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DYNAMIC ANALYSIS OF A TIMOSHENKO BEAM USING A FINITE ELEMENT APPROACH

by

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ABSTRACT

The finite element method was used in this investigation in order to determine the static and dynamic behaviour of non-uniform Timoshenko beams subjected to various boundary conditions. When using this approach, the general displacement functions are based on the cubic polynomial expansion of three principal quantities as degrees of freedom: transverse displacement, rotation of the cross section and shear deformation. The stiffness and mass matrices, however, are derived from expressions of the strain and kinetic energy which are based on the assumptions of curved, slightly curved and straight beams.

In addition, a formula was obtained from the general equation for a slightly curved beam and as a result, frequency equations for a straight beam were obtained for ten common types, taking into account all the forced and natural boundary conditions. Good agreement was achieved by comparing the results of these models with exact solutions, as well as with other numerical methods for straight and tapered beams. In order to test the rate of convergence, the natural frequencies of typical cases were calculated as a function of the number of elements. Results showed that the model converged rapidly and required only a small number of elements to achieve good results. Finally, the present analysis demonstrated the reliability of the model chosen by including the transverse shear and rotary inertia effects at the higher modes.

RÉSUMÉ

Dans cette thèse, la méthode des éléments finis est utilisée pour analyser dynamiquement et statiquement les poutres de Timoshenko non-uniformes et soumises à des conditions aux rives différentes. Dans une telle méthode, les fonctions de déplacement générales sont basées sur le développement en polynôme cubique des trois quantités principales comme degrés de liberté: le déplacement transversal, la rotation de la section transversale et la déformation de cisaillement. Cependant, les matrices de raideur et de masse sont dérivées des expressions de l'énergie de déformation et de l'énergie cinétique qui sont basées sur les hypothèses des poutres courbes, légèrement courbées et droites.

De plus, une formule est obtenue de l'équation générale pour une poutre légèrement courbée et par conséquent, les équations de fréquence d'une poutre droite sont obtenues pour dix cas courants, en tenant compte de toutes les conditions aux limites en charge et naturelles. Un bon accord a été réalisé en comparant les résultats de ces modèles avec des solutions exactes, ainsi qu'avec les autres méthodes numériques pour des poutres droites et diminuées (tapered beams). Afin de tester le taux de convergence, nous avons calculé les fréquences naturelles des cas typiques en fonction du nombre d'éléments. Ce calcul indique que le modèle converge rapidement et ne demande qu'un petit nombre d'éléments pour obtenir de bons résultats. Finalement, la présente analyse démontre la fiabilité du modèle sélectionné en incluant l'effet de cisaillement transversal et d'inertie rationnelle à des modes plus élevés.

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NOMENCLATURE

$a_1, a_2, \ldots a_8$	:	Constants determined in (eq. i) and (eq. j)
$A_1, A_2, \ldots A_8$	:	Constants determined in (2.23) and (2.24)
A	:	Area of cross section
A_a, A_b	:	Area of section at 1st end and 2nd end of a tapered beam
b	:	Beam Width
b_1, b_2	:	Width between 1st end and 2nd end of tapered beam
c	:	Quantity determined in Table 5b
c_1, c_2	:	Upper and lower beam thicknesses
C	:	Quantity determined in Tables 5a and 5b
E	:	Young's modulus
F	:	Quantities determined in Tables 5a and 5b
f	:	Quantities determined in Tables 5b
F_e	:	External force (time-related) by unit length
G	:	Stiffness modulus
H	:	Initial curvature of a curved or slightly curved beam
h	:	Beam thickness
h_1, h_2	:	Thickness between 1st and 2nd end of tapered beam

I_y	:	Moment of inertia around axis y
I_{ya}, I_{yb}	:	Moment of inertia between 1st and 2nd ends of tapered beam
J	:	Number of boundary conditions
J^*	:	Number of lines and/or columns to eliminate in [K] and [M]
k	:	Shear deformation coefficient of a Timoshenko beam
k^*	:	Distortion coefficient of a Timoshenko beam
L	:	Beam length
M_e	:	External moment (time-related) by unit length
M_y	:	Bending moment
N	:	Number of finite elements
NDF	:	Number of degrees of freedom for a uniform element
N_x	:	Normal constraint
n_y, n_z	:	Components following y and z for a normal unit vector on a cross section boundary
P	:	Curvature parameter determined in Tables, 5a, 5b
p	:	Quantity determined in Table 5b

Q	:	Rotary inertia parameter
$q\ (x,t)$	:	Transverse load by unit length applied to the beam
q	:	Quantity determined in Tables 5a, 5b
r	:	Radius of curvature
s	:	Quantity determined in Tables 5a, 5b
S	:	Shear deformation parameter
S_a, S_b	:	Quantity determined in Tables 5a, 5b
t	:	Time
τ	:	Kinetic energy
U, u	:	Axial displacement functions
$\mathcal{U}$	:	Internal deformation energy
$\mathcal{U}_F$	:	Potential energy due to body forces
$\mathcal{U}_R$	:	Potential energy due to external nodal forces
$\mathcal{U}_S$	:	Potential energy due to surface tension
$\mathcal{U}_{SD}$	:	Deformation energy due to shear distortion
V, v	:	Displacement function in direction y
V	:	Quantity determined in Tables 5c, 5d
ϑ	:	Quantity determined in Table 5d
V_Z	:	Vertical force
ω	:	Rotary vibration frequency

W, w	:	Transverse displacement function
w_0	:	Initial form of beam axis
w_a, w_b	:	Quantity determined in (6.7) and (6.8)
$\bar{\bar{W}}$	:	Normal function of w (solution for slightly curved beam)
$\bar{\bar{W}}_0$	:	Solution for straight beam
θ	:	Rotation of cross section
Θ	:	Normal function of (solution for slightly curved beam)
Θ_0	:	Solution for straight beam
θ_a, θ_b	:	Quantities determined in (6.7) and (6.8)
x, y, z	:	Global system coordinates
$\bar{x}, \bar{y}, \bar{z}$	:	Local system coordinates
y_0	:	Non-dimensional quantity determined by (2.19)
ρ	:	Density
λ^2	:	Frequency parameter
η, y	:	Non-dimensional displacements
ν	:	Poisson's ratio
$\chi(z, y)$	:	Function determined in (5.15)
$\varepsilon(x, z, t)$	:	Correction function determined in
ψ	:	Shear deformation

ψ	:	Total shear angle
α_0	:	Angle determined in (A.5.4)
α_z	:	Angle determined in
α	:	Quantity determined in (2.23) and (2.24)
β	:	Quantity determined in (2.23) and (2.24)
β_e	:	Angle determined in
δ_1, δ_2	:	Integration quantities determined by (5.19) and (5.20)
γ	:	Non-dimensional quantity determined in Tables 1 and 2

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CHAPTER 1

INTRODUCTION

1.1 General

In the present research, my colleagues and I were mainly interested in the dynamic behaviour of Timoshenko beams. This study was first undertaken by Rayleigh, who explored the effect of rotational inertia on the beams. After this first analysis, Timoshenko later included cross section and shear deformation effects. These two effects, with hyperbolic characteristics, led to minor modifications in the Bernouilli-Euler theory, where elliptic characteristics were used to calculate the lowest modes of long and thin beams.

Kruszewski [1, 1945] obtained frequency equations for "free-free" and "clamped-free" beams, by solving a complete deflection differential equation with non-homogeneous boundary conditions. These equations, however, were limited to solution for the above-mentioned beams, both because of complex boundary conditions and the time-consuming nature of the task.

Traill-Nash and Collar [2, 1953] presented a relatively complete theoretical solution for the lateral vibration problem with a uniform beam that included shear deflection (but not the delayed shear effect) and rotary inertia for nine cases (the variation in these boundary conditions was a combination of three types of end supports: free, simply supported and clamped). They also demonstrated the consequences of shear deflection and rotary inertia which are essential to a modern understanding of the compact beam.

Similar to the problem mentioned above, Dolph [2, 1954] presented a derivation for several of the findings in Timoshenko's theory, including general solutions and a complete analysis of a simply supported and uniform beam.

Furthermore, Boley and Chaos [4, 1955] studies of the behaviour of transverse beams included the effect of shear and rotary inertia and used an approximation method from the Laplante transformation procedure to solve for the four types of loads applied to a semi-infinite beam step 1: zero velocity and bending moment step 2: zero bending moment and displacements; step 3: zero velocity and force; step 4: zero force and rotation.

Huang [5, 1961] produced a typical analysis for Timoshenko beam convergence such that the frequency equations and the normal free vibration mode for a uniform beam in the different cases (for six ordinary types of simple and finite beams) were a product of homogeneous boundary conditions. Solutions obtained through elastic analysis were produced for two complete differential equations; one for total deflection and the other for rotation of the cross section.

Leckie and Lindberg [6, 1963] offered a alternate system of analysis and validated the accuracy of these methods. This study, which included the important effects mentioned above, was based on exact differential equations for an infinitesimal element in static equilibrium. The first area of rotation chosen, which was the cross section, allowed for the correct and individual boundary conditions at the end of a "free" or "clamped" beam.

Recently, Hurty and Rubinstein [7, 1964] used an approximation method to calculate frequency energy in a simply supported beam. The most common model was the one in Kapur [8, 1966]. This model was based on expansion of the cubic displacement in both cases: for bending (attributed to flexural deformation) and shear (caused by shear deformation). An important finding was that displacement at each node included displacement and rotation.

These displacements were considered separately for stiffness and mass matrices using, as nodal coordinates, flexural and shear deformation, finite deflection and shear slopes. A few of these nodal variables could be eliminated by using a "condensation" method and the size of the system could be reduced without any significant loss of accuracy. The resultant element matrices were (8 x 8), and they could be used evenly between the flexural and shear deformations. Frequency parameters were obtained for "clamped-free" and "simply supported" beams. The frequencies obtained with this method matched the theory for two particular types of beams, but proved to be unsatisfactory in relation to general structure (non-colinear structure). Complexity increased when forces were added to displacements. Each node had then to be specifically studied. The rate of convergence with this method was much faster than with the others.

Another model with a matrix of order (8 x 8) was presented by Carnégie, J. Thomas and Dokumaci [9, 1969]. The element they used, with an internal node, had bending and deflection slopes as coordinates at the two terminal nodes. The middle nodes were used for assignment of the rate of convergence for a beam with rotary inertia and shear deformation. However, some difficulties could occur when the natural free-end boundary conditions were established.

Nickel and Secor [10, 1972] used, as nodal coordinates, total transverse displacement "w", total slope "$\partial w/dx$" and rotation "θ" as a result of bending in the beam-end and mid-beam nodes. Transverse displacement was expressed as a cubic function and rotation, as a quadratic function for the axial coordinates of the beam. The matrix was of the order (7 x 7) for the element, labelled TIM7. It was then reduced to (4 x 4) with the help of a constraint suggested by Egle, labelled TIM4. The two elements produced an unvaried convergence of variables, according to the degrees of freedom in the system.

Then, Davis, Henshell and Waburton [11, 1972] presented an solution approach based on the cubic polynomial for total deflection and rotation. They derived the stiffness matrix for static equilibrium conditions with two degrees of freedom at each node. The boundary conditions were not satisfied in the cases of free-end and simply supported systems.

In 1973, an element developed by Thomas, J.J. Wilson and R.R. Wilson [12] gave an acceptable rate of convergence for calculating the natural frequencies of a simple element as waell as for the "clamped-free" beam.

1.2 Research objectives

This thesis is an attempt to study the vibration problem in a more general model than that of the Timoshenko beam, i.e. in uniform or non-uniform, slightly curved or straight beams. The finite element chosen had two nodes with four degrees of freedom at each node (in the particular case where the element forms an angle with the x axis, there were five degrees of freedom). The nodal coordinates of each element were: total deflection, rotation, deflection slope and the first derivative of total slope. All boundary conditions (forced or natural) could be imposed and rate of convergence was sufficiently rapid.

The problem was studied with all possible displacement functions, including the case of a slightly curved beam. The results obtained for these two types of beam agree very well with results obtained through exact solutions. Comparisons with other numerical methods in use among different authors are also performed.

1.3 Brief summary of the report

This study contains nine chapters, the separate contents of which are summarized below in order to set out the global aspect of the problem.

CHAPTER 1: General history of the Timoshenko beam.

CHAPTER 2: A review of the basic theory concerning the Timoshenko beam, from curved configurations to straight, including finite element analysis of their dynamic behaviour. Analytical solutions of equations of motion derived for different boundary conditions will be presented in this chapter. As to the finite element method, the choice of displacement function that will make for the best analytic model in terms of the convergence criteria will be discussed.

CHAPTER 3: Detailed information regarding the displacement functions that were selected from among three different possibilities.

CHAPTER 4: The development of the matrices: the finite element method is used to build the stiffness and mass matrices of the beam elements. The analysis will also include a development for tapered beams.

CHAPTER 5: Presentation of a graph for determination of a form factor k and allowance for distortion of the cross section of the problem.

CHAPTER 6: Study of free vibration problems. The main purpose of this chapter is to determine the free vibration for all boundary conditions. A static analysis with arbitrary end loadings on the beam nodes will also be presented.

CHAPTER 7: Description of the procedure for computing free vibrations and eigen vectors corresponding to the ten commonest cases.

CHAPTER 8: Calculations and general discussion of results obtained by the computer program.

CHAPTER 9: Conclusions

CHAPTER II

BASIC THEORY

2.1 Slightly curved and straight beam theory

Using Timoshenko's model as a guideline, the general definitive solution of the vibration problem is only obtainable from the equations of motion derived from slightly curved beams (with a simplified case being the straight beam). The formula is based upon the assumption that the large ratio of elastic modulus over shear modulus (E/G) is E/G values may vary from 20 to 50.

First, the assumption is made that the beam under investigation is slightly curved, e.g. the neutral axis of the beam is initially quadratic in form, so that initial curvature "H" then has a constant value (the beam where H varies along the axis and the tapered beam will be discussed in Chapter IV). In this particular case, the beam then becomes straight and this is considered to be the two major effects of Timoshenko theory (rotary inertia and transverse shear). It is assumed that the beam is homogeneous, isotropic, uniform and prismatic (Fig. 1).

2.1.1 Equations of motion

The length of the beam as a uniform element will now be considered and mentioned hereafter as (slightly curved) vibrating in the main plane. The y and z axes will act as coordinates: they refer to the main axes of the cross section, with the x axis being the "centroidal" axis (Fig. 2).

The basic equations of motion are:

$$\frac{\partial V_z}{\partial x} = \rho \frac{\partial^2 w}{\partial t^2} \tag{2.1}$$

$$\frac{\partial N_x}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \tag{2.2}$$

$$\frac{\partial M_y}{\partial x} + V_z = \frac{\rho I_y}{A} \frac{\partial^2 \theta}{\partial t^2} \tag{2.3}$$

$$q(x,t) + \frac{\partial V_z}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial w_o}{\partial x} N_x\right) = \rho \frac{\partial^2 w}{\partial t^2} \tag{2.4}$$

where V_z and M_y are the vertical forces and bending moments acting on the ends of the element; u and w are the axial and transverse displacements of the beam, $w_o(x)$ is the initial axis of the beam; N_x is the normal constraint on the cross section at the 2 ends; q(x,t) is the external transverse load on the beam per unit of length: P is the density of the element; I_y is the moment of inertia with respect to axis y: A is the element's cross sectional area and t is the time variable.

The bending moment, vertical force and normal constraint are given by:

$$M_y = \int_A z\sigma_x dA = \int_{-c_1}^{c_2} bz\sigma_x dz \tag{2.5}$$

$$V_z = \int_A \tau_{xz} dA = \int_{-c_1}^{c_2} b\tau_{xz} dz \tag{2.6}$$

$$N_x = \int_A \sigma_x dA \tag{2.7}$$

where $dA = dydz$, the distance measured from axis x of the beam along axis z; c_1 and c_2 are the upper and lower boundaries, respectively, of beam thickness; b is the total length of the cross section (generally b is a function of x); σ_x and τ_{xz}, respectively, are the total normal and shear stresses of the beam.

Note that $\tau_{xz} = 0$ when $z = -c_1$ or $z = -c_2$. Two functions are proposed:

$$U(x,z,t) = u(x,t) - z\theta(x,t)$$

and

$$W(x,z,t) = w(x,t)$$

Where (U,u) and (W,w) are displacements in the direction of x and z, respectively; θ is the rotation of the cross section around the y axis.

Transverse displacement $w(x,t)$ can be considered as vertical displacement of the "centroid" of each cross section. It could also be assumed that for infinitesimal motion in the y direction (non-coupled motion), σ_z and $\partial b/\partial z$ are very small: the products which include these terms, however, should not be ignored.

Thus, the appropriate components of the stress tensor for these assumptions can be written (see b) of Table 3):

$$\begin{aligned} \sigma_x &= E\varepsilon_x = E\,\frac{\partial U}{\partial x} = E\left(\frac{\partial u}{\partial x} - z\,\frac{\partial \theta}{\partial x}\right) \\ \tau_{xz} &= G\gamma_{xz} = G\left(\frac{\partial W}{\partial x} + \frac{\partial U}{\partial z}\right) = G\left(\frac{\partial w}{\partial x} - \theta\right) \\ \sigma_z &= 0. \end{aligned} \qquad (2.8)$$

where ε_x and γ_{xz} represent normal and shear deformation.

$$M_y = E \int_{-C_1}^{C_2} \left(\frac{\partial u}{\partial x} - z \frac{\partial \theta}{\partial x}\right) bz\,dz = EI_y \frac{\partial \theta}{\partial x} \tag{2.9}$$

and $$V_z = G \left(\frac{\partial w}{\partial x} - \theta\right) \int_{-C_1}^{C_2} b\,dz = AG \left(\frac{\partial w}{\partial x} - \theta\right) \tag{2.10}$$

The term $\frac{\partial w}{\partial x} - \theta$ represents the reference shear deformation at each cross section.

A correction factor is required in order to compensate for the assumption that xxxx is a constant at each cross section. Thus, the numerical factor "k" is introduced into (2.10) in such a way that:

$$V_z = kGA \left(\frac{\partial w}{\partial x} - \theta\right) \tag{2.11}$$

k is called the shear deformation coefficient of the beam (it will be discussed in detail in Chapter V).

All natural boundary conditions require that the bending moments and vertical forces specified at a point along the beam must satisfy equations (2.9) and (2.11), whereas all those that involve rotation and transverse displacements are cancelled out.

With the influence of the bending moment, we obtain, in the absence of shear:

$$\tan^{-1} \frac{\partial w}{\partial x} \simeq \frac{\partial w}{\partial x} = \theta$$

If the transverse shear deformation at the neutral axis of the beam is taken into accounts, the following relationshp can be obtained:

$$\frac{\partial w}{\partial x} = \theta + \psi \tag{2.12}$$

Where $\psi = \gamma_{xz}$ is the shear deformation of the beam. Finally, the normal constraint N_x can be represented as:

$$N_x = EA\left(\frac{\partial u}{\partial x} + \frac{\partial w_o}{\partial x} \cdot \frac{\partial w}{\partial x}\right) \tag{2.13}$$

Substituting (2.9) and (2.11) into (2.1), (2.2), (2.3), (2.4), we obtain:

$$EI_y \frac{\partial^2 \theta}{\partial x^2} + kGA\left(\frac{\partial w}{\partial x} - \theta\right) - \rho I_y \frac{\partial^2 \theta}{\partial t^2} = 0 \tag{2.14}$$

$$q(x,t) + kGA\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial x}\left(N_x \frac{\partial w_o}{\partial x}\right) - \rho A \frac{\partial^2 w}{\partial t^2} = 0 \tag{2.15}$$

For sufficiently large curvatures, longitudinal inertia could be ignored by setting $\frac{\partial^2 u}{\partial t^2} = 0$ in (2.2), with the normal constraint N_x taking a constant value N_o.

(2.13) can be integrated by considering u = w = 0 and x = 0 to x = L for problems in which beam ends are fixed, the expression obtained for N_x will be:

$$N_x = N_o = -\frac{EA}{L}\int_0^L \frac{\partial^2 w_o}{\partial x^2}\, w\, dx \qquad (2.16)$$

Where L is the total length of the beam.

Substituting (2.16) into (2.15), the equations of equilibrium for a uniform beam, in terms of W and θ, can be written:

$$EI_y \frac{\partial^2\theta}{\partial x^2} + kGA\left(\frac{\partial w}{\partial x} - \theta\right) = \rho I_y \frac{\partial^2\theta}{\partial t^2} \qquad (2.17)$$

$$q(x,t) + kGA\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\theta}{\partial x}\right) + \frac{\partial^2 w_o}{\partial x^2}\cdot\frac{EA}{L}\int_0^L \frac{\partial w_o}{\partial x}\,\frac{\partial w}{\partial x}\, dx = \rho A \frac{\partial^2 w}{\partial t^2} \qquad (2.18)$$

These two equations are valid for for $\frac{\partial w_o}{\partial x}^2 \leqslant 1$ (that is to say, for slightly curved beam), where $\frac{\partial w_o}{\partial x}$ represents the initial slope of the neutral axis of the beam.

Then write: $w_o = Hy_o$

with $y_o = 4\left(\frac{x}{L}\right)\left(1 - \frac{x}{L}\right)$ (2.19)

Where H is the maximal value of w_o and y_o is a non-dimensional quantity.

Also obtained is:

$$w = h\,\bar{W} \sin \omega t$$

and $\theta = \frac{h}{L}\,\theta \sin \omega t$

Where beam thickness (h = [oc_1] + [oc_2]; W and θ represent the normal functions of w and θ, respectively, and are the rotational vibration frequency of the beam.

The non-dimensional quantities are generated by substituting $\eta = \frac{x}{L}$, hence (2.19) becomes:

$$y_o = 4\eta\ (1 - \eta)$$

Then $$\frac{\partial^2 y_o}{\partial \eta^2} = -8$$

(2.17) and (2.18) could then be written in the absence of q(x,t) in the free vibration problem:

$$\frac{\partial^2 \theta}{\partial x^2} + S\left(\frac{\partial \bar{\bar{W}}}{\partial x} - \theta\right) + \lambda^2 Q\theta = 0 \qquad (2.20)$$

$$S\frac{\partial}{\partial x}\left(\frac{\partial \bar{\bar{W}}}{\partial x} - \theta\right) + \lambda^2 \bar{\bar{W}} - 64P\int_0^1 \bar{\bar{W}}\, d\eta = 0 \qquad (2.21)$$

All the non-dimensional parameters in (2.20) and (2.21) are determined as follows: $S = KGAL^2/EI_y$: shear deformation parameter; $\lambda^2 = pAL^4\omega^2/EI_y$: frequency parameter; $Q = I_y/AL^2$: rotary inertia parameter; $P = AH^2/I_y$: deviation parameter.

2.1.2 Analytical solution of the equations of motion

The derived equations (2.20) and (2.21) are used for a slightly curved beam; their solutions are discussed in [25], and are written as:

$$\bar{\bar{W}} = \bar{\bar{W}}_o + \frac{64P}{\lambda^2 - 64P}\int_0^L \bar{\bar{W}}_o\, d\eta \qquad (2.22)$$

$$\theta = \theta_o \qquad (2.23)$$

where w_o and θ_o are the theoretical solutions for straight Timoshenko beams. They have the form:

$$\bar{\bar{W}}_o = A_1(t)\cosh\lambda\alpha\eta + A_2(t)\sinh\lambda\alpha\eta + A_3(t)\cos\lambda\beta\eta + A_4(t)\sin\lambda\beta\eta \tag{2.24}$$

$$\theta_o = A_5(t)\sinh\lambda\alpha\eta + A_6(t)\cosh\lambda\alpha\eta + A_7(t)\sin\lambda\beta\eta + A_8(t)\cos\lambda\beta\eta \tag{2.25}$$

α and β are the non-dimensional quantities which can be expressed in the following two cases:

Case a) When $\sqrt{\left(Q - \frac{1}{S}\right)^2 + \frac{4}{\lambda^2}} > Q + \frac{1}{S}$; $\frac{\lambda^2 Q}{S} > 1$

$$\alpha = \frac{\sqrt{2}}{2}\sqrt{-\left(Q + \frac{1}{S}\right) + \sqrt{\left(Q - \frac{1}{S}\right)^2 + \frac{4}{\lambda^2}}}$$

$$\beta = \frac{\sqrt{2}}{2}\sqrt{\left(Q + \frac{1}{S}\right) + \sqrt{\left(Q - \frac{1}{S}\right)^2 + \frac{4}{\lambda^2}}} \tag{2.26a}$$

Case b) When $\sqrt{\left(Q - \frac{1}{S}\right) + \frac{4}{\lambda^2}} < Q + \frac{1}{S}$; $\frac{\lambda^2 Q}{S} < 1$

$$\alpha = j\,\frac{\sqrt{2}}{2}\sqrt{\left(Q + \frac{1}{S}\right) - \sqrt{\left(Q - \frac{1}{S}\right)^2 + \frac{4}{\lambda^2}}}$$

$$\beta = \frac{\sqrt{2}}{2}\sqrt{\left(Q + \frac{1}{S}\right) + \sqrt{\left(Q - \frac{1}{S}\right)^2 + \frac{4}{\lambda^2}}} \tag{2.26b}$$

Constants $A_1(t)$, $A_2(t)$ up to and including $A_8(t)$ in (2.24) and (2.25) must be determined in each of the particular cases separate from the boundary conditions on the beam ends. These constants can also be replaced by the following relationships for purposes of simplification:

$$A_5 = \lambda \frac{\alpha^2 + \frac{1}{S}}{\alpha} A_1 = \lambda \left(\frac{\alpha^2 S + 1}{\alpha S}\right) A_1$$

$$A_6 = \lambda \frac{\alpha^2 + \frac{1}{S}}{\alpha} A_2 = \lambda \left(\frac{\alpha^2 S + 1}{\alpha S}\right) A_2$$

$$A_7 = -\lambda \frac{\beta^2 - \frac{1}{S}}{\beta} A_3 = -\lambda \left(\frac{\beta^2 S - 1}{\beta S}\right) A_3$$

$$A_8 = \lambda \frac{\beta^2 - \frac{1}{S}}{\beta} A_4 = \lambda \left(\frac{\beta^2 S - 1}{\beta S}\right) A_4 \qquad (2.27)$$

Substituting (2.27) into (2.22) and (2.23), the equations of equilibrium for a slighty curved Timoshenko beam will be:

$$\bar{W} = A_1(t) \cosh \lambda\alpha\eta + A_2(t) \sinh \lambda\alpha\eta + A_3(t) \cos\lambda\beta\eta + A_4(t) \sin\lambda\beta\eta$$
$$+ \frac{64}{\lambda^2 - 64}\left[\frac{A_1(t)}{\lambda\alpha} \sinh \lambda\alpha + \frac{A_2(t)}{\lambda\alpha} (\cosh\lambda\alpha - 1) + \frac{A_3(t)}{\lambda\beta} \sin\lambda\beta + \frac{A_4(t)}{\lambda\beta} (1 - \cos\lambda\beta)\right] \qquad (2.28)$$

$$\Theta = \frac{\lambda(\alpha S^2 + 1)}{\alpha S}\left[A_1(t) \sinh \lambda\alpha\eta + A_2(t) \cosh \lambda\alpha\eta\right]$$
$$+ \frac{\alpha(\beta^2 S - 1)}{\beta S}\left[A_3(t) \sin\lambda\beta\eta + A_4(t) \cos\lambda\beta\eta\right] \qquad (2.29)$$

This report will only present the theoretical equations for a typical straight beam (curved or slightly curved beams are excluded). We will, however, expand on the use of a finite element approach to generate the three types of beams.

In regard to straight beams in particular, certain theoretical equations derived from some common cases presented here were established by applying them to (2.28) and (2.29) with a (2.29) with appropriate boundary conditions, as well as constant relation integration in (2.27).

Each equation is solved by means of a computer program (iteration method) that yields the exact frequency parameters, which can then be compared with parameters obtained through the finite element method.

The common types of beams can be separately integrated with respect to conditions at both ends: (to $\eta = 0$ and $\eta = 1$), "clamped-clamped", "clamped-free", "clamped-supported", "free-free", "supported-free", "supported-supported", "clamped-simply supported", "simply supported", "simply supported-simply supported", and "free-simply supported". The corresponding frequency equations are derived from a uniform beam and are summarized in Table 1; the corresponding normal modes appear in Table 2, Appendix D.

2.2 Finite element method

2.2.1 Choice of the model

The first point to consider in solving the problem is what model to choose. The model chosen must be applicable to all types of beams and must satisfy all boundary conditions in all the cases. This method presupposes that the nodal displacements are the unknowns in the problem and that the compatibility conditions within and among the elements must be first satisfied. (A quick review: compatible elements are elements in which the longitudinal boundary displacements are fully accounted for by displacements of the nodal point that includes this boundary).

The work of determining the stiffness matrix is then based on the assumption of virtual displacement, rather than the virtual constraint that has been previously discussed, in [20]. Some terms for derivations of deformation can be ignored in most practical cases. Furthermore, the elements the assumptions are based upon, the field displacements, are the most universal ones. The deformation for a structure comprised of elements is given by its nodes. The stiffness matrix for the structure must be compatible with the displacements and with the corresponding load.

As was earlier mentioned, the type of element used throughout the analysis was a line segment which includes two nodes at ends to adjust to the various types of boundary conditions. The displacement vector for each node is a combination of the following quantities: transverse displacement and its slope, w and w', respectively; rotation of the cross section, θ', and finally transverse shear deformation, ψ. These quantities, w', θ, ψ, are linked by the relation in (2.12); therefore if w is the cubic polynomial, θ and ψ can be represented in the same way as w, and then we obtain:

$$\begin{Bmatrix} \theta \\ \psi \end{Bmatrix} = A_4(t) + A_3(t)\,x + A_2(t)\,x^2 + A_1(t)\,x^3$$

- $A_1(t)\,x^3$ — mode symmetry
- $A_2(t)\,x^2$ — state of uniform deformation
- $A_3(t)\,x$ — rigid-body rotation
- $A_4(t)$ — principal mode of rigid-body motion

(state of uniform deformation, rigid-body rotation, principal mode of rigid-body motion: Mandatory terms)

where A_1, A_2, A_3, A_4 are the constants or time functions for the dynamic case which are to be determined.

Two distinct models are derived from these assumptions:

1) Model "i": in which transverse displacement, w, and shear deformation ψ, are assumed to be cubic polynomials:

$$w = a_1(t)\ x^3 + a_2(t)\ x^2 + a_3(t)\ x + a_4(t)$$

Eq. i

$$\psi + a_5(t)\ x^3 + a_6(t)\ x^2 + a_7(t)\ x + a_8(t)$$

1) Model "j": in which transverse displacement, w, and cross sectional rotation, θ, are assumed to be cubic polynomials:

$$w = a_1(t)\ x^3 + a_2(t)\ x^2 + a_3(t)\ x + a_4(t)$$

$$\theta = a_5(t)\ x^3 + a_6(t)\ x^2 + a_7(t)\ x + a_8(t) \qquad \text{(Eq. j)}$$

where $a_1(t)$, ..., $a_8(t)$ are linked by (2.27).

It is apparent that in both "i" and "j" models, transvrse displacement, w, is always a cubic polynomial. With both models, the formulation for the stiffness and lmass matrices of a uniform Timoshenko beam can be used for: a) curved beams, b) slightly curved beams and c) straight beams (see also Table 3, Appendix D).

Three types of beams with different displacement functions, as indicated in Table 4, were chosen by means of (equation i) and (equation j). For comparison purposes, Table 4 also gives a summary of several models of Timoshenko beams various authors have studied. Two models in particular called number I and number III, satisfy all the normally applied boundary conditions as well as their tests for convergence.

Model number II satisfies only three cases: "supported-supported", "free-free", and "supported-free". The explanation for this particular occurrence is found in equation (2.12). The existence of rotation θ, in the cross section, in this situation is a necessary and sufficient condition for assuming the displacement functions; shear, ψ, although an important factor in Timoshenko theory, is only secondary when compared to the effects of displacement, w, and rotation, θ. (All previous studies have shown that this effect has greater influence at higher modes). Of these three models, only two (I and III) theoretically satisfy the above mentioned conditions and are consequently able to satisfy all strain-displacement boundary conditions. The rotation slope, θ', and shear, ψ, in model II are necessary conditions, but are insufficient to cover all possible boundary conditions.

Of the two models (I and III), model III is considered to be the typical model in this analysis, because of the shear assumption, ψ, which makes the problem more symmetrical.

2.2.2 General Procedure

Since the finite element method is well known, only some particular guidelines will need to be repeated for the purposes of this analysis.

Consider a uniform segment of a beam defined by two nodes, "i" and "j", with the boundaries at the nodal reference (Fig. 3).

The displacement function chosen can be determined by:

$$\left\{u(x,t)\right\} = [N] \begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} \tag{2.31}$$

where $[N]$ is a matrix for the general position function and $\begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix}$ represents the nodal displacements.

Once the displacement function is known, the deformation matrix can be stated as:

$$\{\varepsilon\} = [B] \begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} \tag{2.32}$$

Then, the one-dimensional stress matrix can be written:

$$\{\sigma\} = [ST] \begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} \tag{2.33}$$

Using the expressions strain for and kinetic energy, we derive:

$$\mathcal{U} = \frac{1}{2} \int_{volume} [B^T] [B] \, d \, (volume)$$

$$\mathcal{T} = \frac{1}{2} \rho \int_{volume} [N^T] [N] \, d \, (volume) \tag{2.34}$$

Finally, the stiffness and mass matrices [K] and [M] within the local system, associated with the finite elements are, respectively:

$$[\bar{K}] = \int_{volume} [B^T] [B] \quad d(volume)$$

$$[\bar{M}] = \rho \int_{volume} [N^T] [N] \quad d(volume) \tag{2.35}$$

where $\bar{K}_{ij} = \dfrac{\partial^2 \mathcal{U}(\Delta)}{\partial \Delta_i \, \partial \Delta_j}$

and $\bar{M}_{ij} = \dfrac{\partial^2 \mathcal{T}(\bar{\Delta})}{\partial \bar{\Delta}_i \, \partial \bar{\Delta}_j}$

2.2.3 Convergence Criteria

Accuracy depends upon the criteria for convergence in the method mentioned previously. The criterion discussed in the particular case of a simple element is the monotonic convergence boundary. The two main points of convergence are as follows:

a) Displacement function

The function chosen is the one that does not allow deflection in an element to occur when displacements at its nodes are caused by displacement of the rigid body.

b) Complete element

The function chosen must be capable of representing the displacements of a rigid body (this will enable all points of an element to go through similar displacements) and the states of constant deformation (those necessary for assembling several elements. This prerequisite can be explained physically if we express an infinitimal element for which deformation will be approached as a constant value). This criterion can also be considered as a sufficient condition for convergence. Note that with this method, the number of terms for a complete polynomial of order n, is therefore (n + 1) for a one-dimensional case and 1/2 (n + 2) for two dimensions.

c) Continuity of displacement

The function must be chosen in such a way that the deformation interfaces of the elements are finite (otherwise the displacement would be continued within the element), and is dependent upon the selection of continuous polynomials as general displacement functions. As has previously been found, if there is discontinuity between the elements, a constant deformation condition will automatically guarantee continuity of the displacement.

The coordinates and displacements at the beam element interface (adjacent elements) are the same. Furthermore, the strain energy integrals used in this case are evaluated precisely; the boundary characteristics and the properties of monotonic convergence can be applied.

CHAPTER III

DISPLACEMENT FUNCTIONS

3.1 Selection and determination of displacement functions

The established general formulation is valid for all three models- I, II and III. In particular, with a matrix development for potential energy and kinetic energy, the displacement function used was the one for model III. This model has already been considered as a typical model for later expansion (see Appendix D, Table 4).

When considering a uniform element of the beam (Fig. 4), it may be assumed, in simplistic terms, that $\{u(x,t)\}^T$ resembles the displacement functions, with respect to parameters $\{\alpha(t)\}$.

Where: $$\{u(x,t)\} = [L(x)]\ \{\alpha(t)\} \qquad (3.1)$$

$[L(x)]$ is a matrix containing the cubic displacement functions, and:

$$\{\alpha(t)\}^T = \{a_1(t)\ a_2(t)\ a_3(t)\ a_4(t)\ a_5(t)\ a_6(t)\ a_7(t)\ a_8(t)\}$$

The first term is this potential energy expression is the integral for the energy that is caused by strain; the second term is the shear energy. The same applies to kinetic energy, where the first term represents the integral for the linear variation term and the second represents the integral for rotary inertia. To integrate these terms explicit by transverse displacement w, rotation θ, or shear deformation ϕ must be represented. For the sake of compatibility with the finite element analysis, w, θ or ϕ are assumed to be of the form: (see Table a)

Type III_i: $\begin{Bmatrix} w \\ \psi \end{Bmatrix} = [L(x)] \{\alpha\}$

Type III_j: $\begin{Bmatrix} w \\ \theta \end{Bmatrix} = [L(x)] \{\alpha\}$

The elements in matrix $[L(x)]$ are, generally speaking, the structure's coordinates; and the constant vector $\{\alpha\}$ is generally a time function. The nodal displacements can therefore be assumed to be:

Where $[A]$ = (Cte), interpolation matrix for the nodal displacements.

