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DYNAMIC ANALYSIS OF SPHERICAL SHELLS: A HYBRID METHOD

Aouni A. Lakis and Nguyen Quoc Tuy

gratuit

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Department of Mechanical Engineering
Ecole Polytechnique de Montréal
Campus de l'Université de Montréal
C.P. 6079, Succ.-A-Montréal
Québec, Canada, H3C 3A7

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CONSUL

RESUME

Le but de ce mémoire est de présenter une nouvelle théorie pour l'analyse dynamique des coques sphériques, minces, élastiques et non-uniformes. C'est une méthode hybride basée sur la théorie classique des coques minces et la méthode des éléments finis. Les fonctions de déplacement déduites de la théorie des coques sont adaptées pour un élément fini de forme sphérique conduisant aux matrices de rigidité et de masse de cet élément. Le problème des valeurs propres et des vecteurs propres est solutionné par la méthode standard. Un programme en Fortran V est développé. Les caractéristiques de la vibration libre des coques sphériques, de diverses dimensions, sous différentes conditions aux rives et différents nombres de modes circonférentiels sont déterminées.

Les résultats nous permettent de conclure qu'en général, il y a une bonne concordance entre les fréquences calculées pour la nouvelle théorie et celles obtenues par d'autres auteurs.

ABSTRACT

The objective of this study is to present a new theory for the dynamic analysis of non-uniform thin elastic spherical shells. The theory is a hybrid of classical shell theory and the finite element method. The displacement functions derived from classical shell theory were adapted to a spherical finite element, leading to stiffness and mass matrices for the element. The eigenvalue and eigenvector problem is solved by the standard method. A Fortran V computer program was developed. The free vibration characteristics of spherical shells under different boundary conditions and different circumferential wave numbers were determined.

The results allow the conclusion that there is, in most cases, good agreement between the frequencies calculated by the present theory and the results obtained by other authors.

A comparison of results for the case of a clamped spherical shell showed good similarities between the natural frequencies calculated by the new theory and the values obtained by other authors.

ACKNOWLEDGEMENTS

This research is supported by the Natural Sciences and Engineering Research Council of Canada (Grant No. A8814) and by F.C.A.C. du Québec (Grant No. CRP-2060), whose assistance is hereby gratefully acknowledged.

TABLE OF CONTENTS

LIST OF SYMBOLS

A_α^n	Arbitrary constants defined by (3.1.1)
A_1^n, A_2^n, A_3^n, A_4^n	Arbitrary constants defined by (3.2.1) and (3.2.5)
a_1, a_2+ia_3, a_2-ia_3	r_α roots with α = 1,3
a_{1P}, a_{2P}, a_{3P}, a_{4P}, a_{1Q}, a_{2Q}, a_{3Q}, a_{4Q}	Coefficients defined by (II.2.1)
B_α^n	Arbitrary constants defined by (3.1.1)
B_1^n, B_2^n, B_3^n, B_4^n	Arbitrary constants defined by (3.2.1) and (3.2.5)
$B_{2\nu}$	Bernoulli number
b_1, b_2+ib_3, b_2-ib_3	μ_α roots with α = 1.3
b_{1P}, b_{2P}, b_{3P}, b_{4P}, b_{1Q}, b_{2Q}, b_{3Q}, b_{4Q}	Coefficients defined by (II.2.2)
b_1^w, b_2^w, b_3^w, b_4^w, b_5^w, b_6^w, b_7^w	Constants
$\left\{ C \right\}$	Vector for arbitrary constants defined by (3.4.2)

$$C_r^m = \frac{\cos^m(\phi/2)}{\sin^r(\phi)}$$

C_{ρ}^{n}	Arbitrary constants defined by (3.1.2)
C_1, C_2, C_3, C_r	Constants
C_{1P}, C_{2P}, C_{3P}, C_{4P}, C_{1Q}, C_{2Q}, C_{3Q}, C_{4Q}	Coefficients defined by (II.2.3)
CER1(k,i), CER2(k,i), CER3(k,i), CER4(k,i)	Constants defined by (III.1.40)
CER1(k), CER2(k), CER3(k), CER4(k)	Constants defined by (III.2.34)
CIR1(k,i), CIR2(k,i)	Constants defined by (III.1.40)
CIR1(k), CIR2(k)	Constants defined by (III.2.34)
Chyø	Hyperbolic function of yø
D	Membrane stiffness
D_{ρ}^{n}	Arbitrary constants defined by (3.1.2)
$d(\ell,n)$	Quantity defined by (I.3.7)
det | |	Determinant
d_1	Constant
E	Young's modulus
E_{α}^{u}	Constants defined by (3.1.2)
$E_n(x,y,\phi)$, $E_n^{c}(x,y,\phi)$	Integration functions used in Appendix IV

$\{F\}_{e}$	External force vector
$\{F_j\}_k$, $\{F_i\}_{k+1}$	Internal force vectors at node j of element k and node i of element (k+1)
$F_n(x,y,\phi)$, $F_n^c(x,y,\phi)$	Integration functions used in Appendix IV
F_p^v	Constants defined by (3.1.2)
$f(\ell,n)$	Quantity defined by (I.3.7)
g_1, g_2, g_3	Constants defined by (3.3.1), (3.3.2) and (3.3.3)
HN1, HN2	Constants defined by (III.2.27)
HNR, HNR2	Constants defined by (III.2.28)
HRR(k,j), HRR1(k,j), HRR2(k,j)	Constants defined by (III.1.35)
HRRL(k,j)	Constant defined by (III.1.36)
H12, H22	Constants defined by (III.2.31)
$H_r(m,x,y,\phi)$, $H_r^c(m,x,y,\phi)$	Integration functions used in Appendix IV
$I_n(x,y,\phi)$, $I_n^c(x,y,\phi)$	Integration functions used in Appendix IV
$I_m(P_{\mu 2}^n(\cos\phi))$	Imaginary part of $P_{\mu 2}^n(\cos\phi)$

$i = \sqrt{-1}$	
J	Number of constraints applied at nodes
$J_n(x,y,\phi)$, $J_n^c(x,y,\phi)$	Integration functions used in Appendix IV
K	Bending stiffness
k_1, k_2, k_r	Constants
k	Summation factor
k_s	Tracer for identifying transverse shear deformation effect
ℓ, ℓ_r, ℓ_s	Summation factor
M_ϕ, M_θ, $M_{\phi\theta}$	Bending stress resultants
m_β	Roots of equation (3.15)
N	Number of finite elements
N_ϕ, N_θ, $N_{\phi\theta}$	Membrane stress resultants
n	Number of circumferential modes
$P_{\mu\alpha}^n\ (\cos\phi)$	Associated Legendre function of the first kind of order n and degree $\mu\alpha$
$P_\mu^n = P_\mu^n\ (\cos\phi)$	Ex: $P_{\mu 2}^n = P_{\mu 2}^n\ (\cos\phi)$ $P_1^n = P_1^n\ (\cos\phi)$

p	Circular frequency of applied forces
$Q_{\mu\alpha}^{n}$ $(\cos\phi)$	Associated Legendre function of the second kind of order n and degree μ_α
$Q_{\mu}^{n} = Q_{\mu}^{n}$ $(\cos\phi)$	Ex: $Q_{\mu 2}^{n} = Q_{\mu 2}^{n}$ $(\cos\phi)$ $Q_{1}^{n} = Q_{1}^{n}$ $(\cos\phi)$
Q_{ϕ}, Q_{θ}	Shear forces
R (r)	Radius of the reference surface
R (p,n,ϕ)	Integration function used in Appendix IV
Re $(P_{\mu 2}^{n}$ $(\cos\phi))$	Real part of $P_{\mu 2}^{n}$ $(\cos\phi)$
r	Summation factor
r_α	Roots of equation (3.1.4)
$S_{r}^{m} = \dfrac{\sin^{m}(\phi/2)}{\sin^{r}(\phi)}$	
Shyϕ	Hyperbolic function of yϕ
s	Summation factor
TN1, TN2	Constants defined by (III.2.27)
TNR, TNR2	Constants defined by (III.2.28)
TRR(k,j), TRR1(k,j), TRR2(k,j)	Constants defined by (III.1.35)
TRRL(k,j)	Constants defined by (III.1.36)

$T_r(m,x,y,\phi), T_r^c(m,x,y,\phi)$	Integration functions used in Appendix IV
T12, T22	Constants defined by (III.2.31)
t	Thickness of the spherical shell
$t = tg \dfrac{\phi}{2}$	Used in formulas in Appendices I and III
U_ϕ (or U)	Axial displacement
$(U_\phi)max$	Maximum value of U_ϕ
U_v	Vertical displacement parallel to X3 axis
$U_{\phi n}$	Amplitude of U_ϕ
U_θ	Circumferential displacement
$(U_\theta)max$	Maximal value of U_θ
$U_{\theta n}$	Amplitude of U_θ
UN1, UN2, UN22, UN23 UNR, UNR2, UNR3, UNR4	Constants defined by (III.2.37)
$U_r(m,x,y,\phi)$, $U_r^c(m,x,y,\phi)$	Integration functions used in Appendix IV
URR(k,i), URR1(k,i), URR2(k,i)	Constants defined by (III.1.39)
V	Circumferential displacement
VN1, VN2, VN22, VN23 VNR, VNR2, VNR3, VNR4	Constants defined by (III.2.37)
VRR(k,i), VRR1(k,i), VRR2(k,i)	Constants defined by (III.1.39)

$V_r(m,x,y,\phi)$, $V_r^c(m,x,y,\phi)$	Integration functions used in Appendix IV
W	Radial displacement
$\left(\dfrac{\partial W}{\partial \phi}\right)$	Rotation
$(W)max$	Maximal value of W
W_h	Horizontal displacement parallel to X2 axis
W_α^n	Amplitude of W
W_α^n, ϕ	Derivative of ϕ with respect to W_α^n
$(X1, X2, X3)$	Global Cartesian coordinate system
(R, θ, ϕ)	Local spherical coordinate system
$x_\ell(n) + iy = 2\ell - n - \mu_2 + 1$	
β = 1 and 2	for the improved theory
1	for classical theory
β_n	Constant defined by (3.3.12)
$\Gamma(Z)$	Gamma function
$\Gamma(m,n,x)$	Integration function used in Appendix IV

γ_n — Constant defined by (3.3.11)

$\{\gamma_r\} = (\gamma_1, \gamma_2, \gamma_3, \gamma_4)^T = (x_\ell(n),\ x_\ell(n)+1, x_\ell(n)-1, x_\ell(n)+3)^T$

$\gamma(n,x)$ — Integration function used in Appendix IV

$\Delta(m,x)$ — Integration function used in Appendix IV

$\{\delta\} = (U_\phi \quad W \quad \dfrac{\partial W}{\partial \phi} \quad U_\theta)^T$ — Displacement vector

$\{\delta\}_{loc} = \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix}$ — Vector for displacement at finite element nodes i and j in the local system

$\{\delta\}_{glo} = (U_v,\ W_n,\ \dfrac{\partial W}{\partial \phi},\ U_\theta)^T$ — Vector for displacement at finite element nodes i and j in the global system

$\{\delta\}_T$ — Global displacement vector for the sphere

$\{\ddot{\delta}\}_T$ — Second-order derivative with respect to time of vector δ_T

$\{\delta_j\}_k,\ \{\delta_i\}_{k+1}$ — Generalized displacement vectors at node j of element k and node i of element (k+1)

$\{\epsilon\} = (\epsilon_\phi \ \epsilon_\theta \ 2\epsilon_{\phi\theta} \ K_\phi \ K_\theta \ 2K_{\phi\theta})^T$ — Deformation vector

η_p — Legendre function degree (or index)

Quantity defined by (I.3.1)

$\eta_\ell(n)$ or η_ℓ

$\eta_{n-1} = \eta_\ell(n-1)$

θ	Azimuthal coordinate
$\lambda_\ell(n)$ or λ_n	Quantity defined by (I.3.1)
$\lambda_{n-1} = \lambda_\ell(n-1)$	
λ	Constant
Λ	Auxiliary function introduced by Wilkinson and Kalnins [37], [38]
μ_α	Legendre function degree (or index)
ν	Poisson's ratio
$\zeta = \dfrac{12R^2}{t}$	
ρ	Density
$\left\{\sigma\right\} = (N_\phi \; N_\theta \; N_{\phi\theta} \; M_\phi \; M_\theta \; M_{\phi\theta})^T$	Resultant stress vector
ϕ	Meridional coordinate
ϕ_o	Half-angle at sphere centre
Ψ	Auxiliary function introduced by Wilkinson and Kalnins [37, [38]
ψ_p^n	Constant defined by (3.1.2)
Ω	Natural non-dimensional frequency
ω	Natural angular frequency

Matrices

[A]	Matrix defined by (3.4.7)
[Aø]	Matrix defined by (3.4.5)
[Aøs]	Matrix derived from
$[A_r]$, $[A_r^c]$, $[A_r^v]$	Matrices defined by relations (III.1.2) to (III.1.5)
[B]	Deformation matrix
$[B_r]$, $[B_r^c]$, $[B_r^v]$	Matrices defined by relations (III.1.2) to (III.1.5)
$[C_r]$, $[C_r^c]$, $[C_r^v]$	Matrices defined by (III.1.10)
$[D_r]$, $[D_r^c]$, $[D_r^v]$	Matrices defined by (III.1.10)
[DLAM]	Coordinate transformation matrix
[G]	Matrix defined by (5.1.2)
$[G_{pp}]$, $[G_{pq}]$, $[G_{qp}]$, $[G_{qq}]$	Partitioned [G] matrices
[K]	Global stiffness matrix for the spherical shell
$[K]_{red}$	Reduced [K] matrix
$[k]_{loc}$	Stiffness matrix for a finite element in the local system
[k]	Stiffness matrix for a finite element in the global system

$[M]$	Global mass matrix for the spherical shell
$[M]_{red}$	Reduced [M] matrix
$[m]_{loc}$	Mass matrix for the finite element in the local system
$[m]$	Mass matrix for the finite element in the global system
$[N]$	Displacement matrix
$[P]$	Elasticity matrix
$[P_p]$, $[P_q]$	Partitioned $[Q_e]^T$ $[P]$ matrices
$[Q]$	Matrix defined by (4.1.8)
$[Q_e]$	Matrix derived from [Q]
$[Q_{ep}]$, $[Q_{eq}]$	Partitioned $[Q_e]$ matrix
$[R]$	Matrix defined by (3.4.1)
$[R_e]$	Matrix derived from [R]
$[R_{ep}]$, $[R_{eq}]$	Partitioned $[R_e]$ matrices
$[S]$	Matrix defined by (5.2.2)
$[S_{pp}]$, $[S_{pq}]$, $[S_{qp}]$, $[S_{qq}]$	Partitioned [S] matrices
$[ST]$	Stress resultants matrix
$[T]$	Matrix defined by (3.4.1)
$[\alpha_r]$	Matrix defined by (III.1.10)

$[\Gamma_r]$, $[\Gamma_r^{=}]$, $[\Gamma_r^{\vee}]$	Matrices defined by relations (III.2.2) to (III.2.5)
$[\Delta_r]$, $[\Delta_r^{=}]$, $[\Delta_r^{\vee}]$	Matrices defined by relations (III.2.2) to (III.2.5)
$[\tau]$	Matrix defined by (3.4.5)

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CHAPTER I

INTRODUCTION

1.1) General

Since the 19th century, thin shells have been used in the manufacture of certain machinery. Their practical benefits have been increasing steadily in current mechanical applications as well. Widely used as structural components in aircraft, boats, pipelines, nuclear reactors, etc., they are also prime research topics in a number of published works, from static to dynamic investigations.

Many theories covering thin shells have been put forth. The first attempt to formulate a theory for thin shells under bending on the basis of the general elasticity equations was made by Aron in 1874. Love [1] in 1888 derived a series of basic equations describing the behaviour of thin elastic shells. These equations and the assumptions they are based on are often referred to as Love's "first approximation".

Since then, thin shell theory has been reexamined on several occasions ([2] to [18]). Reissner and Knowles ([2],[3]) did a new derivation of the equations for orthogonal coordinates. Certain other theories have been put forth as well: Sanders ([5],[6]) maintained Love's original hypotheses but developed a system of equations producing zero deformation due to rigid-body motion. These equations are identical to the ones Koitier [7] derived. Flügge [8], Lu're [9] and Byrne [10] independently presented higher-order theories in which the assumption relating to shell thickness is differentiated in the formulations. The effects of transverse shear constraints and shear deformations were explored by Hildebrand, Reissner and Thomas [11], Green and Zerna [12], Reissner [13] and Naghdi ([14] to [17]). Naghdi's results [15] also included the effect of inertial forces.

These general theories have been widely applied to static studies of thin shells of various geometries, particularly spherical shells ([19] to [22]). Novojilov [19] used the Love equations as a basis for his study of tensions in spherical shells subjected to arbitrary loads. Reissner, in different investigations ([20] to [22]), established the equations describing the behaviour of shallow spherical shells(*).

In his dynamic analysis of thin shells, Love [23] developed the first general theory which took into account the effects of bending and tension (membrane) deformations. The formulation contains individual studies of spherical shells involving cases of non-extensional mode vibrations done by Lord Rayleigh [24] and tension (membrane) vibrations done by Lamb [25]. Since Love's work [23], little attention has been paid to the problem of shell vibration (in the area of bending theory). Towards the fifties, some specific geometries were investigated and it was not until the sixties that more extensive studies of different shell geometries began to appear.

Many methods have been developed, particularly for the dynamic analysis of thin elastic spherical shells. These methods could be

---

(*) A spherical shell is said to be shallow when the height-to-base diameter ratio is less than 1/8, that is, the half-angle at the centre is equal to or less than 30° ([18] and [20]).

summarized as follows:

1) The Exact Method:

The general idea here is to reduce the basic equations to a system of 2 differential equations with 2 unknowns. Then, these two unknowns are assumed to be separable into spatial and temporal components. The system can be solved in terms of Bessel or Legendre functions. Certain mathematical artifices have been invented ([26] to [41]) to reduce the general equations, depending on the type of spherical shell involved (shallow or non-shallow), the nature of the vibrations (axisymmetric or non-axisymmetric) and on the effects under consideration (shear constraints, longitudinal inertia, etc.). Reissner ([26],[27]), Johnson and Reissner [28], Kalnins and Naghdi ([31], [32]) and Kalnins [33], studying vibrations in shallow spherical shells, selected transverse displacement and the constraint function as the two unknowns. Kalnins ([34],[35]) used two auxiliary variables for the axial and circumferential displacements while considering the effects of longitudinal, transverse and rotary inertia as well as transverse shear deformation on the non-axisymmetric vibration of shallow spherical shells. The arbitrary vibrations of non-shallow spherical shells including the effects of transverse shear constraints and rotary inertia have been analyzed by Prasad [36], Wilkinson [37] and Wilkinson and Kalnins [38]. They introduced four auxiliary functions, two for the axial and circumferential displacements and two others for the particular effects considered. The Kraus text in [18] gives a concise summary of these techniques. Gau Feng Lin [40] investigated vibration in open non-shallow spherical shells using Flügge's reduction technique [8], ignoring the two effects considered by Prasad, Wilkinson and Kalnins.

2) The Variational Method:

The equations governing the shells are combined into a single variational equation. The several techniques that are used in this method differ mainly in terms of the way they build to the approximate solutions. The most widely used technique is the Rayleigh-Ritz, where the solutions chosen are in two parts, functions and arbitrary constants: the former must satisfy the boundary conditions and the latter, the variational equation thus assuming minimum potential energy. Reissner [42] used this method to compute the lowest frequency in a simply supported spherical shell. There are several other examples given in reference [18].

3) Finite Element Method:

This is a numerical method, whose popularity is attributable in large part to its flexibility in handling continuous media with complex geometry or boundary conditions. There is an abundant literature on it ([43],[44]). The method could be defined as a means for analyzing a structural system. The finite elements make it possible to model a structure by an idealized system, one that is simpler than the original structure. On each finite element, the stress or displacement fields are generally described by polynomial functions. The stiffness and mass matrices are then determined for each element. The complete structure is built up by assembling a finite number of continuous, discrete elements interconnected by a finite number of nodes. This assembly technique yields the global mass and stiffness matrices and the boundary conditions can be incorporated into them. Finally, the stresses, displacements or deformations and the reactions at the defined nodes on the structure are calculated. If we are dealing with a vibration problem, the natural frequencies and modes are also calculated. The finite element method has been used in vibration studies of thin spherical shells by Navaratna [45], Webster [46], Greene et al [47] and Sen and Gould [48].

It should be noted, incidentally, that the literature also makes reference to a certain number of other methods, such as: the Stodola [49], Holzer [50], asymptotic [51] and edge function methods [52]. These methods are in less common use in thin shell vibration studies and will therefore not be dealt with here.

The three main methods mentioned above have both advantages and disadvantages. One of the criteria of a successful method is its ability to predict the high as well as the low frequencies and their corresponding natural modes, and with good accuracy. In the exact method, the procedures developed by Kalnins' school ([31] to [38]) may be considered as the most adequate and demand laborious and often highly complicated analytical manipulations to reduce the initial differential equation system. Furthermore, solving the final system calls upon Legendre functions, and tables of Legendre values are rarely available in the literature, particularly when the degree is a complex number, which is generally the case with spherical shell vibration. The algebraic operations in these functions within the final system can entail losses in precision. Ross [53], using the method indicated by Kalnins and Naghdi [32], working on the same example of an open non-shallow spherical shell with a free edge, and using asymptotic formulas for the Legendre functions, found frequencies comparable to Kalnins and Naghdi's findings for n = 1 (n being the number of circumferential modes). For n = 2 and n = 3, however, there was a good match only for certain frequencies. According to Gau Feng Lin [40], the basic solutions to the open non-shallow spherical shell vibration problem are generally neither analytically exact or explicitly formulated.

The Rayleigh-Ritz variational method and the finite element method have met the success criterion. In addition, they generally lead to symmetric eigenvalue problems which are easy to solve on a computer. The finite element method also has further advantages in that it is easy to formulate and numerical convergence is not sensitive to boundary conditions, in contrast to the Rayleigh-Ritz method [54].

Nevertheless the finite element method with a strain - displacement formulation, where the displacements field is described by polynomial functions [47], also requires certain improvements. In fact, these polynomials cannot completely satisfy the convergence criteria, unless a large number of elements is used. In order to satisfy the condition of zero deformation due to rigid-body motion therefore, Webster [46] and Sen and Gould [48] used additional non-modal functions [44]. To circumvent this hurdle, Lakis and Paidoussis ([55],[56]) developed a hybrid method where the displacement functions in the finite element method were supplied by Sanders' classical shell theory. After using this method for the dynamic analysis of axially non-uniform cylindrical shells, they concluded that the convergence criteria in the finite element method were completely and naturally satisfied. Good low as well as high frequencies were obtained.

1.2) Research Objectives

All shells of revolution are primarily an assembly of cylindrical, conical and spherical shell elements. The research group, working under Dr. Lakis, extensively developed the hybrid method for dynamic linear and non-linear studies of thin cylindrical and conical isotropic and anisotropic shells, both empty and liquid-filled ([57] to [60]), with good results. Using the hybrid method, Lakis also developed a theory and even a computer program for studying free vibration in thin elastic spherical shells. The finite element chosen was spherical, thereby making possible integral use of shell theory. The displacement functions Wilkinson and Kalnins derived ([37],[38]) were adapted to the finite element. These functions were developed from the general system of equations in Naghdi [15], where the theory allows zero deformation due to rigid-body displacement [16].

The present report is a follow-up to the work begun by the Lakis team. The main objectives are somewhat different:

- In terms of the theory: verify and modify (where necessary):
  a) The displacement functions adapted
  b) With number of circumferential modes $n \geq 2$, the free vibration formulation of the following matrices:
  * Stiffness matrix, mass matrix, stress resultants matrix for a finite element.
  * Corresponding global matrices for the complete shell.

- In terms of the programming: adapt the SPHERE program Lakis developed to the particular conditions of the University of Montreal Computer Centre. This was to be done by means of a new program architecture, to reduce CPU time as well as memory space. Finalize the program as a result of modifications to the theory.

- Use the program to compute the free vibration characteristics of shallow and non-shallow uniform thin elastic spherical shells, for circumferential mode number n ≥ 2. The results obtained will be compared with any findings that are available in the literature.

1.3) Report Outline

This next section contains an overview of the nine chapters into which the present report is divided.

Chapter 1: Introduction

Chapter 2: Basic equations for thin elastic spherical shells

Chapter 3: Displacement functions

Chapter 4: Deformation matrix and stress resultants matrix for a finite element

Chapter 5: Stiffness matrix and mass matrix for a finite element

Chapter 6: Equations of motion

Chapter 7: The algorithm

Chapter 8: Numerical calculations and discussion

Chapter 9: Conclusions

Appendices I, II, III, IV, V and VI contain the necessary data, formulas, matrices, tables and figures referred to in these coming chapters.

CHAPTER II

BASIC EQUATIONS FOR THIN ELASTIC SPHERICAL SHELLS

2.1) Equilibrium Equations

In the spherical coordinate system (R, θ, ϕ) shown in Figure 1, five out of the six equations of equilibrium derived in reference [15] for spherical shells under zero external load are written as follows:

$$\frac{\partial N_\phi}{\partial \phi} + \frac{\partial N_{\phi\theta}}{\partial \theta} \operatorname{cosec} \phi + (N_\phi - N_\theta) \operatorname{cotg} \phi + Q_\phi = 0 \qquad (2.1.1)$$

$$\frac{\partial N_{\phi\theta}}{\partial \phi} + \frac{\partial N_\theta}{\partial \theta} \operatorname{cosec} \phi + 2N_{\phi\theta} \operatorname{cotg} \phi + Q_\theta = 0 \qquad (2.1.2)$$

$$\frac{\partial Q_\phi}{\partial \phi} + \frac{\partial Q_\theta}{\partial \theta} \operatorname{cosec} \phi + Q_\phi \operatorname{cotg} \phi - (N_\phi + N_\theta) = 0 \qquad (2.1.3)$$

$$\frac{\partial M_\phi}{\partial \phi} + \frac{\partial M_{\phi\theta}}{\partial \theta} \operatorname{cosec} \phi + (M_\phi - M_\theta) \operatorname{cotg} \phi - RQ_\phi = 0 \qquad (2.1.4)$$

$$\frac{\partial M_{\phi\theta}}{\partial \phi} + \frac{\partial M_\theta}{\partial \theta} \operatorname{cosec} \phi + 2M_{\phi\theta} \operatorname{cotg} \phi - RQ_\theta = 0 \qquad (2.1.5)$$

where N_ϕ, N_θ, $N_{\phi\theta}$ are the membrane stress resultants; M_ϕ, M_θ, $M_{\phi\theta}$, the bending stress resultants and Q_ϕ, Q_θ the shear forces (Figures 2 and 3). The sixth equation, which is an identity equation for spherical shells, is not presented here.

2.2) Strain-displacement Relationships

Strain and displacements are related as follows [15]:

$$\epsilon_\phi = \frac{1}{R}\left(\frac{\partial U_\phi}{\partial \phi} + W\right) \tag{2.2.1}$$

$$\epsilon_\theta = \frac{1}{R}\left(\operatorname{cosec}\phi \frac{\partial U_\theta}{\partial \theta} + U_\phi \operatorname{cotg}\phi + W\right) \tag{2.2.2}$$

$$\gamma_{\phi\theta} = 2\epsilon_{\phi\theta} = \frac{1}{R}\left(\frac{\partial U_\theta}{\partial \phi} - U_\theta \operatorname{cotg}\phi + \operatorname{cosec}\phi \frac{\partial U_\phi}{\partial \theta}\right) \tag{2.2.3}$$

$$K_\phi = \frac{1}{R^2}\left(\frac{\partial U_\phi}{\partial \phi} - \frac{\partial^2 W}{\partial \phi^2}\right) \tag{2.2.4}$$

$$K_\theta = \frac{1}{R^2}\left(U_\phi \operatorname{cotg}\phi + \operatorname{cosec}\phi \frac{\partial U_\theta}{\partial \theta} - \operatorname{cosec}^2\phi \frac{\partial^2 W}{\partial \theta^2} - \operatorname{cotg}\phi \frac{\partial W}{\partial \phi}\right) \tag{2.2.5}$$

$$\tau_\phi + \tau_\theta = 2K_{\phi\theta} = \frac{1}{R^2}\left(\frac{\partial U_\theta}{\partial \phi} - U_\theta \operatorname{cotg}\phi + \operatorname{cosec}\phi \frac{\partial U_\phi}{\partial \theta} + 2\operatorname{cosec}\phi \operatorname{cotg}\phi \frac{\partial W}{\partial \theta} - 2\operatorname{cosec}\phi \frac{\partial^2 W}{\partial \phi \partial \theta}\right) \tag{2.2.6}$$

where U_ϕ is the axial displacement, W the radial and U_θ the circumferential.

2.3) Stress-strain Relationships

For elastic isotropic material, the constituent relationships linking stress resultants to strains are:

$$N_\phi = D(\epsilon_\phi + \nu\epsilon_\theta) \qquad (2.3.1)$$

$$N_\theta = D(\epsilon_\theta + \nu\epsilon_\phi) \qquad (2.3.2)$$

$$N_{\phi\theta} = \frac{D(1 - \nu)}{2}(2\epsilon_{\phi\theta}) \qquad (2.3.3)$$

$$M_\phi = K(K_\phi + \nu K_\theta) \qquad (2.3.4)$$

$$M_\theta = K(K_\theta + \nu K_\phi) \qquad (2.3.5)$$

$$M_{\phi\theta} = \frac{K(1 - \nu)}{2}(2K_{\phi\theta}) \qquad (2.3.6)$$

where

D is the membrane stiffness = $\dfrac{Et}{(1 - \nu^2)}$ (2.3.7)

K is the bending stiffness = $\dfrac{Et^3}{12(1 - \nu^2)}$ (2.3.8)

with:

E is the modulus of elasticity

t is the spherical element thickness

ν is Poisson's ratio.

CHAPTER III

DISPLACEMENT FUNCTIONS

3.1) General Displacement Functions

Wilkinson and Kalnins ([37],[38]), in solving the system of differential equations from Naghdi's theory [15] for spherical shells, introduced 4 auxiliary functions: U, V, ψ and Λ and used the technique for separating spatial and temporal variables. The general expressions for the displacement vector components were presented in references [37] and [38]. The three displacement components W, U_ϕ and U_θ can be written:

$$W = \sum_{n=0}^{\infty} \sum_{\alpha=1}^{3} W_\alpha^n \cos n\theta \cos pt \tag{3.1.1}$$

$$U_\phi = \sum_{n=0}^{\infty} U_{\phi n} \cos n\theta \cos pt \tag{3.1.2}$$

$$U_\theta = \sum_{n=1}^{\infty} U_{\theta n} \sin n\theta \cos pt \tag{3.1.3}$$

with:

$$W_\alpha^n = A_\alpha^n P_{\mu\alpha}^n (\cos \phi) + B_\alpha^n Q_{\mu\alpha}^n (\cos \phi)$$

$$U_{\phi n} = E_\alpha^u W_{\alpha,\phi}^n + n^2 F_\beta^u \psi_\beta^n \operatorname{cosec} \phi$$

$$U_{\theta n} = - n [E_\alpha^u W_\alpha^n \operatorname{cosec} \phi + F_\beta^u \psi_{\beta,\phi}^n]$$

$$\psi_\beta^n = C_\beta^n P_{n\beta}^n (\cos \phi) + D_\beta^n Q_{n\beta}^n (\cos \phi)$$

$$E_\alpha^u = \frac{b_4^w r_\alpha + [b_3^w b_1^w - b_6^w b_4^w + 2k_m (1 + \nu) b_2^w b_4^w/(1 - \nu)]}{d_1 r_\alpha/(1 - \nu) - C_3}$$

$$F_{\beta}^{u} = -\frac{(1 - \nu)}{2C_3}(\eta_{\beta} k_s + b_4^w + b_3^w - 2k_s b_5^w)$$

where:

n is the number of circumferential modes. Figure 3a shows the pattern of some vibration modes.

A_{α}^{n}, B_{α}^{n}, C_{β}^{n}, D_{β}^{n} are arbitrary constants.

$P_{\mu\alpha}^{n}$ $(\cos \phi)$ and $Q_{\mu\alpha}^{n}$ $(\cos \phi)$ are, respectively, the associated Legendre functions of the first and second kind of order n and degree $\mu\alpha$.

$b_1^w = 1 + \nu - k_2 \lambda^2$

$b_2^w = 2 - k_1 \lambda^2/(1 + \nu)$

$b_3^w = \xi - 2k_s - 2k_s k_r \lambda^2/(1 - \nu)$

$b_4^w = 1 + k_s k_2 \lambda^2/(1 - \nu)$

$b_5^w = 1 + (k_1 + k_2)\lambda^2/(1 - \nu)$

$b_6^w = 1 - \nu + k_r \lambda^2$

$b_7^w = 1 + (C_r + k_r)\lambda^2/(1 - \nu)$

$\xi = 12R^2/t^2$

$\lambda^2 = \rho R^2 p^2 (1 - \nu^2)/E$

$C_1 = 4 - 2k_s (1 + \nu) + [k_1 + k_r + 2k_s k_1/(1 - \nu)]\lambda^2$

$C_2 = (1 - \nu)(b_5^w b_6^w + b_1^w b_7^w) + (1 + \nu)(b_4^w b_2^w - b_1^w b_3^w)$
$\quad + b_2^w b_3^w + b_4^w b_6^w - 2k_s b_5^w b_2^w$

$C_3 = (1 - \nu)(b_4^w b_7^w + b_3^w b_5^w)$

$k_1 = 1 + t^2/12R^2$

$k_2 = t^2/6R^2$

$k_r = 1 + 3t^2/20R^2$

$C_r = 2$

$d_1 = 2k_s\ (1 + \nu)b_4^w + (1 - \nu)(b_4^w + b_3^w)$

k_s is the tracer for identifying the transverse shear deformation effect. According to reference [15], k_s is equal to 6/5.
If $k_s = k_r = k_2 = C_r = 0$ and $k_1 = 1$: the secondary effects are neglected and the system of equations reduces to classical shell theory.

β = 1 and 2 for the improved theory;
1 for classical theory.

μ_α = factor given by: $\mu_\alpha = (\frac{1}{4} + r_\alpha)^{1/2} - \frac{1}{2}$

where r_α ($\alpha = 1, 2, 3$) are the roots of the cubic equation:

$$r^3 - C_1r^2 + C_2r - [2(1 + \nu) - \lambda^2k_1]C_3 = 0 \qquad (3.1.4)$$

η_β = factor given by: $\eta_\beta = (\frac{1}{4} + m_\beta)^{1/2} - \frac{1}{2}$

where m_β ($\beta = 1, 2$) are the roots of the quadratic equation:

$$k_a m_p^2 + [b_3^w + b_4^w - 2k_a b_5^w] m_p - \frac{2C_3}{1 - \nu} = 0 \qquad (3.1.5)$$

ν is Poisson's ratio

t is the thickness of the spherical shell

R is the radius of the reference surface

E is Young's modulus

p is the circular frequency of the applied forces

ρ is the density

3.2) <u>Adaptation</u>

The general displacement functions, (3.1.1) to (3.1.3), can be used to derive displacement functions for the finite element method. The modification is done by setting $k_a = k_r = k_z = C_r = 0$, $k_1 = 1$ and $p = 0$. Thus:

$b_1^w = 1 + \nu$

$b_3^w = \zeta = 12R^2/t^2$

$b_4^w = 1$

$b_5^w = 1$

$b_6^w = 1 - \nu$

$b_7^w = 1$

$$C_3 = (1 - \nu)(1 + \zeta) = (1 - \nu)(1 + \frac{12R^2}{t^2})$$

$$d_1 = (1 - \nu)(1 + \zeta) = (1 - \nu)(1 + \frac{12R^2}{t^2})$$

$$E_\alpha^U = \frac{r_\alpha + \zeta\ (1 + \nu) - (1 - \nu)}{(1 + \zeta)(r_\alpha - 1 + \nu)}$$

β has a single value equal to 1

$$m_\beta = m_1 = 1$$

$$F_\beta^U = F_1^U = -\frac{1}{2}$$

As $1 + \zeta \cong \zeta$, the E_α^U expression becomes:

$$E_\alpha^U = \frac{r_\alpha + \zeta\ (1 + \nu)}{\zeta\ [r_\alpha - (1 - \nu)]}$$

The expressions for displacements W, U_ϕ and U_θ are:

$$W = \sum_{n=0}^{\infty} \left\{ A_1^n P_{\mu 1}^n(\cos\phi) + A_2^n P_{\mu 2}^n(\cos\phi) + A_3^n P_{\mu 3}^n(\cos\phi) + B_1^n Q_{\mu 1}^n(\cos\phi) + B_2^n Q_{\mu 2}^n(\cos\phi) + B_3^n Q_{\mu 3}^n(\cos\phi) \right\} \cos n\theta \tag{3.2.1}$$

$$U_\phi = \sum_{n=0}^{\infty} \left\{ \sum_{\alpha=1}^{3} \left[\frac{r_\alpha + \zeta\ (1 + \nu)}{\zeta\ [r_\alpha - (1 - \nu)]}\right] \frac{\partial W_\alpha^n}{\partial \phi} - \frac{n^2}{2}\psi_1^n \operatorname{cosec}\phi \right\} \cos n\theta \tag{3.2.2}$$

$$U_\theta = \sum_{n=0}^{\infty} \left\{ \sum_{\alpha=1}^{3} - n \left[\frac{r_\alpha + \zeta\ (1 + \nu)}{\zeta\ [r_\alpha - (1 - \nu)]}\right] W_\alpha^n \operatorname{cosec}\phi + \frac{n}{2}\frac{\partial \psi_1^n}{\partial \phi} \right\} \sin n\theta \tag{3.2.3}$$

where:

$$W_\alpha^n = A_\alpha^n P_{\mu\alpha}^n(\cos\phi) + B_\alpha^n Q_{\mu\alpha}^n(\cos\phi) \qquad \alpha = 1 \quad 3 \tag{3.2.4}$$

$$\psi_1^n = A_4^n P_1^n(\cos\phi) + B_4^n Q_1^n(\cos\phi) \tag{3.2.5}$$

3.3) Explicit Displacement Expressions

In this section, displacement functions (3.2.1) to (3.2.3) will be made explicit and simplified for the purposes of determining the mass and stiffness matrices.

Within the realm of classical theory, references [32] and [33] demonstrate that the cubic equation in (3.14) possesses at least one real root and two complex conjugate roots. The three roots of the equation are then r_1, r_2 and r_3:

$$r_1 = a_1$$

$$r_2 = a_2 + ia_3$$

$$r_3 = a_2 - ia_3$$

This allows us to write:

$$\frac{r_1 + \zeta\,(1 + \nu)}{\zeta\,[r_1 - (1 - \nu)]} = g_1 \tag{3.3.1}$$

$$\frac{r_2 + \zeta\ (1 + \nu)}{\zeta\ [r_2 - (1 - \nu)]} = g_2 - ig_3 \qquad (3.3.2)$$

$$\frac{r_3 + \zeta\ (1 + \nu)}{\zeta\ [r_3 - (1 - \nu)]} = g_2 + ig_3 \qquad (3.3.3)$$

The degrees $\mu\alpha$ of the associated Legendre functions are:

$$\mu_1 = b_1 \qquad (3.3.4)$$

$$\mu_2 = b_2 + ib_3 \qquad (3.3.5)$$

$$\mu_3 = b_2 - ib_3 \qquad (3.3.6)$$

General treatments of the associated Legendre functions of the first and second kind can be found in references [61] and [62]. Appendix I gives a brief summary of them as well as their developed expressions.

Using relations (I.2.3) and (I.2.4), we can write:

$$\begin{aligned} W_\alpha{}^n = W_n &= A_1{}^n P_{\mu 1}{}^n(\cos \phi) + B_1{}^n Q_{\mu 1}{}^n(\cos \phi) \\ &+ (A_2{}^n + A_3{}^n)\ Re[P_{\mu 2}{}^n(\cos \phi)] + (B_2{}^n + B_3{}^n)\ Re[Q_{\mu 2}{}^n(\cos \phi)] \\ &+ i\,(A_2{}^n - A_3{}^n)\, Im[(P_{\mu 2}{}^n(\cos \phi)] + i\,(B_2{}^n - B_3{}^n)\ Im[Q_{\mu 2}{}^n(\cos \phi)] \end{aligned} \qquad (3.3.7)$$

where R_e and I_m are the real and imaginary parts

i is the imaginary number

$$\psi_1^n = \psi_n = A_4^n P_1^n(\cos \phi) + B_4^n Q_1^n(\cos \phi) \tag{3.3.8}$$

Setting:

$$(n - \mu_2 - 1)(n + \mu_2) = \gamma_n + i\beta_n \tag{3.3.9}$$

$$(n - \mu_3 - 1)(n + \mu_3) = \gamma_n - i\beta_n \tag{3.3.10}$$

We find:

$$\gamma_n = (n - b_2 - 1)(n + b_2) + b_3^2 \tag{3.3.11}$$

$$\beta_n = b_3 \ (-2b_2 - 1) \tag{3.3.12}$$

Using (3.2.4), (3.2.5), 3.3.7), (3.3.8), (II.1.1), (II.1.2), (II.1.3) in (3.2.1), (3.2.2) and (3.2.3), we obtain:

$$W = \sum_{n=o}^{\infty} \Big\{ P_{\mu 1}^n \ A_1^n + Q_{\mu 1}^n \ B_1^n + \mathrm{Re} \ (P_{\mu 2}^n) \ (A_2^n + A_3^n)$$
$$+ \mathrm{Im} \ (P_{\mu 2}^n) \ i \ (A_2^n - A_3^n) + \mathrm{Re} \ (Q_{\mu 2}^n) \ (B_2^n + B_3^n)$$
$$+ \mathrm{Im} \ (Q_{\mu 2}^n) \ i \ (B_2^n - B_3^n) \Big\} \cos n\theta \tag{3.3.13}$$

$$U_\phi = \sum_{n=o}^{\infty} \Big\{ g_1[(n - \mu_1 - 1)(n + \mu_1)P_{\mu 1}^{n-1} - n \cot g \ \phi \ P_{\mu 1}^n]A_1^n$$
$$+ g_1 \ [(n - \mu_1 - 1)(n + \mu_1)Q_{\mu 1}^{n-1} - n \cot g \ \phi \ Q_{\mu 1}^n]B_1^n$$
$$+ [(g_2 \ \gamma_n + g_3 \ \beta_n) \ \mathrm{Re} \ (P_{\mu 2}^{n-1}) + (g_3 \ \gamma_n - g_2 \ \beta_n) \ \mathrm{Im} \ (P_{\mu 2}^{n-1})$$
$$- ng_2 \cot g \ \phi \ \mathrm{Re} \ (P_{\mu 2}^n) - ng_3 \cot g \ \phi \ \mathrm{Im} \ (P_{\mu 2}^n)](A_2^n + A_3^n)$$
$$+ [(g_2 \ \gamma_n + g_3 \ \beta_n) \ \mathrm{Im} \ (P_{\mu 2}^{n-1}) + (g_3 \ \beta_n - g_3 \ \gamma_n) \ \mathrm{Re} \ (P_{\mu 2}^{n-1})$$

$$- ng_3 \cot g\, \phi\, \mathrm{Im}\, (P_{\mu 2}{}^{n}) + ng_3 \cot g\, \phi\, \mathrm{Re}\, (P_{\mu 2}{}^{n})]\; i\,(A_2{}^{n} - A_3{}^{n})$$

$$+ [(g_2\, \gamma_n + g_3\, \beta_n)\, \mathrm{Re}\, (Q_{\mu 2}{}^{n-1}) + (g_3\, \gamma_n - g_2\, \beta_n)\, \mathrm{Im}\, (Q_{\mu 2}{}^{n-1})$$

$$- ng_2 \cot g\, \phi\, \mathrm{Re}\, (Q_{\mu 2}{}^{n}) - ng_3 \cot g\, \phi\, \mathrm{Im}\, (Q_{\mu 2}{}^{n})]\; (B_2{}^{n} + B_3{}^{n})$$

$$+ [(g_2\, \gamma_n + g_3\, \beta_n)\, \mathrm{Im}\, (Q_{\mu 2}{}^{n-1}) + (g_2\, \beta_n - g_3\, \gamma_n)\, \mathrm{Re}\, (Q_{\mu 2}{}^{n-1})$$

$$- ng_2 \cot g\, \phi\, \mathrm{Im}\, (Q_{\mu 2}{}^{n}) + ng_3 \cot g\, \phi\, \mathrm{Re}\, (Q_{\mu 2}{}^{n})]\; i\,(B_2{}^{n} - B_3{}^{n})$$

$$\left. - \frac{n^2}{2} \operatorname{cosec} \phi\, P_1{}^{n}\, A_4{}^{n} - \frac{n^2}{2} \operatorname{cosec} \phi\, Q_1{}^{n}\, B_4{}^{n} \right\} \cos n\theta \qquad (3.3.14)$$

$$U_\theta = \sum_{n=0}^{\infty} \left\{ - ng_1 \operatorname{cosec} \phi\, P_{\mu 1}{}^{n}\, A_1{}^{n} - ng_1 \operatorname{cosec} \phi\, Q_{\mu 1}{}^{n}\, B_1{}^{n} \right.$$

$$-[ng_2\, \mathrm{Re}\, (P_{\mu 2}{}^{n}) + ng_3\, \mathrm{Im}\, (P_{\mu 2}{}^{n})] \operatorname{cosec} \phi\, (A_2{}^{n} + A_3{}^{n})$$

$$-[ng_2\, \mathrm{Im}\, (P_{\mu 2}{}^{n}) - ng_3\, \mathrm{Re}\, (P_{\mu 2}{}^{n})] \operatorname{cosec} \phi\; i\; (A_2{}^{n} - A_3{}^{n})$$

$$-[ng_2\, \mathrm{Re}\, (Q_{\mu 2}{}^{n}) + ng_3\, \mathrm{Im}\, (Q_{\mu 2}{}^{n})] \operatorname{cosec} \phi\, (B_2{}^{n} + B_3{}^{n})$$

$$-[ng_2\, \mathrm{Im}\, (Q_{\mu 2}{}^{n}) - ng_3\, \mathrm{Re}\, (Q_{\mu 2}{}^{n})] \operatorname{cosec} \phi\; i\; (B_2{}^{n} - B_3{}^{n})$$

$$+[\frac{n}{2} (n - 2)(n + 1)\; P_1{}^{n-1} - \frac{n^2}{2} \cot g\, \phi\, P_1{}^{n}]\; A_4{}^{n}$$

$$\left. +[\frac{n}{2} (n - 2)(n + 1)\; Q_1{}^{n-1} - \frac{n^2}{2} \cot g\, \phi\, Q_1{}^{n}]\; B_4{}^{n} \right\} \sin n\theta \qquad (3.3.15)$$

where $P_\mu{}^{n}$ and $Q_\mu{}^{n}$ represent, respectively, $P_\mu{}^{n}\,(\cos\phi)$ and $Q_\mu{}^{n}\,(\cos\phi)$

3.4) <u>Matrix Form of Displacement Functions</u>

Using matrix formulation, the displacement functions can be expressed as follows [55]:

$$\begin{Bmatrix} U_x \\ W \\ U_\theta \end{Bmatrix} = [T]\,[R] \begin{Bmatrix} C \end{Bmatrix} \qquad (3.4.1)$$

where

[T] is the (3 x 3) matrix

[R] is the (3 x 8) matrix

Matrices [T] and [R] are given in Appendix V.

