}$$

$$\left(\frac{1+\upsilon}{2}\right)\dot{U}' + \left(\frac{3-\upsilon}{2}\right)\frac{\dot{U}}{r} + \left(\frac{1-\upsilon}{2}\right) r V'' + \left(\frac{1-\upsilon}{2}\right) V' + \frac{\ddot{V}}{r} - \left(\frac{1-\upsilon}{2}\right)\frac{V}{r} = 0 \quad (2.4.2)$$

$$\text{(b)}$$

$$W'''' + 2\frac{\ddot{W}''}{r^2} + \frac{\ddddot{W}}{r^4} + 2\frac{W'''}{r} - 2\frac{\ddot{W}'}{r^3} - \frac{W''}{r^2} + 4\frac{\dot{W}'}{r^4} + \frac{W'}{r^3} = 0 \qquad \text{(c)}$$

The terms (') and (•) represent [∂( )/ ∂r] and [∂( )/ ∂θ] respectively.

By solving these equations it is possible to derive the displacement functions in terms of the nodal displacements.

CHAPTER III

DISPLACEMENT FUNCTIONS

3.1 Introduction

Two types of finite element will be developed, the first being an element of the circular plate type and the second an element of the annular plate type (Figure 3.1). In this way, circular plate theory can be used to determine displacement functions.

The nodal displacements are (U, W, $\frac{dW}{dr}$, V) where <U> is the radial displacement, <W> is the transversal displacement, $\langle\frac{dW}{dr}\rangle$ is the rotation and <V> is the tangential displacement.

As the plate is circular, the displacements are periodic as a function of θ and can be developed in a Fourier series:

$$
\begin{aligned}
U(r,\theta) &= u_n(r) \cos n\theta \\
W(r,\theta) &= w_n(r) \cos n\theta \qquad (3.1.1) \\
V(r,\theta) &= v_n(r) \sin n\theta
\end{aligned}
$$

n is the number of the circumferential mode

u_n, w_n, v_n are functions solely of r.

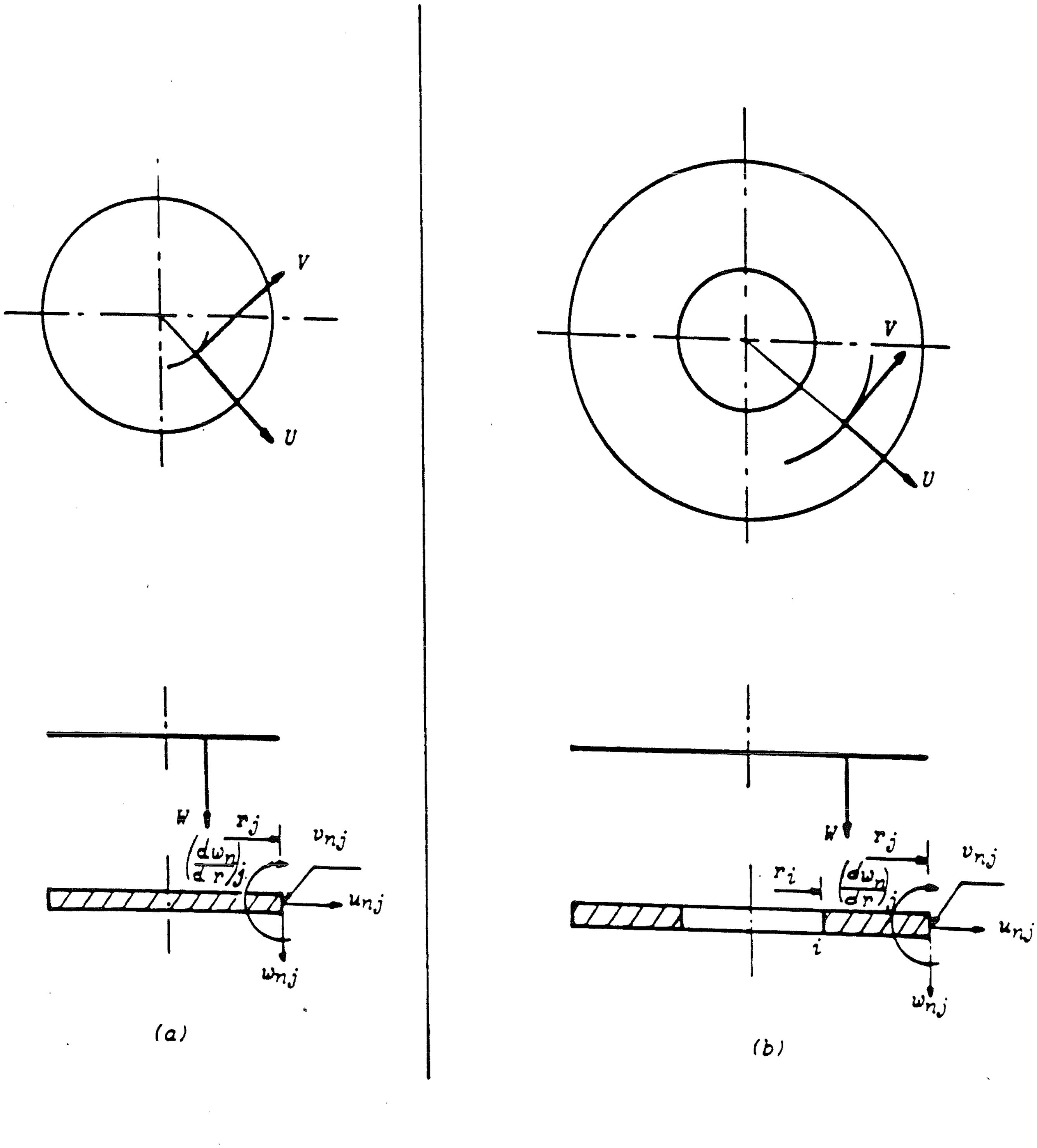


Figure 3.1: Displacements and degrees of freedom for one node
(a) Finite element of the 'circular plate' type
(b) Finite element of the 'annular plate' type.

By analysis of the equilibrium equations (2.2.1), it can be seen that (a) and (b) show the membrane effect while (c) shows the bending effect. These two groups of equations (a, b) and (c) are independent of each other and can be solved separately.

The generalised forms of the displacements u_n and v_n are:

$$u_n = C\, y^{\frac{\lambda - 1}{2}}$$

$$v_n = B\, y^{\frac{\lambda - 1}{2}} \qquad (3.1.2)$$

where λ is the solution to the characteristic equation

$y = \frac{r}{a}$; r is the radius of the plate

a is the outside radius of the plate

By substituting (3.1.2) is equations (a) and (b) of system (2.2.1), we obtain a system of homogeneous equations in B and C of the form

$$[H] \begin{Bmatrix} C \\ B \end{Bmatrix} = \{0\} \qquad (3.1.3)$$

where [H] is a second-order square matrix, the terms of which are functions of λ. This matrix is given in Table (5) of Appendix (A-2). For a non-trivial solution to system (3.1.3), the determinant of [H] should be zero, giving the following characteristic equation:

$$\lambda^4 - 4\lambda^3 - 2(1+4n^2)\ \lambda^2 + 4(3+4n^2)\ \lambda + (9-40n^2 + 16n^4) = 0 \qquad (3.1.4)$$

The characteristic equation has 4 roots, all of which are real.

From the sum of the 4 values for λ_j, we can obtain the complete solution for U and V. Each value of λ_j involves two constants, B_j and C_j. As the values for B_j and C_j are not independent, one can express B_j as a function of C_j.

$$B_j = \alpha_j C_j \qquad (3.1.5)$$

By substituting (3.1.5) in (3.1.3), we can find α_j.

Thus we obtain

$$\alpha_j = \frac{-\frac{1}{4}(\lambda_j - 1)^2 + (\frac{1-\upsilon}{2})\ n^2 + 1}{(\frac{1-\upsilon}{4})\ n\ (\lambda_j - 1) - n\ (\frac{3-\upsilon}{2})} \qquad (3.1.6)$$

3.2 <u>Displacement functions for a finite element of the annular plate type</u>

3.2.1 <u>Circumferential mode n ≧ 2</u>

In matrix form, displacements U, V and W can be written as follows:

$$\begin{Bmatrix} U \\ W \\ V \end{Bmatrix} = [T_n]\ [R_n]\ \{C\} \tag{3.2.1}$$

where $[T_n]$ is the (3 x 3) matrix given in Table (1) of Appendix (A-2).

$[R_n]$ is the (3 x 8) matrix given in Table (3) of Appendix (A-2).

The vector {C} is given by:

$$\{C\} = \begin{Bmatrix} \bar{C}_1 \\ \vdots \\ \bar{C}_4 \end{Bmatrix} \tag{3.2.2}$$

Constants $\bar{C}_j$ are eliminated in favour of displacements at the nodes of the elements.

The displacement field at the node can be defined as:

$$\{\delta_i\} = \{u_{ni},\ w_{ni},\ (\frac{dw_n}{dr})_i\ ,\ v_{ni}\}^t \quad \text{at node } i \tag{3.2.3}$$

$$\{\delta_j\} = \{u_{nj},\ w_{nj},\ (\frac{dw_n}{dr})_j\ ,\ v_{nj}\}^t \quad \text{at node } j$$

$\{\delta_i\}$ and $\{\delta_j\}$ can be expressed as a function of constants $\{C_j\}$ in the following manner:

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [A_n] \{C\} \tag{3.2.4}$$

$[A_n]$ is the (8 x 8) matrix given in Table (2) of Appendix (A-2).

Thus we have $$\{C\} = [A_n]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{3.2.5}$$

Equation (3.2.1) becomes

$$\begin{Bmatrix} U \\ W \\ V \end{Bmatrix} = [T_n] \ [R_n] \ [A_n]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [N_n] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{3.2.6}$$

where $[N_n]$ is the displacement matrix.

3.2.2 Circumferential mode n = 1

In this case, the plate behaves like a beam and is not deflected in the tangential direction. We therefore obtain:

$$\begin{Bmatrix} U \\ W \end{Bmatrix} = [T_1] \ [R_1] \ [A_1]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{3.2.7}$$

where $\{\delta i\} = \{u_{1i},\ w_{1i},\ (\frac{dw_1}{dr})_i\}^t$

$\{\delta j\} = \{u_{1j},\ w_{1j},\ (\frac{dw_1}{dr})_j\}^t$

$[T_1]$ is the (2 x 2) matrix given in Table (2) of Appendix (A-2).

$[R_1]$ is the (2 x 6) matrix given in Table (12) of Appendix (A-2).

$[A_1]$ is the (6 x 6) matrix given in Table (5) of Appendix (A-2).

3.2.3 Circumferential mode n = 0

In the particular case of axisymmetric vibration, the displacements are functions solely of r, the tangential displacement V is zero since $V = v_n \sin n\theta$ where n = 0.

The two other displacements can be written as follows:

$$\begin{Bmatrix} U \\ W \end{Bmatrix} = [T_0]\ [R_0]\ [A_0]^{-1} \begin{Bmatrix} \delta i \\ \delta j \end{Bmatrix} \qquad (3.2.8)$$

where

$\{\delta i\} = \{u_{0i},\ w_{0i},\ (\frac{dw_0}{dr})_i\}$

$$\{\delta j\} = \{u_{0j},\ w_{0j},\ (\frac{dw_0}{dr})_j\}$$

$[T_0]$ is the (2 x 2) matrix given in Table (1) of Appendix (A-2).

$[R_0]$ is the (2 x 6) matrix given in Table (3) of Appendix (A-2).

$[A_0]$ is the (6 x 6) matrix given in Table (2) of Appendix (A-2).

3.3 Displacement functions for a finite element of circular plate type

3.3.1 Circumferential mode $n \geq 2$

Since the circular plate element has a single node, the displacement field is defined by

$$\{\delta j\} = \{u_{nj},\ w_{nj},\ (\frac{dw_n}{dr})_j\ ,\ v_{nj}\}^t$$

The three displacements U, V and W can therefore be written as follows:

$$\begin{Bmatrix} U \\ W \\ V \end{Bmatrix} = [T_{nc}]\ [R_{nc}]\ [A_{nc}]^{-1} \begin{Bmatrix} \delta_i \\ \\ \delta_j \end{Bmatrix} \qquad (3.3.1)$$

where $[T_{nc}]$ is the (3 x 3) matrix given in Table (1) of Appendix (A-2).

$[R_{nc}]$ is the (3 x 4) matrix given in Table (3) of Appendix (A-2).

$[A_{nc}]$ is the (4 x 4) matrix given in Table (2) of Appendix (A-2).

3.3.2 Circumferential mode n = 1

As with the annular plate type element, we do not take the tangential displacement V into account. Displacements U and W can therefore be written as follows:

$$\begin{Bmatrix} U \\ W \end{Bmatrix} = [T_{1c}]\ [R_{1c}]\ [A_{1c}]^{-1}\ \{\delta j\} \qquad (3.3.2)$$

where $\{\delta j\} = \{u_{1j},\ w_{1j},\ (\frac{dw_1}{dr})_j\}^t$

$[T_{1c}]$ is the (2 x 2) matrix given in Table (2) of Appendix (A-2).

$[R_{1c}]$ is the (2 x 3) matrix given in Table (3) of Appendix (A-2).

$[A_{1c}]$ is the (3 x 3) matrix given in Table (2) of Appendix (A-2).

3.3.3 Circumferential mode n = 0

For the same reasons as those cited in paragraph (3.2.3), the displacement V is zero; the displacements U and W can therefore be written as follows:

$$\begin{Bmatrix} U \\ W \end{Bmatrix} = [T_{0c}]\ [R_{0c}]\ [A_{0c}]^{-1}\ \{\delta_j\} \tag{3.3.3}$$

where $\{\delta_j\} = \{u_{0j},\ w_{0j},\ (\frac{dw_0}{dr})_j\}^t$

$[T_{0c}]$ is the (2 x 2) matrix given in Table (1) of Appendix (A-2).

$[R_{0c}]$ is the (2 x 3) matrix given in Table (3) of Appendix (A-2).

$[A_{0c}]$ is the (3 x 3) matrix given in Table (2) of Appendix (A-2).

CHAPTER IV

DEFORMATION AND STRESS COMPONENT MATRICES

4.1 Deformation matrix for a finite element of the annular plate type

4.1.1 Circumferential mode $n \geqq 2$

By substituting equation (3.2.6) in the kinematic equations (2.3.1), we obtain

$$\{\epsilon\} = \left\{\begin{matrix} \epsilon_r \\ \epsilon_\theta \\ 2\overline{\epsilon_{r\theta}} \\ \chi_r \\ \chi_\theta \\ 2\overline{\chi_{r\theta}} \end{matrix}\right\} = \begin{bmatrix} [T_n] & [0] \\ [0] & [T_n] \end{bmatrix} [Q_n]\ [A_n]^{-1} \left\{\begin{matrix} \delta_i \\ \delta_j \end{matrix}\right\} \tag{4.1.1}$$

Matrices $[T_n]$, $[Q_n]$ and $[A_n]$ are given in Appendix (A-2).

The deformation matrix is defined as follows:

$$[BB_n] = \begin{bmatrix} [T_n] & [0] \\ [0] & [T_n] \end{bmatrix} [Q_n]\ [A_n]^{-1} \tag{4.1.2}$$

The deformation vector can be written:

$$\{\in\} = [BB_n] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.1.3}$$

4.1.2 Circumferential mode n = 1

By substituting equation (3.2.7) in the kinematic equations (2.3.1), we obtain:

$$\{\in\} = \begin{Bmatrix} \in_r \\ \in_\theta \\ 2\overline{\in_{r\theta}} \\ \chi_r \\ \chi_\theta \\ 2\overline{\chi_{r\theta}} \end{Bmatrix} = \begin{bmatrix} [T_1] & [0] \\ [0] & [T_1] \end{bmatrix} [Q_n]\ [A_n]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [BB_1] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.1.4}$$

Matrices $[T_1]$, $[Q_1]$ and $[A_1]$ are given in Appendix (A-2).

The deformation matrix is defined as follows:

$$[BB_1] = \begin{bmatrix} [T_1] & [0] \\ [0] & [T_1] \end{bmatrix} [Q_1]\ [A_1]^{-1} \tag{4.1.5}$$

4.1.3 Circumferential mode n = 0

By substituting equation (3.2.8) in the kinematic equations (2.3.1), we obtain:

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \chi_r \\ \chi_\theta \end{Bmatrix} = \begin{bmatrix} [T_0] & [0] \\ [0] & [T_0] \end{bmatrix} [Q_0]\ [A_0]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [BB_0] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.1.6}$$

Matrices $[T_0]$, $[Q_0]$ and $[A_0]$ are given in Appendix (A-2).

The deformation matrix is defined as follows:

$$[BB_0] = \begin{bmatrix} [T_0] & [0] \\ [0] & [T_0] \end{bmatrix} [Q_0]\ [A_0]^{-1} \tag{4.1.7}$$

4.2 Deformation matrix for a finite element of the circular plate type

4.2.1 Circumferential mode $n \geqq 2$

By substituting equations (3.3.1) in the kinematic equations (2.3.1), we obtain:

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ 2\overline{\epsilon_{r\theta}} \\ \chi_r \\ \chi_\theta \\ 2\overline{\chi_{r\theta}} \end{Bmatrix} = \begin{bmatrix} [T_{nc}] & [0] \\ [0] & [T_{nc}] \end{bmatrix} [Q_{nc}]\ [A_{nc}]^{-1} \{\delta_j\} \tag{4.2.1}$$

Matrices $[T_{nc}]$, $[Q_{nc}]$ and $[A_{nc}]$ are given in Appendix (A-2).

The deformation matrix is defined as follows:

$$[BB_{nc}] = \begin{bmatrix} [T_{nc}] & [0] \\ [0] & [T_{nc}] \end{bmatrix} [Q_{nc}]\ [A_{nc}]^{-1} \tag{4.2.2}$$

4.2.2 <u>Circumferential mode n = 1</u>

By substituting equation (3.3.2) in the kinematic equations (2.3.1), we obtain:

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ 2\overline{\epsilon_{r\theta}} \\ \chi_r \\ \chi_\theta \\ 2\overline{\chi_{r\theta}} \end{Bmatrix} = \begin{bmatrix} [T_{1c}] & [0] \\ [0] & [T_{1c}] \end{bmatrix} [Q_{1c}]\ [A_{1c}]^{-1}\{\delta_j\} \tag{4.2.3}$$

Matrices $[T_{1c}]$, $[Q_{1c}]$ and $[A_{1c}]$ are given in Appendix (A-2).

The deformation matrix is defined as follows:

$$[BB_{1c}] = \begin{bmatrix} [T_{1c}] & [0] \\ [0] & [T_{1c}] \end{bmatrix} [Q_{1c}]\ [A_{1c}]^{-1} \tag{4.2.4}$$

4.2.3 Circumferential mode n = 0

By substituting equation (3.3.3) in the kinematic equations (2.3.1), we obtain:

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_r \\ \epsilon_\theta \\ \chi_r \\ \chi_\theta \end{Bmatrix} = \begin{bmatrix} [T_{0c}] & [0] \\ [0] & [T_{0c}] \end{bmatrix} [Q_{0c}]\ [A_{0c}]^{-1} \{\delta_j\} \qquad (4.2.5)$$

Matrices $[T_{0c}]$, $[Q_{0c}]$ and $[A_{0c}]$ are given in Appendix (A-2).

The deformation matrix is defined as follows:

$$[BB_{0c}] = \begin{bmatrix} [T_{0c}] & [0] \\ [0] & [T_{0c}] \end{bmatrix} [Q_{0c}]\ [A_{0c}]^{-1} \qquad (4.2.6)$$

4.3 Stress component matrix for a finite element of the annular plate type

4.3.1 Circumferential mode n ≧ 2

By replacing the vector $\{\varepsilon\}$ of equation (2.4.1) by the expression in (4.1.3), we obtain:

$$\{\sigma\} = [P]\ [BB_n] \begin{Bmatrix} \delta_i \\ \\ \delta_j \end{Bmatrix} \tag{4.3.1}$$

The stress component matrix for the group of nodes i and j of the finite element is produced:

$$\begin{Bmatrix} \sigma_i \\ \\ \sigma_j \end{Bmatrix} = \begin{bmatrix} [T_n] & & & [0] \\ & [T_n] & & \\ & & [T_n] & \\ [0] & & & [T_n] \end{bmatrix} \begin{bmatrix} [P]\ [Q_{ni}]\ [A_n]^{-1} \\ [P]\ [Q_{nj}]\ [A_n]^{-1} \end{bmatrix} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.3.2}$$

$$= [ST_n] \begin{Bmatrix} \delta_i \\ \\ \delta_j \end{Bmatrix} \tag{4.3.3}$$

where the matrices $[Q_{ni}]$ and $[Q_{nj}]$ are obtained from matrix $[Q_n]$ (Table 4, Appendix A-2), by substituting $r = a_o$ and $r = a$ respectively.

Matrices $[T_n]$, $[P]$ and $[A_n]$ are given in Appendix (A-2).

4.3.2 Circumferential mode n = 1

The stress component matrix is

$$\begin{Bmatrix} \sigma_i \\ \sigma_j \end{Bmatrix} = [ST_1] \begin{Bmatrix} \delta_i \\ \\ \delta_j \end{Bmatrix} \tag{4.3.4}$$

where

$$[ST_1] = \begin{bmatrix} [T_1] & & & \\ & [T_1] & & \\ & & \cdot \ [T_1] & \\ & & & [T_1] \end{bmatrix} \begin{bmatrix} [P] \ [Q_{1i}] \ [A_1]^{-1} \\ [P] \ [Q_{1j}] \ [A_1]^{-1} \end{bmatrix} \qquad (4.3.5)$$

where matrices $[Q_{1i}]$ and $[Q_{1j}]$ are obtained from matrix $[Q_1]$ (Table 4, Appendix A-2), by substituting $r = a_o$ and $r = a$ respectively.