3.2 Internal displacements

The displacement functions can be specified by examining each node of an element for models I and III (Fig. 3). The components for transverse displacement w and its slope w', rotation θ and its slope θ', and shear ψ will be ordinates. The node "i" "displacement", for each model, can then be determined by the vector:

Model I (types I_i and I_j):

$$\{\Delta_i\} = \begin{Bmatrix} w_i \\ w'_i \\ \theta_i \\ \theta'_i \end{Bmatrix} \qquad (3.3a)$$

Model III (types III_i and III_j):

$$\{\Delta_i\} = \begin{Bmatrix} w_i \\ \theta_i \\ \theta'_i \\ \psi_i \end{Bmatrix} \qquad (3.3b)$$

For elements having two nodes and 8 degrees of nodal freedom, we obtain from (3.3a) and (3.3b):

Model I:

$$\begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} = \begin{Bmatrix} w_i \\ w'_i \\ \theta_i \\ \theta'_i \\ w_j \\ w'_j \\ \theta_j \\ \theta'_j \end{Bmatrix} = \begin{bmatrix} [A]_{Ii} \\ [A]_{Ij} \end{bmatrix} \{\alpha\} \tag{3.4a}$$

Model III:

$$\begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} = \begin{Bmatrix} w_i \\ \theta_i \\ \theta'_i \\ \psi_i \\ w_j \\ \theta_j \\ \theta'_j \\ \psi_j \end{Bmatrix} = \begin{bmatrix} [A]_{IIIi} \\ [A]_{IIIj} \end{bmatrix} \{\alpha\} \tag{3.4b}$$

Multiplying (3.4a) and (3.4b) by $[A^{-1}]$ IIIi or $[A^{-1}]$ IIIj, respectively, we obtain the general form:

$$\{\alpha(t)\} = [A^{-1}] \{\Delta(t)\}$$

By substituting into (3.1) we end up with:

$$\{u(x,t)\} = [L(x)][A^{-1}] \{\Delta(t)\} = [N]\begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} \tag{3.5}$$

This equation defines the displacement functions.

CHAPTER IV

MATRIX CONSTRUCTION

4.1 Internal deformations

The assumption is made that a finite element of the beam is deformed according to two displacement functions as previously determined in Table 3. Curved beams will be dealt with first, followed by slightly curved beams and, finally, straight beams.

Deformations of an element are expressed as general displacement functions (3.5); non-extension (or curved) beams are being neglected for the purposes of this particular project.

The basic "strain-displacement" relationship for plafne deformation produces:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \partial U/\partial x \\ \partial V/\partial y \\ \partial W/\partial z \\ \partial U/\partial y + \partial V/\partial x \\ \partial V/\partial z + \partial W/\partial y \\ \partial U/\partial z + \partial W/\partial x \end{Bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} - \frac{rz}{r+z}\frac{\partial \theta}{\partial x} \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{\partial w}{\partial x} - \frac{r^2}{(r+z)^2} \quad \theta \end{bmatrix}^T [A^{-1}] \begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} = [B] \begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} \tag{4.1}$$

$$\text{Oũ} \quad \frac{\partial u}{\partial x} = \frac{H}{L}\int_0^L \left(\frac{\partial y_o}{\partial x}\right)\left(\frac{\partial w}{\partial x}\right) dx - H\left(\frac{\partial y_o}{\partial x}\right)\left(\frac{\partial w}{\partial x}\right) \tag{4.2}$$

It is important to note that in this analysis the "strain-displacement" equations contain rigid-body motion. It has been eliminated from resultant expressions, however, in order to determine the deformations, but is is included in the displacement calculations.

4.2 Internal stresses

The "stress-strain" relationship for one-dimensional isotropic materials can be written:

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{Bmatrix} E\varepsilon_x \\ 0 \\ 0 \\ 0 \\ 0 \\ kG\gamma_{xz} \end{Bmatrix} = \begin{bmatrix} E\left[\frac{\partial u}{\partial x} - \frac{rz}{r+z}\frac{\partial \theta}{\partial x}\right] \\ 0 \\ 0 \\ 0 \\ 0 \\ kG\left[\frac{\partial w}{\partial x} - \frac{r^2}{(r+z)^2}\theta\right] \end{bmatrix}^T [A^{-1}]\begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} = [ST]\begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} \qquad (4.3)$$

Where: $\left(\frac{\partial u}{\partial x}\right)$ have been determined in (4.2).

4.3 Strain energy-kinetic energy

As has been mentioned previously, this method uses unknown nodal displacements for the problem; later displacements will be determined by solving the system of linear equations generated to satisfy the equilibrium conditions at the nodes.

By using the original total potential energy, we obtain:

$$\mathcal{U}_{total} = \mathcal{U} - \mathcal{U}_F - \mathcal{U}_R - \mathcal{U}_S$$

Where:

$\mathcal{U}$: Internal deformation energy.

$\mathcal{U}_F$: Potential energy due to body forces.

$\mathcal{U}_R$: Potential energy due tu external nodal forces.

$\mathcal{U}_S$: Potential energy due to surface tension.

With regard to the free vibration modes, the last three energy terms, $\mathcal{U}_F$, $\mathcal{U}_R$, and $\mathcal{U}_S$, can be neglected, therefore giving $\mathcal{U}_{total} = \mathcal{U}$.

- The internal strain energy expression for length "L" of a Timoshenko beam element can be written:

$$\mathcal{U} = \frac{1}{2}\int_{volume} [\sigma_x]^T [\varepsilon_x] \, d(volume) + \frac{1}{2}\int_{volume} [\tau_{xz}]^T [\gamma_{xz}] \, d(volume) \quad (4.4)$$

~~flexion~~ bending term — shear term

- The kinetic energy expression for length L of a Timoshenko beam element can also be given by:

$$\mathcal{T} = \frac{1}{2}\int_{volume} \rho \left[\frac{\partial u}{\partial t}\right]_L^2 d(volume) + \frac{1}{2}\int_{volume} \rho \left[\frac{\partial u}{\partial t}\right]_R^2 d(volume) \quad (4.5)$$

term of linear variation — term of rotary inertia

By substituting (4.1), (4.3) into (4.4) and (4.5), respectively, and then by non-dimensionalizing through the substitution of and assuming that W and θ are cubic polynomials of the form of (2.30), we eventually arrive at the general expressions for internal strain energy $\mathcal{U}$ $\eta = \frac{x}{L}$, $y = \frac{W}{L}$ expression and kinetic energy $\mathcal{T}$, , for a curved ((A.1.5) and (A.1.6)), slightly curved ((A.2.1 and (A.2.6)), or straight ((A.3.1) and (A.3.5)) beam.

4.4 Stiffness and Mass matrices

The expression of potential energy $\mathcal{U}$ in (A.1.5), (A.2.1) or (A.3.1) for the stiffness matrix can be given by:

$$\mathcal{U} = \tfrac{1}{2} \{\Delta\}^T [\bar{K}] \{\Delta\} \tag{4.6}$$

where $[\bar{K}]$ is the stiffness matrix of the beam within the local system.

Kinetic energy $\mathcal{T}$ in (A.1.6), (A.2.6) or (A.3.5) in the mass matrix, can also be expressed by:

$$\mathcal{T} = \tfrac{1}{2} \{\dot{\Delta}\}^T [\bar{M}] \{\dot{\Delta}\} \tag{4.7}$$

where $[\bar{M}]$ is the mass matrix of the beam within the local system. Both matrices $[\bar{K}]$ and $[\bar{M}]$ depend upon the geometric and physical form of the beam and are derived for three types of beams. The elements in $[\bar{K}]$ and $[\bar{M}]$ can be written as follows:

$$\bar{K}_{ij} = \frac{\partial^2 \mathcal{U}(\Delta)}{\partial \Delta_i \partial \Delta_j}$$

And

$$\bar{M}_{ij} = \frac{\partial^2 \mathcal{T}(\dot{\Delta})}{\partial \dot{\Delta}_i \partial \dot{\Delta}_j} \tag{4.8}$$

for

The stiffness and mass matrices for a Timoshenko beam were derived for the following cases:

A) Curved beam

* Stiffness matrix

$$[\bar{K}_c] = \frac{EI_y}{L} [A^{-1}]^T [Y^*_c] [A^{-1}] \tag{4.9}$$

where $[\bar{K}_c]$ is computed in Appendix A (equation A.1.10) and listed in Table 5a (Appendix E).

In the global system, we can derive:

$$[K_c] = [\Lambda^T][\bar{K}_c]\;[\Lambda] \qquad (4.10)$$

where $[\Lambda]$ is a transformation matrix for the coordinates, given by equation (A.5.4)

*Mass matrix

$$[\bar{M}_c] = \rho A L^3\;[A^{-1}]^T\;[\lambda^*_c][A^{-1}] \qquad (4.11)$$

where $[\bar{M}_c]$ is calculated by means of equation (A.1.16) in Appendix A and listed in Tabale 5b of Appendix E.

In the global system, the mass matrix will be in the form:

$$[M_c] = [\Lambda^T][\bar{M}_c][\Lambda] \qquad (4.12)$$

Where $[\Lambda]$ is the same matrix as in (4.10)

B) Slightly curved beam

*Stiffness matrix

$$[\bar{K}_{sc}] = \frac{EI_y}{L}\;[A^{-1}]^T\;[\gamma^*_{sc}][A^{-1}] \qquad (4.13)$$

where $[\bar{K}_{sc}]$ is computed in Appendix A by equation (A.2.4) and listed in Table 5c (Appendix E).

In the global system, we obtain:

$$[K_{sc}] = [\Lambda^T][\bar{K}_{sc}][\Lambda] \tag{4.14}$$

where $[\Lambda]$ is given by equation (A.5.4)

*Mass matrix

$$[\bar{M}_{sc}] = \rho A L^3 \, [A^{-1}]^T \, [\lambda^*{sc}][A^{-1}] \tag{4.15}$$

$[\bar{M}_{sc}]$ is calculated in Appendix A (equation A.2.9) and listed in Table 5d (Appendix E).

In the global system we have:

$$[M_{sc}] = [\Lambda]^T \, [\bar{M}_{sc}] \, [\Lambda] \tag{4.16}$$

where $[\Lambda]$ is the same matrix as in (4.14).

C) Straight beam

*Stiffness matrix

$$[\bar{K}_d] \quad \frac{EI_y}{L} [A^{-1}]^T \, [\gamma^*_d][A^{-1}] \tag{4.17}$$

where $[\bar{K}_d]$ is determined by equation (A.3.4) in Appendix A and listed in Tables 5e, 5g, 5i of Appendix E.

In the global system, the stiffness matrix of a straight beam will be:

$$[K_d] = [\Lambda^T][\bar{K}_d][\Lambda] \tag{4.18}$$

where $[\Lambda]$ is given by equation (A.5.4).

*Mass matrix

$$[\bar{M}_d] = \rho A L^3 \; [A^{-1}]^T \; [\lambda^*_d][A^{-1}] \tag{4.19}$$

where $[\bar{M}_d]$ is calculated using equation (A.3.7) in Appendix A and listed in Tables 5f, 5h, 5j of Appendix E.

In the global system, the mass matrix of a straight beam will be:

$$[M_d] = [\Lambda]^T \; [\bar{M}_d][\Lambda] \tag{4.20}$$

where $[\Lambda]$ is the same matrix as in (4.18).

D) Tapered beam

*Stiffness matrix

$$[\bar{K}_{dt}] = [A^{-1}]^T \; [\gamma^*_{dt}][A^{-1}] \tag{4.21}$$

where $[\bar{K}_{dt}]$ is determined by equations (A.4.3) and (A.4.4) in Appendix A, and listed in Table 9a (Appendix E).

In the global system, it will be:

$$[K_{dt}] = [\Lambda^T][\bar{K}_{dt}][\Lambda] \tag{4.22}$$

where $[\Lambda]$ is given by equation (A.5.4).

*<u>Mass matrix</u>

$$[\bar{M}_{dt}] = [A^{-1}]^T [\lambda^*_{dt}][A^{-1}] \tag{4.23}$$

where $[\bar{M}_{dt}]$ is calculated by equations (A.4.8) and (A.4.9) in Appendix A, and listed in Table 9b (Appendix E).

In the global system, the mass matrix for a tapered beam will be:

$$[M_{dt}] = [\Lambda^T][\bar{M}_{dt}][\Lambda] \tag{4.24}$$

where $[\Lambda]$ is the same matrix used in (4.21).

4.5 Vibration of variable section beams (tapered beams)

4.5.1 Introduction

Upon firmly establishing the formulas for the stiffness and mass matrices, determination of the eigenvalues and eigenvectors for a straight uniform beam produced successful convergence. The development of non-uniform or tapered beams can now be proposed.

The only change in procedure for this specific type of beam is due to its geometric properties (the physical properties are also involved in beams having non-uniform physical characteristics): e.g. the depths of square or rectangular sections, where diameters may vary over a row of circular or elliptical sections. Depth, thickness and diameter can be linear, quadratic, cubic, exponential or tapered functions. The beam may also have different conical forms such as cones and truncated cones.

4.5.2 Past studies and present research

The problems of non-uniform or tapered beams have been previously studied by various authors. Kirchkoff [39, 1879] conducted research into edge vibrations with a cone fixed at one end and free at the other, Ward [40, 1913] continued investigation of a beam with parallel dimensions at the y and z axes which varied between $\frac{X}{L}^m$ and $\frac{X}{L}^n$, respectively, with m and n here representing whole numbers. Nickleson [31, 1961] reported on the lateral vibration problem with a cantilever bar of variable section. Thereafter, Mononobe [41] investigated lateral vibration of thin cantilever bars with variable transverse and boundary conditions. Granch and Alder [18, 1956] used simple beam theory to calculate natural frequencies for several types of beams. Several cases of "clamped-free" and double-edged free beams with various variations in depth were also studied. Martin [44, 1956] looked at free vibration of a beam for different size ratios: The depth could be expressed in decreasing series; the high power terms could be neglected. Lee [47, 1963] considered the problem of shear effects and rotary inertia of an edge by suing Timoshenko theory. This same problem was investigated by both Housner and Keightley [33, 1962] and Rissone and Williams [45, 1965]; they studied the vibrations of a "clamped-free" beam with a small thickness ratio using Myklestad and Stodola methods.

Conway, Becker and Dubil [46, 1964] also calculated the free vibrations of a "clamped-free" truncated cone - shaped cantilever for a certain number of boundary conditions. Rao [32, 1965] determined the fundamental flexural vibration of a "clamped-free" beam with linearly varying rectangular sections, using Galerkin's technique.

Gaines and Volterra [21, 1966] presented cross-sectional free vibration at the upper and lower eigenvalues at the boundaries of a "clamped-free" variable section cantilever bar (in the form of edges or truncated edges, cones or truncated cones), by both neglecting rather paradoxically and then also taking into account two specific effects from Timoshenko's theory.

Carnegie and Thomas [22, 1967] also obtained the eigenvalues and eigenvectors for a long tapered beam by means of an iteration procedure, using the finite difference method from the equations derived from Euler-Bernouilli. Using the finite element method, Thomas, Wilson and Wilson [12, 1973] applied Timoshenko's theory to their model in order to calculate vibrations for a tapered beam.

The most recent study was by Downs [48, 1977], who obtained excellent results with a dynamic discretization technique which included the first eight vibrations of edge of all geometries, and the first four (or six) modes for the stress distribution models.

In the present study, the derivation for a straight beam of different forms was generally accomplished by combining the variable depths and thicknesses. A number of conical beams (cones or truncated cones, edges or truncated edges) of variable section (rectangular, square, and elliptical) were obtained (see Table 8).

For a beam with varying sections, we get:

$$\begin{cases} A(x) = A_a + \Delta A \left(\frac{x}{L}\right) \\ I_y(x) \quad I_{ya} + \Delta I_y \left(\frac{x}{L}\right) \end{cases} \tag{4.25}$$

where $\Delta A = A_b - A_a$

and $\Delta I_y = I_{yb} - I_{ya}$

A_a, I_{ya} and A_b, I_{yb} are cross sections and inertia a and b, respectively, of a tapered beam element. Substituting (4.25) into (A.3.4), (A.3.4) and (A.3.8) will yield the stiffness and mass matrices for a tapered Timoshenko beam.

In this case, the frequency parameters therefore become:

$$\lambda^2 = \frac{\rho A_a L^4 \omega^2}{E I_{ya}}$$

CHAPTER V

DETERMINATION OF THE FORM FACTOR - DISTORTION PROBLEM

5.1 Calculation of factor k

The ambiguous nature of the equations of motion requires proper choice of a shear deformation coordinate, k, when considering shear deformation for several beam problems. This coefficient has a non-dimensional quantity. Several authors have suggested that the usual values of k lead to unsatisfactory results when these equations of motion are used to determine the high-frequency vibration spectrum of the beam. For better results, the coefficient should be arbitrarily adjusted. The cause of the error originates in the Timoshenko equation, from where an efficient transverse shear deformation is selected equal to the shear constraint of the reference cross section, divided by G * k where G is the shear modulus.

That k depends upon the vibration mode and the form of the cross section is a well-known fact, introduced to demonstrate that the distribution of shear stresses as well as shear deformations on the section is not uniform. However, k also depends on the assumptions made concerning the type of end conditions.

Therefore, k is the reference shear deformation for a section with respect to centroidal shear deformation.

The methods two well-known authors [23] and [19] used to determine k will now be discussed. First, Sutherland and Goodman [23], based their assumption on the fact that the shear stress distribution of a cantilever beam with rectangular sections is such that the choice of k does not particularly affect the vibration results for the cantilever beam.

When considering a prismatic length of beam, δx, the deformation energy increment due to shear distortion $\mathcal{U}_{SD}$, will be:

$$d(\delta\mathcal{U}_{SD}) = \tfrac{1}{2}\,(\gamma_{xz}\delta x)\,(\tau_{xz}dA)$$

$$= \tfrac{1}{2}\,\frac{\tau_{xz}}{G}\,\tau_{xz}dA\delta x \qquad \frac{\tau^2_{xz}}{2G}\,\delta x dA$$

$$\delta\mathcal{U}_{SD} = \frac{\delta x}{2G}\int_A \tau^2_{xz}dA \tag{5.1}$$

In addition, the work imposed by forces on the element is:

$$\delta(\text{ work }) = \tfrac{1}{2}\,V_z\beta_e\delta x \tag{5.2}$$

Setting (5.1) equal to (5.2), we obtain:

$$\tfrac{1}{2}\,V_z\beta_e\delta x = \frac{1}{2G}\,\delta x\int_A \tau^2_{xz}dA$$

with $\beta_e = \dfrac{V_z}{kAG}$

$$\rightarrow \quad \frac{V_z^2}{kA} = \int_A \tau^2_{xz}dA$$

$$\boxed{k = \frac{V_z^2}{A\displaystyle\int_A \tau_{xz}dA}} \tag{5.3}$$

With this formula, coefficient k will be determined for each value of cross sectional over A.

The second investigator we shall consider here, G.R. Cowper [19], stated a more general assumption arising from derivations of Timoshenko beam equations, and dependent upon a function X, which is specified for each type of section.

With this method, the two quantities in beam theory mentioned earlier can be determined either i.e. transverse displacement W or rotation θ (θ may be interpreted as being the reference angle of rotation of the cross section around the neutral axis), such that:

$$W = \frac{1}{A} \int_A u_z dA$$
$$\theta = \frac{1}{I_y} \int_A z u_z dA \tag{5.4}$$

where $dA = dydz$.

W is chosen as centroïdal displacement. Accurate definition is a prerequisite, since the beam cross section inevitably is deformed, leading to a small elongation. All points on the cross section do not have the same displacements.

θ is chosen as the inclination angle of a plane surface which usually coincides with the left section.

Starting with the two equations of equilibrium which concern forces in directions z and x, respectively, we get:

$$\frac{\partial\sigma_z}{\partial z} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\tau_{xz}}{\partial x} + F_z = \rho\,\frac{\partial^2 u_z}{\partial t^2} \tag{5.5}$$

$$\frac{\partial\tau_{xz}}{\partial z} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\sigma_x}{\partial x} = \rho\,\frac{\partial^2 u_x}{\partial t^2} \tag{5.6}$$

Integrating (5.5) and (5.6) yields the following expression:

$$V_z = \int_A \tau_{xz}\,dA$$

$$M_y = \int_A z\sigma_x\,dA$$

Then:

$$\frac{\partial V_z}{\partial x} + q(x,t) = \rho A\,\frac{\partial^2 W}{\partial t^2} \tag{5.7}$$

$$\frac{\partial M_y}{\partial x} - V_z = \rho I_y\,\frac{\partial^2\theta}{\partial t^2} \tag{5.8}$$

Where $q(x,t)$ is the total transverse load per unit length, applied to the beam; $q(x,t)$ can also be expressed by:

$$q(x,t) = \oint (n_z\sigma_z + n_y\tau_{yz})\,dS + \int_A F_z\,dA \tag{5.9}$$

n_z, n_y represent the normal unit components of the cross section boundary; dS is the element for the boundary arc; I_y is the moment of inertia of the beam around axis y.

In accordance with residual displacement assumptions v_z, v_x:

$$\int_A v_z\,dA = \int_A v_x\,dA = \int_A z v_x\,dA = 0 \tag{5.10}$$

We can express the "stress-strain" relationship as:

$$\frac{\partial W}{\partial z} + \theta = \frac{\tau_{xz}}{G} - \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \tag{5.11}$$

$$E\,\frac{\partial u_x}{\partial x} = \sigma_x - \nu(\sigma_z + \sigma_y) \tag{5.12}$$

Multiply (5.12) by x, then integrate it over the entire section. With regard to (5.10), we obtain:

$$\frac{\partial W}{\partial z} + \theta = \frac{1}{AG}\int_A \left(\tau_{xz} - G\,\frac{\partial v_x}{\partial z}\right) dzdy \tag{5.13}$$

and $$EI_y\,\frac{\partial \theta}{\partial x} = M_y - \nu\int_A z(\sigma_z + \sigma_y)\,dzdy \tag{5.14}$$

The motion of the beam must therefore satisfy equations (5.7), (5.8), (5.13) and (5.14).

Finally, with the introduction of function X, we can relate stresses τ_{xz}, τ_{xy} and displacements u_x by assuming that the normal stresses σ_z, σ_y, are negligible in comparison with σ_x, as well as with the transverse shear stress distribution in a uniformly loaded beam (V_z varies linearly along the beam and has a constant value for an "extremal clamped-loaded" beam).

$$\begin{aligned}
\tau_{xz} &= -\frac{V_z}{2(1+\nu)I_y}\left[\frac{\partial \chi}{\partial z} + \frac{\nu z^2}{2} + \frac{(2-\nu)y^2}{2}\right] \\
\tau_{yx} &= -\frac{V_z}{2(1+\nu)I_y}\left[\frac{\partial \chi}{\partial y} + (2+\nu)\,zy\right] \\
u_x &= zf(x) - \frac{V_z}{EI_y}(\chi + zy^2)
\end{aligned} \tag{5.15}$$

X in (5.15) represents the harmonic functions which satisfy the boundary conditions of the cross section boundary:

$$\frac{\partial \chi}{\partial n} = -n_z \left[\frac{\nu z^2}{2} + \frac{(2-\nu)y^2}{2}\right] - n_y (2+\nu)zy$$

f(x) is a polynomial of exact form which depends upon the extremal conditions of a beam.

From (5.15) and (5.10), the value Vx, cross-sectional displacement, is calculated as:

$$v_x = \frac{V_z}{EI_y}\left[-\chi - zy^2 + \frac{1}{A}\int_A (\chi + zy^2)dzdy + \frac{z}{I_y}\int_A z(\chi + zy^2)dzdy\right]$$

Substituting the v_x value into (5.13), we obtain:

$$\frac{\partial W}{\partial x} + \theta = \frac{V_z}{kAG}$$

where

$$k = \frac{2(1+\nu)I_y}{\frac{\nu(I_z - I_y)}{2} - \frac{A}{I_y}\int_A z(\chi + zy^2)dzdy} \qquad (5.16)$$

I_x is the moment of inertia for the cross section about the Z axis:

$$I_z = \int_A y^2 dzdy.$$

For the different beam sections, the various corresponding X(z,y) functions will be obtained. Cross-sectional area A will be evaluated in order to determine shear deformation coordinate k. The final results for a number of section forms are given in Table 6 of Appendix D.

5.2 Consideration of section distortions

As was previously noted in our derivations of the equations of motion, the assumption we made is that the plane section remains plane. In general, this assumption does not completely meet all the requirements, when the fact is being ignored that the planor cross sections are distorted during vibration. There is another consideration, which with the help of Arnolds [35] and Barrs [28] research, makes it possible to establish an equation on a more general and satisfactory basis than is usually the case here.

The only change is in the value of deformation coefficient k, which then has a different meaning.

The correction for axial U(x,z,t) and transverse W(x,z,t) displacement functions applied as follows:

$$U(x,z,t) = u(x,t) - z\theta(x,t) + \varepsilon(x,z,t)$$

where $\varepsilon(x,z,t)$ is a correction function added to shear deformation.

Consider a uniform element ABCD (Fig. 8) (we assume that the section is rectangular and that the beam is subjected to bending and shear loads in plane oz, where oz represents the shear displacement axis, and ox represents the neutral axis of the non-loaded beam).

The cross section of elements ABCD initially pivoted from oz to axis [1] with angle θ as the bending slope (bending problem), combined with shear loads (shear problem), resulting in distortion of cross-sectional element ϕ at axis [2]. Similar effects should occur for all the elements in the same section while, simultaneously, the face of the elements forms an angle α_z with a line parallel to the axis [1].

The total shear angle, Ψ, for any thickness z may now be written:

$$\Psi = \phi + \alpha_z$$

where shear ϕ is:

$$\psi = \frac{\partial w}{\partial x} - \theta$$

and the correction function $\varepsilon(z)$:

$$\varepsilon(z) = \int_0^z \alpha_z dz$$

(2.16) and (2.17) now become:

$$EI_y \frac{\partial^2\theta}{\partial x^2} + AG\left(\frac{\partial w}{\partial x} - \theta\right) - E\int_z \frac{\partial^2\varepsilon}{\partial x^2} bzdz - G\int_z \frac{\partial\varepsilon}{\partial z} bdz = \rho I_y \frac{\partial^2\theta}{\partial t^2} - \rho\int_z \frac{\partial^2\varepsilon}{\partial t^2} bzdz \tag{5.17}$$

$$q(x,t) + AG\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\theta}{\partial x}\right) + \frac{\partial^2 w_o}{\partial x^2} \cdot \frac{EA}{L}\int_0^L \frac{\partial w_o}{\partial x} \cdot \frac{\partial w}{\partial x} dx + G\int_z \frac{\partial^2\varepsilon}{\partial x\partial z} bdz = \rho A \frac{\partial^2 w}{\partial t^2} \tag{5.18}$$

We then find two quantities:

$$\delta_1 = \frac{1}{\psi I_y}\int_z b\varepsilon zdz \tag{5.19}$$

$$\delta_2 = \frac{1}{\psi A}\int_z b \frac{\partial\varepsilon}{\partial z} dz \tag{5.20}$$

By substituting (5.19) and (5.20) into (5.17) and (5.18) we observe that $\frac{1+\delta_2}{1+\delta_1}$ identifies with k.

or that $$k^* = \frac{1+\delta_2}{1+\delta_1} \tag{5.21}$$

k* is called the distortion coefficient.

In other words, we may now write:

$$k^* = \frac{I_y}{A} \frac{\int_{-h/2}^{h/2} b\Psi \, dz}{\int_{-h/2}^{h/2} \left[\int_o^z \Psi \, dz \right] bz\, dz} \tag{5.22}$$

Total shear angle can also be represented by a Fourier series as:

$$\Psi = \sum_{k=1,3,5}^{\infty} a_k \frac{\cos k\pi z}{h}$$

Where a_k are constants.

This gives:

$$k^* = \left(\frac{\pi}{h}\right)^2 \frac{I_y}{A} \frac{a_1 - \frac{a_3}{3} + \frac{a_5}{5} - \ldots}{a_1 - \frac{a_3}{3^3} + \frac{a_5}{5^3} - \ldots} \tag{5.23}$$

is

k* is verified and compared to the value of deformation coefficient k. For a rectangular section with a tapered shear deformation distribution, the value of k* can be evaluated as follows: We have:

$$\frac{I_y}{A} = \frac{h^2}{12} \rightarrow k^* = \frac{\pi^2}{h^2} \cdot \frac{h^2}{12} \frac{1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots}{1 + \frac{1}{3^6} + \frac{1}{5^6} + \ldots} \simeq \frac{5}{6}$$

Note that with certain types of cross sections, e.g. circular, square or rectangular, the value of k* is equal to the value of k. This does not necessarily always follow in other cases, because of the existence of α_z within the correction function $\varepsilon(x,z,t)$.

According to our computer calculations, if the beam section is rectangular, circular or square, then $k^* = k$. If the section is elliptical, however, the k formula will be used instead of (5.22), in order to simplify the problem.

CHAPTER VI

FREE VIBRATION

The present chapter deals with free vibration bending problem characteristics and their treatment, for a number of uniform and axially non-uniform beams. With this method, the beam must be subdivided into a number of finite elements. The stiffness and mass matrices for each element are first established; the bending (and subsequently, the free vibration) and eigenvalues, eigenvectors are then determined.

The discrete differential equations of motion for a finite element, without any absorbing effect, can be expressed in the form:

$$[K]\{\Delta\} + [M]\{\ddot{\Delta}\} = F(t) \qquad (6.1)$$

where $[\Delta]$ is the displacement vector F(t) is the external force $[K]$ and $[M]$ are, respectively, the global stiffness and mass matrices.

6.1 Free vibrations

If there are no external forces in operation, equation (6.1) can be rewritten as:

$$[K]\begin{Bmatrix}\Delta_i\\ \Delta_j\end{Bmatrix} + [M]\begin{Bmatrix}\ddot{\Delta}_i\\ \ddot{\Delta}_j\end{Bmatrix} = \{0\} \qquad (6.2)$$

By introducing:

$$\begin{Bmatrix}\Delta_i\\ \Delta_j\end{Bmatrix} = \begin{Bmatrix}\Delta_{io}\\ \Delta_{jo}\end{Bmatrix} \left\{\sin(\omega t + \phi)\right\}$$

(6.2) then becomes:

$$\{[K] - \omega^2 [M]\} \begin{Bmatrix} \Delta_{io} \\ \Delta_{jo} \end{Bmatrix} = \{0\} \tag{6.3}$$

$$\begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} = \begin{Bmatrix} w_i \\ \theta_i \\ \theta'_i \\ \psi_i \\ w_j \\ \theta_j \\ \theta'_j \\ \psi_j \end{Bmatrix} \quad \text{(for model III)}$$

And ω is the rotary vibration frequency.

$[K]$ and $[M]$ are real matrices, symmetrical and always finite. $[M]$ is often called a consistent mass matrix, since the weighting functions used in determination of the mass matrix are identical to the ones used to form the stiffness matrix of an element. As for the dynamic problem, the consistent formulation will be accurate if the actual deformation mode is included in the weighting functions of $\left[\frac{\partial u}{\partial t}\right]_L$ and $\left[\frac{\partial u}{\partial t}\right]_R$ (or w, θ in model I and w, ψ in model III, respectively.

Equation (6.3) leads to a standard eigenvalues problem.

If we cancel the determinant in (6.3), the values are obtained automatically. Due to different boundary conditions for x = 0 and x = L (or n = 0 and n = 1), the number of vibrations and weighting modes obtained will differ depending on the number of displacements terms applied. In our case, the real number of vibrations obtained was equal to 5(N+1) - J, where N is the total number of finite elements and J is the number of natural boundary conditions specified for each case.

Each free vibration will be associated with a particular eigenvector $\begin{Bmatrix} \Delta_{io} \\ \Delta_{jo} \end{Bmatrix}$

6.2 [K] and [M] for the entire system

To develop a model of the entire system, the stiffness and mass matrices are assembled so as to satisfy eqfuilibrium forces and continuiuty of the displacement at the interfaces.

The two global stiffness and mass matrices obtained are called $[K]_{syst}$ and $[M]_{syst}$, respectively. They are shown in Figure 9. $[K]_{syst}$ and $[M]_{syst}$ are square matrices of order 5(N+1), where N is the number of elements.

Note that the stiffness and mass matrices are individually established for an element whose the nodal points are at both ends. For beams, the element used is simpler than for other cases (plate and shell); triangular or isoparametric elements are unnecessary. Displacements $\{\Delta_i\}$ and $\{\Delta_j\}$ corresponding to both ends of af finite element must be continuous (when the system considered is continuous) with those that allow for super overlaying position of the stiffness and mass matrices for each element within the global system.

Moreover, in the free vibration procedure, the beam system is subject to inertial forces and the corresponding constraint and deformation states are determined by using an odd function for the displacement, as was discussed in Chapter II (e.g. $4\left(\frac{x}{L}\right)\left(1 - \frac{x}{L}\right)_{\alpha}$ and representing the sum of the forces and moments at a particular node, noting that must be equal to external forces which, in our case, are mass inertial forces and the moments applied to this node). Then, if $\{F_i\}$ and $\{F_j\}$ represent the internal nodal forces acting upon nodes i and j, respectively, we get:

$$\{F_t\} = \{F_i\} + \{F_j\}$$

and $\{\Delta_i\} = \{\Delta_j\}$

The condition of deformation compatibility will also be satisfied. Assuming the displacement of the basic system, we obtain:

$$w = h\,\overline{W} \sin \omega t$$

$$\theta = \frac{h}{L}\,\theta \sin \omega t$$

where ω is the system's natural frequency.

6.3 Static analysis - constant loads

This section deals with the beam problem as it relates to external forces and time-related moments. It is assumed that an external force, $F_e(x,t)$, and an external moment, $M_e(s,t)$, are acting upon a non-uniform beam; F_e and M_e are forces and moments, respectively, per unit length.

The equations of equilibrium, in this case, can then be rewritten:

$$\begin{aligned} \frac{\partial M_y}{\partial x} + V_z - \frac{\rho I_y}{A}\frac{\partial^2\theta}{\partial t^2} &= M_e(x,t) \\ \frac{\partial V_z}{\partial x} - \rho\frac{\partial^2 w}{\partial t^2} &= F_e(x,t) \end{aligned} \tag{6.4}$$

By eliminating M_y and V_z in (2.1), (2.2), (2.3) and (2.4) with respect to (2.8), (2.10) and (2.12), we obtain:

$$L_1(w,\theta) = EI_y\frac{\partial^2\theta}{\partial x^2} + kGA\left(\frac{\partial w}{\partial x} - \theta\right) - \rho I_y\frac{\partial^2\theta}{\partial t^2} = 0 \tag{6.5}$$

$$L_2(w,\theta) = \rho A\frac{\partial^2 w}{\partial t^2} + kGA\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial x}\left(N_x\frac{\partial w_o}{\partial x}\right) = 0 \tag{6.6}$$

If w and θ are partitioned by the relationship in

$$\begin{cases} w = w_a + w_b \\ \theta = \theta_a + \theta_b \end{cases}$$

we

we can then obtain from (6.5) and (6.6):

$$L_1 (w_a, \theta_a) = -L_1 (w_b, \theta_b) \quad (6.7)$$

$$L_2 (w_a, \theta_a) = -L_2 (w_a, \theta_b) \quad (6.8)$$

And

$$M_y = M_{ya} + M_{yb}$$

$$V_z = V_{za} + V_{zb}$$

Furthermore, reference [34] describes the procedure used to reduce the eigenvalues problem at boundaries W_a, θ_a, V_{za} to a problem where boundary conditions become time-independent. It is then possible to formulate it as follows:

$$W_b = f_1(t)\ h_1(x) + f_2(t)\ h_2(x)$$

$$\theta_b = g_1(t)\ k_1(x) + g_2(t)\ k_2(x)$$

Where $f_1, \ldots . k_2$ are arbitrary time functions of t and of displacement x.

This program may be solved by super position to obtain a general solution.