$\{C\}$ is the (8 x 1) vector for arbitrary parameters

$$\begin{Bmatrix} \\ \\ \\ C \\ \\ \\ \\ \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{Bmatrix} = \begin{Bmatrix} A_1^n \\ A_2^n + A_3^n \\ i(A_2^n - A_3^n) \\ A_4^n \\ B_1^n \\ B_2^n + B_3^n \\ i(B_2^n - B_3^n) \\ B_4^n \end{Bmatrix} \qquad (3.4.2)$$

In the finite element method, parameters C_j are eliminated in favour of displacements at the elements' nodes.

At each finite element node (Figure 4), the three displacements - axial, tangential and circumferential - and the rotation are applied. The nodal displacement field can be defined by the vector:

$$\begin{Bmatrix} \delta \end{Bmatrix} = \begin{Bmatrix} U_\varnothing \\ W \\ \left(\dfrac{\partial W}{\partial \varnothing}\right) \\ U_\theta \end{Bmatrix} \qquad (3.4.3)$$

The rotation expression $(\partial W/\partial \varnothing)$ is:

$$\frac{\partial W}{\partial \varnothing} = A_1^n \, [(n - \mu_1 - 1)(n + \mu_1) \; P_{\mu 1}^{n-1} - n \cot g \, \varnothing \; P_{\mu 1}^{n}]$$

$$+(A_2^n + A_3^n)[\gamma_n Re(P_{\mu 2}^{n-1}) - \beta_n Im(P_{\mu 2}^{n-1}) - n \cot g \, \varnothing \; Re(P_{\mu 2}^{n})]$$

$$+i\,(A_2^n - A_3^n)[\gamma_n Im(P_{\mu 2}^{n-1}) + \beta_n Re(P_{\mu 2}^{n-1}) - n \cot g \, \varnothing \; Im(P_{\mu 2}^{n})]$$

$$+ B_1^n \, [(n - \mu_1 - 1)(n + \mu_1) Q_{\mu 1}^{n-1} - n \cot g \, \varnothing \; Q_{\mu 1}^{n}]$$

$$+(B_2^n + B_3^n)[\gamma_n Re(Q_{\mu 2}^{n-1}) - \beta_n Im(Q_{\mu 2}^{n-1}) - n \cot g \, \varnothing \; Im(Q_{\mu 2}^{n})]$$

$$+i\,(B_2^n - B_3^n)[\gamma_n Im(Q_{\mu 2}^{n-1}) + \beta_n Re(Q_{\mu 2}^{n-1}) - n \cot g \, \varnothing \; Im(Q_{\mu 2}^{n})]$$

(3.4.4)

In matrix form, the displacement field at the node is:

$$\begin{Bmatrix} \delta \end{Bmatrix} = \begin{Bmatrix} U_\varnothing \\ W \\ \left(\dfrac{\partial W}{\partial \varnothing}\right) \\ U_\theta \end{Bmatrix} = [\tau] \; [A_\varnothing] \begin{Bmatrix} C \end{Bmatrix} \qquad (3.4.5)$$

where

$[\tau]$ is the (4 x 4) matrix

$[A_\varnothing]$ is the (4 x 8) matrix

Matrices $[\tau]$ and $[A_\varnothing]$ are shown in Appendix V. Each finite element, having two nodes and four degrees of freedom per node, shows a nodal displacement field as follows:

$$\begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} = [A] \begin{Bmatrix} C \end{Bmatrix} \qquad (3.4.6)$$

where:

δ_i, δ_j are displacement vectors defined by (3.4.5) at nodes i and j, respectively.

[A] is the (8 x 8) matrix

$$[A] = \begin{bmatrix} [A_\varnothing]_i \\ [A_\varnothing]_j \end{bmatrix}$$

where:

$[A_\varnothing]_i$ is the $[A_\varnothing]$ matrix calculated at node i.

$[A_\varnothing]_j$ is the $[A_\varnothing]$ matrix calculated at node j.

Multiplying the two sides of (3.4.6) by $[A]^{-1}$, we obtain:

$$\begin{Bmatrix} C \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (3.4.8)$$

Inserting (3.4.8) into (3.4.1), we find:

$$\begin{Bmatrix} U_\phi \\ W \\ U_\theta \end{Bmatrix} = [T]\ [R]\ [A]^{-1} \begin{Bmatrix} \delta_i \\ \\ \delta_j \end{Bmatrix} \qquad (3.4.9)$$

The displacement matrix is defined as follows:

$$[N] = [R]\ [A]^{-1} \qquad (3.4.10)$$

Relations (3.4.9) become:

$$\begin{Bmatrix} U_\phi \\ W \\ U_\theta \end{Bmatrix} = [T]\ [N] \begin{Bmatrix} \delta_i \\ \\ \delta_j \end{Bmatrix} \qquad (3.4.11)$$

3.5) Development of the Displacement Matrices

Displacement matrices [R] and $[A_\phi]$ as they stand make it difficult to use a computer for the calculations. We therefore need more development on the coefficients involved in the elements of these matrices.

Solving equations (3.1.4) and (3.1.5), using relations (3.3.1) to (3.3.6) and noting that $1 + \zeta \cong \zeta$, we obtain:

$$g_1 = 1 \tag{3.5.1}$$

$$g_2 = 0 \tag{3.5.2}$$

$$g_3 = \frac{1 + \nu}{\sqrt{\zeta \ (1 + \nu^2)}} \tag{3.5.3}$$

$$\mu_1 = b_1 = 1 \tag{3.5.4}$$

$$b_2 = \frac{\sqrt{2}}{2} [\zeta \ (1 - \nu^2)]^{1/4} - \frac{1}{2} \tag{3.5.5}$$

$$b_3 = \frac{\sqrt{2}}{2} [\zeta \ (1 - \nu^2)]^{1/4} \tag{3.5.6}$$

$$\beta_n = - \sqrt{\zeta \ (1 - \nu^2)} \tag{3.5.7}$$

$$\gamma_n = n^2 - n + 0.25 \tag{3.5.8}$$

The new matrices for $[R]$ and $[A_\infty]$ are obtained from these coefficients. They are designated as $[R_-]$ and $[A_{\infty -}]$ and are given in Appendix V.

Expression (3.4.7) becomes:

$$[A] = \begin{bmatrix} [A_{em}]_i \\ [A_{em}]_j \end{bmatrix} \qquad (3.5.9)$$

where

$[A_{em}]_i$ is the $[A_{em}]$ matrix calculated at node i.

$[A_{em}]_j$ is the $[A_{em}]$ matrix calculated at node j.

CHAPTER IV

DEFORMATION MATRIX AND STRESS RESULTANTS MATRIX

FOR A FINITE ELEMENT

4.1) Deformation Matrix

In this section, the deformation vector will be expressed as a function of the three displacements, W, U_ϕ and U_θ defined, respectively, by (3.3.13), (3.3.14) and (3.3.15). Using relations (3.3.13) to (3.3.15) and (II.2.1) to (II.2.7) in the strain-displacement relations (2.2.1) to (2.2.6), we find:

$$\left\{ \epsilon \right\} = \left\{ \begin{matrix} \epsilon_\phi \\ \epsilon_\theta \\ 2\epsilon_{\phi\theta} \\ K_\phi \\ K_\theta \\ 2K_{\phi\theta} \end{matrix} \right\} = \begin{bmatrix} [T] & [O] \\ [O] & [T] \end{bmatrix} \quad [Q] \left\{ C \right\} \qquad (4.1.1)$$

where:

- [T] and {C} are the matrix and the column vector, respectively, defined by (3.4.1).
- [Q] is the (6 x 8) matrix shown in Appendix V. The elements of the matrix are then developed by means of relations (3.5.1) to (3.5.8). The matrix thus obtained is designated as $[Q_*]$ and is given in Appendix V.

The deformation vector becomes:

$$\left\{ \epsilon \right\} = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix} [Q_m] \left\{ C \right\} \tag{4.1.2}$$

Replacing vector {C} with its expression in (3.4.8), we get:

$$\left\{ \epsilon \right\} = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix} [Q_m] \, [A]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.1.3}$$

The deformation matrix is defined as follows:

$$[B] = [Q_m] \, [A]^{-1} \tag{4.1.4}$$

The deformation vector is:

$$\left\{ \epsilon \right\} = \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix} [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \tag{4.1.5}$$

4.2) Stress Resultants Matrix

The spherical shells under investigation are assumed to be isotropic and elastic. The stress resultants as a function of the deformations are:

$$\left\{ \sigma \right\} = \begin{Bmatrix} N_\phi \\ N_\theta \\ N_{\phi\theta} \\ M_\phi \\ M_\theta \\ M_{\phi\theta} \end{Bmatrix} = [P] \left\{ \epsilon \right\} = [P] \begin{Bmatrix} \epsilon_\phi \\ \epsilon_\theta \\ 2\epsilon_{\phi\theta} \\ K_\phi \\ K_\theta \\ 2K_{\phi\theta} \end{Bmatrix} \tag{4.2.1}$$

where [P] is the elasticity matrix shown in Appendix V.

Substituting $\{\epsilon\}$ by its relations (4.1.3) or (4.1.5) into (4.2.1), we get:

$$\begin{Bmatrix} \sigma \end{Bmatrix} = [P] \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix} [Q_m]\,[A]^{-1} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (4.2.2)$$

$$\begin{Bmatrix} \sigma \end{Bmatrix} = [P] \begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix} [B] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (4.2.3)$$

The matrix for stress resultants for all finite element i and j nodes is given by:

$$\begin{Bmatrix} \sigma_i \\ \sigma_j \end{Bmatrix} = \begin{bmatrix} \begin{matrix} [T] & [0] \\ [0] & [T] \end{matrix} & [0] \\ [0] & \begin{matrix} [T] & [0] \\ [0] & [T] \end{matrix} \end{bmatrix} \begin{bmatrix} [P]\,[Q_m]_i\,[A]^{-1} \\ [P]\,[Q_m]_j\,[A]^{-1} \end{bmatrix} \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (4.2.4)$$

where

$[Q_m]_i$ is the $[Q_m]$ matrix calculated at node i.

$[Q_m]_j$ is the $[Q_m]$ matrix calculated at node j.

Setting:

$$[ST] = \begin{bmatrix} \begin{matrix} [T] & [0] \\ [0] & [T] \end{matrix} & [0] \\ [0] & \begin{matrix} [T] & [0] \\ [0] & [T] \end{matrix} \end{bmatrix} \begin{bmatrix} [P]\,[Q_m]_i\,[A]^{-1} \\ [P]\,[Q_m]_j\,[A]^{-1} \end{bmatrix} \qquad (4.2.5)$$

Relations (4.2.4) become:

$$\begin{Bmatrix} \sigma_i \\ \sigma_j \end{Bmatrix} = [ST] \begin{Bmatrix} \delta_i \\ \delta_j \end{Bmatrix} \qquad (4.2.6)$$

CHAPTER V

STIFFNESS AND MASS MATRICES FOR A FINITE ELEMENT

5.1) Stiffness Matrix

In the local system (R, ϕ, θ) , the stiffness matrix is defined as follows ([43],[44]):

$$[k]_{loc} = \int_V [B]^T [P] [B] \, dV \qquad (5.1.1)$$

where:

dV is the volume element = $tR^2 \, d\phi \, d\theta$

[B], [P] are matrices defined by (4.1.4) and (4.2.1.), respectively.

Inserting these relations into (5.1.1), the stiffness matrix is:

$$[k]_{loc} = t \int_S [\ [A]^{-1}\ ]^T [Q_m]^T [P] [Q_m] [A]^T R^2 \, d\phi \, d\theta$$

Integration with respect to θ yields:

$$[k]_{loc} = t\ [\ [A]^{-1}\ ]^T \left\{ \pi R^2 \int_{\phi_i}^{\phi_j} [Q_m]^T [P] [Q_m] \, d\phi \right\} [A]^{-1}$$

Setting

$$[G] = \pi R^2 \int_{\phi_1}^{\phi_j} [Q_m]^T [P] [Q_m] \, d\phi \qquad (5.1.2)$$

The stiffness matrix is expressed:

$$[k]_{loc} = t \, [\, [A]^{-1} \,]^T [G] \, [A]^{-1} \qquad (5.1.3)$$

Matrices [G] and $[k]_{loc}$ are (8 x 8). The development of matrix [G] is given in Appendix III.

5.2) Mass Matrix

In the local system (R, ø, θ) , the mass matrix is defined as follows ([43],[44]):

$$[m]_{loc} = \int_V [N]^T \rho \, [N] \, dV \qquad (5.2.1)$$

where:

dV is the volume element = $t \, R^2 \, d\phi \, d\theta$

ρ is the density of the element.

[N] is the matrix defined by (3.4.10)

Inserting (3.4.10) into (5.2.1) and integrating with respect to θ, we obtain:

$$[m]_{loc} = \rho t \, [\, [A]^{-1} \,]^T \left\{ \pi R^2 \int_{\phi_1}^{\phi_j} [R_m]^T [R_m] \, d\phi \right\} [A]^{-1}$$

Setting:

$$[S] = \pi R^2 \int_{\phi_1}^{\phi_j} [R_m]^T [R_m] \, d\phi \qquad (5.2.2)$$

The mass matrix can be written:

$$[m]_{loc} = \rho t \, [\, [A]^{-1} \,]^T [S] \, [A]^{-1} \qquad (5.2.3)$$

Matrices [S] and $[m]_{loc}$ are (8 x 8). Development of the [S] matrix is shown in Appendix III.

5.3) Coordinate Transformation

The stiffness and mass matrices defined by (5.1.3) and (5.2.3), respectively, were determined within the local system associated with each finite element. During assembly of the finite elements in the global system (X1, X2, X3) (Fig. 1), transformation of the coordinates is required:

$$\left\{ \delta \right\}_{glo} = [DLAM] \left\{ \delta \right\}_{loc} \qquad (5.3.1)$$

where:

$\{\delta\}_{loc}$ is the displacement vector at finite element nodes i and j in the local system

$$= \left\{ U_{\phi}^{i},\ W^{i},\ \left(\frac{\partial W}{\partial \phi}\right)^{i},\ U_{\theta}^{i},\ U_{\phi}^{j},\ W^{j},\ \left(\frac{\partial W}{\partial \phi}\right)^{j},\ U_{\theta}^{j} \right\}^{T}$$

$\{\delta\}_{glo}$ is the displacement vector at finite element nodes i and j in the global system

$$= \left\{ U_{v}^{i},\ W_{h}^{i},\ \left(\frac{\partial W}{\partial \phi}\right)^{i},\ U_{\theta}^{i},\ U_{v}^{j},\ W_{h}^{j},\ \left(\frac{\partial W}{\partial \phi}\right)^{j},\ U_{\theta}^{j} \right\}^{T}$$

with:

U_v is the vertical displacement parallel to axis X3

W_h is the horizontal displacement parallel to axis X2 (see Fig. 5)

[DLAM] is the (8 x 8) coordinate transformation matrix, as given in Appendix V.

In the global system, the mass and stiffness matrices are:

$$[k] = t\ [DLAM]^{T}\ [\ [A]^{-1}\ ]^{T}\ [G]\ [A]^{-1}\ [DLAM] \qquad (5.3.2)$$

$$[m] = \rho t\ [DLAM]^{T}\ [\ [A]^{-1}\ ]^{T}\ [S]\ [A]^{-1}\ [DLAM] \qquad (5.3.3)$$

where

t is the thickness of the spherical shell

ρ is the density

[A] is the matrix defined by (3.5.9)

[G] is the matrix defined by (5.1.2)

[S] is the matrix defined by (5.2.2)

Matrices [k] and [m] are (8 x 8) matrices.

CHAPTER VI

EQUATIONS OF MOTION

6.1) Assembling the Finite Elements

As has already been mentioned, the complete shell is partitioned into a finite number of spherical elements, the positions of which can be chosen arbitrarily. Each finite element has 2 nodes at the ends (Fig. 5).

Having obtained the stiffness matrix (5.3.2) and mass matrix (5.3.3) for each finite element, we then construct the global matrices for the total shell using the usual finite element assembly technique. The technique ensures that:

- the generalized internal forces balance out the external forces at the node common to two adjacent elements [55]:

$$\{F\}^{e} = \{F_j\}_k + \{F_i\}_{k+1}$$

where:

$\{F\}$ - is the external force vector. In the free vibration problem, these forces are zero.

$\{F_j\}_k$ is the internal force vector at node j of element k.

$\{F_i\}_{k+1}$. is the internal force vector at node i of element (k+1).

- There is continuity in the generalized displacements corresponding to those forces at the node considered.

$$\{\delta_j\}_k = \{\delta_i\}_{k+1}$$

Figure 5 gives a schematic view of the construction of the global stiffness [K] and mass [M] matrices. These are square matrices of order 4(N + 1) where N is the number of finite elements.

6.2) Equations of Motion

For free vibration, the equations of motion are:

$$[M]\ \{\ddot{\delta}\}_T + [K]\ \{\delta\}_T = \{0\} \qquad (6.2.1)$$

where:

[M], [K] are the global mass and stiffness matrices

$\left\{\delta\right\}_T$ is the global displacement vector for the complete shell.

$$\left\{\delta\right\}_T = \left\{\delta_o\right\}_T \sin(\omega t + \varphi) \qquad (6.2.2)$$

N is the number of finite elements

The vibration is harmonic:

$$\left\{\delta\right\}_T = \left\{\delta_1, \ \delta_2 \ \ldots \quad \ldots \ \delta_{N+1}\right\}^T$$

where:

ω is the natural angular frequency

φ is the phase angle

Introducing equation (6.2.2) into (6.2.1), we obtain:

$$([K] - \omega^2 [M]) \left\{\delta_o\right\}_T = \left\{0\right\} \qquad (6.2.3)$$

This is only possible for certain values of ω for which the matrix determinant in parentheses is zero. As such a determinant is of degree 4(N + 1), there are generally 4(N + 1) real roots ω^2.

These define the structure's natural angular frequencies and we recognize that this is now a typical eigenvalue problem:

$$\det \mid [K] - \omega^2 [M] \mid = 0 \qquad (6.2.4)$$

For every natural frequency for which equation (6.2.3) has been validated, there is a corresponding vector $\{\delta_o\}$ whose components are defined down to one arbitrary multiplying factor. Such vectors are called natural modes (or eigenvectors) of the system.

6.3) Boundary Conditions

If the shell has boundary constraints such as being simply supported, clamped, etc., the appropriate lines and columns in [K] and [M] are eliminated to satisfy these constraints. Consequently, matrices [K] and [M] reduce to square matrices of order 4(N + 1) - J where J is the number of constraints applied. Thus, for a shell:

- free: J = 0
- with 2 simply supported edges; ($U_\infty = W = 0$): J = 4

– with 2 edges clamped: $(U_\phi = W = \dfrac{\partial W}{\partial \phi} = U_\theta = 0)$: $J = 8$

The form of (6.2.4) is unaffected, except that the reduced matrices $[K]_{red.}$ and $[M]_{red.}$ are finite and positive.

CHAPTER VII

THE ALGORITHM

The behaviour of spherical shells under free vibration will be studied using the new hybrid method, which combines classical shell and finite element theory [55].

In order to determine the eigenvalues (natural angular frequencies) and the eigenvectors (natural modes) of a given spherical shell, the shell must first be partitioned into a sufficient number of spherical elements; the particular number of them will be discussed later on. The SPHERE program does the calculations. Upon acceptance of the data supplied, the program draws up the stiffness and mass matrices for each finite element. It then builds up the global matrices for the total shell with or without boundary constraints. It ends with calculating the eigenvalues and eigenvectors for the shell. The input data needed are the average radius, the thickness, Young's modulus, Poisson's ratio, density, the ϕ coordinates for each individual finite element, the number of circumferential modes as well as the boundary constraints applied at the nodes (if any). It should be mentioned here that Dr. Lakis also developed the SPHERE program to apply to non-uniform spherical shells where thickness and elastic properties may vary.

For each element, the program works as follows:

1. Constants $\lambda_\ell(n)$, $\eta_\ell(n)$, $d(\ell,n)$ and $f(\ell,n)$, defined by (I.3.2) and I.3.7), respectively, are computed.

2. The intermediate matrices are determined, that is, [DLAM], $[R_m]$, $[A_{om}]$, $[Q_m]$, [P], $[A_r]$, $[B_r]$, $[A_r^c]$, $[B^{rc}]$, $[A_r^v]$, $[B_r^v]$, $[\alpha_r]$, $[C_r]$, $[D_r]$, $[C_r^c]$, $[D_r^c]$, $[C_r^v]$, $[D_r^v]$, $[\Gamma_r]$, $[\Delta_r]$, $[\Gamma_r^c]$, $[\Delta_r^c]$, $[\Gamma_r^v]$, $[\Delta_r^v]$, [G] and [S].

3. Displacement matrix [N], stiffness matrix [k], mass matrix [m] and stress resultants matrix [ST] are constructed.

4. Once the mass and stiffness matrices for each finite element are obtained, the global mass [M] and stiffness [K] matrices are constructed by the assembly technique.

5. If there are any constraints at the shell edges, the reduced mass $[K]_{red}$ and stiffness $[M]_{red}$ matrices are constructed.

6. With the global matrices developed, the program follows standard procedure to compute the eigenvalues (natural angular frequencies) and corresponding eigenvectors (natural modes).

CHAPTER VIII

NUMERICAL CALCULATIONS AND DISCUSSION

8.1) Data

In order to test its efficiency and its accuracy, we used the theory developed in the course of this research to calculate the natural frequencies and modes of uniform thin elastic spherical shells, which were both shallow and non-shallow, of various dinmensions and under different boundary conditions. These cases have already been investigated by other authors using other methods. Table 1 in Appendix VI summarizes their findings.

The SPHERE program did all the calculations required in our theory, that is, for:

- number of circumferential modes $n = 2$
- density $\rho = 0.75183 \times 10^{-3}$ Lb-sec^2/in^4
- Poisson's ratio $\nu = 0.3$.
- Young's modulus $E = 0.295 \times 10^8$ Lb/in^2

For purposes of comparisons among the natural frequencies obtained, it is eminently practical at this stage to introduce the non-dimensional natural frequency:

$$\Omega = \omega R \left(\frac{\rho}{E}\right)^{1/2}$$

where:

- ω is the natural angular frequency
- R is the radius of the reference surface
- ρ is the density
- E is the modulus of elasticity

8.2) <u>Case 1</u>: Spherical shell $\phi_o = 60°$, under the three boundary conditions: hinged, free and clamped.

The free, θ-independent vibration of the spherical shell in this case was studied by Kalnins [33], Navaratna [45], Webster [46], Greene et al [47] and Cohen [49]. In the present investigation, the shell was studied for $n = 2$, with 3, 6 and 10 elements for the first boundary condition: with only 10 elements for the last two. Only the non-dimensional natural frequencies, which were calculated with 10 elements, appear in Table 2. The natural modes corresponding to the lowest shell frequencies under the three boundary conditions are illustrated in normalized form in Figures 6, 7 and 8.

It is easy to see that all three displacements U_ϕ, W and U_θ are all zero at the top ($\phi = 0$) of the spherical shell. The displacement ratios, $(U_\phi)_{max}/(W)_{max}$ and $(U_\theta)_{max}/(W)_{max}$ are presented in Table 3. They reveal that at the lowest frequency, spherical shell motion is generally radial. In the case of the hinged shell, we observed convergence of the solution by plotting the number of finite elements against the first four natural frequencies (Figure 9). It became apparent that the first two frequencies can be considered as converging with six elements. As to the higher-order frequencies, ten elements are required for greater precision of results.

8.3) Case 2: Clamped spherical shell with $\phi_o = 30°$

This case was investigated analytically by Kalnins [35] using classical theory and transverse vibration theory. With the new theory, we used 8 finite elements to study the spherical shell, with the results shown in Table 4. The frequencies we obtained with our method are very comparable to Kalnins' values. The displacements corresponding to the first frequency, in normalized form, are shown in Figure 10. The maximum displacement values were:

$(U_\phi)_{max} = 0.1010$

$(W)_{max} = 0.2123$

$(U_\theta)_{max} = 0.0965$

It was observed that at the lowest frequency, motion of the spherical shell is mainly dominated by its radial component.

8.4) Case 3: Simply supported spherical shell with $\phi_0 = 90°$.

Kraus [18] investigated this case using a general theory which included the effects of transverse shear stress and rotational inertia. For cases both with and without these effects, he determined the natural frequencies for the shell motion that was independent of θ for circumferential mode number $n = 0$. With our theory and using 10 finite elements, the natural frequencies were computed for $n = 2$. The results are shown in Table 5. The displacements corresponding to the lowest frequency are given, in normalized form, in Figure 11. The maximal displacement values were:

$(U_{\phi})max = 0.3318$

$(W)max = 0.2317$

$(U_{\theta})max = 0.0854$

The result is that at the lowest frequency, the motion of the spherical shell is predominately influenced by the axial component.

CHAPTER IX

CONCLUSIONS

The purpose of this ~~thesis~~ work was to present a new theory for the dynamic analysis of non-uniform thin elastic spherical shells. It is a hybrid method, based on classical thin shell theory and the finite element method.

The principle objectives as stated at the outset have been accomplished. The displacement functions as adapted were verified. The finite element mass, stiffness and stress resultants matrices and the corresponding global matrices for the complete shell were established. The "SPHERE" computer program which arose from this new theory was restructured, thereby reducing both CPU time and storage space. It was applied to determinations of the free vibration characteristics of uniform spherical shells under various boundary conditions for number of circumferential modes $n \geq 2$.

There is a relative dearth of data in the literature on free vibrations of spherical shells, particularly with $n \geq 2$. We are unable, therefore, to make any direct comparisons. A comparison of the findings for clamped spherical shells has revealed good similarities between the natural frequencies calculated by the new theory and the values found by others in other investigations. The variances observed could be laid at the door of differences in the methods used. In the present state of affairs and in the absence of published articles on the subject, it would be difficult to check the accuracy of the results obtained. Only through future experimental investigation could the true frequencies be confirmed. A look at convergence revealed that working with a number of finite elements greater than or equal to six can provide the low frequencies with good accuracy. As to the higher-order frequencies, ten or more elements appear to be required. Use of a large number of smaller finite elements would certainly push the results we computed with the new theory - the asymptotic solutions - towards the real solutions.

A few cases of uniform spherical shells were presented. The SPHERE program was developed and is now available. This program, in combination with the other programs our group has already developed for cylindrical and conical shells, can lay claim to being a good vehicle for studying the dynamic behaviour of the majority of shells of varying geometry that are used in industry: they could have every kind of discontinuity and every boundary condition conceivable. This new theory is certainly the first stage in subsequent research into spherical shells containing stationary or flowing fluid.

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APPENDIX I

SUMMARY OF ASSOCIATED LEGENDRE FUNCTIONS

I.1) Definition

Associated Legendre functions are solution functions of the Legendre differential equation (see [61] and [62]):

$$(1-z^2)\frac{d^2f}{dz^2} - 2z\frac{df}{dz} + \left[\nu(\nu+1) - \mu^2(1+z^2)^{-1}\right]f = 0 \qquad (\mathrm{I}.1.1)$$

The ν and μ parameters are generally complex numbers. This differential equation has three singular points at $z = 1$, $z = -1$ and $z = \infty$. There are two distinct cases:

1) The real z variable is between −1 and +1:

The basic functions of the solutions are designated as $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ (or also as $P_\nu^\mu(\cos\phi)$ and $Q_\nu^\mu(\cos\phi)$ where $0 \leq \phi \leq \pi$).

2) The z variable is outside the −1, +1 range:

In this case, the functions are designated as $\tilde{P}_\nu^\mu(z)$ and $\tilde{Q}_\nu^\mu(z)$.

These functions are called associated Legendre functions of the first and second kind, respectively. They are special cases of the hypergeometric functions F(a, b; c; z) where parameters a, b and c are such that the quadratic transformation exists.

For $-1 < z < +1$:

$$P_{\nu}^{\mu}(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{1+z}{1-z}\right)^{\mu/2} F\left(-\nu, \nu+1, 1-\mu, \frac{1-z}{2}\right)$$

(I.1.2)

For z outside of the −1, +1 range:

$$\tilde{P}_{\nu}^{\mu}(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} F\left(-\nu, \nu+1, 1-\mu, \frac{1-z}{2}\right)$$

(I.1.3)

where:

Γ = **Gamma function:** $\Gamma(z) = z^{-1} \prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{z} \left(1+\frac{z}{n}\right)^{-1}$

(I.1.4)

F = **hypergeometric function:**

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\,\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

(I.1.5)

1.2) Useful Formulas

a) Relations between the Wronskian solutions:

$$P_\mu^{-n}(X) = \frac{\Gamma(\mu+1-n)}{\Gamma(\mu+1+n)} (-1)^n P_\mu^n(X) \qquad (I.2.1)$$

$$Q_\mu^{-n}(X) = \frac{\Gamma(\mu+1-n)}{\Gamma(\mu+1+n)} (-1)^n Q_\mu^n(X) \qquad (I.2.2)$$

b) Real and imaginary parts of the functions:

$P_\mu^n(\cos\phi)$ et $Q_\mu^n(\cos\phi)$:

$$P_\mu^n(\cos\phi) = \mathrm{Re}\left[P_\mu^n(\cos\phi)\right] + i\mathrm{I_m}\left[P_\mu^n(\cos\phi)\right] \qquad (I.2.3)$$

$$Q_\mu^n(\cos\phi) = \mathrm{Re}\left[Q_\mu^n(\cos\phi)\right] + i\mathrm{I_m}\left[Q_\mu^n(\cos\phi)\right] \qquad (I.2.4)$$

c) Derivatives with respect to ϕ:

$$\frac{dP_\mu^n}{d\phi}(\cos\phi) = (n-\mu-1)(n+\mu)P_\mu^{n-1}(\cos\phi) - n\,\mathrm{cotg}\,\phi\, P_\mu^n(\cos\phi) \qquad (I.2.5)$$

$$\frac{dQ_\mu^n}{d\phi}(\cos\phi) = (n-\mu-1)(n+\mu)Q_\mu^{n-1}(\cos\phi) - n\,\mathrm{cotg}\,\phi\, Q_\mu^n(\cos\phi) \qquad (I.2.6)$$

d) Expressions in trigonometric series:

$$\Gamma\left(\tfrac{3}{2}+\mu\right)P_\mu^n(\cos\phi) = \pi^{-1/2}\, 2^{n+1}(\sin\phi)^n\,\Gamma(n+\mu+1)$$
$$\times \sum_{\ell=0}^{\infty} \frac{\left(\tfrac{1}{2}+n\right)_\ell \left(1+\mu+n\right)_\ell}{\ell!\,\left(\tfrac{3}{2}+\mu\right)_\ell} \sin\left[(2\ell+\mu+n+1)\phi\right] \qquad (I.2.7)$$

where:

$$\left(\frac{1}{2}+n\right)_\ell = \frac{\Gamma\left(\frac{1}{2}+n+\ell\right)}{\Gamma\left(\frac{1}{2}-n\right)}$$

$$(1+\mu+n)_\ell = \frac{\Gamma(1+\mu+n+\ell)}{\Gamma(1+\mu+n)}$$

$$\left(\frac{3}{2}+\mu\right)_\ell = \frac{\Gamma\left(\frac{3}{2}+\mu+\ell\right)}{\Gamma\left(\frac{3}{2}+\mu\right)}$$

$$\Gamma\left(\frac{3}{2}+\mu\right) Q^n_\mu(\cos\phi) = \pi^{\frac{1}{2}}\ 2^n (\sin\phi)^n\ \Gamma(n+\mu+1)$$
$$\times \sum_{\ell=0}^{\infty} \frac{\left(\frac{1}{2}+\mu\right)_\ell\ (1+\nu+\mu)_\ell}{\ell!\ \left(\frac{3}{2}+\mu\right)_\ell} \cos\left[(2\ell+\mu+n+1)\phi\right] \qquad (\text{I}.2.8)$$

e) Expressions for n = -1, 0, 1:

$$P_1^{-1}(\cos\phi) = \frac{\sin\phi}{2} \qquad (\text{I}.2.9)$$

$$P_1^{0}(\cos\phi) = \cos\phi \qquad (\text{I}.2.10)$$

$$P_1^{1}(\cos\phi) = -\sin\phi \qquad (\text{I}.2.11)$$

$$Q_1^{-1}(\cos\phi) = \frac{\sin\phi}{4} \ln\left(\frac{1+\cos\phi}{1-\cos\phi}\right) + \frac{1}{2}\operatorname{cotg}\phi \qquad (\text{I}.2.12)$$

$$Q_1^{0}(\cos\phi) = \frac{\cos\phi}{2} \ln\left(\frac{1+\cos\phi}{1-\cos\phi}\right) - 1 \qquad (\text{I}.2.13)$$

$$Q_1^{1}(\cos\phi) = -\frac{\sin\phi}{2} \ln\left(\frac{1+\cos\phi}{1-\cos\phi}\right) - \operatorname{cotg}\phi \qquad (\text{I}.2.14)$$

1.3) Summary of expressions developed

With the formulas in [61] and [62], we derive the expressions for the Legendre functions used here:

$$P^{n}_{\mu_2} = \sin^{-n}\phi \sum_{\ell=0}^{\infty} [\lambda_\ell(n) + i\eta_\ell(n)][\sin x_\ell(n)\,\phi\ \mathrm{ch}y\phi + i\cos x_\ell(n)\,\phi\ \mathrm{sh}y\phi] \qquad (\mathrm{I}.3.1)$$

where:

$$\lambda_\ell(n) + i\eta_\ell(n) = \frac{(-1)^n \quad \Gamma(\frac{1}{2}-n+\ell)\times\Gamma(\mu_2+1+n)\times\Gamma(1+\mu_2-n+\ell)}{(\pi^{1/2})(2^{n-1})\ \Gamma(\frac{1}{2}-n)\times\Gamma(\mu_2+1-n)\times\Gamma(\frac{3}{2}+\mu_2+\ell)\times\ell!} \qquad (\mathrm{I}.3.2)$$

$$x_\ell(n) + iy = 2\ell - n + \mu_2 + 1 \qquad (\mathrm{I}.3.3)$$

$$\mathrm{Re}\,(P^{n}_{\mu_2}) = \sin^{-n}\phi\left\{\sum_{\ell=0}^{\infty}\lambda_\ell(n)\sin x_\ell(n)\,\phi\ \mathrm{ch}y\phi - \eta_\ell(n)\cos x_\ell(n)\,\phi\ \mathrm{sh}y\phi\right\} \qquad (\mathrm{I}.3.4)$$

$$\mathrm{Im}\,(P^{n}_{\mu_2}) = \sin^{-n}\phi\left\{\sum_{\ell=0}^{\infty}\eta_\ell(n)\sin x_\ell(n)\,\phi\ \mathrm{ch}y\phi + \lambda_\ell(n)\cos x_\ell(n)\,\phi\ \mathrm{sh}y\phi\right\} \qquad (\mathrm{I}.3.5)$$

Another expression for $P^{n}_{\mu_2}$:

$$P^{n}_{\mu_2} = \sum_{\ell=0}^{\infty}[d(\ell,n) + if(\ell,n)]\frac{\sin^{n+2\ell}\ (\phi/2)}{\cos^{n}\ (\phi/2)} \qquad (\mathrm{I}.3.6)$$

with

$$d(\ell,n) + if(\ell,n) = (-1)^n\frac{\Gamma(\mu_2+1+n)\times\Gamma(-\mu_2+\ell)\times\Gamma(1+\mu_2+\ell)}{\Gamma(\mu_2+1-n)\times\Gamma(1+\mu_2)\times\Gamma(-\mu_2)\times\Gamma(1+n+\ell)\times\ell!} \qquad (\mathrm{I}.3.7)$$

$$\mathrm{Re}\,(P^{n}_{\mu_2}) = \sum_{\ell=0}^{\infty} d(\ell,n)\,\frac{\sin^{n+2\ell}\ (\phi/2)}{\cos^{n}\ (\phi/2)} \qquad (\mathrm{I}.3.8)$$

$$I_m\,(P^n_{\mu_2}) = \sum_{\ell=0}^{\infty} j(\ell, n)\ \frac{\sin^{n+2\ell}\ (\phi/2)}{\cos^{n}\ (\phi/2)}$$

(I.3.9)

$$Q^n_{\mu_2} = \sin^{-n}\phi \sum_{\ell=0}^{\infty} \frac{\pi}{2}\left[\lambda_\ell(n) + i\eta_\ell(n)\right]\left[\cos x_\ell(n)\,\phi\ \mathrm{ch}y\phi - i\sin x_\ell(n)\phi\ \mathrm{sh}y\phi\right] \quad (I.3.10)$$

$$Re\,(Q^n_{\mu_2}) = \frac{\pi/2}{\sin^n\phi}\left\{\sum_{\ell=0}^{\infty}\lambda_\ell(n)\cos x_\ell(n)\,\phi\ \mathrm{ch}y\phi + \eta_\ell(n)\sin x_\ell(n)\,\phi\ \mathrm{sh}y\phi\right\} \quad (I.3.11)$$

$$I_m\,(Q^n_{\mu_2}) = \frac{\pi/2}{\sin^n\phi}\left\{\sum_{\ell=0}^{\infty}\eta_\ell\ (n)\cos x_\ell(n)\phi\ \mathrm{ch}y\phi - \lambda_\ell\ (n)\ \sin x_\ell\ (n)\,\phi\ \mathrm{sh}y\phi\right\} \quad (I.3.12)$$

$$P^n_1 = \frac{(-1)^{-n}}{\Gamma(2-n)}\left\{nt^n + (\cos\phi)\,t^n\right\} \quad (I.3.13)$$

with $t = tg\ (\phi/2)$

$$Q^n_1 = \frac{(n-2)!}{2}\left\{(-1)^n\times\Gamma(2-n)\times P^n_1 + \left[(-1)^n\times n - (-1)^n\cos\phi\right]\frac{1}{t^n}\right\}$$

(I.3.14)

<u>Note:</u> Tables for associated Legendre functions of the first and second kind are seldom available in the literature, especially when the degree μ is a fractional or complex number. Gray's work [63] gives results with a fractional $\bar{\mu}$ degree. Recently, Singh and Mirza [64] proposed an automatic calculation method using the Mehler-Dirichlet integral representation of the Legendre functions

with the restriction that $Re\ (\mu) < 1/2$. In the present research, the $P_{\mu 2}^{n}$ and $Q_{\mu 2}^{n}$ functions are represented analytically by an infinite series $\sum_{\ell=0}^{\infty}$. In the course of the numerical calculations on the computer, this infinite series is replaced by the finite series $\sum_{\ell=0}^{10}$. This is the only approximation used to evaluate Legendre functions and there is no restriction on the subscript.