Matrices $[T_1]$, $[P]$ and $[A_1]$ are given in Appendix (A-2).

4.3.3 Circumferential mode n = 0

The stress component matrix is

$$\begin{Bmatrix} \sigma_i \\ \sigma_j \end{Bmatrix} = [ST_0] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (4.3.6)$$

where

$$[ST_0] = \begin{bmatrix} [T_0] & & & [0] \\ & [T_0] & & \\ & & [T_0] & \\ [0] & & & [T_0] \end{bmatrix} \begin{bmatrix} [P_0] \ [Q_{0i}] \ [A_0]^{-1} \\ [P_0] \ [Q_{0j}] \ [A_0]^{-1} \end{bmatrix} \qquad (4.3.7)$$

Matrices $[Q_{0i}]$ and $[Q_{0j}]$ are obtained from matrix $[Q_0]$ (Table 4, Appendix A-2) by substituting $r = a_o$ and $r = a$ respectively.

Matrices $[T_0]$, $[P_0]$ and $[A_0]$ are given in Appendix (A-2).

4.4 Stress component matrix for a finite element of the circular plate type

4.4.1 Circumferential mode $n \geqq 2$

Since the finite element of the circular plate type has only one node, the stress component matrix is given by

$$\{\sigma j\} = [ST_{nc}] \{\delta j\} \tag{4.4.1}$$

where

$$[ST_{nc}] = \begin{bmatrix} [T_{nc}] & [0] \\ [0] & [T_{nc}] \end{bmatrix} [P] [Q_{ncj}] [A_{nc}]^{-1} \tag{4.4.2}$$

$[Q_{ncj}]$ is the matrix $[Q_{nc}]$ (Table 4, Appendix A-2) calculated for $r = a$ ($y = r$).

$[T_{nc}]$, $[P]$ and $[A_{nc}]$ are the matrices given in Appendix (A-2).

4.4.2 Circumferential mode $n = 1$

The stress component matrix is given by:

$$\{\sigma j\} = [ST_{1c}]\ \{\delta j\} \tag{4.4.3}$$

where

$$[ST_{1c}] = \begin{bmatrix} [T_{1c}] & [0] \\ & \\ [0] & [T_{1c}] \end{bmatrix} [P]\ [Q_{1cj}]\ [A_{1c}]^{-1} \tag{4.4.4}$$

$[Q_{1cj}]$ is the matrix $[Q_{1c}]$ (Table 4, Appendix A-2), calculated for r = a (y = 1).

$[T_{1c}]$, [P] and $[A_{1c}]$ are the matrices given in Appendix (A-2).

4.4.3 <u>Circumferential mode n = 0</u>

The stress component matrix is given by:

$$\{\sigma j\} = [ST_{0c}]\ \{\delta j\} \tag{4.4.5}$$

where

$$[ST_{0c}] = \begin{bmatrix} [T_{0c}] & [0] \\ & \\ [0] & [T_{0c}] \end{bmatrix} [P]\ [Q_{0cj}]\ [A_{0c}]^{-1} \tag{4.4.6}$$

$[Q_{0cj}]$ is the matrix $[Q_{0c}]$ (Table 4, Appendix A-2) calculated for $r = a$ $(y = 1)$.

$[T_{0c}]$, $[P]$ and $[A_{0c}]$ are the matrices given in Appendix (A-2).

CHAPTER V

STIFFNESS AND MASS MATRICES

5.1 Annular Plate

5.1.1a Stiffness matrix for circumferential mode $n \geq 2$

In a system of local axes (r, θ), the stiffness matrix is obtained as follows [15]:

$$[k] = \int_S [BB_n]^t \; [P] \; [BB_n] \; dS \tag{5.1.1}$$

where dS is the surface element = r dr dθ

[P] is the elasticity matrix given in Table (6) of Appendix (A-2).

$[BB_n]$ is the matrix defined by equation (4.1.2).

By replacing $[BB_n]$ by its expression in (5.1.1), the stiffness matrix can be written as follows:

$$[k] = \int_S \; [[A_n]^{-1}]^t \; [Q_n]^t \; [P] \; [Q_n] \; [A_n]^{-1} \; r \; dr \; d\theta \tag{5.1.2}$$

The integration with respect to θ gives:

$$[k] = [[A_n]^{-1}]^t \; \{\pi \int_{a_o}^{a} [Q_n]^t \; [P] \; [Q_n] \; r \; dr \} \; [A_n]^{-1} \tag{5.1.3}$$

where a and a_o are the outside and inside radius of the annular plate element. Expressing

$$[G] = \pi \int_{a_o}^{a} [Q_n]^t \quad [P] \quad [Q_n] \quad r \, dr \tag{5.1.4}$$

The stiffness matrix can be expressed as:

$$[k] = [[A_n]^{-1}]^t \quad [G_n] \quad [A_n]^{-1} \tag{5.1.5}$$

Before integration equation (5.1.4), it is advisable to change the variable in the following manner:

$$y = \frac{r}{a}$$

After the change of variable and the integration of (5.1.4), we obtain:

$$Gn(i,j) = \pi a^2 \sum_{k=1}^{6} \frac{Dn(i,k)\ Bn(k,j)}{En(i,k) + Cn(k,j) + 2} \left[1 - yo^{En(i,k)+Cn(k,j)+2}\right]$$
$$(i=1,8 \text{ et } j=1,8)$$

if $En(i,k) + Cn(k,j) + 2 \neq 0$

$$Gn(i,j) = -\pi a^2 \sum_{k=1}^{6} Dn(i,k)\ Bn(k,j)\ Ln\ yo \tag{5.1.6}$$
$$(i=1,8 \text{ et } j=1,8)$$

if $En(i,k) + Cn(k,j) + 2 = 0$

where $[Dn] = [Bn]^t\ [P]$

$$[En] = [Cn]^t \tag{5.1.7}$$

$$y_o = \frac{a}{a_o}$$

$[B_n]$ and $[C_n]$ are the two (6 x 8) matrices given in Table (4) of Appendix (A-2) and Ln is the Neperian Logarithm.

[P] is the elasticity matrix given in Table (6) in Appendix (A-2).

5.1.1.b Mass matrix for circumferential mode $n \geqq 2$

In a system of local axes (r, θ), the mass matrix is defined as follows [15]:

$$[m] = \rho t \int_S [N_n]^t \, [N_n] \, dS \tag{5.1.8}$$

where

dS is the surface element = $r \, dr \, d\theta$

ρ is the density of the finite element

t is the thickness of the finite element

$[N_n]$ is the matrix defined by (3.2.6)

By inserting (3.2.6) in (5.1.8) and integrating it with respect to θ, we obtain:

$$[m] = \rho t \, [[A_n]^{-1}]^t \, \{\pi \int_{a_o}^{a} [R_n]^t \, [R_n] \, r \, dr \} \, [A_n]^{-1} \tag{5.1.9}$$

where $[A_n]$ and $[R_n]$ are given in Appendix (A-2).

Letting

$$[S_n] = \pi \int_{a_o}^{a} [R_n]^t \; [R_n] \; r \, dr \tag{5.1.10}$$

The mass matrix can then be written as follows:

$$[m] = \rho t \; [[A_n]^{-1}]^t \; [S_n] \; [A_n]^{-1} \tag{5.1.11}$$

After integration of equation (5.1.10), we obtain:

$$Sn(i,j) = \pi a^2 \sum_{k=1}^{3} \frac{Fn(i,k)\;Fn(k,j)}{Hn(k,i) + Hn(k,j) + 2} \left[1 - yo^{Hn(k,i)+Hn(k,j)+2}\right]$$
$$(i=1,8 \text{ et } j=1,8)$$

if $Hn(k,i) + Hn(k,j) + 2 \neq 0$

$$Sn(i,j) = -\pi a^2 \sum_{k=1}^{3} Fn(k,i)\;Fn(k,j)\;\mathrm{Ln}\;yo \tag{5.1.12}$$
$$(i=1,8 \text{ et } j=1,8)$$

if $Hn(k,i) + Hn(k,j) + 2 = 0$

where $[F_n]$ and $[H_n]$ are the (3 x 8) matrices given in Table (3) of Appendix (A-2).

5.1.2a <u>Stiffness matrix for circumferential mode n = 1</u>

The stiffness matrix is given by

$$[k] = [[A_1]^{-1}]^t \; [Q_1] \; [A_1]^{-1} \tag{5.1.13}$$

where $[A_1]$ is the matrix given in Table (2) of Appendix (A-2).

$$[G_1] = \pi \int_{a_o}^{a} [Q_1]^t \quad [P] \quad [Q_1] \quad r \, dr \tag{5.1.14}$$

After integration the result obtained is:

$$G_1(i,j) = \pi a^2 \sum_{k=1}^{6} \frac{D_1(i,k)\, B_1(k,j)}{E_1(i,k) + C_1(k,j) + 2} \left[1 - y_o^{E_1(i,k)+C_1(k,j)+2}\right]$$
$$(i=1,6 \text{ et } j=1,6)$$

if $E_1(i,k) + C_1(k,j) + 2 \neq 0$

$$G_1(i,j) = -\pi a^2 \sum_{k=1}^{6} D_1(i,k)\, B_1(k,j) \, \mathrm{Ln}\, y_o \tag{5.1.15}$$
$$(i=1,6 \text{ et } j=1,6)$$

if $E_1(i,k) + C_1(k,j) + 2 = 0$

where $[D_1] = [B_1]^t \quad [P]$

$$[E_1] = [C_1]^t \tag{5.1.16}$$

$$y_o = \frac{a}{a_o}$$

$[B_1]$ and $[C_1]$ are the two (6 x 6) matrices given in Table (4) of Appendix (A-2).

[P] is the elasticity matrix given in Table (6) of Appendix (A-2).

5.1.2b Mass matrix for circumferential mode n = 1

The mass matrix is given by

$$[m] = \rho t \quad [[A_1]^{-1}]^t \quad [S_1] \quad [A_1]^{-1} \tag{5.1.17}$$

where $[A_1]$ is the matrix given in Table (2) of Appendix (A-2).

$$[S_1] = \pi \int_{a_o}^{a} [R_1]^t \quad [R_1] \quad r \quad dr \tag{5.1.18}$$

The elements of matrix $[S_1]$ are determined analytically and listed in Appendix (A-3).

5.1.3a Stiffness matrix for circumferential mode n = 0

The stiffness matrix is given by

$$[k] = [[A_0]^{-1}]^t \quad [G_0] \quad [A_0]^{-1} \tag{5.1.19}$$

where $[A_0]$ is the matrix given in Table (2) of Appendix (A-2).

$$[G_0] = \pi \int_{a_o}^{a} [Q_0]^t \quad [P_0] \quad [Q_0] \quad r \; dr \tag{5.1.20}$$

For this case, $[P_0]$ is the (4 x 4) elasticity matrix given in Table (6) of Appendix (A-2).

The elements of matrix $[G_0]$ are determined analytically and listed in Appendix (A-3).

5.1.3b Mass matrix for circumferential mode $n = 0$

The mass matrix is given by

$$[m] = \rho t \; [[A_0]^{-1}]^t \; [S_0] \; [A_0]^{-1} \tag{5.1.21}$$

where $[A_0]$ is the matrix given in Table (2) of Appendix (A-2).

$$[S_0] = \pi \int_{a_o}^{a} [R_0]^t \; [R_0] \; r \, dr \tag{5.1.22}$$

The elements of matrix $[S_0]$ are determined analytically and listed in Appendix (A-3).

5.2 Circular Plate

5.2.1a Stiffness matrix for circumferential mode $n \geqq 2$

As the circular plate element has only one node, the stiffness matrix will be of the order (4 x 4). It is given by:

$$[k] = [[A_{nc}]^{-1}]^t \; [G_{nc}] \; [A_{nc}]^{-1} \tag{5.2.1}$$

where $[A_{nc}]$ is the (4 x 4) matrix given in Table (2) of Appendix (A-2).

$$[G_{nc}] = \pi \int_0^a [Q_{nc}]^t \; [P] \; [Q_{nc}] \; r \, dr \tag{5.2.2}$$

"a" being the outside radius of the circular plate element.

After integration we obtain:

$$Gnc(i,j) = \pi a^2 \sum_{k=1}^{6} \frac{Dnc(i,k)\ Bnc(k,j)}{Enc(i,k) + Cnc(k,j) + 2}$$
$$(i=1,4 \text{ et } j=1,4)$$

if $Enc(i,k) + Cnc(k,j) + 2 \neq 0$ (5.2.3)

$$Gnc(i,j) = - \pi a^2 \sum_{k=1}^{6} Dnc(i,k)\ Bnc(k,j)\ Ln\ \epsilon \quad ; \quad \epsilon = 10^{-9}$$
$$(i=1,4 \text{ et } j=1,4)$$

if $Enc(i,k) + Cnc(k,j) + 2 = 0$

where $[D_{nc}]\ ]\ [B_{nc}]^t\ [P]$

(5.2.4)

$[E_{nc}] = [C_{nc}]^t$

$[B_{nc}]$ and $[C_{nc}]$ are two (6 x 4) matrices given in Table (4) of Appendix (A-2).

[P] is the elasticity matrix given in Table (6) of Appendix (A-2).

5.2.1b Mass matrix for circumferential mode $n \geq 2$

The mass matrix is given by

$$[m] = \rho t\ [[A_{nc}]^{-1}]^t\ \ [S_{nc}]\ \ [A_{nc}]^{-1} \qquad (5.2.5)$$

where $[A_{nc}]$ is the matrix given in Table (2) of Appendix (A-2).

$$[S_{nc}] = \pi \int_{a_0}^{a} [R_{nc}]^t \; [R_{nc}] \; r \, dr \tag{5.2.6}$$

After integration we obtain:

$$Snc(i,j) = \pi a^2 \sum_{k=1}^{3} \frac{Fnc(k,i) \; Fnc(k,j)}{Hnc(k,i) + Hnc(k,j) + 2}$$
$$(i=1,4 \text{ et } j=1,4)$$

$$\text{if} \quad Hnc(k,i) + Hnc(k,j) + 2 \neq 0 \tag{5.2.7}$$

$$Snc(i,j) = - \pi a^2 \sum_{k=1}^{3} Fnc(k,i) \; Fnc(k,j) \; \mathrm{Log} \, \epsilon \quad ; \quad \epsilon = 10^{-9}$$
$$(i=1,4 \text{ et } j=1,4)$$

$$\text{if} \quad Hnc(k,i) + Hnc(k,j) + 2 = 0$$

where $[F_{nc}]$ and $[H_{nc}]$ are the (3 x 4) matrices given in Table (3) of Appendix (A-2).

5.2.2a Stiffness matrix for circumferential mode n = 1

The stiffness matrix is given by:

$$[k] = [[A_{1c}]^{-1}]^t \; [G_{1c}] \; [A_{1c}]^{-1} \tag{5.2.8}$$

where $[A_{1c}]$ is the (3 x 3) matrix given in Table (2) of Appendix (A-2).

$$[G_{1c}] = \pi \int_{0}^{a} [Q_{1c}]^t \; [P] \; [Q_{1c}] \; r \, dr \tag{5.2.9}$$

After integration we obtain:

$$G_{1c}(i,j) = \pi a^2 \sum_{k=1}^{6} \frac{D_{1c}(i,k)\ B_{1c}(k,j)}{E_{1c}(i,k) + C_{1c}(k,j) + 2}$$

$(i=1,3 \text{ et } j=1,3)$

if $E_{1c}(i,k) + C_{1c}(k,j) + 2 \neq 0$ (5.2.10)

$$G_{1c}(i,j) = -\pi a^2 \sum_{k=1}^{6} D_{1c}(i,k)\ B_{1c}(k,j)\ \mathrm{Ln}\ \epsilon \quad ; \quad \epsilon = 10^{-9}$$

$(i=1,3 \text{ et } j=1,3)$

if $D_{1c}(i,k) + C_{1c}(k,j) + 2 = 0$

where $[D_{1c}] = [B_{1c}]^t\ [P]$

$[E_{1c}] = [C_{1c}]^t$ (5.2.11)

Matrices $[B_{1c}]$ and $[C_{1c}]$ are given in Table (4) of Appendix (A-2).

$[P]$ is the elasticity matrix given in Table (6) of Appendix (A-2).

"a" is the outside radius of the plate.

5.2.2b Mass matrix for circumferential mode n = 1

The mass matrix is given by:

$$[m] = \rho t\ [[A_{1c}]^{-1}]^t\ [S_{1c}]\ [A_{1c}]^{-1} \quad (5.2.12)$$

where $[A_{1c}]$ is the matrix given in Table (2) of Appendix (A-2).

$$[S_{1c}] = \pi \int_{a_0}^{a} [R_{1c}]^t \; [R_{1c}] \; r \, dr \tag{5.2.13}$$

After integration we obtain:

$$S_{1c}(i,j) = \pi a^2 \sum_{k=1}^{3} \frac{F_{1c}(i,k)\, F_{1c}(k,j)}{H_{1c}(i,k) + H_{1c}(k,j) + 2}$$
$$(i=1,3 \text{ et } j=1,3)$$

if $H_{1c}(k,i) + H_{1c}(k,j) + 2 \neq 0$ (5.2.14)

$$S_{1c}(i,j) = - \pi a^2 \sum_{k=1}^{3} F_{1c}(k,i)\, F_{1c}(k,j) \, \mathrm{Ln}\, \epsilon \quad ; \quad \epsilon = 10^{-9}$$
$$(i=1,3 \text{ et } j=1,3)$$

if $H_{1c}(k,i) + H_{1c}(k,j) + 2 = 0$

where $[F_{1c}]$ and $[H_{1c}]$ are given in Table (3) of Appendix (A-2).

5.2.3a <u>Stiffness matrix for circumferential mode n = 0</u>

The stiffness matrix is given by

$$[k] = [[A_{0c}]^{-1}]^t \; [G_{0c}] \; [A_{0c}]^{-1} \tag{5.2.15}$$

where $[A_{0c}]$ is the (3 x 3) matrix given in Table (2) of Appendix (a-2).

$$[G_{oc}] = \pi \int_0^a [Q_{0c}]^t \; [P_0] \; [Q_{0c}] \; r \; dr \tag{5.2.16}$$

For this case $[P_0]$ is the (4 x 4) elasticity matrix given in Table (6) of Appendix (A-2).

The elements of the matrix $[G_{0c}]$ are determined analytically and listed in Appendix (A-3).

5.2.3b Mass matrix for circumferential mode n = 0

The mass matrix is given by

$$[m] = \rho t \;\; [[A_{0c}]^{-1}]^{t} \;\; [S_{0c}] \;\; [A_{0c}]^{-1} \qquad (5.2.17)$$

where $[A_{0c}]$ is the matrix given in Table (2) of Appendix (A-2).

The elements of the matrix $[S_{0c}]$ are determined analytically and listed in Appendix (A-3).

CHAPTER VI

STATIC AND DYNAMIC ANALYSIS

6.1 Assembly of the finite elements

As has already been mentioned, the complete plate is divided into a finite number of circular and annular elements, the positions of which can be selected arbitrarily. Each finite element of the circular plate type has a node at the circumference end, while the elements of the annular plate type have one node at each extremity (Figure 3.1).

Once the stiffness and mass matrices have been obtained, it is possible to construct the global matrices for the complete plate using the usual finite element assembly technique.

This technique takes the following two conditions into account [15]:

- The continuity of the nodal displacements from one element to the next, as:

$$\{\delta_i + 1\} = \{\delta_j\}$$

- The generalized internal forces counterbalance the external forces at the node common to two adjacent elements:

$$\{F\}^e = \{Fj\}_k + \{Fi\}_{k+1}$$

where:

$\{F\}^e$ is the vector of the external forces

$\{Fj\}_k$ is the vector of the internal forces at node j of element k

$\{Fi\}_{k+1}$ is the vector of the internal forces at node i of element (k+1)

The assembly is carried out by superimposition, as shown in Figure (6.1). If N is the number of finite elements, [M] and [K] are two matrices of the order 4(N+1) for an annular plate and of the order 4N for a circular plate.

These matrices are symmetrical and semi-defined; they are also band matrices of which the half-width of the band is equal to 8.

