DYNAMIC ANALYSIS OF ANISOTROPIC CONICAL SHELLS

by

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ABSTRACT

This report presents an analysis of the static and dynamic equilibrium of thin anisotropic and non-uniform conical shells, subjected to different boundary conditions. The procedure used was a hybrid one, based on Sander's shell theory and on the finite element method as applied to thin shells having variable thickness and anisotropic elastic properties, but limited to cases of linear elasticity.

The displacement functions were derived from Sander's equations. The stiffness and mass matrices were derived by exact analytical integration. The free-vibration problem of conical shells was reduced to an eigen values and eigen vectors problem.

The convergence of the method proved to be satisfactory and the frequencies and displacements were compared with existing results.

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LIST OF SYMBOLS

a_1 ; a_2	:	real roots of characteristic equations
A_1 ; A_2	:	Lame parameters
A_j (j= 1,...,8)	:	constant for axial displacement, U
B_j (j= 1,...,8)	:	constant for tangential displacement, V
C_j (j= 1,...,8)	:	constant for radial displacement, W
$\bar{C}_j$	:	element of vector $\{\bar{C}\}$
B_{ij}	:	elements of matrix of elasticity (i = 1,...6; j = 1...6)
D	:	membrane stiffness, Et /(1 - ν^2)
E	:	Young's modulus of elasticity
G	:	shear modulus of elasticity
J		number of boundary conditions imposed
K	:	bending stiffness Et /12(1 - ν^2)
k	:	geometric parameter
ℓ	:	length determined by equation (3.3)
ℓ_n	:	Napierian logarithm
m	:	axial wave number
$M_1, M_2, M_{12}, M_{21}, \bar{M}_{12}$	:	resultant moments
$M_x, M_o, \bar{M}_{xo}$	:	resultant couples for a conical shell
$\bar{\bar{M}}_1$	:	boundary couple value
n	:	number of circumferential modes

N_1, N_2, N_{12}	:	resultant constraints
$N_x, N_o, \bar{N}_{xo}$	:	resultant constraints for a conical shell
Q_1, Q_2	:	resultant shear constraints
r	:	average radius of the shell
R_1, R_2	:	curvature radius of surface of reference
t	:	wall thickness
$\bar{\bar{T}}_{12}$	:	resultant boundary shear
u, v, w	:	displacements: axial, tangential, radial
u_n, v_n, w_n	:	amplitudes, of u,v,w associated with n^{th} number of circumferential modes
x	:	coordinate following the cone generator
y	:	coordinate determined by equation (3.3)
y_i	:	value of y for $x = \ell_i$
y_j	:	value of y for $x = \ell_j$
α	:	half-angle to cone apex
α_j, β_j	:	determined by (3.9)
$\bar{\alpha}_j, \bar{\beta}_j$	:	determined by (3.9)
δ	:	determined by (2.8)
$\varepsilon_x, \varepsilon_o, \bar{\varepsilon}_{xo}$	:	deformation of surface of reference
θ	:	circumferential coordinates
k_1, k_2	:	real parts of λ_j
k_x, k_o, k_{xo}	:	changes in curvature and torsion of surface of reference
λ_j	:	complex roots of characteristic equations
μ_1, μ_2	:	imaginary parts of λ_j

ν	:	Poisson's ratio
ρ	:	density of shell material
ω	:	vibration frequency
ω_a	:	no-dimension frequency determined by equation (6.4)
ζ_1, ζ_2	:	surface of reference coordinate

LIST OF MATRICES

$[A]$	:	determined by relation (3.13)
$\{\bar{C}\}$	:	vector of arbitrary constants
$\{d\}$	:	vector of degrees of freedom for complete shell
$\{di\}$	:	vector of degrees of freedom for finite elements
$[D]$	:	determined by relation (3.5)
$\{Fi\}$	:	force vector applied to node i
$\{F\}$	:	external force vector
$[G]$	:	determined by relation (4.5)
$[k]$	:	elementary stiffness matrix
$[K]$	:	global stiffness matrix
$[m]$	:	elementary mass matrix
$[M]$	:	global mass matrix
$[N]$	:	determined by equation (3.15)
$[P]$	:	matrix of elasticity
$[Q]$	:	determined by relation (3.17)
$[R]$	:	determined by relation (3.11)
$[S]$	:	determined by equation (4.6)
$[T]$	:	determined by relation (3.1)
$[T_1]$	:	determined by relation (3.17)
$\{\delta i\}$	:	vector of degrees of freedom for node i
$\{\varepsilon\}$	:	deformation vector
$\{\sigma\}$	:	stress vector

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 General

Thin shells are structures that are used in various fields of engineering. Some of these are: the aerospace and aeronautical industry (aircraft fuselage, space ship intermediate connections), the nuclear industry (reactor vessel walls), the naval industry (submarines and shipbuilding), the petroleum industry (tanks, piping) and civil engineering (domes and thin wall construction). Knowledge of the free vibration characteristics of these structures is just as important for the researcher, who is seeking to understand their behaviour, as it is for the concerned engineer, who is trying to avoid all destructive effects during their industrial life.

The following classification factors are normally considered in studies of thin shell vibrations: curvature, anisotropy, residual stresses, thickness variations, large displacements, rotational inertia, influence of surroundings, the form of the shell edges and the type of boundary conditions involved.

It is generally difficult to obtain an analytical solution of the equations of motion for thin shells; hence only approximating methods are used. Among these we could mention interactive methods of the Stodola type [1], the finite difference method [2], Rayleigh-Ritz's method [3], [4], Galerkin's method ([5], [6]), the finite element method [7], as well as numerical integration methods where the problem of shell equation of motion is reduced to an eigen value problem [8].

In the finite difference and numerical integration methods, the initial frequency value is assumed. These two procedures require lengthy processing time. Further, it has been indicated [1] that interactive methods do not define the complete vibration spectrum. the same applies to the interactions of the Galerkin's method, which decline in accurary because of the shell's high frequencies.

The best test of a method would be that it be able to determine both the frequency set and vibration modes with the same accuracy and within reasonable computing time. The Rayleigh-Ritz and finite element methods both meet this criterion. The vibration problem then becomes an eigen values eigenvectors problem.

Rayleigh-Ritz's method, however, has the following disadvantages ([5]):

(1) A large number of terms must be retained to express the displacement functions.

(2) There is a lack of consistency between the accuracy of the displacement terms and of the energy and deformation terms.

(3) The choice of displacement functions must be related to the boundary conditions.

The finite element method, on the other hand, satisfies these prerequisites. The foregoing is only a brief overview of the available thin shell methods, since more complete bibliography on the application of the shell method cannot be feasibly included in this paper.

The shell is described by an assembly of finite elements. The accuracy of the method is dependent on both the nature of these elements and the degrees of freedom used to simulate shell behaviour.

Thus, for triangular and flat quadrilateral elements, polynomial functions are chosen, and plate theory is used to derive the equations. To increase accuracy, curved elements which better reflect the shell geometry are chosen [8]. The analytical formulation of these elements is very complicated. The same applies to the numerical choice of reference conditions.

The methods used in this work were part of a research project, directed by Dr. Lakis, which has been the subject of several publications. In this context then, reference [10] is a report on the free vibrations of non-uniform cylindrical shells subjected to different boundary conditions. In reference [18], the natural frequencies of the Pickering steam generator were numerically studied. In [19], the free vibrations of cylindrical shells partially or completely filled with a liquid, under steady-state conditions, were analyzed. The effects of laminar flow and turbulence and one or two phases in the cylindrical structure containing flowing fluids have also been analyzed [20], [21], and [22]. Geometric non-linearities in an empty cylindrical structure were investigated in reference [23]. Reference [24] provided the numerical solution of the characteristic equation for an anisotropic conical shell.

1.2 Research objectives

The present project investigated the static and dynamic equilibrium of thin conical shells, as well as effects upon the membrane and on bending. It was hypothesized that:

- The cone was incomplete: the cone apex and its surroundings were not analyzed.

- The cone was closed: it was without openings.

- The cone was circular: the reference surface of the shell was created by the rotation of a generator around a fixed axis keeping a angle constant.

- The cone was straight: the base of the cone was circular. There were no results for non-circular conical shell bases.

- The shell thickness was variable and was assumed to be a linear function of the actual coordinates along the generator.

- The shell was composed of one or more isotropic or orthotropic layers.

- Displacements were sufficiently small to give geometric linearity.

- The rotational inertia terms were negligible.

- The static equilibrium constraints were zero.

- The shell was assumed to be empty: later experience revealed an interaction of the fluid structure with the normal atmosphere.

- Several cases of boundary conditions were considered. The method used combined thin shell theory and the finite element method. The basic equations of shell theory, based on Love's First Approximation and given by Sanders [11], were reduced to three differential equations, as a function of axial, radial and circumferential displacements of the reference surface.

The finite element chosen was conical and bounded by two circular nodes. There were four degrees of freedom for each node: axial, radial, circumferential displacement and rotation. The geometry of the finite element allowed for use of the complete form of the equations of motion to determine the form functions in terms of nodal displacements. This method was more accurate than the usual choice of polynomial functions. The mass and stiffness matrices were then determined and by assembly of elements, matrices for the entire shell were determined. The free vibrations of the shell involved solution of an eigenvectors problem. The vibration modes were also determined.

1.3 Contents of the report

The following is a description of the seven chapters of this paper.

In Chapter 2, the three equations of motion, thogether with displacements and elasticity matrix are described by me equations for their stress-strain.

In Chapter 3, the format of the displacement funct starting with solution of the characteristic equation. The tions, the deformations and stresses linked to nodal displacem determined.

Analysis of the mass of stiffness matrices as well as the tion is done in Chapter 4.

In Chapter 5, the algorithm used in the solution and in the tion of the computer program are described.

The different tests which were used to validate this me outlined in Chapter 6. Comparisons between present results and obtained by other authors are also discussed.

Finally, conclusions and recommendations regarding this metl given in Chapter 7.

CHAPTER 2

BASIC THEORY

2.1 Equations of motion

To study the equilibrium of a conical shell and consider the effect both of the membrane and the extension of the reference surface, Sander's [11] equation of motion were used. As was mentioned previously, these equations are based upon the First Love Approximation [14]. Zero deformation in the rigid-body motion was obtained with this method, which is not the case with other formulations.

The geometry of the reference surface of the shell studied and the coordinates used are indicated in Figure 2.