With a finite element approach, the results obtained will be in matrix format; for a beam element, the loads applied to the element are determined by:

$$\{F_\alpha\} = [K]_{syst} \{\Delta_\alpha\} \quad (6.9)$$

Where $\{F_\alpha\}$ is the vector for the nodal load, the $\{\Delta_\alpha\}$ is the vector for all nodal displacements.

The preceding equation can be divided as follows:

$$\left\{ \begin{array}{c} F_A \\ \vdots \\ F_B \end{array} \right\} = \left[\begin{array}{c|c} K_{AA} & K_{AB} \\ \hline K_{BA} & K_{BB} \end{array} \right] \left\{ \begin{array}{c} \Delta_A \\ \vdots \\ \Delta_B \end{array} \right\} \tag{6.10}$$

Where $\{F_A\}$ represents the load applied to the beam and $\{F_B\}$ is the unknown reaction at the points where the displacements are specified.

$\{\Delta_A\}$ and $\{\Delta_B\}$ are also unknowns and specified displacements, respectively.

The preceding equation (6.10) can be solved to produce the following results:

$$\begin{aligned} \{\Delta_A\} &= [K_{AA}]^{-1} (\{F_A\} - [K_{AB}]\{\Delta_B\}) \\ \{F_B\} &= [K_{BA}] \quad \{\Delta_A\} + [K_{BB}]\{\Delta_B\} \end{aligned} \tag{6.11}$$

The stresses are eventually identified, through the use of a particular type of relationship, from the displacements:

$$\{\sigma\} = [ST]\{\Delta_A\} \tag{6.12}$$

A computer program with appropriate subroutines will aid in solution of the problem, and will elaborate upon the displacements and unknown reactions at each pre-divided point on the beam (when the program was initiated, only displacements and reactions for a "clamped-free" beam subjected to a constant extremal force were calculated).

6.4 Boundary end conditions

Only in the case of a uniform element with boundary conditions, there are no constraint forces considered. In the case of forced vibrations of a structure connected to an element, the relationship between the "force-deformation" properties, and the contact points where these forces are acting must be known.

Four possible combinations of boundary conditions can be established as follows:

- free end: $\begin{cases} \theta' = 0 \\ \psi = 0 \end{cases}$

- supported end: $\begin{cases} w = 0 \\ \theta' = 0 \end{cases}$

- clamped end: $\begin{cases} w = 0 \\ \theta = 0 \end{cases}$

- simply supported end: $\begin{cases} \theta = 0 \\ \psi = 0 \end{cases}$

Note that these assumptions are only valid for uniform cases, for non-uniform beams, the results are only approximate.

The four possible combinations above generate ten general cases into which all boundary conditions may be grouped in pairs, with each pair sharing the same frequency equation (Table 7).

For the cases of "free-free", "free-simply supported", "simply supported-simply supported" and "supported-free", the zero frequencies reflect a state of stable equilibrium (or rigid-body motion).

For cases of "supported-supported", "simply supported-supported" and "supported-simply supported", the boundary conditions are completely periodic, starting with the first mode. However, for the other cases, the eigenvalues do not belong to periodic intervals, as previously demonstrated.

Particularly for the "simply supported-supported" and "clamped-free" cases, the eigenvalues are non-zero and the frequency pairs are similar at the higher modes.

As regards the bending problem, displacement w and rotation θ are the two most important parameters describing motion of the body. In many cases, it would be advisable therefore to choose Models I and III to satisfy all the boundary conditions, because they contain the displacement parameters as a function of these two quantities.

CHAPTER VII

CALCULATION METHOD

7. Calculation method

The calculation method consists of subdividing a uniform or non-uniform element of a beam into a sufficient number of finite elements: then, by means of a computer program, we do the following:

1) Enter all necessary data for each beam element (geometric and physical properties): length of an element, modulus of elasticity, Poisson's ratio, density, the radius of curvature and the deviation angle.

2) Calculate the stiffness $[\overline{K}]$ and mass $[\overline{M}]$ matrices for each element in the local (8 x 8) system (listed in Tales (5a, 5b,....5i, 5j of Appendix D). Transform to a global system by a coordinate transformation matrix of order (8 x 10) as given by eq. (A.5.4) to produce the new (10 x 10) matrices $[K]$ and $[M]$.

3) Group together all individual $[K]$ and $[M]$ matrices to yield two global matrices $[K]_{syst}$ and $[M]_{syst}$ as described in Chapter VI.

4) Application of static of kinetic boundary conditions.

5) Calculate the eigenvalues (free vibrations) and the corresponding eigenvectors.

For free vibrations, the equations of motion may be written:

$$[K]_{syst} \{\Delta\} + [M]_{syst} \{\ddot{\Delta}\} = \{0\} \tag{7.1}$$

where $\{\Delta^T\} = \{\Delta_1, \Delta_2, \ldots\ldots.\Delta_{N+1}\}$, N being the number of finite elements, $[K]_{syst}$ and $[M]_{syst}$ are real and symmetric square matrices of the dimensions $\left((N+1)\ NDF \times (N+1)\ NDF\right)$.

The nodal displacement vector $\{\Delta_i\}$ is in the form:

$$\{\Delta_i\}^T = \{W_{xi}\ W_{zi}\ \theta_i\ (d\theta/dx)_i\ \psi_i\} \qquad \text{(model III)}$$

where W_{xi}, W_{zi}, θ_i, $(d\theta/dx)_i$, ψ_i are, respectively, transverse displacements in the direction of x and z, cross section rotations, derived rotation and shear deformations at node i.

With $\{\Delta\} = \{\Delta_0\} \sin(\omega t + \phi)$

(7.1) may be written:

$$([K]_{syst} - \omega^2[M]_{syst}) \{\Delta_0\} = \{0\} \tag{7.2}$$

where (7.2) is a standard eigenvector problem.

In calculating the eigenvectors, if J is the number of imposed constraints, then $[K]_{syst}$ and $[M]_{syst}$ are reduced to square matrices of order $[(N + 1)\ NDF] - j$, with the appropriate lines and columns of $([K]_{syst} - \omega^2\ [M]_{syst})$ being then reduced to satisfy the constraints. The form and characteristics of (7.2) are not affected, except for reduced matrices $[K]_{red}$ nd $[M]_{red}$ (of $[K]_{syst}$ and $[M]_{syst}$) which are finite positive, instead of semi-finite positive. The $[M^{-1}]_{red}\ [K]$ system is formed to determine the free vibrations, where $i = i, 2, \ldots, [(N + 1)\ NDF] - J$ and the corresponding eigenvectors.

The eigenvalues problem produces N* real solutions

$$(\omega_1^2\ ,\ \Delta_{0_1}),\ (\omega_2^2\ ,\ \Delta_{0_2}),\ \ldots\ldots(\omega_{N*}^2,\ \Delta_{0_{N*}}) \text{where} N^* = [(N+1)\ NDF] - J$$

with $0 \leq \omega_1^2 \leq \omega_2^2 \leq \ldots\ldots \leq \omega_{N*}^2$

and $\{\Delta_{oi}\}^T\ [M]\ \{\Delta_{oj}\} = \begin{cases} 0 & \text{si } i \neq j \\ 1 & \text{si } i = j \end{cases}$

Vector $\{\Delta_{oi}\}$ is called the wrighting vector of mode i. Solution of (7.2) gives:

$$[K]\ \{\phi\} = [M]\ \{\phi\}\ [\ \Omega^2] \tag{7.3}$$

where $\{\phi\}$ is a matrix in which the columns are eigenvectors Δ_{oi}, and $[\Omega^2]$ is a diagonal matrix which contains eigenvalues $\omega_i^{\ 2}$ at the ith on a diagonal, with the following rotation:

$$\{\phi\} = \{\Delta_{01},\ \Delta_{02},\ldots.\Delta_{0_{N*}}\}$$

and

$$[\ \Omega^2] = \begin{bmatrix} \omega_1^2 & & & \bigcirc \\ & \omega_2^2 & & \\ & & \ddots & \\ \bigcirc & & & \omega_{N*}^2 \end{bmatrix} \tag{7.4}$$

7.2 Computer program

The finite element program is composed of a series of standard modules. The modules appear as subroutines with different uses in different contexts. The entire program is written in Fortran IV and represents an extremely simple application of the subroutines.

The flow chart used in this analysis is given in Fig. 11.

A. Input is the number of finite elements used, the width and thickness of the element (whether the section is rectangular or square), the diameters (if the section is circular or elliptical), the geometric properties, the length of the element, Young's modulus for each element, Poisson's ratio, density of the element material, the rotation angle, the radius of curvature and the angle between two plane faces (see also Table 15a).

B. For each finite element, the program essentially proceeds as follows:

1) Determines the stiffness and mass matrices of the global system.

2) Groups all the matrices for all the systems as described in δ6.2.

3) Imposes all boundary conditions; $[K]_{syst}$ and $[M]_{syst}$ are now reduced to square matrices of order $[5(N + 1)] - J$, where J is the number of equations for constraints imposed.

4) With $[M]_{syst}$ and $[M]_{syst}$ reduced, the free vibrations can be found by solving a group of equations in the form of $[M^{-1}]_{red}[K]_{red}$, where $[K]_{red}$ and $[M]_{red}$ are real symmetric matrices, and $[M]_{red}$ is finite positive; the calculation is done with the help of the EIGZF subroutines in the IMSL catalogue (this subroutine automatically normalizes all eigenvectors).

5) The final step consists of calculating, as necessary, the static reactions for a constant force applied to the beam. The procedure is based on equations (6.10) and (6.11).

All calculation are duplicated on an IBM 360/70.

In conclusion, the computer program used for the present study calculates free vibrations, the corresponding vibration modes, as well as static reactions for the static forces of a straight, curved or slightly curved Timoshenko beam, or of tapered, rectangular, square, circular and elliptical beams, and for different boundary conditions.

CHAPTER VIII

CALCULATIONS AND DISCUSSION

8.1 General discussion

The essential steps in constructing the stiffness and mass matrices with the finite element method have already been discussed. The following calculations corroborate the results we obtained with our model, and compare them with results from the theoretical equations. A Fortran IV computer program with double-precision arithmetic did the computation with relative ease.

For the first group of calculations, the initial prerequisite was to determine the frequency parameters for a straight, uniform Timoshenko beam having the following properties: ratio of E/G = 8/3, rotary inertia parameter $Q = (0.08)^2$, shear deformation coefficient $k = 2/3$, Poissons's ratio $\nu = 0.3$, density $\rho = 7.3236 \times 10^{-4} \, \frac{\text{lb-sec}^2}{\text{po}^4}$; cold-drawn steel. The beam is subdivided into five uniform finite elements and the square roots of frequency parameter (λ) are obtained for 10 general cases using models I, II, and III. The final results demonstrated that two models, I and III, satisfy all boundary conditions and that their λ_s are justified. This was done by comparing model results with their exact solutions (Table 10).

The exact solutions were obtained by using classical Timoshenko beam equations (Table 1). The results of the classical equations are listed in Table 2 for the ten cases.

The comparison of results between the classical method and the finite element approach is shown in Table 12. For the same number of finite elements, models I and III are closest to the theoretical solutions, e.g. there is less percentage of error in the square roots.

Model II satisfies only the three simple cases: "free-free", "supported-free" and "supported-supported"; for the other cases, there are negative frequencies, the theoretical explanation for which is in δ II.2.1.

Once the two models, I and III, are considered typical, then one of the two can be chosen, preferably model III (see δ II.2.1), as the principal model for the following verification work.

As to the second group of calculations, they involved determining frequency parameters for a tapered beam. The results obtained are listed in Tables 14 (14a,...14h) and were compared with results found by other authors. Bernouilli-Euler results are also presented in this table, to corroborate the importance of the two effects from Timoshenko theory for higher modes.

Testing is done with several variations between 0 and 1 derived from the ratio of length "B" to thickness "H". For the same number of finite elements (8 elements), the present results are almost identical to results obtained in reference [1], involving exact solutions for lower modes. When B and H are set close to 1, the differences are approximately 0.02% to 0.03% for the higher modes (6, 7, 8...). The evidence seems to be clear that up to now, for straight beams (tapered or non-tapered), models I and III can be considered as about the best there are for the analysis of vibration problem with Timoshenko beams.

The third example verified free vibrations (HZ) in a tapered beam but using the stiffness and mass matrices for straight beams.

In this example, the solutions in [27] using the matrix transfer method together with the assumption of displacements U, V, W (in three directions), were developed for the constant prismatic curve of the plane. The free vibrations were calculated approximately for a fifteen-element subdivision of the same arc width.

As for the results obtained in the present study (eight finite elements), involving use of approximation theory for a straight beam and its application to a curved beam, the outcome proved to be scientifically acceptable in the light of the process applied (see Table 13).

8.2 Test of convergence

The following section on tests of convergence is an analytical study of the beam with the coordinates shown in Table 10. The problem consists of ascertaining the number of finite elements required to correspond to the appropriate free vibrations in order to satisfy the convergence conditions mentioned in δII.2.3.

For a given beam with subdivided elements varying from two to ten, the three models I, II and III are used to corroborate the six standard cases. Vibrations (HZ) are obtained for the first 4 modes.

For model II, only the three cases of "supported-supported", "supported-free" and "free-free" are dealt with. In addition to models I and III, the "simply supported-supported", "clamped-clamped" and "clamped-free" cases are also investigated.

First, when investigating model II (Fig. 12a,...12d), it can be shown that vibrations for four modes are easily determined for an adequate number of elements (N = 6). These cases are common and simple, with the rate of convergence being extremely rapid for the first four modes.

Second, in the first and third models in Figure 12e (1st mode), the vibrations for "supported-supported" converge at a quicker rate than where N=6. "Clamped-free" and "clamped-clamped" converge more slowly, and this is even true for "simply supported-supported". The rate of convergence is slower still for a number of required elements up to 10, due to the complexity of the structure.

Similarly, with the second and third modes (Fig. 12f, 12g), the vibrations in a beam with a simply supported end display convergence with a slightly more rapid decline. As demonstrated in Figure 12h (4th mode), the simple end conditions, such as in the "supported", "free" and "clamped" conditions, always represent favourable boundary conditions which enhance the rate of convergence, while the other conditions require a greater number of elements to satisfy the convergence criteria. In general, the number of finite elements required to satisfy the convergence criteria adequately vary between six and eight.

Although the rate of convergence also depends upon the order of the polynomials chosen, this effect however does not simplify conditioning of the stiffness matrix.

Furthermore, the formulation where all elements are assumed to have a cubically varying transverse displacement and a better representation of shear, such as in models I and III, (having already satisfied the two previous criteria) allows greater opportunities for rapid convergence and yields more precise results, unlike the other numerical models with fewer degrees of freedom.

CHAPTER IX

CONCLUSION

The construction of stiffness and mass matrices for a uniform Timoshenko beam has been presented very generally and can be readily applied to non-uniform beams (straight and slightly curved). These matrices mainly apply to the vibration problems, which include the effects of shear and rotary inertia. The present elements (models I and III) with transverse displacement, rotation of the cross section and shear deformation, as demonstrated in exprerssion (2.12), were nodal types of variables. This could be used to calculate the vibration characteristics of a simple structure or complex structures with discontinuities in cross section and angular shapes (particularly tapered beams, and of non-uniform geometric and physical beams, in general. This method converges faster and yields accurate predictions of free vibrations. The modes obtained were in accordance with wave form and Timoshenko effects.

The method of analysis described contained an adequate number of cases of boundary conditions. The numerical results (eigenvalues) obtained from the computer program with this method correlated well with the exact solutions, which were obtained by using a final general solution for the equations of motion.

The foregoing procedure was indicative of the elements convergence toward an exact solution of the elasticity equations for a "supported-supported" beam. The assumption made was that the value of the shear deformation coefficient used was correct, and that it therefore could be applied to non-uniform beams with different boundary conditions.

Finally, the elements with cubic variation in shear demonstrated a better rate of convergence than linear variation, as presented in [12].

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APPENDIX A

FORMULATION OF STIFFNESS AND MASS MATRICES OF A BEAM ELEMENT

- TRANSFORMATION MATRIX -

A.1 Curved beam

From (4.4), the internal deformation energy expression for a curved beam gives:

$$\mathcal{U} = \frac{1}{2} E \int_{Volume} \left\{ H\left[\frac{1}{L}\int_0^L \left(\frac{\partial y_o}{\partial x}\right)\left(\frac{\partial w}{\partial x}\right) dx - \left(\frac{\partial y_o}{\partial x}\right)\left(\frac{\partial w}{\partial x}\right)\right] - \frac{z}{1+\frac{z}{r}}\left(\frac{\partial \theta}{\partial x}\right)\right\}^2 d(\text{volume})$$

$$+ \frac{1}{2} kG \int_{volume} \left(\frac{\partial w}{\partial x} - \frac{r^2}{(r+z)^2}\,\theta\right)^2 d(\text{volume}) \qquad \text{(A.1.1)}$$

From (4.5), the kinetic energy expression for a curved beam gives:

$$\mathcal{T} = \frac{1}{2} \rho \int_{volume} \left\{ H\left[\frac{1}{L}\int_0^L \left(\frac{\partial y_o}{\partial x}\right)\left(\frac{\partial w}{\partial t}\right) dx - \left(\frac{\partial y_o}{\partial x}\right)\left(\frac{\partial w}{\partial t}\right)\right] - \frac{z}{1+\frac{z}{r}}\left(\frac{\partial \theta}{\partial t}\right)\right\}^2 d(\text{volume})$$

$$+ \frac{1}{2} \rho \int_{volume} \left(\frac{\partial w}{\partial t}\right)^2 d(\text{volume}) \qquad \text{(A.1.2)}$$

Non-dimensionalizing by substituting:

$$\eta = \frac{x}{L} \implies dx = d\eta \cdot L$$

$$y = \frac{w}{L} \implies dw = dy \cdot L \qquad \text{(A.1.3)}$$

And according to the expansion of the cubical polynomials W and θ, we set W and θ as:

$$\begin{cases} w = a_1\eta^3 + a_2\eta^2 + a_3\eta + a_4 \\ \theta = a_5\eta^3 + a_6\eta^2 + a_7\eta + a_8 \end{cases}$$

Then, we obtain:

$$W_o = H y_o = 4H\left(\frac{x}{L}\right)\left(1-\frac{x}{L}\right)$$

$$\text{Ou: } y_o = 4\left(\frac{x}{L}\right)\left(1-\frac{x}{L}\right) = 4\eta\,(1-\eta) \tag{A.1.4}$$

$$\text{Et } \quad dy_o = 4\,(1-2\eta)\,d\eta$$

The shear relation $\frac{\partial}{\partial x} = \theta + \psi$ then becomes:

$$\psi = \frac{\partial y}{\partial \eta} - \theta$$

Substituting (A.1.3), (A.1.4) into (A.1.1) and (A.1.2), we will obtain:

$$\mathcal{U} = \frac{1}{2} E \int_{volume} \left\{ \left[\frac{4H}{L}\int_0^1 (1-2\eta)\left[\frac{\partial y}{\partial \eta}\right] d\eta - \frac{4H}{L}(1-2\eta)\left[\frac{\partial y}{\partial \eta}\right]\right] - \frac{z}{1+\frac{z}{r}}\,\frac{1}{L}\left[\frac{\partial \theta}{\partial \eta}\right] \right\}^2 d(\text{volume})$$

$$+ \frac{1}{2} kG \int_{volume} \left[\frac{\partial y}{\partial \eta} - \frac{r^2}{(r+z)^2}\,\theta\right]^2 d\,(\text{volume}) \tag{A.1.5}$$

and

$$\mathcal{T} = \frac{1}{2} \rho \int_{volume} \left\{ \left[4H\int_0^1 (1-2\eta)\left[\frac{\partial y}{\partial t}\right] d\eta - 4H\,(1-2\eta)\left[\frac{\partial y}{\partial t}\right]\right] - \frac{z}{1+\frac{z}{r}} \left[\frac{\partial \theta}{\partial t}\right] \right\}^2 d(\text{volume})$$

$$+ \frac{1}{2} \rho A L^3 \int_0^1 \left[\frac{\partial y}{\partial t}\right] d\eta \tag{A.1.6}$$

A.1.a Stiffness matrix

Model III was chosen as a typical model for the job. The matrix format (A.1.5) can be written as:

$$\mathcal{U} = \frac{1}{2} E \int_{Volume} \left\{ \left[\frac{4H}{L} [B_{3b}] \; \frac{4H}{L} [B_{3a}] \right] - \frac{z}{1+\frac{z}{r}} \frac{1}{L} [B_1] \right\}^2 d(\text{volume}) + \frac{1}{2} kG \int_{Volume} [B_2]^2 \, d(\text{volume})$$

$$= \frac{1}{2} E \int_{volume} \left(\frac{4H}{L}\right)^2 [B_3^T][B_3] \, d(\text{volume}) + \frac{1}{2} E \int_{volume} \frac{z^2}{(1+\frac{z}{r})^2} \frac{1}{L^2} [B_1^T][B_1] \, d(\text{volume})$$

$$+ \frac{1}{2} E \int_{volume} \left(\frac{4H}{L^2}\right)\left(-\frac{z}{1+\frac{z}{r}}\right) \left[[B_1^T][B_3] + [B_3^T][B_1] \right] d(\text{volume}) + \frac{1}{2} kG \int_{volume} [B_2^T][B_2] \, d(\text{volume}) \quad (A.1.7)$$

Where $[B_{3a}]$, $[B_{3b}]$, $[B_1]$, $[B_2]$ & $[B_3]$ are listed in Appendix B.

$$\mathcal{U} = \frac{1}{2} \frac{EI_y}{L} \left\{ \begin{aligned} & \frac{AH^2}{I_y} 16 \int_0^1 [U_1] \, d\eta + \frac{Ar^2}{I_y} (Z' - 2Z - 1) \int_0^1 [U_2] \, d\eta + \frac{rHAz}{I_y} 4 \int_0^1 [U_3] \, d\eta \\ & + \frac{kGAL^2}{EI_y} \left[\frac{64 L^2 r^4}{Z''} [Q + q^2] \int_0^1 [U_{41}] d\eta + Z' \int_0^1 [U_{42}] d\eta + \int_0^1 [U_{43}] \, d\eta \right] \end{aligned} \right\}$$

$$= \frac{1}{2} \frac{EI_y}{L} \Big\{ \underbrace{P[U_1]^* + F[U_2]^* + C[U_3]^* + S_a[U_{41}]^* + S_b[U_{42}]^* + S[U_{43}]^*}_{[\gamma_c^*]} \Big\} \quad (A.1.8)$$

Where: $[U_1]^* = 16 \int_0^1 [U_1] \, d\eta$; $[U_2]^* = \int_0^1 [U_2] \, d\eta$; $[U_3]^* = 4 \int_0^1 [U_3] \, d\eta$

$[U_{41}]^* = \int_0^1 [U_{41}] \, d\eta$; $[U_{42}]^* = \int_0^1 [U_{42}] \, d\eta$; $[U_{43}]^* = \int_0^1 [U_{43}] \, d\eta$.

$[U_1]^*$, $[U_2]^*$, $[U_3]^*$, $[U_{41}]^*$, $[U_{42}]^*$ & $[U_{43}]^*$ are listed in Appendix B.

And: $P = s^2/Q$; $s = H/L$; $Q = I_y/AL^2$; $F = q^2(Z' - 2Z - 1)/Q$; $q = r/L$;

$Z = \frac{1}{12}\left(\frac{h}{r}\right)^2$ for rectangular sections; $Z' = \frac{1}{1-3Z}$; $C = sqZ/Q$;

$S_a = 64 L^2 r^4 (Q + q^2) S / Z''$; $Z'' = \frac{1}{(2h+r)^3 (2h-r)^3}$; $S_b = Z'S$; $S = \frac{kG}{EQ}$.

$[A]$ is given in Appendix B and $[\bar{K}_c]$ is the stiffness matraix of a curved beam in a local system.

To determine $[\gamma_c^*]$ of (A.1.8), the calculations were derived analytically. The elements of $[\bar{K}_c]$ have been determined and are listed in Table 5a (Appendix D).

In a global system of coordinates, the stiffness matrix of a curved Timoshenko beam will then be:

$$\underset{(10\times10)}{[K]_c} = \underset{(10\times8)}{[\Lambda^T]}\ \underset{(8\times8)}{[\bar{K}_c]}\ \underset{(8\times10)}{[\Lambda]} \qquad \text{(A.1.11)}$$

where $[\Lambda]$ is the transformation matraix of the coordinate systems, given by eq. (A.5.4).

A.1.b Mass matrix

In the same way from (A.1.6), we derived:

$$\begin{aligned}\mathcal{C} &= \frac{1}{2}\rho\int_{volume}\left\{\left[4H\,[L_{2b}] - 4H\,[L_{2a}]\right] - \frac{Z}{1+\frac{Z}{r}}\,[L_1]\right\}^2 d(\text{volume}) + \frac{1}{2}\rho A L^3\int_0^1 [L_2]^2\, d\eta \\ &= \frac{1}{2}\rho\int_{volume}(4H)^2\,[L_3^T][L_3]\, d(\text{volume}) + \frac{1}{2}\rho\int_{volume}\frac{Z^2}{(1+\frac{Z}{r})^2}\,[L_1^T][L_1]\, d(\text{volume}) \\ &\quad + \frac{1}{2}\rho\int_{volume}(4H)\left(-\frac{Z}{1+\frac{Z}{r}}\right)\left[[L_1^T][L_3] + [L_3^T][L_1]\right] d(\text{volume}) + \frac{1}{2}\rho A L^3\int_0^1 [L_2^T][L_2]\, d\eta\end{aligned} \qquad \text{(A.1.12)}$$

Where L_{2a}, L_{2b}, L_1, L_2 & L_3 are listed in Appendix B.

$$\begin{aligned}\mathcal{C} &= \frac{1}{2}\rho A L^3\left\{\left(\frac{H}{L}\right)^2 16\int_0^1 [T_1]\, d\eta + \left(\frac{r}{L}\right)^2 (Z' - 2Z - 1)\int_0^1 [T_2]\, d\eta + \frac{rHZ}{L^2}4\int_0^1 [T_3]\, d\eta + \int_0^1 [T_4]\, d\eta\right\} \\ &= \frac{1}{2}\rho A L^3\left\{\underbrace{p\,[T_1]^* + f\,[T_2]^* + c\,[T_3]^* + [T_4]^*}_{[\lambda_c^*]}\right\}\end{aligned} \qquad \text{(A.1.13)}$$

Where $[T_1]^* = 16\int_0^1 [T_1]\, d\eta$; $[T_2]^* = \int_0^1 [T_2]\, d\eta$; $[T_3]^* = 4\int_0^1 [T_3]\, d\eta$; $[T_4]^* = \int_0^1 [T_4]\, d\eta$

$[T_1]^*$, $[T_2]^*$, $[T_3]^*$ & $[T_4]^*$ are listed in Appendix B, and $p = QP$; $f = QF$; $c = QC$.

Likewise, we could write, for $\mathcal{C}$: $\mathcal{C} = \frac{1}{2} \rho A L^3 \{\ddot{\alpha}^T\} [\lambda^*] \{\ddot{\alpha}\}$ (A.1.14)

Where $[\lambda^*] \equiv [\lambda^*_c]$ for a curved beam.

$[\lambda^*] \equiv [\lambda^*_{sc}]$ for a slightly curved beam. (A.1.15)

$[\lambda^*] \equiv [\lambda^*_d]$ for a straight beam.

$[\lambda^*] \equiv [\lambda^*_{dt}]$ for a tapered beam.

And $\{\ddot{\alpha}\} = [A^{-1}] \{\ddot{\Delta}\}$

The kinetic energy of a curved beam can be expressed as follows:

$$\mathcal{C} = \frac{1}{2} \{\ddot{\Delta}^T\} \underbrace{\rho A L^3 [A^{-1}]^T [\lambda^*_c] [A^{-1}]}_{[\bar{M}]_c} \{\ddot{\Delta}\} \qquad \text{(A.1.16)}$$

where $[\bar{M}]_C$ represents the mass matrix in a local system.

The mass matrix of a curved beam in a global system will then be:

$$\underset{(10\times10)}{[M]_C} = \underset{(10\times8)}{[\Lambda^T]} \underset{(8\times8)}{[\bar{M}]_C} \underset{(8\times10)}{[\Lambda]} \qquad \text{(A.1.17)}$$

$[\Lambda]$ is given by eq. (A.5.a), $[\lambda^*_C]$ is analytically determined in the same way as $[\gamma^*_C]$ and $[\bar{M}_C]$ and are listed in Table 5b (Appendix D).

A.1.c Resultant stress matrix

The stress matrix can, finally, be determined from (4.3):

$$[\sigma]_c = \begin{Bmatrix} E\left[\frac{4H}{L}\left([B_{3b}] - [B_{3a}]\right) - \frac{z}{1+\frac{z}{r}} \frac{1}{L} [B_1]\right] \\ 0 \\ 0 \\ 0 \\ 0 \\ kG\,[B_2] \end{Bmatrix} \{\alpha\} = \begin{bmatrix} \frac{4EH}{L} [B_3] - \frac{E}{L} \frac{z}{1+\frac{z}{r}} [B_1] \\ 0 \\ 0 \\ 0 \\ 0 \\ kG\,[B_2] \end{bmatrix}^T [A^{-1}] \{\Delta\} = [ST]_c \begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix}$$

(A.1.18)

where $[ST]_C$ is the stress matrix of a curved beam ($[B_1]$, $[B_2]$ and $[B_3]$ are given in Appendix B).

A.2 Slightly curved beam

A.2.a Stiffness matrix

For this type of beam, (A.1.5) becomes:

$$\mathcal{U} = \frac{1}{2} E \int_{volume} \left\{ \left[\frac{4H}{L} \int_0^1 (1-2\eta) \left[\frac{\partial y}{\partial \eta}\right] d\eta - \frac{4H}{L} (1-2\eta) \left[\frac{\partial y}{\partial \eta}\right] \right] - \frac{z}{L} \left[\frac{\partial \theta}{\partial \eta}\right] \right\}^2 d(volume)$$
$$+ \frac{1}{2} kG \int_{volume} \left[\frac{\partial y}{\partial \eta} - \theta \right]^2 d(volume) \qquad \text{(A.2.1)}$$

Or, from (A.1.7), we have:

$$\mathcal{U} = \frac{1}{2} E \int_{volume} \left(\frac{4H}{L}\right)^2 [B_3^T][B_3] \, d(volume) + \frac{1}{2} E \int_{volume} \left(\frac{z}{L}\right)^2 [B_1^T][B_1] \, d(volume)$$
$$+ \frac{1}{2} E \int_{volume} \frac{4H}{L^2}(-z) \Big[[B_1^T][B_3] + [B_3^T][B_1] \Big] d(volume) + \frac{1}{2} kG \int_{volume} [B_{2c}^T][B_{2c}] \, d(volume) \qquad \text{(A.2.2)}$$

Where $[B_3]$, $[B_1]$ & $[B_{2c}]$ are presented in Appendix B.

$$\mathcal{U} = \frac{1}{2} \frac{EI_y}{L} \left\{ \frac{AH^2}{I_y} 16 \int_0^1 [U_1] d\eta + \int_0^1 [U_2] d\eta + \frac{-HM_z}{I_y} 4 \int_0^1 [U_3] d\eta + \frac{kGAL^2}{EI_y} \int_0^1 [U_{4c}] d\eta \right\}$$

Where $I_y = \int_A z^2 dA$, et $M_z = \int_A z \, dA$: 1st moment following z due to shear.

$$\mathcal{U} = \frac{1}{2} \frac{EI_y}{L} \Big\{ \underbrace{P [U_1]^* + [U_2]^* + V [U_3]^* + S [U_{4c}]^*}_{[\gamma_{sc}^*]} \Big\} \qquad \text{(A.2.3)}$$

Where $[U_1]^* = 16 \int_0^1 [U_1] d\eta$; $[U_2]^* = \int_0^1 [U_2] d\eta$; $[U_3]^* = 4 \int_0^1 [U_3] d\eta$; $[U_{4c}]^* = \int_0^1 [U_{4c}] d\eta$

$[U_1]^*$, $[U_2]^*$, $[U_3]^*$ & $[U_{4c}]^*$ are listed in Appendix B.

And $P = s^2/Q$; $V = -s M_z / ALQ$; $S = kG/EQ$; $Q = I_y / AL^2$.

Therefore: $$\mathcal{U} = \frac{1}{2} \{\Delta^T\} \underbrace{\frac{EI_y}{L} [A^{-1}]^T [\gamma_{sc}^*] [A^{-1}]}_{[\overline{K}]_{sc}} \{\Delta\} \qquad \text{(A.2.4)}$$

Where $[K]_{sc}$ is the stiffness matrix of a slightly curved beam in the local system (Table 5c, Appendix D).

In the global system, we have:

$$\underset{(10\times10)}{[K]_{sc}} = \underset{(10\times8)}{[\Lambda^T]}\ \underset{(8\times8)}{[\bar{K}]_{sc}}\ \underset{(8\times10)}{[\Lambda]} \tag{A.2.5}$$

Where $[\Lambda]$ is given by eq. (A.5.4).

A.2.b Mass matrix

In the same way (A.1.6) becomes:

$$\mathcal{C} = \frac{1}{2}\rho\int_{volume}\left\{\left[4H\int_0^1(1-2\eta)\left[\frac{\partial y}{\partial \eta}\right]d\eta - 4H(1-2\eta)\left[\frac{\partial y}{\partial t}\right]\right] - z\left[\frac{\partial \theta}{\partial t}\right]\right\}^2 d(volume)$$

$$+\frac{1}{2}\rho A L^3\int_0^1\left[\frac{\partial y}{\partial t}\right]^2 d\eta \tag{A.2.6}$$

Or, from (A.1.12), we have:

$$\mathcal{C} = \frac{1}{2}\rho\int_{volume}(4H)^2\left[L_3^T\right]\left[L_3\right]d(volume) + \frac{1}{2}\rho\int_{volume}z^2\left[L_1^T\right]\left[L_1\right]d(volume)$$

$$+\frac{1}{2}\rho\int_{volume}(4H)(-z)\left[\left[L_1^T\right]\left[L_3\right]+\left[L_3^T\right]\left[L_1\right]\right]d(volume) + \frac{1}{2}\rho A L^3\int_0^1\left[L_2^T\right]\left[L_2\right]d\eta \tag{A.2.7}$$

Where L_1, L_2, L_3 are listed in Appendix B

$$\mathcal{C} = \frac{1}{2}\rho A L^3\left\{\underbrace{p\left[T_1\right]^* + Q\left[T_2\right]^* + \vartheta\left[T_3\right]^* + \left[T_4\right]^*}_{[\lambda^*_{sc}]}\right\} \tag{A.2.8}$$

Where $[T_1]^* = 16\int_0^1 [T_1]\,d\eta$; $[T_2]^* = \int_0^1 [T_2]\,d\eta$; $[T_3]^* = 4\int_0^1 [T_3]\,d\eta$; $[T_4]^* = \int_0^1 [T_4]\,d\eta$

$[T_1]^*$, $[T_2]^*$, $[T_3]^*$ & $[T_4]^*$ are listed in Appendix B, and: $p = QP$, $Q = I_y/AL^2$; $\vartheta = QV$.

Therefore:

$$\mathcal{C} = \frac{1}{2}\left\{\ddot{\Delta}^T\right\} \underbrace{\rho A L^3 \left[A^{-1}\right]^T \left[\Lambda^*_{sc}\right] \left[A^{-1}\right]}_{\left[\bar{M}\right]_{sc}} \left\{\ddot{\Delta}\right\} \qquad \text{(A.2.9)}$$

Where $[\bar{M}]_{sc}$ is the mass matrix of a slightly curved beam in the local system (Table 5d, Appendix D).

In the global system, we have:

$$\underset{(10\times10)}{\left[M\right]_{sc}} = \underset{(10\times8)}{\left[\Lambda^T\right]} \underset{(8\times8)}{\left[\bar{M}\right]_{sc}} \underset{(8\times10)}{\left[\Lambda\right]} \qquad \text{(A.2.10)}$$

Where $[\Lambda]$ is given by eq. (A.5.4).

A.2.c Stress matrix

(A.1.18) may be written:

$$[\sigma]_{sc} = \begin{Bmatrix} \frac{4EH}{L}\left[B_3\right] - \frac{E}{L} z \left[B_1\right] \\ 0 \\ 0 \\ 0 \\ 0 \\ kG\left[B_{2c}\right] \end{Bmatrix} \{\alpha\} = \begin{bmatrix} \frac{4EH}{L}\left[B_3\right] - \frac{E}{L} z \left[B_1\right] \\ 0 \\ 0 \\ 0 \\ 0 \\ kG\left[B_{2c}\right] \end{bmatrix}^T \left[A^{-1}\right] \{\Delta\}$$

$$= \underset{(8\times8)}{\left[ST\right]_{sc}} \begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} \qquad \text{(A.2.11)}$$

where $[ST]_{sc}$ is the stress matrix of a slightly curved beam ($[B_1]$, $[B_2]$ and $[B_3]$ are given in Appendix B).