6.2 Static Forces

The study of the static equilibrium is carried out in the following manner:

When: {FA} is the vector of the forces applied to the nodes of the plate

{FB} is the vector of unknown reactions

{dA} is the vector of unknown nodal displacements

{dB} is the vector of displacements defined by the boundary conditions

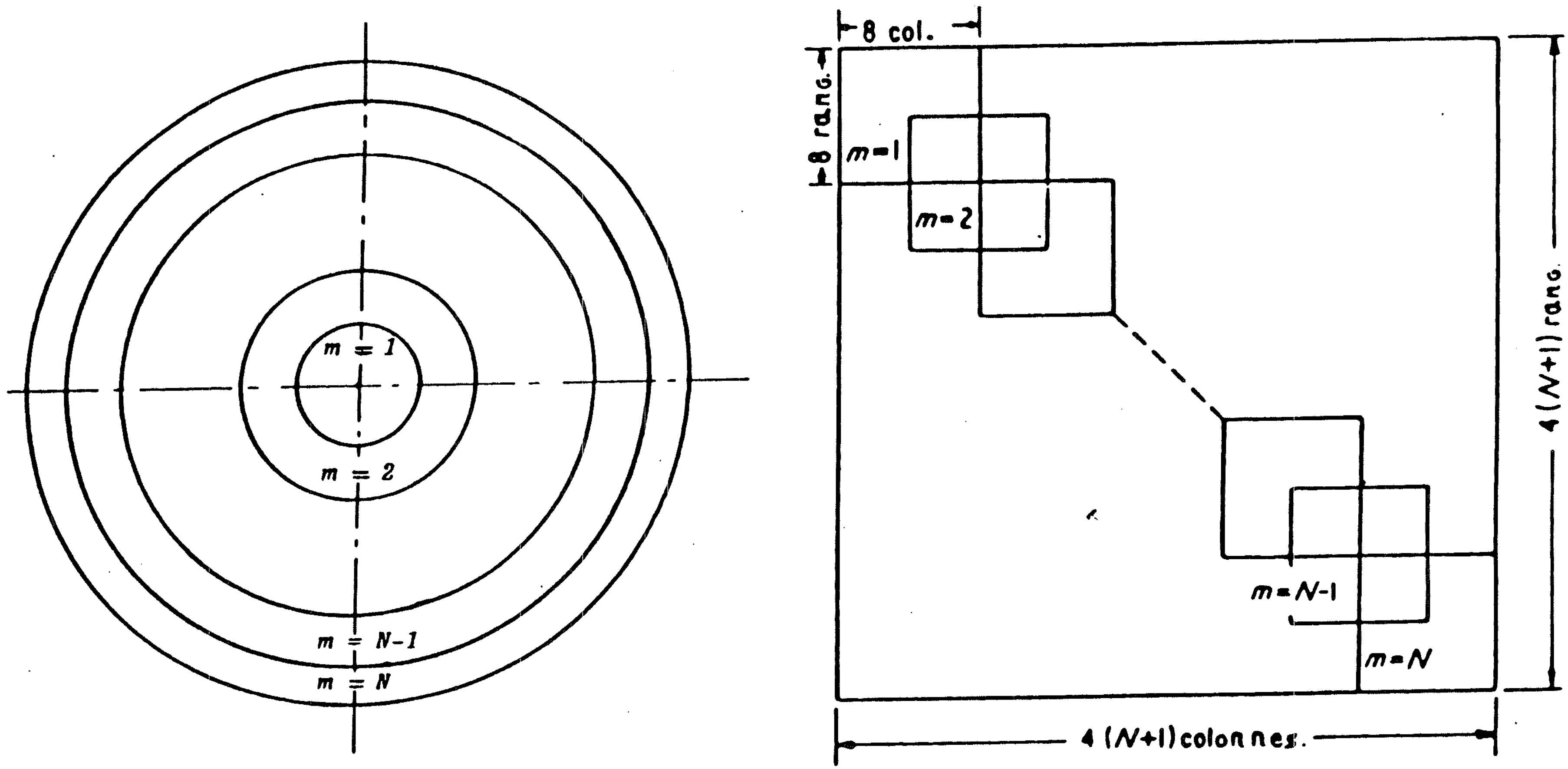


Figure 6.1: Diagram of assembly of the mass and stiffness matrices

The static equilibrium equation $[K]\ \{d\} = \{F\}$ becomes

$$\begin{bmatrix} KAA & KAB \\ KBA & KBB \end{bmatrix} \begin{bmatrix} dA \\ dB \end{bmatrix} = \begin{bmatrix} FA \\ FB \end{bmatrix} \qquad (6.2.1)$$

We have therefore

$$\begin{aligned} \{dA\} &= [KAAA]^{-1}\ \{FA\} - [KAB]\ \{dB\} \\ \{FB\} &= [KBA]\ \{dA\} + [KBB]\ \{dB\} \end{aligned} \qquad (6.2.2)$$

6.3 Free Vibrations

In the free vibrations, the equations of motion are

$$[M]\ \{\ddot{\delta}\}_T + [K]\ \{\delta\}_T = \{0\} \qquad (6.3.1)$$

where $[M]$, $[K]$ are the global mass and stiffness matrices

$\{\delta_T\}$ is the vector for the global displacements of the whole plate

$$\{\delta_T\} = \{\delta_1, \delta_2, \ldots, \delta_{N+1}\}^T$$

N being the number of finite elements.

The vibration is harmonic

$$\{\delta_T\} = \{\delta_0\}_T \sin(\omega t + \psi) \qquad (6.3.2)$$

where ω is the natural angular frequency

ψ is the phase angle.

By introducing equation (6.3.2) in (6.3.1), we obtain

$$([K] - \omega^2 [M]) \{\delta_0\}_T \ \{0\} \tag{6.3.3}$$

This relation holds only for certain values of ω where the determinant of the matrix in parentheses is zero. These values define the natural angular frequencies of the structure and give rise to a typical problem of eigenvalues and eigenvectors.

$$\det [[K] - \omega^2 [M]] = 0 \tag{6.3.4}$$

Corresponding to each natural frequency for which equation (6.3.3) has been verified is a vector $\{\delta_0\}$ of which the components are defined to one close arbitrary multiplying constant. Such vectors are called the natural modes (or eigenvectors) of the system.

6.4 Boundary Conditions

If the plate has boundary constraints such as being simply-supported, clamped, etc., the appropriate lines and columns in [K] and [M] are eliminated to satisfy these constraints. Consequently, matrices [K] and [M] reduce to square matrices of order 4(N+1)-J for an annular plate and 4N-J for a circular plate, where J is the number of constraints applied.

Thus, for a plate:

- free: $J = 0$
- with one edge simply-supported ($u = v = w = 0$): $J = 3$
- with two edges clamped ($u = v = w = \frac{dw}{dr} = 0$) : $J = 8$

CHAPTER VII

NUMERICAL CALCULATIONS AND DISCUSSION

7.1 The Algorithm

7.1.1 Introduction

A computer programme has been written to apply the analytical formula set out in the preceding chapters. We give below the characteristics of the numerical formulation.

The circular or annular plate is first divided into a certain number of finite elements. This number is governed by the convergence of the method. We show that, for the cases studied, a division into 6 elements is sufficient for low frequencies.

The geometric and mechanical characteristics may vary from one element to another. Thus, it is possible to accommodate the case of a structure formed from different pieces of plate.

7.1.2 Organization of the programme

An organization chart is given in Figure (7.1).

7.1.2a Essential data

i) the number of finite elements

ii) the number of circumferential modes

iii) the geometry of each finite element: inside radius, outside radius, thickness

iv) the mechanical properties of each finite element: Young's modulus, Poisson's ratio and density

v) boundary conditions

7.1.2b Steps in the calculation

For each harmonic n, we determine:

i) the coefficients of the characteristic equation (equation 3.1.4)

ii) the four roots (λ_i) $i = 1,4$ of the characteristic equation
Sub-programme ZPOLR of IMSL is used for this

iii) the coefficients (α_i) $i = 1,4$ defined by equation (3.1.6)

iv) matrix [A]

v) The inverse of matrix [A] by sub-programme LINV2F of IMSL

vi) matrices [G] and [S]

vii) the elementary stiffness and mass matrices
Steps (iv) to (vii) are repeated for each finite element

viii) the stiffness and mass matrices for the whole plate

ix) the stiffness and mass matrices reduced by the application of boundary conditions

x) natural frequencies and the corresponding modes

7.2 Convergence

The finite element method permits us to reach an approximate solution to the problem of elasticity. This solution is marred by errors which fall into two categories:

- The discretisation error which stems from the replacement of the initial physical problem by an approximate model.

- The truncation error stemming from the numerical calculation.

From the convergence of the finite element method, we see that the solution to the problem is a function of the number of finite elements used to model the structure under consideration, that is to say, it is a function of the fineness of the net.

The calculation was carried out with one particular plate for a number of circumferential modes 'n' varying from 0 to 6, with a number of finite elements N = 2,4,6,8,10,15 and 20.

The results for m = 1 are given in Figure 7.2, for m = 2 in Figure 7.3, and for m = 3 in Figure 7.4.

We conclude that the convergence of the system demands 6 finite elements for the relatively low radial modes "m", but 20 for radial mode m = 10.

7.3 Rigid-body motion

We shall study rigid-body motion with the aim of finding out whether the selected displacement functions satisfy the convergence criteria of the finite element method.

For this, we study the free vibrations of a free annular finite element (with no boundary conditions).

If no exterior force is applied, the finite element motion equation can be written as follows:

$$[m] \begin{Bmatrix} \ddot{\delta}_i \\ \ddot{\delta}_j \end{Bmatrix} + [k] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \{0\} \tag{7.3.1}$$

We consider

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = \begin{Bmatrix} \delta_{io} \\ \delta_{jo} \end{Bmatrix} \sin(\omega t + \psi) \tag{7.3.2}$$

equation (7.3.1) then becomes:

$$([k] - \omega^2 [m]) \begin{Bmatrix} \delta_{io} \\ \\ \delta_{jo} \end{Bmatrix} = \{0\} \qquad (7.3.3)$$

We have here a standard eigenvalue problem, where "ω" is the natural frequency, [k] and [m] are respectively the stiffness and mass matrices.

The solution to (7.3.3) gives us 6 natural frequencies (for n = 0 and n = 1), each one linked to an eigenvector $\begin{Bmatrix} \delta_{io} \\ \delta_{jo} \end{Bmatrix}$.

For a free finite element, the solution to (3) also gives us the vibration modes for rigid-body motion. For the same finite element, a numerical calculation has been made for circumferential mode n = 0.

After solving the problem of the eigenvalues, we have the following six non-dimensional natural frequencies:

$\Omega_1 = 0$ $\qquad\qquad$ $\Omega_4 = 328.43$

$\Omega_2 = 9.317$ $\qquad\qquad$ $\Omega_5 = 375.37$

$\Omega_3 = 111.50$ $\qquad\qquad$ $\Omega_6 = 1733.0$

We note that the first natural frequency is zero, and show that, for the first mode:

$$u_{ni} = u_{nj} = 0$$

$$w_{ni} = w_{nj} = 1$$

$$\left(\frac{dw_n}{dr}\right)_i = \left(\frac{dw_n}{dr}\right)_j = 10^{-12}$$

This mode involves a pure translation in the transversal direction (w).

If we consider $\frac{dW}{dr} = \frac{w_{nj} - w_{ni}}{r_j - r_i}$, etc., ... and employ the deformation equations (2.3.1) and the values of the first mode, the deformation vector obtained is of the order of 10^{-6}, which is not true for the eigenvectors corresponding to frequencies Ω_2 and those higher.

In making the same type of analysis for circumferential mode n = 1, we find that there is also rigid-body motion for this vibration mode.

7.4 Free Vibrations of Circular Plates

7.4.1 Uniform circular plates

The natural frequencies of a circular plate can unquestionably be calculated by simpler methods than these. Our principal objective, however, has been to verify the accuracy of the mass and stiffness matrices in their general forms, as outlined in Chapter (6).

For this reason, we have compared the non-dimensional natural frequencies determined by this method with those obtained by other authors, both for different boundary conditions (plate clamped, simply-supported, free) and for different values of the number of circumferential mode "n" and of the number of radial mode "m".

We have obtained very good agreement both for high and low frequencies (Tables 1, 2, 3). Figures 7.5, 7.6 and 7.7 show the non-dimensional natural frequency curves as a function of the number of circumferential mode "n" for different boundary conditions and for different values of the number of radial mode "m". In Figure 7.8 we show several vibration modes.

7.4.2 Plate simply-supported along an arbitrary circle

The non-dimensional natural frequencies of a plate which is simply-supported along an arbitrary circle have been obtained by the present method and compared with those obtained by Bodine [31] for the number of circumferential mode $n = 0$ and radial mode $m = 1$, and good agreement has been obtained (Figure 7.9).

7.4.3 Non-uniform circular plates

Two types of non-uniform circular plate have been studied, the first is a clamped circular plate with a discontinuity of thickness, the second is a plate with a linear thickness variation in the radial direction.

7.4.3a Circular plate with a discontinuity of thickness

The plate is of uniform thickness "h_o" as far as radius "b" and of thickness "h_1" from radius "b" to exterior radius "a" (Figure 7.10).

The natural frequencies of this type of plate have been established recently by Irie [9] for the circumferential mode n = 0 and radial mode m = 1, with a h_o/h_1 thickness ratio varying from 0.5 to 2. The results obtained by our method are sufficiently close to those obtained by Irie [9] (Figure 7.10).

By the present method it is possible to determine the natural frequencies of this type of plate for any thickness ratio h_o/h_1, and particularly for high circumferential and radial modes.

7.4.3b Circular plate with a linear thickness variation

The non-dimensional natural frequencies are determined for circumferential mode n = 0 and radial mode m = 1 for different values of

ALPHA = $(h_e - h_c)/h_c$ (Figure 7.11), where h_c is the thickness at the centre of the plate and h_e is the thickness at the outside edge of the plate.

The calculation has been carried out for two boundary conditions: a clamped plate and a simply-supported plate (Figure 7.11). The results obtained are compared with those obtained by the transfer matrix method [10] and good agreement has been obtained.

We note that the frequencies vary linearly with ALPHA for both boundary conditions.

The equations for these straight lines are:

- Clamped plate

  $\Omega = 8.6$ ALPHA $+ 10.2$ for $n = 0,\ m = 1$

- Simply-supported plate

  $\Omega = 3$ ALPHA $+ 4.9$ for $n = 0,\ m = 1$

This method can also give the natural frequencies for a circular plate of any non-uniform thickness, whether the modes are high or low.

7.5 Free Vibrations of Annular Plates

Our method is remarkable for the fact that it enables us to determine with equal precision both low and high frequencies.

The results obtained in the literature are only for relatively low modes ($n = 0, 1, 2$ and $m = 1, 2, 3$). To remedy this lack of results, Tables 4-12 show part of the results obtained for $n = 0, 1, 2$ and $m = 1$-10 for different boundary conditions and various dimensions of the annular plate. Another part of these results is to be found in Figures 7.12 to 7.16.

7.6 Influence of Poisson's ratio

Analysis of the results of Table (13) shows that the effect of Poisson's ratio on the natural frequencies of circular plates is only important for weak vibration modes.

This effect is evident in Figure 7.17, which shows the natural frequencies of a circular plate for the number of circumferential mode $n = 0$ and the first radial mode.

We conclude that the effect is more pronounced for a simply-supported circular plate (17% difference between the frequency calculated with $\nu = 0$

and that calculated with $\nu = 0.5$); for a free plate the error is 16%, while for a clamped plate Poisson's ratio has no effect.

For a simply-supported annular plate the effect of the ratio is not very pronounced (3%), while it is negligible for other boundary conditions.

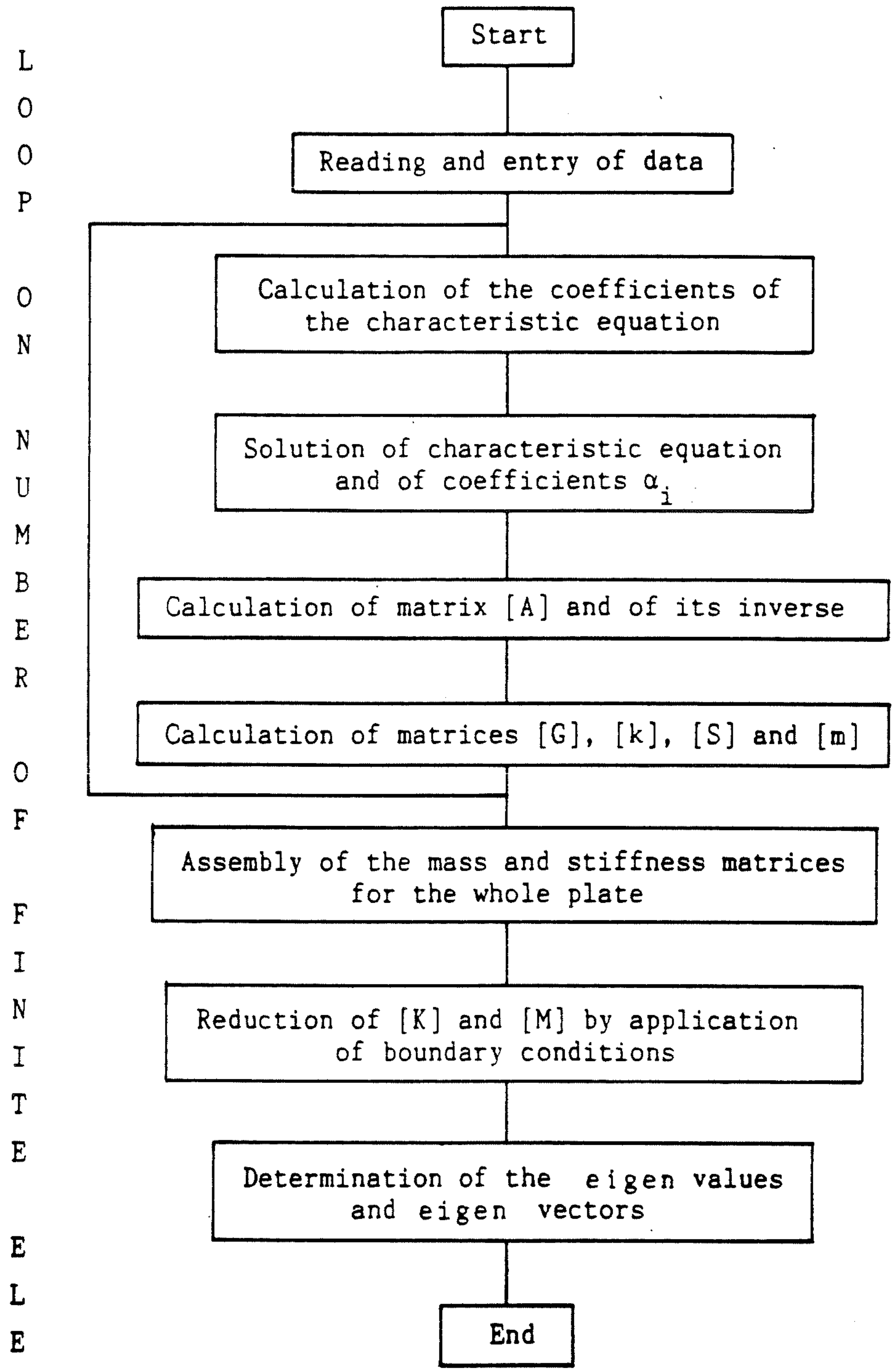


Figure 7.1: Condensed organization chart of computer programme

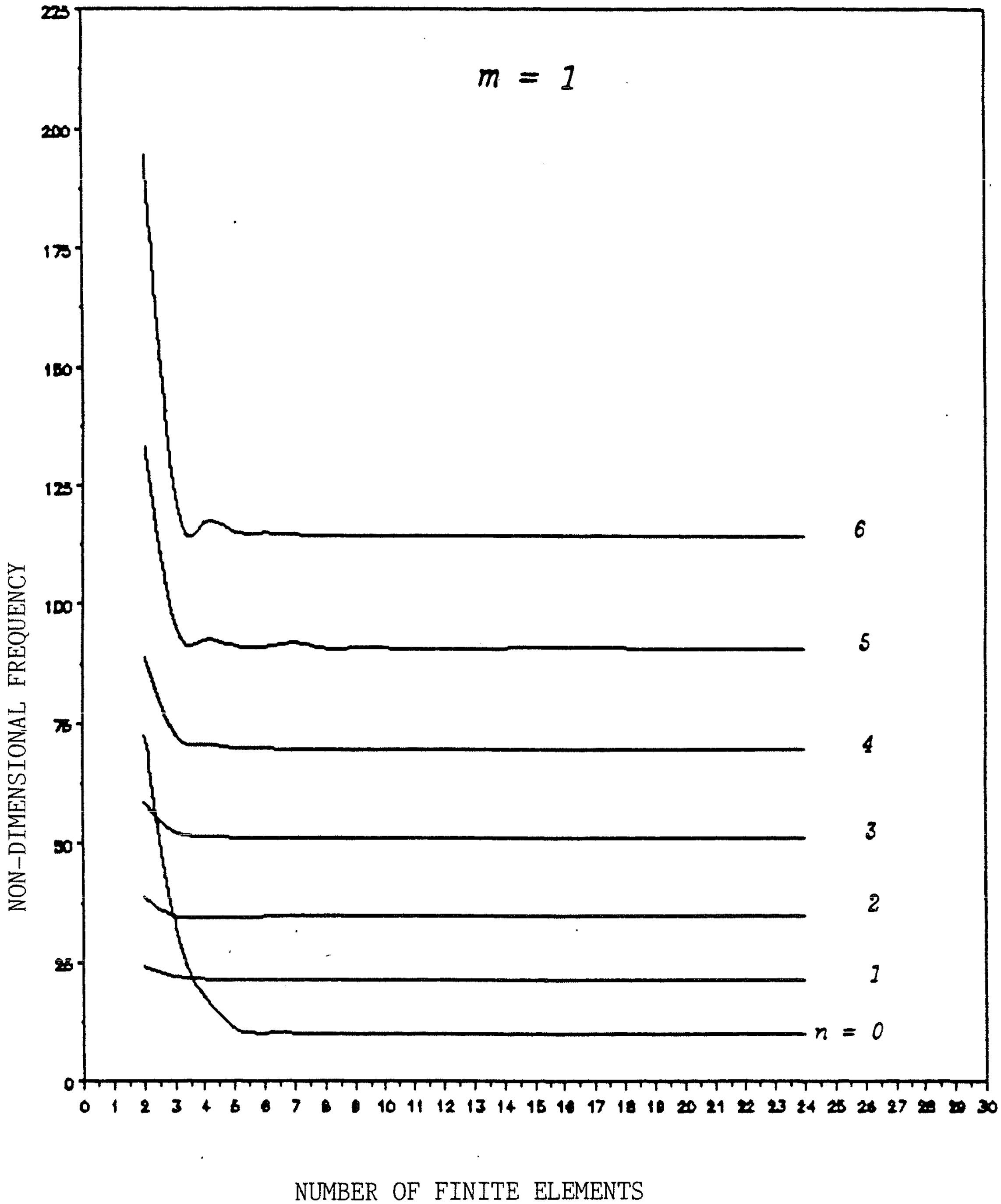
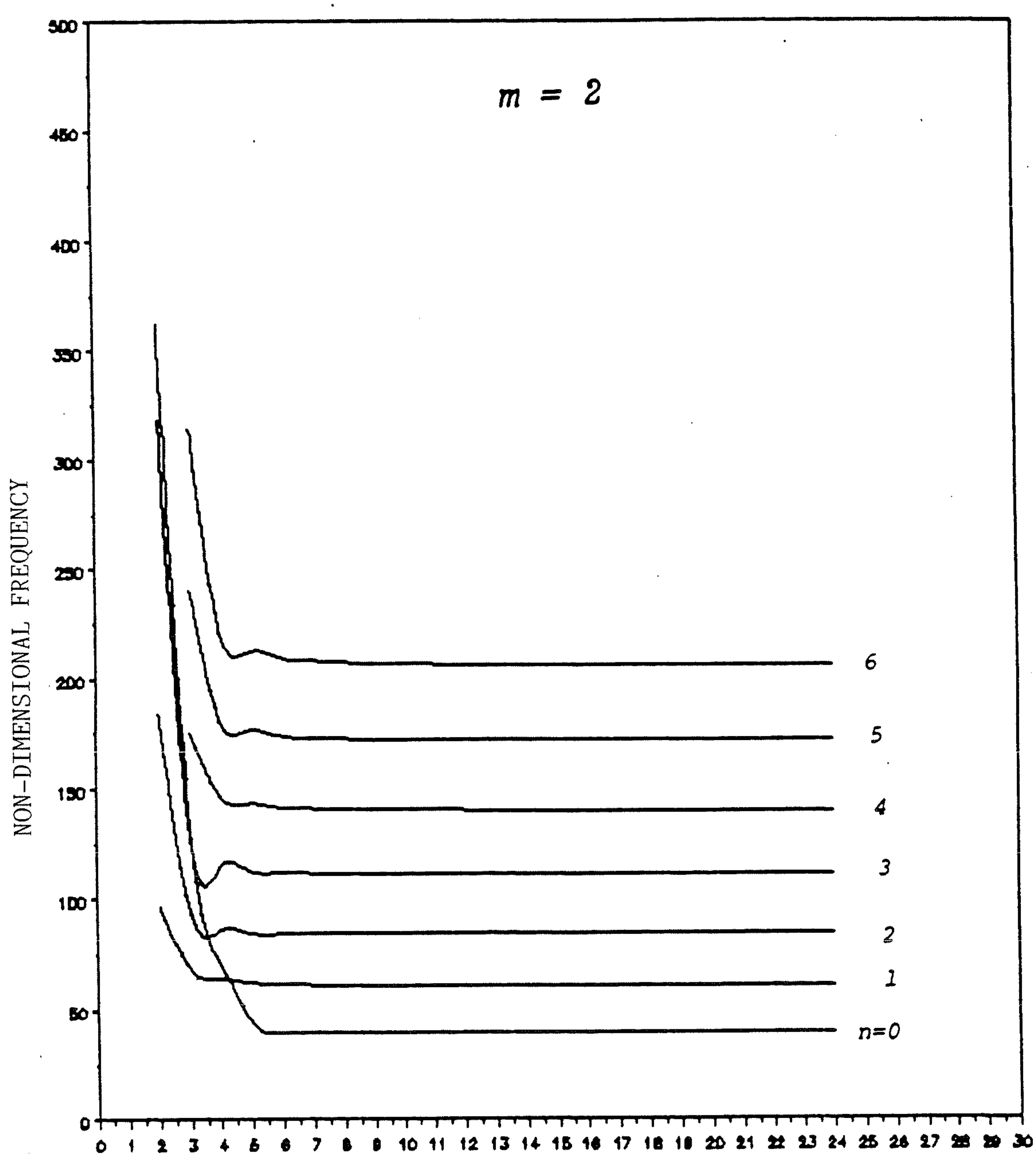