By eliminating the shear forces Q_x and Q_θ by means of equations (A-1.5d,e) in Appendix A, we obtained, in the absence of external forces:

$$r\frac{\partial N_x}{\partial x} + N_x \sin\alpha + \frac{\partial N_{x\theta}}{\partial \theta} - N_\theta \sin\alpha - \frac{\cos\alpha}{2r}\frac{\partial M_{x\theta}}{\partial \theta} = 0$$

$$r\frac{\partial N_{x\theta}}{\partial x} + 2N_{x\theta}\sin\alpha + \frac{\partial N\theta}{\partial \theta} + \frac{3}{2}\cos\alpha\frac{\partial M_{x\theta}}{\partial x} + \frac{3}{4}\frac{\sin 2\alpha}{r}M_{x\theta} + \frac{\cos\alpha}{r}\frac{\partial M_\theta}{\partial \theta} = 0 \quad (2.1)$$

$$r\frac{\partial^2 M_x}{\partial x^2} + 2\sin\alpha + \frac{\partial M_x}{\partial x} + 2\frac{\partial^2 M_{x\theta}}{\partial x \partial \theta} + 2\frac{\sin\alpha}{r}\frac{\partial M_{x\theta}}{\partial \theta} + \frac{1}{r}\frac{\partial^2 M_\theta}{\partial \theta^2} - \sin\alpha\frac{\partial M_\theta}{\partial x} - N_\theta\cos\alpha = 0$$

where N_x, N_θ, M_x, M_θ and $M_{x\theta}$ are resultant constraints. x and θ designate the axial and circumferential coordinates, α is the half-angle to the cone apex.

The displacements of a point on the shell are indicated in Figure 3. They are connected to the deformation vector for this relation.

$$\{\epsilon\} = \left\{ \begin{array}{l} \frac{\partial u}{\partial x} \\ \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\sin\alpha + \frac{w}{r}\cos\alpha \\ \frac{\partial v}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\sin\alpha \\ -\frac{\partial^2 w}{\partial x^2} \\ \frac{\partial v}{\partial \theta}\frac{\cos\alpha}{r^2} - \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{\partial w}{\partial x}\frac{\sin\alpha}{r} \\ -\frac{2}{r}\frac{\partial^2 w}{\partial x \partial \theta} + \frac{3}{2}\frac{\cos\alpha}{r}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial \theta}\frac{\cos\alpha}{2r^2} + \frac{\partial w}{\partial \theta}\frac{2\sin\alpha}{r^2} - \frac{3}{4}\frac{\sin 2\alpha \cdot v}{r^2} \end{array} \right\} \quad (2.2)$$

with

$$\{\epsilon\} = \{\,\epsilon_x\,,\ \epsilon_\theta\,,\ 2\overline{\epsilon_{x\theta}}\,,\ k_x\,,\ k_\theta\,,\ 2\overline{k_{x\theta}}\,\}^T \quad (2.3)$$

where K_x, K_θ and $2\overline{k_{x\theta}}$ express the bending of the reference surface, ϵ_x, ϵ_θ and $2\overline{\epsilon_{x\theta}}$ account for its elasticity, and u_1, v and w are the axial, tangential and radial displacements, respectively. The deformation vector is linked to the stress vector by

$$\{\sigma\} = [P]\,\{\epsilon\} \quad (2.4)$$

[P] is the elasticity matrix which is designated by the general term p_{ij}. The equations of equilibrium in conjunction with the displacements are obtained by using (2.2) and (2.4). They are expressed as:

$$L_1(u,v,w,p_{ij}) = 0$$
$$L_2(u,v,w,p_{ij}) = 0 \qquad (2.5)$$
$$L_3(u,v,w,p_{ij}) = 0$$

where (L_1) i=1,2,3 are three linear differential operators, the form of which is fully explained in Appendix A-2.

2.2 Matrix of elasticity and shell thickness

The matrix of elasticity [P] depends only upon the mechanical characteristics of the material of the shell. When dealing with anisotropic material, however, and limiting it to the more frequent case of anisotropic revolution called orthotrophy, the mechanical properties are invariant when rotating around a fixed axis.

In general [12], [P] can be written for anisotropic material as:

$$[P] = \begin{bmatrix} P_{11} & P_{12} & 0 & P_{14} & P_{15} & 0 \\ P_{21} & P_{22} & 0 & P_{24} & P_{25} & 0 \\ 0 & 0 & P_{33} & 0 & 0 & P_{36} \\ P_{41} & P_{42} & 0 & P_{44} & P_{45} & 0 \\ P_{51} & P_{52} & 0 & P_{54} & P_{55} & 0 \\ 0 & 0 & P_{63} & 0 & 0 & P_{66} \end{bmatrix} \qquad (2.6)$$

For an isotropic conical shell, having a constant thickness t, we obtain:

$$[P] = \begin{bmatrix} D & \nu D & 0 & 0 & 0 & 0 \\ \nu D & D & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-\nu)D}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & K & \nu K & 0 \\ 0 & 0 & 0 & \nu K & K & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(1-\nu)K}{2} \end{bmatrix} \quad (2.7)$$

with

D = membrane stiffness
K = bending stiffness

$$D = \frac{Et}{1-\nu^2} \qquad K = \frac{Et^3}{12(1-\nu^2)}$$

For shells of variable thickness [P], however, (x,θ) will be the functional coordinates. If thickness is independent of θ and proportional to x, set:

$$t = \delta x \quad (2.8)$$

The $[P]$ matrix then takes the form:

$$[P] = \begin{bmatrix} xB_{11} & xB_{12} & 0 & x^2B_{14} & x^2B_{15} & 0 \\ xB_{21} & xB_{22} & 0 & x^2B_{24} & x^2B_{25} & 0 \\ 0 & 0 & xB_{33} & 0 & 0 & x^2B_{36} \\ x^2B_{41} & x^2B_{42} & 0 & x^3B_{44} & x^3B_{45} & 0 \\ x^2B_{51} & x^2B_{52} & 0 & x^3B_{54} & x^3B_{55} & 0 \\ 0 & 0 & x^2B_{63} & 0 & 0 & x^3B_{66} \end{bmatrix} \tag{2.9}$$

Below, the expression of B_{ij} from reference [12] is given for a shell composed of a number of symmetric isotropic or orthotropic layers, as indicaded in Figure 4.

For an even number of layers 2n, we obtain:

$$B_{ij} = 2 \sum_{s=1}^{n} Z^s_{ij}(\delta_s - \delta_{s+1}) \; ; \; i = 1 \text{ to } 3 \text{ and } j = 1 \text{ to } 3$$

$$B_{ij} = \frac{2}{3} \sum_{s=1}^{n} Z^s_{i-3,j-3}(\delta^3_s - \delta^3_{s+1}) ; i = 4 \text{ to } 6 \text{ and } j = 4 \text{ to } 6 \tag{2.10}$$

$$B_{ij} = 0 \; (i = 1 \text{ to } 3 \text{ and } j = 4 \text{ to } 6) \text{ and } (i = 4 \text{ to } 6 \text{ and } j = 1 \text{ to } 3)$$

For an odd number of layers 2n + 1:

$$B_{ij} = 2 \left[Z_{ij}^{n+1} \delta_{n+1} + \sum_{s=1}^{n} Z_{ij}^{s} (\delta_s - \delta_{s+1}) \right]$$

i = 1 to 3 and j = 1 to 3

$$B_{ij} = \frac{2}{3} \left[Z_{i-3,j-3}^{n+1} \delta_{n+1}^{3} + \sum_{s=1}^{n} Z_{i-3,j-3}^{s} (\delta_s^3 - \delta_{s+1}^3) \right]$$

i = 4 to 6 and j = 4 to 6

$B_{ij} = 0$ (i = 1 to 3, j = 4 to 6) or (i = 4 to 6, j = 1 to 3) with

$$Z^s_{11} = E^s_1/(1 - \nu^s_1\ \nu^s_2)$$

$$Z^s_{22} = E^s_2/(1 - \nu^s_1\ \nu^s_2)$$

$$Z^s_{12} = Z^s_{21} = E^s_1\ \nu^s_2/(1 - \nu^s_1\ \nu^s_2)$$

$$Z^s_{33} = \frac{1}{2}\ G^s_{12}$$

where

E^S_1, ν^S_1 (E^S_2, ν^S_2) are young's modulus and Poisson's ratio in conjunction with direction x (and θ), respectively.

G^S_{12}: modulus of elasticity in shear

δ_S is the coefficient of proportionality of thickness t_S at the s^{th} layer with x.

$$t_S = \delta_S x$$

t_S is measured relative to the reference surface.

Substituting (2.9) in (2.5), we obtain a new form for th eequations of motion, which are described in more detail in Appendix A-2.

$$S_1(u,v,w,B_{ij}) = 0$$

$$S_2(u,v,w,B_{ij}) = 0 \qquad (2.12)$$

$$S_3(u,v,w,B_{ij}) = 0$$

The next chapter will demonstrate a coherent method of obtaining displacements u, v and w, in terms of nodal displacements.

CHAPTER 3

ANALYTICAL FORMULATION

3.1 Introduction

The equilibrium of the conical shell as a continuous medium has an infinite number of degrees of freedom and is governed by equations (2.12). They constitute three partially derived linear equations with variable coefficients that make for difficult analytical solution.

The finite element method was chosen to solve this problem.

A brief recapitulation of the principles of this method might be in order. The continuous medium, representing the original shell, is replaced by a system having a finite number of degrees of freedom and governed by a system of linear albegraic equations. Discretization consists of dividing the shell into a number of finite elements and then selecting the displacement functions.

3.2 Displacement functions

The chosen finite element is a cone bounded by circular nodes (Fig. (Fig. 5). This choice allows use of thin shell theory to determine the displacement functions.

The nodal displacements are:

u : axial displacement

w : radial displacement

$\frac{\partial w}{\partial x}$: rotation

v : tangential displacement

The displacement are periodic, therefore, they may be developed into a Fourier series.

$$\{u(x,\theta),\ w(x,\theta),\ v(x,\theta)\}^t = \sum_{n=0}^{\infty} [T(n,\theta)]\ \{u_n(x),\ w_n(x),\ v_n(x)\}^T \qquad (3.1)$$

n : number of circumferential modes

U_n, W_n, V_n are functions of x only. $[T(n,\theta)]$ is a square diagonal (3 (3x3) matrix given in Appendix A-3. (3.1) is carried over into (2.12). By working out the derivation in conjunction with θ, three ordinary differential homogeneous equations are obtained. The left side of each of these equations is a Fourier series and must be cancelled for all values of θ. This is possible only if the coefficients of this series are all zero. For all values of n, three ordinary linear differential equations with variable coefficients in U_n, W_n and V_n are obtained, having the following properties:

In each term, exponent x is equal to the number of derivations in conjunction with x linked to U_n, W_n and V_n. Solutions of these ordinary differential equations are of the form [13]:

U_n, W_n or V_n = constant x^{λ}

λ is a complex root.