A.3 Straight beam

A.3.a Stiffness matrix

The expression of $\mathcal{U}$ in (A.2.1) becomes:

$$\mathcal{U} = \frac{1}{2} E \int_{Volume} \left\{ \frac{z}{L} \frac{\partial \theta}{\partial \eta} \right\}^2 d(\text{volume}) + \frac{1}{2} k G \int_{Volume} \left\{ \frac{\partial y}{\partial \eta} - \theta \right\}^2 d(\text{volume}) \tag{A.3.1}$$

or, from (A.2.3), the P and V terms cancel out and we obtain:

$$P = V = 0$$

and

$$\mathcal{U} = \frac{1}{2} \frac{EI_y}{L} \left\{ \underbrace{[U_2]^* + S\,[U_{4c}]^*}_{[\gamma_d^*]} \right\} \tag{A.3.2}$$

or

$$\mathcal{U} = \frac{1}{2} \{\Delta^T\} \underbrace{\frac{EI_y}{L} [A^{-1}]^T [\gamma_d^*] [A^{-1}]}_{[\bar{K}]_d} \{\Delta\} \tag{A.3.3}$$

where $[\bar{K}_d]$ is the stiffness matrix of a straight Timoshenko beam, in a local system.

$[A]$ is given in appendix B

$[\gamma_d^*]$ is analytically calculated from (A.3.2) in the same way as $[\gamma_c^*]$ and $[\gamma_{sc}^*]$.

In a global system, the stiffness matrix will then be:

$$\underset{(10\times 10)}{[K]_d} = \underset{(10\times 8)}{[\Lambda]} \; \underset{(8\times 8)}{[\bar{K}]_d} \; \underset{(8\times 10)}{[\Lambda]} \tag{A.3.4}$$

where $[\Lambda]$ is given by eq. (A.5.4)

and $[\bar{K}_d]$ is listed in: Table 5e (model I)
Table 5g (model III)
Table 5i (model II)
of Appendix D.

A.3.b Mass matrix

(A.2.6) can be written as follows:

$$\mathcal{T} = \frac{1}{2}\rho \int_{volume} \left\{ z \left[\frac{\partial \theta}{\partial t}\right] \right\}^2 d(\text{volume}) + \frac{1}{2}\rho A L^3 \int_0^1 \left[\frac{\partial y}{\partial t}\right]^2 d\eta \tag{A.3.5}$$

or from (A.2.8) we have: $p = \vartheta = 0$

Therefore

$$\mathcal{T} = \frac{1}{2}\rho A L^3 \{ \underbrace{Q\,[T_2]^* + [T_4]^*}_{[\lambda_d^*]} \} \tag{A.3.6}$$

In addition,

$$\mathcal{T} = \frac{1}{2}\{\ddot{\Delta}^T\} \underbrace{\rho A L^3 \, [A^{-1}]^T \, [\lambda_d^*] \, [A^{-1}]}_{[\bar{M}]_d} \{\ddot{\Delta}\} \tag{A.3.7}$$

where $[\bar{M}_d]$ is the mass matrix in a local system (Table 5g, model I), Table 5h (model III), Table 5j (model II) in Appendix D).

In a global system, the mass matrix of a straight Timoshenko beam will be:

$$\boxed{\underset{(10\times10)}{[M]_d} = \underset{(10\times8)}{[\Lambda^T]}\ \underset{(8\times8)}{[\bar{M}]_d}\ \underset{(8\times10)}{[\Lambda]}} \qquad \text{(A.3.8)}$$

$[\Lambda]$ is given by equation (A.5.4).

A.3.c Stress matrix

Finally, (A.2.11) becomes:

$$[\sigma]_d = \begin{Bmatrix} -\frac{E}{L} z [B_1] \\ 0 \\ 0 \\ 0 \\ 0 \\ kG[B_{2c}] \end{Bmatrix} \{\alpha\} = \begin{bmatrix} -\frac{E}{L} z [B_1] \\ 0 \\ 0 \\ 0 \\ 0 \\ kG[B_{2c}] \end{bmatrix}^T [A^{-1}] \{\Delta\} = \underset{(8\times8)}{[ST]_d} \begin{Bmatrix} \Delta_i \\ \Delta_j \end{Bmatrix} \qquad \text{(A.3.9)}$$

where $[ST]_d$ is the stress matrix of a straight beam ($[B_1]$ and $[B_{2C}]$ are given in Appendix B).

NOTE: a) On the assumption that the beam is linear, elastic, homogeneous and isotropic, when the axes' origin coincides with the centroid of the section, the fist moment following z is cancelled out ($M_z = 0$).

Indeed, if we consider a beam element with a rectangular section with h and b being width and depth, respectively, we could write:

$$M_z = \int_A z\,dA = \int_{-h/2}^{h/2}\int_{-b/2}^{b/2} z\,dy\,dz = \int_{-h/2}^{h/2} zy\Big|_{-b/2}^{b/2} dz$$

$$= b\int_{-h/2}^{h/2} z\,dz = \frac{b}{2} z^2\Big|_{-h/2}^{h/2} = 0$$

That is to say, the two terms "V" and "v" in $[\bar{K}]_c$ and $[\bar{M}]_c$ vanish. The two matrices $[\bar{K}]_c$ and $[\bar{M}]_c$ therefore become the matrices of a curved, homogeneous and isotropic Timoshenko beam.

b) If initial curvature H in "P" and "p" is zero, $[\bar{K}]_c$ and $[\bar{M}]_c$ will become matrices $[\bar{K}]_d$ and $[\bar{K}]_d$ of a straight Timoshenko beam.

A.4 Tapered beam

As has already been mentioned in previous chapters, we assume the area of surface A and the moment of inertia of a tapered beam varying aflong the x axis to be:

$$A(x) = A_a + (A_b - A_a)\eta = A_a + \Delta A\eta$$

$$I_y(x) = I_{ya} + (I_{yb} - I_{ya})\eta = I_{ya} + \Delta I_y\eta$$

The analytical method used here was developed particularly for a straight beam, using Model III. The same development could effectively apply to the other models. (see Table 4 of Appendix D).

A.4.a Stiffness matrix

From (A.3.1) we can write:

$$\mathcal{U} = \frac{1}{2}\frac{E}{L}\int_0^1 I_y(x)\left[\frac{\partial\theta}{\partial\eta}\right]^2 d\eta + \frac{1}{2}kGL\int_0^1 A(x)\left[\frac{\partial y}{\partial\eta}-\theta\right]^2 d\eta \tag{A.4.1}$$

$$= \frac{1}{2}\frac{E}{L}\int_0^1 \left(I_{ya}+\Delta I_y\eta\right)\left[\frac{\partial\theta}{\partial\eta}\right]^2 d\eta + \frac{1}{2}kGL\int_0^1 \left(A_a+\Delta A\eta\right)\left[\frac{\partial y}{\partial\eta}-\theta\right]^2 d\eta$$

$$= \frac{1}{2}\frac{E}{L}\left\{ I_{ya}\left[U_A\right] + I_{yb}\left[U_{2T}\right]^* \right\} + \frac{1}{2}kGL\left\{ A_a\left[U_B\right] + A_b\left[U_{4CT}\right]^* \right\} \tag{A.4.2}$$

Where $\left[U_A\right]$, $\left[U_{2T}\right]^*$, $\left[U_B\right]$ and $\left[U_{4CT}\right]^*$ are listed in Appendix B.

$$\mathcal{U} = \frac{1}{2}\left[\gamma^*_{dt}\right]$$

Or:

$$\mathcal{U} = \frac{1}{2}\left\{\Delta^T\right\}\underbrace{\left[A^{-1}\right]^T\left[\gamma^*_{dt}\right]\left[A^{-1}\right]}_{\left[\bar{K}\right]_{dt}}\left\{\Delta\right\} \tag{A.4.3}$$

$$\mathcal{U} = \frac{1}{2}\left\{\Delta^T\right\}\left\{\frac{E}{L}\left[A^{-1}\right]^T\left\{I_{ya}\left[U_A\right]+I_{yb}\left[U_{2T}\right]^*\right\}\left[A^{-1}\right] + kGL\left[A^{-1}\right]^T\left\{A_a\left[U_B\right]+A_b\left[U_{4CT}\right]^*\right\}\left[A^{-1}\right]\right\}\left\{\Delta\right\} \tag{A.4.4}$$

where $[\bar{K}]_{dt}$ is the stiffness matrix ;of a straight tapered beam in a local system; $[\gamma^*_{dt}]$ is determined analytically as $[\gamma^*_c]$, $[\gamma^*_{sc}]$ and $[\gamma^*_d]$.

In a global system, we have:

$$\underset{(10\times 10)}{[K]_{dt}} = \underset{(10\times 8)}{[\Lambda]}\ \underset{(8\times 8)}{[\bar{K}]_{dt}}\ \underset{(8\times 10)}{[\Lambda]} \tag{A.4.5}$$

where $[\Lambda]$ is given by equation (A.5.4)
and $[\bar{K}_{dt}]$ is given in Table 9a (Appendix D).

A.4.b Mass matrix

In the same way, from (A.3.5) we have:

$$\mathcal{C} = \frac{1}{2}\rho L \int_0^1 I_y(x)\left[\frac{\partial\theta}{\partial\eta}\right]^2 d\eta + \frac{1}{2}\rho L^3 \int_0^1 A(x)\left[\frac{\partial y}{\partial t}\right]^2 d\eta \tag{A.4.6}$$

$$= \frac{1}{2}\rho L \int_0^1 (I_{ya} + \Delta I_y \eta)\left[\frac{\partial\theta}{\partial t}\right]^2 d\eta + \frac{1}{2}\rho L^3 \int_0^1 (A_a + \Delta A\eta)\left[\frac{\partial y}{\partial t}\right]^2 d\eta$$

$$= \frac{1}{2}\rho L \left\{ I_{ya}[T_A] + I_{yb}[T_{2T}]^* \right\} + \frac{1}{2}\rho L^3 \left\{ A_a[T_B] + A_b[T_{4T}]^* \right\} \tag{A.4.7}$$

Or $[T_A]$, $[T_{2T}]^*$, $[T_B]$ & $[T_{4T}]^*$ are listed in Appendix B.

$$\mathcal{C} = \frac{1}{2}\ [\lambda^*_{dt}]$$

Similarly:

$$\mathcal{C} = \frac{1}{2} \left\{\overset{\circ\circ}{\Delta}^T\right\} \underbrace{\left[\bar{A}^{-1}\right]^T \left[\lambda^*_{dt}\right] \left[\bar{A}^{-1}\right]}_{\left[\bar{M}\right]_{dt}} \left\{\overset{\circ\circ}{\Delta}\right\} \tag{A.4.8}$$

$$\mathcal{C} = \frac{1}{2} \left\{\overset{\circ\circ}{\Delta}^T\right\} \left\{ \rho L \left[\bar{A}^{-1}\right]^T \left\{ I_{ya}\left[T_A\right] + I_{yb}\left[T_{2T}\right]^* \right\} \left[\bar{A}^{-1}\right] + \rho L^3 \left[\bar{A}^{-1}\right]^T \left\{ A_a\left[T_B\right] + A_b\left[T_{4T}\right]^* \right\} \left[\bar{A}^{-1}\right] \right\} \left\{\overset{\circ\circ}{\Delta}\right\} \tag{A.4.9}$$

where $[\bar{M}]_{dt}$ is the mass matrix of a local system. In a global system, the mass matrix of a straight tapered beam will be:

$$\underset{(10\times 10)}{\left[M\right]_{dt}} = \underset{(10\times 8)}{\left[\Lambda\right]} \; \underset{(8\times 8)}{\left[\bar{M}\right]_{dt}} \; \underset{(8\times 10)}{\left[\Lambda\right]} \tag{A.4.10}$$

where $[\Lambda]$ is given by eq. (A.5.4)

$[\lambda^*_{dt}]$ is calculated analytically from (A.4.7)

and $[\bar{M}_{dt}]$ is listed in Table 9b (Appendix D).

A.5 Transformation matrix

Les us assume a finite element of a beam that is initially vibrations in a system of Cartesian coordinates $\bar{x}\bar{y}\bar{z}$, having an angle with the global system xyz, all the calculations are now referred to the new system. For our models I, II and III, transverse displacement is the only parameter that changes in system xyz. The other four parameters labelled w', θ, θ' and ψ which represent rotation around the y axis do not change (Figures 5a and 5b).

We therefore arrive at:

$$W = \overline{W}_x \sin\alpha_0 + \overline{W}_z \cos\alpha_0$$

where $\bar{w}_x$ and $\bar{w}_z$ are, respectively, the components of w at x and z in the local $\bar{x}\bar{y}\bar{z}$ axis system. For an element defined by two ends "i" and "j", the transformation matrix will be:

Model I

$$\begin{Bmatrix} W_i \\ W_i' \\ \theta_i \\ \theta_i' \\ W_j \\ W_j' \\ \theta_j \\ \theta_j' \end{Bmatrix} = [\Lambda]_{(8\times 10)} \begin{Bmatrix} \overline{W}_{xi} \\ \overline{W}_{zi} \\ \overline{W}_i' \\ \overline{\theta}_i \\ \overline{\theta}_i' \\ \overline{W}_{xj} \\ \overline{W}_{zj} \\ \overline{W}_j' \\ \overline{\theta}_j \\ \overline{\theta}_j' \end{Bmatrix} \qquad (A.5.1)$$

Degree of freedom in the global system

Degree of freedom in the local system

Model II

$$\begin{Bmatrix} W_i \\ W_i' \\ \theta_i' \\ \psi_i \\ W_j \\ W_j' \\ \theta_j' \\ \psi_j \end{Bmatrix} = [\Lambda]_{(8\times 10)} \begin{Bmatrix} \overline{W}_{xi} \\ \overline{W}_{zi} \\ \overline{W}_i' \\ \overline{\theta}_i' \\ \overline{\Psi}_i \\ \overline{W}_{xj} \\ \overline{W}_{zj} \\ \overline{W}_j' \\ \overline{\theta}_j' \\ \overline{\Psi}_j \end{Bmatrix} \qquad (A.5.2)$$

Model III:

$$\begin{Bmatrix} w_i \\ \theta_i \\ \theta_i' \\ \psi_i \\ w_j \\ \theta_j \\ \theta_j' \\ \psi_j \end{Bmatrix} = [\Lambda]_{(8\times10)} \begin{Bmatrix} \bar{w}_{xi} \\ \bar{w}_{zi} \\ \bar{\theta}_i \\ \bar{\theta}_i' \\ \bar{\psi}_i \\ \bar{w}_{xj} \\ \bar{w}_{zj} \\ \bar{\theta}_j \\ \bar{\theta}_j' \\ \bar{\psi}_j \end{Bmatrix} \qquad (A.5.3)$$

where $[\Lambda]$ is the transformation matrix for the three models, I, II and III (see Table 4 of Appendix D).

$$[\Lambda]_{(8\times10)} = \begin{bmatrix} \begin{matrix} \sin\alpha_o & \cos\alpha_o & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix} & \bigcirc \\ \bigcirc & \begin{matrix} \sin\alpha_o & \cos\alpha_o & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix} \end{bmatrix}$$

(A.5.4)

APPENDIX B

LIST OF MATRICES

1) [A] MATRICES:

a) Model I : $\{w, w', \theta, \theta'\}$:

Type "i"

$$[A]_{\mathrm{I}i} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & -1 & -1 & -1 & -1 \\ 6 & 2 & 0 & 0 & -3 & -2 & -1 & 0 \end{bmatrix}$$

Type "j"

$$[A]_{\mathrm{I}j} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 & 0 \end{bmatrix}$$

b) Model II : $\{w, w', \theta', \psi\}$:

Type "i"

$$[A]_{\mathrm{II}i} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 & -3 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Type "j"

$$[A]_{\mathrm{II}j} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & -1 & -1 & -1 & -1 \end{bmatrix}$$

c) Model III : $\{w, \theta, \theta', \psi\}$:

Type "i"

$$[A]_{\overline{\overline{III}}\,i} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & -1 & -1 & -1 & -1 \\ 6 & 2 & 0 & 0 & -3 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Type "j"

$$[A]_{\overline{\overline{III}}\,j} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & -1 & -1 & -1 & -1 \end{bmatrix}$$

d) Model IV : $\{w, w', \theta, \psi\}$:

Type "i"

$$[A]_{\overline{\overline{IV}}\,i} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Type "j"

$$[A]_{\overline{\overline{IV}}\,j} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 & -1 & -1 & -1 & -1 \end{bmatrix}$$

Note: The inverse matrices $[A^{-1}]$ do not exist in model IV.

2) THE MATRICES USED IN APPENDIX A (model III, type "i")

a) The matrices in (A.1.a), (A.1.b), (A.2.a), (A.2.b), (A.2.c), (A.3.a), (A.3.b) and (A.3.c)

$$[y] = \begin{bmatrix} \eta^3 & \eta^2 & \eta & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\psi] = \begin{bmatrix} 0 & 0 & 0 & 0 & \eta^3 & \eta^2 & \eta & 1 \end{bmatrix}$$

$$[\theta] = [y] - [\psi] = \begin{bmatrix} 3\eta^2 & 2\eta & 1 & 0 & -\eta^3 & -\eta^2 & -\eta & -1 \end{bmatrix}$$

$$[B_1] = \left[\frac{\partial \theta}{\partial \eta}\right] = \begin{bmatrix} 6\eta & 2 & 0 & 0 & -3\eta^2 & -2\eta & -1 & 0 \end{bmatrix}$$

$$[B_2] = \left[\frac{\partial y}{\partial \eta} - \frac{r^2}{(r+z)^2}\,\theta\right] = \begin{bmatrix} 3\eta^2(1-X) & 2\eta(1-X) & 1-X & 0 & \eta^3 X & \eta^2 X & \eta X & X \end{bmatrix} \; ; \; X = \frac{r^2}{(r+z)^2}$$

$$[B_{2c}] = \left[\frac{\partial y}{\partial \eta} - \theta\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & \eta^3 & \eta^2 & \eta & 1 \end{bmatrix}$$

$$[B_{3a}] = (1-2\eta)\left[\frac{\partial y}{\partial \eta}\right] = \begin{bmatrix} 3\eta^2-6\eta^3 & 2\eta-4\eta^2 & 1-2\eta & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[B_{3b}] = \int_0^1 (1-2\eta)\left[\frac{\partial y}{\partial \eta}\right] d\eta = \begin{bmatrix} -1/2 & -1/3 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[B_3] = [B_{3b}] - [B_{3a}] = \begin{bmatrix} 6\eta^3-3\eta^2-\frac{1}{2} & 4\eta^2-2\eta-\frac{1}{3} & 2\eta-1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[L_1] = \left[\frac{\partial \theta}{\partial t}\right] = \begin{bmatrix} 3\eta^2 & 2\eta & 1 & 0 & -\eta^3 & -\eta^2 & -\eta & -1 \end{bmatrix}$$

$$[L_2] = \left[\frac{\partial y}{\partial t}\right] = \begin{bmatrix} \eta^3 & \eta^2 & \eta & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[L_{2a}] = (1-2\eta)\left[\frac{\partial y}{\partial t}\right] = \begin{bmatrix} \eta^3-2\eta^4 & \eta^2-2\eta^3 & \eta-2\eta^2 & 1-2\eta & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[L_{2b}] = \int_0^1 (1-2\eta)\left[\frac{\partial y}{\partial t}\right] d\eta = \begin{bmatrix} -3/20 & -1/6 & -1/6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[L_3] = [L_{2b}] - [L_{2a}] = \begin{bmatrix} 2\eta^4-\eta^3-\frac{3}{20} & 2\eta^3-\eta^2-\frac{1}{6} & 2\eta^2-\eta-\frac{1}{6} & 2\eta-1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[U_1\right]^* = \frac{1}{6300}\begin{bmatrix} 69840 & 53760 & 40320 & 0 & 0 & 0 & 0 & 0 \\ & 42560 & 33600 & 0 & 0 & 0 & 0 & 0 \\ & & 33600 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & \text{SYM.} & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

$$\left[U_2\right]^* = \frac{1}{6300}\begin{bmatrix} 75600 & 37800 & 0 & 0 & -28350 & -25200 & -18900 & 0 \\ & 25200 & 0 & 0 & -12600 & -12600 & -12600 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 11340 & 9450 & 6300 & 0 \\ & & & & & 8400 & 6300 & 0 \\ & & \text{SYM.} & & & & 6300 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

$$\left[U_3\right]^* = \frac{1}{6300}\begin{bmatrix} 60480 & 25200 & 25200 & 0 & -17640 & -10080 & 0 & 0 \\ & 0 & 0 & 0 & -31080 & -8400 & 0 & 0 \\ & & 0 & 0 & -12600 & -8400 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & \text{SYM.} & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

$$[U_{41}]^* = \frac{1}{6300}\begin{bmatrix} 11340 & 9450 & 6300 & 0 & -3150 & -3780 & -4725 & -6300 \\ & 8400 & 6300 & 0 & -2520 & -3150 & -4200 & -6300 \\ & & 6300 & 0 & -1575 & -2100 & -3150 & -6300 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 900 & 1050 & 1260 & 1575 \\ & & & & & 1260 & 1575 & 2100 \\ & & \text{SYM.} & & & & 2100 & 3150 \\ & & & & & & & 6300 \end{bmatrix}$$

$$[U_{42}]^* = \frac{1}{6300}\begin{bmatrix} -22680 & -18900 & -12600 & 0 & 3150 & 3780 & 4725 & 6300 \\ & -16800 & -12600 & 0 & 2520 & 3150 & 4200 & 6300 \\ & & -12600 & 0 & 1575 & 2100 & 3150 & 6300 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & \text{SYM.} & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

$$[U_{43}]^* = \frac{1}{6300}\begin{bmatrix} 11340 & 9450 & 6300 & 0 & 0 & 0 & 0 & 0 \\ & 8400 & 6300 & 0 & 0 & 0 & 0 & 0 \\ & & 6300 & 0 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & \text{SYM.} & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

$$\left[U_{1C}\right]^{*} = \frac{1}{6300}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 900 & 1050 & 1260 & 1575 \\ & & & & & 1260 & 1575 & 2100 \\ & & \text{SYM.} & & & & 2100 & 3150 \\ & & & & & & & 6300 \end{bmatrix}$$

$$\left[T_{1}\right]^{*} = \frac{1}{6300}\begin{bmatrix} 6532 & 7080 & 8040 & 11760 & 0 & 0 & 0 & 0 \\ & 7760 & 8960 & 13440 & 0 & 0 & 0 & 0 \\ & & 10640 & 16800 & 0 & 0 & 0 & 0 \\ & & & 33600 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & \text{SYM.} & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

$$\left[T_{2}\right]^{*} = \left[U_{11}\right]^{*}.$$

$$
\left[T_3\right]^* = \frac{1}{6300}
\begin{bmatrix}
10440 & 8820 & 7140 & 12600 & -1755 & -1740 & -1470 & 0 \\
 & 6720 & 4200 & 8400 & -1950 & -1960 & -1680 & 0 \\
 & & 0 & 0 & -2310 & -2380 & -2100 & 0 \\
 & & & 0 & -3780 & -4200 & -4200 & 0 \\
 & & & & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 \\
 & & \text{SYM.} & & & & 0 & 0 \\
 & & & & & & & 0
\end{bmatrix}
$$

$$
\left[T_4\right]^* = \frac{1}{6300}
\begin{bmatrix}
900 & 1050 & 1260 & 1575 & 0 & 0 & 0 & 0 \\
 & 1260 & 1575 & 2100 & 0 & 0 & 0 & 0 \\
 & & 2100 & 3150 & 0 & 0 & 0 & 0 \\
 & & & 6300 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 \\
 & & \text{SYM.} & & & & 0 & 0 \\
 & & & & & & & 0
\end{bmatrix}
$$

b) <u>The matrices in (A.4.a) and (A.4.b) (tapered beam)</u>

$$[U_{2T}]^* = \frac{1}{420}\begin{bmatrix} 3780 & 1680 & 0 & 0 & -1512 & -1260 & -840 & 0 \\ & 840 & 0 & 0 & -630 & -560 & -420 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 630 & 504 & 315 & 0 \\ & & & & & 420 & 280 & 0 \\ & & \text{SYM.} & & & & 210 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

$$[U_{4CT}]^* = \frac{1}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 52.5 & 60 & 70 & 84 \\ & & & & & 70 & 84 & 105 \\ & & \text{SYM.} & & & & 105 & 140 \\ & & & & & & & 210 \end{bmatrix}$$

$$[U_{A}] = \frac{1}{420}\begin{bmatrix} 1260 & 840 & 0 & 0 & -378 & -420 & -420 & 0 \\ & 840 & 0 & 0 & -210 & -280 & -420 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 126 & 126 & 105 & 0 \\ & & & & & 140 & 140 & 0 \\ & & \text{SYM.} & & & & 210 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

$$[U_B] = \frac{1}{420}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 7.5 & 10 & 14 & 21 \\ & & & & & 14 & 21 & 35 \\ & & \text{SYM.} & & & & 35 & 70 \\ & & & & & & & 210 \end{bmatrix}$$

$$[T_{2T}]^* = \frac{1}{420}\begin{bmatrix} 630 & 504 & 315 & 0 & -180 & -210 & -252 & -315 \\ & 420 & 280 & 0 & -140 & -168 & -210 & -280 \\ & & 210 & 0 & -84 & -105 & -140 & -210 \\ & & & 0 & 0 & 0 & 0 & 0 \\ & & & & 52.5 & 60 & 70 & 84 \\ & & & & & 70 & 84 & 105 \\ & & \text{SYM.} & & & & 105 & 140 \\ & & & & & & & 210 \end{bmatrix}$$

$$[T_{4T}]^* = \frac{1}{420}\begin{bmatrix} 52.5 & 60 & 70 & 84 & 0 & 0 & 0 & 0 \\ & 70 & 84 & 105 & 0 & 0 & 0 & 0 \\ & & 105 & 140 & 0 & 0 & 0 & 0 \\ & & & 210 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & & \text{SYM.} & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

$$
\left[T_A\right] = \frac{1}{420}
\begin{bmatrix}
126 & 126 & 105 & 0 & -30 & -42 & -63 & -105 \\
 & 140 & 140 & 0 & -28 & -42 & -70 & -140 \\
 & & 210 & 0 & -21 & -35 & -70 & -210 \\
 & & & 0 & 0 & 0 & 0 & 0 \\
 & & & & 7.5 & 10 & 14 & 21 \\
 & & & & & 14 & 21 & 35 \\
 & & \text{SYM.} & & & & 35 & 70 \\
 & & & & & & & 210
\end{bmatrix}
$$

$$
\left[T_B\right] = \frac{1}{420}
\begin{bmatrix}
7.5 & 10 & 14 & 21 & 0 & 0 & 0 & 0 \\
 & 14 & 21 & 35 & 0 & 0 & 0 & 0 \\
 & & 35 & 70 & 0 & 0 & 0 & 0 \\
 & & & 210 & 0 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 \\
 & & \text{SYM.} & & & & 0 & 0 \\
 & & & & & & & 0
\end{bmatrix}
$$

LIST OF MATRICES

$[A]$	Nodal displacement interpolation matrix
$[B]$	Matrix defined in (4.1)
$[L(x)]$	Displacement function matrix
$[\bar{K}]$	Stiffness matrix in the local system
$[K]$	Stiffness matrix in the global system
$[\bar{M}]$	Mass matrix in the local system
$[M]$	Mass matrix in the global system
$\left[\frac{\partial u}{\partial t}\right]_L$, $\left[\frac{\partial u}{\partial t}\right]_R$	Equivalent to $\lfloor w \rfloor$ and $\lfloor \theta \rfloor$, or $\lfloor w \rfloor$ and $\lfloor \phi \rfloor$
$[\Lambda]$	Transformation matrix, defined by δ A.4
$[\lambda^*]$	Matrix defined by (A.1.15) (for mass)
$[\gamma^*]$	Matrix defined by (A.1.9) (for stiffness)
$[\Omega^2]$	Diagonal matrix defined by (7.4)
$[ST]$	Matrix defined by (4.3)
$\{\varepsilon\}$	Deformation vector
$\{\sigma\}$	Stress vector
$\{F_t\}$	Force vector
$\{F_\alpha\}$	Nodal load vector
$\{F_i\}$, $\{F_j\}$	Internal nodal force vector acting on nodes i and j, respectively
$\{u(x,t)\}$	Displacement vector defined by (3.1)
$\{\Delta(t)\}$	Nodal displacement vector defined by (3.2)
$\{\alpha(t)\}$	Vector with parameters defined by (3.1)
$\{\phi\}$	Vector of eigenvectors

APPENDIX C

FORMULATION OF THEORETICAL FREQUENCY EQUATIONS

1) METHOD:

Starting from equations (2.24) and (2.25), the parameters representing nodal displacements can be written:

$$W = A_1 \cosh \lambda\alpha\eta + A_2 \sinh \lambda\alpha\eta + A_3 \cos \lambda\beta\eta + A_4 \sin \lambda\beta\eta \qquad (C.1)$$

$$\Theta = \frac{\lambda}{L}\left(\frac{\alpha^2+\frac{1}{S}}{\alpha}\right)\sinh \lambda\alpha\eta\, A_1 + \frac{\lambda}{L}\left(\frac{\alpha^2+\frac{1}{S}}{\alpha}\right)\cosh \lambda\alpha\eta\, A_2$$
$$+\frac{\lambda}{L}\left(\frac{\frac{1}{S}-\beta^2}{\beta}\right)\sin \lambda\beta\eta\, A_3 - \frac{\lambda}{L}\left(\frac{\frac{1}{S}-\beta^2}{\beta}\right)\cos \lambda\beta\eta \qquad (C.2)$$

$$W' = \lambda\alpha \sinh \lambda\alpha\eta\, A_1 + \lambda\alpha \cosh \lambda\alpha\eta\, A_2 - \lambda\beta \sin \lambda\beta\eta\, A_3 + \lambda\beta \cos \lambda\beta\eta\, A_4 \qquad (C.3)$$

$$\Theta' = \frac{\lambda^2}{L}\left(\alpha^2+\frac{1}{S}\right)\cosh \lambda\alpha\eta\, A_1 + \frac{\lambda^2}{L}\left(\alpha^2+\frac{1}{S}\right)\sin \lambda\alpha\eta\, A_2$$
$$+\frac{\lambda^2}{L}\left(\frac{1}{S}-\beta^2\right)\cos \lambda\beta\eta\, A_3 + \frac{\lambda^2}{L}\left(\frac{1}{S}-\beta^2\right)\sin \lambda\beta\eta\, A_4 \qquad (C.4)$$

$$\Psi = \frac{\lambda}{L}\left(\alpha - \frac{\alpha^2+\frac{1}{S}}{\alpha}\right)\sinh \lambda\alpha\eta\, A_1 + \frac{\lambda}{L}\left(\alpha - \frac{\alpha^2+\frac{1}{S}}{\alpha}\right)\cosh \lambda\alpha\eta\, A_2$$
$$-\frac{\lambda}{L}\left(\beta + \frac{\frac{1}{S}-\beta^2}{\beta}\right)\sin \lambda\beta\eta\, A_3 + \frac{\lambda}{L}\left(\beta + \frac{\frac{1}{S}-\beta^2}{\beta}\right)\cos \lambda\beta\eta\, A_4 \qquad (C.5)$$

Substituting the boundary conditions (Table 7, Appendix D) in the equations (considering $w_x \equiv w_z \equiv w$), the theoretical frequency equations for different cases are obtained corresponding to each group of appropriate boundary conditions. Only two cases will be developed here:

2) SAMPLE CASES:

a) "Supported-Supported" case

Boundary conditions: (@ $x=0$ & $x=L$ (or $\eta=0$ & $\eta=1$))

$$W(0) = \theta'(0) = 0$$
$$W(L) = \theta'(L) = 0 \qquad (c.a.1)$$

Substituting (c.a.1) in (c.1) & (c.4), we obtain:

@ $x=0$: $A_1 = A_3 = 0$

@ $x=L$: $\underbrace{\begin{bmatrix} 1 & 1 \\ \alpha^2+\frac{1}{S} & -\left(\beta^2-\frac{1}{S}\right) \end{bmatrix}}_{[C]} \sinh\lambda\alpha \sin\lambda\beta \begin{Bmatrix} A_2 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Since the determinant of $[C]$ cannot be zero, we therefore have:

$$\underline{\underline{\sinh\lambda\alpha \, \sin\lambda\beta = 0}} \qquad (c.a.2)$$

This equation is valid for $\left[\left(Q-\frac{1}{S}\right)^2+\frac{4}{\lambda^2}\right]^{1/2} > Q+\frac{1}{S}$ and

$\frac{\lambda^2 Q}{S} > 1$, where λ, Q & S are defined in Table 1 (Appendix E).

* If $\left[\left(Q-\frac{1}{S}\right)^2+\frac{4}{\lambda^2}\right]^{1/2} < Q+\frac{1}{S}$ & $\frac{\lambda^2 Q}{S} < 1$, we will have:

$$\underline{\underline{j \sin\lambda\alpha' \sin\lambda\beta = 0}} \qquad \text{(c.a.3)}$$

Where j is an imaginary quantity $(j^2=-1)$ & $\alpha = j\alpha'$.

b) <u>"Free-simply supported" case</u>

Boundary conditions:

$$\theta'(0) = \psi(0) = 0 \qquad \text{(c.b.1)}$$

$$\theta(L) = \psi(L) = 0 \qquad \text{(c.b.2)}$$

Substituting (c.b.1) in (c.4) & (c.5) we have:

$$@\ x=0: \quad A_1 = -\frac{\frac{1}{S}-\beta^2}{\frac{1}{S}+\alpha^2} A_3$$

$$A_2 = \frac{\beta + \frac{\frac{1}{S}-\beta^2}{\beta L}}{\alpha - \frac{\frac{1}{S}+\alpha^2}{\alpha L}} A_4 \qquad \text{(c.b.3)}$$

Similarly, (c.b.2) in (c.2) & (c.5) produces:

$$@\ x=L: \quad \left(\frac{\alpha^2+\frac{1}{S}}{\alpha}\right)\sin\lambda\alpha A_1 + \left(\frac{\alpha^2+\frac{1}{S}}{\alpha}\right)\cos\lambda\alpha A_2$$

$$+\left(\frac{\frac{1}{S}-\beta^2}{\beta}\right)\sin\lambda\beta A_3 - \left(\frac{\frac{1}{S}-\beta^2}{\beta}\right)\cos\lambda\beta A_4 = 0$$

and
$$\left(\alpha-\frac{\alpha^2+\frac{1}{S}}{\alpha L}\right)\sinh\lambda\alpha\,A_1+\left(\alpha-\frac{\alpha^2+\frac{1}{S}}{\alpha L}\right)\cosh\lambda\alpha\,A_2$$
$$-\left(\beta+\frac{\frac{1}{S}-\beta^2}{\beta L}\right)\sin\lambda\beta\,A_3+\left(\beta+\frac{\frac{1}{S}-\beta^2}{\beta L}\right)\cos\lambda\beta\,A_4=0 \qquad \text{(c.b.4)}$$

Substituting (c.b.3) in (c.b.4) and setting:

$$a=\frac{\frac{1}{S}+\alpha^2}{\alpha} \quad ; \quad b=\frac{\frac{1}{S}-\beta^2}{\beta}$$

We will obtain:

$$\begin{bmatrix} b\sin\lambda\beta-b\left(\frac{\beta}{\alpha}\right)\sin\lambda\alpha & -a\left(\frac{\beta+\frac{b}{L}}{\alpha-\frac{a}{L}}\right)\cos\lambda\alpha-b\cos\lambda\beta \\ \left(\beta+\frac{b}{L}\right)\sin\lambda\beta+\left(\alpha-\frac{a}{L}\right)\frac{b}{a}\frac{\beta}{\alpha}\sinh\lambda\alpha & \left(\beta+\frac{b}{L}\right)\cos\lambda\beta-\left(\alpha-\frac{a}{L}\right)\left(\frac{\beta+\frac{b}{L}}{\alpha-\frac{a}{L}}\right)\cosh\lambda\alpha \end{bmatrix}\begin{Bmatrix}A_3\\A_4\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \qquad \text{(c.b.5)}$$

Developing (c.b.5), the final equation will be:

$$-\frac{1}{\gamma}\sin\lambda\alpha\cos\lambda\beta-\frac{1}{\xi}\sin\lambda\beta\cos\lambda\alpha+\left[\frac{1}{\gamma}\sin\lambda\alpha-\sin\lambda\beta\right]\cosh\lambda\alpha-\frac{1}{\gamma}\left[\cos\lambda\alpha+\xi\cos\lambda\beta\right]\sinh\lambda\alpha=0 \qquad \text{(c.b.6)}$$

Equation (c.b.6) is valid for $\left[\left(Q-\frac{1}{S}\right)^2+\frac{4}{\lambda^2}\right]^{1/2}>Q+\frac{1}{S}$ & $\frac{\lambda^2 Q}{S}>1$

* If $\left[\left(Q-\frac{1}{S}\right)^2+\frac{4}{\lambda^2}\right]^{1/2}<Q+\frac{1}{S}$ & $\frac{\lambda^2 Q}{S}<1$, we can write:

$$-\frac{1}{\gamma'}\sinh\lambda\alpha'\cos\lambda\beta-\frac{1}{\xi}\sin\lambda\beta\cosh\lambda\alpha'+\left[\frac{1}{\gamma'}\sinh\lambda\alpha'-\sin\lambda\beta\right]\cos\lambda\alpha'-\frac{1}{\gamma'}\left[\cosh\lambda\alpha'+\xi\cos\lambda\beta\right]\sin\lambda\alpha'=0 \qquad \text{(c.b.7)}$$

Where $\gamma=\alpha/\beta$; $\alpha=j\alpha'$; $\gamma=j\gamma'=\alpha'/\beta$; $\xi=-\left(\frac{\frac{1}{S}-\beta^2}{\frac{1}{S}+\alpha^2}\right)$

And $\cosh j\alpha'=\cos\alpha'$; $\sinh j\alpha'=j\sin\alpha'$

$\cos j\alpha'=\cosh\alpha'$; $\sin j\alpha'=j\sinh\alpha'$.