Figure 7.2: Non-dimensional natural frequency of a clamped circular plate as a function of the number of finite elements.

n: Number of circumferential mode
m: Number of radial mode



NUMBER OF FINITE ELEMENTS

Figure 7.3: Non-dimensional natural frequency of a clamped circular plate as a function of the number of finite elements.

n: Number of circumferential mode
m: Number of radial mode

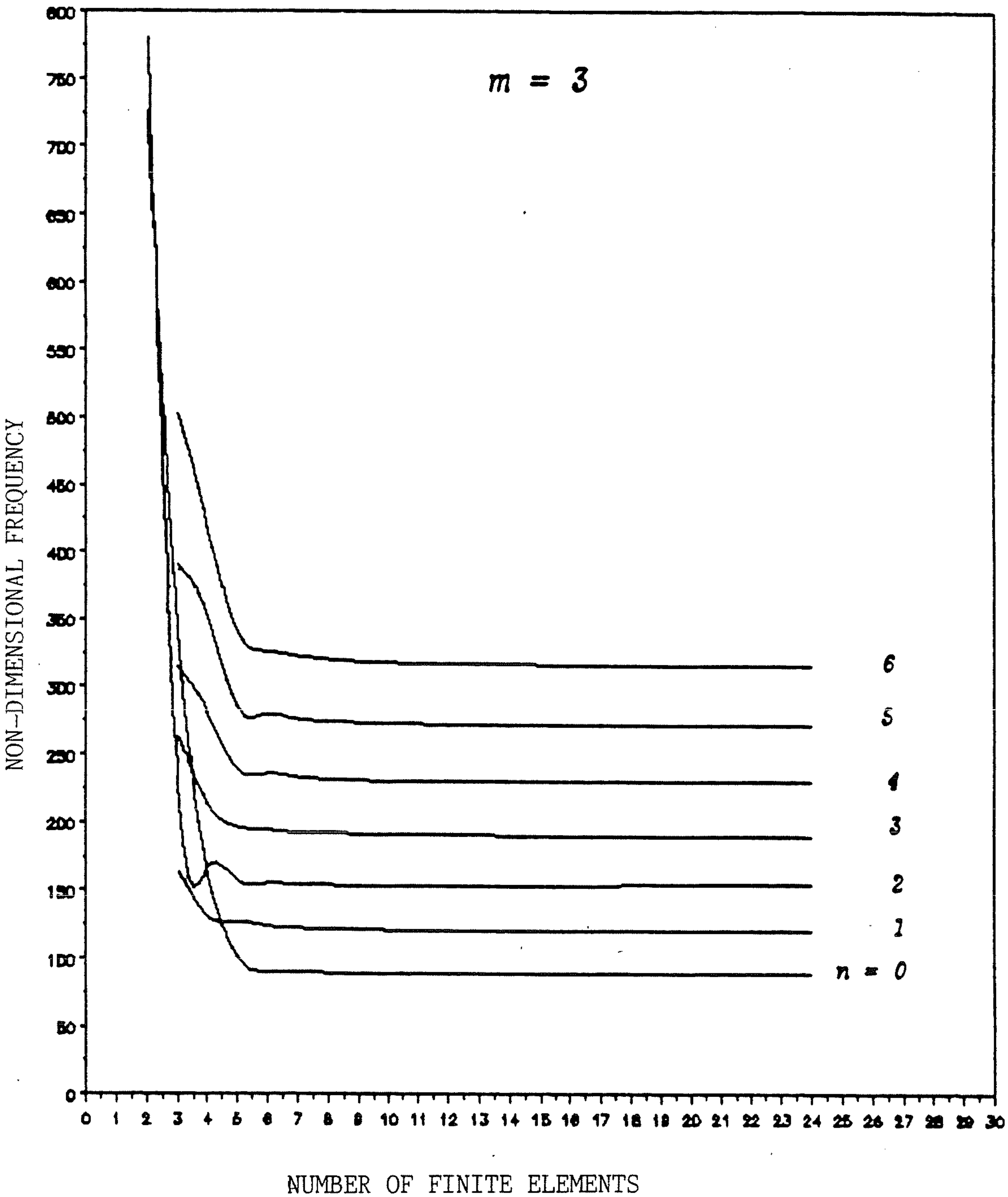


Figure 7.4: Non-dimensional natural frequency of a clamped circular plate as a function of the number of finite elements.

n: Number of circumferential mode
m: Number of radial mode

TABLE 1: NON-DIMENSIONAL NATURAL FREQUENCIES OF A CLAMPED CIRCULAR PLATE

	n = 0		
m	Irie [6]	Leissa [4]	This method
1	10.216	10.216	10.216
2	39.771	39.771	39.771
3	89.104	89.104	89.108
4	158.184	158.183	158.20
5	-	247.005	247.08
6	-	355.568	355.80
7	-	483.872	484.45
8	-	631.914	633.22
9	-	799.702	802.21
10	-	987.216	992.21
	n = 1		
1	21.260	21.26	21.261
2	60.829	60.82	60.838
3	120.079	120.08	120.13
4	199.053	199.06	199.23
5	-	297.77	298.24
6	-	416.20	417.32
7	-	554.37	556.66
8	-	712.30	716.53
9	-	889.95	897.25
10	-	1087.4	1099.2
	n = 2		
1	34.877	34.88	34.861
2	84.583	84.58	84.527
3	153.815	153.81	153.69
4	242.721	242.71	242.49
5	-	351.38	351.01
6	-	479.65	479.34
7	-	627.75	627.62
8	-	795.52	796.09
9	-	983.07	985.09
10	-	1190.4	1195.1

TABLE 2: NON-DIMENSIONAL NATURAL FREQUENCIES OF A SIMPLY-SUPPORTED CIRCULAR PLATE, $\nu = 0.3$

m	Irie [6]	Leissa [5]	This method
	n = 0		
1	4.934	4.935	4.935
2	29.720	29.720	29.720
3	74.156	74.156	74.158
4	138.318	138.318	138.33
5	-	222.215	222.27
6	-	325.849	326.02
7	-	449.222	449.68
8	-	592.332	593.40
9	-	755.182	757.41
10	-	937.771	942.05
	n = 1		
1	13.898	13.898	13.898
2	48.479	48.479	48.484
3	102.733	102.773	102.81
4	176.801	176.801	176.93
5	-	270.566	270.95
6	-	384.069	384.98
7	-	517.310	519.23
8	-	670.290	673.93
9	-	843.009	849.40
10	-	1035.47	1046.0
	n = 2		
1	25.613	25.613	25.603
2	70.117	70.117	70.074
3	134.298	134.298	134.19
4	218.203	218.202	218.00
5	-	321.841	321.53
6	-	445.215	444.86
7	-	588.328	588.11
8	-	751.179	751.49
9	-	933.768	935.30
10	-	-	1140.0

TABLE 3: NON-DIMENSIONAL NATURAL FREQUENCIES OF A FREE PLATE, $\nu = 0.33$

m	Itao et Crandall [7]	Leissa [4]	This method
	n = 0		
1	-	-	-
2	9.068	9.084	9.068
3	38.507	38.55	38.507
4	87.813	87.80	87.816
5	156.88	157.0	156.90
6	245.70	245.9	245.77
7	354.25	354.6	354.48
8	482.55	483.1	483.12
9	630.59	631.0	631.87
10	798.37	798.6	800.97
	n = 1		
1	-	-	-
2	20.513	20.41	20.514
3	59.859	59.74	59.868
4	119.01	118.88	119.06
5	197.92	196.67	198.10
6	296.59	296.46	297.08
7	415.01	414.86	416.12
8	553.17	553.00	555.43
9	711.07	710.92	715.27
10	888.72	888.58	895.94
	n = 2		
1	5.262	5.253	5.259
2	35.243	35.25	35.228
3	84.376	83.9	84.326
4	153.33	154.0	153.21
5	242.07	242.7	241.86
6	350.57	350.8	350.28
7	478.80	479.2	478.55
8	626.84	627.0	626.79
9	794.59	794.7	795.22
10	982.09	981.6	984.17

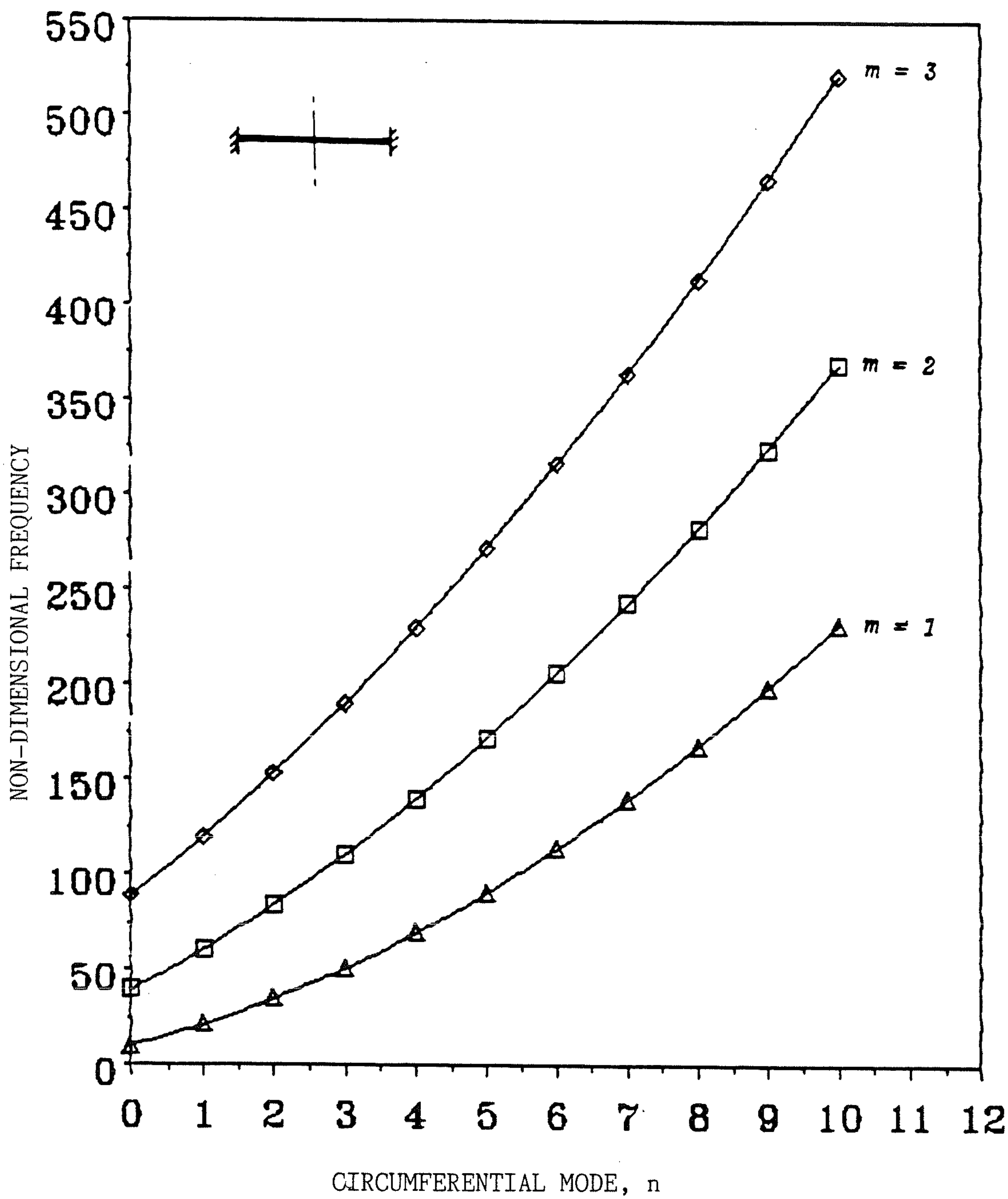


Figure 7.5: Non-dimensional natural frequency of a clamped circular plate as a function of the number of circumferential mode n.

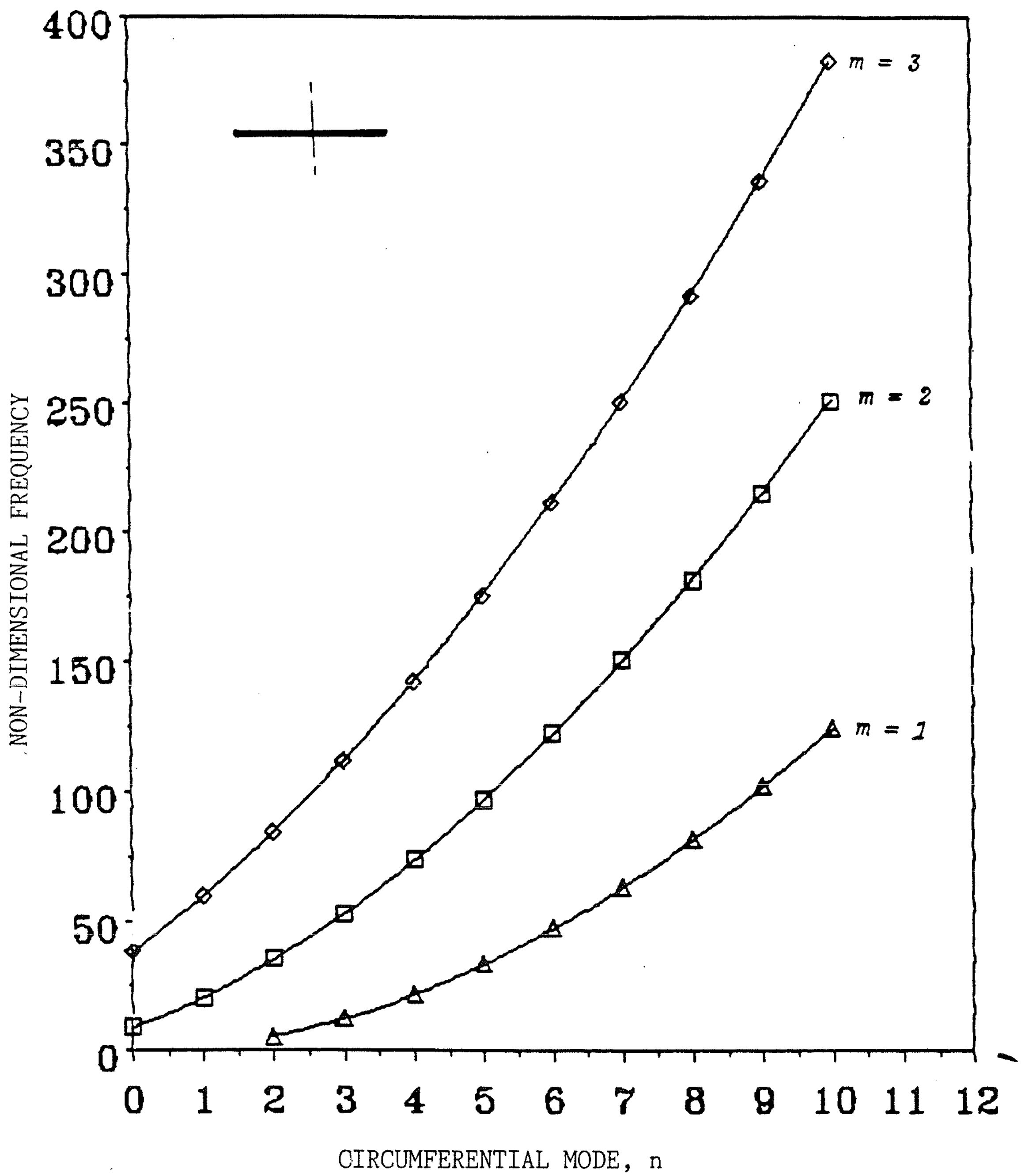


Figure 7.6: Non-dimensional natural frequency of a free circular plate as a function of the number of circumferential mode n.

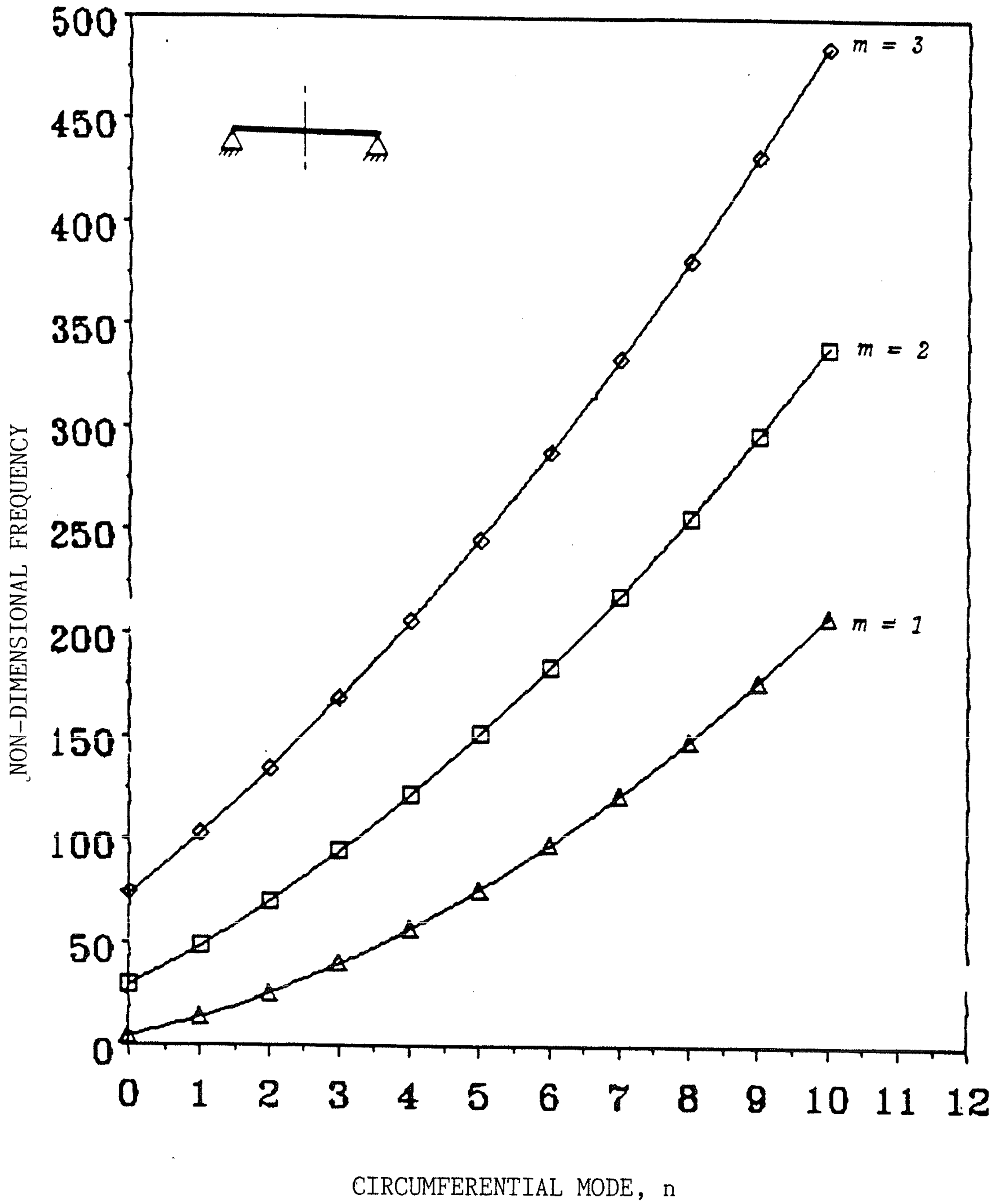


Figure 7.7: Non-dimensional natural frequency of a simply-supported circular plate as a function of the number of circumferential mode n.