I.4) Derivatives of the $Q_{\mu 2}^{n}$ (cosø), $P_{\mu 3}^{n}$ (cosø), $Q_{\mu 2}^{n}$ (cosø) and $Q_{\mu 3}^{n}$ (cosø) functions:

$$\frac{dP_{\mu_2}^{n}(\cos\phi)}{d\phi} = \gamma_n \mathrm{Re}\left(P_{\mu_2}^{n-1}\right) - \beta_n \mathrm{I_m}\left(P_{\mu_2}^{n-1}\right) - n\,\mathrm{cotg}\,\phi\, \mathrm{Re}\left(P_{\mu_2}^{n}\right) + i\left[\gamma_n \mathrm{I_m}\left(P_{\mu_2}^{n-1}\right) + \beta_n \mathrm{Re}\left(P_{\mu_2}^{n-1}\right) - n\,\mathrm{cotg}\,\phi\, \mathrm{I_m}\left(P_{\mu_2}^{n}\right)\right] \quad (I.4.1)$$

$$\frac{dP_{\mu_3}^{n}(\cos\phi)}{d\phi} = \gamma_n \mathrm{Re}\left(P_{\mu_2}^{n-1}\right) - \beta_n \mathrm{I_m}\left(P_{\mu_2}^{n-1}\right) - n\,\mathrm{cotg}\,\phi\, \mathrm{Re}\left(P_{\mu_2}^{n}\right) - i\left[\gamma_n \mathrm{I_m}\left(P_{\mu_2}^{n-1}\right) + \beta_n \mathrm{Re}\left(P_{\mu_2}^{n-1}\right) - n\,\mathrm{cotg}\,\phi\, \mathrm{I_m}\left(P_{\mu_2}^{n}\right)\right] \quad (I.4.2)$$

$$\frac{dQ_{\mu_2}^{n}(\cos\phi)}{d\phi} = \gamma_n \mathrm{Re}\left(Q_{\mu_2}^{n-1}\right) - \beta_n \mathrm{I_m}\left(Q_{\mu_2}^{n-1}\right) - n\,\mathrm{cotg}\,\phi\, \mathrm{Re}\left(Q_{\mu_2}^{n}\right) + i\left[\gamma_n \mathrm{I_m}\left(Q_{\mu_2}^{n-1}\right) + \beta_n \mathrm{Re}\left(Q_{\mu_2}^{n-1}\right) - n\,\mathrm{cotg}\,\phi\, \mathrm{I_m}\left(Q_{\mu_2}^{n}\right)\right] \quad (I.4.3)$$

$$\frac{dQ_{\mu_3}^{n}(\cos\phi)}{d\phi} = \gamma_n \mathrm{Re}\left(Q_{\mu_2}^{n-1}\right) - \beta_n \mathrm{I_m}\left(Q_{\mu_2}^{n-1}\right) - n\,\mathrm{cotg}\,\phi\, \mathrm{Re}\left(Q_{\mu_2}^{n}\right) - i\left[\gamma_n \mathrm{I_m}\left(Q_{\mu_2}^{n-1}\right) + \beta_n \mathrm{Re}\left(Q_{\mu_2}^{n-1}\right) - n\,\mathrm{cotg}\,\phi\, \mathrm{I_m}\left(Q_{\mu_2}^{n}\right)\right] \quad (I.4.4)$$

APPENDIX II

USEFUL EXPRESSIONS IN DETERMINING DISPLACEMENTS AND DEFORMATIONS

II.1) Determination of the terms in parentheses in relations (3.2.2) and (3.2.3)

The calculations below were done from relations (3.3.1) to (3.3.3), (3.3.9) to (3.3.12) and (I.4.1) to (I.4.4):

$$\sum_{\alpha=1}^{3} \left\{ \frac{r_\alpha + \varsigma(1+\nu)}{\varsigma\,[r_\alpha - (1+\nu)]} \right\} \frac{\partial W_\alpha^n}{\partial \phi} = \tag{II.1.1}$$

$$g_1 A_1^n \left\{ (n-\mu_1-1)(n+\mu_1) P_{\mu_1}^{n-1}(\cos\phi) - n\cot g\phi\, P_{\mu_1}^{n}(\cos\phi) \right\}$$

$$+ g_1 B_1^n \left\{ (n-\mu_1-1)(n+\mu_1) Q_{\mu_1}^{n-1}(\cos\phi) - n\cot g\phi\, Q_{\mu_1}^{n}(\cos\phi) \right\}$$

$$+ (g_2 - ig_3) A_2^n \Big\{ \gamma_n R_e(P_{\mu_2}^{n-1}) - \beta_n I_m(P_{\mu_2}^{n-1}) - n\cot g\phi\, R_e(P_{\mu_2}^{n})$$
$$+ i\left[\gamma_n I_m(P_{\mu_2}^{n-1}) + \beta_n R_e(P_{\mu_2}^{n-1}) - n\cot g\phi\, I_m(P_{\mu_2}^{n})\right]\Big\}$$

$$+ (g_2 - ig_3) B_2^n \Big\{ \gamma_n R_e(Q_{\mu_2}^{n-1}) - \beta_n I_m(Q_{\mu_2}^{n-1}) - n\cot g\phi\, R_e(Q_{\mu_2}^{n})$$
$$+ i\left[\gamma_n I_m(Q_{\mu_2}^{n-1}) + \beta_n R_e(Q_{\mu_2}^{n-1}) - n\cot g\phi\, I_m(Q_{\mu_2}^{n})\right]\Big\}$$

$$+ (g_2 + ig_3) A_3^n \Big\{ \gamma_n R_e(P_{\mu_2}^{n-1}) - \beta_n I_m(P_{\mu_2}^{n-1}) - n\cot g\phi\, R_e(P_{\mu_2}^{n})$$
$$- i\left[\gamma_n I_m(P_{\mu_2}^{n-1}) + \beta_n R_e(P_{\mu_2}^{n-1}) - n\cot g\phi\, I_m(P_{\mu_2}^{n})\right]\Big\}$$

$$+ (g_2 + ig_3) B_3^n \Big\{ \gamma_n R_e(Q_{\mu_2}^{n-1}) - \beta_n I_m(Q_{\mu_2}^{n-1}) - n\cot g\phi\, R_e(Q_{\mu_2}^{n})$$
$$- i\left[\gamma_n I_m(Q_{\mu_2}^{n-1}) + \beta_n R_e(Q_{\mu_2}^{n-1}) - n\cot g\phi\, I_m(Q_{\mu_2}^{n})\right]\Big\}$$

$$\sum_{\alpha=1}^{3} \left\{ \frac{r_\alpha + \varsigma \ (1+\nu)}{\varsigma \ [r_\alpha - (1-\nu)]} \right\} W_\alpha^n =$$

$$g_1 A_1^n P_{\mu_1}^n + g_1 B_1^n Q_{\mu_1}^n + (g_2 - i g_3) A_2^n \left[Re\,(P_{\mu_2}^n) + i Im\,(Q_{\mu_2}^n) \right]$$

$$+ (g_2 - i g_3) B_2^n \left[Re\,(Q_{\mu_2}^n) + i Im\,(Q_{\mu_2}^n) \right]$$

$$+ (g_2 + i g_3) A_3^n \left[Re\,(P_{\mu_2}^n) - i Im\,(P_{\mu_2}^n) \right]$$

$$+ (g_2 + i g_3) B_3^n \left[Re\,(Q_{\mu_2}^n) - i Im\,(Q_{\mu_2}^n) \right] \qquad (II.1.2)$$

$$\left(\frac{n}{2}\right) \frac{\partial \Psi_1^n}{\partial \phi} = A_4^n \left[\frac{n}{2} (n-2)(n+1) P_1^{n-1} - \frac{n^2}{2} \operatorname{cotg} \phi \ P_1^n \right]$$

$$+ B_4^n \left[\frac{n}{2} (n-2)(n+1) Q_1^{n-1} - \frac{n^2}{2} \operatorname{cotg} \phi \ Q_1^n \right]$$

$$(II.1.3)$$

II.2) Displacement derivatives

$$1) \quad \frac{\partial U_\phi}{\partial \phi} = a_{1P} A_1^n + a_{2P} (A_2^n + A_3^n) + a_{3P}\, i\, (A_2^n - A_3^n) + a_{4P} A_4^n$$

$$+ a_{1Q} B_1^n + a_{2Q} (B_2^n + B_3^n) + a_{3Q}\, i\, (B_2^n - B_3^n) + a_{4Q} B_4^n$$

$$(II.2.1)$$

where:

$$a_{1P} = g_1\left[(n-\mu_1-1)(n+\mu_1) + n(\operatorname{cosec}^2\phi + n\operatorname{cotg}^2\phi)\right] P_{\mu_1}^{n}$$

$$- g_1 (n-\mu_1-1)(n+\mu_1) P_{\mu_1}^{n-1}$$

$$a_{2P} = \left[(g_2\gamma_n + g_3\beta_n) + ng_2(\operatorname{cosec}^2\phi + n\operatorname{cotg}^2\phi)\right] R_e(P_{\mu_2}^{n})$$

$$+ \left[(g_3\gamma_n - g_2\beta_n) + ng_3(\operatorname{cosec}^2\phi + n\operatorname{cotg}^2\phi)\right] I_m(P_{\mu_2}^{n})$$

$$- (g_2\gamma_n + g_3\beta_n)\operatorname{cotg}\phi\, R_e(P_{\mu_2}^{n-1}) - (g_3\gamma_n - g_2\beta_n)\operatorname{cotg}\phi\, I_m(P_{\mu_2}^{n-1})$$

$$a_{3P} = -\left[(g_3\gamma_n - g_2\beta_n) + ng_3(\operatorname{cosec}^2\phi + n\operatorname{cotg}^2\phi)\right] R_e(P_{\mu_2}^{n})$$

$$+ \left[(g_2\gamma_n + g_3\beta_n) + ng_2(\operatorname{cosec}^2\phi + n\operatorname{cotg}^2\phi)\right] I_m(P_{\mu_2}^{n})$$

$$+ (g_3\gamma_n - g_2\beta_n)\operatorname{cotg}\phi\, R_e(P_{\mu_2}^{n-1})$$

$$- (g_2\gamma_n + g_3\beta_n)\operatorname{cotg}\phi\, I_m(P_{\mu_2}^{n-1})$$

$$a_{4P} = -\frac{n^2}{2}(n-2)(n+1)\operatorname{cosec}\phi\, P_1^{n-1} + \frac{n^2}{2}(n+1)\operatorname{cosec}\phi\operatorname{cotg}\phi\, P_1^{n}$$

Coefficients a_{1Q}, a_{2Q}, a_{3Q}, a_{4Q} are derived from a_{1P}, a_{2P}, a_{3P}, a_{4P} by replacing the symbol P in the associated Legendre function of the first kind with Q of the second kind.

$$2)\quad \frac{\partial^2 W}{\partial \phi^2} = b_{1P} A_1^n + b_{2P}(A_2^n + A_3^n) + b_{3P}\, i\, (A_2^n - A_3^n) + b_{4P} A_4^n$$
$$+ b_{1Q} B_1^n + b_{2Q}(B_2^n + B_3^n) + b_{3Q}\, i\, (B_2^n - B_3^n) + b_{4Q} B_4^n \qquad (\text{II}.\ 2.\ 2)$$

where:

$$b_{1P} = \left[(n-\mu_1-1)(n+\mu_1) + n\,(\operatorname{cosec}^2\phi + n \operatorname{cotg}^2\phi)\right] P_{\mu_1}^{n}$$
$$- (n-\mu_1-1)(n+\mu_1)\, P_{\mu_1}^{n-1}$$

$$b_{2P} = \left[\gamma_n + n\,(\operatorname{cosec}^2\phi + n\operatorname{cotg}^2\phi)\right] R_e\,(P_{\mu_2}^{n}) - \beta_n I_m\,(P_{\mu_2}^{n})$$
$$- \gamma_n \operatorname{cotg}\phi\; R_e\,(P_{\mu_2}^{n-1}) + \beta_n \operatorname{cotg}\phi\; I_m\,(P_{\mu_2}^{n-1})$$

$$b_{3P} = \beta_n R_e\,(P_{\mu_2}^{n}) + \left[\gamma_n + n\,(\operatorname{cosec}^2\phi + n\operatorname{cotg}^2\phi)\right] I_m(P_{\mu_2}^{n})$$
$$- \beta_n \operatorname{cotg}\phi\; R_e\,(P_{\mu_2}^{n-1}) - \gamma_n \operatorname{cotg}\phi\; I_m\,(P_{\mu_2}^{n-1})$$

$$b_{4P} = 0$$

Coefficients b_{1Q}, b_{2Q}, b_{3Q} are derived from b_{1P}, b_{2P}, b_{3P} by replacing the symbol P in the associated Legendre function of the first kind by the Q of the second kind.

$b_{4Q} = 0$

$$3)\quad \frac{\partial U_\phi}{\partial \phi} = c_{1P} A_1^n + c_{2P}(A_2^n + A_3^n) + c_{3P}\, i\, (A_2^n - A_3^n) + c_{4P} A_4^n$$
$$+ c_{1Q} B_1^n + c_{2Q}(B_2^n + B_3^n) + c_{3Q}\, i\, (B_2^n - B_3^n) + c_{4Q} B_4^n \qquad (\text{II}.\ 2.3)$$

where:

$$c_{1P} = ng_1 (n+1) \cot g\phi \operatorname{cosec}\phi \, P^{n}_{\mu_1} - ng_1 (n-\mu_1-1)(n+\mu_1) \operatorname{cosec}\phi \, P^{n-1}_{\mu_1}$$

$$c_{2P} = ng_1 (n+1) \cot g\phi \operatorname{cosec}\phi \, Re\,(P^{n}_{\mu_2}) + ng_3 (n+1) \cot g\phi \operatorname{cosec}\phi \, I_m\,(P^{n}_{\mu_2})$$
$$- n(g_2\gamma_n + g_3\beta_n) \operatorname{cosec}\phi \, Re\,(P^{n-1}_{\mu_2}) - n(g_3\gamma_n - g_2\beta_n) \operatorname{cosec}\phi \, I_m\,(P^{n-1}_{\mu_2})$$

$$c_{3P} = ng_2 (n+1) \cot g\phi \operatorname{cosec}\phi \, I_m\,(P^{n}_{\mu_2}) - ng_3 (n+1) \cot g\phi \operatorname{cosec}\phi \, Re\,(P^{n}_{\mu_2})$$
$$- n(g_2\gamma_n + g_3\beta_n) \operatorname{cosec}\phi \, I_m\,(P^{n-1}_{\mu_2}) + n(g_3\gamma_n - g_2\beta_n) \operatorname{cosec}\phi \, Re\,(P^{n-1}_{\mu_2})$$

$$c_{4P} = \frac{n}{2}\left[(n-2)(n+1) + n\left(\operatorname{cosec}^2\phi + n\cot g^2\phi\right)\right] P^{n}_{1}$$
$$- \frac{n}{2}(n-2)(n+1) \cot g\phi \, P^{n-1}_{1}$$

Coefficients c_{1Q}, c_{2Q}, c_{3Q}, c_{4Q} are derived from c_{1P}, c_{2P}, c_{3P}, c_{4P} by replacing the symbol P in the associated Legendre function of the first kind by the Q of the second kind.

4) $$\frac{\partial U_\phi}{\partial \theta} = -nU_{\phi n} \sin n\theta \qquad (\mathrm{II}.2.4)$$

5) $$\frac{\partial W}{\partial \theta} = -nW_n \sin n\theta \qquad (\mathrm{II}.2.5)$$

6) $$\frac{\partial^2 W}{\partial \theta^2} = -n^2 W_n \cos n\theta \qquad (\mathrm{II}.2.6)$$

7) $$\frac{\partial U_\theta}{\partial \theta} = nU_{\theta n} \cos n\theta \qquad (\mathrm{II}.2.7)$$

where W_n, $U_{\phi n}$ and $U_{\theta n}$ are the terms within parentheses in (3.3.13), (3.3.14) and (3.3.15), respectively.

APPENDIX III

DETERMINING THE ELEMENTS IN [G] AND [S]

III.1) [G] Elements

Before integrating equation (5.1.2) with respect to ϕ , it is useful to express the product $[Q_s]^T [P] [Q_s]$ as follows:

$$[Q_s]^T \; [P] \; [Q_s] = \left[\begin{array}{c} P_P \\ \hline P_Q \end{array}\right] \left[\begin{array}{c|c} Q_{SP} & Q_{SQ} \end{array}\right]$$

$$[Q_s]^T \; [P] \; [Q_s] = \left[\begin{array}{c|c} P_P Q_{SP} & P_P Q_{SQ} \\ \hline P_Q Q_{SP} & P_Q Q_{SQ} \end{array}\right] \qquad (\text{III}.1.1)$$

$$[Q_s]^T \; [P] \cdot [Q_s] = \left[\begin{array}{c|c} G_{PP} & G_{PQ} \\ \hline G_{QP} & G_{QQ} \end{array}\right]$$

where:

$$Q_{SP}(i,j) = \sum_{r=1}^{3} \left[A_r(i,j) + B_r(i,j)\cos\phi\right] t^{n+2r-4} \qquad (\text{III}.1.2)$$

$i = 1, 6$

$j = 1$ and 4

$$Q_{SP}(i,j) = \sum_{\ell=0}^{\infty} \sum_{r=1}^{3} A_r(i,j) \frac{\sin^{n+2\ell+2r-4}(\phi/2)}{\cos^{n+2r-4}(\phi/2)}$$

$i = 1, 6$

$j = 2, 3$ $\qquad (\text{III}.1.3)$

$$Q_{SQ}(i,j) = \frac{(n-2)!}{2} \sum_{r=1}^{3} \left\{ (-1)^n \left[A_r^c(i,j) + B_r^c(i,j)\cos\phi\right] t^{n+2r-4} + \left[A_r^v(i,j) - B_r^v(i,j)\cos\phi\right] \frac{1}{t^{n+2r-4}} \right\} \qquad (\text{III}.1.4)$$

$i = 1, 6$

$j = 1$ and 4

$$Q_{SQ}(i,j) = \frac{\pi}{2}\sum_{\ell=0}^{\infty}\sum_{r=1}^{4}\Big[A_r^c(i,j)\cos\alpha_r(i,j)\phi\,\mathrm{ch}y\phi$$
$i = 1,6$
$$j = 2,3 \qquad - B_r^c(i,j)\sin\alpha_r(i,j)\phi\,\mathrm{sh}y\phi\Big]\sin^{-n-2}\phi \tag{III.1.5}$$

$$P_P(i,j) = \sum_{r=1}^{3}\left[C_r(i,j) + D_r(i,j)\cos\phi\right]t^{n+2r-4} \tag{III.1.6}$$
$i = 1$ and 4
$j = 1,6$

$$P_P(i,j) = \sum_{\ell=0}^{\infty}\sum_{r=1}^{3} C_r(i,j)\,\frac{\sin^{n+2\ell+2r-4}(\phi/2)}{\cos^{n+2r-4}(\phi/2)} \tag{III.1.7}$$
$i = 2,3$
$j = 1,6$

$$P_Q(i,j) = \frac{(n-2)!}{2}\sum_{r=1}^{3}\Big\{(-1)^n\left[C_r^c(i,j) + D_r^c(i,j)\cos\phi\right]t^{n+2r-4}$$
$i = 1$ and 4
$$j = 1,6 \qquad + \left[C_r^v(i,j) - D_r^v(i,j)\cos\phi\right]\frac{1}{t^{n+2r-4}}\Big\} \tag{III.1.8}$$

$$P_Q(i,j) = \frac{\pi}{2}\sum_{\ell=0}^{\infty}\sum_{r=1}^{4}\Big[C_r^c(i,j)\cos\alpha_r(i,j)\phi\,\mathrm{ch}y\phi$$
$i = 2,3$
$$j = 1,6 \qquad - D_r^c(i,j)\sin\alpha_r(i,j)\phi\,\mathrm{sh}y\phi\Big]\sin^{-n-2}\phi \tag{III.1.9}$$

where:

$$\begin{array}{lcllr}
t & = & tg\,\phi/2 & & \\
[C_r] & = & [A_r]^T\,[P] & r = 1,3 & \\
[D_r] & = & [B_r]^T\,[P] & r = 1,3 & \\
[C_r^c] & = & [A_r^c]^T\,[P] & r = 1,3 & \text{(III.1.10)} \\
[D_r^c] & = & [B_r^c]^T\,[P] & r = 1,3 & \\
[C_r^v] & = & [A_r^v]^T\,[P] & r = 1,4 & \\
[D_r^v] & = & [B_r^v]^T\,[P] & r = 1,4 & \\
[\alpha_r] & & & r = 1,4 &
\end{array}$$

Matrices $[A_r]$, $[B_r]$, $[A_r^c]$, $[B_r^c]$, $[A_r^v]$, $[B_r^v]$ and $[\alpha_r]$ are given in Appendix V. These matrices are independent of the variable ϕ. We therefore derive:

$$Gpp\,(i,j) = \sum_{r=1}^{3} \sum_{s=1}^{3} \sum_{k=1}^{6} \sum_{\ell_r=0}^{\infty} \sum_{\ell_s=0}^{\infty} C_r\,(i,k)\,A_s\,(k,j)$$
$i = 2,3$
$j = 2,3$
$$\times \frac{\sin^{2n+2\ell_r+2\ell_s+2r+2s-8}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)} \qquad (\text{III}.1.11)$$

$$Gpp\,(i,j) = 2\sum_{k=1}^{6} \sum_{r=1}^{3} \sum_{s=1}^{3} \Big\{ [C_r\,(i,k) + D_r\,(i,k)][A_s\,(k,j) + B_s\,(k,j)]$$
$i = 1$ and 4
$j = 1$ and 4
$$\times \frac{\sin^{2n+2r+2s-8}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$
$$- [C_r\,(i,k)\,B_s\,(k,j) + D_r\,(i,k)\,B_s\,(k,j)]\; \frac{\sin^{2n+2r+2s-6}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$
$$- [D_r\,(i,k)\,A_s\,(k,j) + D_r\,(i,k)\,B_s(k,j)]\; \frac{\sin^{2n+2r+2s-6}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$
$$+ 2\,D_r\,(i,k)\,B_s\,(k,j)\; \frac{\sin^{2n+2r+2s-4}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)} \Big\} \qquad (\text{III}.1.12)$$

$$Gpp\,(i,j) = \sum_{k=1}^{6} \sum_{r=1}^{3} \sum_{s=1}^{3} \sum_{\ell_r=0}^{\infty} \sum_{\ell_s=0}^{\infty} C_r\,(i,k)\,A_s\,(k,j)$$
$i = 2,3$
$j = 1$ and 4
$$\times \frac{\sin^{2n+2\ell_r+2\ell_s+2r+2s-8}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)} \qquad (\text{III}.1.13)$$

$$Gpp\,(i,j) = \sum_{k=1}^{6} \sum_{\ell_s=0}^{\infty} \sum_{r=1}^{3} \sum_{s=1}^{3} \Big\{ [C_r\,(i,k)\,A_s\,(k,j) + D_r\,(i,k)\,A_s\,(k,j)]$$
$i = 1$ and 4
$j = 2,3$
$$\times \frac{\sin^{2n+2\ell_s+2r+2s-8}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$
$$- 2\,D_r\,(i,k)\,A_s\,(k,j)\; \frac{\sin^{2n+2\ell_s+2r+2s-6}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)} \Big\} \qquad (\text{III}.1.14)$$

$$G_{PQ}(i,j) = \frac{\pi}{2} \sum_{k=1}^{6} \sum_{\ell_s=0}^{\infty} \sum_{r=1}^{3} \sum_{s=1}^{3} \Big\{ 2\left(C_r(i,k) + D_r(i,k)\right)$$
$i = 1 \;\&\; 4$
$j = 2, 3$

$$\times A_S^c(k,j) \cos\alpha_s(k,j)\,\phi\, \mathrm{chy}\phi \;\frac{\sin^{2r-6}(\phi/2)}{\cos^{2n+2r-2}(\phi/2)}$$

$$-2\left(C_r(i,k) + D_r(i,k)\right) B_S^c(k,j) \sin\alpha_s(k,j)\,\phi\, \mathrm{shy}\phi \;\frac{\sin^{2r-6}(\phi/2)}{\cos^{2n+2r-2}(\phi/2)}$$

$$-4D_r(i,k)\, A_S^c(k,j) \cos\alpha_s(k,j)\,\phi\, \mathrm{chy}\phi \;\frac{\sin^{2r-4}(\phi/2)}{\cos^{2n+2r-2}(\phi/2)}$$

$$+4D_r(i,k)\, B_S^c(k,j) \sin\alpha_s(k,j)\,\phi\, \mathrm{shy}\phi \;\frac{\sin^{2r-4}(\phi/2)}{\cos^{2n+2r-2}(\phi/2)} \Big\} \qquad (\text{III}.1.15)$$

$$G_{PQ}(i,j) = \frac{(n-2)!}{2} \sum_{k=1}^{6} \sum_{r=1}^{3} \sum_{s=1}^{3} \Big\{ (-1)^n \left[C_r(i,k) + D_r(i,k)\right]\left[A_S^c(k,j) + B_S^c(k,j)\right]$$
$i = 1 \;\&\; 4$
$j = 1 \;\&\; 4$

$$\times \frac{\sin^{2n+2r+2s-8}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$

$$-2(-1)^n\left[C_r(i,k) B_S^c(k,j) + 2D_r(i,k) B_S^c(k,j) + D_r(i,k) A_S^c(k,j)\right]$$
$$\times \frac{\sin^{2n+2r+2s-6}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$

$$+4(-1)^n D_r(i,k)\, B_S^c(k,j) \frac{\sin^{2n+2r+2s-4}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$

$$+\left[\left(C_r(i,k) + D_r(i,k)\right)\left(A_S^v(k,j) - B_S^v(k,j)\right)\right] \frac{\sin^{2r-2s}(\phi/2)}{\cos^{2r-2s}(\phi/2)}$$

$$+2\left[C_r(i,k)\, B_S^v(k,j) + 2D_r(i,k) B_S^v(k,j) - D_r(i,k)\, A_S^v(k,j)\right]$$
$$\times \frac{\sin^{2r-2s+2}(\phi/2)}{\cos^{2r-2s}(\phi/2)}$$

$$-4D_r(i,k)\, B_S^v(k,j) \frac{\sin^{2r-2s+4}(\phi/2)}{\cos^{2r-2s}(\phi/2)} \Big\} \qquad (\text{III}.1.16)$$

$$G_{PQ}(i,j) = \frac{(n-2)!}{2} \sum_{k=1}^{6} \sum_{\ell_r=0}^{\infty} \sum_{r=1}^{3} \sum_{s=1}^{3} \Big\{ (-1)^n \big[C_r(i,k) A_s^c(k,j) + C_r(i,k) B_s^c(k,$$

$i = 2,3$

$j = 1$ and 4

$$\times \frac{\sin^{2n+2\ell_r+2r+2s-8}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$

$$-2(-1)^n C_r(i,k) B_s^c(k,j) \frac{\sin^{2n+2\ell_r+2r+2s-6}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$

$$+\big[C_r(i,k) A_s^v(k,j) - C_r(i,k) B_s^v(k,j)\big] \frac{\sin^{2\ell_r+2r+2s+2}(\phi/2)}{\cos^{2r-2s}(\phi/2)}$$

$$+2C_r(i,k) B_s^v(k,j) \frac{\sin^{2\ell_r+2r-2s+2}(\phi/2)}{\cos^{2r-2s}(\phi/2)} \Big\} \qquad (\text{III}.1.17)$$

$$G_{PQ}(i,j) = \pi \sum_{k=1}^{6} \sum_{\ell_r=0}^{\infty} \sum_{\ell_s=0}^{\infty} \sum_{r=1}^{4} \sum_{s=1}^{3} \big[C_r(i,k) A_s^c(k,j) \cos\alpha_s(k,j)\phi \, \mathrm{ch}\, y\phi$$

$i = 2,3$

$j = 2,3$

$$- C_r(i,k) B_s^c(k,j) \sin\alpha_s(k,j)\phi \, \mathrm{sh}\, y\phi \big] \frac{\sin^{2\ell_r+2r-6}(\phi/2)}{\cos^{2n+2r-2}(\phi/2)} \qquad (\text{III}.1.18)$$

$$G_{QP}(i,j) = \frac{(n-2)!}{2} \sum_{k=1}^{6} \sum_{r=1}^{3} \sum_{s=1}^{3} \Big\{ (-1)^n \big[C_r^c(i,k) + D_r^c(i,k)\big]\big[A_s(k,j) + B_s(k,j)\big]$$

$i = 1$ & 4

$j = 1$ & 4

$$\times \frac{\sin^{2n+2r+2s-8}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$

$$-2(-1)^n \big[C_r^c(i,k) B_s(k,j) + 2D_r^c(i,k) B_s(k,j) + D_r^c(i,k) A_s(k,j) \big]$$

$$\times \frac{\sin^{2n+2r+2s-6}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$

$$+4(-1)^n D_r^c(i,k) B_s(k,j) \frac{\sin^{2n+2r+2s-4}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$

$$+\big[C_r^v(i,k) - D_r^v(i,k)\big]\big[A_s(k,j) + B_s(k,j)\big] \frac{\sin^{2s-2r}(\phi/2)}{\cos^{2s-2r}(\phi/2)}$$

$$+2\left[D_r^V(i,k)\,A_s(k,j)+2D_r^V(i,k)\,B_s(k,j)-C_r^V(i,k)\,B_s(k,j)\right] \times \frac{\sin^{2s-2r+2}(\phi/2)}{\cos^{2s-2r}(\phi/2)}$$

$$\left.-4D_r^V(i,k)\,B_s(k,j)\,\frac{\sin^{2s-2r+4}(\phi/2)}{\cos^{2s-2r}(\phi/2)}\right\} \qquad (\text{III}.1.19)$$

$$\underset{\substack{i=1\text{ and }4\\ j=2,3}}{G_{QP}(i,j)} = \frac{(n-2)!}{2}\sum_{k=1}^{6}\sum_{\ell_s=0}^{\infty}\sum_{r=1}^{3}\sum_{s=1}^{3}\left\{(-1)^n\left[C_r^c(i,k)\,A_s(k,j)+D_r^c(i,k)\,A_s(k,j)\right] \times \frac{\sin^{2n+2\ell_s+2r+2s-8}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}\right.$$

$$-2(-1)^n D_r^c(i,k)\,A_s(k,j)\,\frac{\sin^{2n+2\ell_s+2r+2s-6}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$

$$+\left[C_r^V(i,k)\,A_s(k,j)-D_r^V(i,k)\,A_s(k,j)\right]\frac{\sin^{2\ell_s-2r+2s}(\phi/2)}{\cos^{2s-2r}(\phi/2)}$$

$$\left.+2D_r^V(i,k)\,A_s(k,j)\,\frac{\sin^{2\ell_s-2r+2s+2}(\phi/2)}{\cos^{2s-2r}(\phi/2)}\right\} \qquad (\text{III}.1.20)$$

$$\underset{\substack{i=2,3\\ j=1\text{ and }4}}{G_{QP}(i,j)} = \pi\sum_{k=1}^{6}\sum_{\ell_r=0}^{\infty}\sum_{r=1}^{4}\sum_{s=1}^{3}\left\{\left[C_r^c(i,k)\cos\alpha_r(i,k)\phi\ \mathrm{ch}\gamma\phi\right.\right.$$

$$\left.+D_r^c(i,k)\sin\alpha_r(i,k)\phi\ \mathrm{sh}\gamma\phi\right]\left[A_s(k,j)+B_s(k,j)\right]\frac{\sin^{2r-6}(\phi/2)}{\cos^{2n+2r-2}(\phi/2)}$$

$$+4\left[-C_r^c(i,k)\cos\alpha_r(i,k)\phi\ \mathrm{ch}\gamma\phi+D_r^c(i,k)\sin\alpha_r(i,k)\phi\ \mathrm{sh}\gamma\phi\right]$$

$$\left.\times B_s(k,j)\,\frac{\sin^{2r-4}(\phi/2)}{\cos^{2n+2r-2}(\phi/2)}\right\} \qquad (\text{III}.1.21)$$

$$G_{QP}(i,j) = \pi \sum_{k=1}^{6} \sum_{l_r=0}^{\infty} \sum_{l_s=0}^{\infty} \sum_{r=1}^{4} \sum_{s=1}^{3} \left[C_r^c(i,k) \cos\alpha_r(i,k)\phi \, \mathrm{ch}\gamma\phi - D_r^c(i,k) \sin\alpha_r(i,k)\phi \, \mathrm{sh}\gamma \right.$$

$i = 2,3$

$j = 2,3$

$$\times A_s(k,j) \frac{\sin^{2l_s + 2s - 6}(\phi/2)}{\cos^{2n+2s-2}(\phi/2)} \qquad (\text{III}.1.22)$$

$$G_{QQ}(i,j) = \left[\frac{(n-2)!}{2}\right]^2 \sum_{k=1}^{6} \sum_{r=1}^{3} \sum_{s=1}^{3} \left\{ \left[C_r^c(i,k) + D_r^c(i,k)\right]\left[A_s^c(k,j) + B_s^c(k,j)\right] \right.$$

$i = 1 \;\&\; 4$

$j = 1 \;\&\; 4$

$$\times \frac{\sin^{2n+2r+2s-8}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$

$$- 2\left[C_r^c(i,k) + D_r^c(i,k)\right] \times B_s^c(k,j) \times \frac{\sin^{2n+2r+2s-6}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$

$$- 2 D_r^c(i,k)\left[A_s^c(k,j) + B_s^c(k,j)\right] \times \frac{\sin^{2n+2r+2s-6}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$

$$+ 4 D_r^c(i,k) B_s^c(k,j) \times \frac{\sin^{2n+2r+2s-4}(\phi/2)}{\cos^{2n+2r+2s-8}(\phi/2)}$$

$$+ (-1)^n \left[C_r^c(i,k) + D_r^c(i,k)\right]\left[A_s^v(k,j) - B_s^v(k,j)\right] \frac{\sin^{2r+2s}(\phi/2)}{\cos^{2r+2s}(\phi/2)}$$

$$+ 2(-1)^n \left[C_r^c(i,k) + D_r^c(i,k)\right] \times B_s^v(k,j) \frac{\sin^{2r+2s+2}(\phi/2)}{\cos^{2r+2s}(\phi/2)}$$

$$- 2(-1)^n \times D_r^c(i,k)\left[A_s^v(k,j) + B_s^v(k,j)\right] \frac{\sin^{2r+2s+2}(\phi/2)}{\cos^{2r+2s}(\phi/2)}$$

$$- 4(-1)^n \times D_r^c(i,k) \times B_s^v(k,j) \frac{\sin^{2r+2s+4}(\phi/2)}{\cos^{2r+2s}(\phi/2)}$$

$$+ (-1)^n \left[C_r^v(i,k) - D_r^v(i,k)\right]\left[A_s^c(k,j) + B_s^c(k,j)\right] \frac{\sin^{2s-2r}(\phi/2)}{\cos^{2s-2r}(\phi/2)}$$

$$- 2(-1)^n \left[C_r^v(i,k) - D_r^v(i,k)\right] \times B_s^c(k,j) \frac{\sin^{2s-2r+2}(\phi/2)}{\cos^{2s-2r}(\phi/2)}$$

$$+ 2(-1)^n \times D_r^v(i,k)\left[A_s^c(k,j) + B_s^c(k,j)\right] \frac{\sin^{2s-2r+2}(\phi/2)}{\cos^{2s-2r}(\phi/2)}$$

$$- 4 D_r^v(i,k) \times B_s^c(k,j) \frac{\sin^{2s-2r+4}(\phi/2)}{\cos^{2s-2r}(\phi/2)}$$

$$+\left[C_r^V(i,k)-D_r^V(i,k)\right]\left[A_s^V(k,j)-B_s^V(k,j)\right]\frac{\cos^{2n+2r+2s-8}(\phi/2)}{\sin^{2n+2r+2s-8}(\phi/2)}$$

$$+2\left[C_r^V(i,k)-D_r^V(i,k)\right]\times B_s^V(k,j)\frac{\cos^{2n+2r+2s-8}(\phi/2)}{\sin^{2n+2r+2s-10}(\phi/2)}$$

$$+2D_r^V(i,k)\left[A_s^V(k,j)-B_s^V(k,j)\right)\frac{\cos^{2n+2r+2s-8}(\phi/2)}{\sin^{2n+2r+2s-10}(\phi/2)}$$

$$\left.+4D_r^V(i,k)\,B_s^V(k,j)\frac{\cos^{2n+2r+2s-8}(\phi/2)}{\sin^{2n+2r+2s-14}(\phi/2)}\right\}\qquad(\text{III}.1.23)$$

$$\underset{\substack{i=1\text{ and }4\\ j=2,3}}{G_{QQ}(i,j)}=\frac{\pi}{2}(n-2)!\sum_{k=1}^{6}\sum_{r=1}^{3}\sum_{s=1}^{4}\left\{(-1)^n\left[C_r^c(i,k)+D_r^c(i,k)\right]\right.$$

$$\times A_r^c(k,j)\cos\alpha_r(k,j)\phi\,\mathrm{ch}y\phi\frac{\sin^{2r-6}(\phi/2)}{\cos^{2n+2r-2}(\phi/2)}$$

$$-2D_r^c(i,k)A_r^c(k,j)\cos\alpha_r(k,j)\phi\,\mathrm{ch}y\phi\frac{\sin^{2r-4}(\phi/2)}{\cos^{2n+2r-2}(\phi/2)}$$

$$-(-1)^n\left[C_r^c(i,k)+D_r^c(i,k)\right]B_r^c(k,j)\sin\alpha_r(k,j)\phi\,\mathrm{sh}y\phi\frac{\sin^{2r-6}(\phi/2)}{\cos^{2n+2r-2}(\phi/2)}$$

$$+2(-1)^n D_r^c(i,k)B_r^c(k,j)\sin\alpha_r(k,j)\phi\,\mathrm{sh}y\phi\frac{\sin^{2r-4}(\phi/2)}{\cos^{2n+2r-2}(\phi/2)}$$

$$+\left[C_r^V(i,k)-D_r^V(i,k)\right]A_r^c(k,j)\cos\alpha_r(k,j)\phi\,\mathrm{ch}y\phi\frac{\cos^{2r-6}(\phi/2)}{\sin^{2n+2r-2}(\phi/2)}$$

$$+2D_r^V(i,k)A_r^c(k,j)\cos\alpha_r(k,j)\phi\,\mathrm{ch}y\phi\frac{\cos^{2r-6}(\phi/2)}{\sin^{2n+2r-4}(\phi/2)}$$

$$-\left[C_r^V(i,k)-D_r^V(i,k)\right]B_r^c(k,j)\sin\alpha_r(k,j)\phi\,\mathrm{sh}y\phi\frac{\cos^{2r-6}(\phi/2)}{\sin^{2n+2r-2}(\phi/2)}$$

$$\left.-2D_r^V(i,k)B_r^c(k,j)\sin\alpha_r(k,j)\phi\,\mathrm{sh}y\phi\frac{\cos^{2r-6}(\phi/2)}{\sin^{2n+2r-4}(\phi/2)}\right\}\qquad(\text{III}.1.24)$$

$$G_{QQ}(i,j) = \left(\frac{\pi}{2}\right)^2 \sum_{k=1}^{6}\sum_{r=1}^{3}\sum_{s=1}^{4} \Big\{ C_r^c\,(i,k)\, A_s^c\,(k,j) \cos\alpha_r\,(i,k)\phi \cos\alpha_s(k,j)\phi$$
$i = 2,3$
$j = 2,3$

$$\times\, \mathrm{ch}y^2\,\phi\ \sin^{-2n-4}\phi$$

$$-\, C_r^c\,(i,k)\, B_s^c\,(k,j) \cos\alpha_r\,(i,k)\phi\ \sin\alpha_s\,(k,j)\phi\ \mathrm{ch}y\phi\ \mathrm{sh}y\phi\ \sin^{-2n-4}\phi$$

$$-\, D_r^c\,(i,k)\, A_s^c\,(k,j) \sin\alpha_r\,(i,k)\phi\ \cos\alpha_s\,(k,j)\phi\ \mathrm{sh}y\phi\ \mathrm{ch}y\phi\ \sin^{-2n-4}\phi$$

$$+\, D_r^c\,(i,k)\, B_s^c\,(k,j) \sin\alpha_r\,(i,k)\phi\ \sin\alpha_s\,(k,j)\phi\ \mathrm{sh}y^2\phi\ \sin^{-2n-4}\phi \Big\}$$

(III . 1.25)

$$G_{QQ}(i,j) = \frac{\pi}{2}\,(n-2)! \sum_{k=1}^{6}\sum_{r=1}^{4}\sum_{s=1}^{3} \Big\{ (-1)^n\, C_r^c\,(i,k)\big[A_s^c\,(k,j) + B_s^c\,(k,j)\big]$$
$i = 2,3$
$j = 1\ \&\ 4$

$$\times \cos\alpha_r\,(i,k)\phi\ \mathrm{ch}y\phi\ \frac{\sin^{2s-6}(\phi/2)}{\cos^{2n+2s-2}(\phi/2)}$$

$$-\, 2(-1)^n\, C_r^c\,(i,k)\ B_s^c\,(k,j) \cos\alpha_r\,(i,k)\phi\ \mathrm{ch}y\phi\ \frac{\sin^{2s-4}(\phi/2)}{\cos^{2n+2s-2}(\phi/2)}$$

$$-\, (-1)^n\, D_r^c\,(i,k)\big[A_s^c\,(k,j) + B_s^c\,(k,j)\big] \sin\alpha_r\,(i,k)\phi\ \mathrm{sh}y\phi\ \frac{\sin^{2s-6}(\phi/2)}{\cos^{2n+2s-2}(\phi/2)}$$

$$+\, 2(-1)^n\, D_r^c\,(i,k)\, B_s^c\,(k,j)\ \sin\alpha_r\,(i,k)\phi\ \mathrm{sh}y\phi\ \frac{\sin^{2s-4}(\phi/2)}{\cos^{2n+2s-2}(\phi/2)}$$

$$+\, C_r^c\,(i,k)\big[A_s^v\,(k,j) - B_s^v\,(k,j)\big] \cos\alpha_r\,(i,k)\phi\ \mathrm{ch}y\phi\ \frac{\cos^{2s-6}(\phi/2)}{\sin^{2n+2s-2}(\phi/2)}$$

$$+\, 2C_r^c\,(i,k)\ B_s^v\,(k,j)\ \cos\alpha_r\,(i,k)\phi\ \mathrm{ch}y\phi\ \frac{\cos^{2s-6}(\phi/2)}{\sin^{2n+2s-4}(\phi/2)}$$

$$- D_r^c\ (i,k)\left[A_s^v\ (k,j) - B_s^v\ (k,j)\right] \sin \alpha_r\ (i,k)\, \phi\, \mathrm{sh} y \phi$$
$$\times \frac{\cos^{2s-6}(\phi/2)}{\sin^{2n+2s-2}(\phi/2)}$$
$$- 2D_r^c\ (i,k)\ B_s^v\ (k,j) \sin \alpha_r\ (i,k)\, \phi\, \mathrm{sh} y \phi \frac{\cos^{2s-6}(\phi/2)}{\sin^{2n+2s-4}(\phi/2)}$$

(III . 1. 26)

$$\begin{aligned} GQQ(1,2) &= GQQ(2,1) \\ GQQ(1,3) &= GQQ(3,1) \\ GQQ(4,2) &= GQQ(2,4) \\ GQQ(4,3) &= GQQ(3,4) \end{aligned}$$

(III. 1. 27)

$$\begin{aligned} GPP(2,1) &= GPP(1,2) \\ GPP(3,1) &= GPP(1,3) \\ GPP(2,4) &= GPP(4,2) \\ GPP(3,4) &= GPP(4,3) \end{aligned}$$

(III. 1. 28)

With the terms of the product $[Q_-]^T\ [P]\ [Q_-]$ being obtained by applying the integration formulas appearing in Appendix IV, we therefore get:

$$G_{PP}(i,j) = 4\sum_{k=1}^{6}\sum_{r=1}^{3}\sum_{s=1}^{3}\left(C_r(i,k)+D_r(i,k)\right)\left(A_s(k,j)+B_s(k,j)\right)$$

i = 1 and 4

j = 1 and 4

$$\times\left[\Gamma\left(b+2,b,\frac{\phi_1}{2}\right)-\Gamma\left(b+2,b,\frac{\phi_1}{2}\right)\right]$$

$$-2\left(C_r(i,k)B_s(k,j)+D_r(i,k)\left(A_s(k,j)+2B_s(k,j)\right)\right)$$

$$\times\left[\Gamma\left(b+4,b,\frac{\phi_2}{2}\right)-\Gamma\left(b+4,b,\frac{\phi_1}{2}\right)\right]$$

$$+4D_r(i,k)\,B_s(k,j)\times\left[\Gamma\left(b+6,b,\frac{\phi_2}{2}\right)-\Gamma\left(b+6,b,\frac{\phi_1}{2}\right)\right]$$

(III. 1. 29)

with: $b = 2n + 2(r+s) - 9$

$$\Gamma(m_1, n_1, \phi) = \int \frac{\sin^{m_1}(\phi)}{\cos^{n_1}(\phi)}\, d\phi \qquad m_1, n_1 > 0$$

$$G_{PP}(i,j) = 4\sum_{k=1}^{6}\sum_{\ell_r=0}^{\infty}\sum_{r=1}^{3}\sum_{s=1}^{3}\left(C_r(i,k)+D_r(i,k)\right)A_s(k,j)$$

i = 1 and 4

j = 2, 3

$$\times\left[\Gamma\left(c,b,\frac{\phi_2}{2}\right)-\Gamma\left(c,b,\frac{\phi_1}{2}\right)\right]$$

$$-2D_r(i,k)\,A_s(k,j)\times\left[\Gamma\left(c+2,b,\frac{\phi_2}{2}\right)-\Gamma\left(c+2,b,\frac{\phi_1}{2}\right)\right]$$

(III. 1. 30)

with: $c = 2n + 2\ell_r + 2(r+s) - 7$

$$G_{PP}(i,j) = \sum_{k=1}^{6}\sum_{\ell_r=0}^{\infty}\sum_{\ell_s=0}^{\infty}\sum_{r=1}^{3}\sum_{s=1}^{3} C_r(i,k)\,A_s(k,j)$$

i = 2, 3

j = 2, 3

$$\times\left[\Gamma\left(as,b,\frac{\phi_2}{2}\right)-\Gamma\left(as,b,\frac{\phi_1}{2}\right)\right]$$

(III. 1. 31)

with: $as = 2n + 2(\ell_r + \ell_s) + 2(r+s) - 7$

$$\begin{aligned} G_{PP}(2,1) &= G_{PP}(1,2)\\ G_{PP}(3,1) &= G_{PP}(1,3)\\ G_{PP}(2,4) &= G_{PP}(4,2)\\ G_{PP}(3,4) &= G_{PP}(4,3) \end{aligned} \qquad \text{(III. 1. 32)}$$

$$G_{PQ}(i,j) = \frac{(n-2)!}{2} \sum_{k=1}^{6} \sum_{r=1}^{3} \sum_{s=1}^{3} \Big\{ 2(-1)^n \big(C_r(i,k) + D_r(i,k)\big)\big(A_S^c(k,j) + B_S^c(k,j)\big)$$
$i = 1 \text{ and } 4$
$j = 1 \text{ and } 4$
$$\times \left[\Gamma\left(b+2, b, \frac{\phi_2}{2}\right) - \Gamma\left((b+2), b, \frac{\phi_1}{2}\right)\right]$$
$$-4(-1)^n \Big(C_r(i,k)\, B_S^c(k,j) + D_r(i,k)\big(A_S^c(k,j) + 2B_S^c(k,j)\big)\Big)$$
$$\times \left[\Gamma\left(b+4, b, \frac{\phi_2}{2}\right) - \Gamma\left(b+4, b, \frac{\phi_1}{2}\right)\right]$$
$$+8(-1)^n D_r(i,k)\, B_S^c(k,j)\left[\Gamma\left(b+6, b, \frac{\phi_2}{2}\right) - \Gamma\left(b+6, b, \frac{\phi_1}{2}\right)\right]$$
$$+2\big(C_r(i,k) + D_r(i,k)\big)\big(A_S^V(k,j) - B_S^V(k,j)\big)\left[\Gamma\left(u+1, u-1, \frac{\phi_2}{2}\right) - \Gamma\left(u+1, u-1, \frac{\phi_1}{2}\right)\right]$$
$$-4\big(-C_r(i,k) B_S^V(k,j) + D_r(i,k)\big(A_S^V(k,j) - 2B_S^V(k,j)\big)\big)\left[\Gamma\left(u+3, u-1, \frac{\phi_2}{2}\right) - \Gamma\left(u+3, u-1, \frac{\phi_1}{2}\right)\right]$$
$$-8D_r(i,k)\, B_S^V(k,j)\left[\Gamma\left(u+5, u-1, \frac{\phi_2}{2}\right) - \Gamma\left(u+5, u-1, \frac{\phi_1}{2}\right)\right]\Big\} \qquad (\text{III}.1.33)$$

with: $u = 2(r-s)$

$$G_{PQ}(i,j) = \frac{(n-2)!}{2} \sum_{k=1}^{6} \sum_{\ell=0}^{\infty} \sum_{r=1}^{3} \sum_{s=1}^{3} \Big\{ 2(-1)^n C_r(i,k)\big(A_S^c(k,j) + B_S^c(k,j)\big)$$
$i = 2,3$
$j = 1 \text{ and } 4$
$$\times \left[\Gamma\left(c, b, \frac{\phi_2}{2}\right) - \Gamma\left(c, b, \frac{\phi_1}{2}\right)\right]$$
$$-4(-1)^n C_r(i,k)\, B_S^c(k,j)\left[\Gamma\left(c+2, b, \frac{\phi_2}{2}\right) - \Gamma\left(c+2, b, \frac{\phi_1}{2}\right)\right]$$
$$+4\, C_r(i,k)\, B_S^V(k,j)\left[\Gamma\left(u+2\ell+3, u-1, \frac{\phi_2}{2}\right) - \Gamma\left(u+2\ell+3, u-1, \frac{\phi_1}{2}\right)\right]$$
$$+2C_r(i,k)\big(A_S^V(k,j) - B_S^V(k,j)\big)\left[\Gamma\left(u+2\ell+1, u-1, \frac{\phi_2}{2}\right) - \Gamma\left(u+2\ell+1, u-1, \frac{\phi_1}{2}\right)\right]\Big\}$$
$$(\text{III}.1.34)$$

$$G_{PQ}(i,j) = \frac{\pi}{2} \sum_{k=1}^{6} \sum_{\ell=0}^{\infty} \sum_{r=1}^{3} \sum_{s=1}^{3} \Big\{ \left(2^{n+2r-4}\right) \Big[\big(C_r(i,k) \times A_S^c(k,j) \times HRR(k,j)$$
$i = 1 \;\&\; 4$
$j = 2,3$
$$+\frac{1}{2} \times D_r(i,k)\, A_S^c(k,j)\big(HRR1(k,j) + HRR2(k,j)\big)$$
$$-C_r(i,k)\; B_S^c(k,j) \times TRR(k,j) \qquad (\text{III}.1.35)$$
$$-\frac{1}{2} D_r(i,k)\, B_S^c(k,j) \times \big(TRR1(k,j) + TRR2(k,j)\big)\Big]\Big\}$$