APPENDIX D

LIST OF TABLES

Boundary conditions	Cas a) : $\left[(Q-\frac{1}{S})^2+\frac{4}{\lambda^2}\right]^{1/2} > Q+\frac{1}{S}$; $\frac{\lambda^2 Q}{S} > 1$	Cas b) : $\left[(Q-\frac{1}{S})^2+\frac{4}{\lambda^2}\right]^{1/2} < Q+\frac{1}{S}$; $\frac{\lambda^2 Q}{S} < 1$
1- Clamped - Clamped	$2-2\cosh\lambda\alpha\cos\lambda\beta+\frac{\lambda}{(1-\frac{\lambda^2 Q}{S})^{1/2}}\left[\frac{\lambda^2}{S}(Q-\frac{1}{S})^2+(\frac{3}{S}-Q)\right]\sinh\lambda\alpha\sin\lambda\beta=0$	$2-2\cos\lambda\alpha'\cos\lambda\beta+\frac{\lambda}{(-1+\frac{\lambda^2 Q}{S})^{1/2}}\left[\frac{\lambda^2}{S}(Q-\frac{1}{S})^2+(\frac{3}{S}-Q)\right]\sin\lambda\alpha'\sin\lambda\beta=0$
2- Clamped - free	$2+\left[2+\lambda^2(Q-\frac{1}{S})^2\right]\cosh\lambda\alpha\cos\lambda\beta-\frac{\lambda(Q+\frac{1}{S})}{(1-\frac{\lambda^2 Q}{S})^{1/2}}\sinh\lambda\alpha\sin\lambda\beta=0$	$2+\left[2+\lambda^2(Q-\frac{1}{S})^2\right]\cos\lambda\alpha'\cos\lambda\beta-\frac{\lambda(Q+\frac{1}{S})}{(-1+\frac{\lambda^2 Q}{S})^{1/2}}\sin\lambda\alpha'\sin\lambda\beta=0$
3- Clamped - supported	$\gamma\mathcal{L}\tanh\lambda\alpha-\tan\lambda\beta=0$	$\gamma'\mathcal{L}\tan\lambda\alpha'+\tan\lambda\beta=0$
4- Free - Free	$2-2\cosh\lambda\alpha\cos\lambda\beta+\frac{\lambda}{(1-\frac{\lambda^2 Q}{S})^{1/2}}\left[\lambda^2 Q(Q-\frac{1}{S})^2+(3Q-\frac{1}{S})\right]\sinh\lambda\alpha\sin\lambda\beta=0$	$2-2\cos\lambda\alpha'\cos\lambda\beta+\frac{\lambda}{(-1+\frac{\lambda^2 Q}{S})^{1/2}}\left[\lambda^2 Q(Q-\frac{1}{S})^2+(3Q-\frac{1}{S})\right]\sin\lambda\alpha'\sin\lambda\beta=0$
5- Supported - free	$\gamma\tanh\lambda\alpha-\mathcal{L}\tan\lambda\beta=0$	$\gamma'\tan\lambda\alpha'+\mathcal{L}\tan\lambda\beta=0$
6- Supported - Supported	$\sinh\lambda\alpha\sin\lambda\beta=0$	$j\sin\lambda\alpha'\sin\lambda\beta=0$
7- Clamped - Simply supported	$(\sin\lambda\alpha+\gamma\mathcal{L}\sin\lambda\beta)\cosh\lambda\alpha+(\cos\lambda\beta-\cos\lambda\alpha)\sinh\lambda\alpha$ $+\gamma\left[\sin\lambda\beta\cos\lambda\alpha+\frac{1}{\gamma\mathcal{L}}\sin\lambda\alpha\cos\lambda\beta\right]=0$	$(\sinh\lambda\alpha'+\gamma'\mathcal{L}\sin\lambda\beta)\cos\lambda\alpha'+(\cos\lambda\beta-\cosh\lambda\alpha')\sin\lambda\alpha'$ $+\gamma'\left[\sin\lambda\beta\cosh\lambda\alpha'+\frac{1}{\gamma'\mathcal{L}}\sinh\lambda\alpha'\cos\lambda\beta\right]=0$
8- Simply supported-supported	$\cosh\lambda\alpha\cos\lambda\beta=0$	$\cos\lambda\alpha'\cos\lambda\beta=0$
9- Simply supported Simply supported	$\sinh\lambda\alpha\sin\lambda\beta=0$	$j\sin\lambda\alpha'\sin\lambda\beta=0$
10- Free - Simply supported	$-\frac{1}{\gamma}\sin\lambda\alpha\cos\lambda\beta-\frac{1}{\mathcal{L}}\sin\lambda\beta\cos\lambda\alpha+\left[\frac{1}{\gamma}\sin\lambda\alpha-\sin\lambda\beta\right]\cosh\lambda\alpha$ $-\frac{1}{\gamma}\left[\cos\lambda\alpha+\mathcal{L}\cos\lambda\beta\right]\sinh\lambda\alpha=0$	$-\frac{1}{\gamma'}\sinh\lambda\alpha'\cos\lambda\beta-\frac{1}{\mathcal{L}}\sin\lambda\beta\cosh\lambda\alpha'+\left[\frac{1}{\gamma'}\sinh\lambda\alpha'-\sin\lambda\beta\right]\cos\lambda\alpha'$ $-\frac{1}{\gamma'}\left[\cosh\lambda\alpha'+\mathcal{L}\cos\lambda\beta\right]\sin\lambda\alpha'=0$

$$\left\{\begin{matrix}\alpha \\ \beta\end{matrix}\right. = \frac{\sqrt{2}}{2}\left(\mp\left(Q+\frac{1}{S}\right)+\left[\left(Q-\frac{1}{S}\right)^2+\frac{4}{\lambda^2}\right]^{1/2}\right)^{1/2} ;\ \alpha=j\alpha' ;\ \gamma=\frac{\alpha}{\beta}=j\gamma'=\frac{\alpha'}{\beta} ;\ \mathcal{L}=\frac{\alpha^2+Q}{\alpha^2+\frac{1}{S}}=\frac{\beta^2-\frac{1}{S}}{\beta^2-Q}=\frac{\alpha^2+Q}{\beta^2-Q}=\frac{\beta^2-\frac{1}{S}}{\alpha^2+\frac{1}{S}} ;\ \left\{\begin{matrix}S=\frac{kGAL^2}{EI_y} \\ Q=\frac{I_y}{AL^2}\end{matrix}\right.$$

Table 1: Theoretical frequency equations for a straight Timoshenko beam.

Boundary conditions	Cas a) : $[(Q-\frac{1}{S})^2+\frac{4}{\lambda^2}]^{1/2} > Q+\frac{1}{S}$; $\frac{\lambda^2 Q}{S} > 1$	Cas b) : $[(Q-\frac{1}{S})^2+\frac{4}{\lambda^2}]^{1/2} < Q+\frac{1}{S}$; $\frac{\lambda^2 Q}{S} < 1$
Supported - free	$\bar{W}_o = \gamma \frac{\cos\lambda\beta}{\cosh\lambda\alpha} \sinh\lambda\alpha\eta + \sin\lambda\beta\eta$ $\Theta_o = \frac{1}{\gamma}\frac{\sin\lambda\beta}{\sinh\lambda\alpha}\cosh\lambda\alpha\eta + \cos\lambda\beta\eta$	$\bar{W}_o = -\gamma' \frac{\cos\lambda\beta}{\cos\lambda\alpha'} \sin\lambda\alpha'\eta + \sin\lambda\beta\eta$ $\Theta_o = -\frac{1}{\gamma'}\frac{\sin\lambda\beta}{\sin\lambda\alpha'}\cos\lambda\alpha'\eta + \cos\lambda\alpha\eta$
Simply supported-supported	$\bar{W}_o = D_1 \cos\lambda\beta\eta$; $\Theta_o = D_2\cos\lambda\beta\eta$	$\bar{W}_o = D_1 \cos\lambda\beta\eta$; $\Theta_o = D_2\cos\lambda\beta\eta$
Supported-Supported	$\bar{W}_o = D_1 \sin\lambda\beta\eta$; $\Theta_o = D_2\sin\lambda\beta\eta$	$\bar{W}_o = D_1 \sin\lambda\beta\eta$; $\Theta_o = D_2\sin\lambda\beta\eta$
Clamped - Clamped	$\bar{W}_o = D_1\Big[\cosh\lambda\alpha\eta + \gamma\xi\Big(\frac{-\cosh\lambda\alpha+\cos\lambda\beta}{\gamma\xi\sinh\lambda\alpha+\sin\lambda\beta}\Big)\sinh\lambda\alpha\eta - \cos\lambda\beta\eta + \Big(\frac{-\cosh\lambda\alpha+\cos\lambda\beta}{\gamma\xi\sinh\lambda\alpha+\sin\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$ $\Theta_o = D_2\Big[\cosh\lambda\alpha\eta + \frac{1}{\gamma\xi}\Big(\frac{-\cosh\lambda\alpha+\cos\lambda\beta}{\frac{1}{\gamma\xi}\sinh\lambda\alpha+\sin\lambda\beta}\Big)\sinh\lambda\alpha\eta - \cos\lambda\beta\eta + \Big(\frac{-\cosh\lambda\alpha+\cos\lambda\beta}{\frac{1}{\gamma\xi}\sinh\lambda\alpha+\sin\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$	$\bar{W}_o = D_1\Big[\cos\lambda\alpha'\eta - \gamma'\xi\Big(\frac{\cos\lambda\alpha'-\cos\lambda\beta}{\gamma'\xi\sin\lambda\alpha'-\sin\lambda\beta}\Big)\sin\lambda\alpha'\eta - \cos\lambda\beta\eta + \Big(\frac{\cos\lambda\alpha'-\cos\lambda\beta}{\gamma'\xi\sin\lambda\alpha'-\sin\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$ $\Theta_o = D_2\Big[\cos\lambda\alpha'\eta + \frac{1}{\gamma'\xi}\Big(\frac{-\cos\lambda\alpha'+\cos\lambda\beta}{\frac{1}{\gamma'\xi}\sin\lambda\alpha'+\sin\lambda\beta}\Big)\sin\lambda\alpha'\eta - \cos\lambda\beta\eta + \Big(\frac{-\cos\lambda\alpha'+\cos\lambda\beta}{\frac{1}{\gamma'\xi}\sin\lambda\alpha'+\sin\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$
Clamped - Supported	$\bar{W}_o = D_1\big[\cosh\lambda\alpha\eta - \coth\lambda\alpha\sinh\lambda\alpha\eta - \cos\lambda\beta\eta + \cot\lambda\beta\sin\lambda\beta\eta\big]$ $\Theta_o = D_2\Big[\cosh\lambda\alpha\eta - \frac{1}{\gamma\xi}\Big(\frac{\gamma\sinh\lambda\alpha+\sin\lambda\beta}{\frac{1}{\xi}\cosh\lambda\alpha+\cos\lambda\beta}\Big)\sinh\lambda\alpha\eta - \cos\lambda\beta\eta - \Big(\frac{\gamma\sinh\lambda\alpha+\sin\lambda\beta}{\frac{1}{\xi}\cosh\lambda\alpha+\cos\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$	$\bar{W}_o = D_1\big[\cos\lambda\alpha'\eta - \cot\lambda\alpha'\sin\lambda\alpha'\eta - \cos\lambda\beta\eta + \cot\lambda\beta\sin\lambda\beta\eta\big]$ $\Theta_o = D_2\Big[\cos\lambda\alpha'\eta - \frac{1}{\gamma'\xi}\Big(\frac{\gamma'\sin\lambda\alpha'-\sin\lambda\beta}{\frac{1}{\xi}\cos\lambda\alpha'+\cos\lambda\beta}\Big)\sin\lambda\alpha'\eta - \cos\lambda\beta\eta + \Big(\frac{\gamma'\sin\lambda\alpha'-\sin\lambda\beta}{\frac{1}{\xi}\cos\lambda\alpha'+\cos\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$
Clamped - Free	$\bar{W}_o = D_1\Big[\cosh\lambda\alpha\eta - \gamma\xi\Big(\frac{\frac{1}{\gamma}\sinh\lambda\alpha-\sin\lambda\beta}{\xi\cosh\lambda\alpha+\cos\lambda\beta}\Big)\sinh\lambda\alpha\eta - \cos\lambda\beta\eta + \Big(\frac{\frac{1}{\gamma}\sinh\lambda\alpha+\sin\lambda\beta}{\xi\cosh\lambda\alpha+\cos\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$ $\Theta_o = D_2\Big[\cosh\lambda\alpha\eta - \frac{1}{\gamma\xi}\Big(\frac{\gamma\sinh\lambda\alpha+\sin\lambda\beta}{\frac{1}{\xi}\cosh\lambda\alpha+\cos\lambda\beta}\Big)\sinh\lambda\alpha\eta - \cos\lambda\beta\eta - \Big(\frac{\gamma\sinh\lambda\alpha+\sin\lambda\beta}{\frac{1}{\xi}\cosh\lambda\alpha+\cos\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$	$\bar{W}_o = D_1\Big[\cos\lambda\alpha'\eta + \gamma'\xi\Big(\frac{\frac{1}{\gamma'}\sin\lambda\alpha'-\sin\lambda\beta}{\xi\cos\lambda\alpha'+\cos\lambda\beta}\Big)\sin\lambda\alpha'\eta - \cos\lambda\beta\eta + \Big(\frac{\frac{1}{\gamma'}\sin\lambda\alpha'-\sin\lambda\beta}{\xi\cos\lambda\alpha'+\cos\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$ $\Theta_o = D_2\Big[\cos\lambda\alpha'\eta - \frac{1}{\gamma'\xi}\Big(\frac{\gamma'\sin\lambda\alpha'-\sin\lambda\beta}{\frac{1}{\xi}\cos\lambda\alpha'+\cos\lambda\beta}\Big)\sin\lambda\alpha'\eta - \cos\lambda\beta\eta + \Big(\frac{\gamma'\sin\lambda\alpha'-\sin\lambda\beta}{\frac{1}{\xi}\cos\lambda\alpha'+\cos\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$
Free - Free	$\bar{W}_o = D_1\Big[\cosh\lambda\alpha\eta + \gamma\Big(\frac{\cosh\lambda\alpha-\cos\lambda\beta}{\gamma\sinh\lambda\alpha-\xi\sin\lambda\beta}\Big)\sinh\lambda\alpha\eta + \frac{1}{\xi}\cos\lambda\beta\eta + \Big(\frac{\cosh\lambda\alpha-\cos\lambda\beta}{\gamma\sinh\lambda\alpha-\xi\sin\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$ $\Theta_o = D_2\Big[\cosh\lambda\alpha\eta - \frac{1}{\gamma}\Big(\frac{\cosh\lambda\alpha-\cos\lambda\beta}{\gamma\sinh\lambda\alpha-\xi\sin\lambda\beta}\Big)\sinh\lambda\alpha\eta + \xi\cos\lambda\beta\eta + \Big(\frac{\gamma\sinh\lambda\alpha-\xi\sin\lambda\beta}{\cosh\lambda\alpha-\cos\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$	$\bar{W}_o = D_1\Big[\cos\lambda\alpha'\eta + \gamma'\Big(\frac{\cos\lambda\alpha'-\cos\lambda\beta}{\gamma'\sin\lambda\alpha'+\xi\sin\lambda\beta}\Big)\sin\lambda\alpha'\eta + \frac{1}{\xi}\cos\lambda\beta\eta - \Big(\frac{\cos\lambda\alpha'-\cos\lambda\beta}{\gamma'\sin\lambda\alpha'+\xi\sin\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$ $\Theta_o = D_2\Big[\cos\lambda\alpha'\eta + \frac{1}{\gamma'}\Big(\frac{\cos\lambda\alpha'-\cos\lambda\beta}{\gamma'\sin\lambda\alpha'+\xi\sin\lambda\beta}\Big)\sin\lambda\alpha'\eta + \xi\cos\lambda\beta\eta - \Big(\frac{\gamma'\sin\lambda\alpha'+\xi\sin\lambda\beta}{\cos\lambda\alpha'-\cos\lambda\beta}\Big)\sin\lambda\beta\eta\Big]$

D_1, D_2 : Constants connected to any of the equations in (2.27).

Table 2: Normal vibration modes for a straight Timoshenko beam.

TYPE OF BEAM	INITIAL CURVATURE "H"	RADIUS OF CURVATURE "r"	DEFORMATION VECTORS
a) CURVED BEAMS	$H \neq 0$	$r \neq 0$	$\{\varepsilon_x\} = \frac{\partial u}{\partial x} - \frac{1}{1+\frac{z}{r}} z \frac{\partial \theta}{\partial x}$ $\{\gamma_{xz}\} = \frac{\partial w}{\partial x} - \frac{r^2}{(r+z)^2} \theta$
b) SLIGHTLY CURVED	$H \neq 0$	$r > 0, \left(\frac{\partial w_0}{\partial x}\right)^2 < 1$	$\{\varepsilon_x\} = \frac{\partial u}{\partial x} - z \frac{\partial \theta}{\partial x}$ $\{\gamma_{xz}\} = \frac{\partial w}{\partial x} - \theta$
c) STRAIGHT	$H = 0$	$r \to \infty$	$\{\varepsilon_x\} = -z \frac{\partial \theta}{\partial x}$ $\{\gamma_{xz}\} = \frac{\partial w}{\partial x} - \theta$

$\left(\frac{\partial w_0}{\partial x}\right)$: Initial slope of the neutral axis of the beam

<u>Table 3</u>: Deformation assumptions for three types of beams.

$$\left[\bar{K}_{IIIc}\right]_{(8\times 8)} = \frac{EI_y}{420L}$$

$17280P$ $+7560S$	$-1440P$ $-100800C$ $+3150S_b$ $+630S$	$-47880C$ $+630S_b$	$-1440P$ $+630S$	$-17280P$ $-7560S$	$-1440P$ $-100800F$ $+3150S_b$ $+630S$	$-52920F$ $-630S_b$	$-1440P$ $+630S$
	$4880P$ $+7560F$ $-137760C$ $+2340S_a$ $-1260S_b$ $+840S$	$630F$ $-35280C$ $+330S_a$	$4880P$ $-68880C$ $-630S_b$ $+840S$	$1440P$ $+100800C$ $-3150S_b$ $-630S$	$400P$ $-7560F$ $+36960C$ $+810S_a$ $+1260S_b$ $-210S$	$630F$ $-33600C$ $-195S_a$ $-105S_b$	$400P$ $-31920C$ $+630S_b$ $-210S$
		$840F$ $+60S_a$	$-35280C$	$47880C$ $-630S_b$	$-630F$ $-16800C$ $+195S_a$ $+105S_b$	$-210F$ $-45S_a$	$-16800C$ $+105S_b$
			$4880P$ $+840S$	$1440P$ $-630S$	$400P$ $+68880C$ $+630S_b$ $-210S$	$-33600C$ $-105S_b$	$400P$ $-210S$
				$17280P$ $+7560S$	$1440P$ $-100800C$ $-3150S_b$ $-630S$	$52920C$ $+630S_b$	$1440P$ $-630S$
	SYM.				$4880P$ $+7560F$ $+63840C$ $+2340S_a$ $-1260S_b$ $+840S$	$-630F$ $-15120C$ $-330S_a$	$4880P$ $+31920C$ $-630S_b$ $+840S$
						$840F$ $+60S_a$	$-15120C$
							$4880P$ $+840S$

$P = s^2/Q$; $Q = I_y/AL^2$; $F = q^2(Z' - 2Z - 1)/Q$; $Z = \frac{1}{12}(h/r)^2$ for rectangular sections;
$C = sqZ/Q$; $Z' = 1/(1-3Z)$; $S = kG/EQ$; $Z'' = 1/(2h+r)^3(2h-r)^3$; $S_a = 64L^2r^4(Q+q^2)S/Z''$;
$s = H/L$; $S_b = Z'S$; $q = r/L$.

Table 5a: Stiffness matrix of a curved Timoshenko beam (model III)

$$[\overline{M}_{IIIc}]_{(8\times 8)} = \frac{\rho A L^3}{420}$$

11008p +2340	944p −2520c +330	−60c	944p +330	5792p +810	−736p +2520c −195	−360c	−736p −195
	132p +2340f −540c +60	330f −25c	132p −270c +60	736p −2520c +195	−108p +810f +540c −45	−195f −35c	−108p +270c −45
		60f	−25c	−360c	195f +35c	−45f	35c
			132p +60	736p +195	−108p +270c −45	−35c	−108p −45
				11008p +2340	−944p +2520c −330	−60c	−944p −330
	SYM.				132p +2340f −540c +60	−330f +25c	132p −270c +60
						60f	25c
							132p +60

$P = s^2/Q$; $Q = I_y/AL^2$; $F = q^2(Z' - 2Z - 1)/Q$; $Z = \frac{1}{12}\left(\frac{h}{r}\right)^2$ for rectangular sections;
$C = sqZ/Q$; $Z' = 1/(1-3Z)$; $S = kG/EQ$; $Z'' = 1/\left((2h+r)^3(2h-r)^3\right)$; $S_a = 64L^2r^4(Q+q^2)S/Z''$;
$s = H/L$; $S_b = Z'S$; $q = r/L$; $p = QP$; $f = QF$; $c = QC$.

Table 5b: Mass matrix of a curved Timoshenko beam (model III)

$$\left[\bar{K}\right]_{IIIsc} = \frac{EI_y}{420L}$$

(8x8)

17280P +7560S	-1440P -100800V +3780S	-47880V +630S	-1440P +630S	-17280P -7560S	-1440P +100800V +3780S	-52920V -630S	-1440P +630S
	4880P -137760V +1920S +7560	-35280V +330S +630	4880P -68880V +210S	1440P +100800V -3780S	400P +36960V +1860S -7560	-33600V -300S +630	400P -31920V +420S
		60S +840	-35280V	47880V -630S	-16800V +300S -630	45S -210	-16800V +105S
			4880P +840S	1440P -630S	400P +68880V +420S	-33600V -105S	400P -210S
				17280P +7560S	1440P -100800V -3780S	52920V +630S	1440P -630S
	SYM.				4880P +63840V +1920S +7560	-15120V -330S -630	4880P +31920V +210S
						60S +840	-15120V
							4880P +840S

$P = \delta^2/Q$; $\delta = H/L$; $Q = I_y/AL^2$; $V = -\delta M_z/QAL$; $S = kG/EQ$.

Table 5c: Stiffness matrix of a slightly curved Timoshenko beam (model III)

$$\left[\bar{M}_{IIIsc}\right]_{(8\times 8)} = \frac{\rho A L^3}{420}$$

$11008p$ $+2340$	$944p$ -2520ϑ $+330$	-60ϑ	$944p$ $+330$	$5792p$ $+810$	$-736p$ $+2520\vartheta$ -195	-360ϑ	$-736p$ -195
	$132p$ -540ϑ $+2340Q$ $+60$	-25ϑ $+330Q$	$132p$ -270ϑ $+60$	$736p$ -2520ϑ $+195$	$-108p$ $+540\vartheta$ $+810Q$ -45	-35ϑ $-195Q$	$-108p$ $+270\vartheta$ -45
		$60Q$	-25ϑ	-360ϑ	35ϑ $+195Q$	$-45Q$	35ϑ
			$132p$ $+60$	$736p$ $+195$	$-108p$ $+270\vartheta$ -45	-35ϑ	$-108p$ -45
				$11008p$ $+2340$	$-944p$ $+2520\vartheta$ -330	-60ϑ	$-944p$ -330
	SYM.				$132p$ -540ϑ $+2340Q$ $+60$	25ϑ $-330Q$	$132p$ -270ϑ $+60$
						$60Q$	25ϑ
							$132p$ $+60$

$P = s^2/Q$; $s = H/L$; $Q = I_y/AL^2$; $V = -sM_z/QAL$; $S = kG/EQ$; $p = QP$; $\vartheta = QV$

Table 5d: Mass matrix of a slightly curved Timoshenko beam (model III)

$$
\left[\bar{K}_{Id}\right]_{(8\times 8)} = \frac{EI_y}{420L}
\begin{bmatrix}
504S & 42S & 210S & 42S & -504S & 42S & 210S & -42S \\
 & 56S & -42S & 0 & -42S & -14S & 42S & -7S \\
 & & 156S+504 & 22S+42 & -210S & 42S & 54S-504 & -13S+42 \\
 & & & 4S+56 & -42S & 7S & 13S-42 & -3S-14 \\
 & & & & 504S & -42S & -210S & 42S \\
 & & & & & 56S & -42S & 0 \\
 & \text{SYM.} & & & & & 156S+504 & -22S-42 \\
 & & & & & & & 4S+56
\end{bmatrix}
$$

$S = kG/EQ \; ; \quad Q = I_y/AL^2$

Table 5e: Stiffness matrix of a straight Timoshenko beam (model I)

$$\left[\bar{M}_{Id}\right]_{(8\times 8)} = \frac{\rho A L^3}{420} \begin{bmatrix} 156 & 22 & 0 & 0 & 54 & -13 & 0 & 0 \\ & 4 & 0 & 0 & 13 & -3 & 0 & 0 \\ & & 156Q & 22Q & 0 & 0 & 54Q & -13Q \\ & & & 4Q & 0 & 0 & 13Q & -3Q \\ & & & & 156 & -22 & 0 & 0 \\ & & & & & 4 & 0 & 0 \\ & \text{SYM.} & & & & & 156Q & -22Q \\ & & & & & & & 4Q \end{bmatrix}$$

$$Q = I_y / A L^2$$

Table 5f: Mass matrix of a straight Timoshenko beam (model I)

$$
\left[\bar{K}_{\text{IIId}}\right]_{(8\times 8)} = \frac{EI_y}{420L}
\begin{bmatrix}
504S & 252S & 42S & 42S & -504S & 252S & -42S & 42S \\
 & 128S+504 & 22S+42 & 14S & -252S & 124S-504 & -20S+42 & 28S \\
 & & 4S+56 & 0 & -42S & 20S-42 & -3S-14 & 7S \\
 & & & 56S & -42S & 28S & -7S & -14S \\
 & & & & 504S & -252S & 42S & -42S \\
 & & & & & 128S+504 & -22S-42 & 14S \\
 & \text{SYM.} & & & & & 4S+56 & 0 \\
 & & & & & & & 56S
\end{bmatrix}
$$

$$S = kG/EQ \quad ; \quad Q = I_y/AL^2$$

Table 5g: Stiffness matrix of a straight Timoshenko beam (model III)

$$
\left[\overline{M}_{IIId}\right]_{(8\times 8)} = \frac{\rho A L^3}{420}
\begin{bmatrix}
156 & 22 & 0 & 22 & 54 & -13 & 0 & -13 \\
 & 156Q+4 & 22Q & 4 & 13 & 54Q-3 & 13Q & -3 \\
 & & 4Q & 0 & 0 & 13Q & -3Q & 0 \\
 & & & 4 & 13 & -3 & 0 & -3 \\
 & & & & 156 & -22 & 0 & -22 \\
 & \text{SYM.} & & & & 156Q+4 & -22Q & 4 \\
 & & & & & & 4Q & 0 \\
 & & & & & & & 4
\end{bmatrix}
$$

$$Q = I_y / AL^2$$

Table 5h: Mass matrix of a straight Timoshenko beam (model III)

$$
\underset{(8\times 8)}{\left[\bar{K}_{IId}\right]} = \frac{EI_y}{420L}
\begin{bmatrix}
504S & 252S & 42S & -210S & -504S & 252S & -42S & -210S \\
 & 128S+504 & 22S+42 & -114S-504 & -252S & 124S-504 & -20S+42 & -96S+504 \\
 & & 4S+56 & -22S-42 & -42S & 20S-42 & -3S-14 & -13S+42 \\
 & & & 156S+504 & 210S & -96S+504 & 13S-42 & 54S-504 \\
 & & & & 504S & -252S & 42S & 210S \\
 & & & & & 128S+504 & -22S-42 & -114S-504 \\
 & \text{SYM.} & & & & & 4S+56 & 22S+42 \\
 & & & & & & & 156S+504
\end{bmatrix}
$$

$$S = kG/EQ \; ; \quad Q = I_y/AL^2$$

Table 5i: Stiffness matrix of a straight Timoshenko beam (model II)

$$
\underset{(8\times 8)}{\left[\bar{M}_{IId}\right]} = \frac{\rho A L^3}{420}
\begin{bmatrix}
156 & 22 & 0 & 0 & 54 & -13 & 0 & 0 \\
 & 156Q+4 & 22Q & -156Q & 13 & 54Q-3 & -13Q & -54Q \\
 & & 4Q & 22Q & 0 & 13Q & -3Q & -13Q \\
 & & & 156Q & 0 & -54Q & 13Q & 54Q \\
 & & & & 156 & -22 & 0 & 0 \\
 & & & & & 156Q+4 & -22Q & -156Q \\
 & \text{SYM.} & & & & & 4Q & 22Q \\
 & & & & & & & 156Q
\end{bmatrix}
$$

$$Q = I_y / AL^2$$

Table 5j: Mass matrix of a straight Timoshenko beam (model II)

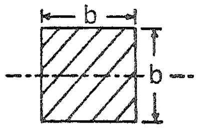
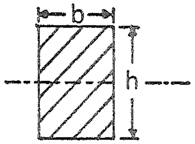
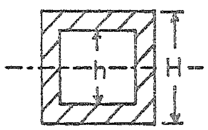
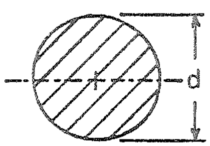
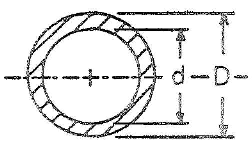
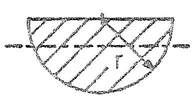
FORM OF THE CROSS SECTION	SECTION	FORMULAS (FROM [19]) AND VALUES FOR SHEAR COEFFICIENT	k (FOR $\nu = 0.3$)
1- Square 	b^2	$\frac{10(1+\nu)}{12+11\nu}$	0.850
2- RECTANGLE 	bh		
3- Thin-walled - Square cylinder 	H^2-h^2	$\frac{20(1+\nu)}{48+39\nu}$	0.435 $(h \simeq H)$
4- CIRCLE 	$\frac{\pi d^2}{4}$	$\frac{6(1+\nu)}{7+6\nu}$	0.900
5- Hollow circle 	$\frac{\pi}{4}(D^2-d^2)$	$\frac{6(1+\nu)(1+m^2)^2}{(7+6\nu)(1+m^2)^2+(20+12\nu)m^2}$ $m=\frac{d}{D}$	0.530 $(d \simeq D)$ 0.620 $(\frac{d}{D}=\frac{1}{2})$ 0.771 $(\frac{d}{D}=\frac{1}{4})$
6- SEMI-CIRCLE 	$\frac{\pi r^2}{4}$	$\frac{1+\nu}{1.305+1.273\nu}$	0.770

Table 6: Values of the shear coefficient (or form factor) of various sections

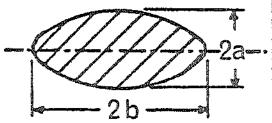
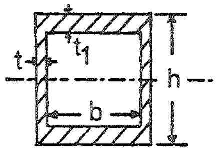
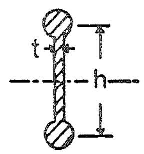
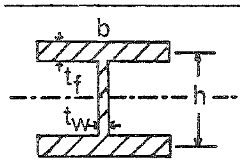
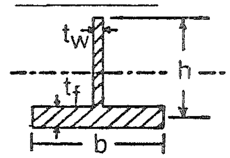
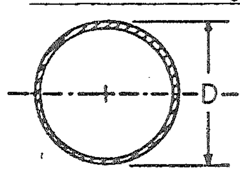
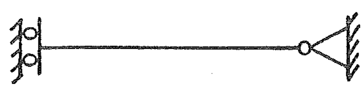
FORM OF THE CROSS SECTION	SECTION	FORMULAS (FROM [19]) AND VALUES FOR SHEAR COEFFICIENT	k (FOR $\nu = 0.3$)
7- ELLIPSE 	πab	$\frac{12(1+\nu)a^2(3a^2+b^2)}{(40+37\nu)a^4+(16+10\nu)a^2b^2+\nu b^4}$; $a<b$ ou (or) $a>b$	0.827 $\left(\frac{b}{a}=2\right)$
8- Thin-walled box section 	$(b+2t)h-(h-2t_1)b$	$m=\frac{bt_1}{ht}$; $n=\frac{b}{h}$ $\frac{10(1+\nu)(1+3m)^2}{(12+72m+150m^2+90m^3)+\nu(11+66m+135m^2+90m^3)+10n^2[3m^2+m(3+\nu)]}$	0.483 $\left(\frac{t_1}{t}=2;\ \frac{b}{h}=\frac{1}{2}\right)$
9- "SPAR-AND-WEB SECTION" 	$2A_s+t(h-d)$ A_s: aire d'un "spar"	$m=\frac{2A_s}{ht}$ $\frac{10(1+\nu)(1+3m)^2}{(12+72m+150m^2+90m^3)+\nu(11+66m+135m^2+90m^3)}$	0.607 $\left(A_s=2;\ t=1;\ h=6\right)$
10- Thin-walled I-section 	$(bh+bt_f)-(b-t_w)(h-t_f)$	$m=\frac{2bt_f}{ht_w}$; $n=\frac{b}{h}$ $\frac{10(1+\nu)(1+3m)^2}{(12+72m+150m^2+90m^3)+\nu(11+66m+135m^2+90m^3)+30n^2(m+m^2)+5\nu n^2(8m+9m^2)}$	0.324 $\left(\frac{b}{h}=\frac{1}{2};\ \frac{t_f}{t_w}=2\right)$
11- T-section 	$bt_f+t_w(h-\frac{1}{2}t_f)$	$m=\frac{bt_f}{ht_w}$; $n=\frac{b}{h}$ $\frac{10(1+\nu)(1+4m)^2}{(12+96m+276m^2+192m^3)+\nu(11+88m+248m^2+216m^3)+30n^2(m+m^2)+10\nu n^2(4m+5m^2+m^3)}$	0.424 $\left(\frac{b}{h}=\frac{1}{2};\ \frac{t_f}{t_w}=2\right)$
12- Thin-walled cylinder 	$\simeq \frac{\pi D^2}{4}$	$\frac{2(1+\nu)}{4+3\nu}$	0.530

Table 6 (cont'd): Values of the shear coefficient (or form factor) of various section