In introducing this form, we obtain equations where the coefficients have no useful dimensions. To avoid this inconvenience, we choose a new non-dimensional variable determined by:

$$y = \sqrt{x/\ell} \qquad (3.3)$$

ℓ is the arbitrary reference length. This choice is dictated by experience with axisymmetric problems.

We then follow with:

$$\{u_n(x)\ ,\ v_n(x)\ ,\ w_n(x)\} = y^{\lambda-1}\{A,\ B,\ C\} \qquad (3.4)$$

λ, A, B and C are complex.

3.3 Characteristic equations

Replacing U_n, V_n and W_n in the (2.12) system with the expressions given in (3.4), we get an algebraic system of equations written as:

$$[D]\ \{A,B,C\}^T = 0 \tag{3.5}$$

where [D] is a square (3x3) matrix, the terms of which are functions of parameter λ and given in Appendix 1-3. The (3.5) system necessitates a solution other than the trivial null solution. The determinant of [D], therefore, must be null. The resultant characteristic equation is:

$$\det(D) = 0$$

Development of the determinant produces a fourth-degree polynomial equation in λ^2 written as:

$$h_8\lambda^8 + h_6\lambda^6 + h_4\lambda^4 + h_2\lambda^2 + h_0 = 0 \tag{3.6}$$

The expression for h_l is given in Appendix A-3.

The characteristic equation provides for eight roots having the following properties:

- if λ is a root, its opposite $(-\lambda)$ is also a root.

- if λ is a complex root, then the (3.6) coefficients are real and the conjugate of λ, (λC), is also a root.

So complex root λ will give three other complex roots:

$$(-\lambda),\ \lambda C \text{ and } (-\lambda C)$$

and a real root will give a second real root:

$$-\lambda$$

Two following cases will then ensure that (3.6) allows only real or non-real complex roots.

3.3.1 Complex roots

If the 8 roots of (3.6) are complex and non-real, they can as a results, be assembled into two groups.

$$\begin{aligned} \lambda_1 &= k_1 + i\mu_1 & \lambda_5 &= k_2 + i\mu_2 \\ \lambda_2 &= k_1 - i\mu_1 & \lambda_6 &= k_2 - i\mu_2 \\ \lambda_3 &= -k_1 + i\mu_1 & \lambda_7 &= -k_2 + i\mu_2 \\ \lambda_4 &= -k_1 - i\mu_1 & \lambda_8 &= -k_2 - i\mu_2 \end{aligned} \qquad (3.7)$$

k_1, μ_1, k_2 and μ_2 are real and positive.

Each root λ_1 produces a displacement $\{U_n(x),\ V_n(x),\ W_n(x)\}_1$. The solution is a linear combination of these eight displacements.

For $U_n(x)$ we then have:

$$u_n(x) = \sum_{i=1}^{8} Ai\ y^{\lambda_1 - 1}$$

$$= A_1 y^{k_1-1}\ y^{i\mu_1} + A_2 y^{k_1-1}\ y^{i\mu_1} + A_3 y^{k_1-1}\ y^{i\mu_1} + A_4 y^{k_1-1}\ y^{i\mu_1} +$$

$$A_5 y^{k_2-1}\ y^{i\mu_2} + A_6 y^{k_2-1}\ y^{-i\mu_2} + A_7 y^{-k_2-1}\ y^{i\mu_2} + A_8 y^{-k_2-1}\ y^{-i\mu_2}$$

which can be expressed as:

$$u_n(x) = y^{k_1-1}(A_1 y^{i\mu_1} + A_2 y^{-i\mu_1}) + y^{-k_1-1}(A_3 y^{i\mu_1} + A_4 y^{-i\mu_1}) + y^{k_2-1}(A_5 y^{i\mu_2} + A_6 y^{-i\mu_2}) + y^{-k_2-1}(A_7 y^{i\mu_2} + A_8 y^{-i\mu_2}) \quad (3.8a)$$

and

$$v_n(x) = y^{k_1-1}(B_1 y^{i\mu_1} + B_2 y^{-i\mu_1}) + y^{-k_1-1}(B_3 y^{i\mu_1} + B_4 y^{-i\mu_1}) + y^{k_2-1}(B_5 y^{i\mu_2} + B_6 y^{-i\mu_2}) + y^{-k_2-1}(B_7 y^{i\mu_2} + B_8 y^{-i\mu_2}) \quad (3.8b)$$

$$w_n(x) = y^{k_1-1}(C_1 y^{i\mu_1} + C_2 y^{-i\mu_1}) + y^{-k_1-1}(C_3 y^{i\mu_1} + C_4 y^{-i\mu_1}) + y^{k_2-1}(C_5 y^{i\mu_2} + C_6 y^{-i\mu_2}) + y^{-k_2-1}(C_7 y^{i\mu_2} + C_8 y^{-i\mu_2}) \quad (3.8c)$$

As for λ_j, A_j, B_j and C_j are not independent. We usually express A_j and B_j in conjunction with C_j, therefore:

$$A_j = \alpha_j C_j$$

$$B_j = \beta_j C_j \tag{3.9}$$

α_j amd β_j depend on the terms in matrix [D], and consequently, on λ_j. Set:

$$\alpha_1 = \bar{\alpha}_1 + i\bar{\alpha}_2$$

$$\beta_1 = \bar{\beta}_1 + i\bar{\beta}_2$$

Since λ_2 is a conjugate root of λ_1 we obtain:

$$\alpha_2 = \bar{\alpha}_1 - i\bar{\alpha}_2$$

$$\beta_2 = \bar{\beta}_1 - i\bar{\beta}_2$$

Likewise, λ_3 and λ_4 being conjugates as well as λ_5 and λ_6, λ_7 and λ_8, this will result in:

$$\alpha_3 = \bar{\alpha}_3 + i\bar{\alpha}_4 \qquad \alpha_4 = \bar{\alpha}_3 - i\bar{\alpha}_4$$

$$\beta_3 = \bar{\beta}_3 + i\bar{\beta}_4 \qquad \beta_4 = \bar{\beta}_3 - i\bar{\beta}_4$$

$$\alpha_5 = \bar{\alpha}_5 + i\bar{\alpha}_6 \qquad \alpha_6 = \bar{\alpha}_5 - i\bar{\alpha}_6$$

$$\beta_5 = \bar{\beta}_5 + i\bar{\beta}_6 \qquad \beta_6 = \bar{\beta}_5 - i\bar{\beta}_6$$

$$\alpha_7 = \bar{\alpha}_7 + i\bar{\alpha}_8 \qquad \alpha_8 = \bar{\alpha}_7 - i\bar{\alpha}_8$$

$$\beta_7 = \bar{\beta}_7 + i\bar{\beta}_8 \qquad \beta_8 = \bar{\beta}_7 - i\bar{\beta}_8$$

Carrying over these formulas into (3.8) we obtain:

$$u_n(x) = y^{k_1-1}[(\alpha_1C_1 + \alpha_2C_2)\cos \mu_1 \ell n\, y + i(\alpha_1C_1 - \alpha_2C_2)\sin \mu_1 \ell n\, y]$$

$$+ y^{-k_1-1}[(\alpha_3C_3 + \alpha_4C_4)\cos \mu_1 \ell n\, y + i(\alpha_3C_3 - \alpha_4C_4)\sin \mu_1 \ell n\, y]$$

$$+ y^{k_2-1}[(\alpha_5C_5 + \alpha_6C_6)\cos \mu_2 \ell n\, y + i(\alpha_5C_5 - \alpha_6C_6)\sin \mu_2 \ell n\, y]$$

$$+ y^{-k_2-1}[(\alpha_7C_7 + \alpha_8C_8)\cos \mu_2 \ell n\, y + i(\alpha_7C_7 - \alpha_8C_8)\sin \mu_2 \ell n\, y]$$

Obtaining:

$$\alpha_1C_1 + \alpha_2C_2 = \bar{\alpha}_1C_1 + i\bar{\alpha}_2C_1 + \bar{\alpha}_1C_2 - i\bar{\alpha}_2C_2$$

$$= \bar{\alpha}_1(C_1 + C_2) + i\bar{\alpha}_2(C_1 - C_2)$$

$U_n(x)$ being real, the following real arbitrary constants are determined:

$$\bar{C}_1 = C_1 + C_2$$

$$\bar{C}_2 = i(C_1 - C_2)$$

We then obtain:

$$\alpha_1 C_1 + \alpha_2 C_2 = \bar{\alpha}_1 \bar{C}_1 + \bar{\alpha}_2 \bar{C}_2$$

$$i(\alpha_1 C_1 - \alpha_2 C_2) = \bar{\alpha}_2 \bar{C}_2 - \bar{\alpha}_2 \bar{C}_1$$

which are real. In the same way, if we determine:

$$\bar{C}_3 = C_3 + C_4$$

$$\bar{C}_4 = i(C_3 + C_4)$$

$$\bar{C}_5 = C_5 + C_6$$

$$\bar{C}_6 = i(C_5 + C_6)$$

$$\bar{C}_7 = C_7 + C_8$$

$$\bar{C}_8 = i(C_7 + C_8)$$

$U_n(x)$ will be given by linear combination with real coefficients for the eight real constants $\overline{C}_k$.

To obtain the expression corresponding to $V_n(x)$ and $W_n(x)$, respectively, we must replace $\overline{\alpha}_k$ by $\overline{\beta}_k$

We then obtain:

$$\{ u_n(x),\ w_n(x),\ v_n(x) \}^t = [R]\ \{\overline{C}\} \tag{3.11}$$

where $[R]$ is a 3x8 matrix as demonstrated in Appendix A-3. $\{\overline{C}\}$ is a column vector for the 8 $\overline{C}_k$ constants

$$\{\overline{C}\}^t = \{\overline{C}_1,\overline{C}_2,\overline{C}_3,\overline{C}_4,\overline{C}_5,\overline{C}_6,\overline{C}_7,\overline{C}_8,\}$$

3.3.2 Complex and real roots

In cases where the characteristic equation allows for complex and real roots, these fall into two groups:

$$\lambda_1 = k_1 + i\mu_1 \qquad \lambda_5 = a_1$$

$$\lambda_2 = k_1 - i\mu_1 \qquad \lambda_6 = a_2$$

$$\lambda_3 = -k_1 + i\mu_1 \qquad \lambda_7 = -a_1$$

$$\lambda_4 = -k_1 - i\mu_1 \qquad \lambda_8 = -a_2$$

where k_1, μ_1, a_1 and a_2 are real and positive.