Terms HRR (k,j), HRR1 (k,j), HRR2 (k,j), TRR (k,j), TRR1 (k,j) and TRR2 (k,j) are given in Appendix IV.

$$G_{PQ}(i,j) = \frac{\pi}{2} \sum_{k=1}^{6} \sum_{\ell_r=0}^{\infty} \sum_{\ell_s=0}^{\infty} \sum_{r=1}^{4} \sum_{s=1}^{3} \Big\{ (2^{n+2r-4}) \Big[C_r(i,k) A_S^c(k,j)\, \mathrm{HRRL}(k,j)$$
$i = 2,3$
$j = 2,3$
$$- C_r(i,k) B_S^c(k,j)\, \mathrm{TRRL}(k,j) \Big] \Big\} \qquad (\text{III}.1.36)$$

Terms HRRL (k,j) and TRRL (k,j) are given in Appendix IV.

$$[G_{QP}] = [G_{PQ}]^T \qquad (\text{III}.1.37)$$

$$G_{QQ}(i,j) = \left[\frac{(n-2)!}{2}\right]^2 \sum_{k=1}^{6} \sum_{r=1}^{3} \sum_{s=1}^{3} \Big\{ 2\left(C_r^c(i,k) + D_r^c(i,k)\right)\left(A_S^c(k,j) + B_S^c(k,j)\right)$$
$i = 1$ and 4
$j = 1$ and 4
$$\times \left[\Gamma\left(b+2, b, \frac{\phi_2}{2}\right) - \Gamma\left(b+2, b, \frac{\phi_1}{2}\right)\right]$$
$$-4\left(C_r^c(i,k) B_S^c(k,j) + D_r^c(i,k)\left(A_S^c(k,j) + 2B_S^c(k,j)\right)\right)\left[\Gamma\left(b+4, b, \frac{\phi_2}{2}\right) - \Gamma\left(b+4, b, \frac{\phi_1}{2}\right)\right]$$
$$+8D_r^c(i,k) B_S^c(k,j) \times \left[\Gamma\left(b+6, b, \frac{\phi_2}{2}\right) - \Gamma\left(b+6, b, \frac{\phi_1}{2}\right)\right]$$
$$+8D_r^v(i,k) B_S^v(k,j) \times \left[\Gamma\left(4-b, -b-2, \frac{\phi_2}{2}\right) - \Gamma\left(4-b, -b-2, \frac{\phi_1}{2}\right)\right]$$
$$+2\left(C_r^v(i,k) - D_r^v(i,k)\right)\left(A_r^v(k,j) - B_r^v(k,j)\right) \times \left[\Gamma\left(-b, -b-2, \frac{\phi_2}{2}\right) - \Gamma\left(-b, -b-2, \frac{\phi_1}{2}\right)\right]$$
$$+4\left(C_r^v(i,k) B_r^v(k,j) + D_r^v(i,k)\left(A_r^v(k,j) - 2B_r^v(k,j)\right)\right)$$
$$\times \left[\Gamma\left(2-b, -b-2, \frac{\phi_2}{2}\right) - \Gamma\left(2-b, -b-2, \frac{\phi_1}{2}\right)\right]$$
$$+2(-1)^n \Big(\left(C_r^c(i,k) + D_r^c(i,k)\right)\left(A_S^v(k,j) - B_S^v(k,j)\right)$$
$$+ \left(C_S^v(i,k) - D_S^v(i,k)\right)\left(A_r^c(k,j) + B_r^c(k,j)\right) \Big)$$
$$\times \left[\Gamma\left(u+1, u-1, \frac{\phi_2}{2}\right) - \Gamma\left(u+1, u-1, \frac{\phi_1}{2}\right)\right]$$

$$- 4(-1)^n\Big(-C_r^c(i,k)\,B_s^v(k,j) + D_r^c(i,k)\big(A_s^v(k,j) - 2B_s^v(k,j)\big)$$
$$+ C_s^v(i,k)\,B_r^c(k,j) - D_s^v(i,k)\big(A_r^c(k,j) - 2B_r^c(k,j)\big)\Big)$$
$$\times\left[\Gamma\left(u+3, u-1, \frac{\phi_2}{2}\right) - \Gamma\left(u+3, u-1, \frac{\phi_1}{2}\right)\right]$$
$$- 8(-1)^n\Big(D_r^c(i,k)\,B_s^v(k,j) + D_s^v(i,k)\,B_r^c(k,j)\Big)$$
$$\times\left[\Gamma\left(u+5, u-1, \frac{\phi_2}{2}\right) - \Gamma\left(u+5, u-1, \frac{\phi_1}{2}\right)\right]\Bigg\}$$

(III . 1.38)

with: $u = 2(r - s)$

$$G_{QQ}(i,j) = \frac{\pi}{4}(n-2)!\sum_{k=1}^{6}\sum_{r=1}^{4}\sum_{s=1}^{3}\Bigg\{\left(2^{n+2r-4}\right)\Big[(-1)^n C_s^c(i,k)\,A_r^c(k,j)\,\mathrm{HRR}(k,i)$$

$i = 2,3$

$j = 1$ and 4

$$+ \frac{1}{2}(-1)^n C_s^c(i,k)\,B_r^c(k,j)\big(\mathrm{HRR1}(k,i) + \mathrm{HRR2}(k,i)\big)$$
$$-(-1)^n D_s^c(i,k)\,A_r^c(k,j)\ \mathrm{TRR}(k,j)$$
$$-\frac{1}{2}(-1)^n D_s^c(i,k)\,B_r^c(k,j)\big(\mathrm{TRR1}(k,i) + \mathrm{TRR2}(k,i)\big)$$
$$+ C_s^c(i,k)\,A_r^v(k,j)\ \mathrm{URR}(k,i)$$
$$-\frac{1}{2} C_s^c(i,k)\,B_r^v(k,j)\big(\mathrm{URR1}(k,i) + \mathrm{URR2}(k,i)\big)$$
$$- D_s^c(k,j)\,A_r^v(k,j)\ \mathrm{VRR}(k,i)$$
$$+\frac{1}{2} D_s^c(k,j)\,B_r^v(k,j)\big(\mathrm{VRR1}(k,i) + \mathrm{VRR2}(k,i)\big)\Big]\Bigg\}$$

(III . 1.39)

Terms URR (k,i), URR1 (k,i), URR2 (k,i), VRR (k,i), VRR1 (k,i) and VRR2 (k,i) are given in Appendix IV.

$$G_{QQ}(i,j) = \left(\frac{\pi}{4}\right)^2 \sum_{k=1}^{6}\sum_{r=1}^{3}\sum_{s=1}^{4}\Big\{\left(C_r^c(i,k)A_s^c(k,j) + D_r^c(i,k)B_s^c(k,j)\right) \times CER1(k,i)$$
$i = 2,3$
$j = 2,3$

$$- \left(D_r^c(i,k)B_s^c(k,j) - C_r^c(i,k)A_s^c(k,j)\right) \times CER2(k,i)$$

$$- \left(C_r^c(i,k)B_s^c(k,j) + D_r^c(i,k)A_s^c(k,j)\right) \times CIR2(k,i)$$

$$+ \left(C_r^c(i,k)B_s^c(k,j) - D_r^c(i,k)A_s^c(k,j)\right) \times CIR1(k,i)$$

$$- \left(D_r^c(i,k)B_s^c(k,j) - C_r^c(i,k)A_s^c(k,j)\right) \times CER3(k,i)$$

$$+ \left(C_r^c(i,k)A_s^c(k,j) + D_r^c(i,k)B_s^c(k,j)\right) \times CER4(k,i)\Big\} \qquad (\text{III}.1.40)$$

Terms CER1 (k,i), CER2 (k,i), CER3 (k,i), CER4 (k,i), CIR1 (k,i) and CIR2 (k,i) are given in Appendix IV.

$$\begin{aligned} G_{QQ}(1,2) &= G_{QQ}(2,1) \\ G_{QQ}(1,3) &= G_{QQ}(3,1) \\ G_{QQ}(4,2) &= G_{QQ}(2,4) \\ G_{QQ}(4,3) &= G_{QQ}(3,4) \end{aligned} \qquad (\text{III}.1.41)$$

III.2) <u>[S] Elements</u>

Similarly, the product of $[R_s]^T\ [R_s]$ in (5.2.2) can be written:

$$[R_S]^T[R_S] = \left[R_{SP} \mid R_{SQ}\right]^T \left[\begin{array}{c} R_{SP} \\ \hline R_{SQ} \end{array}\right] = \left[\begin{array}{c|c} R_{SP}^T R_{SP} & R_{SP}^T R_{SQ} \\ \hline R_{SQ}^T R_{SP} & R_{SQ}^T R_{SQ} \end{array}\right]$$

$$= \left[\begin{array}{c|c} S_{PP} & S_{PQ} \\ \hline S_{QP} & S_{QQ} \end{array}\right] \qquad (\text{III}.2.1)$$

where:

$$R_{SP}(i,j) = \sum_{r=1}^{3} \left[\Gamma_r(i,j) + \Delta_r(i,j)\cos\phi\right] t^{n+r-2} \qquad (\text{III}.\ 2.2)$$
$i = 1,3$
$j = 1$ and 4

$$R_{SP}(i,j) = \sum_{\ell=0}^{\infty}\sum_{r=1}^{3} \Gamma_r(i,j) \frac{\sin^{n+2\ell+r-2}(\phi/2)}{\cos^{n+r-2}(\phi/2)}$$
$i = 1,3$
$j = 2,3$ $\qquad (\text{III}.\ 2.3)$

$$R_{SQ}(i,j) = \frac{(n-2)!}{2} \sum_{r=1}^{3} \left\{(-1)^n \left[\Gamma_r^c(i,j) + \Delta_r^c(i,j)\cos\phi\right] t^{n+r-2} + \left[\Gamma_r^v(i,j) - \Delta_r^v(i,j)\cos\phi\right] \frac{1}{t^{n+r-2}}\right\} \qquad (\text{III}.\ 2.4)$$
$i = 1,3$
$j = 2,3$

$$R_{SQ}(i,j) = \frac{\pi}{2} \sum_{\ell=0}^{\infty}\sum_{r=1}^{4} \left[\Gamma_r^c(i,j)\cos\gamma_r\phi\, \mathrm{ch}y\phi - \Delta_r^c(i,j)\sin\gamma_r\phi\, \mathrm{sh}y\phi\right] \sin^q\phi \qquad (\text{III}.\ 2.5)$$
$i = 1,3$
$j = 2,3$

with $q = -n$ for $i = 2$
$q = -n-1$ otherwise

Matrices $[\Gamma_r]$, $[\Delta_r]$, $[\Gamma_r^c]$, $[\Delta_r^c]$, $[\Gamma_r^v]$, $[\Delta_r^v]$ are given in Appendix V.

$t = tg(\phi/2)$

γ_r with $r = 1, 4$:

$\gamma_1 = x_\ell(n)$

$\gamma_2 = x_\ell(n) + 1$

$\gamma_3 = x_\ell(n) - 1$

$\gamma_4 = x_\ell(n) + 3$

$$S_{PP}(i,j) = \sum_{k=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\sum_{\ell_r=0}^{\infty}\sum_{\ell_s=0}^{\infty} \Gamma_r(k,i)\Gamma_s(k,j) \frac{\sin^{2n+2(\ell_r+\ell_s)+r+s-4}(\phi/2)}{\cos^{2n+r+s-4}(\phi/2)} \qquad (\text{III}.\ 2.6)$$
$i = 2,3$
$j = 2,3$

$$S_{PP}(i,j) = \sum_{k=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\sum_{l_s=0}^{\infty}\left\{\left[\Gamma_r(k,i)+\Delta_r(k,i)\right]\Gamma_s(k,j)\right] \times \frac{\sin^{2n+2l_s+r+s-4}(\phi/2)}{\cos^{2n+r+s-4}(\phi/2)}$$
i = 1 and 4
j = 2, 3

$$-2\Delta_r(k,i)\Delta_s(k,j)\frac{\sin^{2n+2l_s+r+s-4}(\phi/2)}{\cos^{2n+r+s-4}(\phi/2)}\right\} \qquad (\text{III}.2.7)$$

$$S_{PP}(i,j) = \sum_{k=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\left\{\left[\Gamma_r(k,i)+\Delta_r(k,i)\right]\left[\Gamma_s(k,j)+\Delta_s(k,j)\right]\frac{\sin^{m_r}(\phi/2)}{\cos^{m_r}(\phi/2)}\right.$$
i = 1 and 4
j = 1 and 4

$$-2\left[\Gamma_r(k,i)\Delta_s(k,j)+2\Delta_r(k,i)\Delta_s(k,j)+\Delta_r(k,i)\Gamma_s(k,j)\right]\times\frac{\sin^{m_r+2}(\phi/2)}{\cos^{m_r}(\phi/2)}$$

$$\left.+4\Delta_r(k,i)\Delta_s(k,j)\frac{\sin^{m_r+4}(\phi/2)}{\cos^{m_r}(\phi/2)}\right\} \qquad (\text{III}.2.8)$$

with: $m_r = 2n + r + s - 4$

$$S_{PP}(i,j) = \sum_{k=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\sum_{l_r=0}^{\infty}\left\{\Gamma_r(k,i)\left(\Gamma_s(k,j)+\Delta_s(k,j)\right) \times \frac{\sin^{2n+2l+r+s-4}(\phi/2)}{\cos^{2n+r+s-4}(\phi/2)}\right.$$
i = 2, 3
j = 1 and 4

$$\left.-2\Gamma_r(k,i)\Delta_s(k,j)\frac{\sin^{2n+2l+r+s-4}(\phi/2)}{\cos^{2n+r+s-4}(\phi/2)}\right\} \qquad (\text{III}.2.9)$$

$$\begin{aligned} S_{PP}(2,1) &= S_{PP}(1,2) & \qquad S_{QQ}(2,1) &= S_{QQ}(1,2)\\ S_{PP}(3,1) &= S_{PP}(1,3) & S_{QQ}(3,1) &= S_{QQ}(1,3)\\ S_{PP}(2,4) &= S_{PP}(4,2) & S_{QQ}(2,4) &= S_{QQ}(4,2)\\ S_{PP}(3,4) &= S_{PP}(4,3) & S_{QQ}(3,4) &= S_{QQ}(4,3) \end{aligned}$$

$$(\text{III}.2.10)$$

$$[S_{QP}] = [S_{PQ}]^T \qquad (\text{III}.2.11)$$

$$S_{PQ}(i,j) = \frac{(n-2)!}{2}\sum_{k=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\sum_{\ell_r=0}^{\infty}\Big\{(-1)^n\Gamma_r(k,i)[\Gamma_s^c(k,j)+\Delta_s^c(k,j)]$$
$i = 2,3$
$j = 1$ and 4
$$\times\frac{\sin^{2n+2\ell_r+r+s-4}(\phi/2)}{\cos^{2n+r+s-4}(\phi/2)}$$

$$-2(-1)^n\Gamma_r(k,i)\Delta_s^c(k,j)\frac{\sin^{2n+2\ell_r+r+s-2}(\phi/2)}{\cos^{2n+r+s-4}(\phi/2)}$$

$$+\Gamma_r(k,i)[\Gamma_s^v(k,j)-\Delta_s^v(k,j)]\frac{\sin^{2\ell_r+r-s}(\phi/2)}{\cos^{r-s}(\phi/2)}$$

$$+2\Gamma_r(k,i)\Delta_s^v(k,j)\frac{\sin^{2\ell_r+r-s+2}(\phi/2)}{\cos^{r-s}(\phi/2)}\Big\} \qquad (\text{III}.2.12)$$

$$\sin\phi\, S_{PQ}(i,j) = \frac{\pi}{2}\sum_{\substack{k=1\\ \&\,3}}\sum_{s=1}^{4}\sum_{\ell_r=0}^{\infty}\sum_{\ell_s=0}^{\infty}\Big\{\Gamma_s^c(k,j)\big[2^{n-1}\Gamma_1(k,i)\cos\gamma_s\phi\,\mathrm{ch}y\phi\, S_{2n-1}^{2n+2\ell_r-2}(\phi)$$
$i = 2,3$
$j = 2,3$
$$+2^{n+1}\Gamma_3(k,i)\cos\gamma_s\phi\,\mathrm{ch}y\phi\, S_{2n+1}^{2n+2\ell_r-2}(\phi)\big]$$

$$-\Delta_s^c(k,j)\big[2^{n-1}\Gamma_1(k,i)\sin\gamma_s\phi\,\mathrm{sh}y\phi\, S_{2n-1}^{2n+2\ell_r-2}(\phi)+2^{n+1}\Gamma_3(k,i)\sin\gamma_s\phi\,\mathrm{sh}y\phi\, S_{2n+1}^{2n+\ell_r+2}(\phi)\big]\Big\}$$

$$+\pi(2^{n-1})\sum_{\ell_r=0}^{\infty}\sum_{\ell_s=0}^{\infty}\Gamma_2(2,i)\big[\Gamma_1^c(2,j)\cos\gamma_1\phi\,\mathrm{ch}y\phi\, S_{2n-1}^{2n+2\ell_r}(\phi)$$
$$-\Delta_1^c(2,j)\sin\gamma_1\phi\,\mathrm{sh}y\phi\, S_{2n-1}^{2n+2\ell_r}(\phi)\big] \qquad (\text{III}.2.13)$$

with: $$S_r^m(\phi) = \frac{\sin^m(\phi/2)}{\sin^r(\phi)}$$

$$\sin\phi\, S_{PQ}(i,j) = \frac{\pi}{2}\sum_{\substack{k=1\\ \&\,3}}\sum_{r=1}^{2}\sum_{s=1}^{4}\sum_{\ell_s=0}^{\infty}(2^{\frac{m_r}{2}})\Big\{\Gamma_s^c(k,j)[\Gamma_{2r-1}(k,i)+\Delta_{2r-1}(k,i)]$$
$i = 1$ & 4
$j = 2,3$
$$\times\cos\gamma_s\phi\,\mathrm{ch}y\phi\, S_{2n+2r-3}^{m_r}(\phi)$$
$$-2\Gamma_s^c(k,j)\Delta_{2r-1}(k,i)\cos\gamma_s\phi\,\mathrm{ch}y\phi\, S_{2n+2r-3}^{m_r+2}(\phi)$$

$$+2\Delta_s^c(k,j)\Delta_{2r-1}(k,i)\sin\gamma_s\phi\,\mathrm{sh}y\phi\, S_{2n+2r-3}^{m_r+2}(\phi)$$

$$-\Delta_s^c(k,j)[\Gamma_{2r-1}(k,i)+\Delta_{2r-1}(k,i)]\sin\gamma_s\phi\,\mathrm{sh}y\phi\, S_{2n+2r-3}^{m_r+2}(\phi)\Big\}$$

$$+\pi(2^{n-1})\sum_{\ell_s=0}^{\infty}\Big\{\Gamma_1^c(2,j)[\Gamma_2(2,i)+\Delta_2(2,i)]\cos\gamma_1\phi\,\mathrm{ch}y\phi\, S_{2n-1}^{2n}(\phi)$$

$$- 2\Gamma_1^c(2,j)\,\Delta_2(2,i)\cos\gamma_1\phi\ \mathrm{ch}y\phi\ S^{2n+2}_{2n-1}(\phi)$$

$$- \Delta_1^c(2,j)\left[\Gamma_2(2,i) + \Delta_2(2,i)\right]\sin\gamma_1\phi\ \mathrm{sh}y\phi\ S^{2n}_{2n-1}(\phi)$$

$$+ 2\Delta_1^c(2,j)\,\Delta_2(2,i)\sin\gamma_1\phi\ \mathrm{sh}y\phi\ S^{2n+2}_{2n-1}(\phi)\Big\}$$

(III. 2. 14)

with: $m_r = 2n + 4r - 6$

$$\underset{\substack{i=1\ \ 4\\ j=1\ \ 4}}{S_{PQ}(i,j)} = \frac{(n-2)!}{2}\sum_{k=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\Big\{(-1)^n\left[\Gamma_r(k,i) + \Delta_r(k,i)\right] \times \left[\Gamma_s^c(k,j) + \Delta_s^c(k,j)\right]\frac{\sin^m(\phi/2)}{\cos^m(\phi/2)}$$

$$- 2(-1)^n\left[\Gamma_r(k,i) + \Delta_r(k,i)\right]\Delta_s^c(k,j)\frac{\sin^{m+2}(\phi/2)}{\cos^m(\phi/2)}$$

$$- 2(-1)^n\Delta_r(k,i)\left[\Gamma_s^c(k,j) + \Delta_s^c(k,j)\right]\frac{\sin^{m+2}(\phi/2)}{\cos^m(\phi/2)}$$

$$+ 4(-1)^n\Delta_r(k,i)\,\Delta_s^c(k,j)\frac{\sin^{m+4}(\phi/2)}{\cos^m(\phi/2)}$$

$$+ \left[\Gamma_r(k,i) + \Delta_r(k,i)\right]\left[\Gamma_s^v(k,j) - \Delta_s^v(k,j)\right]\frac{\sin^{r-s}(\phi/2)}{\cos^{r-s}(\phi/2)}$$

$$+ 2\left[\Gamma_r(k,i) + \Delta_r(k,i)\right]\Delta_s^v(k,j)\frac{\sin^{r-s+2}(\phi/2)}{\cos^{r-s}(\phi/2)}$$

$$- 2\Delta_r(k,i)\left[\Gamma_s^v(k,j) - \Delta_s^v(k,j)\right]\frac{\sin^{r-s+2}(\phi/2)}{\cos^{r-s}(\phi/2)}$$

$$- 4\Delta_r(k,i)\,\Delta_s^v(k,j)\frac{\sin^{r-s+4}(\phi/2)}{\cos^{r-s}(\phi/2)}\Big\}$$

(III. 2. 15)

with: $m = 2n + r + s - 4$

$$
\begin{aligned}
&\sin\phi \, S_{QQ}\underset{\substack{i\,=\,1\,\&\,4\\ j\,=\,2,3}}{(i,j)} = \frac{\pi}{4}(n-2)!\Bigg\{(2^n)\left[\Gamma_2^c(2,i)+\Delta_2^c(2,i)\right]\\
&\qquad\times\Big[\sum_{\ell=0}^{\infty}\Gamma_1^c(2,j)\cos\gamma_1\phi\,\mathrm{ch}y\phi\, S_{2n-1}^{2n}(\phi)-\sum_{\ell=0}^{\infty}\Delta_1^c(2,j)\sin\gamma_1\phi\,\mathrm{sh}y\phi\, S_{2n-1}^{2n}(\phi)\Big]\\
&-(2^{n+1})\Delta_2^c(2,i)\Big[\sum_{\ell=0}^{\infty}\Gamma_1^c(2,j)\cos\gamma_1\phi\,\mathrm{ch}y\phi\, S_{2n-1}^{2n+2}(\phi)-\sum_{\ell=0}^{\infty}\Delta_1^c(2,j)\sin\gamma_1\phi\,\mathrm{sh}y\phi\, S_{2n-1}^{2n+2}(\phi)\Big]\\
&+2^n\left[\Gamma_2^v(2,i)-\Delta_2^v(2,i)\right]\Big[\sum_{\ell=0}^{\infty}\Gamma_1^c(2,j)\cos\gamma_1\phi\,\mathrm{ch}y\phi\, C_{2n-1}^{2n}(\phi)-\sum_{\ell=0}^{\infty}\Delta_1^c(2,j)\sin\gamma_1\phi\,\mathrm{sh}y\phi\, C_{2n-1}^{2n}(\phi)\Big]\\
&+(2^{n-2})\Delta_2^v(2,i)\Big[\sum_{\ell=0}^{\infty}\Gamma_1^c(2,j)\cos\gamma_1\phi\,\mathrm{ch}y\phi\, C_{2n-1}^{2n-2}(\phi)-\sum_{\ell=0}^{\infty}\Delta_1^c(2,j)\sin\gamma_1\phi\,\mathrm{sh}y\phi\, C_{2n-1}^{2n-2}(\phi)\\
&-\frac{1}{2}\sum_{\ell=0}^{\infty}\Gamma_1^c(2,j)\Big(\cos(\gamma_1+2)\phi\,\mathrm{ch}y\phi\, C_{2n-1}^{2n-2}(\phi)+\cos(\gamma_1-2)\phi\,\mathrm{ch}y\phi\, C_{2n-1}^{2n-2}(\phi)\Big)\\
&+\frac{1}{2}\sum_{\ell=0}^{\infty}\Delta_1^c(2,j)\Big(\sin(\gamma_1+2)\phi\,\mathrm{sh}y\phi\, C_{2n-1}^{2n-2}(\phi)+\sin(\gamma_1-2)\phi\,\mathrm{sh}y\phi\, C_{2n-1}^{2n-2}(\phi)\Big)\Big]\\
&+\sum_{\substack{k=1\\ \&\,3}}\sum_{r=1}^{2}\left(2^{\frac{m_r}{2}}\right)\Big[\Big(\Gamma_{2r-1}^c(k,i)+\Delta_{2r-1}^c(k,i)\Big)\sum_{s=1}^{4}\sum_{\ell_s=0}^{\infty}\Big(\Gamma_s^c(k,j)\cos\gamma_s\phi\,\mathrm{ch}y\phi\, S_{2n+2r-3}^{m_r}(\phi)\\
&\qquad\qquad-\Delta_s^c(k,j)\sin\gamma_s\phi\,\mathrm{sh}y\phi\, S_{2n+2r-3}^{m_r}(\phi)\Big)\\
&-2\Delta_{2r-1}^c(k,i)\sum_{s=1}^{4}\sum_{\ell_s=0}^{\infty}\Big(\Gamma_s^c(k,i)\cos\gamma_s\phi\,\mathrm{ch}y\phi\, S_{2n+2r-3}^{m_r+2}(\phi)-\Delta_s^c(k,j)\sin\gamma_s\phi\,\mathrm{sh}y\phi\, S_{2n+2r-3}^{m_r+2}(\phi)\Big)\\
&+\Big(\Gamma_{2r-1}^v(k,i)-\Delta_{2r-1}^v(k,i)\Big)\sum_{s=1}^{4}\sum_{\ell_s=0}^{\infty}\Big(\Gamma_s^c(k,j)\cos\gamma_s\phi\,\mathrm{ch}y\phi\, C_{2n+2r-3}^{m_r}(\phi)\\
&\qquad\qquad-\Delta_s^c(k,j)\sin\gamma_s\phi\,\mathrm{sh}y\phi\, C_{2n+2r-3}^{m_r}(\phi)\Big)\\
&+\frac{1}{4}\Delta_{2r-1}^v(k,i)\sum_{s=1}^{4}\Bigg(\sum_{\ell_s=0}^{\infty}\Gamma_s^c(k,j)\Big(\cos\gamma_s\phi\,\mathrm{ch}y\phi\, C_{2n+2r-3}^{m_r-2}(\phi)-\frac{1}{2}\cos(\gamma_s+2)\phi\,\mathrm{ch}y\phi\, C_{2n+2r-3}^{m_r-2}(\phi)\\
&\qquad\qquad-\frac{1}{2}\cos(\gamma_s-2)\phi\,\mathrm{ch}y\phi\, C_{2n+2r-3}^{m_r-2}(\phi)\Big)\\
&-\sum_{\ell_s=0}^{\infty}\Delta_s^c(k,j)\Big(\sin\gamma_s\phi\,\mathrm{sh}y\phi\, C_{2n+2r-3}^{m_r-2}(\phi)-\frac{1}{2}\sin(\gamma_s+2)\phi\,\mathrm{sh}y\phi\, C_{2n+2r-3}^{m_r-2}(\phi)\\
&\qquad\qquad-\frac{1}{2}\sin(\gamma_s-2)\phi\,\mathrm{sh}y\phi\, C_{2n+2r-3}^{m_r-2}(\phi)\Big)\Bigg)\Big]\Bigg\}
\end{aligned}
\qquad (\text{III}.2.16)
$$

with: $m_r = 2n + 4r - 6$

$$S_r^m(\phi) = \frac{\sin^m(\phi/2)}{\sin^r(\phi)} \qquad C_r^m(\phi) = \frac{\cos^m(\phi/2)}{\sin^r(\phi)}$$

$$\sin\phi\, S_{QQ}(i,j) = \frac{\pi^2}{16}\sum_{k=1}^{3}\sum_{r=1}^{4}\sum_{s=1}^{4}\sum_{\ell_r=0}^{\infty}\sum_{\ell_s=0}^{\infty}\Big\{\big[\Gamma_r^c(k,i)\Gamma_s^c(k,j)+\Delta_r^c(k,i)\Delta_s^c(k,j)\big]$$

$i = 2,3$

$j = 2,3$

$$\times\cos(\gamma_r-\gamma_s)\phi\ \mathrm{ch}\,2y\phi\ \sin^{2q+1}\phi$$

$$+\big[\Gamma_r^c(k,i)\Gamma_s^c(k,j)-\Delta_r^c(k,i)\Delta_s^c(k,j)\big]\cos(\gamma_r+\gamma_s)\phi\ \mathrm{ch}\,2y\phi\ \sin^{2q+1}\phi$$

$$-\big[\Gamma_r^c(k,i)\Delta_s^c(k,j)+\Delta_r^c(k,i)\Gamma_s^c(k,j)\big]\sin(\gamma_r+\gamma_s)\phi\ \mathrm{sh}\,2y\phi\ \sin^{2q+1}\phi$$

$$+\big[\Gamma_r^c(k,i)\Delta_s^c(k,j)-\Delta_r^c(k,i)\Gamma_s^c(k,j)\big]\sin(\gamma_r-\gamma_s)\phi\ \mathrm{sh}\,2y\phi\ \sin^{2q+1}\phi$$

$$+\big[\Gamma_r^c(k,i)\Gamma_s^c(k,j)+\Delta_r^c(k,i)\Delta_s^c(k,j)\big]\cos(\gamma_r+\gamma_s)\phi\ \sin^{2q+1}\phi$$

$$+\big[\Gamma_r^c(k,i)\Gamma_s^c(k,j)-\Delta_r^c(k,i)\Delta_s^c(k,j)\big]\cos(\gamma_r-\gamma_s)\phi\ \sin^{2q+1}\phi\Big\}$$

(III . 2. 17)

$q = -n \qquad k = 2$

$q = -n - 1$

$$\sin\phi\, S_{QQ}(i,j) = \frac{(n-2)!\,(n-2)!}{2}\sum_{k=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\Big\{\big[\Gamma_r^c(k,i)+\Delta_r^c(k,i)\big]$$

$i = 1\ \&\ 4$

$j = 1\ \&\ 4$

$$\times\big[\Gamma_s^c(k,j)+\Delta_s^c(k,j)\big]\frac{\sin^{m+1}(\phi/2)}{\cos^{m-1}(\phi/2)}$$

$$-2\Big[\Gamma_r^c(k,i)\Delta_s^c(k,j)+\Delta_r^c(k,i)\big(\Gamma_s^c(k,j)+2\Delta_s^c(k,j)\big)\Big]\frac{\sin^{m+3}(\phi/2)}{\cos^{m-1}(\phi/2)}$$

$$+4\Delta_r^c(k,i)\Delta_s^c(k,j)\frac{\sin^{m+5}(\phi/2)}{\cos^{m-1}(\phi/2)}+4\Delta_r^v(k,i)\Delta_s^v(k,j)\frac{\sin^{5-m}(\phi/2)}{\cos^{-m-1}(\phi/2)}$$

$$+\big[\Gamma_r^v(k,i)-\Delta_r^v(k,i)\big]\big[\Gamma_s^v(k,j)-\Delta_s^v(k,j)\big]\frac{\sin^{1-m}(\phi/2)}{\cos^{-m-1}(\phi/2)}$$

$$+2\Big[\Gamma_r^v(k,i)\Delta_s^v(k,j)+\Delta_r^v(k,i)\big(\Gamma_s^v(k,j)-2\Delta_s^v(k,j)\big)\Big]\frac{\sin^{3-m}(\phi/2)}{\cos^{-m-1}(\phi/2)}$$

$$+2(-1)^n\big[\Gamma_r^c(k,i)+\Delta_r^c(k,i)\big]\big[\Gamma_s^v(k,j)-\Delta_s^v(k,j)\big]\frac{\sin^{r-s+1}(\phi/2)}{\cos^{r-s-1}(\phi/2)}$$

$$-8(-1)^n \Delta_r^c(k,i)\Delta_s^v(k,j)\frac{\sin^{r-s+5}(\phi/2)}{\cos^{r-s-1}(\phi/2)}$$

$$+4(-1)^n\left[\Gamma_r^c(k,i)\Delta_s^v(k,j)+\Delta_r^c(k,i)\left(2\Delta_s^v(k,j)-\Gamma_s^v(k,j)\right)\right]$$

$$\left.\times\frac{\sin^{r-s+3}(\phi/2)}{\cos^{r-s-1}(\phi/2)}\right\} \qquad (\text{III}.2.18)$$

with: $m = 2n + r + s - 4$

$$\sin\phi\, S_{QQ}(i,j) = (2^{-n})\sum_{r=2}^{4}\sum_{s=1,3}\Big\{(-1)^n\left[\Gamma_r^c(1,i)\left(\Gamma_s^c(1,j)+\Delta_s^c(1,j)\right)\cos\gamma_r\phi\right.$$

$i = 2,3$

$j = 1 \;\&\; 4$

$$\left.+\Gamma_1^c(3,i)\left(\Gamma_s^c(3,j)+\Delta_s^c(3,j)\right)\cos\gamma_1\phi\right]\mathrm{ch}\gamma\phi$$

$$\times\frac{\sin^{s-2}(\phi/2)}{\cos^{2n+s-2}(\phi/2)}$$

$$-2(-1)^n\left[\Gamma_r^c(1,i)\Delta_s^c(1,j)\cos\gamma_r\phi+\Gamma_1^c(3,i)\Delta_s^c(3,j)\cos\gamma_1\phi\right]\mathrm{ch}\gamma\phi\,\frac{\sin^{s}(\phi/2)}{\cos^{2n+s-2}(\phi/2)}$$

$$-(-1)^n\left[\Delta_r^c(1,i)\left(\Gamma_s^c(i,j)+\Delta_s^c(1,j)\right)\sin\gamma_r\phi+\Delta_1^c(3,i)\left(\Gamma_s^c(3,j)+\Delta_s^c(3,j)\right)\sin\gamma_1\phi\right]$$

$$\times\,\mathrm{sh}\gamma\phi\,\frac{\sin^{s-2}(\phi/2)}{\cos^{2n+s-2}(\phi/2)}$$

$$+2(-1)^n\left[\Delta_r^c(1,i)\Delta_s^c(1,j)\sin\gamma_r\phi+\Delta_1^c(3,i)\Delta_s^c(3,j)\sin\gamma_1\phi\right]\mathrm{sh}\gamma\phi\,\frac{\sin^{s}(\phi/2)}{\cos^{2n+s-2}(\phi/2)}$$

$$+\left[\Gamma_r^c(1,i)\left(\Gamma_s^v(1,j)-\Delta_s^v(1,j)\right)\cos\gamma_r\phi+\Gamma_1^c(3,i)\left(\Gamma_s^v(3,j)-\Delta_s^v(3,j)\right)\cos\gamma_1\phi\right]\mathrm{ch}\gamma\phi\,\frac{\sin^{-2n-s-2}(\phi/2)}{\cos^{-s-2}(\phi/2)}$$

$$+2\left[\Gamma_r^c(1,i)\Delta_s^v(1,j)\cos\gamma_r\phi+\Gamma_1^c(3,i)\Delta_s^v(3,j)\cos\gamma_1\phi\right]\mathrm{ch}\gamma\phi\,\frac{\sin^{-2n-s-4}(\phi/2)}{\cos^{-s+2}(\phi/2)}$$

$$-\left[\Delta_r^c(1,i)\left(\Gamma_s^v(1,j)-\Delta_s^v(1,j)\right)\sin\gamma_r\phi+\Delta_1^c(3,i)\left(\Gamma_s^v(3,j)-\Delta_s^v(3,j)\right)\sin\gamma_1\phi\right]\mathrm{sh}\gamma\phi\,\frac{\sin^{-2n-s+2}(\phi/2)}{\cos^{-s+2}(\phi/2)}$$

$$\left.-2\left[\Delta_r^c(1,i)\Delta_s^v(1,j)\sin\gamma_r\phi+\Delta_1^c(3,i)\Delta_s^v(3,j)\sin\gamma_1\phi\right]\mathrm{sh}\gamma\phi\,\frac{\sin^{-2n-s+4}(\phi/2)}{\cos^{-s+2}(\phi/2)}\right\}$$

$$+(2^{-n+1})\Big\{(-1)^n\Gamma_1^c(2,i)\left(\Gamma_2^c(2,j)+\Delta_2^c(2,j)\right)\cos\gamma_1\phi\,\mathrm{ch}\gamma\phi\,\frac{\sin(\phi/2)}{\cos^{2n-1}(\phi/2)}$$

$$-2(-1)^n\Gamma_1^c(2,i)\Delta_2^c(2,j)\cos\gamma_1\phi\,\mathrm{ch}\gamma\phi\,\frac{\sin^{3}(\phi/2)}{\cos^{2n-1}(\phi/2)}$$

$$-(-1)^n \Delta_1^c(2,i)\left(\Gamma_2^c(2,j)+\Delta_2^c(2,j)\right) \sin\gamma_1\phi \, \frac{\sin \quad (\phi/2)}{\cos^{2n-1}(\phi/2)}$$

$$+2(-1)^n \Delta_1^c(2,i)\Delta_2^c(2,j) \sin\gamma_1\phi \; \mathrm{sh}\gamma\phi \, \frac{\sin^3 \quad (\phi/2)}{\cos^{2n-1}(\phi/2)}$$

$$+\Gamma_1^c(2,i)\Gamma_2^v(2,j)\cos\gamma_1\phi \; \mathrm{ch}\gamma\phi \, \frac{\sin^{-2n+1}(\phi/2)}{\cos^{-1}(\phi/2)}$$

$$+2\Gamma_1^c(2,i)\Delta_2^v(2,j)\cos\gamma_1\phi \; \mathrm{ch}\gamma\phi \, \frac{\sin^{-2n+3}(\phi/2)}{\cos^{-1}(\phi/2)}$$

$$-\Delta_1^c(2,i)\left(\Gamma_2^c(2,j)-\Delta_2^c(2,j)\right)\sin\gamma_1\phi \; \mathrm{sh}\gamma\phi \, \frac{\sin^{-2n+1}(\phi/2)}{\cos^{-1}(\phi/2)}$$

$$\left.-2\Delta_1^c(2,i)\Delta_2^c(2,j)\sin\gamma_1\phi \; \mathrm{sh}\gamma\phi \, \frac{\sin^{-2n+3}(\phi/2)}{\cos^{-1}(\phi/2)}\right\} \qquad (\mathrm{III}.2.19)$$

Integrating matrices [S_{PP}], [S_{PQ}] and [S_{QQ}] with respect to ϕ gives:

$$\underset{\substack{i=1 \text{ and } 4\\ j=1 \text{ and } 4}}{S_{PP}(i,j)} = 4\sum_{k=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\left\{\left(\Delta_r(k,i)+\Gamma_r(k,i)\right)\left(\Delta_s(k,j)+\Gamma_s(k,j)\right)\right.$$
$$\times\left[\Gamma\left(b+2,b,\frac{\phi_2}{2}\right)-\Gamma\left(b+2,b,\frac{\phi_1}{2}\right)\right]$$
$$-2\left[\Gamma_r(k,i)\Delta_s(k,j)+\Delta_r(k,i)\left(2\Delta_s(k,j)+\Gamma_s(k,j)\right)\right]$$
$$\times\left[\Gamma\left(b+4),b,\frac{\phi_2}{2}\right)-\Gamma\left(b+4,b,\frac{\phi_1}{2}\right)\right]$$
$$\left.+4\Delta_r(k,i)\Delta_s(k,j)\left[\Gamma\left(b+6,b,\frac{\phi_2}{2}\right)-\Gamma\left(b+6,b,\frac{\phi_1}{2}\right)\right]\right\} \qquad (\mathrm{III}.2.20)$$

where $b = 2n + r + s - 5$

$$\underset{\substack{i=1 \,\&\, 4\\ j=2,3}}{S_{PP}(i,j)} = 4\sum_{k=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\left\{\left(\Gamma_r(k,i)+\Delta_r(k,i)\right)\Gamma_s(k,j)\right.$$
$$\times\left[\Gamma\left(c,b,\frac{\phi_2}{2}\right)-\Gamma\left(c,b,\frac{\phi_1}{2}\right)\right]$$
$$\left.-2\Delta_r(k,i)\Gamma_s(k,j)\left[\Gamma\left(c+2,b,\frac{\phi_2}{2}\right)-\Gamma\left(c+2,b,\frac{\phi_1}{2}\right)\right]\right\} \qquad (\mathrm{III}.2.21)$$

where $c = 2n + 2\ell + r + s - 3$

$$S_{PP}(i,j) = 4 \sum_{k=1}^{3} \sum_{\ell_r=0}^{\infty} \sum_{\ell_s=0}^{\infty} \sum_{r=1}^{3} \sum_{s=1}^{3} \Gamma_r(k,i)\Gamma_s(k,j)\left[\Gamma\left(as, b, \frac{\phi_2}{2}\right) - \Gamma\left(as, b, \frac{\phi_1}{2}\right)\right]$$
$i = 2,3$
$j = 2,3$ (III. 2. 22)

where $as = 2n + 2(\ell + \ell_s) + r + s - 3$

$$\begin{aligned} S_{PP}(2,1) &= S_{PP}(1,2) \\ S_{PP}(3,1) &= S_{PP}(1,3) \\ S_{PP}(2,4) &= S_{PP}(4,2) \\ S_{PP}(3,4) &= S_{PP}(4,3) \end{aligned} \qquad \text{(III. 2. 23)}$$

$$\begin{aligned} S_{PQ}(i,j) = 2(n-2)! \sum_{k=1}^{3} \sum_{r=1}^{3} \sum_{s=1}^{3} \Big\{ & (-1)^n \big(\Gamma_r(k,i) + \Delta_r(k,i)\big)\big(\Gamma_S^c(k,j) + \Delta_S^c(k,j)\big) \\ & \times \left[\Gamma\left(b+2, b, \frac{\phi_2}{2}\right) - \Gamma\left(b+2, b, \frac{\phi_1}{2}\right)\right] \\ & - 2(-1)^n \big(\Gamma_r(k,i)\Delta_S^c(k,j) + \Delta_r(k,i)\big(\Gamma_S^c(k,j) + 2\Delta_S^c(k,j)\big)\big) \\ & \times \left[\Gamma\left(b+4, b, \frac{\phi_2}{2}\right) - \Gamma\left(b+4, b, \frac{\phi_1}{2}\right)\right] \\ & + 4(-1)^n \Delta_r(k,i)\Delta_S^c(k,j)\left[\Gamma\left(b+6, b, \frac{\phi_2}{2}\right) - \Gamma\left(b+6, b, \frac{\phi_1}{2}\right)\right] \\ & + \big(\Gamma_r(k,i) + \Delta_r(k,i)\big)\big(\Gamma_S^v(k,j) - \Delta_S^v(k,j)\big)\left[\Gamma\left(u+1, u-1, \frac{\phi_2}{2}\right) - \Gamma\left(u+1, u-1, \frac{\phi_1}{2}\right)\right] \\ & + 2\big(\Gamma_r(k,i)\Delta_S^v(k,j) + \Delta_r(k,i)\big(2\Delta_S^v(k,j) - \Gamma_S^v(k,j)\big)\big) \\ & \times \left[\Gamma\left(u+3, u-1, \frac{\phi_2}{2}\right) - \Gamma\left(u+3, u-1, \frac{\phi_1}{2}\right)\right] \\ & - 4\Delta_r(k,i)\Delta_S^v(k,j)\left[\Gamma\left(u+5, u-1, \frac{\phi_2}{2}\right) - \Gamma\left(u+5, u-1, \frac{\phi_1}{2}\right)\right] \Big\} \end{aligned}$$
$i = 1 \,\&\, 4$
$j = 1 \,\&\, 4$

with: $u = r - s$

(III. 2. 24)

$$\begin{aligned} S_{PQ}(i,j) = (n-2)! \sum_{k=1}^{3} \sum_{\ell_r=0}^{\infty} \sum_{r=1}^{3} \sum_{s=1}^{3} \Big\{ & 2(-1)^n \Gamma_r(k,i)\big(\Gamma_S^c(k,j) + \Delta_S^c(k,i)\big) \\ & \times \left[\Gamma\left(c, b, \frac{\phi_2}{2}\right) - \Gamma\left(c, b, \frac{\phi_1}{2}\right)\right] \\ & - 4(-1)^n \Gamma_r(k,i)\Delta_S^c(k,j)\left[\Gamma\left(c+2, b, \frac{\phi_2}{2}\right) - \Gamma\left(c+2, b, \frac{\phi_1}{2}\right)\right] \\ & + 4\Gamma_r(k,i)\Delta_S^v(k,j)\left[\Gamma\left(u+2\ell+3, u-1, \frac{\phi_2}{2}\right) - \Gamma\left(u+2\ell+3, u-1, \frac{\phi_1}{2}\right)\right] \\ & + 2\Gamma_r(k,i)\big(\Gamma_S^v(k,j) - \Delta_S^v(k,j)\big)\left[\Gamma\left(u+2\ell+1, u-1, \frac{\phi_2}{2}\right) - \Gamma\left(u+2\ell+1, u-1, \frac{\phi_1}{2}\right)\right] \Big\} \end{aligned}$$
$i = 2,3$
$j = 1 \,\&\, 4$