STANDARD TYPES OF SUPPORT		LIMITING CONDITIONS @ $x=0$	@ $x=L$	SATISFAC-TORY MODE
1– Clamped-clamped		$w_x=0$, $w_z=0$, $\theta=0$	$w_x=0$, $w_z=0$, $\theta=0$	I, III
2– Clamped-free		$w_x=0$, $w_z=0$, $\theta=0$	$\theta'=0$, $\psi=0$	I, III
3– Clamped-supported		$w_x=0$, $w_z=0$, $\theta=0$	$w_x=0$, $w_z=0$, $\theta'=0$	I, III
4– Free-free		$\theta'=0$, $\psi=0$	$\theta'=0$, $\psi=0$	I, II, III
5– Supported-free		$w_x=0$, $w_z=0$, $\theta'=0$	$\theta'=0$, $\psi=0$	I, II, III
6– Supported-supported		$w_x=0$, $w_z=0$, $\theta'=0$	$w_x=0$, $w_z=0$, $\theta'=0$	I, II, III
7– Clamped-simply supported		$w_x=0$, $w_z=0$, $\theta=0$	$\theta=0$, $\psi=0$	I, III
8– Simply supported-supported		$\theta=0$, $\psi=0$	$w_x=0$, $w_z=0$, $\theta'=0$	I, III
9– Simply supported-simply supported		$\theta=0$, $\psi=0$	$\theta=0$, $\psi=0$	I, III
10– Free-simply supported		$\theta'=0$, $\psi=0$	$\theta=0$, $\psi=0$	I, III

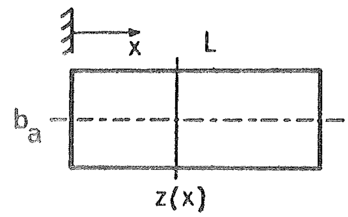
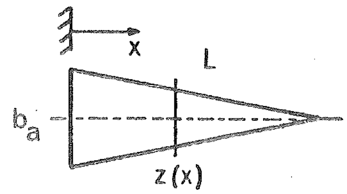
Model I: $\{w, w', \theta, \theta'\}$; Model II: $\{w, w', \theta', \psi\}$; Model III: $\{w, \theta, \theta', \psi\}$

Table 7: Natural boundary conditions applying in the standard cases

<table>
<tr><th>SECTION</th><th>BEAM SHAPE</th><th>VARIATION IN THICKNESS, DEPTH, DIAMETER</th></tr>
<tr><td rowspan="2">RECTANGULAR

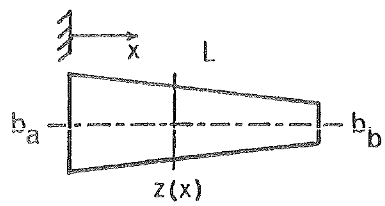
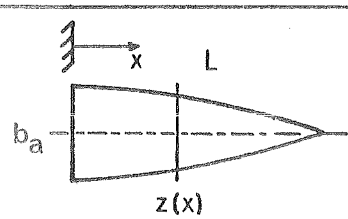
Largeur $b = b_a\left(\frac{x}{L}\right)^{n-1}$

Epaisseur $h = h_a\left(\frac{x}{L}\right)$

$n = 1,2,3\ldots$</td><td>
</td><td>$Z(x) = b_a$</td></tr>
<tr><td>
</td><td>$Z(x) = b_a\left(1-\frac{x}{L}\right)$</td></tr>
<tr><td rowspan="2">CIRCULAR

Rayon $r = r_a\left(\frac{x}{L}\right)$

Aire $A_a = \pi r_a^2$

Moment d'inertie $I_{ya} = \frac{\pi}{4} r_a^4$</td><td>
</td><td>$Z(x) = b_a + (b_b - b_a)\frac{x}{L}$</td></tr>
<tr><td>
</td><td>$Z(x) = b_a\left(1-\frac{x}{L}\right)^{1/2}$</td></tr>
<tr><td>ELLIPTICAL

$b = b_a\left(\frac{x}{L}\right)^{n-1}$

$h = h_a\left(\frac{x}{L}\right)$

$A_a = \frac{\pi}{4} b_a h_a$

$I_{ya} = \frac{\pi}{64} b_a h_a^3$

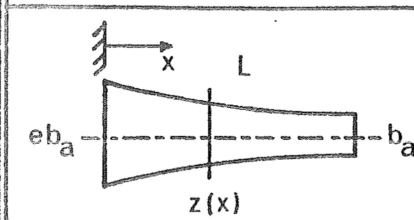
$n = 1,2,3\ldots$</td><td>
</td><td>$Z(x) = b_a e^{\left(1-\frac{x}{L}\right)}$</td></tr>
</table>

Table 8: Basic formulas for calculating width, thickness or diameter variations in a tapered beam.

$$\left[\overline{K}_{IIIdt}\right] = \frac{1}{420}\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & -k_{11} & k_{16} & k_{17} & k_{18} \\ & k_{22} & k_{23} & k_{24} & -k_{12} & k_{26} & k_{27} & k_{28} \\ & & k_{33} & k_{34} & k_{35} & k_{36} & k_{37} & k_{38} \\ & & & k_{44} & -k_{14} & k_{46} & k_{47} & k_{48} \\ & & & & k_{11} & -k_{16} & -k_{17} & -k_{18} \\ & & & & & k_{66} & k_{67} & k_{68} \\ & \text{SYM.} & & & & & k_{77} & k_{34} \\ & & & & & & & k_{88} \end{bmatrix}$$

$k_{11} = 252\,(A_a + A_b)\,kGL$

$k_{12} = 4\,(33\,A_a + 30\,A_b)\,kGL$

$k_{13} = 6\,(4\,A_a + 3\,A_b)\,kGL$

$k_{14} = 42\,A_b\,kGL$

$k_{16} = 4\,(30\,A_a + 33\,A_b)\,kGL$

$k_{17} = -6\,(3\,A_a + 4\,A_b)\,kGL$

$k_{18} = 42\,A_a\,kGL$

$k_{22} = 252\,(I_{ya} + I_{yb})\frac{E}{L} + 2\,(35\,A_a + 29\,A_b)\,kGL$

$k_{23} = 42\,I_{yb}\frac{E}{L} + (13\,A_a + 9\,A_b)\,kGL$

$k_{24} = -2\,(2\,A_a - 9\,A_b)\,kGL$

$k_{26} = -252\,(I_{ya} + I_{yb})\frac{E}{L} + 62\,(A_a + A_b)\,kGL$

$k_{27} = 42\,I_{ya}\frac{E}{L} - (9\,A_a + 11\,A_b)\,kGL$

$k_{28} = 4\,(6\,A_a + A_b)\,kGL$

$k_{33} = 14\,(3\,I_{ya} + I_{yb})\frac{E}{L} + (2.5\,A_a + 1.5\,A_b)\,kGL$

$k_{34} = 2\,(A_b - A_a)\,kGL$

$k_{35} = -6\,(4\,A_a + 3\,A_b)\,kGL$

$k_{36} = -42\,I_{yb}\frac{E}{L} + (11\,A_a + 9\,A_b)\,kGL$

$k_{37} = -7\,(I_{ya} + I_{yb})\frac{E}{L} - 1.5\,(A_a + A_b)\,kGL$

$k_{38} = (5\,A_a + 2\,A_b)\,kGL$

$k_{44} = 14\,(3\,A_a + A_b)\,kGL$

$k_{46} = 4\,(A_a + 6\,A_b)\,kGL$

$k_{47} = -(2\,A_a + 5\,A_b)\,kGL$

$k_{48} = -7\,(A_a + A_b)\,kGL$

$k_{66} = 252\,(I_{ya} + I_{yb})\frac{E}{L} + (58\,A_a + 70\,A_b)\,kGL$

$k_{67} = -42\,I_{ya}\frac{E}{L} - (9\,A_a + 13\,A_b)\,kGL$

$k_{68} = 2\,(9\,A_a - 2\,A_b)\,kGL$

$k_{77} = 14\,(I_{ya} + 3\,I_{yb})\frac{E}{L} + (1.5\,A_a + 2.5\,A_b)\,kGL$

$k_{88} = 14\,(3\,A_b + A_a)\,kGL$

Table 9a: Stiffness matrix of a straight tapered Timoshenko beam (model III)

$$
\left[\bar{M}_{III,4t} \right] = \frac{1}{420}
\begin{bmatrix}
m_{11} & m_{12} & 0 & m_{12} & m_{15} & m_{16} & 0 & m_{16} \\
 & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} & m_{27} & m_{28} \\
 & & m_{33} & 0 & 0 & m_{36} & m_{37} & 0 \\
 & & & m_{24} & m_{25} & m_{28} & 0 & m_{28} \\
 & & & & m_{55} & m_{56} & 0 & m_{56} \\
 & & & & & m_{66} & m_{67} & m_{68} \\
 & \text{SYM.} & & & & & m_{77} & 0 \\
 & & & & & & & m_{68}
\end{bmatrix}
$$

$m_{11} = 12(10A_a + 3A_b)\rho L^3$

$m_{12} = (15A_a + 7A_b)\rho L^3$

$m_{15} = 27(A_a + A_b)\rho L^3$

$m_{16} = -(7A_a + 6A_b)\rho L^3$

$m_{22} = 12(10I_{ya} + 3I_{yb})\rho L + (2.5A_a + 1.5A_b)\rho L^3$

$m_{23} = (15I_{ya} + 7I_{yb})\rho L$

$m_{24} = (2.5A_a + 1.5A_b)\rho L^3$

$m_{25} = (6A_a + 7A_b)\rho L^3$

$m_{26} = 27(I_{ya} + I_{yb})\rho L - 1.5(A_a + A_b)\rho L^3$

$m_{27} = -(7I_{ya} + 6I_{yb})\rho L$

$m_{28} = -1.5(A_a + A_b)\rho L^3$

$m_{33} = (2.5I_{ya} + 1.5I_{yb})\rho L$

$m_{36} = (6I_{ya} + 7I_{yb})\rho L$

$m_{37} = -15(I_{ya} + I_{yb})\rho L$

$m_{55} = 12(3A_a + 10A_b)\rho L^3$

$m_{56} = -(7A_a + 15A_b)\rho L^3$

$m_{66} = 12(3I_{ya} + 10I_{yb})\rho L + (1.5A_a + 2.5A_b)\rho L^3$

$m_{67} = -(7I_{ya} + 15I_{yb})\rho L$

$m_{68} = (1.5A_a + 2.5A_b)\rho L^3$

$m_{77} = (1.5I_{ya} + 2.5I_{yb})\rho L$

Table 9b: Mass matrix of a straight tapered Timoshenko beam (model III)

LIMITING CONDITIONS	EXACT SOLUTIONS	MODEL II	% ERROR	MODELS I & III	% ERROR
1 Clamped-clamped	14.694 30.894 49.909 69.309	8.64 26.96 47.70 68.83	-41.173 -12.734 - 4.426 - 0.691	14.693 30.970 49.904 69.468	-0.006 0.246 -0.010 0.229
2 Clamped-free	3.284 15.488 34.301 53.652	0.154×10^{-5} 12.83 32.27 52.94	-100.000 - 17.161 - 5.921 - 1.327	3.284 15.487 34.237 53.673	0.000 - 0.006 - 0.011 0.039
3 Clamped - supported	11.639 29.157 48.796 69.110	8.64 26.96 47.70 68.83	- 25.732 - 7.535 - 2.259 - 0.405	11.639 29.159 48.81 69.21	0.000 0.006 0.028 0.144
4 Free-free	0.00 0.00 17.85 37.25	0.181×10^{-5} 0.651×10^{-6} 17.85 37.45	0.000 0.000 0.000 0.536	0.285×10^{-5} 0.892×10^{-6} 17.85 37.45	0.000 0.000 0.000 0.536
5 Supported-free	0.00 12.77 32.23 52.82	0.183×10^{-5} 12.83 32.27 52.94	0.000 0.469 0.124 0.227	0.307×10^{-5} 12.83 32.27 52.94	0.000 0.469 0.124 0.227
6 Supported - supported	8.645 26.960 47.680 68.726	8.644 26.957 47.696 68.826	-0.011 -0.011 0.033 0.145	8.644 26.957 47.696 68.826	-0.011 -0.011 0.033 0.145
7 Clamped - simply supported	——	0.213×10^{-5} 12.83 32.27 52.94	——	4.843 19.98 38.83 58.97	——
8 Simply supported supported	2.376 17.18 37.21 58.02	0.356×10^{-5} 12.83 32.27 52.94	-100.000 - 25.320 - 13.276 - 8.755	2.376 17.22 37.22 58.25	0.000 0.232 0.026 0.396
9 Simply supported simply supported	0.000 8.645 26.960 47.680	0.254×10^{-5} 0.250×10^{-5} 17.85 37.45	0.000 -100.000 - 33.790 - 21.455	0.191×10^{-5} 8.644 26.957 47.696	0.000 - 0.011 - 0.011 0.033
10 Free-simply supported	0.000 5.224 22.624 45.065	0.113×10^{-5} 0.681×10^{-6} 17.85 37.45	0.000 -100.000 - 21.101 - 16.897	0.231×10^{-5} 5.209 22.186 42.623	0.000 - 0.287 - 1.936 - 5.418
OBSERVATION		VALID FOR 4, 5, 6		VALID FOR ALL CASES	

Cold-drawn steel, $k = 2/3$, $G/E = 3/8$, $Q = (0.08)^2$, $\rho = 7.3236 \times 10^{-4} \frac{lbm\text{-}sec^2}{IN^4}$, 5 finite elements used

Table 10: Theoretical verification of the frequency parameter roots in the three models I, II and III and for the cases analyzed

LIMITING CONDITIONS	MODELS I & III	MODEL II
FREE - FREE	0.430×10^{-3} 0.134×10^{-3} 2,690.458 5,643.709	0.274×10^{-3} 0.981×10^{-4} 2,690.458 5,643.708
FREE - SIMPLY SUPPORTED	0.349×10^{-3} 785.059 3,343.492 6,423.254	
SIMPLY SUPPORTED-SIMPLY SUPPORTED	0.288×10^{-3} 1,302.559 4,062.366 7,187.769	
SUPPORTED-FREE	0.462×10^{-3} 1,933.449 4,863.540 7,978.242	0.276×10^{-3} 1,933.449 4,863.539 7,978.242
SIMPLY SUPPORTED-SUPPORTED	358.409 2,595.470 5,608.583 8,778.486	
CLAMPED-FREE	494.819 2,333.633 5,168.567 8,088.483	
CLAMPED-SIMPLY SUPPORTED	729.867 3,011.182 5,851.182 8,885.976	
SUPPORTED-SUPPORTED	1,302.559 4,062.366 7,187.769 10,371.965	1,302.559 4,062.366 7,187.769 10,371.966
CLAMPED-SUPPORTED	1,753.679 4,392.837 7,354.957 10,429.416	
CLAMPED-CLAMPED	2,213.679 4,667.587 7,520.385 10,468.697	

Cold-drawn steel, $k = 2/3$, $G/E = 3/8$, $Q = (0.08)^2$, $\rho = 7.3236 \times 10^{-4} \frac{\text{lbm-sec}^2}{\text{in}^4}$, 5 elements.

Table 11: Frequency (HZ) of a straight uniform Timoshenko beam in the three models I, II, and III (ascending order)

$k = 0.650$, $Q = (0.05)^2$

MODE	2 ELEMENTS [9] 12 d.o.f	2 ELEMENTS [13] 8 d.o.f	2 ELEMENTS I AND III 8 OR 10 d.o.f	3 ELEMENTS	4 ELEMENTS	5 ELEMENTS
1 ($\lambda_1 = 3.419$)	0.000	0.000	0.058	—	—	—
2 ($\lambda_2 = 18.613$)	0.208	0.317	0.021	—	—	—
3 ($\lambda_3 = 44.624$)	3.284	5.219	4.638	—	—	—

$k = 2/3$, $Q = (0.02)^2$

MODE	2 ELEMENTS [9], 12 d.o.f	2 ELEMENTS I & III, 8 OR 10 d.o.f	3 ELEMENTS MC CALLEY 6 d.o.f	3 ELEMENTS [12] 9 d.o.f	3 ELEMENTS [13] 12 d.o.f	3 ELEMENTS I & III 12 OR 16 d.o.f	4 ELEMENTS	5 ELEMENTS I AND III 20 OR 25 d.o.f
1 ($\lambda_1 = 3.500$)	0.024	0.057	0.018	0.010	0.000	0.000	—	0.000
2 ($\lambda_2 = 21.353$)	0.498	0.641	0.586	0.334	0.178	0.196	—	0.028
3 ($\lambda_3 = 57.474$)	11.106	12.072	2.372	2.219	0.699	0.739	—	0.104

$k = 2/3$, $Q = (0.08)^2$

MODE	2 EL. MC CALLEY 4 d.o.f	2 EL. [9] 12 d.o.f	2 EL. [13] 8 OR 9 d.o.f	2 EL. I & III 8 OR 10 d.o.f	3 EL. [12], 8 OR 9 d.o.f	3 EL. I & III, 12 OR 16 d.o.f	4 EL. MC CALLEY 8 OR 9 d.o.f	4 EL. [9] 24 d.o.f	4 EL. [13] 16 OR 18 d.o.f	4 EL. I & III 16 OR 20 d.o.f	5 EL. I AND III 20 OR 25 d.o.f
1 ($\lambda_1 = 3.284$)	0.30	0.00	0.00	0.000	0.00	0.000	0.06	0.00	0.00	0.000	0.000
2 ($\lambda_2 = 15.488$)	4.25	0.12	0.19	0.200	0.50	0.038	1.44	0.00	0.01	0.012	0.006
3 ($\lambda_3 = 34.301$)	59.00	2.19	2.80	2.819	6.83	0.215	6.45	0.03	0.05	0.064	0.011

Cold-drawn steel, $G/E = 3/8$, $\rho = 7.3236 \times 10^{-4}$ lbm-sec^2/in^4, d.o.f: degrees of freedom. Note: The values of   for I & III are obtained by double precision.

Table 12: Error percentages in the frequency parameter roots for a straight uniform "clamped-free" Timoshenko beam obtained for different values of k (form factor) and Q (rotational inertia parameter) in various numerical methods.

REF. [27], matrix transfer method (15 elements used)	PRESENT METHOD (Model III, strai straight beam theory) (8 elements used)
7.6	9.812
43.5	54.928
126.9	144.393

$E = 28 \times 10^6\ lb/in^2$, $L = 19.593\ in$, $\nu = 0.29$, $b = 1\ in$, $h = 0.09$,

$\rho = 7.5078 \times 10^{-4}\ \frac{lbm\text{-}sec^2}{in^4}$; 301 stainless steel bar.

Table 13: Frequency (HZ) of a tapered "clamped-free" Timoshenko beam (Fig. 13)

Table 14 (a,...h): Frequency parameters roots for a tapered beam (Fig. 10) using the Bernoulli-Euler and Timoshenko theories.

Notation: 1) Given: $\nu = 0.3$

$k = 0.85$

$Q = (0.08^2)$

$\rho = 7.3236 \times 10^{-4} \frac{\text{lbm-sec}^2}{\text{po}^4}$

$G/E = 3/8$

2) $H = h_2/h_1$: thickness ratio

$B = b_2/b_1$: depth ratio

a: exact solutions wrinch equation [42]

b: numerical solutions for upper limits (Rayleigh-Ritz) [21]

c: numerical solutions for lower limits

d: exact solutions of [46]

e: exact solutions converted equations from [48]

f: exact solutions standard manual

g: exact solutions for equations in [1]

h: numerical solutions finite element (10 elements) [49]

i: solutions devised by dynamic discretization technique (8 elements) [50]

p: present method (model III, 8 (elements)

3) Frequency parameter roof for a tapered beam:

$$\lambda = \left(\frac{\rho A, L^4 w^2}{EI_{y1}} \right)^{\frac{1}{2}}$$

note: The results were obtained for every combination of B and H:

- The upper portin: Bernoulli-Euler theory.
- The lower portion: Timoshenko theory.

H \ B	0	0.1	0.2	0.4	0.7	1.0
0	8.71926[a] 8.7193[b] 8.6628[c] 8.71924[h] 8.71926[l] 8.13372[i] 7.92671[p]					5.31510[a] 5.3151[b] 5.2998[c] 5.31507[h] 5.31511[i] 5.08622[i] 5.09357[p]
0.1		7.2049[b] 7.1827[c] 7.20486[d] 7.20487[i] 6.78850[i] 6.71569[p]				4.6307[b] 4.6246[c] 4.63072[e] 4.63072[i] 4.44487[l] 4.44118[p]
0.2			6.1964[b] 6.1863[c] 6.19634[d] 6.19639[i] 5.86287[i] 5.82727[p]			4.2925[b] 4.2891[c] 4.29249[e] 4.29249[i] 4.11769[i] 4.10743[p]
0.4				5.0090[b] 5.0056[c] 5.00906[d] 5.00903[i] 4.74979[i] 4.72813[p]		3.9343[b] 3.9326[c] 3.93427[e] 3.93428[i] 3.76228[i] 3.74688[p]
0.7					4.0669[b] 4.0658[c] 4.06287[d] 4.06693[i] 3.85346[i] 3.83439[p]	3.6667[b] 3.6659[c] 3.66675[e] 3.66675[i] 3.48689[i] 3.46974[p]
1.0						3.5160[b] 3.5155[c] 3.51602[f] 3.51602[h] 3.51602[i] 3.32405[g] 3.32405[i] 3.32405[p]

Table 14a: Frequency parameter root for the first mode

H \ B	0	0.1	0.2	0.4	0.7	1.0
0	21.1457^{a} 21.1457^{b} 20.3766^{c} 21.1472^{h} 21.1457^{i} — 18.5758^{i} 17.60511^{p}					15.2072^{a} 15.2072^{b} 14.8603^{c} 15.2078^{h} 15.2076^{i} — 13.7798^{i} 13.99704^{p}
0.1		18.6868^{b} 18.1268^{c} 18.6802^{d} 18.6802^{i} — 16.4421^{i} 16.34043^{p}				14.9314^{b} 14.7291^{c} 14.9308^{e} 14.9308^{i} — 13.3372^{i} 13.36937^{p}
0.2			18.3866^{b} 18.1268^{c} 18.3855^{d} 18.3855^{i} — 15.9230^{i} 15.86075^{p}			15.7442^{b} 15.5782^{c} 15.7427^{e} 15.7427^{i} — 13.7575^{i} 13.71479^{p}
0.4				19.0657^{b} 18.8807^{c} 19.0649^{d} 19.0649^{i} — 15.9107^{i} 15.81825^{p}		17.4882^{b} 17.3449^{c} 17.4879^{e} 17.4879^{i} — 14.6449^{i} 14.55824^{p}
0.7					20.5555^{b} 20.4104^{c} 20.5555^{d} 20.5555^{i} — 16.1766^{i} 16.07537^{p}	19.8806^{b} 19.7493^{c} 19.8806^{e} 19.8806^{i} — 15.6411^{i} 15.54511^{p}
1.0						22.0345^{b} 21.9072^{c} 22.0345^{f} 22.0345^{h} 22.0345^{i} — 16.2890^{g} 16.2890^{i} 16.28910^{p}

Table 14b: Frequency parameter root for the second mode

H \ B	0	0.1	0.2	0.4	0.7	1.0
0	38.4538 [a] 38.4540 [b] 34.3348 [c] 38.4788 [h] 38.4539 [i] 31.6431 [l] 29.86669 [p]					30.0198 [a] 30.0199 [b] 27.5880 [c] 30.0341 [h] 30.0241 [i] 25.5528 [l] 26.56245 [p]
0.1		37.2195 [b] 34.4834 [c] 37.1238 [d] 37.1238 [i] 29.8463 [l] 29.92889 [p]				32.8574 [b] 30.8563 [e] 32.8331 [f] 32.8331 [i] 26.6811 [l] 26.83637 [p]
0.2			39.8509 [b] 37.3952 [c] 39.8336 [d] 39.8336 [i] 30.7268 [l] 30.65150 [p]			36.9200 [b] 34.9201 [c] 36.8846 [e] 36.8846 [i] 28.6363 [l] 28.53802 [p]
0.4				45.7917 [b] 43.3934 [c] 45.7384 [d] 45.7384 [i] 32.7692 [l] 32.54886 [p]		44.0567 [b] 41.9052 [c] 44.0248 [e] 44.0248 [i] 31.6243 [l] 31.40320 [p]
0.7					54.0172 [b] 51.5382 [c] 54.0152 [d] 54.0152 [i] 35.1026 [l] 34.86169 [p]	53.3259 [b] 50.9370 [c] 53.3222 [e] 53.3222 [i] 34.6625 [l] 34.42902 [p]
1.0						61.7151 [b] 59.0746 [c] 61.6972 [f] 61.7129 [h] 61.6972 [i] 36.7078 [g] 36.7078 [l] 36.70851 [p]

Table 14c: Frequency parameter root for the third mode

H \ B	0	0.1	0.2	0.4	0.7	1.0
0	60.6801[a] 60.8589[h] 60.6814[i] 46.7586[i] 44.97782[p]					49.7633[a] 49.8868[h] 49.7864[l] 39.7040[i] 42.11457[p]
0.1		63.5049[d] 63.5049[i] 46.2748[i] 46.70384[p]				58.9170[e] 58.9171[i] 43.2814[i] 43.61563[p]
0.2			71.2418[d] 71.2418[i] 48.7398[i] 48.66961[p]			68.1163[e] 68.1164[i] 46.8294[i] 46.67096[p]
0.4				85.3438[d] 85.3438[l] 52.6316[i] 52.26383[p]		83.5541[e] 83.5541[i] 51.6232[i] 51.24365[p]
0.7					103.975[d] 103.975[i] 56.3081[i] 55.91184[p]	103.267[e] 103.267[l] 55.9280[i] 55.53893[p]
1.0						120.902[f] 121.017[h] 120.902[i] 58.2788[g] 58.2788[i] 58.28341[p]

Table 14d: Frequency parameter root for the fourth mode

H \ B	0	0.1	0.2	0.4	0.7	1.0
0	87.8340[a] 87.8399[i]					74.4400[a] 74.5244[i]
	63.5175[i] 62.35627[p]					55.7378[i] 59.85039[p]
0.1		98.1657[d] 98.1657[i]				93.3881[e] 93.3881[i]
		64.9540[i] 65.78303[p]				62.2042[i] 62.72251[p]
0.2			112.828[d] 112.828[i]			109.594[e] 109.594[i]
			68.9018[i] 68.88619[p]			67.1993[i] 67.01899[p]
0.4				138.035[d] 138.035[i]		136.203[e] 136.203[i]
				74.2587[i] 73.82221[p]		73.4140[i] 72.92246[p]
0.7					170 577[d] 170 577[i]	169.862[e] 169.862[i]
					78.7486[i] 78.24661[p]	78.4163[i] 77.91895[p]
1.0						199.860[f] 199.860[i]
						80.2126[g] 80.2127[i] 80.23055[p]

Table 14e: Frequency parameter root for the fifth mode

H \ B	0	0.1	0.2	0.4	0.7	1.0
0	119.919[a] 119.940[i] 81.6222[i] 81.62042[p]					104.051[a] 104.289[i] 73.2848[i] 79.32176[p]
0.1		141.233[d] 141.233[i] 85.2855[i] 86.46655[p]				136.321[e] 136.321[i] 82.7708[i] 83.42960[p]
0.2			164.668[d] 164.668[i] 90.4981[i] 90.56551[p]			161.360[e] 161.359[i] 88.9730[i] 88.79999[p]
0.4				203.845[d] 203.845[i] 96.8403[i] 96.40018[p]		201.986[e] 201.986[i] 96.0680[i] 95.58165[p]
0.7					253.820[d] 253.820[i] 100.592[i] 100.12127[p]	253.100[e] 253.099[i] 100.271[i] 99.80173[p]
1.0						298 556[f] 298 556[i] 94.4517[g] 94.4520[i] 94.46971[p]

Table 14f: Frequency parameter root for the sixth mode

H \ B	0	0.1	0.2	0.4	0.7	1.0
0	156.936^{a} 157.001^{i} 101.216^{i} 103.06881^{p}					138.596^{a} 139.153^{i} 92.1666^{i} 100.77608^{p}
0.1		192.764^{d} 192.764^{i} 107.568^{i} 108.27332^{p}				187.753^{e} 187.753^{i} 105.078^{i} 105.32828^{p}
0.2			226.796^{d} 226.796^{i} 113.738^{i} 113.04987^{p}			223.436^{e} 223.436^{i} 112.267^{i} 111.35501^{p}
0.4				282.789^{d} 282.789^{i} 119.904^{i} 119.28533^{p}		280.911^{e} 280.911^{i} 119.190^{i} 118.53447^{p}
0.7					353.706^{d} 353.706^{i} 113.912^{i} 113.95597^{p}	352.982^{e} 352.982^{i} 113.792^{i} 113.85648^{p}
1.0						416.991^{f} 416.991^{i} 106.836^{g} 106.836^{i} 106.87304^{p}

Table 14g: Frequency parameter root for the seventh mode

H \ B	0	0.1	0.2	0.4	0.7	1.0
0	198.887[a] 199.052[i] 123.450[i] 127.16003[p]					178.075[a] 179.201[i] 113.180[i] 124.94138[p]
0.1		252.788[d] 252.788[i] 132.014[i] 131.08608[p]				247.701[e] 247.701[i] 129.869[i] 128.56251[p]
0.2			299.231[d] 299.231[i] 134.722[i] 134.09727[p]			295.831[e] 295.831[i] 133.631[i] 132.83913[p]
0.4				374.872[d] 374.879[i] 128.763[i] 128.42260[p]		372.980[e] 372.987[i] 128.204[i] 128.00331[p]
0.7					470.237[d] 470.237[i] 121.735[i] 121.40250[p]	469.511[e] 469.511[i] 121.403[i] 121.07336[p]
1.0						555.165[f] 555.165[i] 114.722[g] 114.732[i] 114.78133[p]

Table 14h: Frequency parameter root for the eighth mode

Table 15a: Instructions for data entry

1. Execution menu (I2)

   col. 1-2 .... Number of problems to be executed (NGXC)

2. Control menu

   col. 1 .... 1: for a straight beam (linear form)
   2: for a straight beam (tapered form) (KZ)
   3: for a curved or slighty curved beam
   4: for a curved or (linear form)
   slighty curved beam (tapered form)

   col. 2 .... 1: rectangular section
   2: square section (KW)
   3: circular section
   4: elliptical section

3. Control menu for number of elements (I2)

   col. 1-2 ...... number of elements used (NEL)

4. Instructions menu for beam width and thickness (4D10.4)

col. 1-10	width at 1st end		(Z1)
col. 11-20	width at 2nd	for rectangular	(Z2)
col. 21-30	thickness at 1st	and square sections	(H1)
col. 31-40	thickness at 2nd end		(H2)

5. Instructions menu for geometric properties (6D10.4)

col. 1-10	diameter at 1st end	rectangular section	(D1)
col. 11-20	diameter at 2nd end		(D2)
col. 21-30	1st diameter at 1st end		(A1)
col. 31-40	1st diameter at 2nd end	elliptical section	(A2)
col. 41-50	2nd diameter at 1st end		(B1)
col. 51-60	2nd diameter at 2nd end		(B2)

Note: Once a section has been chosen for execution, the other sections take the value of "1".

6. Element menus (3I2, D7.4, D11.3, D7.3, D11.4, D8.3, D8.4, D7.3)
   col. 1-2 .... number of each element (NE)
   col. 3-4 .... 1st node of an element (NODE(NE,J),J=1,2
   col. 5-6 .... 2nd node of an element
   col. 7-13 .... length of an element (SL)
   col. 14-24 .... Young's modulus (E)
   col. 25-31 .... Poisson's ratio (U)
   col. 32-42 .... densities (SM)
   col. 43-50 .... angle of deviation (ALF)
   col. 51-58 .... radius of curvature (R)
   col. 59-65 .... angle between 2 plane surfaces for curved bems) (ALFI)

Note: SL = 1 if ALFI ≠ 0

7. Title menu (15A4)
   col. 1-60 .... location of specified boundary (title)

8. Menu for the boundary conditions (2E2, D6.4)
   col. 1-2 .... specified node (node)
   col. 3-4 .... directional code (see Table 15b, NDF=5)(IDiR)
   col. 5-10 .... Displacement vector (DELP)

9. Blank menu

10. Test menu (I1)
   col. 1 = 1:......... calculation continues
   ≠ 1:......... stop

11. Boundary conditions for static force determination (2I2, D6.4)
   col. 1-2 .... specified node (node)
   col. 3-4 .... directional code (see Table 15b, NDF=4)(IDiR)
   col. 5-10 .... displacement vector (DEE)

12. Blank menu

13. Menu for loads and moments

    col. 1-4 .... loaded node (node)

    col. 5-8 .... directional code (see Table 15b, NDF=4)(IDir)

    col. 9-15 .... force and/or moment vectors (AFORCE)

14. Blank menu

<u>Sample instruction:</u> Calculate the frequency of a tapered beam (using straight beam theory) with a rectangular section, 301 stainless steel (8 elements used).

1)	1						
2)	11						
3)	8						
4)	1.	1.	.09	.09			
5)	1.	1.	1.	1.	1.	1.	
6)	112	2.508		28.00 D+06	.29	7.5078 D-04	23.5
	223	2.328		"	"		25.5
	334	2.147		"	"		27.0
	445	1.972		"	"		31.25
	556	3.441		"	"		35.25
	667	2.828		"	"		44.25
	778	2.332		"	"		59.25
	889	2.039		"	"		78.5

7) Case 02: Timoshenko non-uniform beam: //CLAMPED-FREE//

8)	11	0.0
	12	0.0
	13	0.0
	94	0.0
	95	0.0

9) Blank menu

10) 1

11) 11 0.0
12 0.0
93 0.0
94 0.0

12) Blank menu

13) 51 -100.00

14) Blank menu

MODEL / CASE	I (types "i and j")				II (types "i and j")				III (types "i and j")			
	NDF = 5 $\{w_x\ w_z\ w'\ \theta\ \theta'\}$ 1 2 3 4 5		NDF = 4 $\{w\ w'\ \theta\ \theta'\}$ 1 2 3 4		NDF = 5 $\{w_x\ w_z\ w'\ \theta'\ \psi\}$ 1 2 3 4 5		NDF = 4 $\{w\ w'\ \theta'\ \psi\}$ 1 2 3 4		NDF = 5 $\{w_x\ w_z\ \theta\ \theta'\ \psi\}$ 1 2 3 4 5		NDF = 4 $\{w\ \theta\ \theta'\ \psi\}$ 1 2 3 4	
	x=0	x=L	x=0	x=L	x=0	x=L	x=0	x=L	x=0	x=L	x=0	x=L
1 clamped-clamped	11 12 14	91 92 94	11 13	91 93	11 12	91 92	11	91	11 12 13	91 92 93	11 12	91 92
2 clamped-free	11 12 14	95	11 13	94	11 12	94 95	11	93 94	11 12 13	94 95	11 12	93 94
3 clamped-supported	11 12 14	91 92 95	11 13	91 94	11 12	91 92 94	11	91 93	11 12 13	91 92 94	11 12	91 93
4 free-free	15	95	14	94	14 15	94 95	13 14	93 94	14 15	94 95	13 14	93 94
5 supported-free	11 12 15	95	11 14	94	11 12 14	94 95	11 13	93 94	11 12 14	94 95	11 13	93 94
6 supported-supported	11 12 15	91 92 95	11 14	91 94	11 12 14	91 92 94	11 13	91 93	11 12 14	91 92 94	11 13	91 93
7 clamped-simply supported	11 12 14	94	11 13	93	11 12	95	11	94	11 12 13	93 95	11 12	92 94
8 simply-supported supported	14	91 92 95	13	91 94	15	91 92 94	14	91 93	13 15	91 92 94	12 14	91 93
9 simply-supported simply-supported	14	94	13	93	15	95	14	94	13 15	93 95	12 14	92 94
10 free-simply-supported	15	94	14	93	14 15	95	13 14	94	14 15	93 95	13 14	92 94

(NDF number of degrees of freedom). For two figures, the 1st digit indicates the node, the 2nd indicates the displacement as already numbered above.

Table 15b: Table of boundary conditions used for the computer program (8 elements).