The same procedure is used as in the preceding paragraph to obtain the $[R]$ matrix corresponding to this case.

Displacement at point $M(x,\theta)$ corresponding to the circumferential mode n is then:

$$\{ u(x,\theta),\ w(x,\theta),\ v(x,\theta) \}^t = [T]\ [R]\ \{\bar{C}\} \tag{3.12}$$

This displacement in conjunction with the degrees of freedom $\{\delta_i\}$ and $\{\delta_j\}$ is now expressed with respect to nodes i and j of the finite element to which point M belongs,

giving

$$\{\delta_i\} = \{ u_i,\ w_i,\ \frac{\partial w_i}{\partial x},\ v_i \}^T$$

$$\{\delta_j\} = \{ u_j,\ w_j,\ \frac{\partial w_i}{\partial x},\ v_i \}^T$$

$$\{d_i\} = \begin{bmatrix} \delta_i \\ \delta_j \end{bmatrix}$$

Relation (3.11) produces $\frac{\partial W}{\partial x}$, then x is replaced by x_i for node i and by x_j for node j. $\{\delta_i\}$ and $\{\delta_j\}$ can then be expressed as a function of constants $\overline{C}_k$. This relation is written:

$$\{d_s\} = [A]\{\overline{C}\} \quad (3.13)$$

[A] is a square matrix of order 8 as shown in Appendix A-3 for both cases of the roots of the characteristic equation.

Then obtaining:

$$\{\overline{C}\} = [A^{-1}]\{d_s\}$$

Relation (3.12) is then written as:

$$\{u, w, v\}^T = [T][R][A^{-1}]\{d_s\} \quad (3.14)$$

or

$$\{u, w, v\}^t = [N]\{d_s\} \quad (3.15)$$

with

$$[N] = [T][R][A^{-1}]$$

$[N]$ is a 3x8 matrix and a functiuon of coordinates x and θ from point M; it determines the displacement functions for circumferential mode n. The total displacement is:

$$\{u,w,v\}^e = \sum_{n=0}^{\infty} [N]\{d_i\} \tag{3.16}$$

3.4 Strain-displacement relationships

The deformation vector is determined in relation to (2.2). Using (3.16), $\{\varepsilon\}$ is expressed as part of the nodal displacements.

$$\{\epsilon\} = \sum_{n=0}^{\infty} [T^1][Q][A^{-1}]\{d_i\} \tag{3.17}$$

$[Q]$ is a (6x8) matrix as shown in Appendix A-3.

with

$$[T^1] = \begin{bmatrix} [T] & 0 \\ 0 & [T] \end{bmatrix}$$

3.5 Stress-strain relationships

The resultant stress vector is also expressed in conjunction with nodal displacements.

Referring back to relation (2.4), (3.17) is written either:

$$\{\sigma\} = \{P\} \sum_{n=0}^{\infty} [T^1]\,[Q]\,[A^{-1}]\,\{d_1\}$$

or

$$\{\sigma\} = \sum_{n=0}^{\infty} [T^1]\,[P]\,[Q]\,[A^{-1}]\,\{d_1\} \tag{3.18}$$

In summary, this chapter has illustrated that the finite element method made it possible to express shell motion relative to the nodes of the finite elements and which also became the unknowns of the problem. The following chapter looks at the methods used to determine these unknowns.

CHAPTER 4

MASS AND STIFFNESS MATRICES

4.1 Equations of motion in matrix form

To determine nodal displacements, the Lagrange equations are applied to a finite element.

By definition we have:

$$U_i = \int_{A_i} \{\epsilon\}^t \{\sigma\} dA \tag{4.1a}$$

$$T_i = \int_{A_i} \frac{1}{2} \rho (\dot{u}^2 + \dot{w}^2 + \dot{v}^2) t dA \tag{4.1b}$$

where U_i and T_i indicate, respectively, the deformation energy and the energy of E_i A_i is the area of the reference surface for the conical element.

Using relations (3.17) and (3.18) where $\{d_i\}$ and $\{A^{-1}\}$ are constants of a finite element, we botain the equations of motion for matrix form Ei.

$$[m]\{\ddot{d}_i\} + [k]\{d_i\} = 0 \tag{4.2}$$

$[m]$ et $[k]$ are called mass and stiffness matrices, and are given by:

$$[k] = [A^{-1}]^t\ [G]\ [A^{-1}] \tag{4.3}$$

$$[m] = [A^{-1}]\ [S]\ [A^{-1}] \tag{4.4}$$

with

$$[G] = \int_{A_1} [Q]^t\ [T^1]^t\ [P]\ [T^1]\ [Q]dA \tag{4.5}$$

$$[S] = \frac{1}{2}\int_{A_1} \rho[R]^t\ [T]^t\ [T]\ [R]tdA \tag{4.6}$$

From their definitions $[m]$ and $[k]$, two symmetrical (8x8) matrices are derived, and finite positive.

The static case is governed by the equation:

$$[k]\ \{d_1\} = \{f_1\} \tag{4.7}$$

4.2 Meaning of terms [m] and [k]

For the differential element illustrated in Figure 3, using relations (2.8) and (3.3) we obtain:

$$\begin{aligned} dA &= 2\ell^2 \sin\alpha \; y^3 dy d\theta \\ t dA &= 2\delta\ell^3 \sin\alpha \; y^5 dy d\theta \end{aligned} \qquad (4.8)$$

In relations (4.5) and (4.6), the circumferential variable θ intervenes only when matrices $[T]$ and $[T^{1}]$ are diagonal and expressed by two terms: $\cos^2 n\theta$ and $\sin^2 n\theta$, in which integration occurs immediately:

$$\int_0^{2\pi} \cos^2 n\theta \, d\theta = \int_0^{2\pi} \sin^2 n\theta \, d\theta = \pi$$

The non-dimensional orthogonal coordinate intervenes in:

- matrices [Q] and [R], when the general form of the terms is:

$$y^k [a \cos(\mu\ell n y) + b \sin(\mu\ell n y)]$$

- elasticity matrix [P], when the general form of the terms is:

$$y^k \, B_{ij}$$

Integration of the matrix products of the relations in (4.5) and (4.6) in conjunction with (y), occurs with a term of near constant form:

$$\int y^k[a\cos(\mu\ell ny) + b\sin(\mu\ell ny)][(a'\cos(\mu'\ell ny)+b'\sin(\mu'\ell ny)]dy$$

After manipulation and utilization of the trigonometric relationship, integrals of the following forms are obtained:

$$\int y^k\cos(\mu\ell ny)dy \quad \text{and} \quad \int y^k\sin(\mu\ell ny)dy$$

and integration is immediate:

$$\int y^k\cos(\mu\ell ny)dy = \frac{y^{k+1}}{(k+1)^2+\mu^2}(\mu\sin(\mu\ell ny) + (k+1)\cos(\mu\ell ny)) + \text{constant} \qquad (4.9a)$$

$$\int y^k\sin(\mu\ell ny)dy = \frac{y^{k+1}}{(k+1)^2+\mu^2}((k+1)\sin(\mu\ell ny) - \mu\cos(\mu\ell ny)) + \text{constant} \qquad (4.9b)$$

Matrices $[G]$ and $[S]$ and consequently $[k]$ and $[m]$ are then determined analytically. However, we should specify now that this report does not include analytical developments of the 72 different terms in $[G]$ and $[S]$: the time and space required for their development would have been prohibiture. the calculation was directly programmed from the relations in (4.5) and (4.6).

4.3 Global mass and stiffness matrices

Equation (4.2) relates to a finite element. If $\{d\}$ indicates the degrees of freedom vector for the set of nodes, shell motion is then governed by an analogous equation which can be writen as:

$$[M]\{\ddot{d}\} + [K]\{d\} = \{F\} \qquad (4.10)$$

$[M]$ and $[K]$ are the mass and stiffness matrices of the whole shell. They are obtained by assembling the element matrices $[m]$ and $[k]$ in accordance with the two following conditions [10]:

- there must be continuity of nodal displacements from one element to the next:

$$\{\delta_i+1\} = \{\delta_j\}$$

- the external forces and moments applied to a node must be respectively equal to the forces and internal moments:

$$\{f\}^e = \{f_j\} + \{f_i+1\}$$

Assembly is done by overlaying as indicated in Figure 6. If N is the number of finite elements, then $[M]$ and $[K]$ are two symmetrical and semi-finite square matrices of order 4(N+1). They are also two banded matrices where half band width is equal to 8.

For the static equilibrium of the shell we of course have:

$$[K]\{d\} = \{F\} \qquad (4.11)$$

4.4 Solution

The solution of (4.10) is obtained as follows:

The shell being subjected to a certain number, J, of boundary conditions, matrices [M] and [K] then become square matrices of order 4(N+1)-J and which are symmetrical with positive properties, which was not true of the original matrices. With boundary conditions then being imposed upon the displacements, their reduction produced by eliminating the line and column corresponding to the fixed degrees of freedom.

We then set $\{d\} = \cos(\omega t)\ \{d_0\}$

Relation (4.10) then gives $([K] - \omega^2 [M])\ \{d\} = 0$

Given the 4(N+1)-J order, we then obtain 4(N+1)-J eigenvalues and eigenvectors.

Analysis of the static equilibrium is performed in the following way:

$\{F_a\}$ is the force vector applid to shell nodes
$\{F_d\}$ is the vector for unknown reactions

$\{d_a\}$ is the vector of unknown nodal displacements

$\{d_s\}$ is the vector of displacements specified by the boundary conditions.

Equation (4.11) is written as:

$$\begin{bmatrix} K_{AA} & K_{AB} \\ K_{BA} & K_{BB} \end{bmatrix} \begin{matrix} d_A \\ d_B \end{matrix} = \begin{matrix} F_A \\ F_B \end{matrix}$$

Obtaining:

$$\{d_A\} = [K_{AA}]^{-1} (\{F_A\} - [K_{AB}] \{d_B\})$$

(4.13)

$$\{F_B\} = [K_{BA}] \{d_A\} + [K_{BB}] \{d_B\}$$

CHAPTER 5

THE ALGORITHM

5.1 Introduction

A Fortran V computer program was written and applied to the analytical formulations developed in the preceding chapters for a number of cases. The characteristics of the numerical solutions will now be given.

The original conical shell was subdivided into a certain number of finite elements. This number was dictated by the convergence of the method. It will be demonstrated that for the cases considered, a five-element partition is sufficient. Geometric and mechanical characteristics can vary from one element to another, and therefore one can deal with different shapes of truncated conical shells.