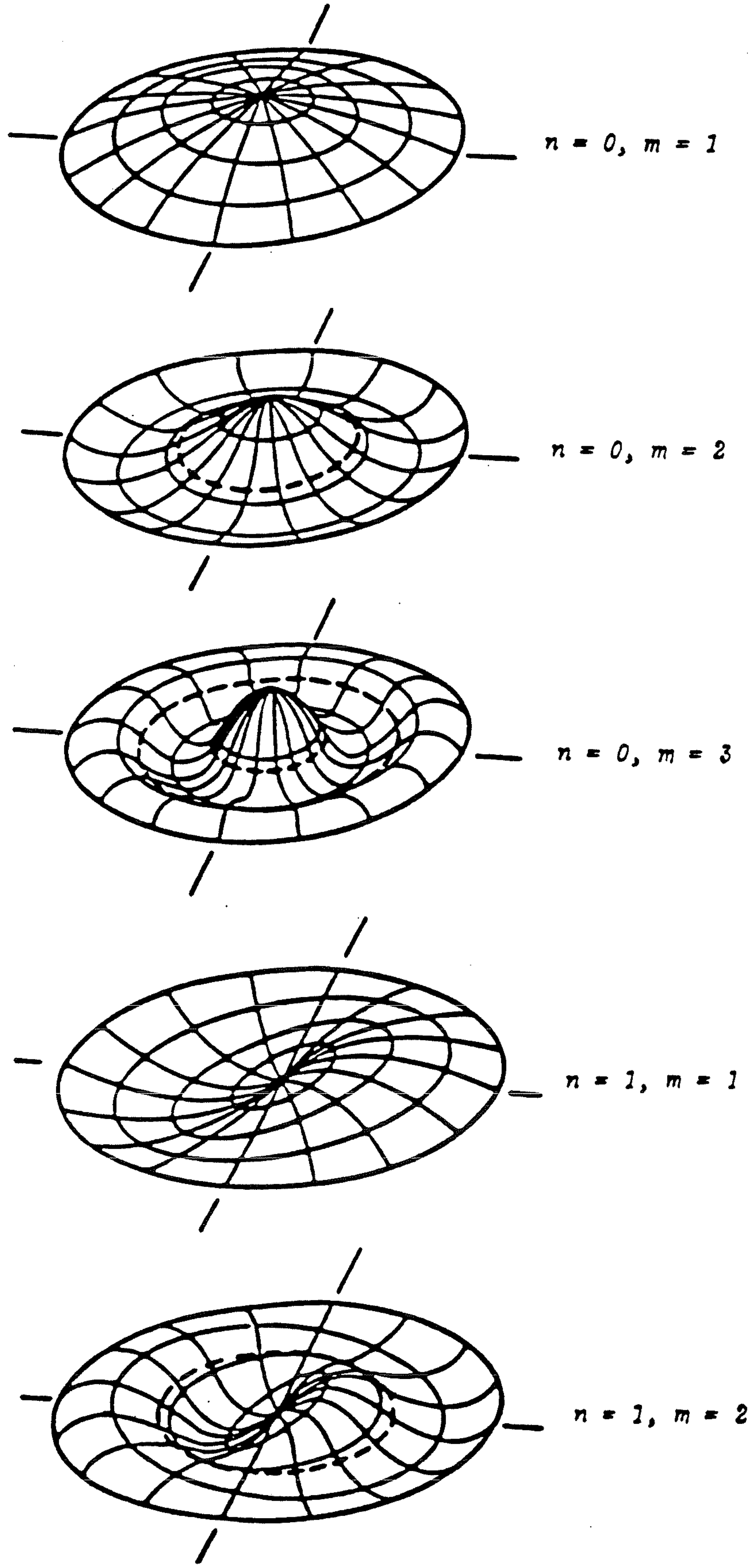


Figure 7.8: Vibration Modes

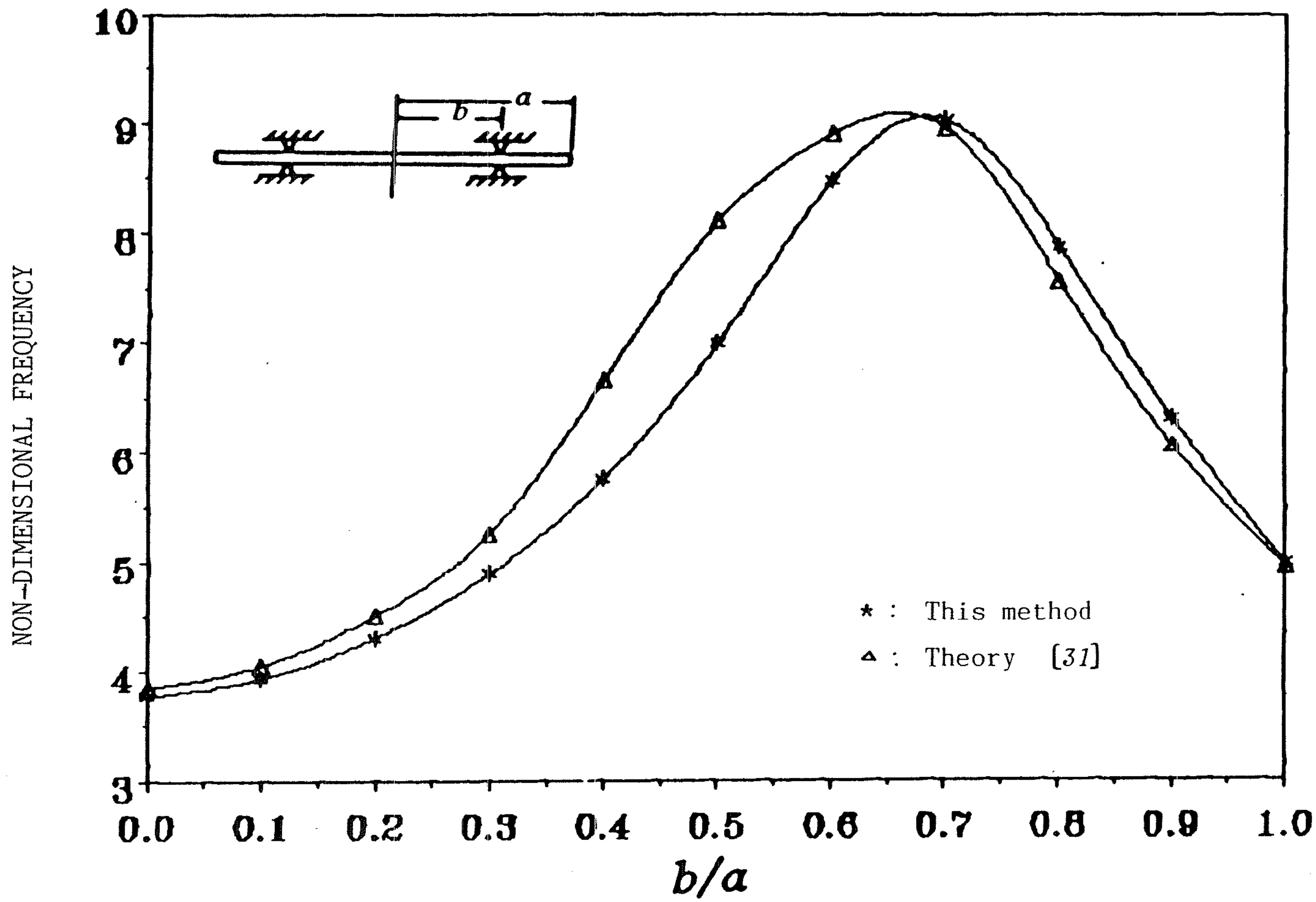


Figure 7.9: Non-dimensional natural frequency of a circular plate simply-supported along an arbitrary circle (n = 0, m = 1).

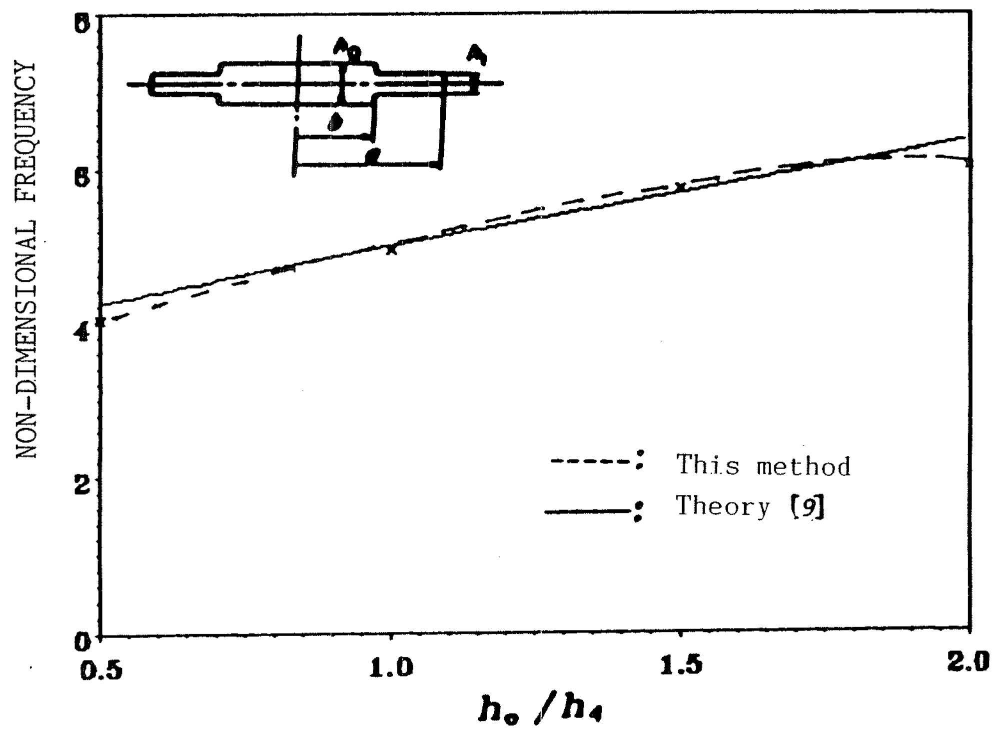


Figure 7.10: Non-dimensional natural frequency of a simply-supported circular plate with a discontinuity of thickness ($n = 0$, $m = 1$, $b/a = 0.5$).

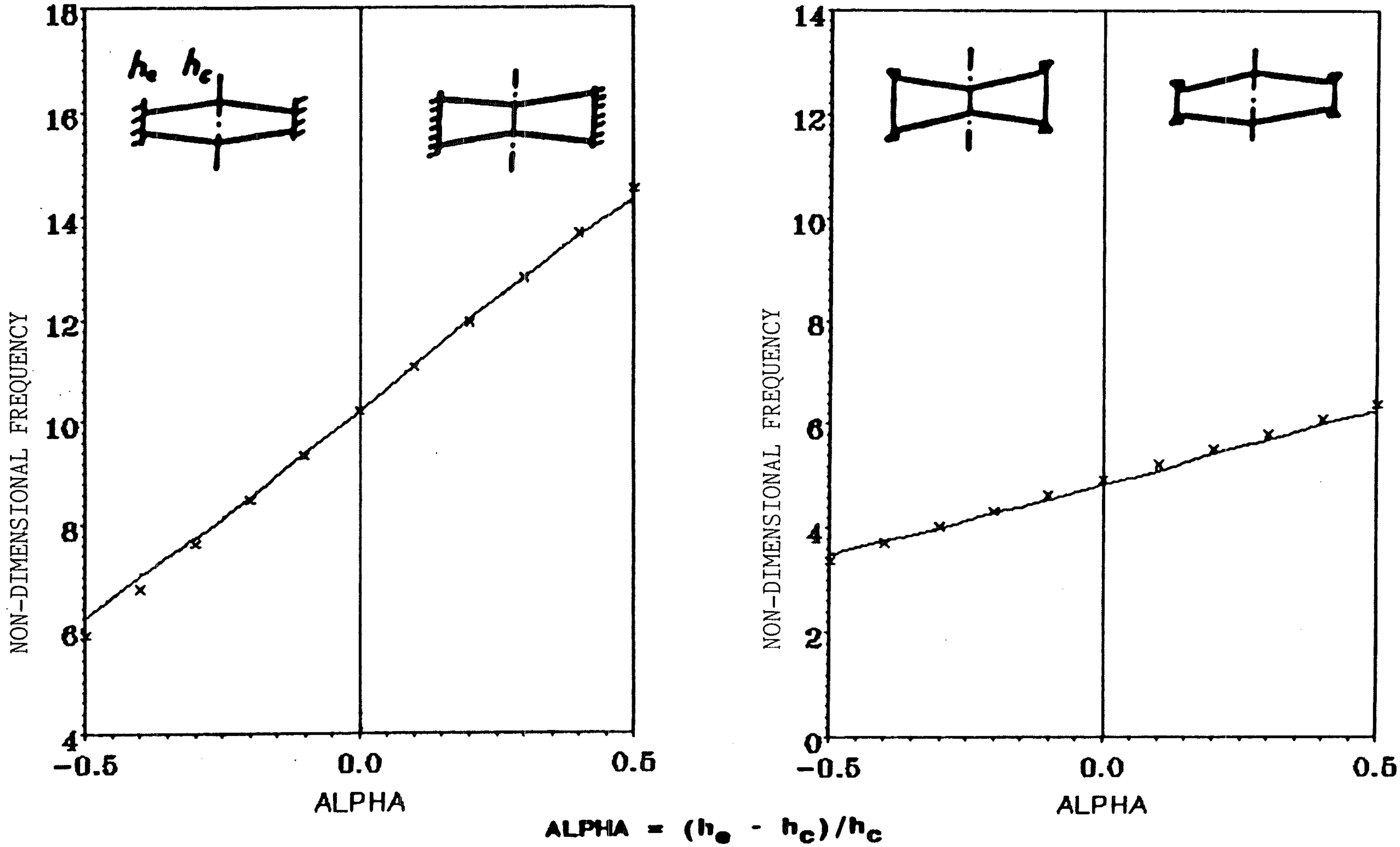


Figure 7.11: Non-dimensional natural frequency of a circular plate with a linear variation of thickness (n = 0, m = 1).

——— : This method
x : Transfer matrix method [10]

TABLE 4: NON-DIMENSIONAL NATURAL FREQUENCIES OF A CLAMPED - CLAMPED ANNULAR PLATE. CIRCUMFERENTIAL MODE n = 0

m \ b/a	0.1	0.3	0.5	0.7	0.9
1	27.281	45.346	89.25	248.43	1986.5
2	75.367	125.36	246.35	685.06	7055.1
3	148.22	246.17	483.25	1343.3	12273
4	245.53	407.31	799.15	2220.9	12907
5	367.31	608.86	1194.2	3318.7	13427
6	513.64	850.93	1668.8	4638.3	15778
7	684.63	1133.7	2223.1	6182.6	17094
8	880.47	1457.6	2657.8	7946.5	20248
9	1101.4	1823.0	3573.9	9945.5	23404
10	1347.9	2230.5	4372.6	12179	24516

TABLE 5: NON-DIMENSIONAL NATURAL FREQUENCIES OF A CLAMPED - CLAMPED ANNULAR PLATE. CIRCUMFERENTIAL MODE n = 1

m \ b/a	0.1	0.3	0.5	0.7	0.9
1	28.916	46.644	90.230	249.16	2258.2
2	78.63	127.38	247.74	686.04	6235.3
3	152.55	248.52	484.81	1344.3	11828
4	250.65	409.86	800.82	2222.0	15690
5	373.04	611.54	1196.0	3319.4	16653
6	519.86	853.71	1670.5	4637.2	18713
7	691.24	1136.6	2224.9	6176.6	19414
8	887.42	1460.5	2859.7	7939.0	19690
9	1108.7	1825.9	3575.9	9926.1	20402
10	1355.4	2233.5	4374.6	12140	21280

TABLE 6: NON-DIMENSIONAL NATURAL FREQUENCIES OF A CLAMPED - CLAMPED ANNULAR PLATE. CIRCUMFERENTIAL MODE n = 2

m \ b/a	0.1	0.3	0.5	0.7	0.9
1	36.617	51.139	93.321	251.48	2246.0
2	90.450	133.68	251.98	689.20	6290.6
3	167.15	255.69	489.52	1347.8	12265
4	267.31	417.57	805.80	2225.6	20121
5	391.30	619.63	1201.1	3323.2	21618
6	539.39	862.08	1675.8	4641.5	29330
7	711.82	1145.2	2230.3	6181.6	29439
8	908.89	1469.3	2865.2	7640.8	35301
9	1130.9	1834.9	3581.4	7944.8	36295
10	1378.4	2242.6	4380.3	9933.1	37643

TABLE 7: NON-DIMENSIONAL NATURAL FREQUENCIES OF A SIMPLY-SUPPORTED - SIMPLY-SUPPORTED ANNULAR PLATE CIRCUMFERENTIAL MODE n = 0

m \ b/a	0.1	0.3	0.5	0.7	0.9
1	14.485	21.079	40.043	110.08	937.95
2	51.782	81.737	158.64	439.21	5276.6
3	112.99	182.54	356.09	987.69	12247
4	198.47	323.61	632.53	1755.6	12490
5	308.31	505.02	988.04	2743.5	12840
6	442.59	726.86	1422.8	3952.3	13980
7	601.42	989.31	1937.2	5382.5	16989
8	784.97	1292.6	2531.6	7031.4	19994
9	993.46	1637.2	3207.0	8912.3	20459
10	1227.3	2023.6	3964.3	11019	22772

TABLE 8: NON-DIMENSIONAL NATURAL FREQUENCIES OF A SIMPLY-SUPPORTED - SIMPLY-SUPPORTED ANNULAR PLATE CIRCUMFERENTIAL MODE n = 1

m \ b/a	0.1	0.3	0.5	0.7	0.9
1	16.776	23.317	41.797	111.44	995.84
2	56.507	84.636	160.57	440.59	3959.4
3	119.33	185.65	358.06	989.00	8726.3
4	205.84	326.81	634.51	1756.8	15426
5	316.35	508.27	990.03	2744.2	15860
6	451.10	730.14	1424.8	3951.9	16978
7	610.26	992.61	1939.0	5380.6	19150
8	794.05	1295.9	2533.7	7031.5	19635
9	1002.7	1640.5	3209.0	8906.3	20004
10	1236.7	2026.9	3966.3	11007	20969

TABLE 9: NON-DIMENSIONAL NATURAL FREQUENCIES OF A SIMPLY-SUPPORTED - SIMPLY-SUPPORTED ANNULAR PLATE CIRCUMFERENTIAL MODE n = 2

m \ b/a	0.1	0.3	0.5	0.7	0.9
1	25.936	30.273	47.089	115.58	998.15
2	71.688	93.418	166.35	444.84	4035.9
3	138.87	195.04	363.96	993.28	8987.7
4	228.37	336.46	640.45	1761.1	15944
5	340.92	518.04	996.00	2748.6	21618
6	477.05	739.99	1430.8	3956.4	24789
7	637.20	1002.5	1945.2	5384.3	29291
8	821.72	1305.9	2539.7	7036.2	35081
9	1031.0	1650.5	3215.0	7640.8	35774
10	1265.4	2037.0	3972.4	8911.5	36295

TABLE 10: NON-DIMENSIONAL NATURAL FREQUENCIES OF A CLAMPED-FREE ANNULAR PLATE CIRCUMFERENTIAL MODE n = 0

m \ b/a	0.1	0.3	0.5	0.7	0.9
1	10.159	11.424	17.714	43.143	347.55
2	39.521	51.745	93.847	251.67	1840.8
3	90.447	132.41	252.20	692.09	6310.8
4	164.32	253.14	488.97	1350.1	12271
5	262.03	414.24	804.85	2227.7	12900
6	383.96	615.75	1199.9	3325.6	13040
7	530.32	857.80	1674.4	4645.2	14114
8	701.31	1140.6	2228.7	6189.6	17094
9	897.12	1464.4	2863.4	6703.8	19087
10	1118.1	1829.7	3579.4	7953.5	20070

TABLE 11: NON-DIMENSIONAL NATURAL FREQUENCIES OF A CLAMPED-FREE ANNULAR PLATE CIRCUMFERENTIAL mode n = 1

m \ b/a	0.1	0.3	0.5	0.7	0.9
1	21.195	19.540	22.015	45.333	355.19
2	60.062	59.760	97.376	253.72	2229.1
3	117.09	138.66	255.06	693.79	6282.4
4	192.62	258.51	419.58	1351.6	11969
5	288.93	419.12	807.33	2229.1	15690
6	408.52	620.33	1202.3	3326.4	16670
7	552.68	862.18	1676.7	4644.2	18351
8	721.87	1144.8	2231.0	6183.5	18777
9	916.30	1468.5	2865.7	6439.8	19453
10	1136.2	1833.8	3581.7	7945.7	19704