A P P E N D I X E

LIST OF FIGURES

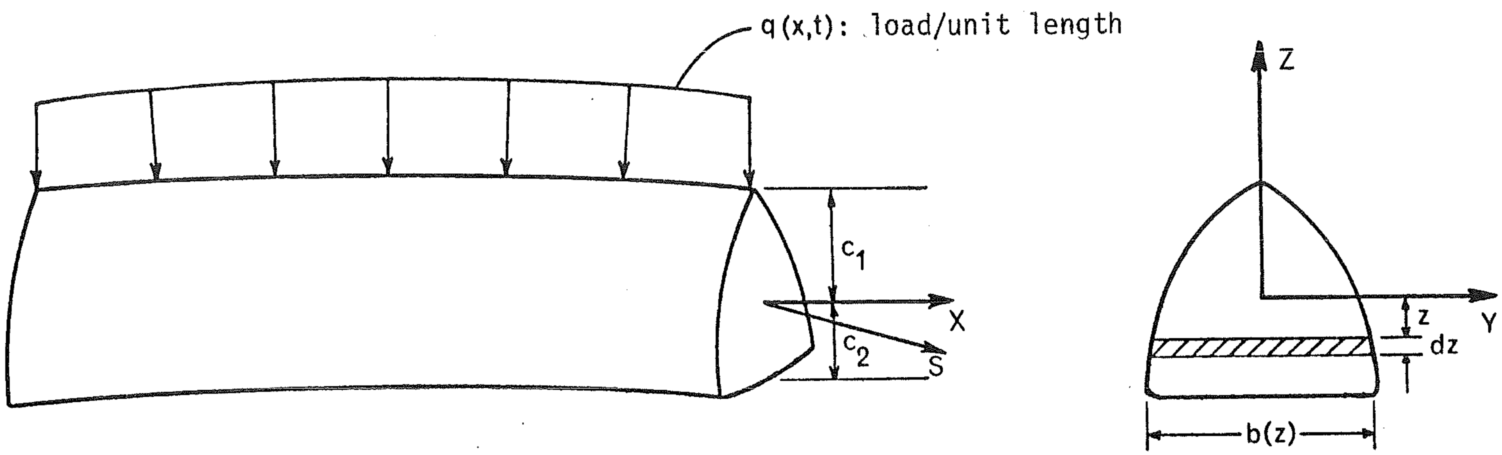


Figure 1: Studied beam: homogeneous, uniform, without initial constraint, displacement or torsion.

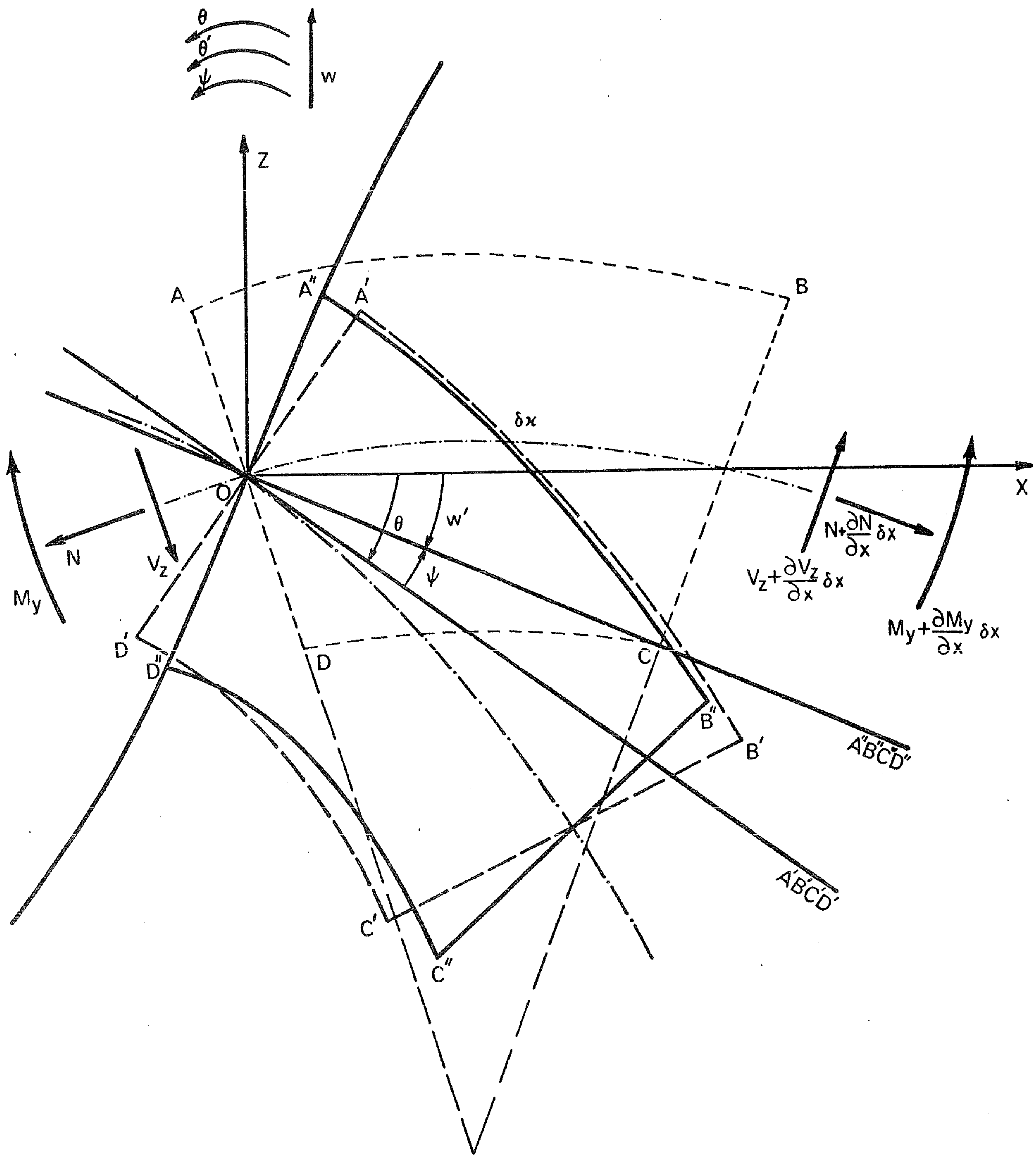


Figure 2: Detailed illustration of the elastic behaviour of a beam segment.

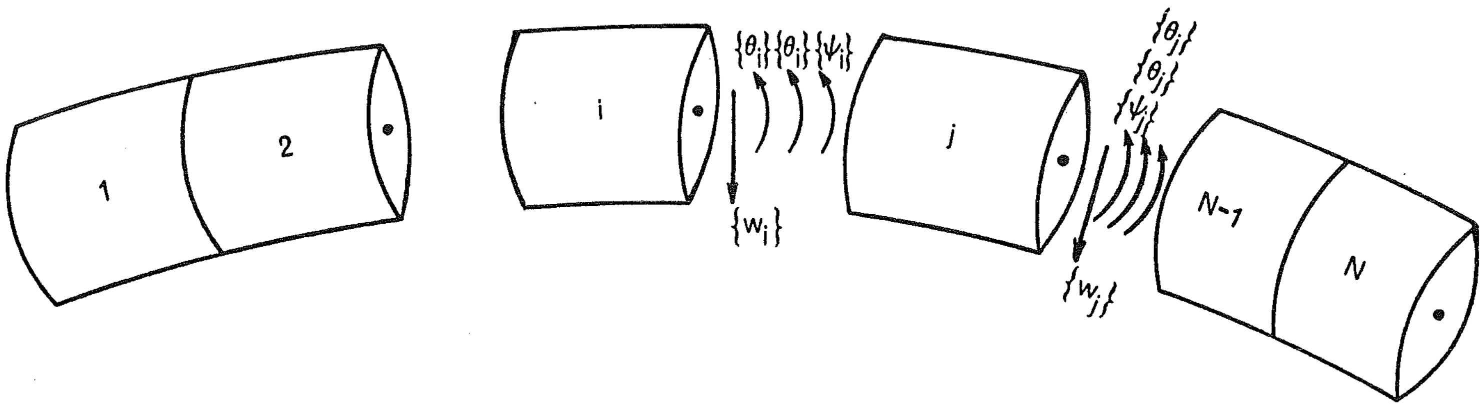


Figure 3: Nodal displacements at points i and j.

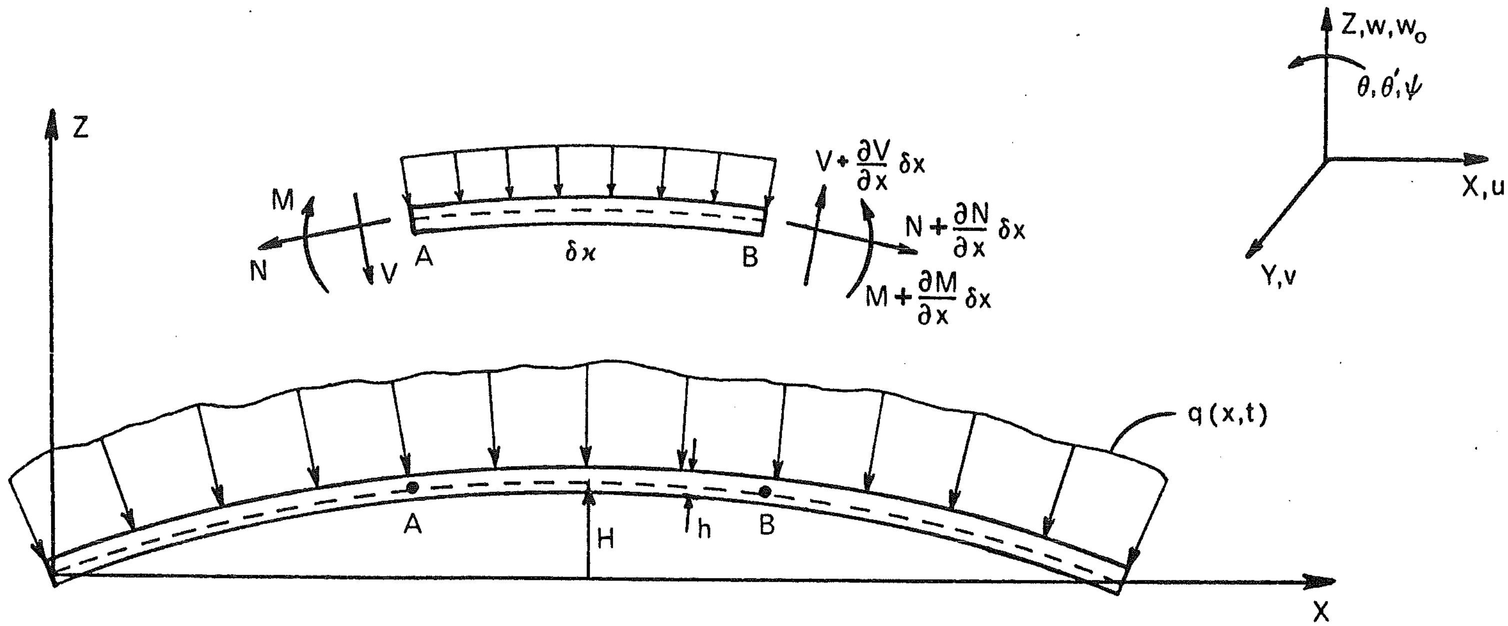


Figure 4: Geometry and notation resultant constraints N_x, V_z, moment M_y rotation and displacement U, V, W on a beam element.

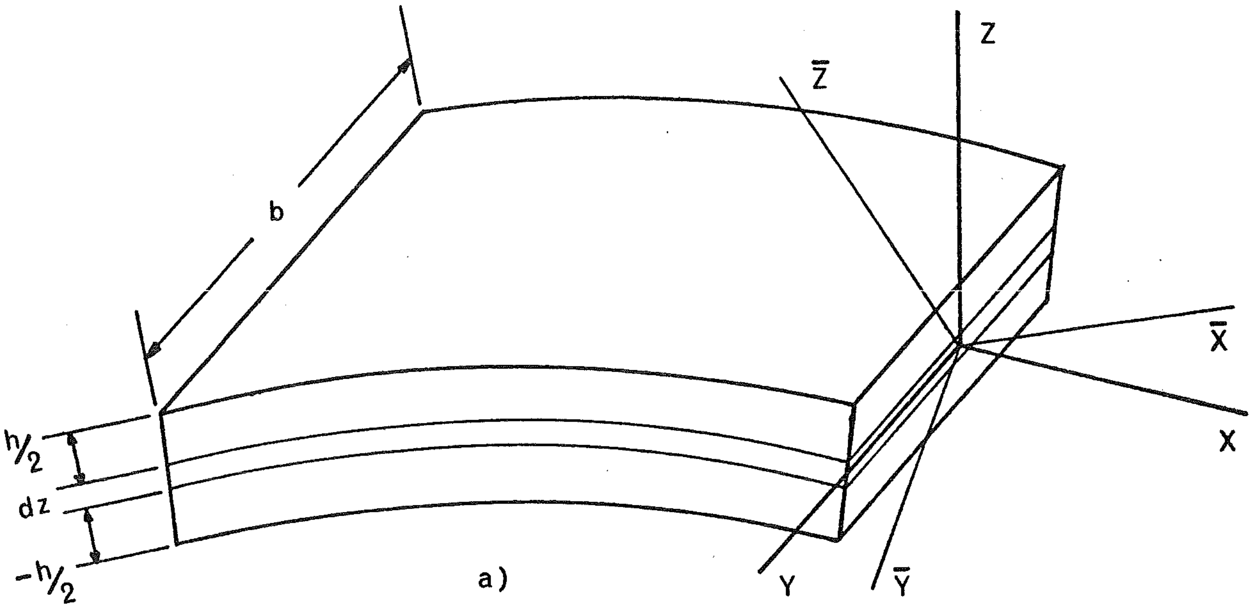


Figure 5a: Local and global coordinates.

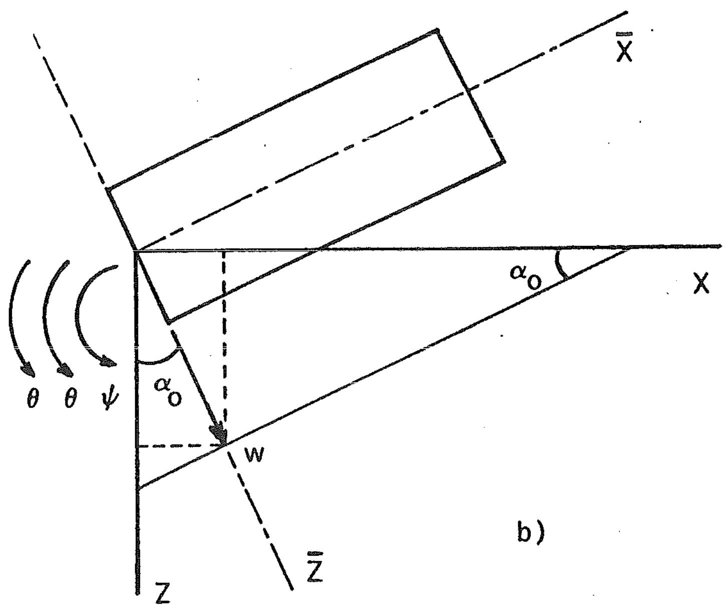


Figure 5b: Transformation between local and global displacement components at a nodal point.

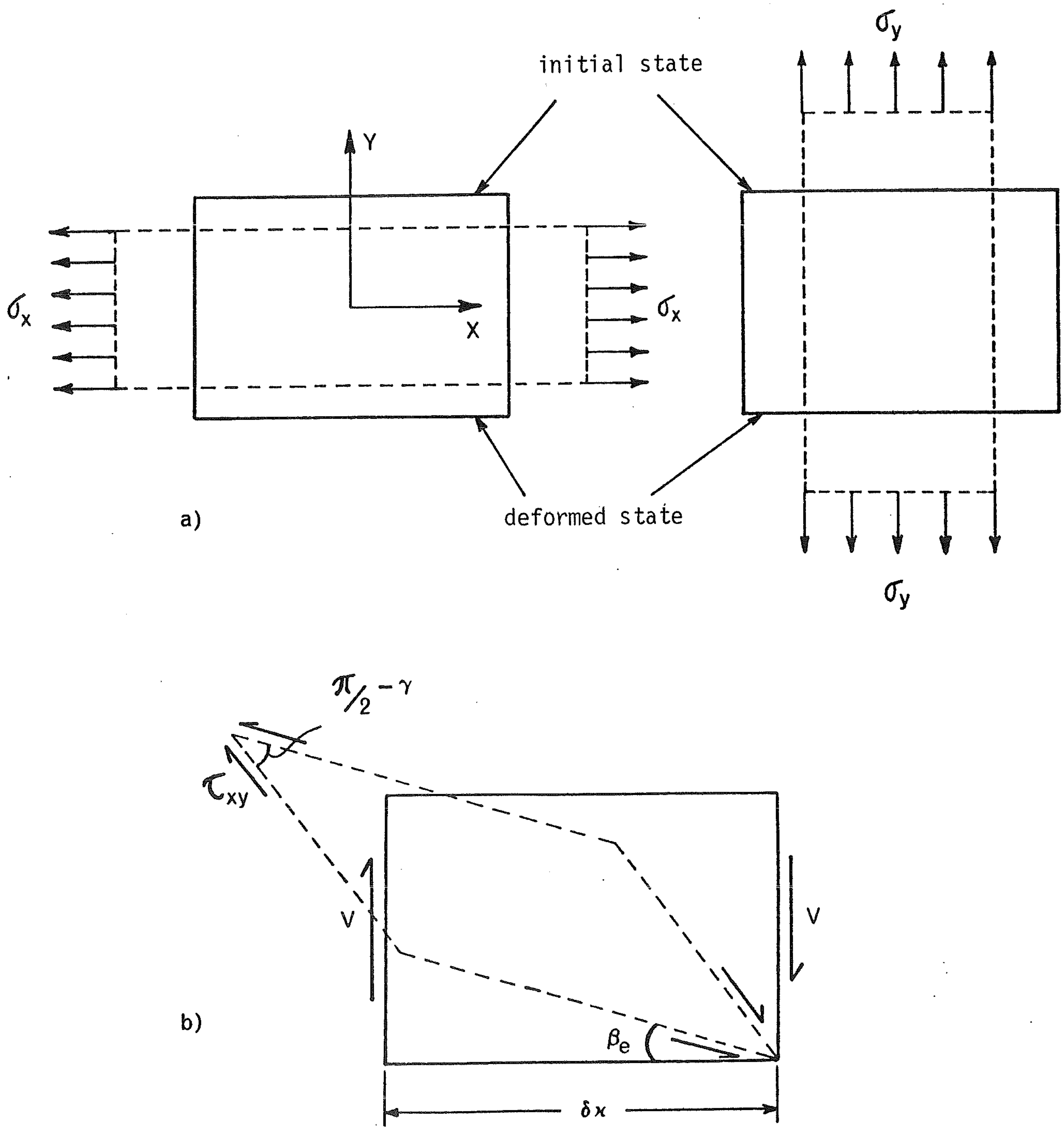


Figure 6: a) Deformation state of the normal constraints (horizontal and vertical).

b) Deformation state of the shear constraints (diagonal).

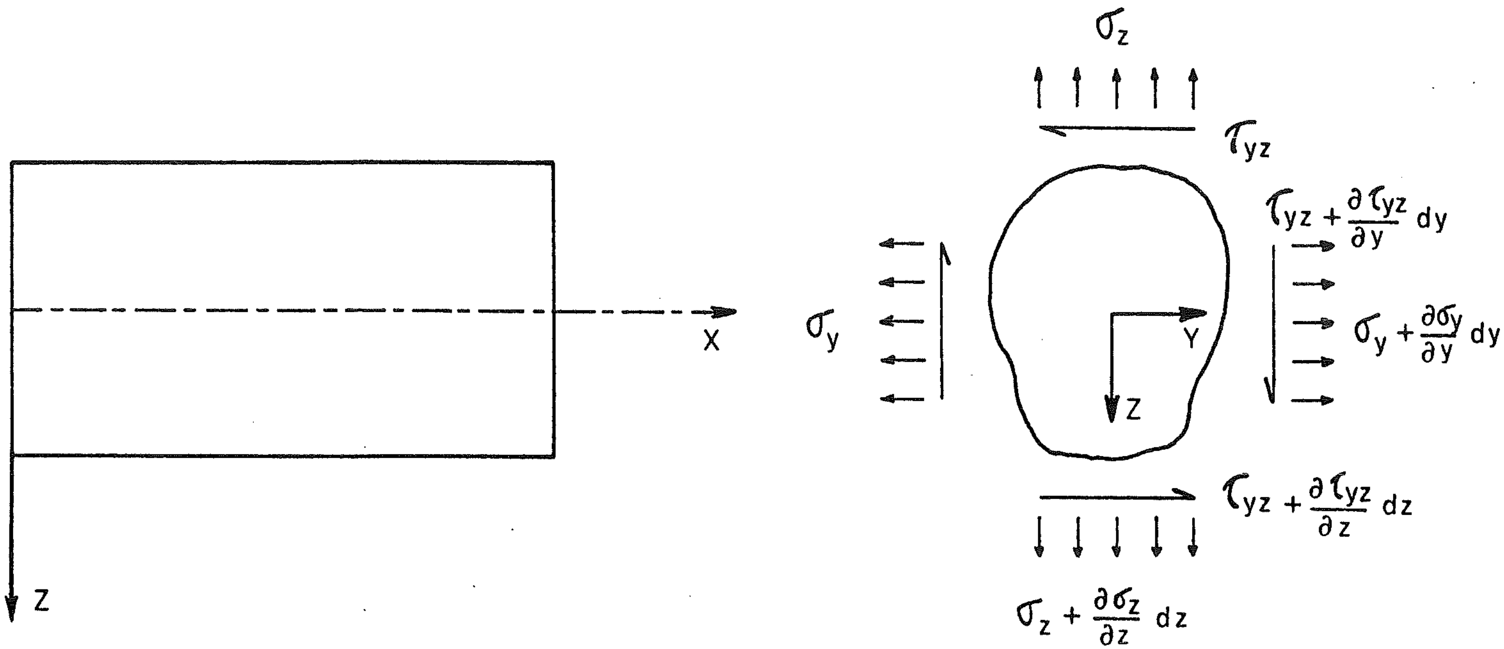


Figure 7: Stress distribution over a beam element.

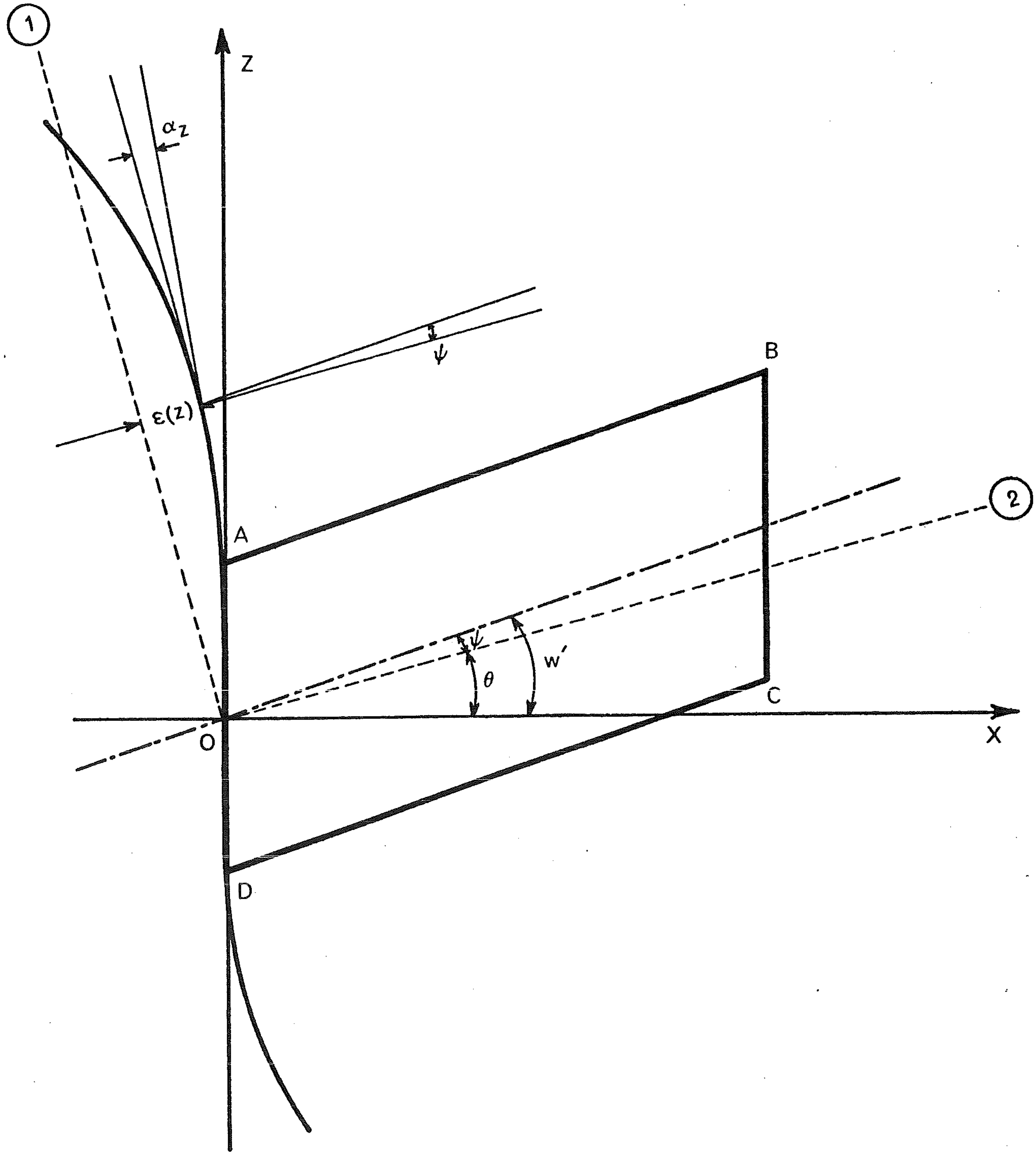


Figure 8: Distortion of a beam section.

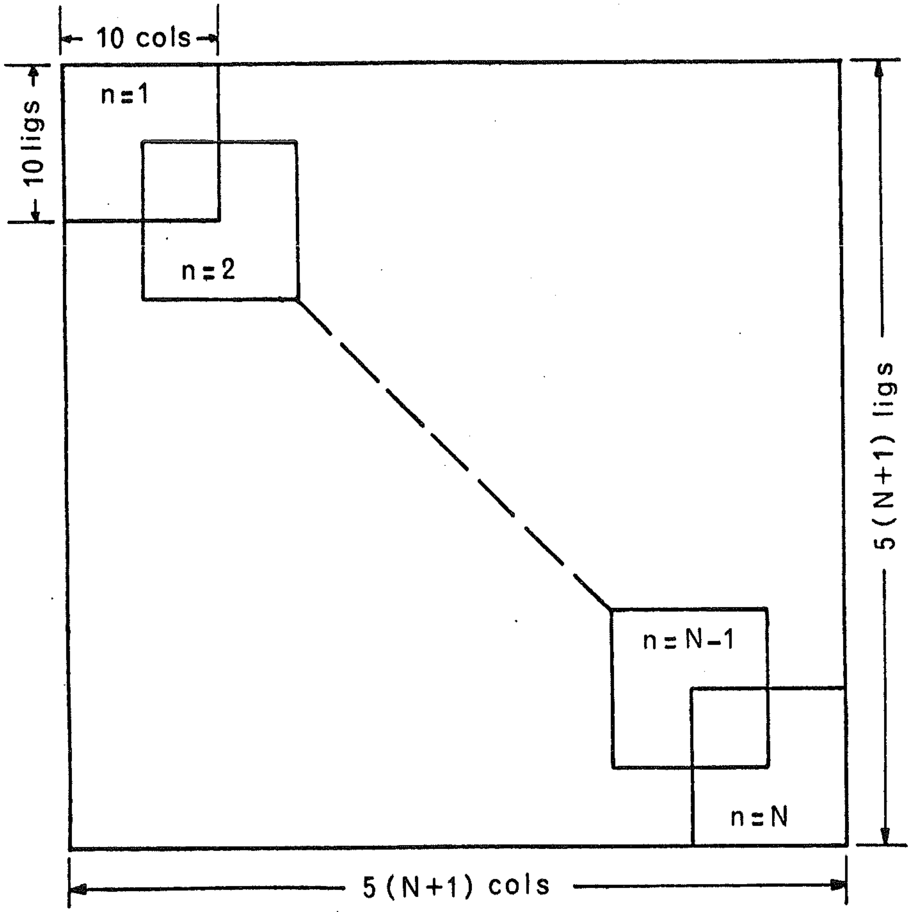


Figure 9: Assembly diagram of stiffness and mass matrices for total system.

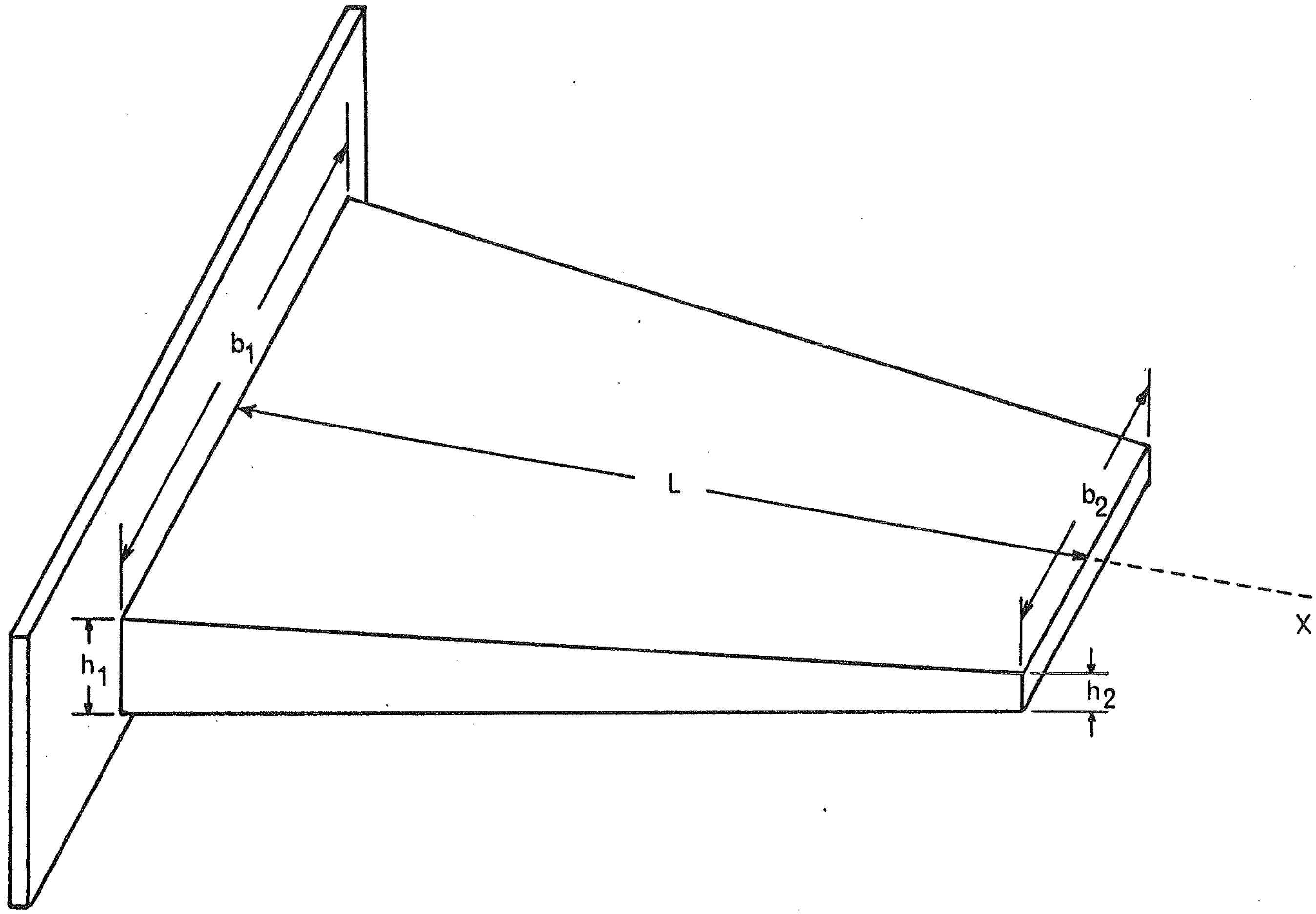
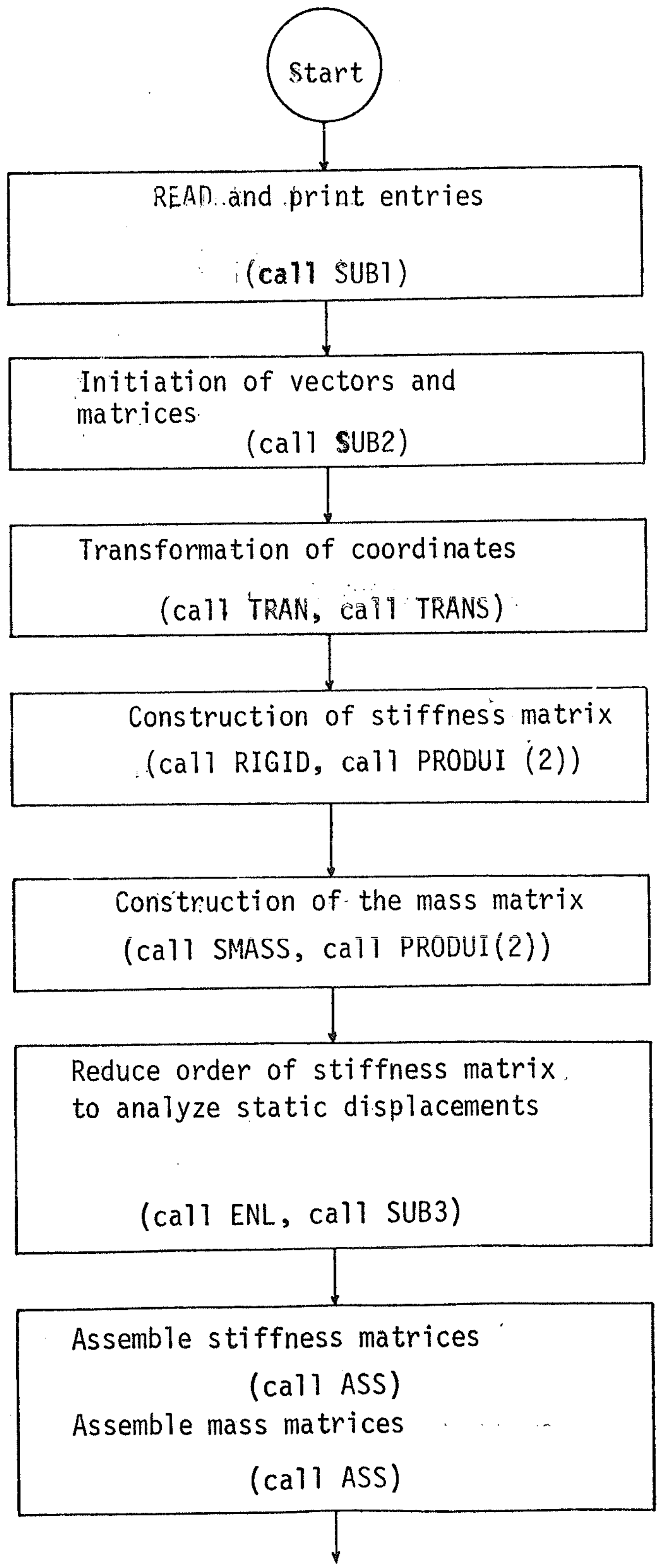


Figure 10: Sample tapered beam (linearly tapered).