(III. 2. 25)

$$S_{PQ}(i,j) = S_{PQ1}(i,j) + S_{PQ2}(i,j) \qquad \text{(III. 2. 26)}$$

$i = 1 \,\&\, 4$	$i = 1 \,\&\, 4$	$i = 1 \,\&\, 4$
$j = 2,3$	$j = 2,3$	$j = 2,3$

where:

$$S_{PQ1}(i,j)=\frac{\pi}{2}(2^n)\sum_{\ell_r=0}^{\infty}\Big\{\big(\Gamma_2(2,i)+\Delta_2(2,i)\big)\big(\Gamma_1^c(2,j)\times HN1-\Delta_1^c(2,j)\times TN1\big)$$
$$-2\Delta_2(2,i)\big(\Gamma_1^c(2,j)\times HN2-\Delta_1^c(2,j)\times TN2\big)\Big\}$$

$i=1$ & 4

$j=2,3$

$$(\text{III}.2.27)$$

Terms HN1, TN1, HN2 and TN2 are given in Appendix IV.

$$S_{PQ2}(i,j)=\frac{\pi}{2}\sum_{\substack{k=1\\ \& 3}}\sum_{r=1}^{2}\sum_{s=1}^{4}\sum_{\ell_r=0}^{\infty}(2^{m_r})\Big\{\big(\Gamma_{2r-1}(k,i)+\Delta_{2r-1}(k,i)\big)\times\big(\Gamma_S^c(k,j)\times HNR-\Delta_S^c(k,j)\times TNR\big)$$
$$-2\Delta_{2r-1}(k,i)\big(\Gamma_S^c(k,j)\times HNR2-\Delta_S^c(k,j)\times TNR2\big)\Big\}$$

$i=1$ & 4

$j=2,3$

$$(\text{III}.2.28)$$

Terms HNR, TNR, HNR2 and TNR2 are given in Appendix IV.

$$\underset{\substack{i=2,3\\ j=2,3}}{S_{PQ}(i,j)} = \underset{\substack{i=2,3\\ j=2,3}}{S_{PQ1}(i,j)} + \underset{\substack{i=2,3\\ j=2,3}}{S_{PQ2}(i,j)} \qquad (\text{III}.2.29)$$

where:

$$S_{PQ1}(i,j)=\frac{\pi}{2}(2^n)\sum_{\ell_r=0}^{\infty}\sum_{\ell_s=0}^{\infty}\Big\{\Gamma_2(2,i)\big(\Gamma_1^c(2,j)\times H11-\Delta_1^c(2,j)\times T11\big)\Big\}$$

$i=2,3$

$j=2,3$

$$(\text{III}.2.30)$$

$$S_{PQ2}(i,j)=\pi(2^n)\sum_{\substack{k=1\\ \& 3}}\sum_{s=1}^{4}\sum_{\ell_r=0}^{\infty}\sum_{\ell_s=0}^{\infty}\Big\{\Gamma_S^c(k,j)\Big(\frac{1}{4}\times\Gamma(k,i)\times H12+\Gamma_3(k,i)\times H22\Big)$$
$$-\Delta_S^c(k,j)\Big(\frac{1}{4}\times\Gamma_1(k,i)\times T12+\Gamma_3(k,i)\times T22\Big)\Big\}$$

$i=2,3$

$j=2,3$

$$(\text{III}.2.31)$$

Terms H12, H22, T12, T22 are given in Appendix IV.

$$[S_{QP}] = [S_{PQ}]^T \qquad (\text{III}.2.32)$$

$$S_{QQ}(i,j) = (n-2)!\,(n-2)! \sum_{k=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} \Big\{ \big(\Gamma_r^c(k,i)+\Delta_r^c(k,i)\big)\big(\Gamma_s^c(k,j)+\Delta_s^c(k,j)\big) \times\left[\Gamma\left(b+2,b,\frac{\phi_2}{2}\right)-\Gamma\left(b+2,b,\frac{\phi_1}{2}\right)\right]$$

$i = 1$ and 4

$j = 1$ and 4

$$-2\big(\Gamma_r^c(k,i)\Delta_s^c(k,j)+\Delta_r^c(k,i)\big(\Gamma_s^c(k,j)+2\Delta_s^c(k,j)\big)\big) \times\left[\Gamma\left(b+4,b,\frac{\phi_2}{2}\right)-\Gamma\left(b+4,b,\frac{\phi_1}{2}\right)\right]$$

$$+4\Delta_r^c(k,i)\Delta_s^c(k,j)\left[\Gamma\left(b+6,b,\frac{\phi_2}{2}\right)-\Gamma\left(b+6,b,\frac{\phi_1}{2}\right)\right]$$

$$+4\Delta_r^v(k,i)\Delta_s^v(k,j)\left[\Gamma\left(4-b^2,-b-2,\frac{\phi_2}{2}\right)-\Gamma\left(4-b,-b-2,\frac{\phi_1}{2}\right)\right]$$

$$+\big(\Gamma_r^v(k,i)-\Delta_r^v(k,i)\big)\big(\Gamma_s^v(k,j)-\Delta_s^v(k,j)\big)\left[\Gamma\left(-b,{}^2-b-2,\frac{\phi_2}{2}\right)-\Gamma\left(-b,-b-2,\frac{\phi_1}{2}\right)\right]$$

$$+2\big(\Gamma_r^v(k,i)\Delta_s^v(k,j)+\Delta_r^v(k,i)\big(\Gamma_s^v(k,j)-2\Delta_s^v(k,j)\big)\big) \times\left[\Gamma\left(2-b,-b-2,\frac{\phi_2}{2}\right)-\Gamma\left(2-b,-b-2,\frac{\phi_1}{2}\right)\right]$$

$$+2(-1)^n\big(\Gamma_r^c(k,i)+\Delta_r^c(k,i)\big)\big(\Gamma_s^v(k,j)-\Delta_s^v(k,j)\big)\left[\Gamma\left(u+1,u-1,\frac{\phi_2}{2}\right)-\Gamma\left(u+1,u-1,\frac{\phi_1}{2}\right)\right]$$

$$-8(-1)^n\Delta_r^c(k,j)\Delta_s^v(k,j)\left[\Gamma\left(u+5,u-1,\frac{\phi_2}{2}\right)-\Gamma\left(u+5,u-1,\frac{\phi_1}{2}\right)\right]$$

$$+4(-1)^n\big(\Gamma_r^c(k,i)\Delta_s^v(k,j)+\Delta_r^c(k,i)\big(2\Delta_s^v(k,j)-\Gamma_s^v(k,j)\big)\big) \times\left[\Gamma\left(u+3,u-1,\frac{\phi_2}{2}\right)-\Gamma\left(u+3,u-1,\frac{\phi_1}{2}\right)\right]\Big\} \qquad (\text{III}.2.33)$$

$$S_{QQ}(i,j) = \frac{\pi^2}{16}\sum_{k=1}^{3}\sum_{r=1}^{4}\sum_{s=1}^{4}\sum_{\ell_r=0}^{\infty}\sum_{\ell_s=0}^{\infty}\Big\{\big(\Gamma_r^c(k,i)\Gamma_s^c(k,j)+\Delta_r^c(k,i)\Delta_s^c(k,j)\big)\times CER1(k)$$

$i = 2,3$

$j = 2,3$

$$+\big(\Gamma_r^c(k,i)\Gamma_s^c(k,j)-\Delta_r^c(k,i)\Delta_s^c(k,j)\big)\times CER2(k)$$

$$-\big(\Gamma_r^c(k,i)\Delta_s^c(k,i)+\Delta_r^c(k,i)\Gamma_s^c(k,j)\big)\times CIR2(k)$$

$$+\big(\Gamma_r^c(k,i)\Delta_s^c(k,i)-\Delta_r^c(k,i)\Gamma_s^c(k,j)\big)\times CIR1(k)$$

$$+\big(\Gamma_r^c(k,i)\Gamma_s^c(k,j)+\Delta_r^c(k,i)\Delta_s^c(k,j)\big)\times CER3(k)$$

$$+\big(\Gamma_r^c(k,i)\Gamma_s^c(k,j)+\Delta_r^c(k,i)\Delta_s^c(k,j)\big)\times CER4(k)\Big\} \qquad (\text{III}.2.34)$$

Terms CER1 (k), CER2 (k), CER3 (k), CER4 (k), CIR1 (k), CIR2 (k) are given in Appendix IV.

$$S_{QQ}(i,j) = S_{QQ1}(i,j) + S_{QQ2}(i,j) \qquad (\text{III}.2.35)$$
$i = 1 \ \& \ 4$, $j = 2,3$

where:

$$S_{QQ1}(i,j) = \frac{\pi}{4}(n-2)!\sum_{\ell=0}^{\infty}\Big\{(2^n)\big(\Gamma_2^c(2,i)+\Delta_2^c(2,i)\big)\big(\Gamma_1^c(2,j)\times HN1-\Delta_1^c(2,j)\times TN1\big)$$
$$-(2^{n+1})\Delta_2^c(2,i)\big(\Gamma_1^c(2,j)\times HN2-\Delta_1^c(2,j)\times TN2\big)$$
$$+(2^n)\big(\Gamma_2^v(2,i)-\Delta_2^v(2,i)\big)\big(\Gamma_1^c(2,j)\times UN1-\Delta_1^c(2,j)\times VN1\big)$$
$$+\frac{1}{4}(2^n)\,\Delta_2^v(2,i)\Big[\Gamma_1^c(2,j)\times UN2-\Delta_1^c(2,j)\times VN2$$
$$-\frac{1}{2}\Gamma_1^c(2,j)(UN22+UN23)+\frac{1}{2}\Delta_1^c(2,j)(VN22+VN23)\Big]\Big\}$$
$i = 1 \ \& \ 4$, $j = 2,3$

$$(\text{III}.2.36)$$

$$S_{QQ2}(i,j) = \frac{\pi}{4}(n-2)!\sum_{k=1\,\&\,3}\sum_{r=1}^{2}\sum_{s=1}^{4}\sum_{\ell_s=0}^{\infty}\left(2^{\frac{mr}{2}}\right)\Big\{$$
$$\big(\Gamma_{2r-1}^c(k,i)+\Delta_{2r-1}^c(k,i)\big)\big(\Gamma_S^c(k,j)\times HNR-\Delta_S^c(k,j)\times TNR\big)$$
$$-2\Delta_{2r-1}^c(k,i)\big(\Gamma_S^c(k,j)\times HNR2-\Delta_S^c(k,j)\times TNR2\big)$$
$$+\big(\Gamma_{2r-1}^v(k,i)-\Delta_{2r-1}^v(k,i)\big)\big(\Gamma_S^c(k,j)\times UNR-\Delta_S^c(k,j)\times VNR\big)$$
$$+\frac{1}{4}\Delta_{2r-1}^v(k,i)\Big[\Gamma_S^c(k,j)\big(UNR2-\tfrac{1}{2}UNR3-\tfrac{1}{2}UNR4\big)$$
$$-\Delta_S^c(k,j)\big(VNR2-\tfrac{1}{2}VNR3-\tfrac{1}{2}VNR4\big)\Big]\Big\}$$
$i = 1 \ \& \ 4$, $j = 2,3$

$$(\text{III}.2.37)$$

Terms HN1, HN2, TN1, TN2, UN1, UN2, UN22, UN23, VN1, VN2, VN22, VN23, HNR, HNR2, TNR, TNR2, UNR, UNR2, UNR3, UNR4, VNR, VNR2, VNR3 and VNR4 are given in Appendix IV.

$$
\begin{aligned}
S_{QQ}(2,1) &= S_{QQ}(1,2) \\
S_{QQ}(3,1) &= S_{QQ}(1,3) \\
S_{QQ}(2,4) &= S_{QQ}(4,2) \\
S_{QQ}(3,4) &= S_{QQ}(4,3)
\end{aligned}
\qquad (\text{III}.2.38)
$$

APPENDIX IV

INTEGRATION FORMULAS

The integration formulas taken from mathematical tables or developed for our own purposes, which have been used directly or indirectly in integrating matrix [G] and [S] elements, are summarized below:

1) $$\gamma(m,x) = \int \frac{dx}{\cos^m x} = \frac{\sin x}{(m-1)\cos^{m-1} x} + \left(\frac{m-2}{m-1}\right)\gamma(m-2,x)$$

where m is a positive integer.

$$\gamma(-m,x) = \int \cos^m dx = \frac{1}{m}\sin x \cos^{m-1} x + \frac{m-1}{m}\gamma(-m+2,x)$$

2) $$\Delta(m,x) = \int \frac{dx}{\sin^m x} = -\frac{\cos x}{(m-1)\sin^{m-1} x} + \left(\frac{m-2}{m-1}\right)\Delta(m-2,x)$$

where m is a positive integer

$$\Delta(-m,x) = \int \sin^m x dx = -\frac{\sin^{m-1} x \cos x}{m} + \left(\frac{m-1}{m}\right)\Delta(-m+2,x)$$

3) $$\Gamma(m,n,x) = \int \frac{\sin^m x}{\cos^n x}\, dx$$

where m , n are positive integers

$m = 2 \quad n \neq 2$:

$$\Gamma(2,n,x) = \int \frac{\sin^2 x}{\cos^n x}\, dx = -\frac{1}{(2-n)}\frac{\sin x}{\cos^{n-1} x} + \frac{\gamma(n,x)}{(2-n)}$$

$m > 2 \quad m = n$:

$$\Gamma(m,n,x) = \int \frac{\sin^m x}{\cos^m x}\, dx = \frac{1}{m-1}\operatorname{tg}^{m-1} x - \Gamma(m-2,m-2,x)$$

$m > 2 \quad m \neq n$:

$$\Gamma(m,n,x) = \int \frac{\sin^m x}{\cos^n x}\, dx$$

$$= -\frac{1}{(m-n)}\frac{\sin^{m-1} x}{\cos^{n-1} x} + \left(\frac{m-1}{m-n}\right)\Gamma(m-2,n,x)$$

4) $$\Gamma(-m,-n,x) = \int \frac{\cos^n x}{\sin^m x}\, dx$$

$n = 2 \quad m \neq 2$:

$$\Gamma(-m,-2,x) = \int \frac{\cos^2 x}{\sin^m x}\, dx = \frac{1}{(2-m)}\frac{\cos x}{\sin^{m-1} x} + \frac{\Delta(m,x)}{(2-m)}$$

$n > 2 \quad m = n$:

$$\Gamma(-m,-n,x) = \int \frac{\cos^n x}{\sin^n x}\, dx$$

$$= \left(\frac{-1}{n-1}\right)\operatorname{cotg}^{n-1} x - \Gamma(-n+2,-n+2,x)$$

$n > 2 \quad m \neq n$:

$$\Gamma(-m,-n,x) = \int \frac{\cos^n x}{\sin^m x}\,dx = \left(\frac{1}{n-m}\right)\frac{\cos^{n-1} x}{\sin^{m-1} x} + \left(\frac{n-1}{n-m}\right)\Gamma(-m,-n+2,x)$$

5) $\Gamma(m,-n,x) = \int \sin^m x \cos^n x \,dx$

$n = 2 \qquad \Gamma(m,-2,x) = \int \sin^m x \cos^2 x\,dx$

$$= \frac{1}{m+2}\sin^{m+1} x \cos x + \frac{\Delta(-m,x)}{m+2}$$

$n > 2 \qquad \Gamma(m,-n,x) = \dfrac{1}{(m+n)}\sin^{m+1} x \cos^{n-1} x + \left(\dfrac{n-1}{m+n}\right)\Gamma(m,-n+2,x)$

6) $\Gamma(-m,n,x) = \displaystyle\int \frac{dx}{\sin^m x \cos^n x}$

$m = 1 \qquad n > 2$:

$$\Gamma(-1,n,x) = \int \frac{dx}{\sin x \cos^n x} = \frac{1}{(n-1)\cos^{n-1} x} + \Gamma(-1,n-2,x)$$

$m = 2 \qquad n > 2$:

$$\Gamma(-2,n,x) = \int \frac{dx}{\sin^2 x \cos^n x} = \frac{-1}{\sin x \cos^{n-1} x} + n\gamma(n,x)$$

$m > 2 \qquad n > 2$:

$$\Gamma(-m,n,x) = \int \frac{dx}{\sin^m x \cos^n x} = \frac{-1}{(m-1)\sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{m-1}\Gamma(-m+2,n,x)$$

7) $$R(p,n,\phi) = \int \frac{\phi^p}{\sin^n \phi}\, d\phi$$

with: $0 < |\phi| < \pi$ $\qquad$ n and p : positive integers or zero

$n = 0$ and/or $p = 0$
$$\begin{cases} R(0,0,\phi) = \phi \\ R(0,n,\phi) = \Delta(n,\phi) \\ R(p,0,\phi) = \dfrac{\phi^{p+1}}{p+1} \end{cases}$$

$n = 1$, $p \geqslant 1$
$$\left\{ R(p,1,\phi) = \frac{\phi^p}{p} + \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1} \times 2 \times (2^{2\nu-1} - 1)}{(p+2\nu)\,(2\nu)!} B_{2\nu}\, \phi^{p+2\nu} \right.$$

$n = 2$, $p \geqslant 1$
$$\begin{cases} R(1,2,\phi) = -\phi \cot g\phi + \log(\sin\phi) \\ R(p,2,\phi) = -\phi^p \cot g\phi + \dfrac{p\,\phi^{p-1}}{(p-1)} + p\displaystyle\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu}\, 2^{2\nu} B_{2\nu}\, \phi^{p+2\nu-1}}{(p+2\nu-1)\,(2\nu)!} \end{cases}$$

$n \geqslant 3$, $p \geqslant 1$
$$\begin{cases} R(p,n,\phi) = -\dfrac{p\phi^{p-1}\sin\phi - (n-2)\,\phi^p \cos\phi}{(n-1)\,(n-2)\,\sin^{n-1}\phi} \\ \qquad + \dfrac{p(p-1)R(p-2,n-2,\phi)}{(n-1)\,(n-2)} + \left(\dfrac{n-2}{n-1}\right) R(p,n-2,\phi) \end{cases}$$

where $B_{2\nu}$ = Bernouilli number

$B_2 = 1/6$

$B_4 = -1/30$

$B_6 = 1/42$

$B_8 = -1/30$

$B_{10} = 5/66$

$B_{12} = -691/2730$

$B_{14} = 7/6$

$B_{16} = -3617/510$

$B_{18} = 54.97117794$

$B_{20} = -529.12424$

$B_{22} = 6.1921232 \times 10^{3}$

$B_{24} = -8.658025311 \times 10^{4}$

$B_{26} = 1.425517167 \times 10^{6}$

$B_{28} = -2.729823107 \times 10^{7}$

$B_{30} = 6.015808739 \times 10^{8}$

$B_{32} = -1.511631577 \times 10^{10}$

$B_{34} = 4.2961464 \times 10^{11}$

$B_{36} = -1.371165 \times 10^{13}$

$B_{38} = 4.883323 \times 10^{14}$

$B_{40} = -1.9296579 \times 10^{16}$

$B_{42} = 8.4169305\ 10^{17}$

$B_{44} = -4.033807 \times 10^{19}$

$B_{46} = 2.1150748 \times 10^{21}$

$B_{48} = -1.20866265 \times 10^{23}$

$B_{50} = 7.500866746 \times 10^{24}$

$B_{52} = -5.0387781 \times 10^{26}$

$B_{54} = 3.6528776 \times 10^{28}$

$B_{56} = -2.849877 \times 10^{30}$

$B_{58} = 2.3865427 \times 10^{32}$

$B_{60} = -2.1399949 \times 10^{34}$

The $R(p, n, \phi)$ function is obtained from the numerical iterations such that:

$$\frac{R(p,n,\phi)_{\nu} - R(p,n,\phi)_{\nu-1}}{R(p,n,\phi)_{\nu}} \leqslant 1 \times 10^{-9}$$

8) $$\cos(x+iy)\phi = 1+\sum_{r=1}^{\infty} Y_r^c(x,y)\phi^{2r} + i\sum_{r=1}^{\infty} Z_r^c(x,y)\phi^{2r}$$

$$\sin(x+iy)\phi = x\phi + \sum_{r=1}^{\infty} Y_r(x,y)\phi^{2r+1}$$
$$+\, i\left\{\sum_{r=1}^{\infty} Z_r(x,y)\phi^{2r+1} + y\phi\right\}$$

with:

$$Y_r^c(x,y) = \sum_{k=0,2,4\ldots}^{2r} \frac{(-1)^{\frac{2r+k}{2}}}{(2r-k)!\,k!}\, x^{2r-k}\, y^k$$

$$Z_r^c(x,y) = \sum_{k=1,3,5\ldots}^{2r-1} \frac{(-1)^{\frac{2r+k+3}{2}}}{(2r-k)!\,k!}\, x^{2r-k}\, y^k$$

$$Y_r(x,y) = \sum_{k=0,2,4\ldots}^{2r} \frac{(-1)^{\frac{2r+k}{2}}}{(2r+1-k)!\,k!}\, x^{2r+1-k}\, y^k$$

$$Z_r(x,y) = \sum_{k=1,3,5\ldots}^{2r+1} \frac{(-1)^{\frac{2r+k+3}{2}}}{(2r+1-k)!\,k!}\, x^{2r+1-k}\, y^k$$

9) $$\int \frac{\cos(x+iy)\phi}{\sin\phi}\,d\phi = \int \frac{x\phi}{\sin\phi}\,d\phi + \sum_{t=1}^{\infty} Y_t^c(x,y)\int \frac{\phi^{2t}}{\sin\phi}\,d\phi + i\sum_{t=1}^{\infty} Z_t^c(x,y)\int\frac{\phi^{2t}}{\sin\phi}\,d\phi$$

$$\int \frac{\sin(x+iy)\phi}{\sin\phi}\,d\phi = \int \frac{x\phi}{\sin\phi}\,d\phi + \sum_{t=1}^{\infty} Y_t(x,y)\int \frac{\phi^{2t+1}}{\sin\phi}\,d\phi$$
$$+\, i\left\{\sum_{t=1}^{\infty} Z_t(x,y)\int \frac{\phi^{2t+1}}{\sin\phi}\,d\phi + \int \frac{y\phi}{\sin\phi}\,d\phi\right\}$$

$$\int \frac{\cos(x+iy)\phi}{\sin^3\phi}\,d\phi = \int \frac{d\phi}{\sin^3\phi} + \sum_{t=1}^{\infty} Y_t^c(x,y)\int \frac{\phi^{2t}}{\sin^3\phi}\,d\phi + i\sum_{t=1}^{\infty} Z_t^c(x,y)\int \frac{\phi^{2t}}{\sin^3\phi}\,d\phi$$

$$\int \frac{\sin(x+iy)\phi}{\sin^3\phi}\,d\phi = \int \frac{x\phi}{\sin^3\phi}\,d\phi + \sum_{t=1}^{\infty} y_t(x,y) \int \frac{\phi^{2t+1}}{\sin^3\phi}\,d\phi$$
$$+\, i\left\{\sum_{t=1}^{\infty} z_t(x,y) \int \frac{\phi^{2t+1}}{\sin^3\phi}\,d\phi + \int \frac{y\phi}{\sin^3\phi}\,d\phi\right\}$$

$$\int \frac{\cos(x+iy)\phi}{\sin^n\phi}\,d\phi = R(0,n,\phi) + \sum_{t=1}^{\infty} y_t^c(x,y)\, R(2t,n,\phi)$$
$$+\, i \sum_{t=1}^{\infty} z_t^c(x,y)\, R(2t,n,\phi)$$

$$\int \frac{\sin(x+iy)\phi}{\sin^n\phi}\,d\phi = xR(1,n,\phi) + \sum_{t=1}^{\infty} y_t(x,y)\, R(2t+1,n,\phi)$$
$$+\, i\left\{\sum_{t=1}^{\infty} z_t(x,y)\, R(2t+1,n,\phi) + yR(1,n,\phi)\right\}$$

10) $\int \sin^n\phi \cos(x+iy)\phi\,d\phi$ $\qquad$ n : positive integer

$n=1$: $$\int \sin\phi\cos(x+iy)\phi\,d\phi = -\frac{\cos(1+x+iy)\phi}{2(1+x+iy)} - \frac{\cos(1-x-iy)\phi}{2(1-x-iy)}$$
$$(x+iy)^2 \neq 1$$

$n=2$: $$\int \sin^2\phi \cos(x+iy)\phi\,d\phi$$
$$= \frac{\sin\phi}{(x+iy)^2-4}\left[2\cos\phi\cos(x+iy)\phi + (x+iy)\sin\phi\sin(x+iy)\phi\right]$$
$$-\frac{2}{(x+iy)^2-4}\int \cos(x+iy)\phi\,d\phi$$

$n>2$: $$\int \sin^n\phi\cos(x+iy)\phi\,d\phi$$
$$= \frac{\sin^{n-1}\phi}{n^2-(x+iy)^2}\left[n\cos\phi\cos(x+iy)\phi + (x+iy)\sin\phi\sin(x+iy)\phi\right]$$
$$-\frac{n(n-1)}{n^2-(x+iy)^2}\int \sin^{n-2}\cos(x+iy)\phi\,d\phi$$

11) m : odd number

11.1) $$\sin^m b\phi = \sum_{\mu=1,3,5\ldots}^{m} a_\mu \sin\mu b\phi$$

with: $$a_\mu = \frac{(-1)^{\frac{2m-\mu-1}{2}}}{2^{m-1}} C_m^{\frac{m-\mu}{2}}$$

where $$C_m^{\frac{m-\mu}{2}} = \frac{m!}{(m-\mu)!\left(\frac{m-\mu}{2}-m\right)!}$$

$$\sin^m b\phi \cos c\phi = \sum_{\mu=1,3,5..}^{\infty} \frac{a_\mu}{2}\left[\sin(\mu b+c)\phi + \sin(\mu b-c)\phi\right]$$

$$\sin^m b\phi \sin c\phi = \sum_{\mu=1,3,5..}^{\infty} \frac{a_\mu}{2}\left[\cos(\mu b-c)\phi - \cos(\mu b+c)\phi\right]$$

11.2) $$\cos^m b\phi = \sum_{\mu=1,3,5\ldots}^{m} b_\mu \cos\mu b\phi \qquad b_\mu = \frac{C_m^{\frac{m-\mu}{2}}}{2^{m-1}}$$

$$\cos^m b\phi \cos c\phi = \sum_{\mu=1,3,5\ldots}^{m} \frac{b_\mu}{2}\left[\cos(\mu b+c)\phi + \cos(\mu b-c)\phi\right]$$

$$\cos^m b\phi \sin c\phi = \sum_{\mu=1,3,5\ldots}^{m} \frac{b_\mu}{2}\left[\sin(\mu b+c)\phi - \sin(\mu b-c)\phi\right]$$

12) m : even number

12.1) $$\sin^m b\phi = \sum_{\mu=0,2,4\ldots}^{m} a_\mu \cos\mu b\phi$$

where $a_0 = \dfrac{C_m^{m/2}}{2^m}$ $\qquad a_\mu = \dfrac{(-1)^{\frac{2m-\mu}{2}} C_m^{\frac{m-\mu}{2}}}{2^{m-1}}$

$$C_m^p = \frac{m!}{p!\,(m-p)!}$$

12.2) $$\cos^m b\phi = \sum_{\mu=0,2,4\ldots}^{m} c_\mu \cos \mu b\phi$$

where $c_0 = a_0$ $\qquad c_\mu = a_\mu \, (-1)^{\frac{2m-\mu}{2}}$

13) $$\int \cos(x+iy)\phi \, S_r^m(\phi)\, d\phi = \int \cos(x+iy)\phi \frac{\sin^m(\phi/2)}{\sin^r(\phi\;)}\, d\phi$$

$$= \underset{r=2,3}{H_r}(m, x, y, \phi) + i \underset{r=2,3}{T_r}(m, x, y, \phi)$$

where

$$\underset{r=2,3}{H_r}(m, x, y, \phi) = \sum_{\mu=0,2,4\ldots}^{\infty} \frac{a_\mu}{2} \Big\{ 2R\,(0, r, \phi) + \sum_{t=1}^{\infty} R\,(2t, r, \phi)\Big[Y_t^c\left(\frac{\mu}{2}+x, y\right) + Y_t^c\left(x-\frac{\mu}{2}, y\right)\Big]\Big\}$$

$$\underset{r=2,3}{T_r}(m, x, y, \phi) = -\sum_{\mu=0,2,4\ldots}^{\infty} \frac{a_\mu}{2} \Big\{ \sum_{t=1}^{\infty} R\,(2t, r, \phi) \times \Big[Z_t^c\left(x+\frac{\mu}{2}, y\right) + Z_t^c\left(x-\frac{\mu}{2}, y\right)\Big]\Big\}$$

14) $$\int \cos(x+iy)\phi \; C_r^m \phi \, d\phi = \underset{r=2,3}{U_r}(m,x,y,\phi) + i \underset{r=2,3}{V_r}(m,x,y,\phi)$$

Terms U_r (m,x,y,ϕ) and V_r (m,x,y,ϕ) are derived from H_r (m,x,y,ϕ) and T_r (m,x,y,ϕ), respectively, replacing a_μ with $(-1)^{\frac{2m-\mu}{2}}$ a_μ.

15) $$\int \sin(x+iy)\phi \; S_r^m(\phi)\, d\phi = \underset{r=2,3}{H_r^c}(m,x,y,\phi) + i \underset{r=2,3}{T_r^c}(m,x,y,\phi)$$

where:

$$\underset{r=2,3}{H_r^c}(m,x,y,\phi) = \sum_{\mu=0,2,4\ldots}^{m} \frac{a}{2}\mu \left\{ 2x\, R(1,r,\phi) + \sum_{t=1}^{\infty} R(2t+1,r,\phi)\left[Y_t\left(x+\frac{\mu}{2},y\right) - Y_t\left(x-\frac{\mu}{2},y\right)\right]\right\}$$

$$\underset{r=2,3}{T_r^c}(m,x,y,\phi) = \sum_{\mu=0,2,4\ldots}^{m} \frac{a}{2}\mu \left\{ 2y\, R(1,r,\phi) + \sum_{t=1}^{\infty} R(2t+1,r,\phi)\left[Z_t\left(x+\frac{\mu}{2},y\right) + Z_t\left(x-\frac{\mu}{2},y\right)\right]\right\}$$

16) $$\int \sin(x+iy)\phi \; C_r^m \phi \, d\phi = U_r^c(m,x,y,\phi) + iV_r^c(m,x,y,\phi)$$

The U_r^c (m,x,y,ϕ) and V_r^c (m,x,y,ϕ) terms are derived from H_r^c (m,x,y,ϕ) and T_r^c (m,x,y,ϕ), respectively, replacing a_μ with a_μ $(-1)^{\frac{2m-\mu}{2}}$.

17) $$H_r(m, x, y, \phi) = \int \cos x\phi \; \mathrm{ch}\, y\phi \; S_r^m(\phi)\, d\phi$$

$$T_r(m, x, y, \phi) = \int \sin x\phi \; \mathrm{sh}\, y\phi \; S_r^m(\phi)\, d\phi$$

$$U_r(m, x, y, \phi) = \int \cos x\phi \; \mathrm{ch}\, y\phi \; C_r^m(\phi)\, d\phi$$

$$V_r(m, x, y, \phi) = \int \sin x\phi \; \mathrm{sh}\, y\phi \; C_r^m(\phi)\, d\phi$$

$$H_r^c(m, x, y, \phi) = \int \sin x\phi \; \mathrm{ch}\, y\phi \; S_r^m(\phi)\, d\phi$$

$$T_r^c(m, x, y, \phi) = \int \cos x\phi \; \mathrm{sh}\, y\phi \; S_r^m(\phi)\, d\phi$$

$$U_r^c(m, x, y, \phi) = \int \sin x\phi \; \mathrm{ch}\, y\phi \; C_r^m(\phi)\, d\phi$$

$$V_r^c(m, x, y, \phi) = \int \cos x\phi \; \mathrm{sh}\, y\phi \; C_r^m(\phi)\, d\phi$$

These functions are obtained using the numerical iterations such that:

$$\frac{H_r(m, x, y, \phi)_k - H_r(m, x, y, \phi)_{k-1}}{H_r(m, x, y, \phi)_k} \leqslant 1 \times 10^{-9}$$

$$\frac{T_r(m, x, y, \phi)_k - T_r(m, x, y, \phi)_{k-1}}{T_r(m, x, y, \phi)_k} \leqslant 1 \times 10^{-9}$$

$$\frac{U_r(m, x, y, \phi)_k - U_r(m, x, y, \phi)_{k-1}}{U_r(m, x, y, \phi)_k} \leqslant 1 \times 10^{-9}$$

$$\frac{V_r(m, x, y, \phi)_k - V_r(m, x, y, \phi)_{k-1}}{V_r(m, x, y, \phi)_k} \leqslant 1 \times 10^{-9}$$

$$\frac{H_r^c(m,x,y,\phi)_k - H_r^c(m,x,y,\phi)_{k-1}}{H_r^c(m,x,y,\phi)_k} \leqslant 1 \times 10^{-9}$$

$$\frac{T_r^c(m,x,y,\phi)_k - T_r^c(m,x,y,\phi)_{k-1}}{T_r^c(m,x,y,\phi)_k} \leqslant 1 \times 10^{-9}$$

$$\frac{U_r^c(m,x,y,\phi)_k - U_r^c(m,x,y,\phi)_{k-1}}{U_r^c(m,x,y,\phi)_k} \leqslant 1 \times 10^{-9}$$

$$\frac{V_r^c(m,x,y,\phi)_k - V_r^c(m,x,y,\phi)_{k-1}}{V_r^c(m,x,y,\phi)_k} \leqslant 1 \times 10^{-9}$$

18) $$H_1(-2,x,y,\phi) = \int \cos x\phi \ \mathrm{ch} y\phi \ S_1^{-2}(\phi)\, d\phi$$

$$= 2\left[H_2\left(0, 2x+1, 2y, \frac{\phi}{2}\right) + H_2\left(0, 2x-1, 2y, \frac{\phi}{2}\right)\right]$$

$$T_1(-2,x,y,\phi) = \int \sin x\phi \ \mathrm{sh} y\phi \ S_1^{-2}(\phi)\, d\phi$$

$$= 2\left[T_2\left(0, 2x+1, 2y, \frac{\phi}{2}\right) + T_2\left(0, 2x-1, 2y, \frac{\phi}{2}\right)\right]$$

19) $$H_2(0,x,y,\phi) = \int \cos x\phi \ \mathrm{ch} y\phi \ \frac{1}{\sin\phi}\, d\phi$$

$$T_2(0,x,y,\phi) = \int \sin x\phi \ \mathrm{sh} y\phi \ \frac{1}{\sin\phi}\, d\phi$$

$$H_2^c(0,x,y,\phi) = \int \sin x\phi \ \mathrm{ch} y\phi \ \frac{1}{\sin\phi}\, d\phi$$

$$T_2^c(0,x,y,\phi) = \int \cos x\phi \ \mathrm{sh} y\phi \ \frac{1}{\sin\phi}\, d\phi$$

20) $$E_n(x,y,\phi) = \int \sin^n\phi \, \sin x\phi \, \mathrm{ch}\, y\phi \, d\phi$$

$$= \frac{\sin^{n-1}\phi}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{ n\cos\phi\big[(x^2-y^2-n^2)\sin x\phi \, \mathrm{ch}\, y\phi + 2xy\cos x\phi \, \mathrm{sh}\, y\phi\big]$$
$$- \sin\phi \cos x\phi \, \mathrm{ch}\, y\phi\,[(x^2-n^2)x + xy^2] + \sin\phi \sin x\phi \, \mathrm{sh}\, y\phi\,[x^2y + y(y^2+n^2)]\Big\}$$

$$- \frac{n(n-1)}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{(x^2-y^2-n^2)E_{n-2}(x,y,\phi) + 2xy\, I_{n-2}(x,y,\phi)\Big\}$$

21) $$I_n(x,y,\phi) = \int \sin^n\phi \, \cos x\phi \, \mathrm{sh}\, y\phi \, d\phi$$

$$= \frac{\sin^{n-1}\phi}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{ n\cos\phi\big[(x^2-y^2-n^2)\cos x\phi \, \mathrm{sh}\, y\phi - 2xy \sin x\phi \, \mathrm{ch}\, y\phi\big]$$

$$+ \sin\phi \cos x\phi \, \mathrm{ch}\, y\phi\,[x^2y + (y^2+n^2)y] + \sin\phi \sin x\phi \, \mathrm{sh}\, y\phi\,[(x^2-n^2)x + xy^2]\Big\}$$

$$- \frac{n(n-1)}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{(x^2-y^2-n^2) I_{n-2} - 2xy\, E_{n-2}(x,y,\phi)\Big\}$$

22) $$E^c_n(x,y,\phi) = \int \sin^n\phi \, \cos x\phi \, \mathrm{ch}\, y\phi \, d\phi$$
$$= \frac{\sin^{n-1}\phi}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{ n\cos\phi\big[(x^2-y^2-n^2)\cos x\phi \, \mathrm{ch}\, y\phi - 2xy \sin x\phi \, \mathrm{sh}\, y\phi\big]$$

$$+ \sin\phi \sin x\phi \, \mathrm{ch}\, y\phi\,[(x^2-n^2)x + xy^2] + \sin\phi \cos x\phi \, \mathrm{sh}\, y\phi\,[x^2y + y(y^2+n^2)]\Big\}$$

$$- \frac{n(n-1)}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{(x^2-y^2-n^2)E^c_{n-2}(x,y,\phi) - 2xy\, I^c_{n-2}(x,y,\phi)\Big\}$$

23) $$I_n^c(x,y,\phi) = \int \sin^n\phi \; \sin x\phi \; \mathrm{sh}\, y\phi \; d\phi$$

$$= \frac{\sin^{n-1}\phi}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{ n\cos\phi \, [(x^2-y^2-n^2)\sin x\phi \; \mathrm{sh}\, y\phi + 2xy \cos x\phi \; \mathrm{ch}\, y\phi]$$

$$+ \sin\phi \, \sin x\phi \; \mathrm{ch}\, y\phi \, [x^2y+(y^2+n^2)y]$$

$$- \sin\phi \, \cos x\phi \; \mathrm{sh}\, y\phi \, [(x^2-n^2)x + xy^2] \Big\}$$

$$- \frac{n(n-1)}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{(x^2-y^2-n^2) I^c_{n-2}(x,y,\phi) + 2xy E^c_{n-2}(x,y,\phi)\Big\}$$

24) $$F_n(x,y,\phi) = \int \cos^n\phi \; \sin x\phi \; \mathrm{ch}\, y\phi \; d\phi$$

$$= \frac{-\cos^{n-1}\phi}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{ n\sin\phi \, [(x^2-y^2-n^2)\sin x\phi \; \mathrm{ch}\, y\phi + 2xy \cos x\phi \; \mathrm{sh}\, y\phi]$$

$$+ \cos\phi \, \cos x\phi \; \mathrm{ch}\, y\phi \, [(x^2-n^2)x + xy^2]$$

$$- \cos\phi \, \sin x\phi \; \mathrm{sh}\, y\phi \, [x^2y + y(y^2+n^2)] \Big\}$$

$$- \frac{n(n-1)}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{(x^2-y^2) F_{n-2}(x,y,\phi) + 2xy J_{n-2}(x,y,\phi)\Big\}$$

25) $$J_n(x,y,\phi) = \int \cos^n\phi \; \cos x\phi \; \mathrm{sh}\, y\phi \; d\phi$$

$$= \frac{-\cos^{n-1}\phi}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{ n\sin\phi \, [(x^2-y^2-n^2)\cos x\phi \; \mathrm{sh}\, y\phi - 2xy \sin x\phi \; \mathrm{ch}\, y\phi]$$

$$- \cos\phi \, \cos x\phi \; \mathrm{ch}\, y\phi \, [x^2y + (y^2+n^2)y]$$

$$- \cos\phi \, \sin x\phi \; \mathrm{sh}\, y\phi \, [(x^2-n^2)x + xy^2] \Big\}$$

$$- \frac{n(n-1)}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{(x^2-y^2-n^2) J_{n-2}(x,y,\phi) - 2xy F_{n-2}(x,y,\phi)\Big\}$$

26) $F_n^c(x,y,\phi) = \int \cos^n\phi \cos x\phi \operatorname{ch} y\phi \, d\phi$

$$= \frac{-\cos^{n-1}\phi}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{ n\sin\phi\,[(x^2-y^2-n^2)\cos x\phi \operatorname{ch} y\phi - 2xy\sin x\phi \operatorname{sh} y\phi]$$
$$- \cos\phi \sin x\phi \operatorname{ch} y\phi\,[(x^2-n^2)x + xy^2]$$
$$- \cos\phi \cos x\phi \operatorname{sh} y\phi\,[x^2y + y(y^2+n^2)]\Big\}$$
$$- \frac{n(n-1)}{[(x^2-y^2-n^2)^2+4x^2y^2}\Big\{(x^2-y^2-n^2)F_{n-2}^c(x,y,\phi) - 2xy\,J_{n-2}^c(x,y,\phi)\Big\}$$

27) $J_n^c(x,y,\phi) = \int \cos^n\phi \sin x\phi \operatorname{sh} y\phi \, d\phi$

$$= \frac{-\cos^{n-1}\phi}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{ n\sin\phi\,[(x^2-y^2-n^2)\sin x\phi \operatorname{sh} y\phi + 2xy\cos x\phi \operatorname{ch} y\phi]$$
$$- \cos\phi \sin x\phi \operatorname{ch} y\phi\,[x^2y + (y^2+n^2)y]$$
$$+ \cos\phi \cos x\phi \operatorname{sh} y\phi\,[(x^2-n^2)x + xy^2]\Big\}$$
$$- \frac{n(n-1)}{[(x^2-y^2-n^2)^2+4x^2y^2]}\Big\{(x^2-y^2-n^2)J_{n-2}^c(x,y,\phi) + 2xy\,F_{n-2}^c(x,y,\phi)\Big\}$$

28) $\int \cos(x+iy)\phi \; S_1^m(\phi)\, d\phi$

$$= \int \sin^m t \sin 2(x+1)t \operatorname{ch} 2yt\,dt - \int \sin^m t \sin 2(x-1)t \operatorname{ch} 2yt\,dt$$

$$+ i\Big[\int \sin^m t \cos 2(x+1)t \operatorname{sh} 2yt\,dt - \int \sin^m t \cos 2(x-1)t \operatorname{sh} 2yt\,dt\Big]$$

where $t = \phi/2$

$$= E_m(2x+2, 2y, \phi/2) - E_m(2x-2, 2y, \phi/2)$$
$$+ i\,[I_m(2x+2, 2y, \phi/2) - I_m(2x-2, 2y, \phi/2)]$$
$$= H_1^s(m, x, y, \phi) + i\,T_1^s(m, x, y, \phi)$$

29) $\int \sin(x+iy)\phi\ S_1^m(\phi)\,d\phi$

$$= \int \cos 2(x-1)t\ \mathrm{ch}2yt\ \sin^m t\,dt - \int \cos 2(x+1)t\,\mathrm{ch}2yt\sin^m t\,dt$$

$$+ i\left[\int \sin 2(x+1)t\ \mathrm{sh}2yt\ \sin^m t\,dt - \int \sin 2(x-1)t\,\mathrm{sh}2yt\sin^m t\,dt\right]$$

$$= E_m^c(2x-2, 2y, \phi/2) - E_m^c(2x+2, 2y, \phi/2)$$
$$+ i\left[I_1^c(2x+2, 2y, \phi/2) - I_m^c(2x-2, 2y, \phi/2)\right]$$
$$= H_1^c(m, x, y, \phi) + i\,T_1^c(m, x, y, \phi)$$

30) $\int \cos(x+iy)\phi\ C_1^m(\phi)\,d\phi$

$$= \int \sin 2(x+1)t\,\mathrm{ch}2yt\ \cos^m t\,dt - \int \sin 2(x+1)t\ \mathrm{ch}2yt\cos^m t\,dt$$
$$+ i\left[\int \cos 2(x+1)t\ \mathrm{sh}2yt\cos^m t\,dt - \int \cos 2(x-1)t\,\mathrm{sh}2yt\cos^m t\,dt\right]$$

$$= F_m(2x+2, 2y, \phi/2) - F_m(2x-2, 2y, \phi/2)$$
$$+ i\left[J_m(2x+2, 2y, \phi/2) - J_m(2x-2, 2y, \phi/2)\right]$$

$$= U_1(m, x, y, \phi) + i\,V_1(m, x, y, \phi)$$

31) $\int \sin(x+iy)\phi\ C_1^m(\phi)\,d\phi$

$$= \int \cos 2(x-1)t\,\mathrm{ch}2yt\cos^m t\,dt - \int \cos 2(x+1)t\,\mathrm{ch}2yt\cos^m t\,dt$$
$$+ i\left[\int \sin 2(x+1)t\,\mathrm{sh}2yt\cos^m t\,dt - \int \sin 2(x-1)t\,\mathrm{sh}2yt\cos^m t\,dt\right]$$

$$= F_m^c(2x-2, 2y, \phi/2) - F_m^c(2x+2, 2y, \phi/2)$$
$$+ i\left[J_m^c(2x+2, 2y, \phi/2) - J_m^c(2x-2, 2y, \phi/2)\right]$$
$$= U_1^c(m, x, y, \phi) + i\,V_1^c(m, x, y, \phi)$$