TABLE 12: NON-DIMENSIONAL NATURAL FREQUENCIES OF A CLAMPED-FREE ANNULAR PLATE CIRCUMFERENTIAL MODE n = 2

m \ b/a	0.1	0.3	0.5	0.7	0.9
1	34.536	32.594	32.116	51.586	378.02
2	83.480	79.061	107.49	259.91	2251.6
3	151.35	156.47	263.56	699.13	6324.3
4	238.07	274.35	499.38	1356.6	11669
5	343.52	433.65	814.72	2234.0	12311
6	468.13	634.01	1209.4	3331.3	18330
7	613.33	875.28	1683.7	4649.3	20219
8	781.10	1157.5	2237.8	5037.0	29409
9	973.13	1480.9	2872.5	6189.1	29535
10	1190.5	1845.9	3565.0	6473.2	32624

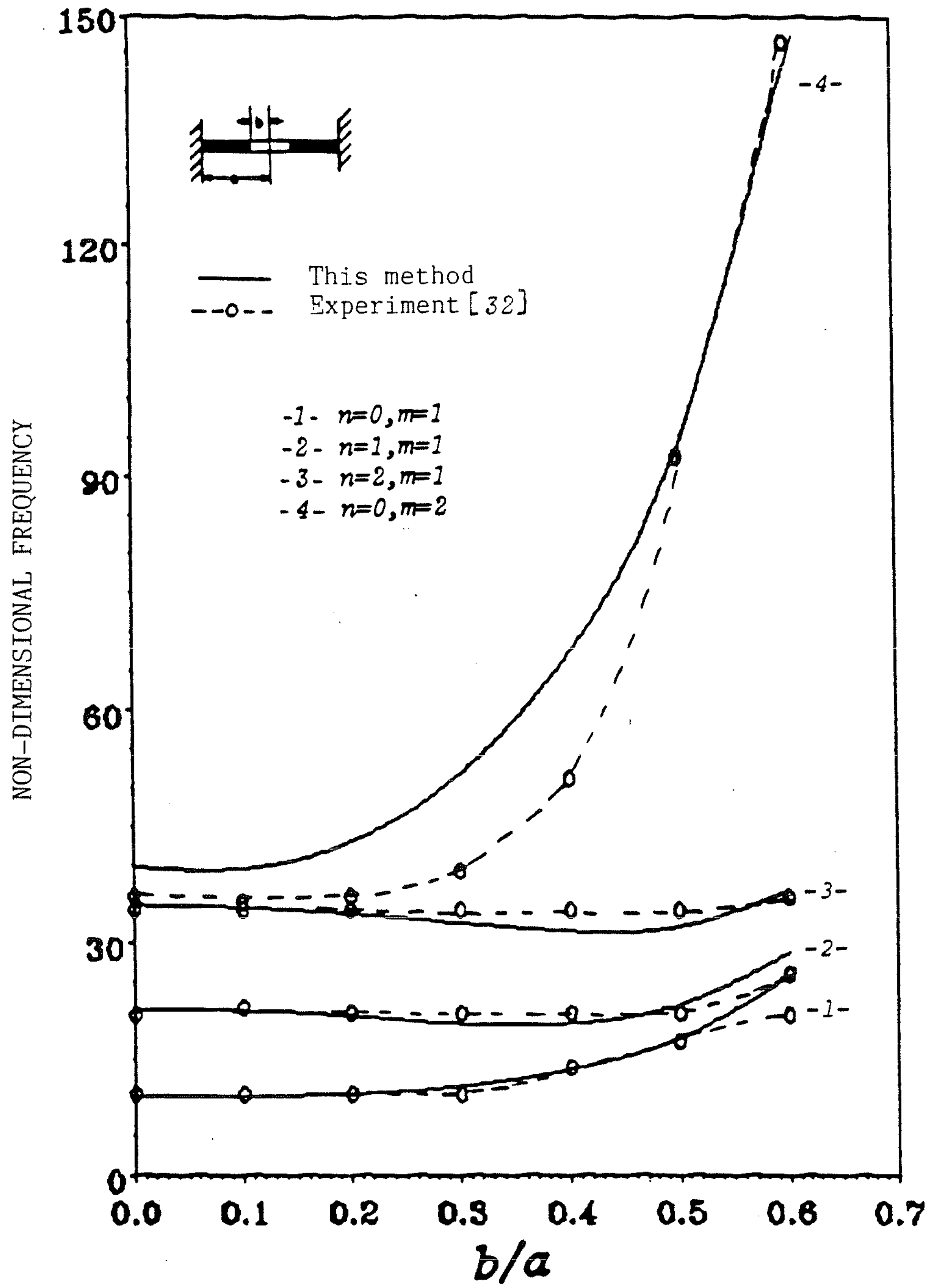


Figure 7.12: Non-dimensional natural frequency of a clamped-free plate.

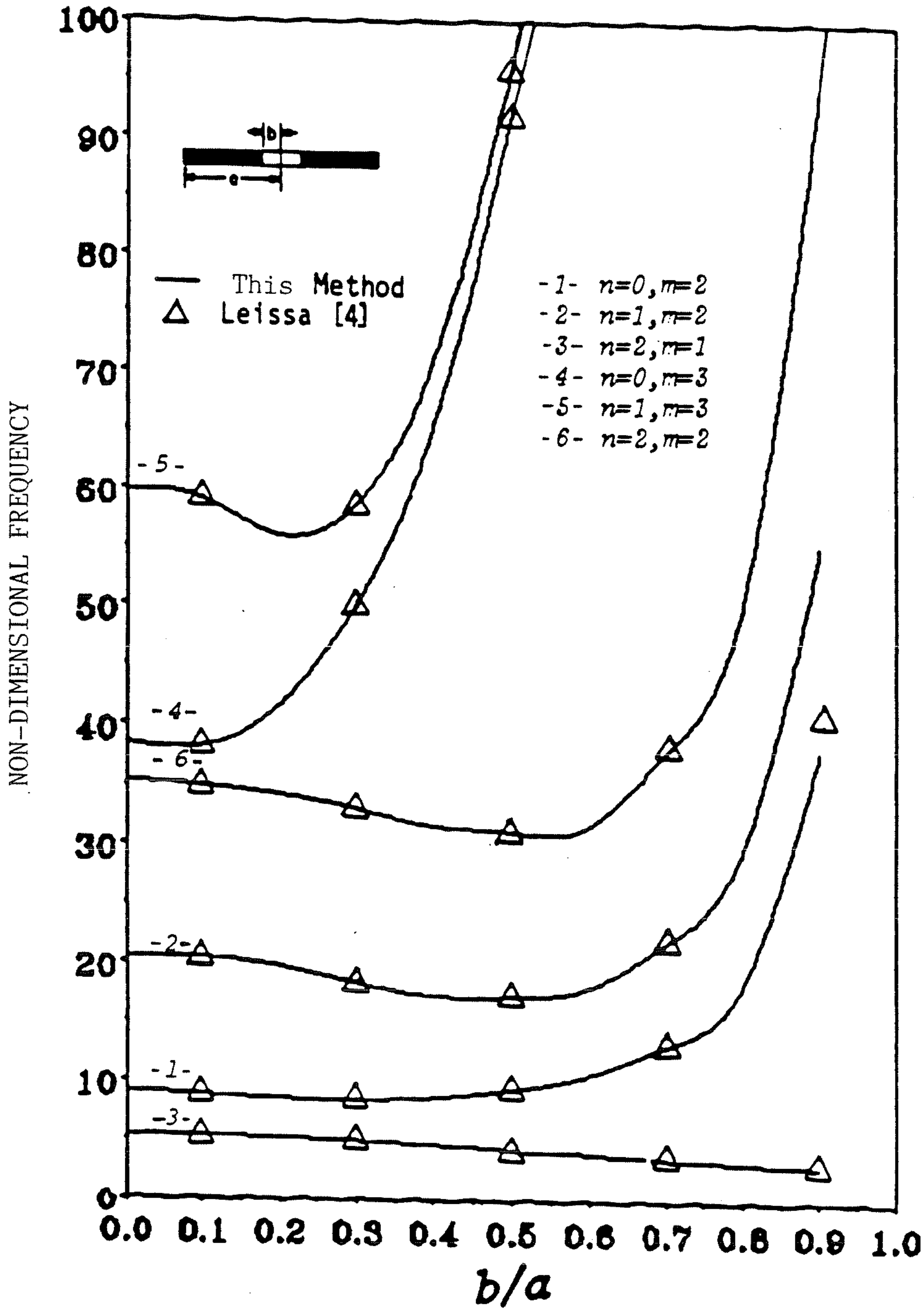


Figure 7.13: Non-dimensional natural frequency of a free-free plate.

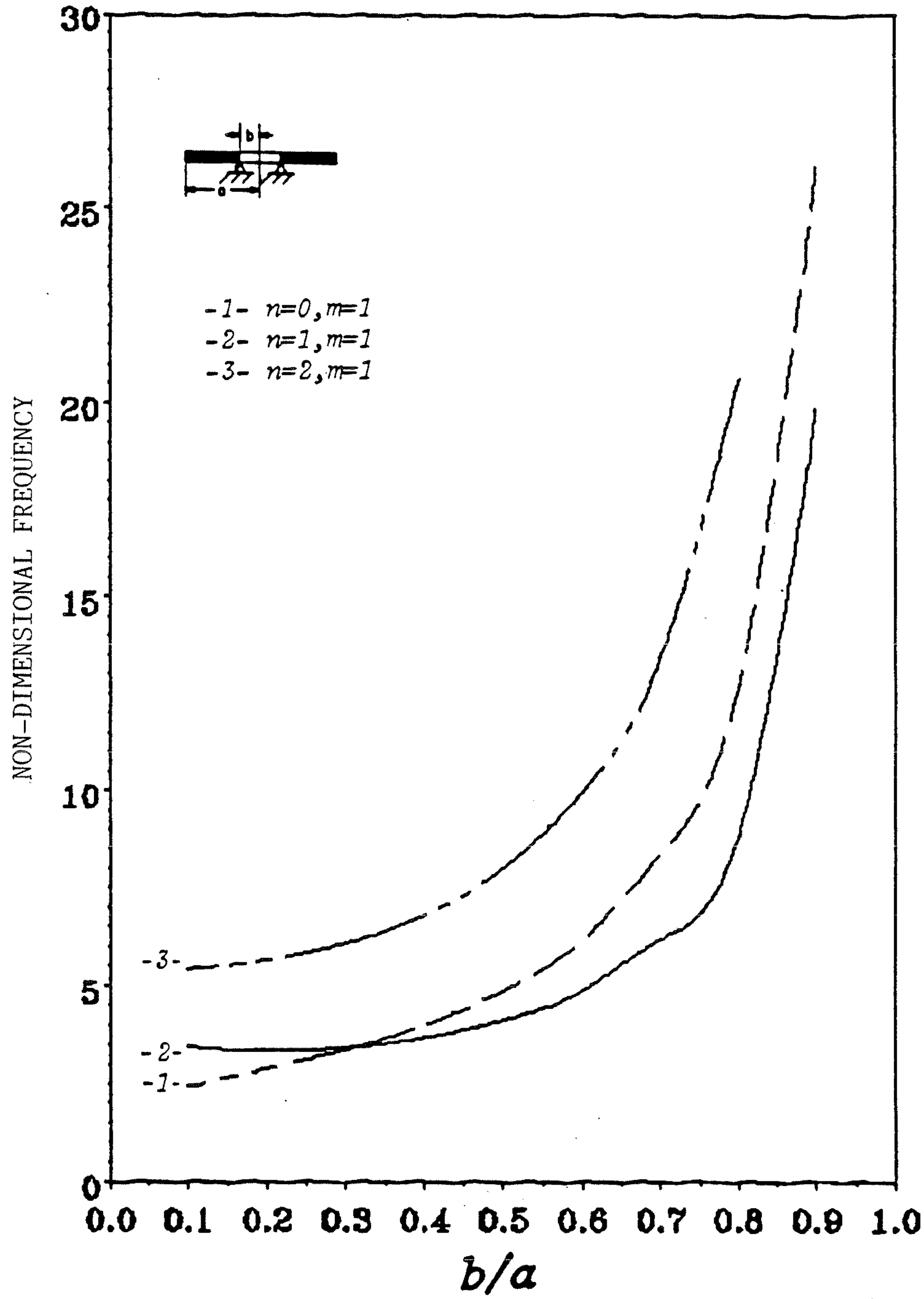


Figure 7.14: Non-dimensional natural frequency of a free - simply-supported plate.

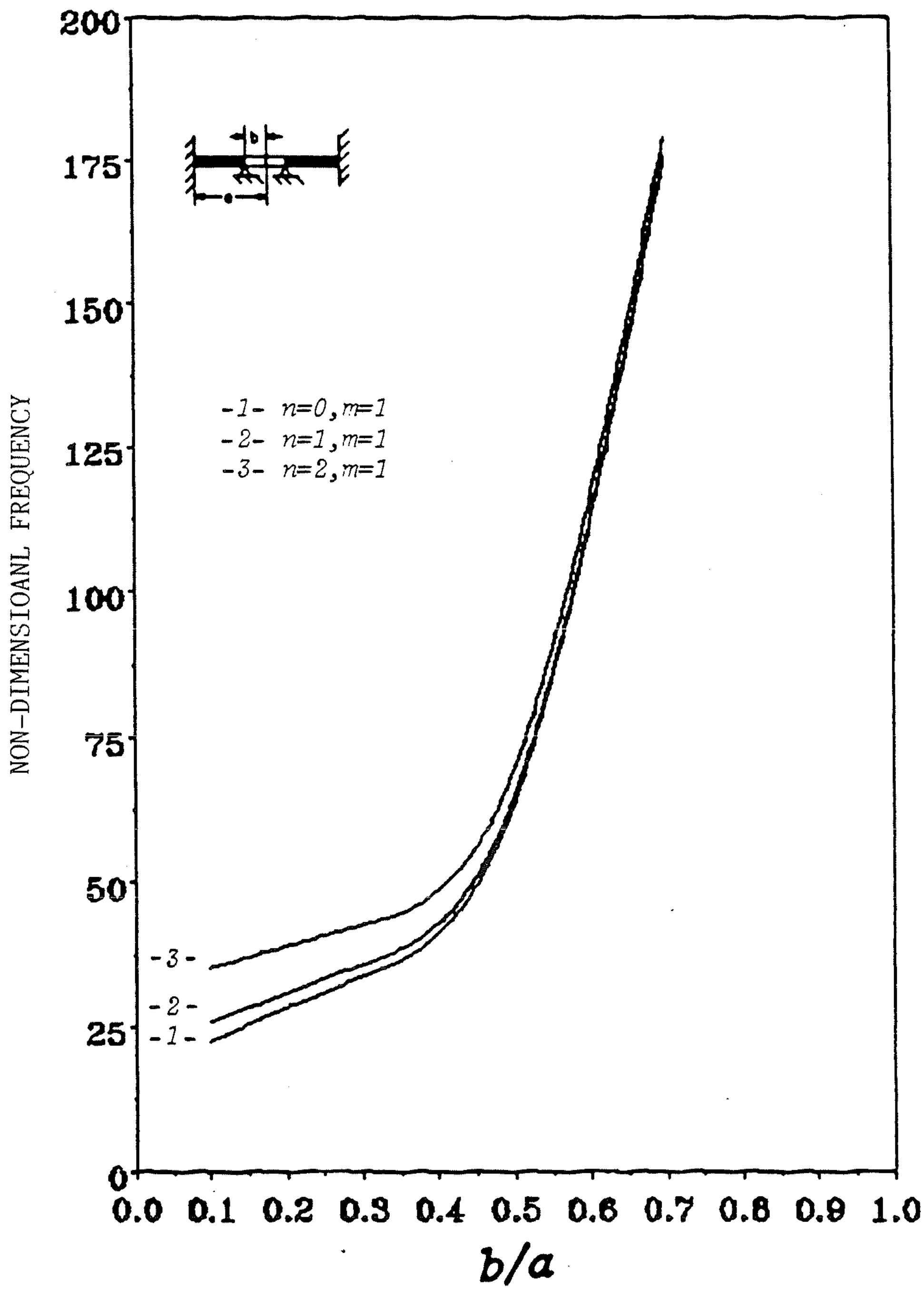


Figure 7.15: Non-dimensional natural frequency of a clamped - simply-supported plate.

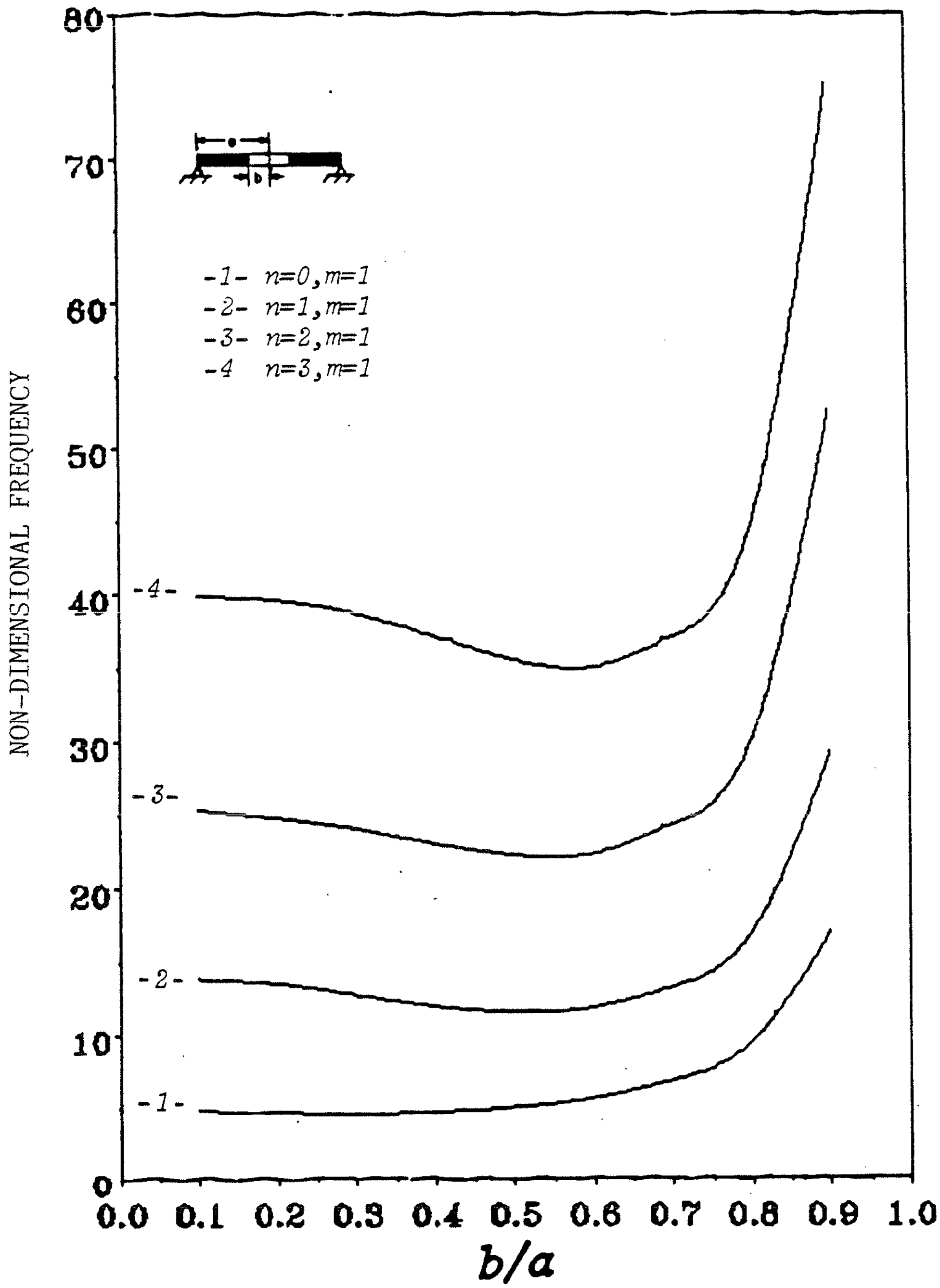


Figure 7.16: Non-dimensional natural frequency of a simply-supported - free plate.