These finite elements are replaced with elements varying linearly in thickness with the x coordinate. Thickness proportionality factor in relation (2.8) is chosen as:

$$t_{pm} = \delta . x_m$$

where t_{om} indicates the initial thickness at the average point of the element at coordinate x_m. It is important to note that a change in thickness will lead to an approximation of the shell's thickness the thinner the mesh, the more accurate the thickness.

5.2 Program organization

Flow chart and a tree diagram of the program are given in Figures 7 and 8.

5.2.1 Program organization

i) Number of finite elements.

ii) The geometry of each finite element: length, the apex angle and coefficent δ.

iii) The mechanical properties of each element: Young's modulus, Poisson's ratios and the density.

iv) The boundary conditions.

v) The inverse of [A]. To achieve this, we must use the LINV2F subprogram from IMSL utilizing a Gaussian method of elimination which searches for the maximal pivot per column and the CROUT algorithm for memory storage. This phase in the processing is very important. We must check the accuracy of the results of this operation inverting matrix [A]. Indeed, with the [A] terms not being of the same order, can be falsified results through truncation errors. The calculation is therefore double checked for accuracy.

vi) Matrices [G] and [S] determined by (4.5) and (4.6) (GCONE and SCONE subprograms). The element integrals enter into this calculation as given by (4.9) and are calculated by the RE11 and RE12 subprograms.

vii) The elements mass and stiffness matrices. Steps i) to vii) are repeated for each element.

viii) The mass and stiffness matrices for the entire shell.

ix) The mass and stiffness matrices are reduced through application of boundary conditions.

x) Free vibrations and the corresponding modes.

CHAPTER 6

CALCULATIONS AND DISCUSSION

6.1 Introduction

For the present chapter, the methods described previously in this report were then applied to a certain number of cases.

Initial calculations were done to demonstrate convergence of the formulation chosen. Then, free vibrations in the four configurations of each of the thin conical shells were determined, all of which have been theoretically and experimentally investigated by the authors of references [6] and [7] and used as examples for comparisons with other studies.

6.2 Convergence

The finite element method was used to obtain a solution to the elasticity problems. The solution has several errors in it which are usually classified into two categories:

- One called discretization error, which historically in the replacement of the initial physical problem by an approximate model. In the present case, this error stemmed from the different simplified hypothesis used to derive the shell methods and the model of the continuous medium, i.e. The shell, having a finite number of degrees of freedom and a choice of linear variation thickness.

- The other, truncation error, originated in the numerical calculations. Numbers in the computer are represented with only a finite set of integers, which leads to limited accuracy of calculation. Accumulated truncation errors can lead to erroneous results even if the original equations are mathematically exact.

Model	α en d°	$\frac{x_2}{x_1}$	$\frac{h}{R_2}$	R_2(in)	ν	δ
1	14.2	2.23	.00166	6.07	0.3	5.623E-4
2	30.2	2.27	.00127	7.95	0.3	8.869E-4
3	45.1	2.25	.00112	8.96	0.3	1.098E-3
4	60.5	2.25	.00101	10.00	0.3	1.217E-3

Table 6.1: Characteristics of four models of shells

In the four examples chosen, the thickness of the shell is constant. This solution method assumes a variable linear thickness with an orthogonal coordinate. The thickness proportionality coefficient is determined such that the initial thickness is found in the middle of the shell, so that:

$$\delta = 2h/(x_2 + x_1)$$

The values for Young's modulus and density are not indicated in references [6] and [7], and so we took:

$$E = 3.10^6 \text{ lb/in}^2$$

$$\rho = 1 \text{ lb s}^2/\text{in}^4$$

Only the finite element, bounded at both ends, was used. Rigid motion corresponds to zero frequency and represents translation, rotation or a combination of the two. For a shell of revolution subjected to axisymmetric forces, there is only one vibration mode corresponding to rigid-body motion [16]. The calculations were done for n = 2.

From the nodal displacements obtained for the two nodes of the element, boundary deformations were determined as follow:

For a number of circumferential modes n, we have:

$$u(x,\theta) = u_n(x)\cos n\theta$$
$$w(x,\theta) = w_n(x)\cos n\theta \qquad (6.1)$$
$$v(x,\theta) = v_n(x)\cos n\theta$$

By using (2.2), it becomes:

$$\epsilon x = \cos n\theta \frac{du_n}{dx}$$

$$\epsilon\theta = \cos n\theta \frac{1}{r}(nv_n + u_n\sin\alpha + w_n\cos\alpha)$$

$$(6.2)$$

$$\epsilon x\theta = \frac{1}{2}\sin n\theta(\frac{dv_n}{dx} - \frac{nu_n}{r} - \frac{v_n\sin\alpha}{r})$$

$$kx = -\cos n\theta \frac{d\beta}{dx}$$

$$k\theta = \cos n\theta . \frac{1}{r}\left(n v_n \frac{\cos\alpha}{r} + n^2 \frac{w_n}{r} - \beta \sin\alpha \right)$$

$$kx\theta = \sin n\theta . \frac{1}{2r}\left(2n\beta + \frac{3}{2}\frac{dv_n}{dx}\cos\alpha + nu_n \frac{\cos\alpha}{2r} - 2nw_n \frac{\sin\alpha}{r} - \frac{3}{4} v \frac{\sin 2\alpha}{r} \right)$$

These formulas give us the maximal boundary constraints, by taking:

$\cos n\hat{\theta} = 1$ or $\sin n\hat{\theta} = 1$

It is evident from (6.2) that $\varepsilon x\theta$ and $kx\theta$ are zero when other components of the deformation vector are maximal in absolute and inverted value.

The first derivations appearing in (6.2) were roughly obtained by:

$$\frac{du_n}{dx} = (u_n(2) - u_n(1))/(x_2 - x_1)$$

$$\frac{dv_n}{dx} = (v_n(2) - v_n(1))/(x_2 - x_1) \qquad (6.3)$$

$$\frac{d\beta}{dx} = (\beta(2) - \beta(1))/(x_2 - x_1)$$

where 1 and 2 refer to the finite elements two nodes. Table 6.2 gives the calculatated values of U_n, W_n, B, V_n and W for the four models and correspond to the first mode where n = 2. The deformation values are indicated in Table 6.3, where it can be seen that one can only write four configurations, obtaining:

$$\{\epsilon\} = 0$$

and this occurs with maximal error equal to 18% and corresponding to $k\theta = 18.274\ 10^{-2}$ for model 1 at x = xl.

The first convergence criterion is thus satisfied. This could have been determined at the outset since the displacement functions chosen validate Sander's equations and give us the zero deformations for rigid-body movement. The difference, in conjunction with the null values in Table 6.3 arise out of the truncation error inherent in the numerical manipulations.

Model		u_n	w_n	β	v_n	ω(rad/s)
1	$x=x_1$	1.752E-4	.450	3.471E-2	-.218	.938
	$x=x_2$	2.196E-4	1.00	4.590E-2	-.484	
2	$x=x_1$	1.057E-5	.440	5.935E-2	-.190	.562
	$x=x_2$	1.611E-5	1.00	6.730E-2	-.432	
3	$x=x_1$	2.633E-6	.444	7.539E-2	-.156	.449
	$x=x_2$	5.153E-6	1.00	8.280E-2	-.353	
4	$x=x_1$	1.111E-6	.444	8.345E-2	-.109	.373
	$x=x_2$	2.993E-5	1.00	9.088E-2	-.246	

Table 6.2: First vibration mode for n = 2

Model		ϵ_x	ϵ_θ	$\epsilon_{x\theta}$	k_x	k_θ	$k_{x\theta}$
1	$x=x_1$	3.253E-6	1.122E-4	-5.364E-5	-8.199E-4	1.827E-1	-4.246E-3
	$x=x_2$	3.253E-6	1.053E-4	-9.720E-5	-8.199E-4	8.122E-2	1.817E-3
2	$x=x_1$	6.265E-7	8.173E-4	-4.220E-5	-8.991E-4	1.082E-1	-2.212E-3
	$x=x_2$	6.265E-7	3.558E-4	-1.929E-5	-8.991E-4	4.721E-2	1.009E-3
3	$x=x_1$	3.586E-7	3.538E-4	-1.439E-4	-1.054E-3	8.469E-2	-1.839E-3
	$x=x_2$	3.586E-7	-1.392E-5	-6.426E-5	-1.054E-3	3.707E-2	8.278E-4
4	$x=x_1$	4.515E-6	1.433E-4	-5.816E-5	-1.164E-3	6.815E-2	-1.588E-3
	$x=x_2$	4.515E-5	4.496E-5	-2.873E-5	-1.164E-3	2.967E-2	7.662E-4

Table 6.3: Boundary deformation vector for n = 2 and corresponding to the first vibration mode

6.2.2 Number of finite elements

Another aspect of convergence with the finite element mothod is the behaviour of the solution relative to the number of finite elements used to model the structure in question, that is to say, in conjunction with mesh fineness.

Now, the results will be given the non-dimensional frequency as determined by:

$$\omega_a = \omega\, R_2(\rho(1-\nu^2)/E)^{1/2} \qquad (6.4)$$

with:

ω: frequency rad/sec

R_2: large-base conical radius

ρ: density

j: Poissons's ratio

E: Young's modulus

and this applies to the model 1 conical shell, which is free at both ends.

Number of finite elements	2	3	5	8	10
Non-dimensional frequency	.329E-2	.332E-2	.333E-2	.333E-2	.335-2

Table 6.4: Frequency ω_a of first mode, corresponding to n = 2.
α, = 14.2°

6.3 Free vibrations of simply supported shells

The results will be given for the four shells described previously, and compared to the results Hu and Lindholm [6] obtained.

The boundary conditions are:

$$u = w = 0 \qquad \text{for} \qquad x = x_1 \quad \text{and} \quad x = x_2$$

The analysis the authors of reference [6] did has these characteristics:

- The stress-strain relations are as given by Naghdi ([17]).
- The equations of motion are 8^{th} order. We consider the effect of shear deformation and rotational inertia in the orthogonal direction only.

Galerkin's method has been used to derive a matrix system where the eigenvalues are calculated. The amplitude of displacements are expressed as:

$$u_n = \sum_{m=1}^{m_1} a_m \sin(m\pi s/L)$$

$$v_n = \sum_{m=1}^{m_2} b_m \sin(m\pi s/L)$$

$$w_n = \sum_{m=1}^{m_3} c_m \sin(m\pi s/L)$$

with

with $s = \ln(x_2/x)$

$L = \ln(x_2/x_1)$

m_1, m_2 and m_3 are integers chosen in with the level of accuracy desired.