32) **Integration functions independent of k, i and j**

$$\Gamma(a, b, \phi_2/2) - \Gamma(a, b, \phi_1/2)$$

$$\begin{cases} \Gamma(c, b, \phi_2/2) - \Gamma(c, b, \phi_1/2) \\ \Gamma(c+2, b, \phi_2/2) - \Gamma(c+2, b, \phi_1/2) \end{cases}$$

$$\begin{cases} \Gamma(b+2, b, \phi_2/2) - \Gamma(b+2, b, \phi_1/2) \\ \Gamma(b+4, b, \phi_2/2) - \Gamma(b+4, b, \phi_1/2) \\ \Gamma(b+6, b, \phi_2/2) - \Gamma(b+6, b, \phi_1/2) \end{cases} \quad \text{independent of } \ell$$

$$\begin{cases} \Gamma(u+1, u-1, \phi_2/2) - \Gamma(u+1, u-1, \phi_1/2) \\ \Gamma(u+3, u-1, \phi_2/2) - \Gamma(u+3, u-1, \phi_1/2) \\ \Gamma(u+5, u-1, \phi_2/2) - \Gamma(u+5, u-1, \phi_1/2) \end{cases} \quad \text{independent of } \ell$$

$$\begin{cases} \Gamma(u+2\ell+3, u-1, \phi_2/2) - \Gamma(u+2\ell+3, u-1, \phi_1/2) \\ \Gamma(u+2\ell+1, u-1, \phi_2/2) - \Gamma(u+2\ell+1, u-1, \phi_1/2) \end{cases}$$

$$\begin{cases} \Gamma(4-b, -b-2, \phi_2/2) - \Gamma(4-b, -b-2, \phi_1/2) \\ \Gamma(-b, -b-2, \phi_2/2) - \Gamma(-b, -b-2, \phi_1/2) \\ \Gamma(2-b, -b-2, \phi_2/2) - \Gamma(2-b, -b-2, \phi_1/2) \end{cases} \quad \text{independent of } \ell$$

33) Integration functions dependent on ℓ, r, s, k, i and j.

$$HRR(k,j) = H_r(m, \alpha_s(k,j), b_3, \phi_2) - H_r(m, \alpha_s(k,j), b_3, \phi_1)$$
$$TRR(k,j) = T_r(m, \alpha_s(k,j), b_3, \phi_2) - T_r(m, \alpha_s(k,j), b_3, \phi_1)$$

$$HRR1(k,j) = H_r(m, \alpha_s(k,j)-1, b_3, \phi_2) - H_r(m, \alpha_s(k,j)-1, b_3, \phi_1)$$
$$TRR1(k,j) = T_r(m, \alpha_s(k,j)-1, b_3, \phi_2) - T_r(m, \alpha_s(k,j)-1, b_3, \phi_1)$$

$$HRR2(k,j) = H_r(m, \alpha_s(k,j)+1, b_3, \phi_2) - H_r(m, \alpha_s(k,j)+1, b_3, \phi_1)$$
$$TRR2(k,j) = T_r(m, \alpha_s(k,j)+1, b_3, \phi_2) - T_r(m, \alpha_s(k,j)+1, b_3, \phi_1)$$

$$HRRL(k,j) = H_r(m+2\ell_r, \alpha_s(k,j), b_3, \phi_2) - H_r(m+2\ell, \alpha_s(k,j), b_3, \phi_1)$$
$$TRRL(k,j) = T_r(m+2\ell_r, \alpha_s(k,j), b_3, \phi_2) - T_r(m+2\ell, \alpha_s(k,j), b_3, \phi_1)$$

with $k = 1,6$ $\quad j = 2,3$

$$URR(k,i) = U_r(m, \alpha_s(k,i), b_3, \phi_2) - U_r(m, \alpha_s(k,i), b_3, \phi_1)$$
$$VRR(k,i) = V_r(m, \alpha_s(k,i), b_3, \phi_2) - V_r(m, \alpha_s(k,i), b_3, \phi_1)$$

$$URR1(k,i) = U_r(m, \alpha_s(k,i)-1, b_3, \phi_2) - U_r(m, \alpha_s(k,i)-1, b_3, \phi_1)$$
$$VRR1(k,i) = V_r(m, \alpha_s(k,i)-1, b_3, \phi_2) - V_r(m, \alpha_s(k,i)-1, b_3, \phi_1)$$

$$URR2(k,i) = U_r(m, \alpha_s(k,i)+1, b_3, \phi_2) - U_r(m, \alpha_s(k,i)+1, b_3, \phi_1)$$
$$VRR2(k,i) = V_r(m, \alpha_s(k,i)+1, b_3, \phi_2) - V_r(m, \alpha_s(k,i)+1, b_3, \phi_1)$$

with $k = 1,6$ $\quad i = 2,3$

$$CER1(k,i) = E^{c}_{2n-3}\left(\alpha_r(k,i) - \alpha_s(k,j), 2b_3, \phi_2\right) - E^{c}_{2n-3}\left(\alpha_r(k,i) - \alpha_s(k,j), 2b_3, \phi_1\right)$$

$$CIR1(k,i) = I^{c}_{2n-3}\left(\alpha_r(k,i) - \alpha_s(k,j), 2b_3, \phi_2\right) - I^{c}_{2n-3}\left(\alpha_r(k,i) - \alpha_s(k,j), 2b_3, \phi_1\right)$$

$$CER2(k,i) = E^{c}_{2n-3}\left(\alpha_r(k,i) + \alpha_s(k,j), 2b_3, \phi_2\right) - E^{c}_{2n-3}\left(\alpha_r(k,i) + \alpha_s(k,j), 2b_3, \phi_1\right)$$

$$CIR2(k,i) = I^{c}_{2n-3}\left(\alpha_r(k,i) + \alpha_s(k,j), 2b_3, \phi_2\right) - I^{c}_{2n-3}\left(\alpha_r(k,i) + \alpha_s(k,j), 2b_3, \phi_1\right)$$

$$CER3(k,i) = E^{c}_{2n-3}\left(\alpha_r(k,i) - \alpha_s(k,j), 0, \phi_2\right) - E^{c}_{2n-3}\left(\alpha_r(k,i) - \alpha_s(k,j), 0, \phi_1\right)$$

$$CER4(k,i) = E^{c}_{2n-3}\left(\alpha_r(k,i) + \alpha_s(k,j), 0, \phi_2\right) - E^{c}_{2n-3}\left(\alpha_r(k,i) + \alpha_s(k,j), 0, \phi_1\right)$$

34)
$$\begin{aligned}
HNR &= H_r\ (m, \gamma_s, b_3, \phi_2) \\
TNR &= T_r\ (m, \gamma_s, b_3, \phi_2) \\
HNR2 &= H_r\ (m+2, \gamma_s, b_3, \phi_2) \\
TNR2 &= T_r\ (m+2, \gamma_s, b_3, \phi_2)
\end{aligned}$$

$$\begin{aligned}
UNR &= U_r\ (m, \gamma_s, b_3, \phi_2) \\
VNR &= V_r\ (m, \gamma_s, b_3, \phi_2) \\
UNR2 &= U_r\ (m-2, \gamma_s, b_3, \phi_2) \\
VNR2 &= V_r\ (m-2, \gamma_s, b_3, \phi_2) \\
UNR3 &= U_r\ (m-2, \gamma_s+2, b_3, \phi_2) \\
VNR3 &= V_r\ (m-2, \gamma_s+2, b_3, \phi_2) \\
UNR4 &= U_r\ (m-2, \gamma_s-2, b_3, \phi_2) \\
VNR4 &= V_r\ (m-2, \gamma_s-2, b_3, \phi_2)
\end{aligned}$$

$$HN_1 = H_1 \; (2n \; , \; \gamma_1 \; , b_3 \; , \phi_2)$$

$$TN_1 = T_1 \; (2n \; , \; \gamma_1 \; , b_3 \; , \phi_2).$$

$$HN_2 = H_1 \; (2n + 2 \; , \gamma_1 \; , b_3 \; , \phi_2)$$

$$TN_2 = T_1 \; (2n + 2 \; , \gamma_1 \; , b_3 \; , \phi_2)$$

$$H12 = H_1 \; (2n + 2\ell_r - 2 \; , \gamma_s \; , b_3 \; , \phi_2)$$

$$T12 = T_1 \; (2n + 2\ell_r - 2 \; , \gamma_s \; , b_3 \; , \phi_2)$$

$$H22 = H_2 \; (2n + 2\ell_r + 2 \; , \gamma_s \; , b_3 \; , \phi_2)$$

$$T22 = T_2 \; (2n + 2\ell_r + 2 \; , \gamma_s \; , b_3 \; , \phi_2)$$

$$H11 = H_1 \; (2n + 2\ell_r \; , \gamma_1 \; , b_3 \; , \phi_2)$$

$$T11 = T_1 \; (2n + 2\ell_r \; , \gamma_1 \; , b_3 \; , \phi_2)$$

$$UN1 = U_1 \; (2n \; , \gamma_1 \; , b_3 \; , \phi_2)$$

$$VN1 = V_1 \; (2n \; , \gamma_1 \; , b_3 \; , \phi_2)$$

$$UN2 = U_1 \; (2n - 2 \; , \gamma_1 \; , b_3 \; , \phi_2)$$

$$VN2 = V_1 \; (2n - 2 \; , \gamma_1 \; , b_3 \; , \phi_2)$$

$$UN22 = U_1 \; (2n - 2 \; , \gamma_1 + 2 \; , b_3 \; , \phi_2)$$

$$VN22 = V_1 \; (2n - 2 \; , \gamma_1 + 2 \; , b_3 \; , \phi_2)$$

$$UN23 = U_1 \; (2n - 2 \; , \gamma_1 - 2 \; , b_3 \; , \phi_2)$$

$$VN23 = V_1 \; (2n - 2 \; , \gamma_1 - 2 \; , b_3 \; , \phi_2)$$

$$CER1(k) = E^{c}_{2q+1}(\gamma_r - \gamma_s, 2b_3, \phi_2) - E^{c}_{2q+1}(\gamma_r - \gamma_s, 2b_3, \phi_1)$$
$$CIR1(k) = I^{c}_{2q+1}(\gamma_r - \gamma_s, 2b_3, \phi_2) - I^{c}_{2q+1}(\gamma_r - \gamma_s, 2b_3, \phi_1)$$

$$CER2(k) = E^{c}_{2q+1}(\gamma_r + \gamma_s, 2b_3, \phi_2) - E^{c}_{2q+1}(\gamma_r + \gamma_s, 2b_3, \phi_1)$$
$$CIR2(k) = I^{c}_{2q+1}(\gamma_r + \gamma_s, 2b_3, \phi_2) - I^{c}_{2q+1}(\gamma_r + \gamma_s, 2b_3, \phi_1)$$

$$CER3(k) = E^{c}_{2q+1}(\gamma_r + \gamma_s, 0, \phi_2) - E^{c}_{2q+1}(\gamma_r + \gamma_s, 0, \phi_1)$$

$$CER4(k) = E^{c}_{2q+1}(\gamma_r - \gamma_s, 0, \phi_2) - E^{c}_{2q+1}(\gamma_r - \gamma_s, 0, \phi_1)$$

where $k = 1$ $\quad q = n - 1$

$= 2$ $\quad = n$

$= 3$ $\quad = n - 1$

APPENDIX V

LIST OF MATRICES

This list contains the matrices used in the foregoing chapters.

Table	Matrix	Dimension
1 :	[T]	(3 x 3)
2 :	[R]	(3 x 8)
3 :	[τ]	(4 x 4)
4 :	[A_{σ}]	(4 x 8)
5 :	[R_{∞}]	(3 x 8)
6 :	[$A_{\sigma\infty}$]	(4 x 8)
7 :	[Q]	(6 x 8)
8 :	[Q_{∞}]	(6 x 8)
9 :	[P]	(6 x 6)
10:	[DLAM]	(8 x 8)
11:	[A_r]	(6 x 4), r = 1,3
12:	[B_r]	(6 x 4), r = 1,3
13:	[$A_r^{\subset}$]	(6 x 4), r = 1,4
14:	[$B_r^{\subset}$]	(6 x 4), r = 1,4
15:	[$A_r^{\cup}$]	(6 x 4), r = 1,3
16:	[$B_r^{\cup}$]	(6 x 4), r = 1,3
17:	[α_r]	(6 x 4), r = 1,4
18:	[Γ_r]	(3 x 4), r = 1,3
19:	[Δ_r]	(3 x 4), r = 1,3
20:	[$\Gamma_r^{\subset}$]	(3 x 4), r = 1,4

21:	$[\Delta_r^{=}]$	(3 x 4), r = 1,4
22:	$[\Gamma_r^{\vee}]$	(3 x 4), r = 1,3
23:	$[\Delta_r^{\vee}]$	(3 x 4), r = 1,3

The other matrices are:

$$[C_r] = [A_r]^T [P]$$
$$[D_r] = [B_r]^T [P]$$
$$[C_r^c] = [A_r^c] [P]$$
$$[D_r^c] = [B_r^c] [P]$$
$$[C_r^v] = [A_r^v] [P]$$
$$[D_r^v] = [B_r^v] [P]$$

$$[A] = \begin{bmatrix} [A\phi]_i & [A\phi s]_i \\ [A\phi]_j & [A\phi s]_j \end{bmatrix}$$

$$[N] = [R] [A]^{-1} \quad [R_s] [A]^{-1}$$
$$[B] = [Q_s] [A]^{-1}$$

$$[ST] = \begin{bmatrix} \begin{matrix} [T] & [0] \\ [0] & [T] \end{matrix} & [0] \\ [0] & \begin{matrix} [T] & [0] \\ [0] & [T] \end{matrix} \end{bmatrix}$$

$$\times \begin{bmatrix} [P] & [Q_s]_i & [A]^{-1} \\ [P] & [Q_s]_j & [A]^{-1} \end{bmatrix}$$

Table 1: Matrix [T]

$$[T] = \begin{bmatrix} \cos n\theta & 0 & 0 \\ 0 & \cos n\theta & 0 \\ 0 & 0 & \sin n\theta \end{bmatrix}.$$

Table 2: Matrix [R]

The elements on the left half are:

$$R(1,1) = g_1 (n - \mu_1 - 1)(n + \mu_1) P_{\mu_1}^{n-1} - n \operatorname{cotg}\phi \, P_{\mu_1}^{n}$$

$$R(2,1) = P_{\mu_1}^{n}$$

$$R(3,1) = -n g_1 \operatorname{cosec}\phi \, P_{\mu_1}^{n}$$

$$R(1,2) = (g_2 \gamma_n + g_3 \beta_n) R_e (P_{\mu_2}^{n-1}) + (g_3 \gamma_n - g_2 \beta_n) I_m (P_{\mu_2}^{n-1}) - n g_2 \operatorname{cotg}\phi \, R_e (P_{\mu_2}^{n}) - n g_3 \operatorname{cotg}\phi \, I_m (P_{\mu_2}^{n})$$

$$R(2,2) = R_e (P_{\mu_2}^{n})$$

$$R(3,2) = -[n g_2 R_e (P_{\mu_2}^{n}) + n g_3 I_m (P_{\mu_2}^{n})] \operatorname{cosec}\phi$$

$$R(1,3) = (g_2 \gamma_n + g_3 \beta_n) I_m (P_{\mu_2}^{n-1}) + (g_2 \beta_n - g_3 \gamma_n) R_e (P_{\mu_2}^{n-1}) - n g_2 \operatorname{cotg}\phi \, I_m (P_{\mu_2}^{n}) + n g_3 \operatorname{cotg}\phi \, R_e (P_{\mu_2}^{n})$$

$$R(2,3) = I_m (P_{\mu_2}^{n})$$

$$R(3,3) = -[n g_2 I_m (P_{\mu_2}^{n}) - n g_3 R_e (P_{\mu_2}^{n})] \operatorname{cosec}\phi$$

$$R(1,4) = -\frac{n^2}{2} \operatorname{cosec}\phi \, P_1^n$$

$$R(2,4) = 0$$

$$R(3,4) = \frac{n}{2}(n-2)(n+1)P_1^{n-1} - \frac{n^2}{2} \operatorname{cotg}\phi \, P_1^n$$

The elements on the right half of [R] are obtained from the left-side elements by replacing the associated Legendre function of the first kind P with the Q function of the second kind.

Table 3: Matrix [τ]

$$[\tau] = \begin{bmatrix} \cos n\theta & 0 & 0 & 0 \\ 0 & \cos n\theta & 0 & 0 \\ 0 & 0 & \cos n\theta & 0 \\ 0 & 0 & 0 & \sin n\theta \end{bmatrix}$$

Table 4: Matrix $[A_\omega]$

The elements on the left half are:

$$A\phi(1,1) = g_1 (n-\mu_1-1)(n+\mu_1) P_{\mu_1}^{n-1} - n\operatorname{cotg}\phi \, P_{\mu_1}^n$$
$$A\phi(2,1) = P_{\mu_1}^n$$
$$A\phi(3,1) = (n-\mu_1-1)(n+\mu_1) P_{\mu_1}^{n-1} - n\operatorname{cotg}\phi \, P_{\mu_1}^n$$
$$A\phi(4,1) = -n g_1 \operatorname{cosec}\phi \, P_{\mu_1}^n$$

$$A\phi\,(1,2) = (g_2\gamma_n + g_3\beta_n)\,Re\,(P_{\mu_2}^{n-1}) + (g_3\gamma_n - g_2\beta_n)\,I_m\,(P_{\mu_2}^{n-1})$$
$$\quad - ng_2\,\mathrm{cotg}\phi\,Re\,(P_{\mu_2}^{n}) - ng_3\,\mathrm{cotg}\phi\,Im\,(P_{\mu_2}^{n})$$

$$A\phi\,(2,2) = Re\,(P_{\mu_2}^{n})$$

$$A\phi\,(3,2) = \gamma_n Re\,(P_{\mu_2}^{n-1}) - \beta_n I_m\,(P_{\mu_2}^{n-1}) - n\,\mathrm{cotg}\phi\,Re\,(P_{\mu_2}^{n})$$

$$A\phi\,(4,2) = -\left[ng_2\,Re\,(P_{\mu_2}^{n}) + ng_3\,I_m\,(P_{\mu_2}^{n})\right]\mathrm{cosec}\phi$$

$$A\phi\,(1,3) = (g_2\gamma_n + g_3\beta_n)\,I_m\,(P_{\mu_2}^{n-1}) - (g_3\gamma_n - g_2\beta_n)\,Re\,(P_{\mu_2}^{n-1})$$
$$\quad - ng_2\,\mathrm{cotg}\phi\,I_m\,(P_{\mu_2}^{n}) + ng_3\,\mathrm{cotg}\phi\,Re\,(P_{\mu_2}^{n})$$

$$A\phi\,(2,3) = I_m\,(P_{\mu_2}^{n})$$

$$A\phi\,(3,3) = \gamma_n I_m\,(P_{\mu_2}^{n-1}) + \beta_n Re\,(P_{\mu_2}^{n-1}) - n\,\mathrm{cotg}\phi\,I_m\,(P_{\mu_2}^{n})$$

$$A\phi\,(4,3) = -\left[ng_2\,I_m\,(P_{\mu_2}^{n}) - ng_3\,Re\,(P_{\mu_2}^{n})\right]\mathrm{cosec}\phi$$

$$A\phi\,(1,4) = -\frac{n}{2}\,\mathrm{cosec}\phi\,P_1^{n}$$

$$A\phi\,(2,4) = 0$$

$$A\phi\,(3,4) = 0$$

$$A\phi\,(4,4) = \frac{n}{2}(n-2)(n+1)P_1^{n-1} - \frac{n^2}{2}\,\mathrm{cotg}\phi\,P_1^{n}$$

The elements on the right half of $[A_\infty]$ are obtained from the left-side elements, by replacing the associated Legendre functions of the first kind P with Q of the second kind.

<u>Table 5</u>: Matrix $[R_s]$

The elements on the left half are:

$$R_S\ (1,1) = (n-2)(n+1)\,P_1^{n-1} - n\cot g\phi\ P_1^{n}$$
$$R_S\ (2,1) = P_1^{n}$$
$$R_S\ (3,1) = -\,n\,\mathrm{cosec}\phi\ P_1^{n}$$

$$R_S\ (1,2) = -(1+\nu)\,\mathrm{Re}\,(P_{\mu_2}^{n-1}) + g_3\,(n-0.5)^2\,\mathrm{Im}\,(P_{\mu_2}^{n-1}) - n g_3 \cot g\phi\ \mathrm{Im}\,(P_{\mu_2}^{n})$$
$$R_S\ (2,2) = \mathrm{Re}\,(P_{\mu_2}^{n})$$
$$R_S\ (3,2) = -\,n g_3\ \mathrm{cosec}\,\phi\ \mathrm{Im}\,(P_{\mu_2}^{n})$$

$$R_S\ (1,3) = -(1+\nu)\,\mathrm{Im}\,(P_{\mu_2}^{n-1}) - g_3\,(n-0.5)^2\,\mathrm{Re}\,(P_{\mu_2}^{n-1}) + n g_3 \cot g\,\phi\ \mathrm{Re}\,(P_{\mu_2}^{n})$$
$$R_S\ (2,3) = \mathrm{Im}\,(P_{\mu_2}^{n})$$
$$R_S\ (3,3) = n g_3\ \mathrm{cosec}\phi\ \mathrm{Re}\,(P_{\mu_2}^{n})$$

$$R_S\ (1,4) = -\frac{n^2}{2}\ \mathrm{cosec}\,\phi\ P_1^{n}$$

$$R_S\ (2,4) = 0$$
$$R_S\ (3,4) = \frac{n}{2}(n-2)(n+1)\,P_1^{n-1} - \frac{n^2}{2}\cot g\phi\ P_1^{n}$$

The elements on the right half of $[R_s]$ are obtained from the left-side elements by replacing the associated Legendre function of the first kind P with the Q of the second kind.

<u>Table 6</u>: Matrix $[A_{\phi s}]$

$$A\phi s\ (1,1) = R_s\ (1,1)$$
$$A\phi s\ (2,1) = R_s\ (2,1)$$
$$A\phi s\ (3,1) = R_s\ (1,1)$$
$$A\phi s\ (4,1) = R_s\ (3,1)$$

$$A\phi s\ (1,2) = R_s\ (1,2)$$
$$A\phi s\ (2,2) = R_s\ (2,2)$$

$$A\phi s\ (3,2) = (n-0.5)^2\, Re\, (P_{\mu_2}^{n-1}) + \sqrt{\vartheta\,(1-\nu^2)}\, Im\, (P_{\mu_2}^{n-1}) - n\,\mathrm{cotg}\,\phi\ Re\ (P_{\mu_2}^{n})$$
$$A\phi s\ (4,2) = R_s\ (3,2)$$

$$A\phi s\ (1,3) = R_s\ (1,3)$$
$$A\phi s\ (2,3) = R_s\ (2,3)$$

$$A\phi s\ (3,3) = (n-0.5)^2\, Im\, (P_{\mu_2}^{n-1}) - \sqrt{\vartheta\,(1-\nu^2)}\, Re\, (P_{\mu_2}^{n-1}) - n\,\mathrm{cotg}\,\phi\ Im\ (P_{\mu_2}^{n})$$
$$A\phi s\ (4,3) = R_s\ (3,3)$$

$$A\phi s\ (1,4) = R_s\ (1,4)$$
$$A\phi s\ (2,4) = R_s\ (2,4)$$
$$A\phi s\ (3,4) = 0$$
$$A\phi s\ (4,4) = R_s\ (3,4)$$

The elements on the right half of $[A_{\phi\infty}]$ are obtained from the left-side elements by replacing the associated Legendre function of the first kind P with the Q function of the second kind.

Table 7: Matrix [Q]

$$Q(1,1) = \frac{g_1}{r}\left[(n-\mu_1-1)(n+\mu_1) + n\left(\operatorname{cosec}^2\phi + n\operatorname{cotg}^2\phi\right) + \frac{1}{g_1}\right] P^n_{\mu_1} - \frac{g_1}{r}(n-\mu_1-1)(n+\mu_1)\operatorname{cotg}\phi\, P^{n-1}_{\mu_1}$$

$$Q(2,1) = \frac{1}{r}\left[1 - ng_1\left(n\operatorname{cosec}^2\phi + \operatorname{cotg}^2\phi\right)\right] P^n_{\mu_1} + \frac{g_1}{r}(n-\mu_1-1)(n+\mu_1)\operatorname{cotg}\phi\, P^{n-1}_{\mu_1}$$

$$Q(3,1) = \frac{2n}{r} g_1 (n+1)\operatorname{cosec}\phi\operatorname{cotg}\phi\, P^n_{\mu_1} - \frac{2n}{r} g_1 (n-\mu_1-1)(n+\mu_1)\operatorname{cosec}\phi\, P^{n-1}_{\mu_1}$$

$$Q(4,1) = \frac{(g_1-1)}{r}\left[(n-\mu_1-1)(n+\mu_1) + n\left(\operatorname{cosec}^2\phi + n\operatorname{cotg}^2\phi\right)\right] P^n_{\mu_1}$$
$$- \frac{(g_1-1)}{r}(n-\mu_1-1)(n+\mu_1)\operatorname{cotg}\phi\, P^{n-1}_{\mu_1}$$

$$Q(5,1) = \frac{n}{r^2}(1-g_1)(\mathrm{cotg}^2\phi + n\,\mathrm{cosec}^2\phi)\,P^n_{\mu_1} + \frac{(g_1-1)}{r^2}(n-\mu_1-1)(n+\mu_1)\,\mathrm{cotg}\phi\,P^{n-1}_{\mu_1}$$

$$Q(6,1) = \frac{2n}{r^2}(n+1)(g_1-1)\,\mathrm{cosec}\phi\;\mathrm{cotg}\phi\,P^n_{\mu_1} + \frac{2n}{r^2}(1-g_1)(n-\mu_1-1)(n+\mu_1)\,\mathrm{cosec}\phi\,P^{n-1}_{\mu_1}$$

$$\begin{aligned} Q(1,2) = {} & \frac{1}{r}\left[(g_2\gamma_n + g_3\beta_n) + n g_2(\mathrm{cosec}^2\phi + n\,\mathrm{cotg}^2\phi) + 1\right] \mathrm{Re}\,(P^n_{\mu_2}) \\ & + \frac{1}{r}\left[g_3\gamma_n - g_2\beta_n) + n g_3(\mathrm{cosec}^2\phi + n\,\mathrm{cotg}^2\phi) + 1\right] \mathrm{Im}\,(P^n_{\mu_2}) \\ & - \frac{1}{r}(g_2\gamma_n + g_3\beta_n)\,\mathrm{cotg}\phi\,\mathrm{Re}\,(P^{n-1}_{\mu_2}) - \frac{1}{r}(g_3\gamma_n - g_2\beta_n)\,\mathrm{cotg}\phi\,\mathrm{Im}\,(P^{n-1}_{\mu_2}) \end{aligned}$$

$$\begin{aligned} Q(2,2) = {} & \frac{1}{r}\left[1 - n g_2(n\,\mathrm{cosec}^2\phi + \mathrm{cotg}^2\phi)\right] \mathrm{Re}\,(P^n_{\mu_2}) - \frac{n g_3}{r}(n\,\mathrm{cosec}^2\phi + \mathrm{cotg}^2\phi)\,\mathrm{Im}(P^{n-1}_{\mu_2}) \\ & + \frac{1}{r}(g_2\gamma_n + g_3\beta_n)\,\mathrm{cotg}\phi\,\mathrm{Re}\,(P^{n-1}_{\mu_2}) + \frac{1}{r}(g_3\gamma_n - g_2\beta_n)\,\mathrm{cotg}\phi\,\mathrm{Im}\,(P^{n-1}_{\mu_2}) \end{aligned}$$

$$\begin{aligned} Q(3,2) = {} & \frac{2n(n+1)}{r} g_2\,\mathrm{cosec}\phi\,\mathrm{cotg}\phi\,\mathrm{Re}\,(P^n_{\mu_2}) + \frac{2n(n+1)}{r} g_3\,\mathrm{cosec}\phi\,\mathrm{cotg}\phi\,\mathrm{Im}\,(P^{n-1}_{\mu_2}) \\ & - \frac{2n}{r}(g_2\gamma_n + g_3\beta_n)\;\mathrm{cosec}\phi\,\mathrm{Re}\,(P^{n-1}_{\mu_2}) - \frac{2n}{r}(g_3\gamma_n - g_2\beta_n)\,\mathrm{cosec}\phi\,\mathrm{Im}\,(P^{n-1}_{\mu_2}) \end{aligned}$$

$$\begin{aligned} Q(4,2) = {} & \frac{1}{r^2}\left[(g_2-1)\gamma_n + g_3\beta_n + n(g_2-1)(\mathrm{cosec}^2\phi + n\,\mathrm{cotg}^2\phi)\right] \mathrm{Re}\,(P^n_{\mu_2}) \\ & + \frac{1}{r^2}\left[g_3\gamma_n - (g_2-1)\beta_n + n g_3(\mathrm{cosec}^2\phi + n\,\mathrm{cotg}^2\phi)\right] \mathrm{Im}\,(P^n_{\mu_2}) \\ & - \frac{1}{r^2}\left[(g_2-1)\gamma_n + g_3\beta_n\right]\mathrm{cotg}\phi\,\mathrm{Re}\,(P^{n-1}_{\mu_2}) - \frac{1}{r^2}\left[g_3\gamma_n - (g_2-1)\beta_n\right]\mathrm{cotg}\phi\,\mathrm{Im}\,(P^{n-1}_{\mu_2}) \end{aligned}$$

$$\begin{aligned} Q(5,2) = {} & \frac{n}{r^2}(1-g_2)(\mathrm{cotg}^2\phi + n\,\mathrm{cosec}^2\phi)\,\mathrm{Re}\,(P^n_{\mu_2}) - \frac{n g_3}{r^2}(\mathrm{cotg}^2\phi + n\,\mathrm{cosec}^2\phi)\,\mathrm{Im}\,(P^{n-1}_{\mu_2}) \\ & + \frac{1}{r^2}\left[\gamma_n(g_2-1) + g_3\beta_n\right]\mathrm{cotg}\phi\,\mathrm{Re}\,(P^{n-1}_{\mu_2}) + \frac{1}{r^2}\left[g_3\gamma_n - \beta_n(g_2-1)\right]\mathrm{cotg}\phi\,\mathrm{Im}\,(P^{n-1}_{\mu_2}) \end{aligned}$$

$$\begin{aligned} Q(6,2) = {} & \frac{2n(n+1)}{r^2}(g_2-1)\,\mathrm{cosec}\phi\;\mathrm{cotg}\phi\,\mathrm{Re}\,(P^n_{\mu_2}) + \frac{2n(n+1)}{r^2} g_3\,\mathrm{cosec}\phi\,\mathrm{cotg}\phi\,\mathrm{Im}\,(P^n_{\mu_2}) \\ & - \frac{2n}{r^2}\left[\gamma_n(g_2-1) + g_3\beta_n\right]\mathrm{cosec}\phi\,\mathrm{Re}\,(P^{n-1}_{\mu_2}) - \frac{2n}{r^2}\left[g_3\gamma_n - (g_2-1)\beta_n\right]\mathrm{cosec}\phi\,\mathrm{Im}(P^{n-1}_{\mu_2}) \end{aligned}$$

$$\begin{aligned} Q(1,3) = {} & -\frac{1}{r}\left[(g_3\gamma_n - g_2\beta_n) + n g_3(\mathrm{cosec}^2\phi + n\,\mathrm{cotg}^2\phi)\right] \mathrm{Re}\,(P^n_{\mu_2}) \\ & + \frac{1}{r}\left[(g_2\gamma_n + g_3\beta_n) + n g_2(\mathrm{cosec}^2\phi + n\,\mathrm{cotg}^2\phi)\right] \mathrm{Im}\,(P^n_{\mu_2}) \\ & + \frac{1}{r}(g_3\gamma_n - g_2\beta_n)\,\mathrm{cotg}\phi\,\mathrm{Re}\,(P^{n-1}_{\mu_2}) - \frac{1}{r}(g_2\gamma_n + g_3\beta_n)\,\mathrm{cotg}\phi\,\mathrm{Im}\,(P^{n-1}_{\mu_2}) \end{aligned}$$

$$\begin{aligned} Q(2,3) = {} & \frac{n g_3}{r}(n\,\mathrm{cosec}^2\phi + \mathrm{cotg}^2\phi)\,\mathrm{Re}\,(P^n_{\mu_2}) + \frac{1}{r}\left[1 - n g_2(n\,\mathrm{cosec}^2\phi + \mathrm{cotg}^2\phi)\right]\mathrm{Im}\,(P^n_{\mu_2}) \\ & + \frac{1}{r}(g_2\gamma_n + g_3\beta_n)\,\mathrm{cotg}\phi\,\mathrm{Im}\,(P^{n-1}_{\mu_2}) + \frac{1}{r}(g_2\beta_n - g_3\gamma_n)\,\mathrm{cotg}\phi\,\mathrm{Re}\,(P^{n-1}_{\mu_2}) \end{aligned}$$

$$Q(3,3) = -\frac{2n(n+1)}{r} g_3 \operatorname{cosec}\phi \operatorname{cotg}\phi \, R_e(P^n_{\mu_2}) + \frac{2n(n+1)}{r} g_2 \operatorname{cosec}\phi \operatorname{cotg}\phi \, I_m(P^n_{\mu_2}) + \frac{2n}{r}(g_3\gamma_n - g_2\beta_n)\operatorname{cosec}\phi \, R_e(P^{n-1}_{\mu_2}) - \frac{2n}{r}(g_2\gamma_n + g_3\beta_n)\operatorname{cosec}\phi \, I_m(P^{n-1}_{\mu_2})$$

$$\begin{aligned} Q(4,3) = & -\frac{1}{r^2}\left[g_3\gamma_n - (g_2-1)\beta_n + n g_3(\operatorname{cosec}^2\phi + n\operatorname{cotg}^2\phi)\right] R_e(P^n_{\mu_2}) \\ & + \frac{1}{r^2}\left[(g_2-1)\gamma_n + g_3\beta_n + n(g_2-1)(\operatorname{cosec}^2\phi + n\operatorname{cotg}^2\phi)\right] I_m(P^n_{\mu_2}) \\ & + \frac{1}{r^2}\left[g_3\gamma_n - (g_2-1)\beta_n\right]\operatorname{cotg}\phi \, R_e(P^{n-1}_{\mu_2}) - \frac{1}{r^2}\left[(g_2-1)\gamma_n + g_3\beta_n\right] \\ & \times \operatorname{cotg}\phi \, I_m(P^{n-1}_{\mu_2}) \end{aligned}$$

$$\begin{aligned} Q(5,3) = & \frac{n g_3}{r^2}(\operatorname{cotg}^2\phi + n\operatorname{cosec}^2\phi) R_e(P^n_{\mu_2}) + \frac{n}{r^2}(1-g_2)(\operatorname{cotg}^2\phi + n\operatorname{cosec}^2\phi) I_m(P^n_{\mu_2}) \\ & + \frac{1}{r^2}\left[-g_3\gamma_n + (g_2-1)\beta_n\right]\operatorname{cotg}\phi \, R_e(P^{n-1}_{\mu_2}) + \frac{1}{r^2}\left[(g_2-1)\gamma_n + g_3\beta_n\right] \\ & \times \operatorname{cotg}\phi \, I_m(P^{n-1}_{\mu_2}) \end{aligned}$$

$$\begin{aligned} Q(6,3) = & -\frac{2n(n+1)}{r^2} g_3 \operatorname{cosec}\phi \operatorname{cotg}\phi \, R_e(P^n_{\mu_2}) + \frac{2n(n+1)}{r^2}(g_2-1)\operatorname{cosec}\phi\operatorname{cotg}\phi \, I_m(P^n_{\mu_2}) \\ & + \frac{2n}{r^2}\left[g_3\gamma_n - \beta_n(g_2-1)\right]\operatorname{cosec}\phi \, R_e(P^{n-1}_{\mu_2}) \\ & - \frac{2n}{r^2}\left[\gamma_n(g_2-1) + g_3\beta_n\right]\operatorname{cosec}\phi \, I_m(P^{n-1}_{\mu_2}) \end{aligned}$$

$$Q(1,4) = -\frac{n^2}{2r}(n-2)(n+1)\operatorname{cosec}\phi \, P_1^{n-1} + \frac{n^2}{2r}(n+1)\operatorname{cosec}\phi\operatorname{cotg}\phi \, P_1^n$$

$$Q(2,4) = -\frac{n^2}{2r}(n+1)\operatorname{cosec}\phi\operatorname{cotg}\phi \, P_1^n + \frac{n^2}{2r}(n-2)(n+1)\operatorname{cosec}\phi \, P_1^{n-1}$$

$$Q(3,4) = \frac{n}{2r}(n+1)\left[(n-2) + n(\operatorname{cosec}^2\phi + \operatorname{cotg}^2\phi)\right]P_1^n - \frac{n}{r}(n-2)(n+1)\operatorname{cotg}\phi \, P_1^{n-1}$$

$$Q(4,4) = \frac{n^2}{2r^2}(n+1)\operatorname{cosec}\phi\operatorname{cotg}\phi \, P_1^n - \frac{n^2}{2r^2}(n-2)(n+1)\operatorname{cosec}\phi \, P_1^{n-1}$$

$$Q(5,4) = -\frac{n^2}{2r^2}(n+1)\operatorname{cosec}\phi\operatorname{cotg}\phi \, P_1^n + \frac{n^2}{2r^2}(n-2)(n+1)\operatorname{cosec}\phi \, P_1^{n-1}$$

$$\begin{aligned} Q(6,4) = & \frac{n}{2r^2}(n+1)\left[(n-2) + n(\operatorname{cosec}^2\phi + \operatorname{cotg}^2\phi)\right]P_1^n \\ & - \frac{n}{r^2}(n-2)(n+1)\operatorname{cotg}\phi \, P_1^{n-1} \end{aligned}$$

The elements on the right half of [Q] are obtained from the left-side elements by replacing the associated Legendre function of the first kind P with the Q function of the second kind.

<u>Table 8</u>: Matrix [Q₋]

$$Q_S(1,1) = -\frac{(n+1)}{r} P_1^n + \frac{n(1+n)}{r} \operatorname{cosec}^2\phi P_1^n - \frac{(n-2)(n+1)}{r} \operatorname{cotg}\phi \, P_1^{n-1}$$

$$Q_S(2,1) = \frac{(n+1)}{r} P_1^n - \frac{n(n+1)}{r} \operatorname{cosec}^2\phi P_1^n + \frac{(n-2)(n+1)}{r} \operatorname{cotg}\phi \, P_1^{n-1}$$

$$Q_S(3,1) = \frac{2n(n+1)}{r} \operatorname{cosec}\phi \operatorname{cotg}\phi P_1^n - \frac{2n(n-2)(n+1)}{r} \operatorname{cosec}\phi \, P_1^{n-1}$$

$$Q_S(4,1) = 0$$

$$Q_S(5,1) = 0$$

$$Q_S(6,1) = 0$$

$$Q_S(1,2) = -\frac{\nu}{r} R_e(P^n_{\mu_2}) - g_3 \frac{(n-0.25)}{r} I_m(P^n_{\mu_2}) + g_3 \frac{n(n+1)}{r} \operatorname{cosec}^2\phi \, I_m(P^n_{\mu_2})$$
$$+ \frac{(1+\nu)}{r} \operatorname{cotg}\phi \, R_e(P^{n-1}_{\mu_2}) - g_3 \frac{(n-0.5)^2}{r} \operatorname{cotg}\phi \, I_m(P^{n-1}_{\mu_2})$$

$$Q(2,2) = \frac{1}{r} R_e(P^n_{\mu_2}) + \frac{n g_3}{r} I_m(P^n_{\mu_2}) - \frac{n(n+1)}{r} g_3 \operatorname{cosec}^2\phi \, I_m(P^n_{\mu_2})$$
$$- \frac{(1+\nu)}{r} \operatorname{cotg}\phi \, R_e(P^{n-1}_{\mu_2}) + g_3 \frac{(n-0.5)^2}{r} \operatorname{cotg}\phi \, I_m(P^{n-1}_{\mu_2})$$

$$Q(3,2) = \frac{2n(n+1)}{r} g_3 \operatorname{cosec}\phi \operatorname{tg}\phi \, I_m(P^n_{\mu_2})$$
$$+ \frac{2n(1+\nu)}{r} \operatorname{cosec}\phi \, R_e(P^{n-1}_{\mu_2})$$
$$- \frac{2n(n-0.5)^2}{r} g_3 \operatorname{cosec}\phi \, I_m(P^{n-1}_{\mu_2})$$

$$Q_S(4,2) = \frac{(n-1.25-\nu)}{r^2}\, Re\,(P^n_{\mu_2}) + \frac{\beta_n}{r^2}\, I_m\,(P^n_{\mu_2})$$
$$- \frac{n\,(n+1)}{r^2} \operatorname{cosec}^2\phi\ Re\,(P^n_{\mu_2}) + g_3\ \frac{n(n+1)}{r^2} \operatorname{cosec}^2\phi\, I_m\,(P^n_{\mu_2})$$
$$+ \frac{(n^2-n+1.25+\nu)}{r^2} \operatorname{cotg}\phi\, Re\,(P^{n-1}_{\mu_2}) - \frac{\beta_n}{r^2} \operatorname{cotg}\phi\ I_m\,(P^{n-1}_{\mu_2})$$

$$Q_S(5,2) = -\frac{n}{r^2}\, Re\,(P^n_{\mu_2}) + \frac{n}{r^2}\, g_3\, I_m\,(P^n_{\mu_2}) + \frac{n\,(n+1)}{r^2} \operatorname{cosec}^2\phi\, R_e\,(P^n_{\mu_2})$$
$$- \frac{n\,(n+1)}{r^2}\, g_3 \operatorname{cosec}^2\phi\, I_m\,(P^n_{\mu_2}) - \frac{(n^2-n+1.25+\nu)}{r^2} \operatorname{cotg}\phi\, Re\,(P^{n-1}_{\mu_2})$$
$$+ \frac{\beta_n}{r^2} \operatorname{cotg}\phi\ I_m\,(P^{n-1}_{\mu_2})$$

$$Q_S(6,2) = -\frac{2n\,(n+1)}{r^2} \operatorname{cosec}\phi\ \operatorname{cotg}\phi\ Re\,(P^n_{\mu_2})$$
$$+ \frac{2n\,(n+1)}{r^2}\, g_3 \operatorname{cosec}\phi\ \operatorname{cotg}\phi\ I_m\,(P^n_{\mu_2})$$
$$- \frac{2n}{r^2}\,(n^2-n+1.25+\nu) \operatorname{cosec}\phi\ Re\,(P^{n-1}_{\mu_2})$$
$$- \frac{2n}{r^2}\, \beta_n \operatorname{cosec}\phi\ I_m\,(P^{n-1}_{\mu_2})$$

$$Q_S(1,3) = -\frac{\nu}{r}\, I_m\,(P^n_{\mu_2}) + g_3\left(\frac{n-0.25}{r}\right) Re\,(P^n_{\mu_2}) - g_3 \frac{n\,(n+1)}{r} \operatorname{cosec}^2\phi\, R_e\,(P^n_{\mu_2})$$
$$+ \frac{(1+\nu)}{r} \operatorname{cotg}\phi\ I_m\,(P^{n-1}_{\mu_2}) + g_3\, \frac{(n-0.5)^2}{r} \operatorname{cotg}\phi\, Re\,(P^{n-1}_{\mu_2})$$

$$Q_S(2,3) = \frac{1}{r}\, I_m\,(P^n_{\mu_2}) - \frac{n g_3}{r}\, Re\,(P^n_{\mu_2}) + \frac{n\,(n+1)}{r}\, g_3 \operatorname{cosec}^2\phi\ Re\,(P^n_{\mu_2})$$
$$- \frac{(1+\nu)}{r} \operatorname{cotg}\phi\, I_m\,(P^{n-1}_{\mu_2}) - g_3\, \frac{(n-0.5)^2}{r} \operatorname{cotg}\phi\ Re\,(P^{n-1}_{\mu_2})$$

$$Q_S(3,3) = -\frac{2n\,(n+1)}{r}\, g_3 \operatorname{cosec}\phi\ \operatorname{cotg}\phi\ Re\,(P^n_{\mu_2})$$
$$+ \frac{2n}{r}\,(n-0.5)^2 g_3 \operatorname{cosec}\phi\, Re\,(P^{n-1}_{\mu_2}) + \frac{2n\,(1+\nu)}{r} \operatorname{cosec}\phi\, I_m\,(P^{n-1}_{\mu_2})$$

$$Q_S(4,3)=\frac{(n-1.25-\nu)}{r^2}\,I_m(P^n_{\mu_2})-\frac{\beta_n}{r^2}\,Re(P^n_{\mu_2})-\frac{n(n+1)}{r^2}\,\mathrm{cosec}^2\phi\, I_m(P^n_{\mu_2})$$
$$-g_3 n\frac{(n+1)}{r^2}\,\mathrm{cosec}^2\phi\, Re\,(P^n_{\mu_2})+\frac{(n^2-n+1.25+\nu)}{r^2}\,\mathrm{cotg}\phi\; I_m\,(P^{n-1}_{\mu_2})$$
$$+\frac{\beta n}{r^2}\,\mathrm{cotg}\phi\; Re\,(P^{n-1}_{\mu_2})$$

$$Q_S(5,3)=-\frac{n}{r^2}\,I_m(P^n_{\mu_2})-\frac{n}{r^2}\,g_3 Re\,(P^n_{\mu_2})+\frac{n(1+n)}{r^2}\,\mathrm{cosec}^2\phi\; I_m\,(P^n_{\mu_2})$$
$$+\frac{n(n+1)}{r^2}\,g_3\,\mathrm{cosec}^2\phi\; Re\,(P^n_{\mu_2})-\frac{(n^2-n+1.25+\nu)}{r^2}\,\mathrm{cotg}\phi\, I_m(P^{n-1}_{\mu_2})$$
$$-\frac{\beta n}{r^2}\,\mathrm{cotg}\phi\; Re\,(P^{n-1}_{\mu_2})$$

$$Q_S(6,3)=-\frac{2n}{r^2}\,(n+1)\,\mathrm{cosec}\phi\;\mathrm{cotg}\phi\; I_m\,(P^n_{\mu_2})$$
$$-\frac{2n}{r^2}\,(n+1)\,\mathrm{cosec}\phi\;\mathrm{cotg}\phi\; Re\,(P^n_{\mu_2})$$
$$+\frac{2n}{r^2}\,(n^2-n+1.25+\nu)\,\mathrm{cosec}\,\phi\; I_m\,(P^{n-1}_{\mu_2})$$
$$+\frac{2n}{r^2}\,\beta_n\,\mathrm{cosec}\phi\; Re\,(P^{n-1}_{\mu_2})$$

$$Q_S(1,4)=\frac{n^2}{2r}(n+1)\,\mathrm{cosec}\,\phi\,\mathrm{cotg}\phi\, P^n_1-\frac{n^2}{2r}\,(n-2)(n+1)\,\mathrm{cosec}\,\phi\; P^{n-1}_1$$

$$Q_S(2,4)=-\frac{n^2}{2r}\,(n+1)\,\mathrm{cosec}\phi\;\mathrm{cotg}\phi\, P^n_1+\frac{n^2}{2r}\,(n-2)(n+1)\,\mathrm{cosec}\phi\; P^{n-1}_1$$

$$Q_S(3,4)=-\frac{n(n+1)}{r}\,P^n_1+\frac{n^2(n+1)}{r}\,\mathrm{cosec}^2\phi\, P^n_1-\frac{n(n-2)(n+1)}{r}\,\mathrm{cotg}\phi\; P^{n-1}_1$$

$$Q_S(4,4)=\frac{n^2}{2r^2}\,(n+1)\,\mathrm{cosec}\,\phi\,\mathrm{cotg}\phi\, P^n_1-\frac{n^2}{2r^2}\,(n-2)(n+1)\,\mathrm{cosec}\,\phi\; P^{n-1}_1$$

$$Q_S(5,4)=-\frac{n^2}{2r^2}\,(n+1)\,\mathrm{cosec}\phi\,\mathrm{cotg}\phi\, P^n_1+\frac{n^2}{2r^2}\,(n-2)(n+1)\,\mathrm{cosec}\,\phi\; P^{n-1}_1$$

$$Q_S(6,4)=\frac{-n(n+1)}{r^2}\,P^n_1+\frac{n^2(n+1)}{r^2}\,\mathrm{cosec}^2\phi\; P^n_1$$
$$-\frac{n(n-2)(n+1)}{r^2}\,\mathrm{cotg}\phi\; P^{n-1}_1$$

The elements on the right half of $[Q_m]$ are obtained from the left-side elements by replacing the associated Legendre function of the first kind P with the Q function of the second kind.