TABLE 13: INFLUENCE OF POISSON'S RATIO ν ON THE NON-DIMENSIONAL NATURAL FREQUENCIES OF A SIMPLY-SUPPORTED CIRCULAR PLATE

n	m	$\frac{\Omega\,(0.5)}{\Omega\,(0)}$ *	
		Leissa [5]	Cette méthode
0	1	1.17307	1.17308
0	2	1.01981	1.01982
0	3	1.00743	1.00742
0	11	1.00450	1.00520
1	1	1.04736	1.04825
2	1	1.02375	1.02470

* $\Omega(0.5)$: non-dimensional natural frequency calculated for $\nu = 0.5$
$\Omega(0)$: non-dimensional natural frequency calculated for $\nu = 0$

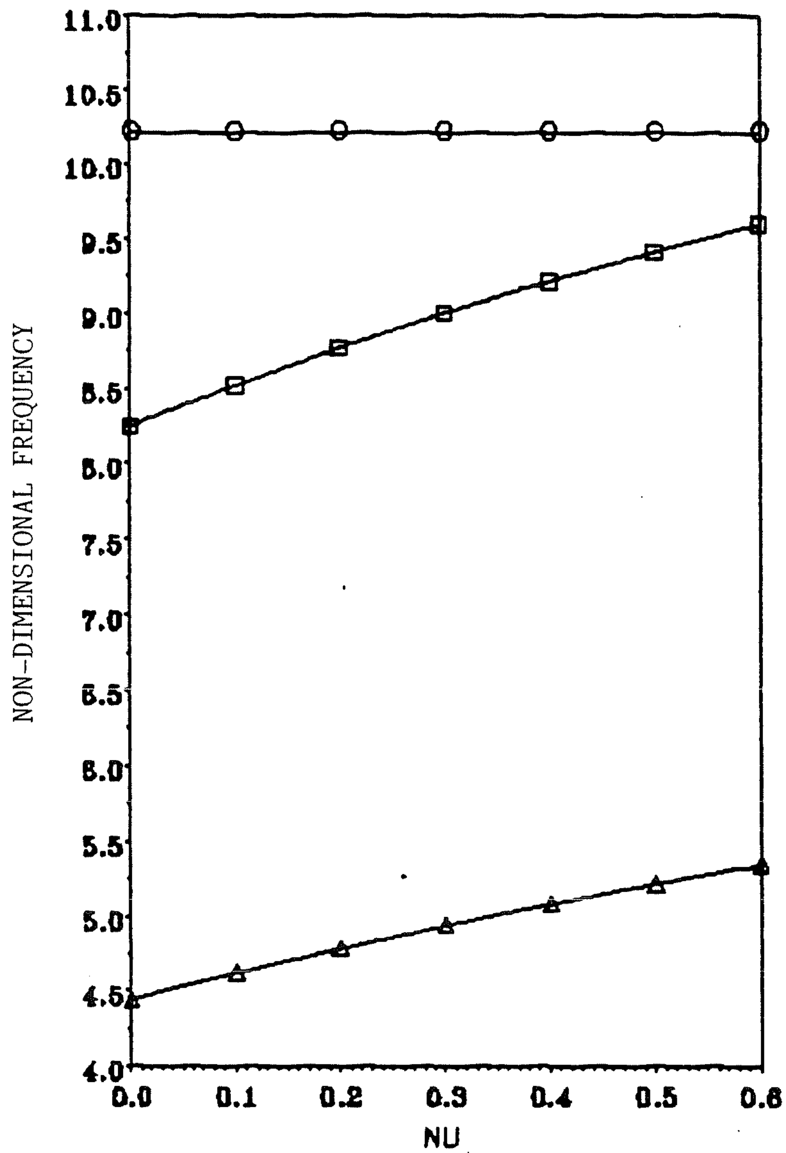


Figure 7.17: Non-dimensional natural frequency as a function of Poisson's ratio (n = 0, m = 1).

O : clamped circular plate
□ : free circular plate
Δ : simply-supported circular plate

CHAPTER VIII

CONCLUSION

The objective of this work was to present a new method for the static and dynamic analysis of thin, elastic, isotropic, non-uniform circular and annular plates. The method combined circular plate theory with finite element analysis.

Two types of finite element were developed, the first was an element of the circular plate type and the second was of the annular plate type. The displacement functions, the mass and stiffness matrices were developed for circumferential modes $n = 0$, $n = 1$ and $n \geqq 2$.

The convergence of the method was established and the natural frequencies were obtained for various boundary conditions and for different circumferential and radial modes. These were compared with the results of other investigations and satisfactory agreement was obtained.

This method combines the advantages of finite element analysis which deals with complex plates (variable thickness, non-uniform materials, various boundary conditions, different types of load), and the precision of formulation which the use of displacement functions derived from circular plate theory contributes.

The method enabled us to supplement the few results available on high natural frequencies associated with high circumferential and radial modes. It also enabled us to determine the natural frequencies and vibration modes of non-uniform annular and circular plates.

We consider that we have, here, a method by which it is possible to predict the vibrationary characteristics of circular and annular plates.

It is not possible, however, to apply this method either to circular or annular plates with non-symmetrical holes or to the study of plates with non-symmetrical boundary conditions.

The overall objective of the research group directed by Dr. Lakis is to find a numerical model for a shell of revolution, when empty or partially or completely filled with water. To this end, the research group has already developed cylindrical [18] to [26], conical [27, 28], and spherical [29] elements, and circular and annular plate elements [this paper].

The next step in this line of work should be the analysis of linear and non-linear anisotropic plates, and of the dynamic stability of circular and annular plates.

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APPENDIX A-1

SANDERS' CONICAL SHELL THEORY

a) General equilibrium equations

In the system of coordinates shown in Figure A-1.1, five of the six equilibrium equations derived from reference [15, 26] for a conical shell with zero exterior load can be written as follows:

$$\frac{\partial(r\ M_x)}{\partial x} + \frac{\partial\bar{N}_{x\theta}}{\partial\theta} - N_\theta \frac{\partial r}{\partial x} - \frac{1}{2}\frac{\partial}{\partial\theta}\left(\frac{\bar{M}_{x\theta}}{r_\theta}\right) = 0 \qquad \text{(a)}$$

$$\frac{\partial(r\ \bar{N}_{x\theta})}{\partial x} + \frac{\partial N_\theta}{\partial\theta} + \bar{N}_{x\theta}\frac{\partial r}{\partial x} + \frac{r}{r_\theta} Q_\theta + \frac{r}{2}\frac{\partial}{\partial x}\left(\frac{\bar{M}_{x\theta}}{r_\theta}\right) = 0 \qquad \text{(b)}$$

$$\frac{\partial(r\ Q_x)}{\partial x} + \frac{\partial Q_\theta}{\partial\theta} - \frac{N_\theta}{r_\theta} r = 0 \qquad \text{(c)}$$

(A-1.1)

$$\frac{\partial(r\ M_x)}{\partial x} + \frac{\partial\bar{M}_{x\theta}}{\partial x} - M_\theta \frac{\partial r}{\partial x} - r\ Q_x = 0 \qquad \text{(d)}$$

$$\frac{\partial(r\ \bar{M}_{x\theta})}{\partial x} + \frac{\partial M_\theta}{\partial\theta} + \bar{M}_{x\theta}\frac{\partial r}{\partial x} - r\ Q_\theta = 0 \qquad \text{(e)}$$

The sixth is verified in the same manner

where N_x, N_θ, $\overline{N_{x\theta}}$, M_x, M_θ and $\overline{M_{x\theta}}$ are the stress components, Q_x and Q_θ are the shear forces (Figure A-1.2), x and θ represent the

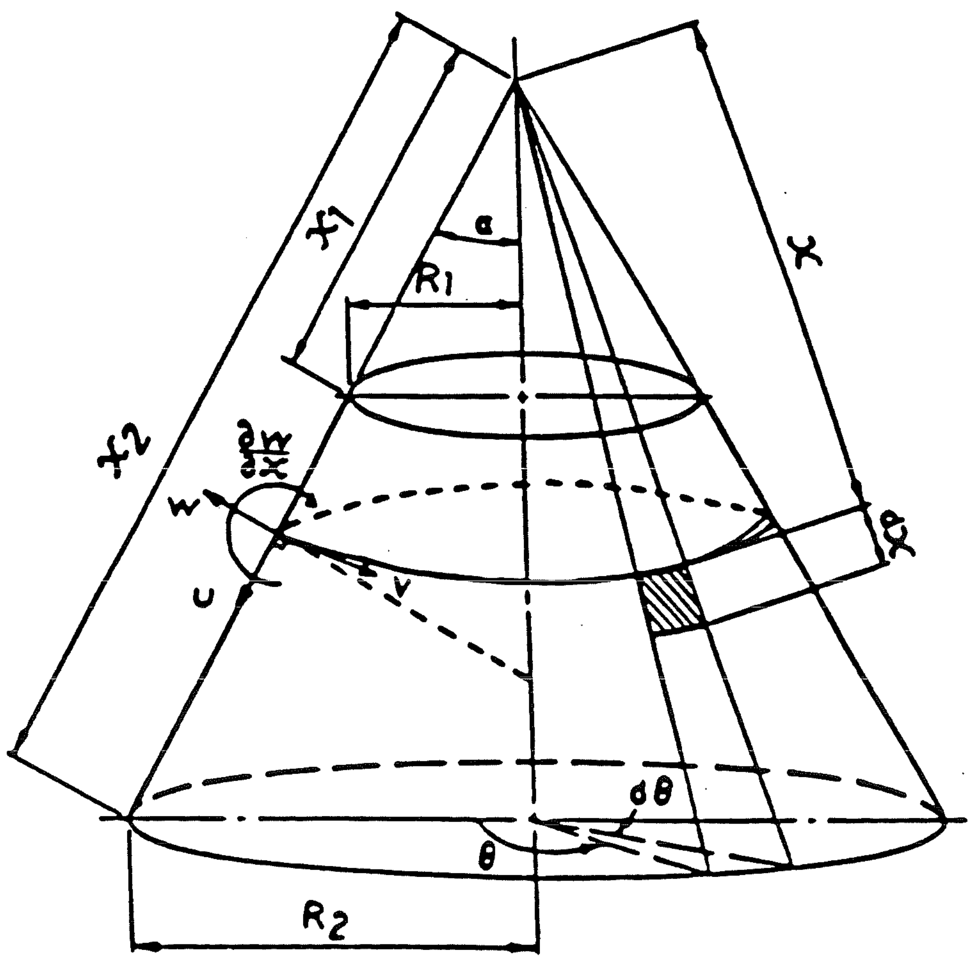

Figure A-1.2: Geometry of the mean surface of a conical shell.

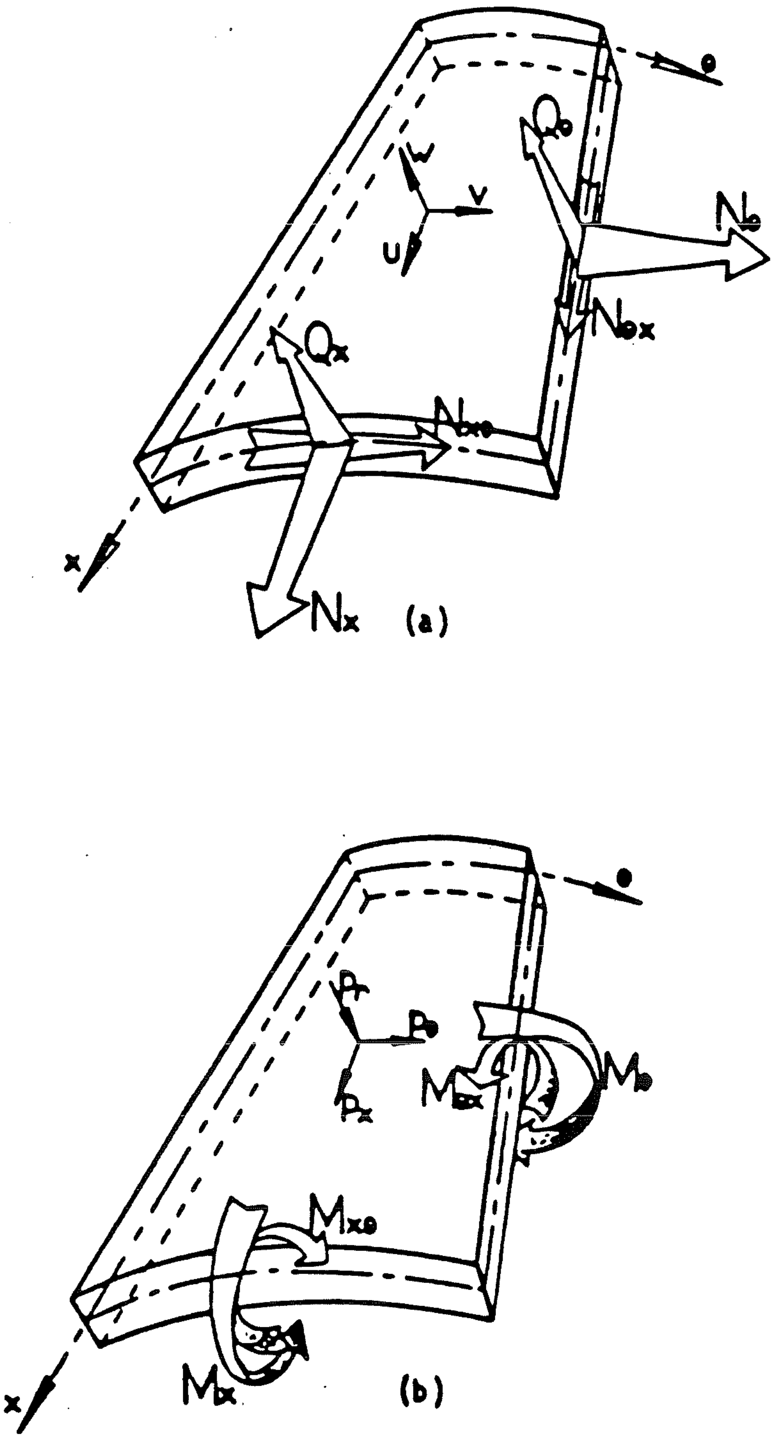

Figure A-1.1: Differential element for a conical shell
(a) Displacements and stress components
(b) Torque components and load factors

meridional and angular coordinates. α is the half-angle at the summit of the cone.

By eliminating the shear forces Q_x and Q_θ in the middle of equations (A-1.1 d, e), we obtain the following three basic equations:

$$r\,\frac{\partial N_x}{\partial x} + N_x \sin\alpha + \frac{\partial \bar{N}_{x\theta}}{\partial \theta} - N_\theta \sin\alpha - \frac{\cos\alpha}{2r}\,\frac{\partial \bar{M}_{x\theta}}{\partial \theta} = 0$$

$$r\,\frac{\partial \bar{N}_{x\theta}}{\partial x} + 2\,\bar{N}_{x\theta}\sin\alpha + \frac{\partial N_\theta}{\partial \theta} + \frac{3\cos\alpha}{2}\,\frac{\partial \bar{M}_{x\theta}}{\partial x} + \frac{3\sin 2\alpha}{4r}\,\bar{M}_{x\theta} + \frac{\cos\alpha}{r}\,\frac{\partial M_\theta}{\partial \theta} = 0$$

$$\text{(A-1.2)}$$

$$r\,\frac{\partial^2 M_x}{\partial x^2} + 2\sin\alpha\,\frac{\partial M_x}{\partial x} + 2\,\frac{\partial^2 \bar{M}_{x\theta}}{\partial x\,\partial \theta} + \frac{2\sin\alpha}{r}\,\frac{\partial \bar{M}_{x\theta}}{\partial \theta} + \frac{1}{r}\,\frac{\partial^2 M_\theta}{\partial \theta^2} - \sin\alpha\,\frac{\partial M_\theta}{\partial x}$$

$$- N_\theta \cos\alpha = 0$$

b) Deformation Vector

$$
\{\epsilon\} = \left\{ \begin{array}{c} \epsilon_x \\[2ex] \epsilon_\theta \\[2ex] 2\overline{\epsilon_{x\theta}} \\[2ex] \chi_x \\[2ex] \chi_\theta \\[2ex] 2\overline{\chi_{x\theta}} \end{array} \right\} = \begin{array}{l} \dfrac{\partial U}{\partial x} \\[2ex] \dfrac{1}{r}\dfrac{\partial V}{\partial \theta} + \dfrac{U}{r}\sin\alpha + \dfrac{W}{r}\cos\alpha \\[2ex] \dfrac{\partial V}{\partial x} + \dfrac{1}{r}\dfrac{\partial U}{\partial \theta} - \dfrac{V}{r}\sin\alpha \\[2ex] -\dfrac{\partial^2 W}{\partial x^2} \\[2ex] \dfrac{\partial V}{\partial \theta}\dfrac{\cos\alpha}{r^2} - \dfrac{1}{r^2}\dfrac{\partial^2 W}{\partial \theta^2} - \dfrac{\partial W}{\partial x}\dfrac{\sin\alpha}{r} \\[2ex] -\dfrac{2}{r}\dfrac{\partial^2 W}{\partial x\,\partial \theta} + \dfrac{\partial V}{\partial x}\dfrac{3\cos\alpha}{2r} - \dfrac{\partial U}{\partial \theta}\dfrac{\cos\alpha}{2r^2} + \dfrac{\partial W}{\partial \theta}\dfrac{2\sin\alpha}{r^2} \\[2ex] -\dfrac{3\sin 2\alpha}{4r^2}V \end{array} \qquad \text{(A-1.3)}
$$

where χ_x, χ_θ, $2\overline{\chi_{x\theta}}$ show the bending of the mean surface, while ε_x, ε_θ, $2\overline{\varepsilon_{x\theta}}$ show its elongation; u, v and w are respectively the axial, tangential and radial displacements.

APPENDIX A-2

MATRICES

This list shows the matrices used in the preceding chapters.

Table 1: Matrices $[T_n]$, $[T_{nc}]$, $[T_1]$, $[T_{1c}]$, $[T_0]$ et $[T_{0c}]$

Table 2: Matrices $[A_n]$, $[A_1]$, $[A_0]$, $[A_{nc}]$, $[A_{1c}]$ et $[A_{0c}]$

Table 3: Matrices $[R_n]$, $[F_n]$, $[H_n]$, $[R_1]$, $[R_0]$, $[R_{nc}]$, $[F_{nc}]$, $[H_{nc}]$, $[R_{1c}]$, $[F_{1c}]$, $[H_{1c}]$ et $[R_{0c}]$

Table 4: Matrices $[Q_n]$, $[B_n]$, $[C_n]$, $[Q_1]$, $[B_1]$, $[C_1]$, $[Q_0]$, $[Q_{nc}]$, $[B_{nc}]$, $[C_{nc}]$, $[Q_{1c}]$, $[B_{1c}]$, $[C_{1c}]$ et $[Q_{0c}]$

Table 5: Matrix $[H]$

Table 6: Matrices $[P]$ et $[P_0]$

Table 1: Matrices $[T_n]$, $[T_{nc}]$, $[T_1]$, $[T_{1c}]$, $[T_0]$ et $[T_{0c}]$

Matrices $[T_n]$ et $[T_{nc}]$

$$[T_n] = [T_{nc}] = \begin{bmatrix} \cos n\theta & 0 & 0 \\ 0 & \cos n\theta & 0 \\ 0 & 0 & \sin n\theta \end{bmatrix}$$

Matrices $[T_1]$ et $[T_{1c}]$

$$[T_1] = [T_{1c}] = \begin{bmatrix} \cos \theta & 0 \\ 0 & \cos \theta \end{bmatrix}$$

Matrices $[T_0]$ et $[T_{0c}]$

$$[T_0] = [T_{0c}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Table 2: Matrices $[A_n]$, $[A_1]$, $[A_0]$, $[A_{nc}]$, $[A_{1c}]$ et $[A_{0c}]$

Matrix $[A_n]$

$$[A_n] = \begin{bmatrix} y_o^{\frac{\lambda_1-1}{2}} & y_o^{\frac{\lambda_2-1}{2}} & y_o^{\frac{\lambda_3-1}{2}} & y_o^{\frac{\lambda_4-1}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y_o^{n} & y_o^{-n} & y_o^{n+2} & y_o^{-n+2} \\ 0 & 0 & 0 & 0 & \frac{n}{a}\, y_o^{n-1} & -\frac{n}{a}\, y_o^{-n-1} & \frac{n+2}{a}\, y_o^{n+1} & -\frac{n+2}{a}\, y_o^{-n+1} \\ \alpha 1\, y_o^{\frac{\lambda_1-1}{2}} & \alpha 2\, y_o^{\frac{\lambda_2-1}{2}} & \alpha 3\, y_o^{\frac{\lambda_3-1}{2}} & \alpha 4\, y_o^{\frac{\lambda_4-1}{2}} & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \frac{n}{a} & -\frac{n}{a} & \frac{n+2}{a} & \frac{-n+2}{a} \\ \alpha 1 & \alpha 2 & \alpha 3 & \alpha 4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix $[A_1]$

$$[A_1] = \left[\begin{array}{c|c|c|c|c|c}
y_o^{\frac{\lambda_1-1}{2}} & y_o^{\frac{\lambda_2-1}{2}} & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & y_o & y_o^3 & y_o^{-1} & y_o \operatorname{Ln} y_o \\ \hline
0 & 0 & \frac{1}{a} & \frac{3}{a} y_o^2 & -\frac{1}{a} y_o^{-2} & \frac{1}{a} (\operatorname{Ln} y_o + 1) \\ \hline
1 & 1 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 1 & 1 & 1 & 0 \\ \hline
0 & 0 & \frac{1}{a} & \frac{3}{a} & -\frac{1}{a} & \frac{1}{a}
\end{array}\right]$$

Matrix $[A_0]$

$$[A_0] = \left[\begin{array}{c|c|c|c|c|c}
y_o^{\frac{\lambda_1-1}{2}} & y_o^{\frac{\lambda_2-1}{2}} & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 1 & y_o^2 & \operatorname{Ln} y_o & y_o^2 \operatorname{Ln} y_o \\ \hline
0 & 0 & 0 & \frac{2}{a} y_o & \frac{1}{a} y_o^{-1} & \frac{y_o}{a} (1 + 2 \operatorname{Ln} y_o) \\ \hline
1 & 1 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 1 & 1 & 0 & 0 \\ \hline
0 & 0 & 0 & \frac{2}{a} & \frac{1}{a} & \frac{1}{a}
\end{array}\right]$$

Matrix [Anc]

$$[A_{nc}] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & \frac{n}{a} & \frac{n+2}{a} \\ \alpha_1 & \alpha_2 & 0 & 0 \end{bmatrix}$$

Matrix [A1c]

$$[A_{1c}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & \frac{2}{a} \end{bmatrix}$$

Matrix [A0c]

$$[A_{0c}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \frac{n}{a} & \frac{n+2}{a} \end{bmatrix}$$

Table 3: Matrices $[R_n]$, $[F_n]$, $[H_n]$, $[R_1]$, $[R_0]$, $[R_{nc}]$, $[F_{nc}]$, $[H_{nc}]$, $[R_{1c}]$, $[F_{1c}]$, $[H_{1c}]$ et $[R_{0c}]$