In the four numerical tests, the number of finite elements chosen is equal to 5. The number of circumferential waves is taken in intervals [2,20].

As was mentioned previously, for each finite element the thickness is a linear variable which must equal the initial thickness of the model in the centre of the mesh. The proportionality coefficient δ then varies from one element to another. Likewise, the arbitrary parameter ℓ from the relationship in (3.3) is applied as well to the orthogonal coordinate of the mid-point of the element. It must be mentioned that if ℓ is an arbitrary length in this analytical formulation, it must in no way influence the results. It is extremely important to the numerical phase of the work, since the order of magnitude of the terms of matrix [A] depends upon it. A side from this problem, the inversion of [A] is sensitive to the algorithm, where a loss of accuracy could result if the difference between the terms in [A] is too great. In fact, Gaussian elimination would lead, in this case, to almost zero pivots. The initial results for the numerical values of different matrices are used in the formulation. These results are qualitative for the four models and the five elements, as thus are the results relative to the first finite element of the first model for n = 10 which will be obtained.

Table 5 of Appendix A-4 shows the 8 roots of the characteristic equation.

Table 6 shows matrix [A] and its inverse. Likewise, matrix [G] and the elementary stiffness matrix [K] are indicated in Table 7. Matrices [s] and [m] are given in Table 8.

These numerical values are comparable with results in the literature. However, we can conclude that:

- The roots of the characteristic equation have the exact same form as was indicated in Chapter 3.

- The elementary mass and stiffness matrices are positive symmetrical and diagonally dominant:

$$m_{ii} \geq m_{ij} \quad (i,j=1,8)$$

$$k_{ii} \geq k_{ij} \quad (i,j=1,8)$$

which confirms the fact that the chosen solution becomes a stable numerical problem.

The relative results concerning the non-dimensional frequency variations, as a function of the number of circumferential waves, are given for the other models in Figures 9 to 12, where the theoretical and experimental results obtained by Hu and Lindholm [6] are recorded. The numerical values obtained with our method were observed to be valid experimental approximation of the whole set of n values. In fact, in our procedure, used in reference [6], the number of terms (m_1=4, m_2=4, m_3=5) used for the displacements led to good accuracy for small values of n only. Free vibration was observed to be minimal for a relatively large number of circumferential waves, $n > 6$.

The vibration modes corresponding to this case ($n > 2$) are called breathing modes by some authors. These modes have greater practical importance, and their characteristics are as follows:

- The radial deformation mode is predominant.

- The corresponding vibrations are the lowest in the spectrum.

- The response to external excitation leads to maximal deformations.

Figures 13 to 20 indicate the variation in the normalized radial vibration mode in conjunction with the orthogonal coordinate. It can be seen that by using 5 finite elements we can correctly predict the structures response. As previously mentioned, the deformations and constraints obtained are determined by an exact calculation of modes. It can be seen that the radial deformation mode is strongly dependent on n, and the point of maximal deformation tends toward the base of the cone as n increases.

6.4 Free vibrations of conical shells with free boundaries

A second series of calculations was performed to examine the free vibrations of four shells with the following boundary condition:

The boundaries were free.

This case has been studied by several authors, who mostly used non-extension shell deformation theory.

However, this method becomes ineffective depending on the number values of the circumferential modes.

Hu, Gromley and Lindholm [7] used displacement functions allowing for the non-extension of the reference surface which had these terms:

$$u_n = A \sin\alpha \cos\alpha$$

$$v_n = (A + B \frac{x}{x^2}) n \cos\alpha$$

$$w_n = A(n^2 - \sin^2\alpha) + Bn^2 \frac{x}{x_2}$$

They used Ritz's method to derive the characteristic equation validated by the frequency. The results they obtained are consistent with the experimental results for low values of $(n < 8)$.

The shell was subdivided into 10 elements for the present calculations. The umber of circumferential modes n varied from 2 to 20.

Figures 21 to 24 show the non-dimensional frequency variation as a function of n for the 4 shell configurations. The normalized shear vibration is indicated for the 11 nodes used for model 3.

It should be specified once again that quantitative comparisons with reference [7] have been made possible.

However, A look at the graphs, led to the following observations:

- The frequencies obtained correlate with our experimental results through the entire n interval.

- The normalized shear deformation modes obtained are the same as the experimental modes obtained by the authors in reference [7].

Thus, for low n values $(n < 10)$ the conical generator remains straight, and starts to depart from this configuration as n increases. This transition corresponds to the changing concavity of the curve $w_a - n$.

The position of the circular nodes $(\omega_n = 0)$ corresponds to the first vibration mode, which varies from the smaller to the larger cross section as n increases. In this way, the maximum mode is always obtained by:

On the other hand, the circular node for the second mode is located near the large cross section for the set of n values.

A divergence between vibration modes was noticed for n = 10. This divergence can be attributed to the fact that it is experimentally impossible to have a conical shell free from all boundary constraints.

n		2	4	6	8	10	12	14	16	18	20
Model 1	m=1	.002	.015	.029	.044	.055	.077	.104	.135	.172	.215
	m=2	.004	.033	.057	.069	.090	.114	.143	.179	.222	.272
Model 2	m=1	.0018	.010	.033	.030	.041	.064	.079	.108	.135	.165
	m=2	.361	.204	.261	.080	.091	.119	.143	.171	.202	.237
Model 3	m=1	1.710^{-3}	.009	.016	.018	.040	.057	.078	.093	.122	.148
	m=2	.023	.025	.043	.067	.101	.121	.149	.171	.199	.231
Model 4	m=1	.003	.008	.015	.022	.036	.052	.072	.090	.111	.138
	m=2	.008	.016	.041	.064	.099	.113	.142	.165	.190	.220

Table 6.5: Non-dimensional frequency ω_a for the 4 models with free boundaries

6.5 Processing time

The computer program was executed on a Cyber 855 in the University of Montreal Computer Centre. This CDC-type computer allows for an internal representation of the number in floating-point mode with 60-bits single precision (48 bits for mantissa, 11 for exponent and 1 for the symbol).

The prcessing time and memory capacity required for the following typical cases will now be given.

The model 3 shell is subdivided into 10 finite elements with free boundaries at both ends.

CPU Time: : 48s

Memory space: 115 700 bytes

Price : 14$

It should be specified, however, that programming was designed so that the actual mode must be compatible with an existing progam that deals with cylindrical shells [10]. But an enhanced structure for these two programs, such as by dynamic memory allocation, will lead to a code requiring less time-consuming processing and memory space.

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CONCLUSION

A method based on Sander's equations for thin shells and using the finite element method was formulated in order to study the free vibrations of anisotropic conical shells with variable thickness. The bending and stiffness of the surface of reference were investigated and calculated in this study.

The finite element was conical. It was geometrically accurate, making possible derivation of the displacement functions of the equations of motion for the shell. Mass and stiffness were determined by analytical integration.

The convergence of this method was established. Rigid-body motion was studied. The deformation modes and free vibration frequencies were obtained for the four configurations of the different conical shells. In light of the different sources of experimental error and the limited accuracy of computation, it could be concluded that the numerical results obtained reflected the real behaviour of the structures under study.

We can therefore suggest that an adequate method has been found to predict the characteristic vibrations of thin conical shells, one that was derived from the initial hypotheses.

The present theory was develop and has been applied to straight circular conical shells as well as thin cylindrical shells [10]. It can also, however, be used to analyze a shell with meridian curvatures by assembling conical elements in gradual approximation of its geometry. Nerverthelless, this theory cannot be applied to open shells.

It would be interesting to apply this method to pressurized or liquid filled shells [20], [21], [22].

It would also be of interest to study the compatibility of the finite conical shell with the finite cylindrical shell by considering the case of a truncated shell that has both these geometries.

The logical extension of this method would be to consider the geometric non-linearities of the walls. In this connection, Sanders-Koiter non linear theory could be used.

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APPENDIX A-1

SANDER'S SHELL THEORY

a) General equations of equilibrium

The development of equations with respect to static equilibrium has been the subject of several books and published articles. We shall therefore restrict ourselves to the five final equations of motion given by Sanders [11] in the form (Figure 1).

$$\frac{\partial A_2 N_1}{\partial \zeta_1} + \frac{\partial A_1 \bar{N}_{12}}{\partial \zeta_2} + \bar{N}_{12} \frac{\partial A_1}{\partial \zeta_2} - N_2 \frac{\partial A_2}{\partial \zeta_1} + \frac{A_1 A_2}{R_1} Q_1 + \frac{A_1}{2} \frac{\partial}{\partial \zeta_2} \left[\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \bar{M}_{12}\right] = 0 \qquad \text{(a)}$$

$$\frac{\partial A_2 \bar{N}_{12}}{\partial \zeta_1} + \frac{\partial A_1 N_2}{\partial \zeta_2} + \frac{\partial A_2}{\partial \zeta_1} \bar{N}_{12} - \frac{\partial A_1}{\partial \zeta_2} N_1 + \frac{A_1 A_2}{R_2} Q_2 + \frac{A_2}{2} \frac{\partial}{\partial \zeta_1} \left[\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \bar{M}_{12}\right] = 0 \qquad \text{(b)}$$

$$\frac{\partial A_2 Q_1}{\partial \zeta_1} + \frac{\partial A_1 Q_2}{\partial \zeta_2} - \left(\frac{N_1}{R_1} + \frac{N_2}{R_2}\right) A_1 A_2 = 0 \qquad \text{(c)} \quad \text{(A-1.1)}$$

$$\frac{\partial A_2 M_1}{\partial \zeta_1} + \frac{\partial A_1 \bar{M}_{12}}{\partial \zeta_2} + \bar{M}_{12} \frac{\partial A_1}{\partial \zeta_2} - M_2 \frac{\partial A_2}{\partial \zeta_1} - A_1 A_2 Q_1 = 0 \qquad \text{(d)}$$

$$\frac{\partial A_2 \bar{M}_{12}}{\partial \zeta_1} + \frac{\partial A_1 M_2}{\partial \zeta_2} + \bar{M}_{12} \frac{\partial A_2}{\partial \zeta_1} - M_1 \frac{\partial A_1}{\partial \zeta_2} - A_1 A_2 Q_2 = 0 \qquad \text{(e)}$$

avec

$$\begin{cases} \bar{N}_{12} = 1/2\,(N_{12} + N_{21}) \\ \bar{M}_{12} = 1/2\,(M_{12} + M_{21}) \end{cases}$$

b) Deformation vector

In addition to the equilibrium conditions, there is a second group of equations determining the state of constraint in the shell, the law of elasticity. To this purpose, deformation vector $\{\varepsilon\}$ was developed:

$$\varepsilon_1 = \frac{1}{A_1}\frac{\partial U_1}{\partial \zeta_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \zeta_2} U_2 + \frac{W}{R_1}$$

$$\varepsilon_2 = \frac{1}{A_2}\frac{\partial U_2}{\partial \zeta_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \zeta_1} U_1 + \frac{W}{R_2}$$

$$\bar{\varepsilon}_{12} = \frac{1}{2A_1 A_2}\left(A_2 \frac{\partial U_2}{\partial \zeta_1} + A_1 \frac{\partial U_1}{\partial \zeta_2} - \frac{\partial A_1}{\partial \zeta_2} U_1 - \frac{\partial A_2}{\partial \zeta_1} U_2\right) \qquad \text{(A-1.2)}$$

$$\kappa_1 = \frac{1}{A_1}\frac{\partial \beta_1}{\partial \zeta_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial \zeta_2} \beta_2$$

$$\kappa_2 = \frac{1}{A_2}\frac{\partial \beta_2}{\partial \zeta_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial \zeta_1} \beta_1$$

$$\bar{\kappa}_{12} = \frac{1}{2A_1 A_2}\left[A_2 \frac{\partial \beta_2}{\partial \zeta_1} + A_1 \frac{\partial \beta_1}{\partial \zeta_2} - \frac{\partial A_1}{\partial \zeta_2} \beta_1 - \frac{\partial A_2}{\partial \zeta 1} \beta_2 + \frac{1}{2}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\left(\frac{\partial A_2 U_2}{\partial \zeta_1} - \frac{\partial A_1 U_1}{\partial \zeta_2}\right)\right]$$

with
$$\begin{cases} \beta_1 = \dfrac{U_1}{R_1} - \dfrac{1}{A_1}\dfrac{\partial W}{\partial \zeta_1} \\ \beta_2 = \dfrac{U_2}{R_2} - \dfrac{1}{A_2}\dfrac{\partial W}{\partial \zeta_2} \end{cases}$$

c) Boundary conditions

The boundary conditions are given by:

$$\frac{M_1}{R_1} + N_1 = \bar{\bar{N}}_1 \qquad \text{or} \qquad U_1 = \bar{\bar{U}}_1$$

$$\bar{N}_{12} + \left(\frac{3}{R_2} - \frac{1}{R_1}\right)\frac{\bar{M}_{12}}{2} = \bar{\bar{T}}_{12} \qquad \text{or} \qquad U_2 = \bar{\bar{U}}_2 \tag{A-1.3}$$

$$Q_1 + \frac{1}{A_2}\frac{\partial \bar{M}_{12}}{\partial \zeta_2} = \bar{\bar{V}}_1 \qquad \text{or} \qquad W = \bar{\bar{W}}$$

$$M_1 = \bar{\bar{M}}_1 \qquad \text{or} \qquad \frac{\partial W}{\partial \zeta_1} = \frac{\partial \bar{\bar{W}}}{\partial \zeta_1}$$

For a a ξ_1 constant boundary, the double-barred quantities correspond to boundary values.

d) Parameters for a conical shell of revolution (Fig. 2 and 3) the quantities are:

$$\begin{array}{llll} \zeta_1 = x & A_1 = r_\phi = \infty & R_1 = r_\phi = \infty & U_1 = U \\ \zeta_2 = \theta & A_2 = r & R_2 = r_\theta & U_2 = V \\ & & & W = W \end{array} \tag{A-1.4}$$

with $\lim\limits_{r_\phi \to \infty} (r_\phi \, d\phi) = dx$

and $\frac{dr}{d\phi} = r_\phi \cos \phi$

Carrying these parameters over into the five equations of equilibrium (A-1.1), we obtain:

$$\frac{\partial(r\,Nx)}{\partial x} + \frac{\partial \bar{N}x\theta}{\partial \theta} - N\theta \frac{\partial r}{\partial x} - \frac{1}{2}\frac{\partial}{\partial \theta}\left(\frac{\bar{M}_{x\theta}}{r_\theta}\right) = 0 \qquad \text{(a)}$$

$$\frac{\partial(r\,\bar{N}_{x\theta})}{\partial x} + \frac{\partial N\theta}{\partial \theta} + \bar{N}x\theta \frac{\partial r}{\partial x} + \frac{r}{r_\theta} Q_\theta + \frac{r}{2}\frac{\partial}{\partial x}\left(\frac{\bar{M}_{x\theta}}{r_\theta}\right) = 0 \qquad \text{(b)}$$

$$\frac{\partial(r\,Q_x)}{\partial x} + \frac{\partial Q_\theta}{\partial \theta} - \frac{N_\theta}{r_\theta} r = 0 \qquad \text{(c)}$$

(A-1.5)

$$\frac{\partial(r\,M_x)}{\partial x} + \frac{\partial \bar{M}_{x\theta}}{\partial x} - M_\theta \frac{\partial r}{\partial x} - r\,Q_x = 0 \qquad \text{(d)}$$

$$\frac{\partial(r\,\bar{M}_{x\theta})}{\partial x} + \frac{\partial M_\theta}{\partial \theta} + \bar{M}_{x\theta} \frac{\partial r}{\partial x} - r\,Q_\theta = 0 \qquad \text{(e)}$$

APPENDIX A-2

EQUATIONS OF MOTION

This appendix contains the equations of motion for a thin conical shell, mentionned in reference to different chapters of this analysis.

$$P_{11}\left[x \sin\alpha \frac{\partial^2 U}{\partial x^2} + \sin\alpha \frac{\partial U}{\partial x}\right] + P_{12}\left[\frac{\partial^2 V}{\partial x\,\partial\theta} + \sin\alpha \frac{\partial U}{\partial x} + \cos\alpha \frac{\partial W}{\partial x}\right]$$

$$- P_{14}\left[x \sin\alpha \frac{\partial^3 W}{\partial x^3} + \sin\alpha \frac{\partial^2 W}{\partial x^2}\right] + P_{15}\left[\frac{-\cot\alpha}{x}\frac{\partial V}{\partial\theta} + \frac{\cot\alpha}{x}\frac{\partial^2 V}{\partial x\,\partial\theta} + \frac{1}{x^2 \sin\alpha}\right.$$

$$\cdot \frac{\partial^2 W}{\partial\theta^2} - \frac{1}{x\sin\alpha}\frac{\partial^3 W}{\partial x\,\partial\theta^2} - \sin\alpha\frac{\partial^2 W}{\partial x^2}\Big] + P_{33}\left[\frac{\partial^2 V}{\partial x\,\partial\theta} + \frac{1}{x\sin\alpha}\frac{\partial^2 U}{\partial\theta^2} - \frac{1}{x}\frac{\partial V}{\partial\theta}\right]$$

$$- P_{21}\sin\alpha\frac{\partial U}{\partial x} + P_{24}\sin\alpha\frac{\partial^2 W}{\partial x^2} + P_{36}\left[-\frac{2}{x\sin\alpha}\frac{\partial^3 W}{\partial x\,\partial\theta^2} + \frac{3\cot\alpha}{x}\frac{\partial^2 V}{\partial x\,\partial\theta}\right.$$

$$- \frac{\cot\alpha}{2x^2\sin\alpha}\frac{\partial^2 U}{\partial\theta^2} + \frac{2}{x^2\sin\alpha}\frac{\partial^2 W}{\partial\theta^2} - \frac{3\cot\alpha}{2x^2}\frac{\partial V}{\partial\theta}\Big] - P_{22}\left[\frac{1}{x}\frac{\partial V}{\partial\theta} + \frac{\sin\alpha}{x}U + \frac{\cos\alpha}{x}W\right]$$

$$+ P_{25}\left[-\frac{\cot\alpha}{x^2}\frac{\partial V}{\partial\theta} + \frac{1}{x^2\sin\alpha}\frac{\partial^2 W}{\partial\theta^2} + \frac{\sin\alpha}{x}\frac{\partial W}{\partial x}\right] - P_{63}\left[\frac{\cot\alpha}{2x}\frac{\partial^2 V}{\partial x\,\partial\theta} + \frac{\cot\alpha}{2x^2\sin\alpha}\right.$$

$$\cdot \frac{\partial^2 U}{\partial\theta^2} + \frac{\cot\alpha}{2x^2}\frac{\partial V}{\partial\theta}\Big] + P_{66}\left[\frac{\cot\alpha}{x^2\sin\alpha}\frac{\partial^3 W}{\partial x\,\partial\theta^2} - \frac{3\cot^2\alpha}{4x^2}\frac{\partial^2 V}{\partial x\,\partial\theta} + \frac{\cot^2\alpha}{4x^3\sin\alpha}\frac{\partial^2 U}{\partial\theta^2}\right.$$

$$\left. - \frac{\cot\alpha}{x^3\sin\alpha}\frac{\partial^2 W}{\partial\theta^2} + \frac{3\cot^2\alpha}{4x^3}\frac{\partial V}{\partial\theta}\right] = 0$$

$$P_{21}\frac{\partial^2 U}{\partial\theta\,\partial x} - P_{24}\frac{\partial^3 W}{\partial\theta\,\partial x^2} + P_{22}\left[\frac{1}{x\sin\alpha}\frac{\partial^2 V}{\partial\theta^2} + \frac{1}{x}\frac{\partial U}{\partial\theta} + \frac{\cot\alpha}{x}\frac{\partial W}{\partial\theta}\right] + P_{25}\left[\frac{\cot\alpha}{x^2\sin\alpha}\right.$$

$$\cdot \frac{\partial^2 V}{\partial\theta^2} - \frac{1}{x^2\sin^2\alpha}\frac{\partial^2 W}{\partial\theta^2} - \frac{1}{x}\frac{\partial^2 W}{\partial\theta\,\partial x}\Big] + P_{36}\left[-2\frac{\partial^3 W}{\partial\theta\,\partial x^2} + \frac{3\cos\alpha}{2}\frac{\partial^2 V}{\partial x^2} - \frac{\cot\alpha}{2x}\right.$$

$$\cdot \frac{\partial^2 U}{\partial\theta\,\partial x} - \frac{4}{x^2}\frac{\partial W}{\partial\theta} - \frac{3\cos\alpha}{2x}\frac{\partial V}{\partial x}\Big] + P_{51}\frac{\cot\alpha}{x}\frac{\partial^2 U}{\partial\theta\,\partial x} + P_{52}\left[\frac{\cot\alpha}{x^2\sin\alpha}\frac{\partial^2 V}{\partial\theta^2}\right.$$