Table 9: Matrix [P]

$$[P] = \begin{bmatrix} D & \nu D & 0 & 0 & 0 & 0 \\ \nu D & D & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{D(1-\nu)}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & K & \nu K & 0 \\ 0 & 0 & 0 & \nu K & K & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{K(1-\nu)}{2} \end{bmatrix}$$

Table 10: Matrix [DLAM]

$$[DLAM] = \begin{bmatrix} \sin\phi_i & -\cos\phi_i & 0 & 0 & 0 & 0 & 0 & 0 \\ \cos\phi_i & \sin\phi_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\phi_j & -\cos\phi_j & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\phi_j & \sin\phi_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where ϕ_i and ϕ_j are the coordinates of the nodes i and j, respectively (see Fig. 5).

Table 11: $[A_r]$ Matrices

$$
[A_1] = \begin{bmatrix}
\frac{(-1)^n}{r}\left[\frac{n^2(n+1)}{4} + \frac{(n-1)(n-2)(n+1)}{2}\right] & \left[g_3\frac{n(n+1)}{4}f(\ell,n) + \frac{(1+\nu)}{2}d(\ell,n-1) - g_3\frac{(n-0.5)^2}{2}f(\ell,n-1)\right]\frac{1}{r} & \left[-g_3\frac{n(n+1)}{4}d(\ell,n) + \frac{(1+\nu)}{2}f(\ell,n-1) + g_3\frac{(n-0.5)^2}{2}d(\ell,n-1)\right]\frac{1}{r} & \frac{n}{4}A_1(3,1) \\
-A_1(1,1) & -A_1(1,2) & -A_1(1,3) & -\frac{n}{4}A_1(3,1) \\
\frac{(-1)^n}{r}\left[\frac{n^2(n+1)}{2} + n(n-1)(n-2)(n+1)\right] & \left[g_3\frac{n(n+1)}{2}f(\ell,n) + n(1+\nu)d(\ell,n-1) - g_3 n(n-0.5)^2 f(\ell,n-1)\right]\frac{1}{r} & \left[-g_3\frac{n(n+1)}{2}d(\ell,n) + n(1+\nu)f(\ell,n-1) + g_3 n(n-0.5)^2 d(\ell,n-1)\right]\frac{1}{r} & nA_1(1,1) \\
0 & \left[-\frac{n(n+1)}{4}d(\ell,n) + g_3\frac{n(n+1)}{4}f(\ell,n) + \frac{(n^2-n+1.25+\nu)}{2}d(\ell,n-1) - \frac{\beta}{2}f(\ell,n-1)\right]\frac{1}{r^2} & \left[-\frac{n(n+1)}{4}f(\ell,n) - g_3\frac{n(n+1)}{4}d(\ell,n) + \frac{(n^2-n+1.25+\nu)}{2}f(\ell,n-1) + \frac{\beta}{2}d(\ell,n-1)\right]\frac{1}{r^2} & \frac{n}{4r}A_1(3,1) \\
0 & -A_1(4,2) & -A_1(4,3) & -\frac{n}{4r}A_1(3,1) \\
0 & \left[-\frac{n(n+1)}{2}d(\ell,n) + g_3\frac{n(n+1)}{2}f(\ell,n) + n(n^2-n+1.25+\nu)d(\ell,n-1) - n\beta f(\ell,n-1)\right]\frac{1}{r^2} & \left[-\frac{n(n+1)}{2}f(\ell,n) - g_3\frac{n(n+1)}{2}d(\ell,n) + n(n^2-n+1.25+\nu)f(\ell,n-1) + n\beta d(\ell,n-1)\right]\frac{1}{r^2} & \frac{n}{r}A_1(1,1)
\end{bmatrix}
$$

Table 11 cont'd: $[A_r]$ Matrices

$[A_2] =$

$\frac{(-1)^n}{2r}(n+1)(n+2)$	$\left[-\nu d(\ell,n)+g_3\left(\frac{n^2-n+0.5}{2}\right)f(\ell,n) - \left(\frac{1+\nu}{2}\right)d(\ell,n-1)+g_3\frac{(n-0.5)^2}{2}f(\ell,n-1)\right]\frac{1}{r}$	$\left[-\nu f(\ell,n)-g_3\left(\frac{n^2-n+0.5}{2}\right)d(\ell,n) - \left(\frac{1+\nu}{2}\right)f(\ell,n-1)-g_3\frac{(n-0.5)^2}{2}d(\ell,n-1)\right]\frac{1}{r}$	$\frac{n}{4}A_2(3,1)$
$-A_2(1,1)$	$\left[d(\ell,n)-g_3\frac{n(n-1)}{2}f(\ell,n)+\left(\frac{1+\nu}{2}\right)d(\ell,n-1)-g_3\frac{(n-0.5)^2}{2}f(\ell,n-1)\right]\frac{1}{r}$	$\left[f(\ell,n)+g_3\frac{n(n-1)}{2}d(\ell,n)+\left(\frac{1+\nu}{2}\right)f(\ell,n-1)+g_3\frac{(n-0.5)^2}{2}d(\ell,n-1)\right]\frac{1}{r}$	$-\frac{n}{4}A_2(3,1)$
$\frac{(-1)^n}{r}n(n-1)(n-2)(n+1)$	$\left[n(1+\nu)d(\ell,n-1)-g_3\frac{n(n-1)}{2}f(\ell,n)+\frac{(1+\nu)}{2}d(\ell,n-1)-g_3\frac{(n-0.5)^2}{2}f(\ell,n-1)\right]\frac{1}{r}$	$\left[n(1+\nu)f(\ell,n-1)+g_3\frac{n(n-1)}{2}d(\ell,n)+\frac{(1+\nu)}{2}f(\ell,n-1)+g_3\frac{(n-0.5)^2}{2}d(\ell,n-1)\right]\frac{1}{r}$	$nA_2(1,1)$
0	$\left[\left(\frac{n}{2}-1.25-\nu-\frac{n^2}{2}\right)d(\ell,n)+\left(\beta+g_3\frac{n(n+1)}{2}\right)f(\ell,n)-\left(\frac{n^2-n+1.25+\nu}{2}\right)d(\ell,n-1)+\frac{\beta}{2}f(\ell,n-1)\right]\frac{1}{r^2}$	$\left[\left(\frac{n}{2}-1.25-\nu-\frac{n^2}{2}\right)f(\ell,n)-\left(\beta+g_3\frac{n(n+1)}{2}\right)f(\ell,n)-\left(\frac{n^2-n+1.25+\nu}{2}\right)f(\ell,n-1)+\frac{\beta}{2}d(\ell,n-1)\right]\frac{1}{r^2}$	$\frac{n}{4r}A_2(3,1)$
0	$\left[\frac{n(n-1)}{2}d(\ell,n)-g_3\frac{n(n-1)}{2}f(\ell,n)+\frac{(n^2-n+1.25+\nu)}{2}d(\ell,n-1)-\frac{\beta}{2}f(\ell,n-1)\right]\frac{1}{r^2}$	$\left[\frac{n(n-1)}{2}f(\ell,n)+g_3\frac{n(n-1)}{2}d(\ell,n)+\frac{(n^2-n+1.25+\nu)}{2}f(\ell,n-1)+\frac{\beta}{2}d(\ell,n-1)\right]\frac{1}{r^2}$	$-\frac{n}{4r}A_2(3,1)$
0	$\left[n(n^2-n+1.25+\nu)d(\ell,n-1) - n\beta f(\ell,n-1)\right]\frac{1}{}$	$\left[n(n^2-n+1.25+\nu)f(\ell,n-1) + n\beta d(\ell,n-1)\right]\frac{1}{}$	$\frac{n}{r}A_2(1,1)$

Table 11 cont'd: $[A_r]$ Matrices

$$
[A_3] = \begin{bmatrix}
(-1)^n \dfrac{n^2(n+1)}{4r} & g_3 \dfrac{n(n+1)}{4} f(\ell,n) \cdot \dfrac{1}{r} & -g_3 \dfrac{n(n+1)}{4} d(\ell,n) \cdot \dfrac{1}{r} & \dfrac{n}{4} A_3(3,1) \\
-A_3(1,1) & -A_3(1,2) & -A_3(1,3) & -\dfrac{n}{4} A_3(3,1) \\
-(-1)^n \dfrac{n^2(n+1)}{2r} & -g_3 \dfrac{n(n+1)}{2} f(\ell,n) \cdot \dfrac{1}{r} & g_3 \dfrac{n(n+1)}{2} d(\ell,n) \cdot \dfrac{1}{r} & n A_3(1,1) \\
0 & \left[-\dfrac{n(n+1)}{4} d(\ell,n) + g_3 \dfrac{n(n+1)}{4} f(\ell,n)\right] \dfrac{1}{r^2} & \left[-\dfrac{n(n+1)}{4} f(\ell,n) - g_3 \dfrac{n(n+1)}{4} d(\ell,n)\right] \dfrac{1}{r^2} & \dfrac{n}{4r} A_3(3,1) \\
0 & -A_3(4,2) & -A_3(4,3) & -\dfrac{n}{4r} A_3(3,1) \\
0 & \left[\dfrac{n(n+1)}{2} d(\ell,n) - g_3 \dfrac{n(n+1)}{2} f(\ell,n)\right] \dfrac{1}{r^2} & \left[\dfrac{n(n+1)}{2} f(\ell,n) + g_3 \dfrac{n(n+1)}{2} d(\ell,n)\right] \dfrac{1}{r^2} & \dfrac{n}{r} A_3(1,1)
\end{bmatrix}
$$

Table 12: $[B_r]$ Matrices

$$[B_1] = \left[\begin{array}{c|c|c|c}
\frac{(-1)^n}{r}\left[\frac{n(n+1)}{4} + \frac{(n-2)(n+1)}{2}\right] & 0 & 0 & \frac{n}{4} B_1(3,1) \\ \hline
-B_1(1,1) & 0 & 0 & -\frac{n}{4} B_1(3,1) \\ \hline
\frac{(-1)^n}{r}\left[\frac{n(n+1)}{2} + n(n-2)(n+1)\right] & 0 & 0 & n B_1(1,1) \\ \hline
0 & 0 & 0 & \frac{n}{4r} B_1(3,1) \\ \hline
0 & 0 & 0 & -\frac{n}{4r} B_1(3,1) \\ \hline
0 & 0 & 0 & \frac{n}{r} B_1(1,1)
\end{array}\right]$$

$$[B_2] = \left[\begin{array}{c|c|c|c}
0 & 0 & 0 & \frac{n}{4} B_2(3,1) \\ \hline
-B_2(1,1) & 0 & 0 & -\frac{n}{4} B_2(3,1) \\ \hline
\frac{(-1)^n}{r} n(n-2)(n+1) & 0 & 0 & n B_2(1,1) \\ \hline
0 & 0 & 0 & \frac{n}{4r} B_2(3,1) \\ \hline
0 & 0 & 0 & -\frac{n}{4r} B_2(3,1) \\ \hline
0 & 0 & 0 & \frac{n}{r} B_2(1,1)
\end{array}\right]$$

Table 12 cont'd: $[B_r]$ Matrices

$$[B_3] = \left[\begin{array}{c|c|c|c}
(-1)^n \dfrac{n(n+1)}{4r} & 0 & 0 & \dfrac{n}{4} B_3(3,1) \\ \hline
-B_3(1,1) & 0 & 0 & -\dfrac{n}{4} B_3(3,1) \\ \hline
-(-1)^n \dfrac{n(n+1)}{2r} & 0 & 0 & n B_3(1,1) \\ \hline
0 & 0 & 0 & \dfrac{n}{4r} B_3(3,1) \\ \hline
0 & 0 & 0 & \dfrac{-n}{4r} B_3(3,1) \\ \hline
0 & 0 & 0 & \dfrac{n}{r} B_3(1,1)
\end{array}\right]$$

Table 13: $\left[A_r{}^c\right]$ Matrices

$$
\left[A_1^c\right] = \left[\begin{array}{c|c|c|c}
\frac{(-1)^n}{r}\left(\frac{bn}{4} + \frac{c\,(n-1)}{2}\right) & \left[\frac{A}{2} - \frac{c}{2} + n\,(n+1)\,g_3\,\eta_n\right]\frac{1}{r} & & \frac{n}{4}A_1^c\,(3,1) \\
\hline
-A_1^c\,(1,1) & \left[\frac{A_2}{2} - \frac{C_2}{2} - n\,(n+1)\,g_3\,\eta_n\right]\frac{1}{r} & & -\frac{n}{4}A_1^c\,(3,1) \\
\hline
\frac{(-1)^n}{r}\left[\frac{an^2}{2} + en\,(n-1)\right] & 0 & 0 & nA_1^c\,(1,1) \\
\hline
0 & \left[\frac{A_3}{2} - \frac{C_3}{2} + n\,(n+1)(-\lambda_n + g_3\eta_n)\right] \times \frac{1}{r^2} & & \frac{n}{4r}A_1^c\,(3,1) \\
\hline
0 & \left[\frac{A_4}{2} - \frac{C_4}{2} + n\,(n+1)(\lambda_n - g_3\eta_n)\right] \times \frac{1}{r^2} & & -\frac{n}{4r}A_1^c\,(3,1) \\
\hline
0 & 0 & 0 & \frac{n}{r}A_1^c\,(1,1)
\end{array}\right]
$$

$$
\left[A_2^c\right] = \left[\begin{array}{c|c|c|c}
\frac{(-1)^n}{r}\left[-an + \frac{bn}{2} - \frac{c(n-1)}{2}\right] & \left[-\frac{A}{2} + \frac{c}{2}\right]\frac{1}{r} & & \frac{n}{4}A_2^c\,(3,1) \\
\hline
-A_2^c\,(1,1) & \left[-\frac{A_2}{2} + \frac{C_2}{2}\right]\frac{1}{r} & & -\frac{n}{4}A_2^c\,(3,1) \\
\hline
\frac{(-1)^n}{r}\,en\,(n-1) & \left[\frac{E}{2} + g_3 n\,(n+1)\,\eta_n\right]\frac{1}{r} & & nA_2^c\,(1,1) \\
\hline
0 & \left[-\frac{A_3}{2} + \frac{C_3}{2}\right]\frac{1}{r^2} & & \frac{n}{4r}A_2^c\,(3,1) \\
\hline
0 & \left[-\frac{A_4}{2} + \frac{C_4}{2}\right]\frac{1}{r^2} & & -\frac{n}{4r}A_2^c\,(3,1) \\
\hline
0 & \left[G + K\right]\frac{1}{r^2} & & \frac{n}{r}A_2^c\,(1,1)
\end{array}\right]
$$

Table 13 cont'd: $\left[A_r{}^c\right]$ Matrices

$$
\left[A_3^c\right] = \left[\begin{array}{c|c|c|c}
\dfrac{(-1)^n}{4r}\, bn & \left[-\dfrac{A}{2}\right]\dfrac{1}{r} & & \dfrac{n}{4} A_3^c\,(3,1) \\ \hline
-A_3^c\,(1,1) & \left[-\dfrac{A_2}{2}\right]\dfrac{1}{r} & & -\dfrac{n}{4} A_3^c\,(3,1) \\ \hline
-\dfrac{(-1)^n}{2r}\, an^2 & \left[g_3 n(n+1)\eta_n - \dfrac{E}{2}\right]\dfrac{1}{r} & & n A_3^c\,(1,1) \\ \hline
0 & \left[-\dfrac{A_3}{2}\right]\dfrac{1}{r^2} & & \dfrac{n}{4r} A_3^c\,(3,1) \\ \hline
0 & \left[-\dfrac{A_4}{2}\right]\dfrac{1}{r^2} & & -\dfrac{n}{4r} A_3^c\,(3,1) \\ \hline
0 & \left[G-K\right]\dfrac{1}{r^2} & & \dfrac{n}{r} A_3^c\,(1,1)
\end{array}\right]
$$

$$
\left[A_4^c\right] = \left[\begin{array}{c|c|c|c}
0 & -\dfrac{C}{2r} & & 0 \\ \hline
0 & -\dfrac{C_2}{2r} & & 0 \\ \hline
0 & -\dfrac{E}{2r} & & 0 \\ \hline
0 & -\dfrac{C_3}{2r^2} & & 0 \\ \hline
0 & -\dfrac{C_4}{2r^2} & & 0 \\ \hline
0 & -\dfrac{K}{2r^2} & & 0
\end{array}\right]
$$

where: $a = (n+1)$

$b = n(n+1)$

$c = -(n+1)$

$g_3 = -\dfrac{(1+\nu)}{\beta}$ $\qquad \beta = \beta_n = -\sqrt{\varsigma\,(1-\nu^2)}$

$CNU = n^2 - n + 1.25 + \nu$

$$A = -\nu\lambda_n - g_3(n-0.25)\eta_n + \left(\frac{1+\nu}{2}\right)\lambda_{n-1} - g_3\left(\frac{n-0.5}{2}\right)^2\eta_{n-1}$$

$$A_2 = \lambda_n + n g_3 \eta_n - \left(\frac{1+\nu}{2}\right)\lambda_{n-1} + g_3\left(\frac{n-0.5}{2}\right)^2\eta_{n-1}$$

$$A_3 = (n - 1.25 - \nu)\lambda_n + \beta\eta_n + \frac{CNU}{2}\lambda_{n-1} - \frac{\beta}{2}\eta_{n-1}$$

$$A_4 = -n\lambda_n + n g_3 \eta_n - \frac{CNU}{2}\lambda_{n-1} + \frac{\beta}{2}\eta_{n-1}$$

$$C = \frac{(1+\nu)}{2}\lambda_{n-1} - g_3\left(\frac{n-0.5}{2}\right)^2\eta_{n-1}$$

$$C_2 = -C$$

$$C_3 = \frac{CNU}{2}\lambda_{n-1} - \frac{\beta}{2}\eta_{n-1}$$

$$C_4 = -C_3$$

$$E = 2n(1+\nu)\lambda_{n-1} - 2g_3 n(n-0.5)^2\eta_{n-1}$$

$$G = n(n+1)(-\lambda_n + g_3\eta_n)$$

$$K = n(CNU)\lambda_{n-1} - n\beta\eta_{n-1}$$

Except where otherwise indicated, the elements in the 3rd column of matrices $[A_1^c]$, $[A_2^c]$, $[A_3^c]$ and $[A_4^c]$ are obtained from the 2nd-column (same-line) elements, by replacing λ_n, η_n, λ_{n-1}, η_{n-1} with η_n, $-\lambda_n$, η_{n-1} and $-\lambda_{n-1}$, respectively.

Table 14: $\left[B_r{}^c\right]$ Matrices

$$
\left[B_1^c\right] = \left[\begin{array}{c|c|c|c}
\frac{(-1)^n}{r}\left(\frac{b}{4} + \frac{c}{2}\right) & \left[-\frac{B}{2} + \frac{D}{2} + g_3 n(1+n)\lambda_n\right]\frac{1}{r} & & \frac{n}{4}B_1^c(3,1) \\
\hline
-B_1^c(1,1) & \left[-\frac{B_2}{2} + \frac{D_2}{2} - g_3 n(1+n)\lambda_n\right]\frac{1}{r} & & -\frac{n}{4}B_1^c(3,1) \\
\hline
\frac{(-1)^n}{r}\left(\frac{an}{2} + cn\right) & 0 & 0 & nB_1^c(1,1) \\
\hline
0 & \left[-\frac{B_3}{2} + \frac{D_3}{2} + n(n+1)(\eta_n + g_3\lambda_n)\right]\frac{1}{r^2} & & \frac{n}{4r}B_1^c(3,1) \\
\hline
0 & \left[-\frac{B_4}{2} + \frac{D_4}{2} - n(n+1)(\eta_n + g_3\lambda_n)\right]\frac{1}{r^2} & & -\frac{n}{4r}B_1^c(3,1) \\
\hline
0 & 0 & 0 & \frac{n}{r}B_1^c(1,1)
\end{array}\right]
$$

$$
\left[B_2^c\right] = \left[\begin{array}{c|c|c|c}
\frac{(-1)^n}{r}\left(-a + \frac{b}{2} - \frac{c}{2}\right) & \left[\frac{B}{2} - \frac{D}{2}\right]\frac{1}{r} & & \frac{n}{4}B_2^c(3,1) \\
\hline
-B_2^c(1,1) & \left[\frac{B_2}{2} - \frac{D_2}{2}\right]\frac{1}{r} & & -\frac{n}{4}B_2^c(3,1) \\
\hline
\frac{(-1)^n}{r}\,cn & \left[-\frac{F}{2} + g_3 n(n+1)\lambda_n\right]\frac{1}{r} & & nB_2^c(1,1) \\
\hline
0 & \left[\frac{B_3}{2} - \frac{D_3}{2}\right]\frac{1}{r^2} & & \frac{n}{4r}B_2^c(3,1) \\
\hline
0 & \left[\frac{B_4}{2} - \frac{D_4}{2}\right]\frac{1}{r^2} & & -\frac{n}{4r}B_2^c(3,1) \\
\hline
0 & \left[-H - L\right]\frac{1}{r^2} & & \frac{n}{r}B_2^c(1,1)
\end{array}\right]
$$

Table 14 cont'd: $\left[B_r^{\,c}\right]$ Matrices

$$
\left[B_3^c\right] = \begin{bmatrix}
\dfrac{(-1)^n\, b}{4r} & \dfrac{B}{2r} & & \dfrac{n}{4} B_3^c\,(3,1) \\
-B_3^c\,(1,1) & \dfrac{B_2}{2r} & & -\dfrac{n}{4} B_3^c\,(3,1) \\
\dfrac{(-1)^n\, an}{2r} & \left[\dfrac{F}{2} + g_3 n\,(n+1)\lambda_n\right]\dfrac{1}{r} & & n B_3^c\,(1,1) \\
0 & \dfrac{B_3}{2r^2} & & \dfrac{n}{4r} B_3^c\,(3,1) \\
0 & \dfrac{B_4}{2r^2} & & -\dfrac{n}{4r} B_3^c\,(3,1) \\
0 & \left[H + L\right]\dfrac{1}{r^2} & & \dfrac{n}{r} B_3^c\,(1,1)
\end{bmatrix}
$$

$$
\left[B_4^c\right] = \begin{bmatrix}
0 & \dfrac{D}{2r} & & 0 \\
0 & \dfrac{D_2}{2r} & & 0 \\
0 & \dfrac{F}{2r} & & 0 \\
0 & \dfrac{D_3}{2r^2} & & 0 \\
0 & \dfrac{D_4}{2r^2} & & 0 \\
0 & \dfrac{L}{r^2} & & 0
\end{bmatrix}
$$

where:

$$a = (n+1)$$

$$b = n\,(n+1)$$

$$c = -\,(n+1)$$

$$g_3 = -\,\frac{(1+\nu)}{\beta} \qquad \beta = \beta_n = -\sqrt{\xi\,(1-\nu^2)}$$

$$CNU = n^2 - n + 1.25 + \nu$$

$$B = -\nu \eta_n + g_3 (n - 0.25) \lambda_n + \frac{(1+\nu)}{2} \eta_{n-1} + g_3 \left(\frac{n-0.5}{2}\right)^2 \lambda_{n-1}$$
$$B_2 = \eta_n - n g_3 \lambda_n - \frac{(1+\nu)}{2} \eta_{n-1} - g_3 \left(\frac{n-0.5}{2}\right)^2 \lambda_{n-1}$$
$$B_3 = (n - 1.25 - \nu) \eta_n - \beta \lambda_n + \frac{CNU}{2} \eta_{n-1} + \frac{\beta}{2} \lambda_{n-1}$$
$$B_4 = -n \eta_n - n g_3 \lambda_n - \frac{CNU}{2} \eta_{n-1} - \frac{\beta}{2} \lambda_{n-1}$$

$$D = \frac{(1+\nu)}{2} \eta_{n-1} + g_3 \left(\frac{n-0.5}{2}\right)^2 \lambda_{n-1}$$

$$D_2 = -D$$
$$D_3 = \frac{CNU}{2} \eta_{n-1} + \frac{\beta}{2} \lambda_{n-1}$$

$$D_4 = -D_3$$
$$F = 2n (1+\nu) \eta_{n-1} + 2 g_3 n (n - 0.5)^2 \lambda_{n-1}$$

$$H = n (n+1) (-\eta_n - g_3 \lambda_n)$$
$$L = n (CNU) \eta_{n-1} + n \beta \lambda_{n-1}$$

Except where otherwise indicated, elements in the 3rd column in matrices $[B1^c]$, $[B_2{}^c]$, $[B_3{}^c]$ and $[B_4{}^c]$ are obtained from the 2nd-column (same-line) elements, by replacing λ_n, η_n, λ_{n-1}, η_{n-1} respectively, with η_n, $-\lambda_n$, η_{n-1} and $-\lambda_{n-1}$.

Table 15: $\left[A_r^V\right]$ Matrices

$$
\left[A_1^V\right] = \begin{bmatrix}
\frac{(-1)^n}{r}\left[\frac{n^2(n+1)}{4} - \frac{(n^2-1)}{2}\right] & 0 & 0 & \frac{n}{4} A_1^V(3,1) \\
-A_1^V(1,1) & 0 & 0 & -\frac{n}{4} A_1^V(3,1) \\
\frac{(-1)^n}{r} n(n+1)\left(\frac{n}{2} - 1\right) & 0 & 0 & n A_1^V(1,1) \\
0 & 0 & 0 & \frac{n}{4r} A_1^V(3,1) \\
0 & 0 & 0 & -\frac{n}{4r} A_1^V(3,1) \\
0 & 0 & 0 & \frac{n}{r} A_1^V(1,1)
\end{bmatrix}
$$

$$
\left[A_2^V\right] = \begin{bmatrix}
\frac{(-1)^n}{2r}(n+1)(n^2-n+1) & 0 & 0 & \frac{n}{4} A_2^V(3,1) \\
-A_2^V(1,1) & 0 & 0 & -\frac{n}{4} A_2^V(3,1) \\
\frac{(-1)^n}{r} n(n^2-1) & 0 & 0 & n A_2^V(1,1) \\
0 & 0 & 0 & \frac{n}{4r} A_2^V(3,1) \\
0 & 0 & 0 & -\frac{n}{4r} A_2^V(3,1) \\
0 & 0 & 0 & \frac{n}{r} A_2^V(1,1)
\end{bmatrix}
$$

Table 15 cont'd: $\left[A_r^V\right]$ Matrices

$$
\left[A_3^V\right] = \begin{bmatrix}
\frac{(-1)^n}{4r}\, n^2\,(n+1) & 0 & 0 & \frac{n}{4}\, A_3^V\,(3,1) \\
-A_3^V\,(1,1) & 0 & 0 & -\frac{n}{4}\, A_3^V\,(3,1) \\
\frac{(-1)^n}{2r}\, n^2\,(n+1) & 0 & 0 & n\, A_3^V\,(1,1) \\
0 & 0 & 0 & \frac{n}{4r}\, A_3^V\,(3,1) \\
0 & 0 & 0 & -\frac{n}{4r}\, A_3^V\,(3,1) \\
0 & 0 & 0 & \frac{n}{r}\, A_3^V\,(1,1)
\end{bmatrix}
$$

Table 16: $\left[B_r^V \right]$ Matrices

$$
\left[B_1^V \right] = \begin{bmatrix}
\frac{(-1)^n}{2r} (n+1)\left(\frac{n}{2} - 1\right) & 0 & 0 & \frac{n}{4} B_1^V (3,1) \\
- B_1^V (1,1) & 0 & 0 & -\frac{n}{4} B_1^V (3,1) \\
\frac{(-1)^n}{2r} n (n+1) & 0 & 0 & n B_1^V (1,1) \\
0 & 0 & 0 & \frac{n}{4r} B_1^V (3,1) \\
0 & 0 & 0 & -\frac{n}{4r} B_1^V (3,1) \\
0 & 0 & 0 & \frac{n}{r} B_1^V (1,1)
\end{bmatrix}
$$

$$
\left[B_2^V \right] = \begin{bmatrix}
\frac{(-1)^n}{2r} (n^2 - 1) & 0 & 0 & \frac{n}{4} B_2^V (3,1) \\
- B_2^V (1,1) & 0 & 0 & -\frac{n}{4} B_2^V (3,1) \\
\frac{(-1)^n}{r} n (n+1) & 0 & 0 & n B_2^V (1,1) \\
0 & 0 & 0 & \frac{n}{4r} B_2^V (3,1) \\
0 & 0 & 0 & -\frac{n}{4r} B_2^V (3,1) \\
0 & 0 & 0 & \frac{n}{r} B_2^V (1,1)
\end{bmatrix}
$$

Table 16 cont'd: $\left[B_r^V \right]$ Matrices

$$[B_3^V] = \begin{bmatrix} \frac{(-1)^n}{4r} n(n+1) & 0 & 0 & \frac{n}{4} B_3^V(3,1) \\ -B_3^V(1,1) & 0 & 0 & -\frac{n}{4} B_3^V(3,1) \\ \frac{(-1)^n}{2r} n(n+1) & 0 & 0 & n B_3^V(1,1) \\ 0 & 0 & 0 & \frac{n}{4r} B_3^V(3,1) \\ 0 & 0 & 0 & -\frac{n}{4r} B_3^V(3,1) \\ 0 & 0 & 0 & \frac{n}{r} B_3^V(1,1) \end{bmatrix}$$

Table 17: $[\alpha_r]$ Matrices

$$[\alpha_1] = \left[\begin{array}{c|c|c|c} 0 & x_\ell(n) & x_\ell(n) & 0 \\ \hline 0 & x_\ell(n) & x_\ell(n) & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & x_\ell(n) & x_\ell(n) & 0 \\ \hline 0 & x_\ell(n) & x_\ell(n) & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \qquad [\alpha_2] = \left[\begin{array}{c|c|c|c} 0 & x_\ell(n)+2 & x_\ell(n)+2 & 0 \\ \hline 0 & x_\ell(n)+2 & x_\ell(n)+2 & 0 \\ \hline 0 & x_\ell(n)+1 & x_\ell(n)+1 & 0 \\ \hline 0 & x_\ell(n)+2 & x_\ell(n)+2 & 0 \\ \hline 0 & x_\ell(n)+2 & x_\ell(n)+2 & 0 \\ \hline 0 & x_\ell(n)+1 & x_\ell(n)+1 & 0 \end{array}\right]$$

$$[\alpha_3] = \left[\begin{array}{c|c|c|c} 0 & x_\ell(n)-2 & x_\ell(n)-2 & 0 \\ \hline 0 & x_\ell(n)-2 & x_\ell(n)-2 & 0 \\ \hline 0 & x_\ell(n)-1 & x_\ell(n)-1 & 0 \\ \hline 0 & x_\ell(n)-2 & x_\ell(n)-2 & 0 \\ \hline 0 & x_\ell(n)-2 & x_\ell(n)-2 & 0 \\ \hline 0 & x_\ell(n)-1 & x_\ell(n)-1 & 0 \end{array}\right] \qquad [\alpha_4] = \left[\begin{array}{c|c|c|c} 0 & x_\ell(n)+4 & x_\ell(n)+4 & 0 \\ \hline 0 & x_\ell(n)+4 & x_\ell(n)+4 & 0 \\ \hline 0 & x_\ell(n)+3 & x_\ell(n)+3 & 0 \\ \hline 0 & x_\ell(n)+4 & x_\ell(n)+4 & 0 \\ \hline 0 & x_\ell(n)+4 & x_\ell(n)+4 & 0 \\ \hline 0 & x_\ell(n)+3 & x_\ell(n)+3 & 0 \end{array}\right]$$

$$: \quad x_\ell(n) = 2\ell - n + b_3 + \frac{1}{2} \quad ; \quad b_3 = \frac{\sqrt{2}}{2}\left[\xi\,(1-\nu^2)\right]^{1/4}$$

Table 18: $\left[\Gamma_r\right]$ Matrices

$$[\Gamma_1] = \begin{bmatrix} (-1)^n\left[-(n-2)(n+1)(n-1) - \frac{n^2}{2}\right] & -(1+\nu)\,d(\ell,n-1) + g_3(n-0.5)^2 f(\ell,n-1) - g_3\frac{n}{2} f(\ell,n) & -(1+\nu)\,f(\ell,n-1) - g_3(n-0.5)^2 d(\ell,n-1) + g_3\frac{n}{2} d(\ell,n) & \frac{n}{2}\Gamma_1(3,1) \\ 0 & 0 & 0 & 0 \\ -(-1)^n\frac{n^2}{2} & -g_3\frac{n}{2} f(\ell,n) & g_3\frac{n}{2} d(\ell,n) & \frac{n}{2}\Gamma_1(1,1) \end{bmatrix}$$

$$[\Gamma_2] = \begin{bmatrix} 0 & 0 & 0 & \frac{n}{2}\Gamma_2(3,1) \\ (-1)^n n & d(\ell,n) & f(\ell,n) & 0 \\ 0 & 0 & 0 & \frac{n}{2}\Gamma_2(1,1) \end{bmatrix}$$

$$[\Gamma_3] = \begin{bmatrix} (-1)^n\frac{n^2}{2} & g_3\frac{n}{2} f(\ell,n) & -g_3\frac{n}{2} d(\ell,n) & \frac{n}{2}\Gamma_3(3,1) \\ 0 & 0 & 0 & 0 \\ -(-1)^n\frac{n^2}{2} & -g_3\frac{n}{2} f(\ell,n) & g_3\frac{n}{2} d(\ell,n) & \frac{n}{2}\Gamma_3(1,1) \end{bmatrix}$$

Table 19: $\left[\Delta_r\right]$ Matrices

$$[\Delta_1] = \left[\begin{array}{c|c|c|c} (-1)^n\left[-(n-2)(n+1)-\frac{n}{2}\right] & 0 & 0 & \frac{n}{2}\Delta_1(3,1) \\ \hline 0 & 0 & 0 & 0 \\ \hline -(-1)^n\ \frac{n}{2} & 0 & 0 & \frac{n}{2}\Delta_1(1,1) \end{array}\right]$$

$$[\Delta_2] = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & \frac{n}{2}\Delta_2(3,1) \\ \hline (-1)^n & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{n}{2}\Delta_2(1,1) \end{array}\right]$$

$$[\Delta_3] = \left[\begin{array}{c|c|c|c} (-1)^n\ \frac{n}{2} & 0 & 0 & \frac{n}{2}\Delta_3(3,1) \\ \hline 0 & 0 & 0 & 0 \\ \hline (-1)^n\ \frac{n}{2} & 0 & 0 & \frac{n}{2}\Delta_3(1,1) \end{array}\right]$$

Table 20: $\left[\Gamma_r^c\right]$ Matrices

$$\left[\Gamma_1^c\right] = \begin{bmatrix} (-1)^n\left(\frac{n^2}{2}-1\right) & 0 & 0 & \frac{n}{2}\Gamma_1^c(3,1) \\ 0 & \lambda_n & \eta_n & 0 \\ (-1)^n\frac{n^2}{2} & -g_3\, n\, \eta_\ell & g_3\, n\, \lambda_\ell & \frac{n}{2}\Gamma_1^c(1,1) \end{bmatrix}$$

$$\left[\Gamma_2^c\right] = \begin{bmatrix} 0 & \left(\frac{A}{2}-g_3\, n\, \eta_n\right) & \left(\frac{B}{2}+g_3\, n\, \lambda_n\right) & \frac{n}{2}\Gamma_2^c(3,1) \\ (-1)^n n & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{n}{2}\Gamma_2^c(1,1) \end{bmatrix}$$

$$\left[\Gamma_3^c\right] = \begin{bmatrix} (-1)^n\frac{n^2}{2} & \left(-\frac{A}{2}-g_3\, n\, \eta_n\right) & \left(-\frac{B}{2}+g_3\, n\, \lambda_n\right) & \frac{n}{2}\Gamma_3^c(3,1) \\ 0 & 0 & 0 & 0 \\ (-1)^n\frac{n^2}{2} & 0 & 0 & \frac{n}{2}\Gamma_3^c(1,1) \end{bmatrix}$$

$$\left[\Gamma_4^c\right] = \begin{bmatrix} 0 & -\frac{A}{2} & -\frac{B}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where:
$$A = -(1+\nu)\,\lambda_{n-1} + g_3\,(n-0.5)^2\,\eta_{n-1}$$
$$B = -(1+\nu)\,\eta_{n-1} - g_3\,(n-0.5)^2\,\lambda_{n-1}$$

Table 21: $\left[\Delta_r^c\right]$ Matrices

$$\left[\Delta_1^c\right] = \begin{bmatrix} (-1)^n\left(\frac{n}{2}+1\right) & 0 & 0 & \frac{n}{2}\Delta_1^c\,(3,1) \\ 0 & -\eta_n & \lambda_n & 0 \\ (-1)^n \; \frac{n}{2} & -g_3\,n\,\lambda_n & -g_3\,n\,\eta_n & \frac{n}{2}\Delta_1^c\,(1,1) \end{bmatrix}$$

$$\left[\Delta_2^c\right] = \begin{bmatrix} 0 & \left(-\frac{B}{2}-g_3\,n\,\lambda_n\right) & \left(\frac{A}{2}-g_3\,n\,\eta_n\right) & \frac{n}{2}\Delta_2^c\,(3,1) \\ (-1)^n & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{n}{2}\Delta_2^c\,(1,1) \end{bmatrix}$$

$$\left[\Delta_3^c\right] = \begin{bmatrix} (-1)^n \; \frac{n}{2} & \left(\frac{B}{2}-g_3\,n\,\lambda_n\right) & \left(-\frac{A}{2}-g_3\,n\,\eta_n\right) & \frac{n}{2}\Delta_3^c\,(3,1) \\ 0 & 0 & 0 & 0 \\ (-1)^n \; \frac{n}{2} & 0 & 0 & \frac{n}{2}\Delta_3^c\,(1,1) \end{bmatrix}$$

$$\left[\Delta_4^c\right] = \begin{bmatrix} 0 & \frac{B}{2} & -\frac{A}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Table 22: $\left[\Gamma_r^V\right]$ **Matrices**

$$\left[\Gamma_1^V\right] = \left[\begin{array}{c|c|c|c} (-1)^n\left[-\dfrac{n^2}{2}+1\right] & 0 & 0 & \dfrac{n}{2}\Gamma_1^V(3,1) \\ \hline 0 & 0 & 0 & 0 \\ \hline (-1)^n\,\dfrac{n^2}{2} & 0 & 0 & \dfrac{n}{2}\Gamma_1^V(1,1) \end{array}\right]$$

$$\left[\Gamma_2^V\right] = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & \dfrac{n}{2}\Gamma_2^V(3,1) \\ \hline (-1)^n\,n & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \dfrac{n}{2}\Gamma_2^V(1,1) \end{array}\right]$$

$$\left[\Gamma_3^V\right] = \left[\begin{array}{c|c|c|c} (-1)^n\,\dfrac{n^2}{2} & 0 & 0 & \dfrac{n}{2}\Gamma_3^V(3,1) \\ \hline 0 & 0 & 0 & 0 \\ \hline -(-1)^n\,\dfrac{n^2}{2} & 0 & 0 & \dfrac{n}{2}\Gamma_3^V(1,1) \end{array}\right]$$

Table 23: $\left[\Delta_r^V\right]$ Matrices

$$\left[\Delta_1^V\right] = \left[\begin{array}{c|c|c|c} (-1)^n\left[-\frac{n}{2}-1\right] & 0 & 0 & \frac{n}{2}\Delta_1^V(3,1) \\ \hline 0 & 0 & 0 & 0 \\ \hline (-1)^n\ \frac{n}{2} & 0 & 0 & \frac{n}{2}\Delta_1^V(1,1) \end{array}\right]$$

$$\left[\Delta_2^V\right] = \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & \frac{n}{2}\Delta_2^V(3,1) \\ \hline (-1)^n & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{n}{2}\Delta_2^V(1,1) \end{array}\right]$$

$$\left[\Delta_3^V\right] = \left[\begin{array}{c|c|c|c} -(-1)^n\ \frac{n}{2} & 0 & 0 & \frac{n}{2}\Delta_3^V(3,1) \\ \hline 0 & 0 & 0 & 0 \\ \hline -(-1)^n\ \frac{n}{2} & 0 & 0 & \frac{n}{2}\Delta_3^V(1,1) \end{array}\right]$$

APPENDIX VI

TABLES AND FIGURES

Table 1: Numerical cases considered

Case	Geometry			Boundary conditions	References
	R (inches)	t (inches)	ϕ_o (degrees)		
1	20	1	60	Hinged shell	[33], [45] [47], [49]
				Free shell	[33], [45] [46]
				Clamped shell	[33], [45] [46]
2	20	1	30	Clamped shell	[35]
3	50	1	90	Simply supported shell	[18]

Table 2: Case I - Non-dimensional natural frequencies

$$\Omega = \omega R \left(\frac{\rho}{E}\right)^{1/2}$$

Freq. no.	Kalnins [33]	Navaratna [45]	Webster [46]	Greene et al [47] (*)	Cohen [49]	Present theory
Hinged shell						
	Axisymmetric vibration (**)					
1	0.962	0.963		0.989	0.959	1.044
2	1.334	1.338		1.338	1.325	1.788
3	-	1.653		1.652	1.646	2.152
4	2.128	2.131		2.162		2.306
5	-	2.141				2.509
6	3.176	3.185				2.967
7	3.988	3.993				3.027
8	-	4.159				3.657
9	4.575	4.601				3.760
10	-	6.031				5.405
11	6.231	6.267				5.657
Free shell						
1	0.931	0.932	0.931			0.418
2	1.088	1.094	1.089			1.010
3	1.533	1.544	1.535			1.284
4	2.348	2.363	2.360			1.994
5	2.544	2.548	2.551			2.339
6	-	2.982	2.985			2.408
7	3.497	3.519	4.023			2.725
8	-	4.971	4.950			2.992
9	4.951	4.980	5.548			3.691
10	5.230	5.543	6.224			3.770
11	6.693	6.734	6.862			5.360
Clamped shell						
1	1.006	1.008	1.007			1.745
2	1.391	1.395	1.391			1.902
3	-	1.702	1.700			2.309
4	-	2.126	2.095			2.384
5	2.375	2.387	2.386			2.700
6	3.486	3.506	3.851			3.199
7	3.991	3.996	4.062			3.691
8	-	4.159	4.151			4.191
9	4.947	5.001	5.962			5.390
10	-	6.037	6.208			5.627
11	6.690	6.729	7.148			6.606