Matrices $[R_n]$, $[F_n]$ et $[H_n]$

$$R_n(i,j) = F_n(i,j)\ y^{H_n(i,j)} \qquad i = 1,3; \quad j = 1,8$$

$$[F_n] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \alpha 1 & \alpha 2 & \alpha 3 & \alpha 4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[H_n] = \begin{bmatrix} \frac{\lambda_1 - 1}{2} & \frac{\lambda_2 - 1}{2} & \frac{\lambda_3 - 1}{2} & \frac{\lambda_4 - 1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n & -n & n+2 & -n+2 \\ \frac{\lambda_1 - 1}{2} & \frac{\lambda_2 - 1}{2} & \frac{\lambda_3 - 1}{2} & \frac{\lambda_4 - 1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrices [Rnc], [Fnc] et [Hnc]

$$R_{nc}(i,j) = F_{nc}(i,j)\ y^{H_{nc}(i,j)} \qquad i = 1,3; \quad j = 4$$

$$[H_{nc}] = \begin{bmatrix} \frac{\lambda_1 - 1}{2} & \frac{\lambda_2 - 1}{2} & 0 & 0 \\ 0 & 0 & n & n+2 \\ \frac{\lambda_1 - 1}{2} & \frac{\lambda_2 - 1}{2} & 0 & 0 \end{bmatrix}$$

$$[F_{nc}] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \alpha_1 & \alpha_2 & 0 & 0 \end{bmatrix}$$

Matrix $[R_1]$

$$[R_1] = \left[\begin{array}{c|c|c|c|c|c} y^{\frac{\lambda_1-1}{2}} & y^{\frac{\lambda_2-1}{2}} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & y & y^3 & y^{-1} & y \operatorname{Ln} y \end{array}\right]$$

Matrix $[R_0]$

$$[R_0] = \left[\begin{array}{c|c|c|c|c|c} y^{\frac{\lambda_1-1}{2}} & y^{\frac{\lambda_2-1}{2}} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & y^2 & \operatorname{Ln} y & y^2 \operatorname{Ln} y \end{array}\right]$$

Matrices $[R_{1c}]$, $[F_{1c}]$ et $[H_{1c}]$

$$R_{1c}(i,j) = F_{1c}(i,j)\; y^{H_{1c}(i,j)} \qquad i = 1,2; \quad j = 1,3$$

$$[H_{1c}] = \left[\begin{array}{c|c|c} \frac{\lambda_1 - 1}{2} & 0 & 0 \\ \hline 0 & n & n+2 \end{array}\right]$$

$$[F_{1c}] = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 1 \end{array}\right]$$

Matrix $[R_{0c}]$

$$[R_{0c}] = \left[\begin{array}{c|c|c} y^{\frac{\lambda_1 - 1}{2}} & 0 & 0 \\ \hline 0 & 1 & y^2 \end{array}\right]$$

Table 4: Matrices $[Q_n]$, $[B_n]$, $[C_n]$, $[Q_1]$, $[B_1]$, $[C_1]$, $[Q_0]$, $[Q_{nc}]$, $[B_{nc}]$, $[C_{nc}]$, $[Q_{1c}]$, $[B_{1c}]$, $[C_{1c}]$ et $[Q_{0c}]$

Matrices $[Q_n]$, $[B_n]$ et $[C_n]$

$$Q_n(i,j) = B_n(i,j)\, y^{C_n(i,j)} \qquad i = 1,6; \quad j = 1,8$$

$$[B_n] = \begin{bmatrix}
\frac{\lambda_1-1}{2a} & \frac{\lambda_2-1}{2a} & \frac{\lambda_3-1}{2a} & \frac{\lambda_4-1}{2a} & 0 & 0 & 0 & 0 \\
\frac{1+\alpha_1 n}{a} & \frac{1+\alpha_2 n}{a} & \frac{1+\alpha_3 n}{a} & \frac{1+\alpha_4 n}{a} & 0 & 0 & 0 & 0 \\
\frac{1}{a}[-n+\alpha_1(\frac{\lambda_1-3}{2})] & \frac{1}{a}[-n+\alpha_2(\frac{\lambda_2-3}{2})] & \frac{1}{a}[-n+\alpha_3(\frac{\lambda_3-3}{2})] & \frac{1}{a}[-n+\alpha_4(\frac{\lambda_4-3}{2})] & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{-n(n-1)}{a^2} & \frac{-n\,(n+1)}{a^2} & \frac{-(n+2)(n+1)}{a^2} & \frac{-(1-n)(2-n)}{a^2} \\
0 & 0 & 0 & 0 & \frac{n\,(n-1)}{a^2} & \frac{n\,(n+1)}{a^2} & \frac{n^2-n-2}{a^2} & \frac{n^2+n-2}{a^2} \\
0 & 0 & 0 & 0 & \frac{2n(n-1)}{a^2} & \frac{-2n\,(n+1)}{a^2} & \frac{2n\,(n+1)}{a^2} & \frac{-2n\,(n-1)}{a^2}
\end{bmatrix}$$

$$[C_n] = \begin{bmatrix} \frac{\lambda_1-3}{2} & \frac{\lambda_2-3}{2} & \frac{\lambda_3-3}{2} & \frac{\lambda_4-3}{2} & 0 & 0 & 0 & 0 \\ \frac{\lambda_1-3}{2} & \frac{\lambda_2-3}{2} & \frac{\lambda_3-3}{2} & \frac{\lambda_4-3}{2} & 0 & 0 & 0 & 0 \\ \frac{\lambda_1-3}{2} & \frac{\lambda_2-3}{2} & \frac{\lambda_3-3}{2} & \frac{\lambda_4-3}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n-2 & -n-2 & n & -n \\ 0 & 0 & 0 & 0 & n-2 & -n-2 & n & -n \\ 0 & 0 & 0 & 0 & n-2 & -n-2 & n & -n \end{bmatrix}$$

Matrices $[Q_1]$, $[B_1]$ et $[C_1]$

$$Q_1(i,j) = B_1(i,j)\, y^{C_1(i,j)} \qquad i = 1,6; \quad j = 1,6$$

$$[B_1] = \begin{bmatrix} \frac{\lambda_1-1}{2a} & \frac{\lambda_2-1}{2a} & 0 & 0 & 0 & 0 \\ \frac{1}{a} & \frac{1}{a} & 0 & 0 & 0 & 0 \\ -\frac{1}{a} & -\frac{1}{a} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{6}{a^2} & -\frac{2}{a^2} & -\frac{1}{a^2} \\ 0 & 0 & 0 & -\frac{2}{a^2} & \frac{2}{a^2} & -\frac{1}{a^2} \\ 0 & 0 & 0 & \frac{4}{a^2} & -\frac{4}{a^2} & \frac{2}{a^2} \end{bmatrix}$$

$$[C_1] = \begin{bmatrix} \frac{\lambda_1-3}{2} & \frac{\lambda_2-3}{2} & 0 & 0 & 0 & 0 \\ \frac{\lambda_1-3}{2} & \frac{\lambda_2-3}{2} & 0 & 0 & 0 & 0 \\ \frac{\lambda_1-3}{2} & \frac{\lambda_2-3}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & -3 & -1 \end{bmatrix}$$

Matrix $[Q_0]$

$$[Q_0] = \begin{bmatrix} \frac{\lambda_1 - 1}{2a}\, y^{(\lambda_1 - 3)/2} & \frac{\lambda_2 - 1}{2a}\, y^{(\lambda_2 - 3)/2} & 0 & 0 & 0 & 0 \\ \frac{1}{a}\, y^{(\lambda_1 - 3)/2} & \frac{1}{a}\, y^{(\lambda_2 - 3)/2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{a^2} & \frac{1}{a^2}\, y^{-2} & -\frac{1}{a^2}(3+2\mathrm{Ln}y) \\ 0 & 0 & 0 & -\frac{2}{a^2} & -\frac{1}{a^2}\, y^{2} & -\frac{1}{a^2}(1+2\mathrm{Ln}y) \end{bmatrix}$$

Matrices $[Q_{nc}]$, $[B_{nc}]$ et $[C_{nc}]$

$$Q_{nc}(i,j) = B_{nc}(i,j)\ y^{C_{nc}(i,j)} \qquad i = 1,6; \quad j = 1,4$$

$$[C_{nc}] = \begin{bmatrix} \frac{\lambda_1 - 3}{2} & \frac{\lambda_2 - 3}{2} & 0 & 0 \\ \frac{\lambda_1 - 3}{2} & \frac{\lambda_2 - 3}{2} & 0 & 0 \\ \frac{\lambda_1 - 3}{2} & \frac{\lambda_2 - 3}{2} & 0 & 0 \\ 0 & 0 & n-2 & n \\ 0 & 0 & n-2 & n \\ 0 & 0 & n-2 & n \end{bmatrix}$$

$$[B_{nc}] = \begin{bmatrix} \frac{\lambda_1 - 1}{2a} & \frac{\lambda_2 - 1}{2a} & 0 & 0 \\ \frac{1+\alpha_1 n}{a} & \frac{1+\alpha_2 n}{a} & 0 & 0 \\ \frac{1}{a}[-n+\alpha_1(\frac{\lambda_1 - 3}{2})] & \frac{1}{a}[-n+\alpha_2(\frac{\lambda_2 - 3}{2})] & 0 & 0 \\ 0 & 0 & \frac{-n(n-1)}{a^2} & \frac{-(n+2)(n+1)}{a^2} \\ 0 & 0 & \frac{n\ (n-1)}{a^2} & \frac{n^2 - n - 2}{a^2} \\ 0 & 0 & \frac{2n(n-1)}{a^2} & \frac{2n\ (n+1)}{a^2} \end{bmatrix}$$

Matrices $[Q_{1c}]$, $[B_{1c}]$ et $[C_{1c}]$

$$Q_{1c}(i,j) = B_{1c}(i,j)\, y^{C_{1c}(i,j)} \qquad i = 1,6; \quad j = 1,3$$

$$[B_{1c}] = \begin{bmatrix} \frac{\lambda_1 - 1}{2a} & 0 & 0 \\ \frac{1+\alpha_1 n}{a} & 0 & 0 \\ -\frac{n}{a} & 0 & 0 \\ 0 & \frac{-n(n-1)}{a^2} & \frac{-(n+2)(n+1)}{a^2} \\ 0 & \frac{n\,(n-1)}{a^2} & \frac{n^2 - n - 2}{a^2} \\ 0 & \frac{2n(n-1)}{a^2} & \frac{2n\,(n+1)}{a^2} \end{bmatrix}$$

$$[C_{1c}] = \begin{bmatrix} \frac{\lambda_1 - 3}{2} & 0 & 0 \\ \frac{\lambda_1 - 3}{2} & 0 & 0 \\ \frac{\lambda_1 - 3}{2} & 0 & 0 \\ 0 & n-2 & n \\ 0 & n-2 & n \\ 0 & n-2 & n \end{bmatrix}$$

Matrix $[Q_{0c}]$

$$[Q_{c0}] = \left[\begin{array}{c|c|c} \frac{\lambda_1 - 1}{2a}\, y^{(\lambda_1 - 3)/2} & 0 & 0 \\ \hline \frac{1}{a}\, y^{(\lambda_1 - 3)/2} & 0 & 0 \\ \hline 0 & 0 & -\frac{2}{a^2} \\ \hline 0 & 0 & -\frac{2}{a^2} \end{array}\right]$$

TABLE 5: MATRIX [H]

$$H(1,1) = \frac{\lambda^2}{4} - \frac{\lambda}{2} - \left(\frac{1 - \upsilon}{2}\right) n^2 - \frac{3}{4}$$

$$H(1,2) = \left(\frac{1 + \upsilon}{4}\right) n \lambda - \frac{n}{4} (7 - \upsilon)$$

$$H(2,1) = \left(\frac{1 + \upsilon}{4}\right) n \lambda - \frac{n}{4} (5 - 3\upsilon)$$

$$H(2,2) = \left(\frac{1 - \upsilon}{4}\right) \lambda^2 - \left(\frac{1 - \upsilon}{4}\right) \lambda - 3\left(\frac{1 - \upsilon}{8}\right) - n^2$$

Table 6: Matrices [P] et [P0]

Matrices [P]

$$[P] = \begin{bmatrix} D & \upsilon D & 0 & 0 & 0 & 0 \\ \upsilon D & D & 0 & 0 & 0 & 0 \\ 0 & 0 & D\left(\frac{1-\upsilon}{2}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & K & \upsilon K & 0 \\ 0 & 0 & 0 & \upsilon K & K & 0 \\ 0 & 0 & 0 & 0 & 0 & K\left(\frac{1-\upsilon}{2}\right) \end{bmatrix}$$

Matrices [P0]

$$[P_0] = \begin{bmatrix} D & \upsilon D & 0 & 0 \\ \upsilon D & D & 0 & 0 \\ 0 & 0 & K & \upsilon K \\ 0 & 0 & \upsilon K & K \end{bmatrix}$$

APPENDIX A-3

DETERMINATION OF THE ELEMENTS OF [G] AND [S]

A-3.1 Matrix $[S_1]$ for an annular plate with n = 1

As matrix $[S_1]$ is symmetric, we give only the right-hand side of it:

$$S_1(1,1) = \left(\frac{\pi a^2}{\lambda_1+1}\right)\left(1 - y_0^{\lambda_1+1}\right)$$

$$S_1(1,2) = \left(\frac{\pi \alpha^2}{\frac{\lambda_1+\lambda_2}{2}+1}\right)\left(1 - y_0^{\frac{\lambda_1+\lambda_2}{2}+1}\right)$$

$$S_1(1,j) = 0 \quad \text{pour } j = 3,4,5,6$$

$$S_1(2,2) = \frac{\pi a^2}{\lambda_2+1}\left(1 - y_0^{\lambda_2+1}\right)$$

$$S_1(2,j) = 0 \quad \text{pour } j = 3,6$$

$$S_1(3,3) = \frac{\pi a^2}{4}\left(1 - y_0^4\right)$$

$$S_1(3,4) = \frac{\pi a^2}{4}\left(1 - y_0^6\right)$$

$$S_1(3,5) = \frac{\pi a^2}{2} (1 - y_0^2)$$

$$S_1(3,6) = \pi a^2 (- \frac{1}{16} - \frac{y_0^4}{4} \text{Ln } Y_0 + \frac{y_0^4}{16})$$

$$S_1(4,4) = \frac{\pi a^2}{8} (1 - y_0^8)$$

$$S_1(4,5) = \frac{\pi a^2}{4} (1 - y_0^4)$$

$$S_1(4,6) = \pi a^2 (- \frac{1}{36} - \frac{y_0^6}{6} \text{Ln } Y_0 + \frac{y_0^6}{36})$$

$$S_1(5,5) = - \pi a^2 \text{ Ln } y_0$$

$$S_1(5,6) = \pi a^2 (- \frac{1}{4} - \frac{y_0^2}{2} \text{Ln } Y_0 + \frac{y_0^4}{4})$$

$$S_1(6,6) = \pi a^2 [- \frac{1}{32} - \frac{y_0^4}{4} (\text{Ln}^2 Y_0 - \frac{1}{2} \text{Ln } Y_0 + \frac{1}{8})]$$

where $y = \frac{a_0}{a}$

a is the outside radius of the annular plate

a_0 is the inside radius

λ_1 and λ_2 are the roots of the characteristic equation

A-3.2 Matrix $[G_0]$ for an annular plate with n = 0

As matrix $[G_0]$ is symmetric, we give only the right-hand side of it:

$$G0(1,1) = \pi\,(p_{11} + 2p_{12} + p_{22})\,(1 - y0^2)$$

$$G0(1,j) = 0 \quad \text{pour } j = 2,3,4,5,6$$

$$G0(2,2) = \pi\,(p_{11} + p_{12} + p_{22})\,(1 - y0^2)$$

$$G0(2,j) = 0 \quad \text{pour } j = 3,4,5,6$$

$$G0(3,j) = 0 \quad \text{pour } j = 3,4,5,6$$

$$G0(4,4) = \left(\frac{4\pi}{a^2}\right)(p_{33} + 2p_{34} + p_{44})\,(1 - y0^2)$$

$$G0(4,5) = \left(\frac{4\pi}{a^2}\right)(p_{33} - p_{44})\,\text{Ln } y0$$

$$G0(4,6) = -\left(\frac{2\pi}{a^2}\right)(p_{33} + 2p_{34} + p_{44})\,[1 - 2y0^2(\text{Ln } y0 - \tfrac{1}{2})]$$

$$+ \left(\frac{2\pi}{a^2}\right)(3p_{33} + 4p_{34} + p_{44})\,(1 - y0^2)$$

$$G_0(5,5) = -\left(\frac{\pi}{a^2}\right) (p_{33} - 2p_{34} + p_{44}) (1 - y_0^2)$$

$$G_0(6,6) = \left(\frac{2\pi}{a^2}\right) (p_{33} + 2p_{34} + p_{44}) [1 + 2y_0^2(\mathrm{Ln}\ y_0^2 - \mathrm{Ln}\ y_0 - \tfrac{1}{2})]$$

$$- \left(\frac{2\pi}{a^2}\right) (3p_{33} + 2p_{34} + p_{44}) [1 - 2y_0^2(\mathrm{Ln}\ y_0 - \tfrac{1}{2})]$$

$$+ \frac{\pi}{a^2} (9p_{33} + 6p_{34} + p_{44}) [1 - y_0^2]$$

where $y_0 = \dfrac{a_0}{a}$

a_0 is the inside radius of the annular plate

a is the outside radius

Terms p_{ij} are the elements of matrix $[P_0]$ given in Table (6) of Appendix (A-2).

A-3.3 Matrix $[S_0]$ for an annular plate with n = 0

As matrix $[S_0]$ is symmetric, we give only the right-hand side of it:

$$S_0(1,2) = \left(\frac{2\pi\ \alpha^2}{\frac{\lambda_1+\lambda_2}{2}+1}\right)\left(1 - y_0^{\frac{\lambda_1+\lambda_2}{2}+1}\right)$$

$$S_0(1,j) = 0 \quad \text{pour } j = 3,4,5,6$$

$$S_0(2,2) = -\ 2\ \pi\ a^2\ \mathrm{Ln}\ y_0$$

$$S_0(2,j) = 0 \quad \text{pour } j = 3,4,5,6$$

$$S_0(3,3) = \pi\ a^2\ (1 - y_0^2)$$

$$S_0(3,4) = \frac{\pi\ a^2}{4}\ (1 - y_0^2)$$

$$S_0(3,5) = 2\pi\ a^2\ \left[-\ \frac{1}{4} - \frac{y_0^2}{2}\left(\mathrm{Ln}\ y_0 - \frac{1}{2}\right)\right]$$

$$S_0(3,6) = 2\pi\ a^2\ \left(-\ \frac{1}{16} - \frac{y_0^4}{4}\ \mathrm{Ln}\ y_0 + \frac{1}{4}\right)$$

$$S_0(4,4) = \frac{\pi\ a^2}{3}\ (1 - y_0^6)$$

$$S_0(4,5) = 2\ \pi\ a^2\ \left[-\ \frac{1}{16} - \frac{y_0^4}{4}\left(\mathrm{Ln}\ y_0 - \frac{1}{4}\right)\right]$$

$$S_0(4,6) = 2\pi\ a^2\ \left(-\ \frac{1}{36} - \frac{y_0^6}{6}\left(\mathrm{Ln}\ y_0 + \frac{1}{6}\right)\right]$$

$$S_0(5,5) = 2\pi a^2 \left[\frac{y_0^2}{2}(\mathrm{Ln}\, y_0 - \mathrm{Ln}^2\, y_0 - \frac{1}{2}) + \frac{1}{4}\right]$$

$$S_0(5,6) = 2\pi a^2 \left(- \frac{y_0^4}{4}\left(\frac{\mathrm{Ln}\, y_0}{2} - \mathrm{Ln}\, y_0 - \frac{1}{8}\right) + \frac{1}{32}\right]$$

$$S_0(6,6) = 2\pi a^2 \left[\frac{y_0^6}{6}\left(\frac{\mathrm{Ln}\, y_0}{3} - \mathrm{Ln}^2\, y_0 - \frac{1}{18}\right) + \frac{1}{108}\right]$$

where $y_0 = \frac{a_0}{a}$

a_0 is the inside radius of the annular plate

a is the outside radius of the annular plate

λ_1 and λ_2 are the roots of the characteristic equation

A-3.4 Matrix $[G_{0c}]$ for a circular plate with n = 0

As Matrix $[G_{0c}]$ is symmetric, we give only the right-hand side of it:

$$G_{0c}(1,1) = \frac{2\pi}{\lambda_1 - 1}\left[\frac{(\lambda_1 - 1)^2}{4} p_{11} + (\lambda_1 - 1) p_{12} + p_{22}\right]$$

$$G_{0c}(1,2) = 0$$

$$G_{0c}(1,3) = \frac{8}{a(\lambda_1 - 1)}\left[\left(\frac{\lambda_1 - 1}{4}\right)(p_{13} + p_{14}) + p_{23} + p_{24}\right]$$

$$G_{0c}(2,j) = 0 \qquad \text{for } j = 2,3$$

$$G_{0c}(3,3) = \frac{4}{a^2} (p_{33} + 2p_{34} + p_{44})$$

where a is the outside radius of the plate

p_{ij} are the elements of the elasticity matrix $[P_0]$ given in Table (6) of Appendix (A-2).

A-3.5 Matrix $[S_{0c}]$ for a circular plate with n = 0

As matrix $[S_{0c}]$ is symmetric, we give only the right-hand side of it:

$$S_{0c}(1,1) = \frac{2 \pi a^2}{\lambda_1 + 1}$$

$$S_{0c}(1,2) = 0$$

$$S_{0c}(1,3) = 0$$

$$S_{0c}(2,2) = \frac{1}{2}$$

$$S_{0c}(2,3) = \frac{1}{4}$$

$$S_{0c}(3,3) = \frac{1}{6}$$

a is the outside radius of the plate

λ_1 is the root of the characteristic equation

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