Figure 11: Flow chart of the principle program



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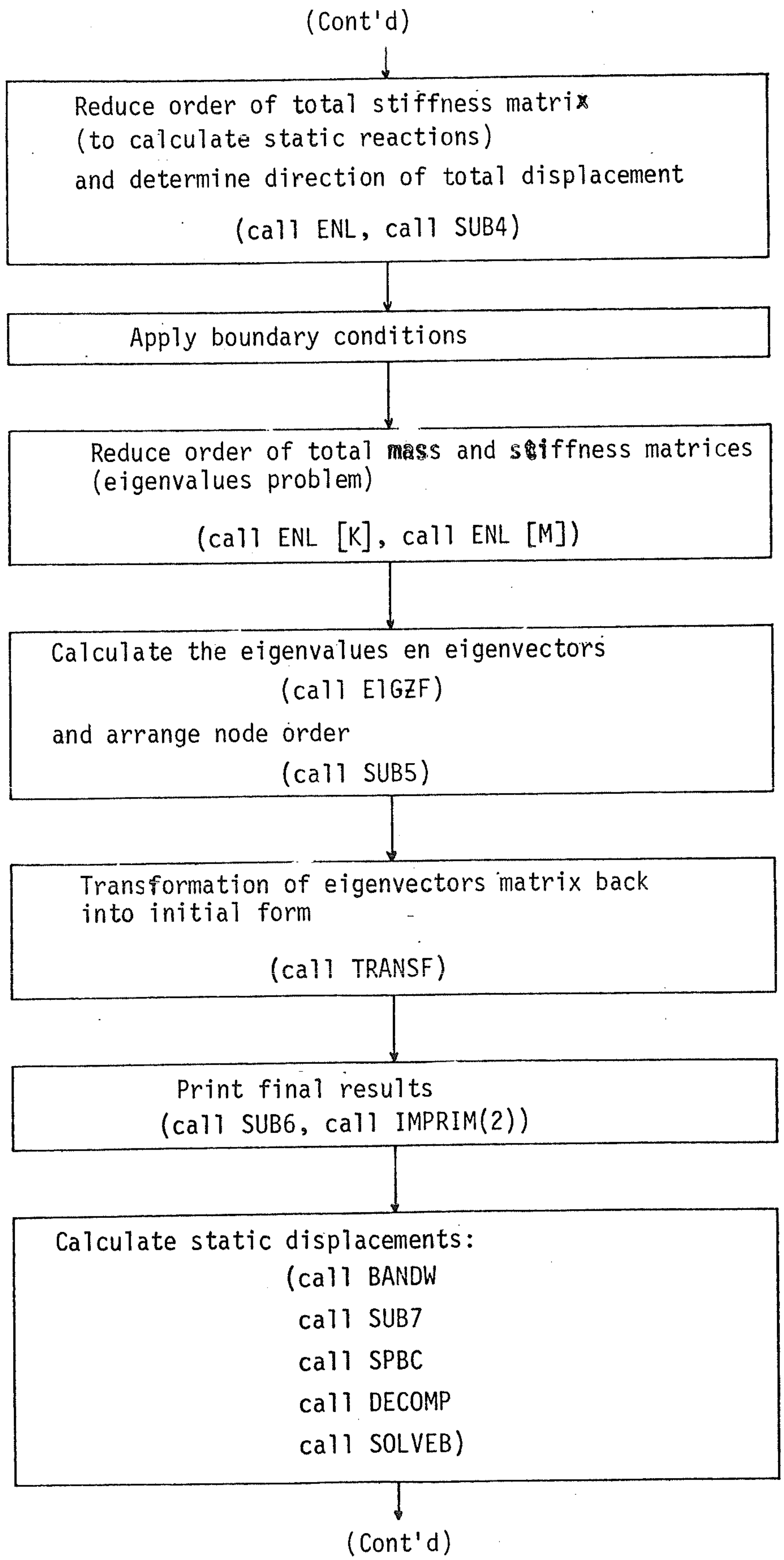
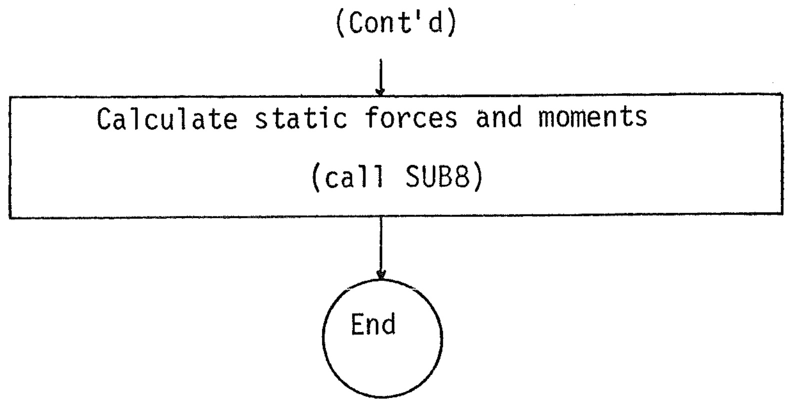

(Cont'd)
Reduce order of total stiffness matrix
(to calculate static reactions)
and determine direction of total displacement
(call ENL, call SUB4)
Apply boundary conditions
Reduce order of total mass and stiffness matrices
(eigenvalues problem)
(call ENL [K], call ENL [M])
Calculate the eigenvalues en eigenvectors
(call EIGZF)
and arrange node order
(call SUB5)
Transformation of eigenvectors matrix back
into initial form
(call TRANSF)
Print final results
(call SUB6, call IMPRIM(2))
Calculate static displacements:
(call BANDW
call SUB7
call SPBC
call DECOMP
call SOLVEB)
(Cont'd)


(Cont'd)
Calculate static forces and moments
(call SUB8)
End

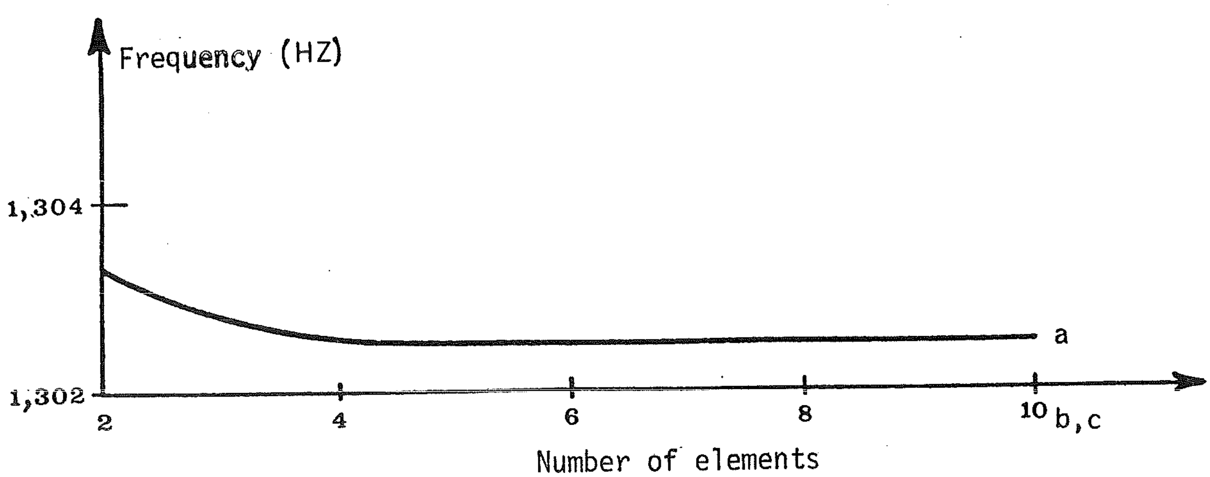


Figure 12a: Convergence test for model II (1st mode)

a) supported-supported

b) free-supported (freq 0)

c) free-free (freq 0)

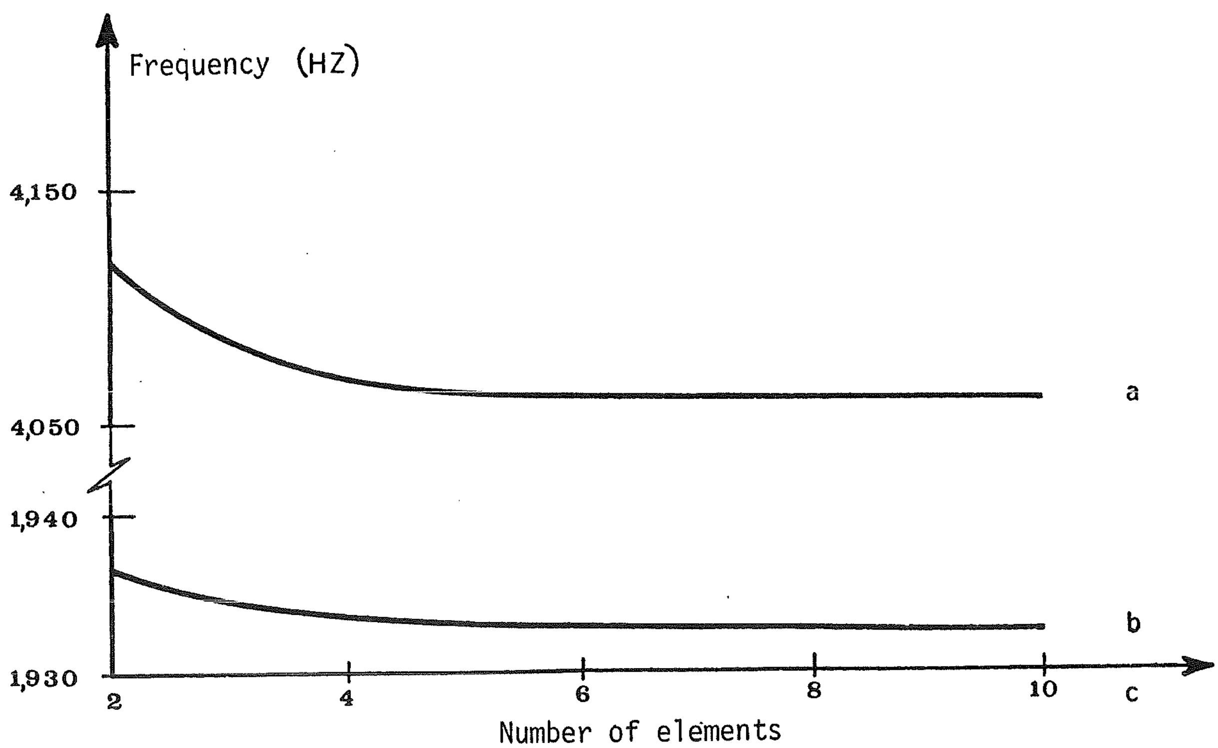
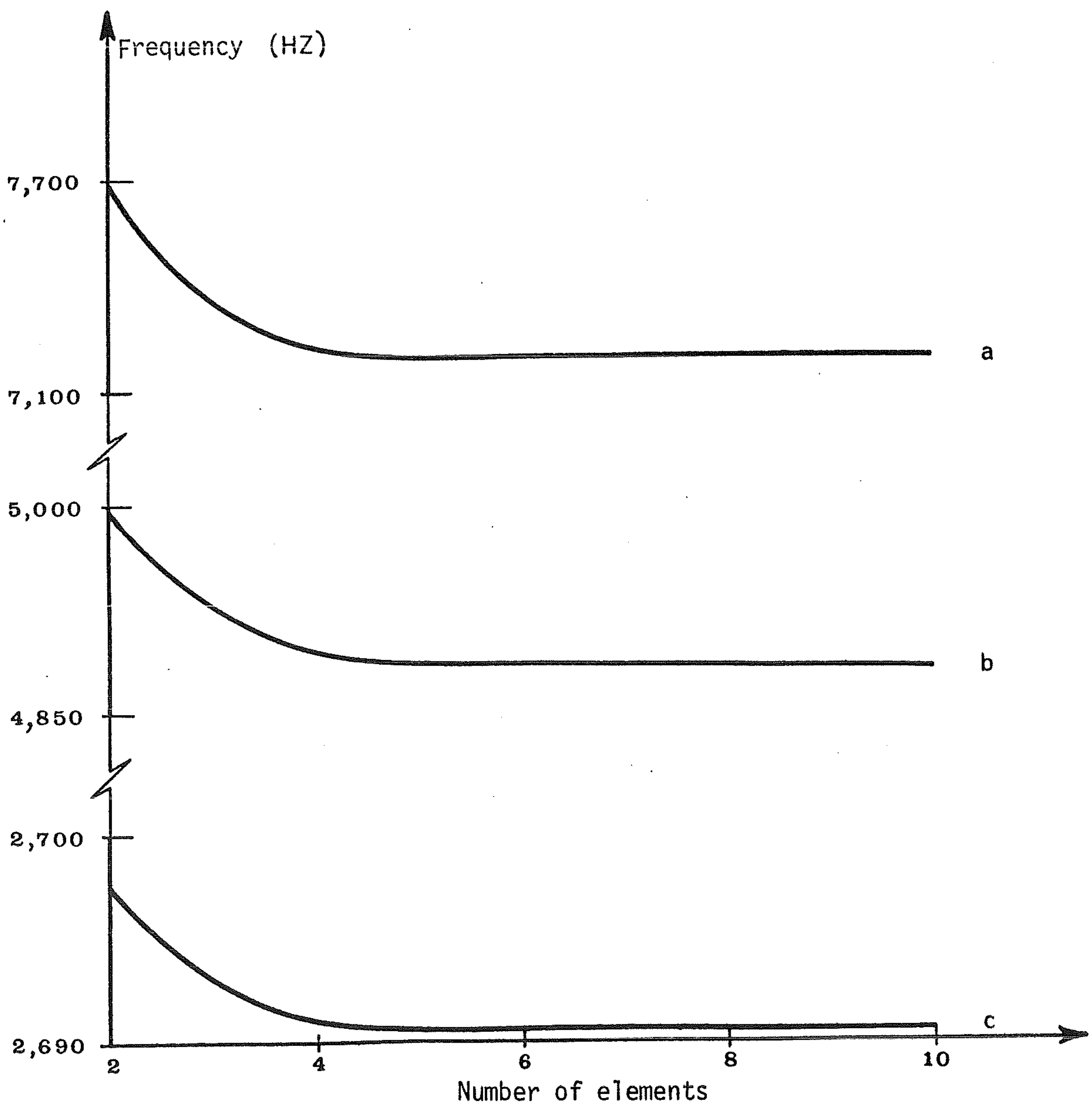


Figure 12b: Convergence test for model II (2nd mode)

a) supported-supported

b) free-supported (freq 0)

c) free-free (freq 0)



<u>Figure 12c</u>: Convergence test for model II (3rd mode)

a) supported-supported

b) free-supported

c) free-free

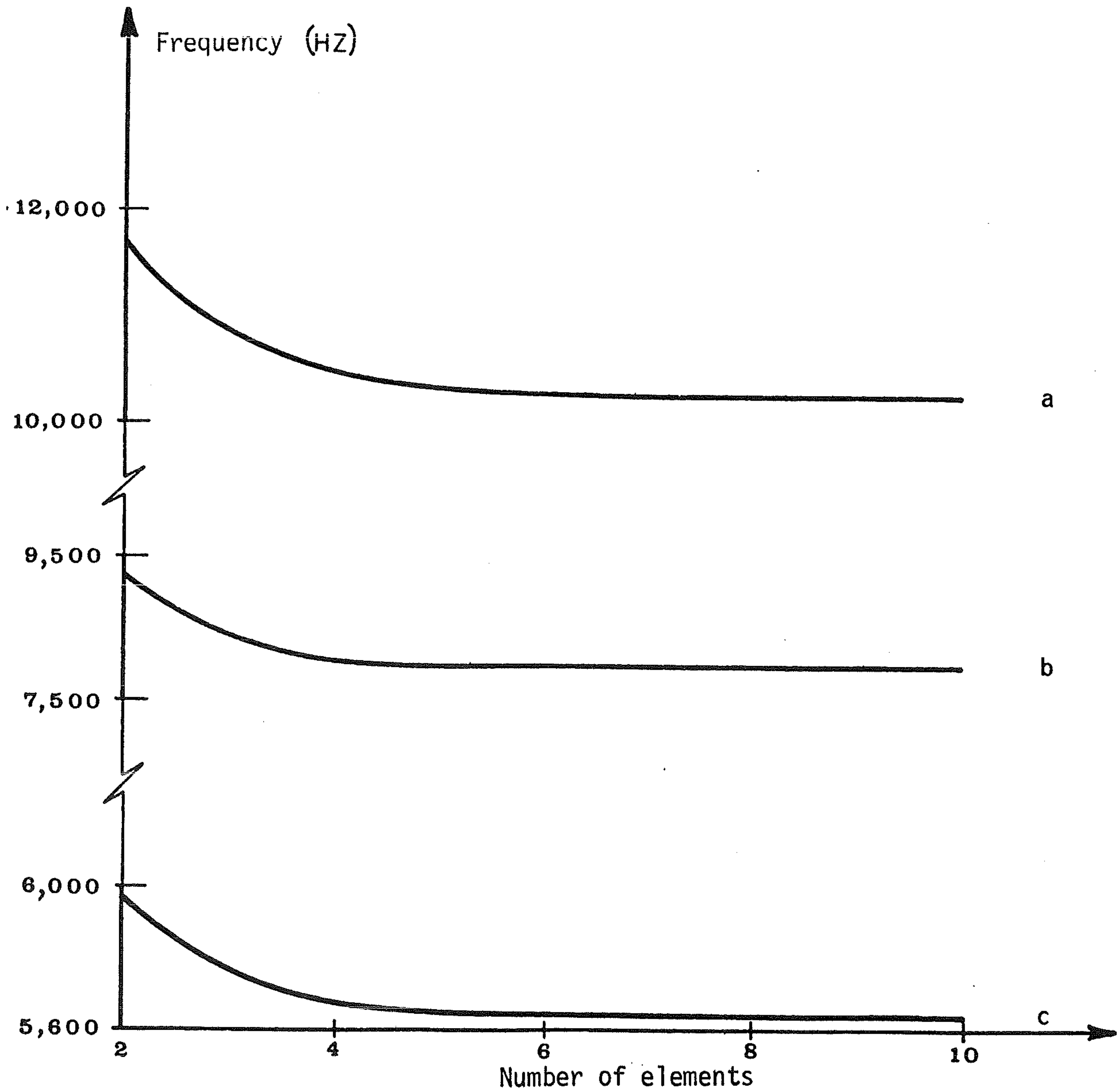


Figure 12d: Convergence test for model II (4th mode)

a) supported-supported

b) free-supported

c) free-free

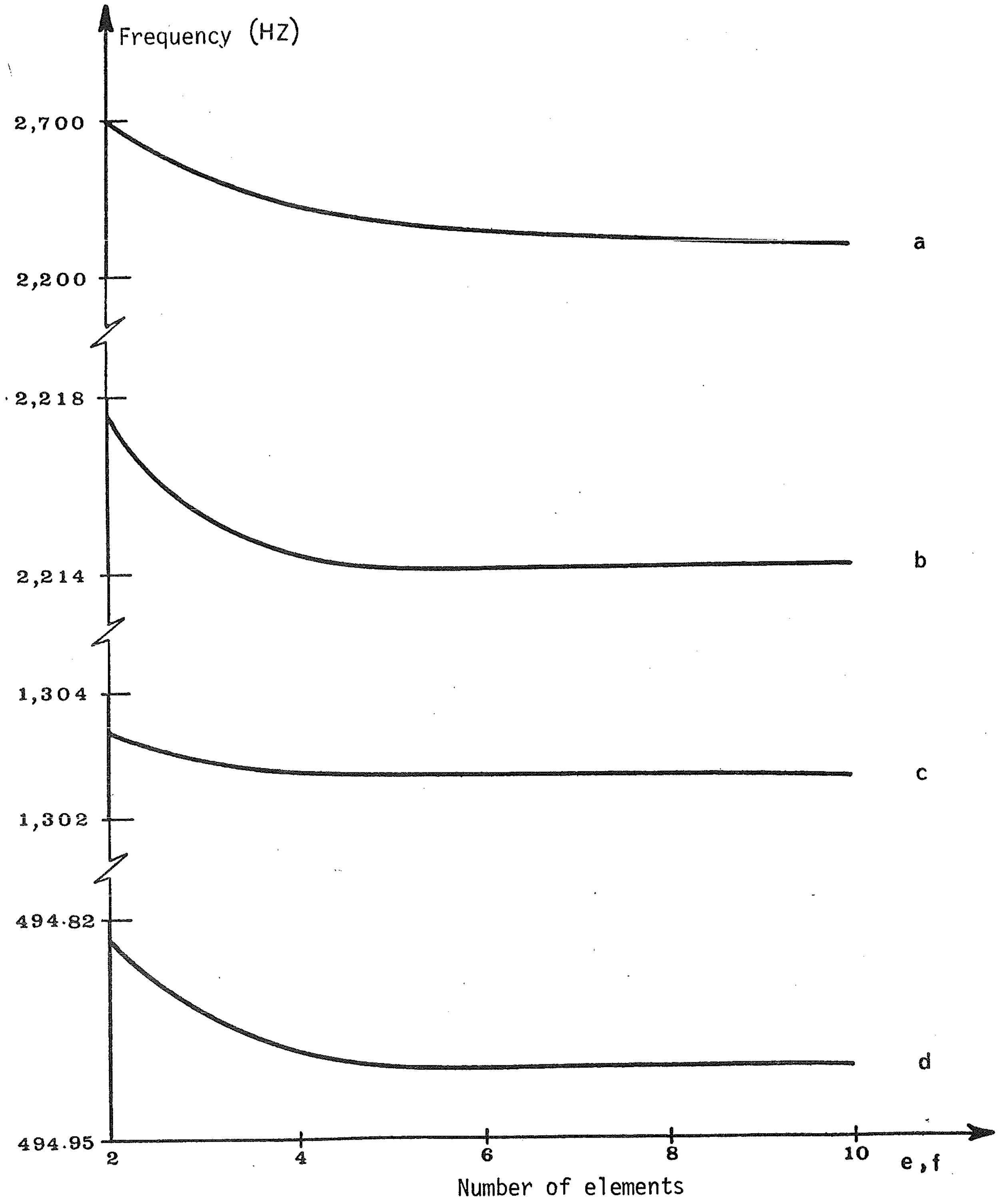


Figure 12e: Convergence test for models I and III (1st mode)

a) simply-supported - supported

b) clamped-clamped ; c) supported-supported
d) clamped-free e) free - simply-supported (freq . 0)
f) free-free (freq 0)

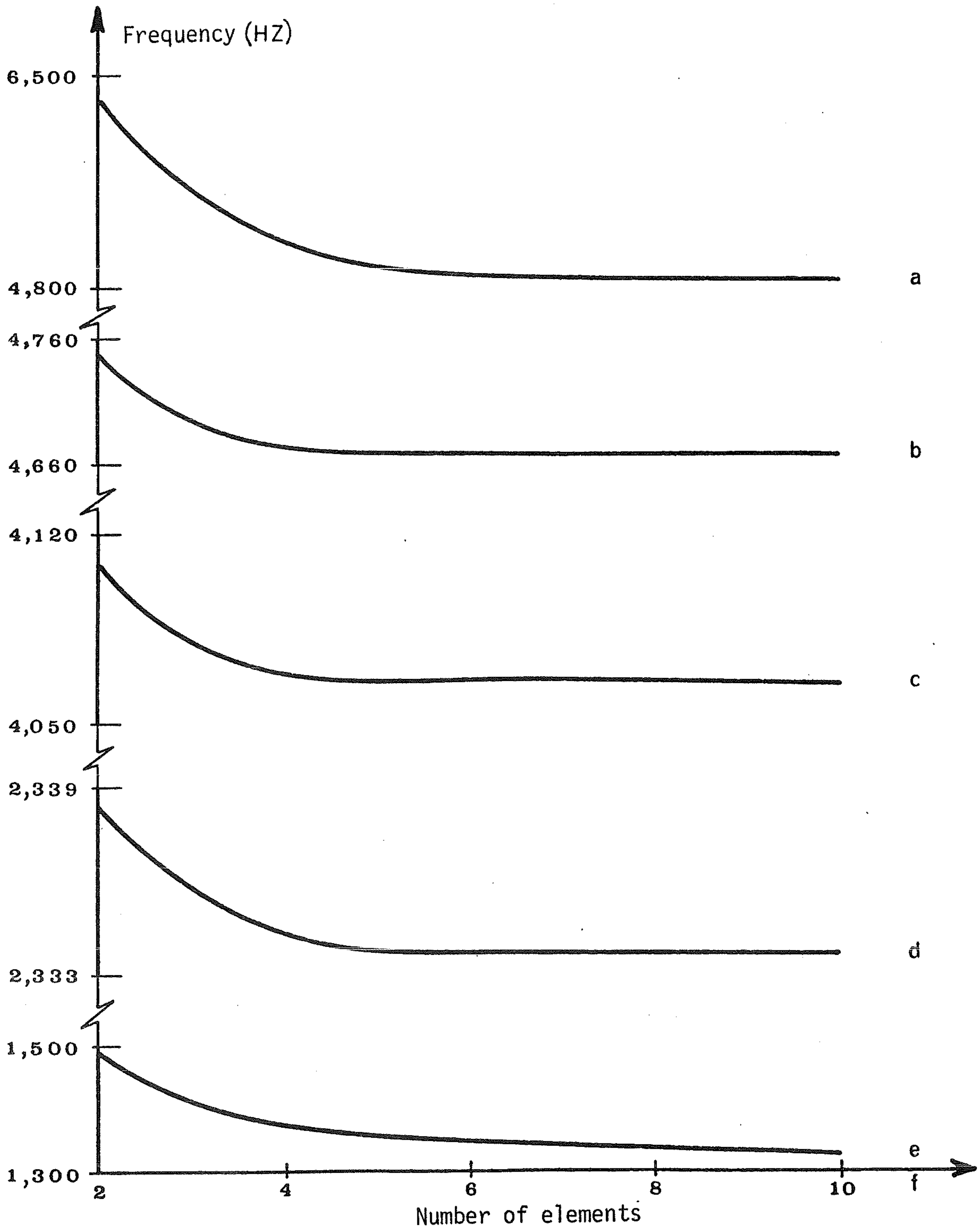


Figure 12f: Convergence test of elements for models I and III (2nd mode)

a) simply-supported-supported

b) clamped-clamped c) supported-supported

d) claped-free e) free-simply-supported
f) free-free (freq 0)

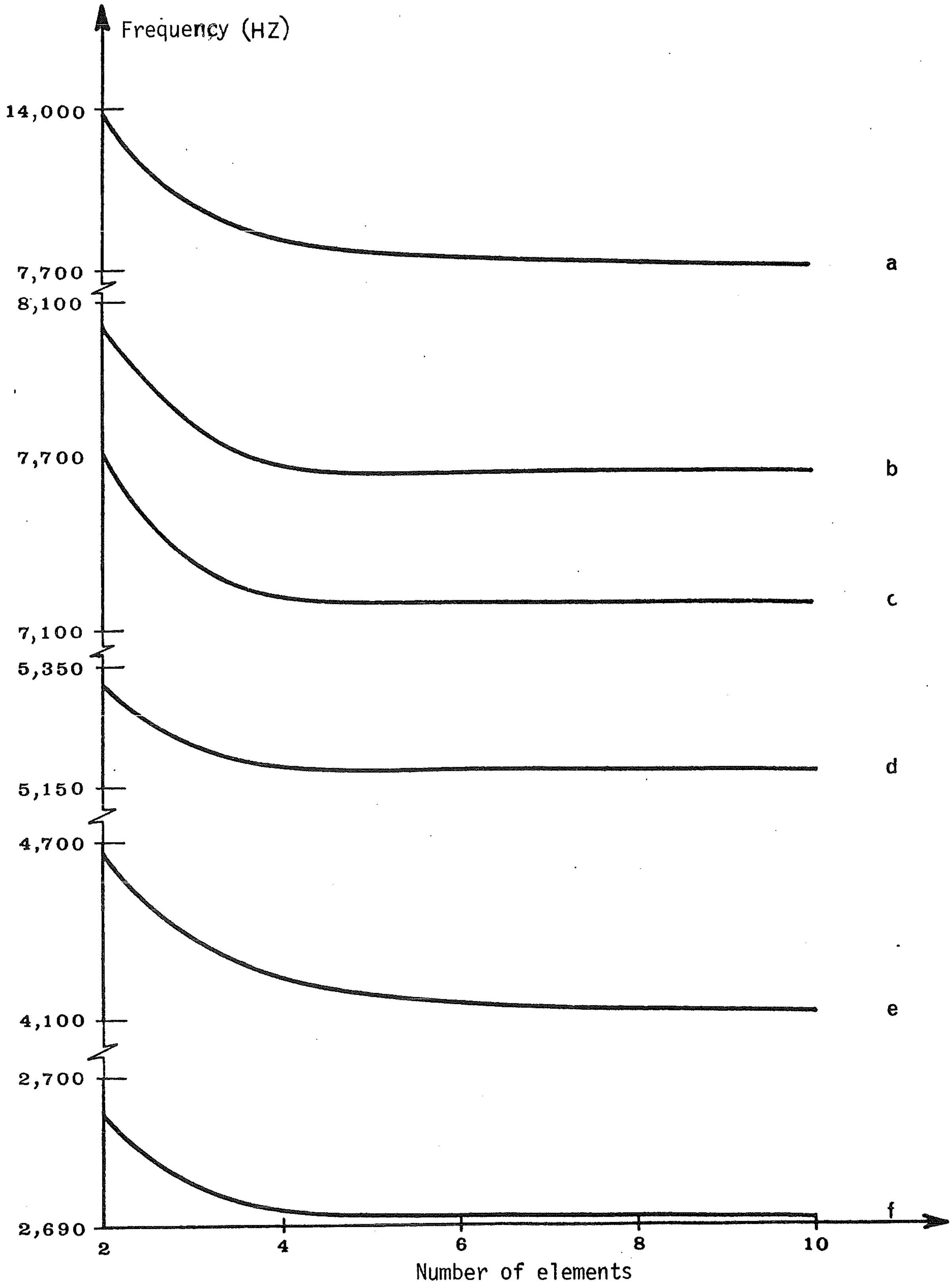


Figure 12g: Convergence test for models I and III (3rd mode)

a) simply-supported-supported b) clamped-clamped

c) supported-supported d) clamped-free

e) free-simply-supported f) free-free

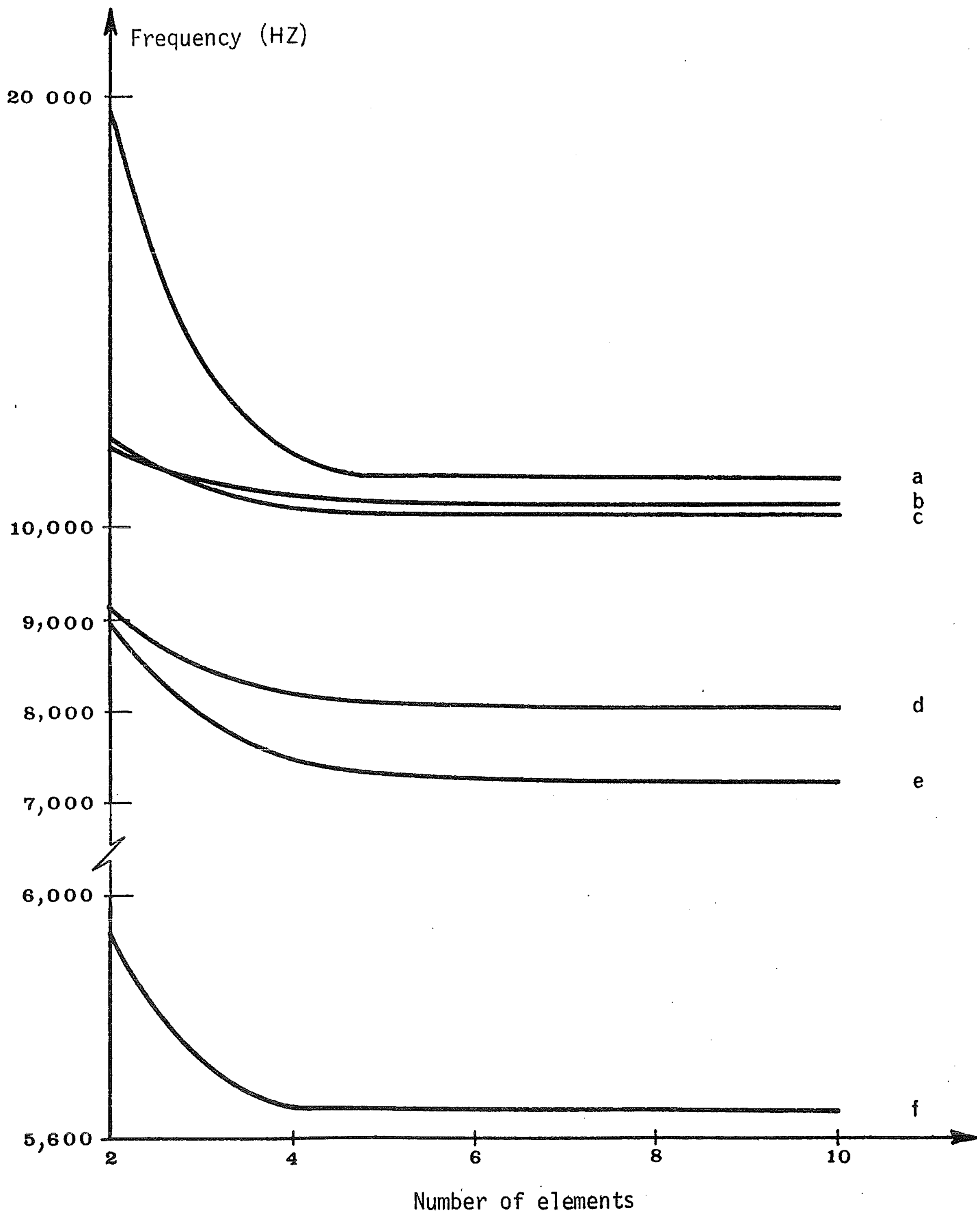


Figure 12h: Convergence test for models I and III (4th mode)

a) simply-supported-supported

b) clamped-clamped c) supported-supported

d) clamped-free e) free-simply-supported

f) free-free

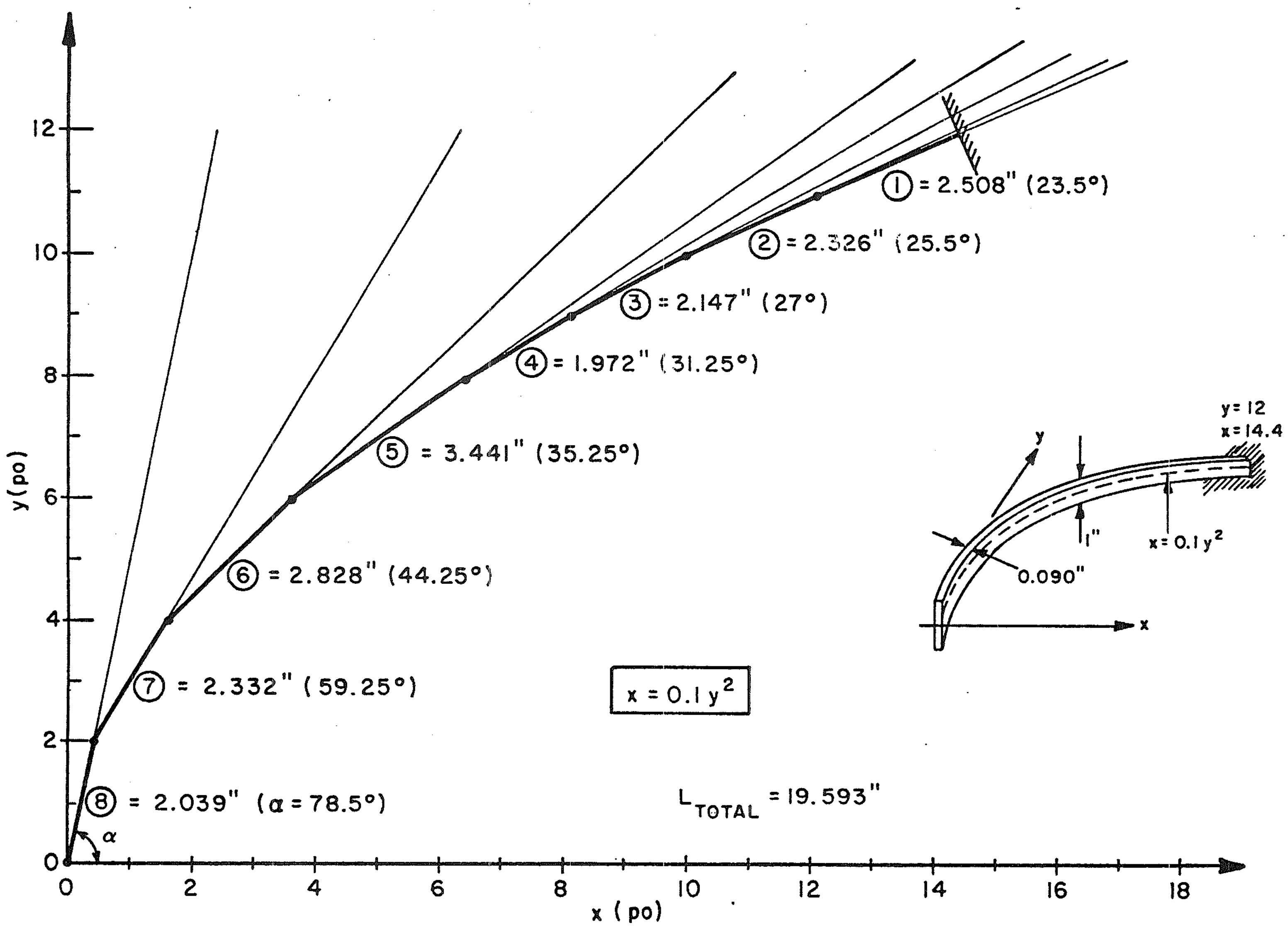


Figure 13: Tapered "clamped-free" Timoshenko beam.