(*) results obtained with 10 elements
(**) for information

Table 3: Case 1 - Displacement ratios

Boundary conditions	$\frac{(U_\alpha)max}{(W)max}$	$\frac{(U_\theta)max}{(W)max}$
Hinged shell	0.0341	0.2361
Free shell	0.3508	0.4880
Clamped shell	0.6554	0.8021

Table 4: Case 2 - Non-dimensional natural frequencies

$$\Omega = \omega R \left(\frac{\rho}{E}\right)^{1/2}$$

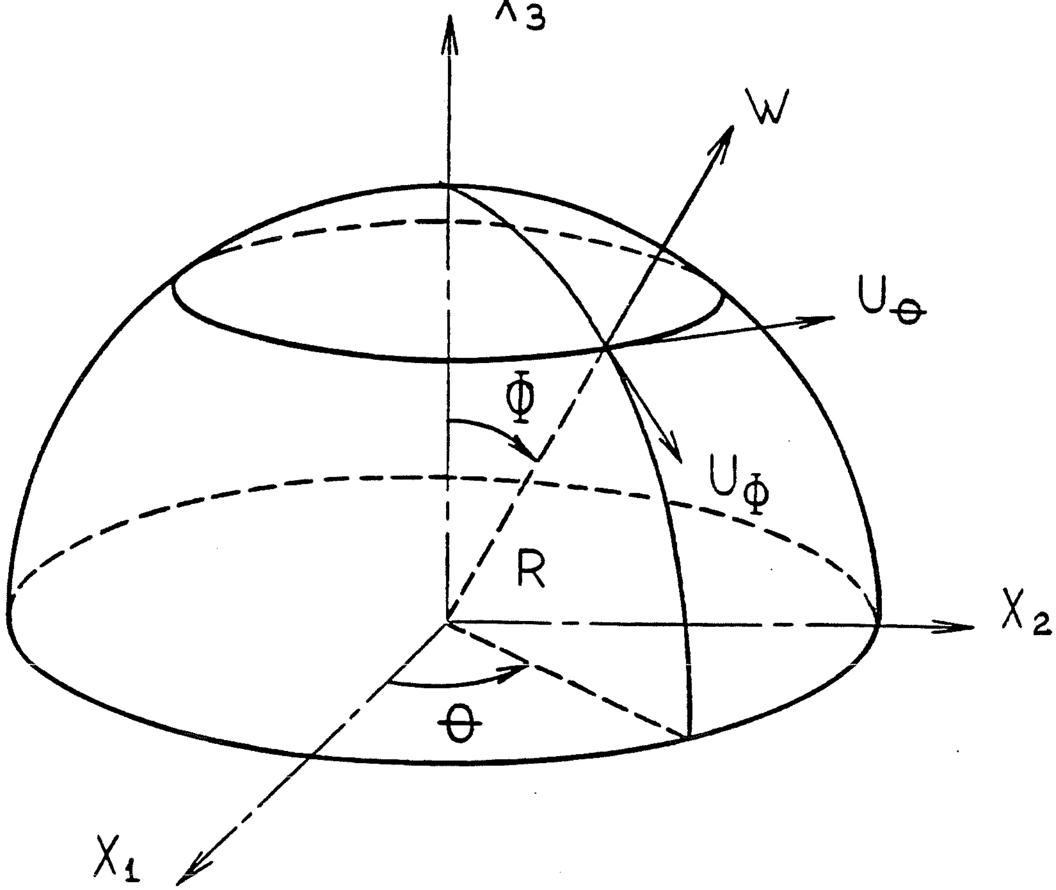
Freq. no.	Kalnins [35]		Present theory
	Classical	Transverse	
1	1.168	1.182	2.393
2	2.589	2.611	2.562
3	3.230	-	2.926
4	4.288	-	3.623
5	4.683	4.683	5.629

Table 5: Case 3 - Non-dimensional natural frequencies

$$\Omega = \omega R \left(\frac{\rho}{E}\right)^{1/2}$$

Freq. no.	Kraus [18] (*) n = 0		Present theory
	Ω_L	Ω_{RN}	n = 2
1	0.7548	0.7547	0.5964
2	0.9432	0.9428	0.6843
3	1.0152	1.0142	0.7441
4	1.1082	1.1056	0.9034
5	1.2523	1.2464	1.2759
6	1.4576	1.4461	1.5841
7	1.6558	1.6502	1.8900
8	1.7636	1.7457	2.2006

(*) For information



<u>Figure 1</u>: Coordinate systems

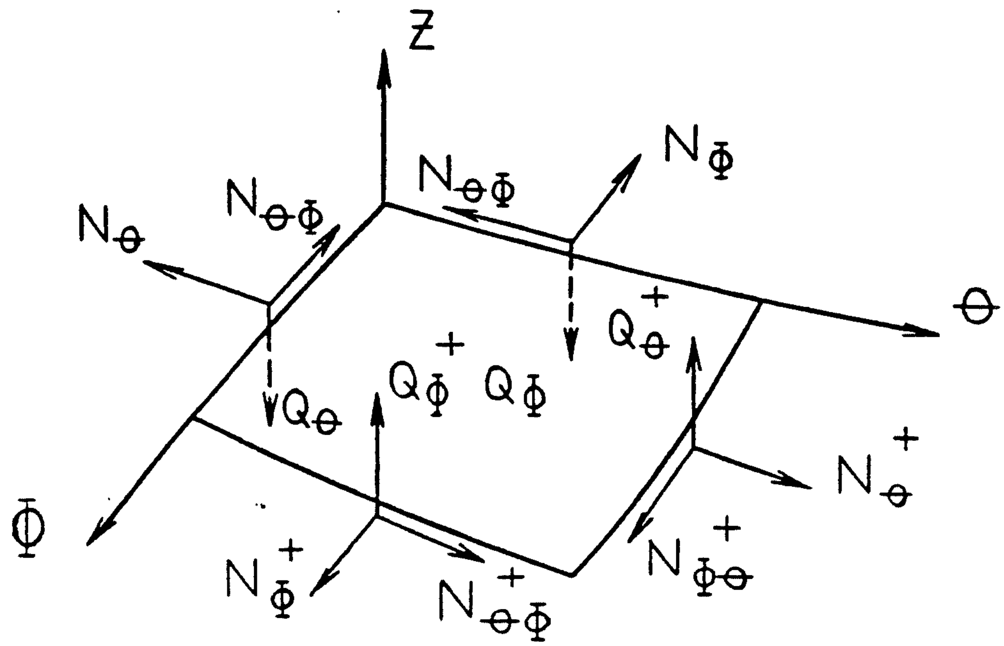


Figure 2: Membrane stress resultants

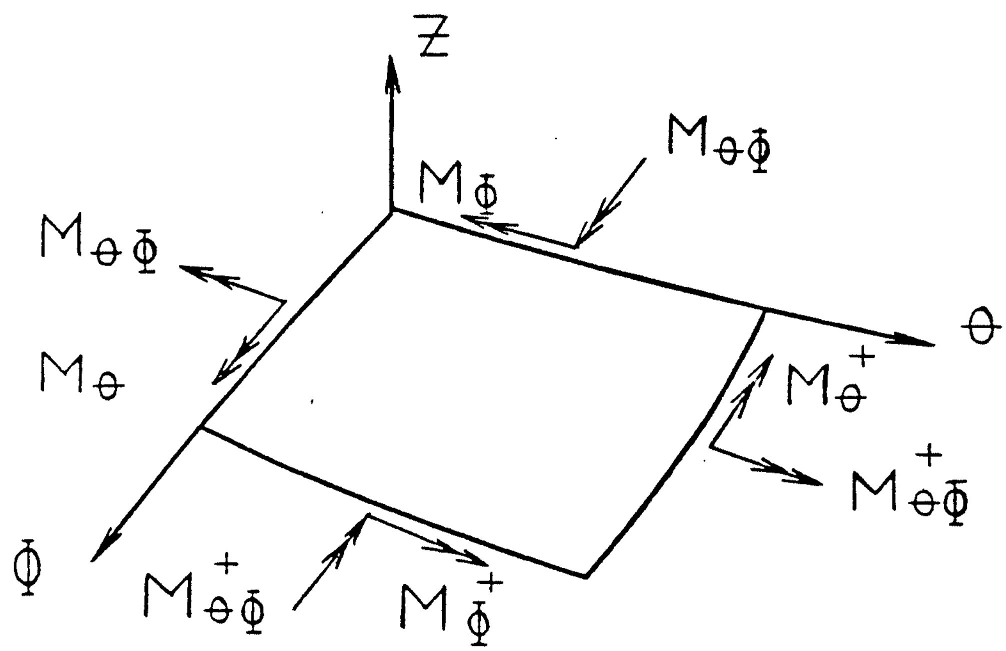


Figure 3: Bending stress resultants

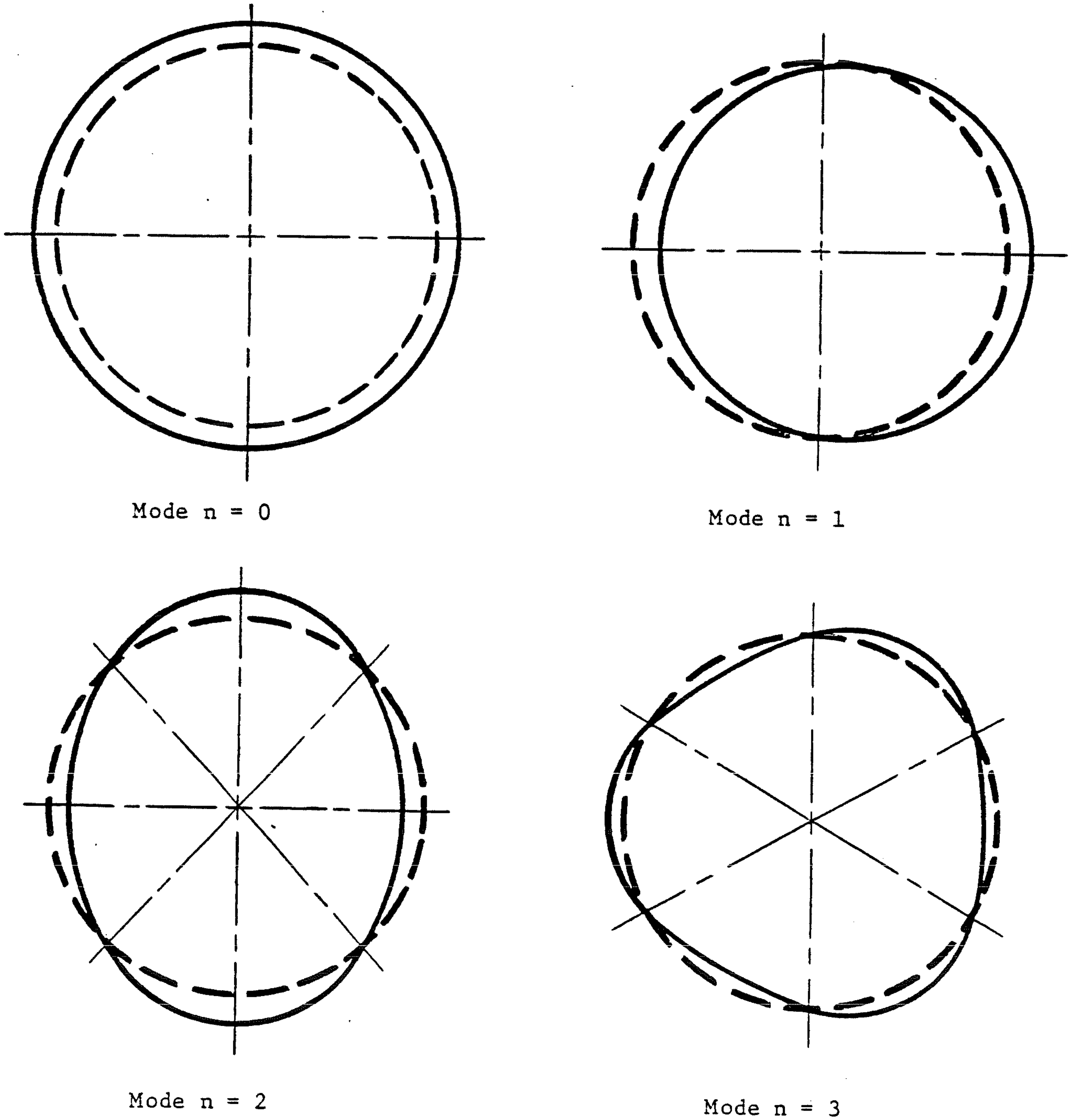
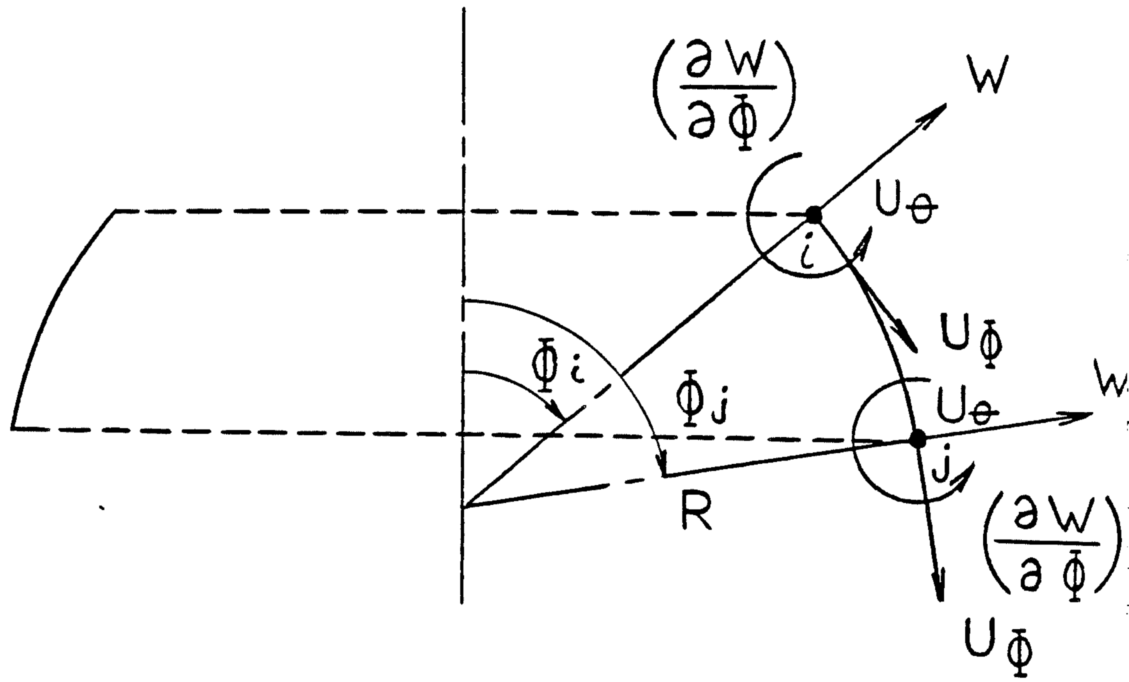
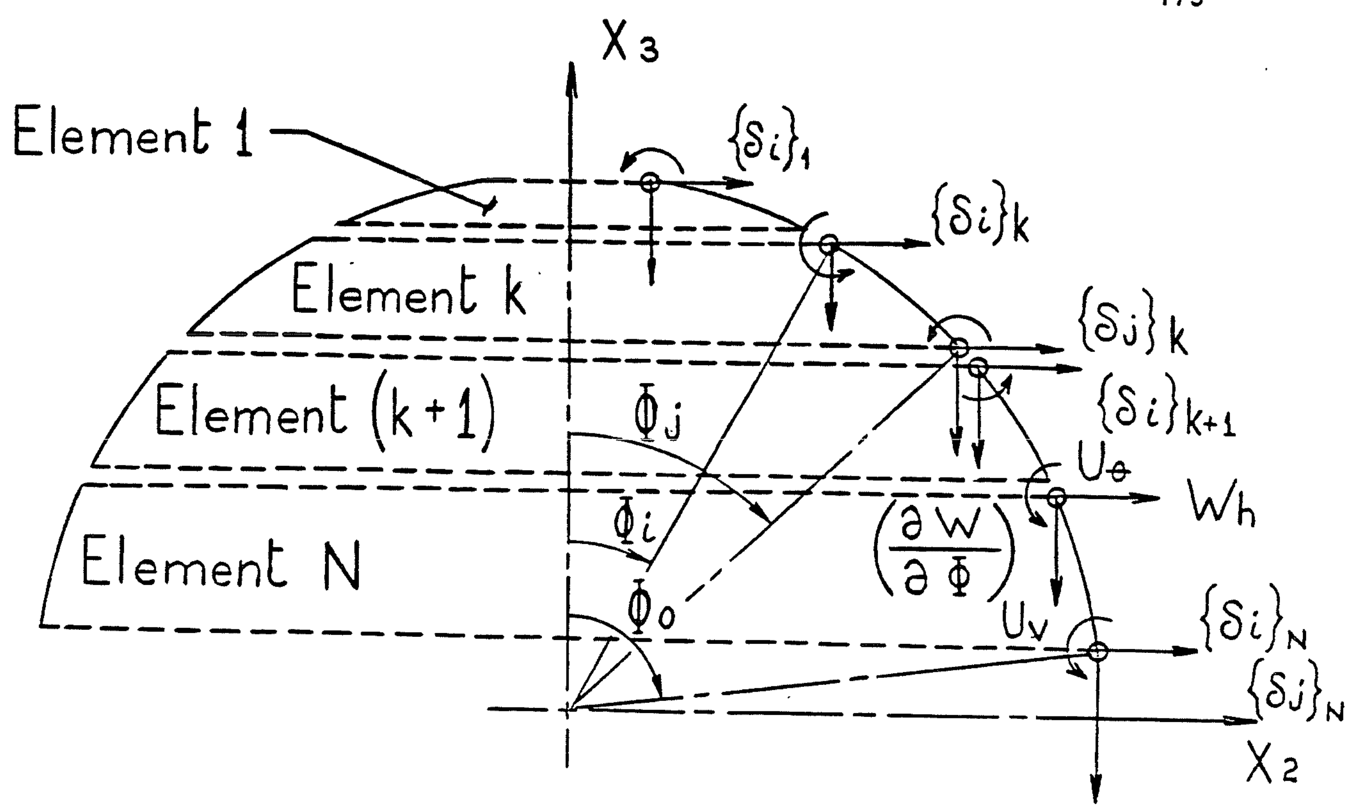


Figure 3a: Pattern of some vibration modes

Figure 4: Displacements at finite element nodes





$$\{\delta\}_{glo} = [DLAM]\ \{\delta\}_{loc}$$

$$\left\{ U_v^i, W_h^i, \left(\frac{\partial w}{\partial \phi}\right)^i, U_\theta^i, U_v^j, W_h^j, \left(\frac{\partial w}{\partial \phi}\right)^j, U_\theta^j \right\}^T$$

$$= [DLAM]\ \left\{ U_\phi^i, W^i, \left(\frac{\partial w}{\partial \phi}\right)^i, U_\theta^i, U_\phi^j, W^j, \left(\frac{\partial w}{\partial \phi}\right)^j, U_\theta^j \right\}^T$$

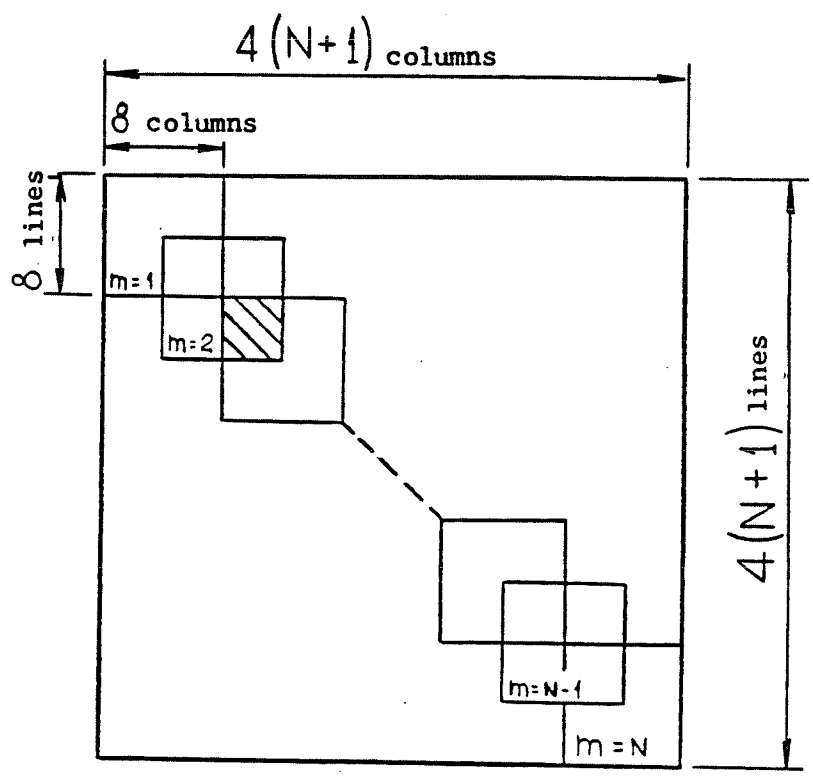


Figure 5: Construction of gobal mass and stiffness matrices

Figure 6: Normalized displacements vs. ϕ coordinate
Case 1 - Hinged shell - Ω = 1.044

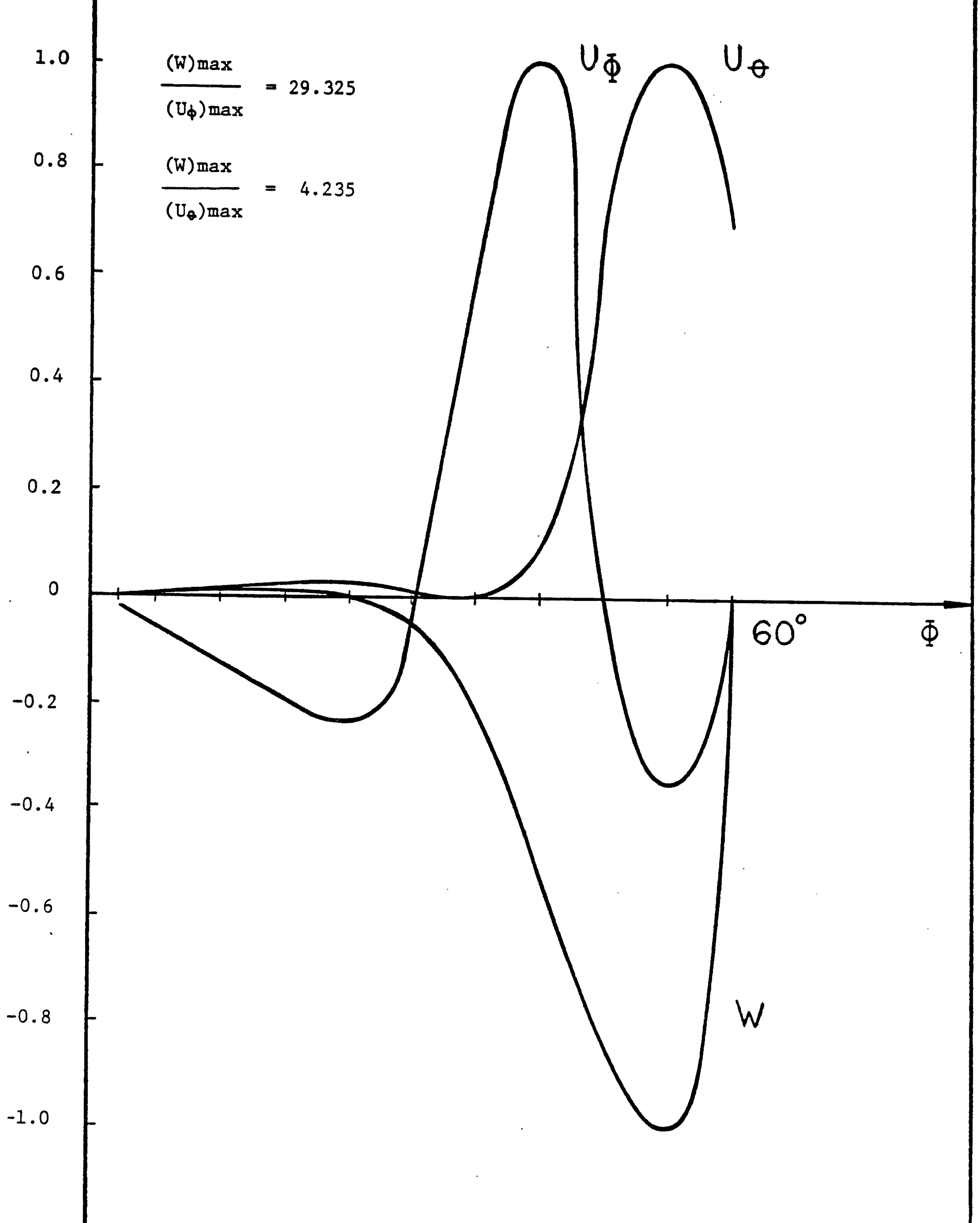
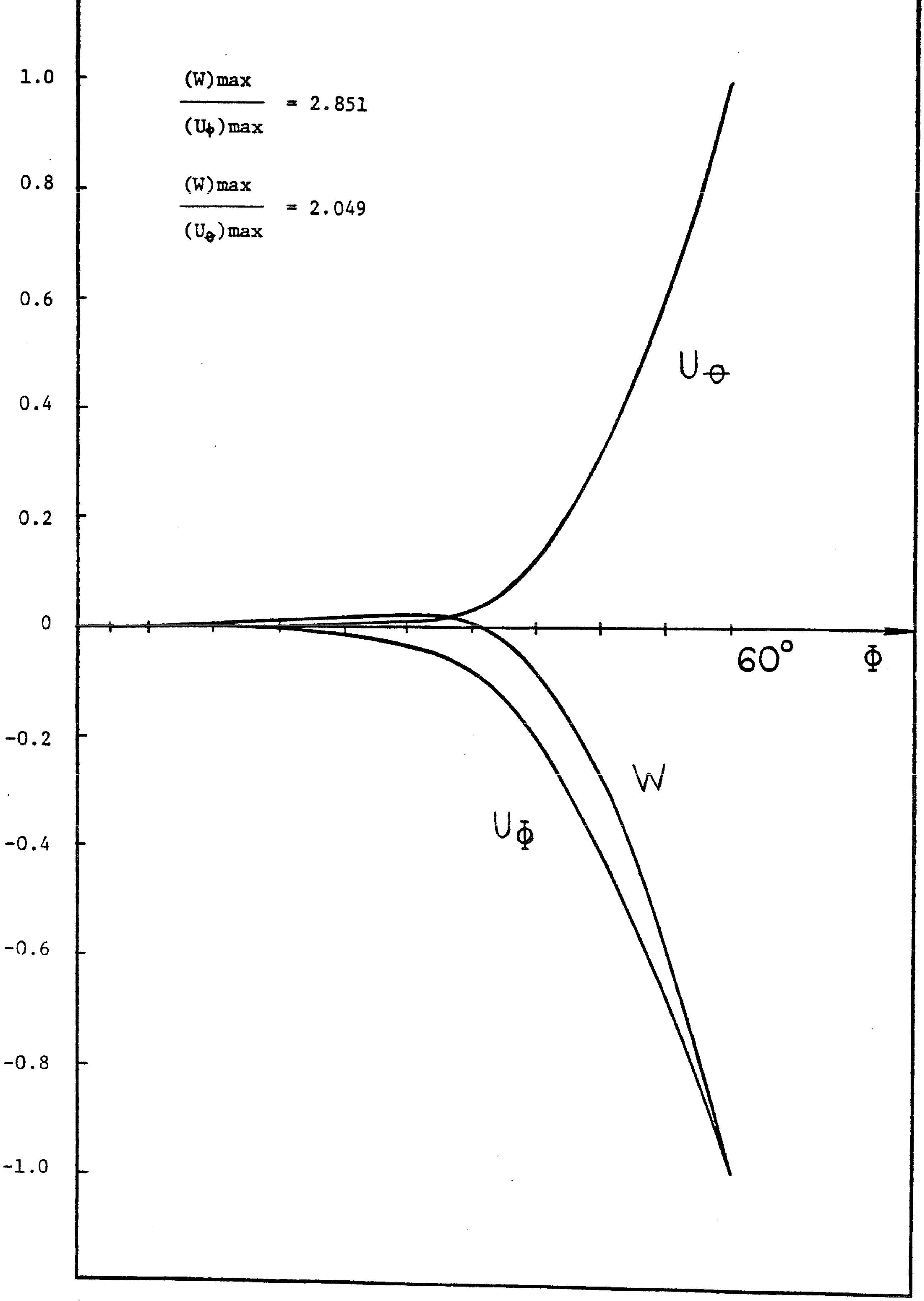


Figure 7: Normalized displacements vs ϕ coordinate
Case 1 - Free shell - Ω = 0.418



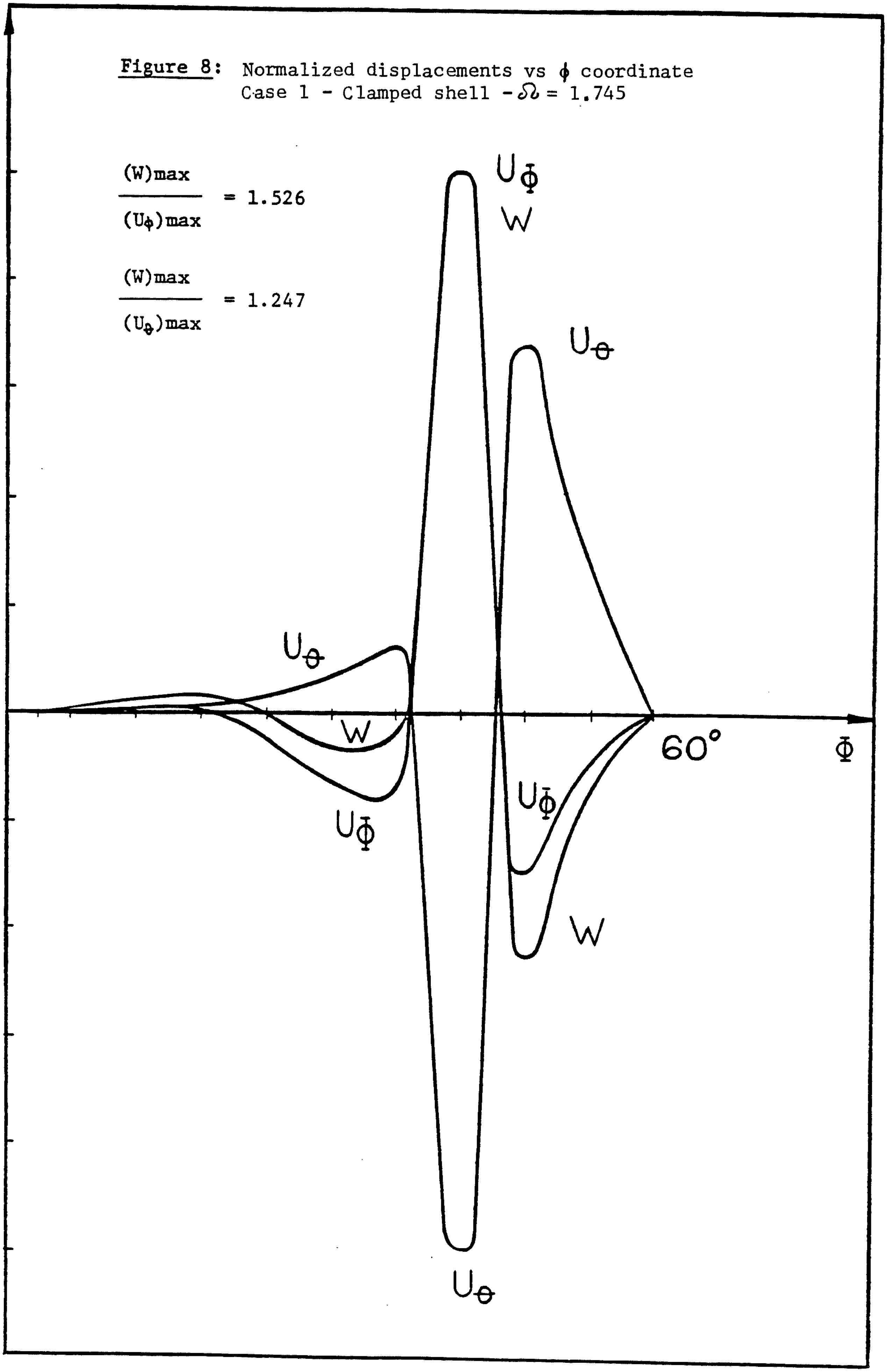


Figure 8: Normalized displacements vs ϕ coordinate
Case 1 - Clamped shell - Ω = 1.745
(W)max / (Uϕ)max = 1.526
(W)max / (Uθ)max = 1.247
1.0
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1.0
Uϕ
W
Uθ
60°
ϕ

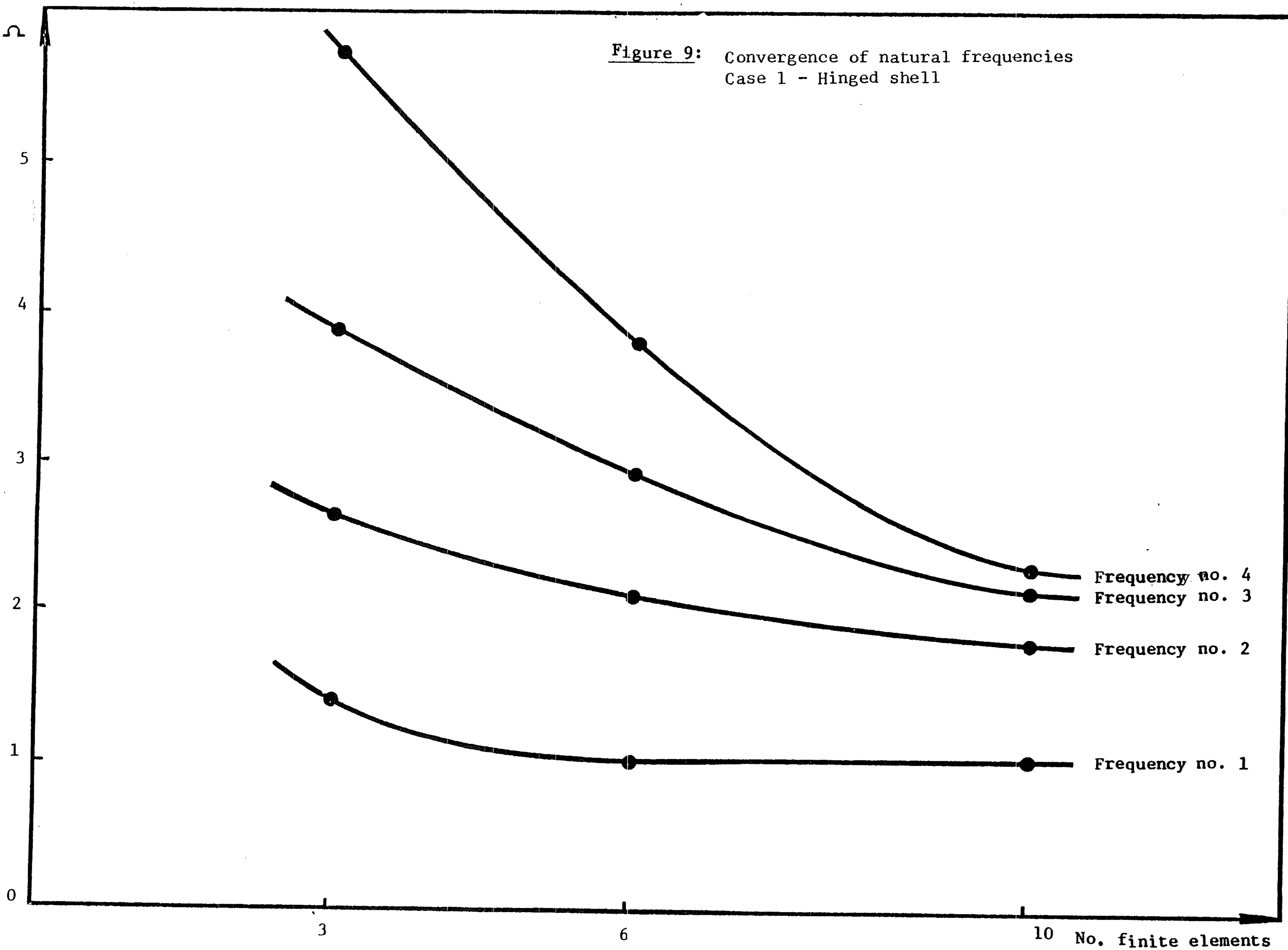


Figure 9: Convergence of natural frequencies
Case 1 - Hinged shell

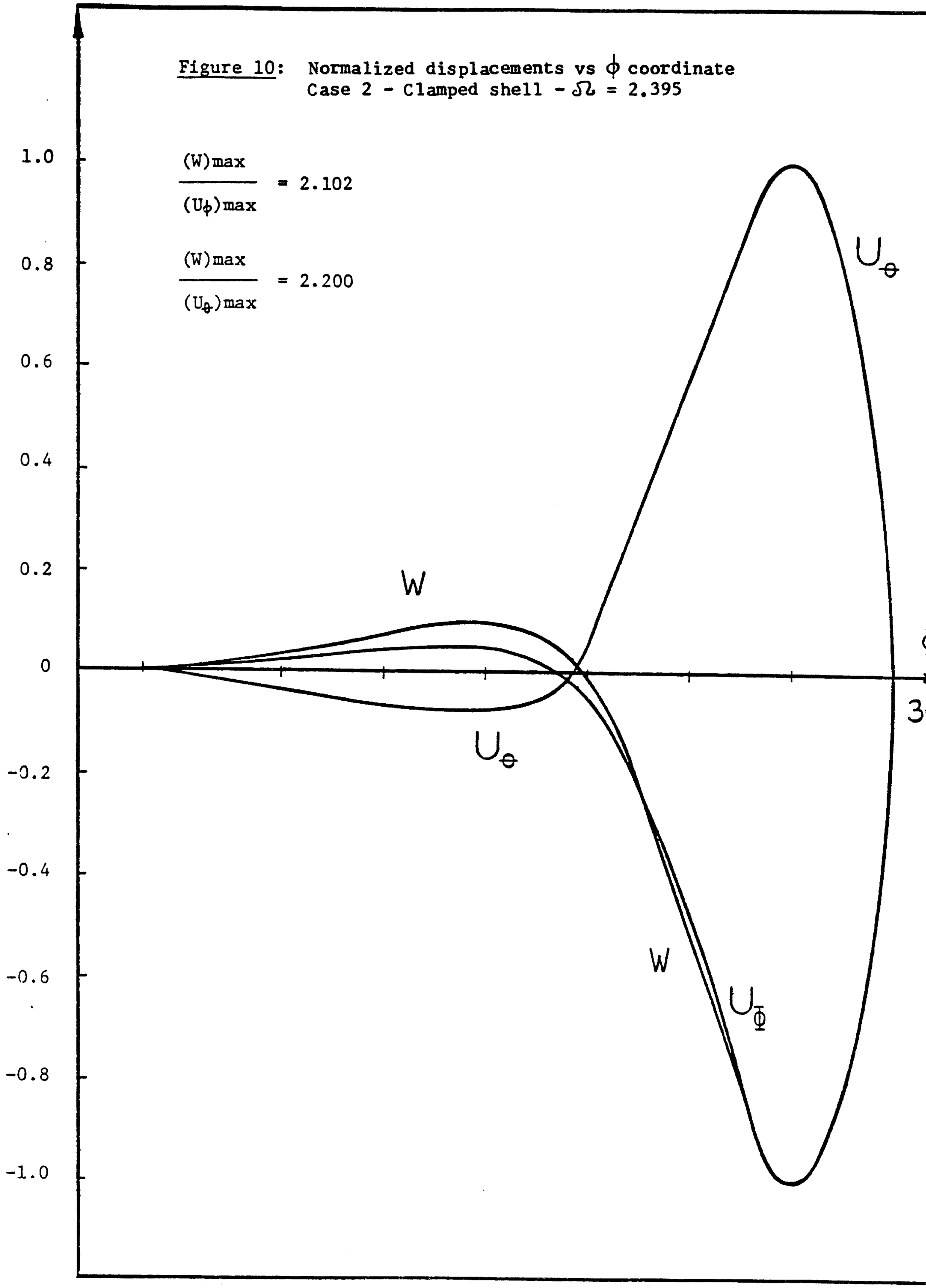
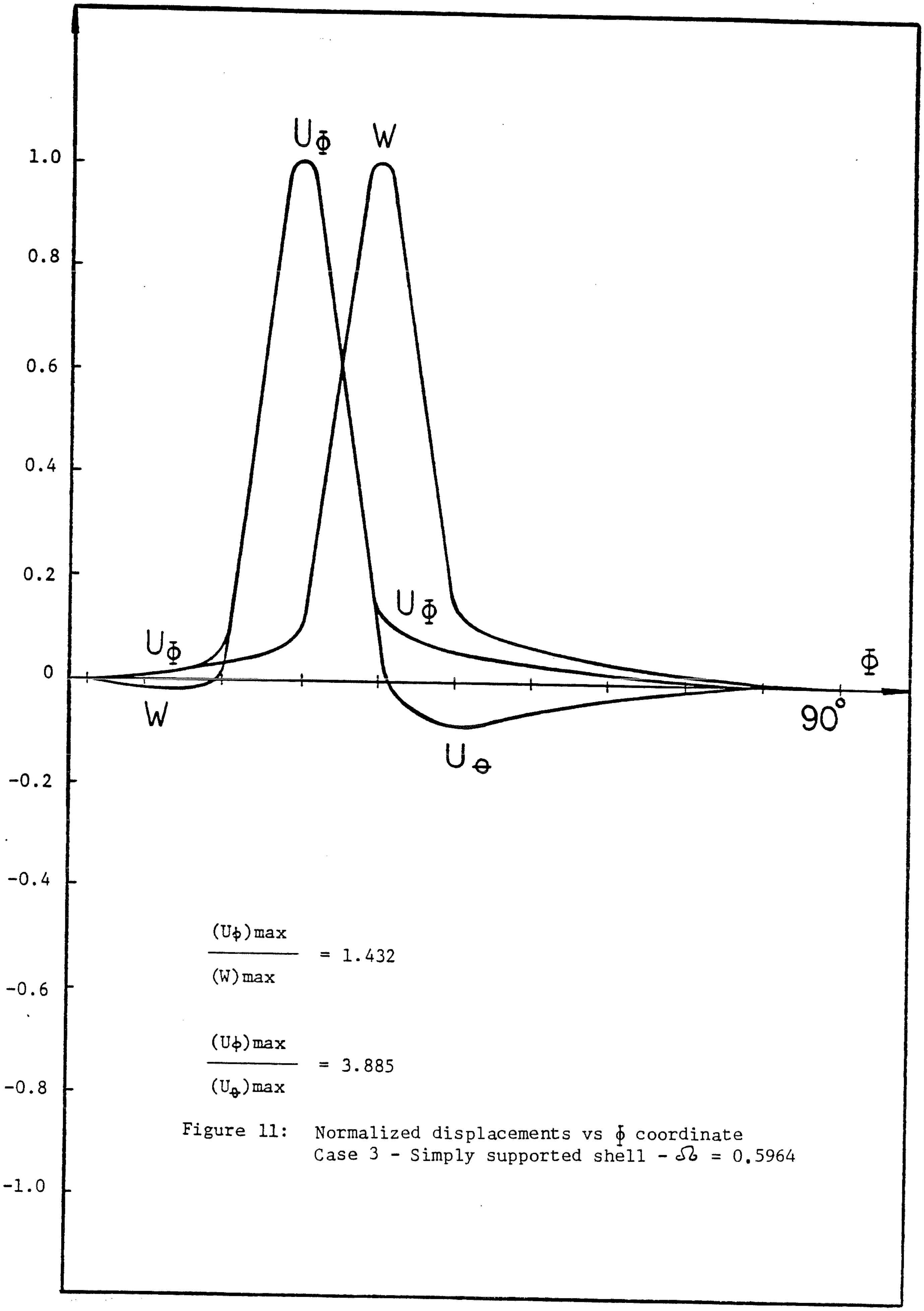


Figure 10: Normalized displacements vs ϕ coordinate
Case 2 - Clamped shell - $\delta\Omega$ = 2.395



$$\frac{(U_\phi)\text{max}}{(W)\text{max}} = 1.432$$

$$\frac{(U_\phi)\text{max}}{(U_\theta)\text{max}} = 3.885$$

Figure 11: Normalized displacements vs ϕ coordinate
Case 3 - Simply supported shell - Ω = 0.5964

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