$$+ \cot\alpha \frac{\partial U}{\partial \theta}] + B_{33} [x^2 \sin\alpha \frac{\partial^2 V}{\partial x^2} + x \frac{\partial^2 U}{\partial \theta\, \partial x} + 2x \sin\alpha \frac{\partial V}{\partial x} + 2 \frac{\partial U}{\partial \theta} - 2 \sin\alpha\, V]$$

$$+ B_{36} [-2x^2 \frac{\partial^3 W}{\partial \theta\, \partial x^2} + 3x^2 \cos\alpha \frac{\partial^2 V}{\partial x^2} + x \cot\alpha \frac{\partial^2 U}{\partial \theta\, \partial x} - 4x \frac{\partial^2 W}{\partial \theta\, \partial x} + 6x \cos\alpha$$

$$\cdot \frac{\partial V}{\partial x} + 2 \cot\alpha \frac{\partial U}{\partial \theta} + 4 \frac{\partial W}{\partial \theta} - 6 \cos\alpha\, V] - B_{45}\, x^2 \cot\alpha \frac{\partial^3 W}{\partial \theta\, \partial x^2} + B_{55} [\frac{\cot^2\alpha}{\sin\alpha}$$

$$\cdot \frac{\partial^2 V}{\partial \theta^2} - \frac{\cot\alpha}{\sin^2\alpha} \frac{\partial^3 W}{\partial \theta^3} - x \cot\alpha \frac{\partial^2 W}{\partial \theta\, \partial x}] + B_{66} [-6x \cot\alpha \frac{\partial^2 W}{\partial \theta\, \partial x} - 3x^2 \cot\alpha \frac{\partial^3 W}{\partial \theta\, \partial x^2}$$

$$+ \frac{9x}{2} \cos\alpha \cot\alpha \frac{\partial V}{\partial x} + \frac{9x^2}{4} \cos\alpha \cot\alpha \frac{\partial^2 V}{\partial x^2} - \frac{3}{2} \cot^2\alpha \frac{\partial U}{\partial \theta} - \frac{3x}{4} \cot^2\alpha \frac{\partial^2 U}{\partial \theta\, \partial x}$$

$$+ 6 \cot\alpha \frac{\partial W}{\partial \theta} - \frac{9}{2} \cos\alpha \cot\alpha\, V] = 0$$

$$-B_{12}\, x \cos\alpha \frac{\partial U}{\partial x} + B_{14} [6x \sin\alpha \frac{\partial U}{\partial x} + 6x^2 \sin\alpha \frac{\partial^2 U}{\partial x^2} + x^3 \sin\alpha \frac{\partial^3 U}{\partial x^3}]$$

$$+ B_{15} [\frac{x}{\sin\alpha} \frac{\partial^3 U}{\partial \theta^2\, \partial x} - 2x \sin\alpha \frac{\partial U}{\partial x} - x^2 \sin\alpha \frac{\partial^2 U}{\partial x^2}] - B_{22} [\cot\alpha \frac{\partial V}{\partial \theta} + \cos\alpha\, U$$

$$+ \cos\alpha \cot\alpha\, W] + B_{24} [4x \frac{\partial^2 V}{\partial \theta\, \partial x} + x^2 \frac{\partial^3 V}{\partial \theta\, \partial x^2} + 4x \sin\alpha \frac{\partial U}{\partial x} + x^2 \sin\alpha \frac{\partial^2 U}{\partial x^2}$$

$$+ 4x \cos\alpha \frac{\partial W}{\partial x} + 2x^2 \cos\alpha \frac{\partial^2 W}{\partial x^2} + 2 \frac{\partial V}{\partial \theta} + 2 \sin\alpha\, U + 2 \cos\alpha\, W]$$

$$+ B_{25} [-\cot^2\alpha \frac{\partial V}{\partial \theta} + \frac{1}{\sin^2\alpha} \frac{\partial^3 V}{\partial \theta^3} + \frac{1}{\sin\alpha} \frac{\partial^2 U}{\partial \theta^2} + 2 \frac{\cot\alpha}{\sin\alpha} \frac{\partial^2 W}{\partial \theta^2} - \frac{\partial V}{\partial \theta} - x \frac{\partial^2 V}{\partial \theta\, \partial x}$$

$$- \sin\alpha\, U - x \sin\alpha \frac{\partial U}{\partial x} - \cos\alpha\, W] + B_{36} [6x \frac{\partial^2 V}{\partial \theta\, \partial x} + 2x^2 \frac{\partial^3 V}{\partial \theta\, \partial x^2} + \frac{4}{\sin\alpha} \frac{\partial^2 U}{\partial \theta^2}$$

$$+ \frac{2x}{\sin\alpha} \frac{\partial^3 U}{\partial \theta^2\, \partial x} - 4 \frac{\partial V}{\partial \theta} - 2x \frac{\partial^2 V}{\partial \theta\, \partial x}] - B_{44} [12x^2 \sin\alpha \frac{\partial^2 W}{\partial x^2} + 8x^3 \sin\alpha \frac{\partial^3 W}{\partial x^3}$$

$$+ x^4 \sin\alpha \frac{\partial^4 W}{\partial x^4}] + B_{45} [4x \cot\alpha \frac{\partial^2 V}{\partial \theta\, \partial x} + x^2 \cot\alpha \frac{\partial^3 V}{\partial \theta\, \partial x^2} - \frac{4x}{\sin\alpha} \frac{\partial^3 W}{\partial \theta^2\, \partial x}$$

$$- \frac{2x^2}{\sin\alpha} \frac{\partial^4 W}{\partial \theta^2\, \partial x^2} - 6x \sin\alpha \frac{\partial W}{\partial x} + 2 \cot\alpha \frac{\partial V}{\partial \theta} - \frac{2}{\sin\alpha} \frac{\partial^2 W}{\partial \theta^2} - 3x^2 \sin\alpha \frac{\partial^2 W}{\partial x^2}]$$

$$\cdot \frac{\partial^4 W}{\partial \theta^4} - \frac{1}{x \sin \alpha} \frac{\partial^3 W}{\partial \theta^2 \partial x} + \frac{2 \cot \alpha}{x^3} \frac{\partial V}{\partial \theta} - \frac{\cot \alpha}{x^2} \frac{\partial^2 V}{\partial \theta \partial x} - \frac{2}{x^3 \sin \alpha} \frac{\partial^2 W}{\partial \theta^2}$$

$$+ \frac{1}{x^2 \sin \alpha} \frac{\partial^3 W}{\partial \theta^2 \partial x} - \frac{\sin \alpha}{x^2} \frac{\partial W}{\partial x} + \frac{\sin \alpha}{x} \frac{\partial^2 W}{\partial x^2}] - P_{21} \cos \alpha \frac{\partial U}{\partial x} - P_{22} [\frac{\cot \alpha}{x} \frac{\partial V}{\partial \theta}$$

$$+ \frac{\cos \alpha}{x} U + \frac{\cot \alpha \cos \alpha}{x} W] + P_{24} \cos \alpha \frac{\partial^2 W}{\partial x^2} + P_{25} [- \frac{\cot^2 \alpha}{x^2} \frac{\partial V}{\partial \theta}$$

$$+ \frac{\cot \alpha}{x^2 \sin \alpha} \frac{\partial^2 W}{\partial \theta^2} + \frac{\cos \alpha}{x} \frac{\partial W}{\partial x}] = 0$$

b) A conical shell with variable thickness (2.12):

$$B_{11} [2x \sin\alpha \frac{\partial U}{\partial x} + x^2 \sin \alpha \frac{\partial^2 U}{\partial x^2}] + B_{12} [x \frac{\partial^2 V}{\partial \theta \partial x} + x \cos \alpha \frac{\partial W}{\partial x} + \sin \alpha U$$

$$+ \cos \alpha W] - B_{14} [3x^2 \sin \alpha \frac{\partial^2 W}{\partial x^2} + x^3 \sin \alpha \frac{\partial^3 W}{\partial x^3}] + B_{15} [x \cot \alpha \frac{\partial^2 V}{\partial \theta \partial x}$$

$$- \frac{x}{\sin \alpha} \frac{\partial^3 W}{\partial \theta^2 \partial x} - 2x \sin \alpha \frac{\partial W}{\partial x} - x^2 \sin \alpha \frac{\partial^2 W}{\partial x^2} + \cot\alpha \frac{\partial V}{\partial \theta} - \frac{1}{\sin \alpha} \frac{\partial^2 W}{\partial \theta^2}]$$

$$- B_{22} [\frac{\partial V}{\partial \theta} + \sin \alpha U + \cos \alpha W] + B_{24} [x^2 \sin \alpha \frac{\partial^2 W}{\partial x^2}] + B_{25} [-\cot \alpha \frac{\partial V}{\partial \theta}$$

$$+ \frac{1}{\sin \alpha} \frac{\partial^2 W}{\partial \theta^2} + x \sin \alpha \frac{\partial W}{\partial x}] + B_{33} [x \frac{\partial^2 V}{\partial \theta \partial x} + \frac{1}{\sin \alpha} \frac{\partial^2 U}{\partial \theta^2} - \frac{\partial V}{\partial \theta}] + B_{36} [- \frac{2x}{\sin \alpha}$$

$$\cdot \frac{\partial^3 W}{\partial \theta^2 \partial x} + x \cot \alpha \frac{\partial^2 V}{\partial \theta \partial x} - \frac{\cot \alpha}{\sin \alpha} \frac{\partial^2 U}{\partial \theta^2} + \frac{2}{\sin \alpha} \frac{\partial^2 W}{\partial \theta^2} - \cot \alpha \frac{\partial V}{\partial \theta}]$$

$$+ B_{66} [x \frac{\cot \alpha}{\sin \alpha} \frac{\partial^3 W}{\partial \theta^2 \partial x} - \frac{3x}{4} \cot^2 \alpha \frac{\partial^2 V}{\partial \theta \partial x} + \frac{\cot^2 \alpha}{4 \sin \alpha} \frac{\partial^2 U}{\partial \theta^2} - \frac{\cot \alpha}{\sin \alpha} \frac{\partial^2 W}{\partial \theta^2} + \frac{3 \cot^2 \alpha}{4}$$

$$\cdot \frac{\partial V}{\partial \theta}] = 0$$

$$B_{12} x \frac{\partial^2 U}{\partial \theta \partial x} + B_{15} x \cot \alpha \frac{\partial^2 U}{\partial \theta \partial x} + B_{22} [\frac{1}{\sin \alpha} \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial U}{\partial \theta} + \cot \alpha \frac{\partial W}{\partial \theta}]$$

$$- B_{24} x^2 \frac{\partial^3 W}{\partial \theta \partial x^2} + B_{25} [\frac{2 \cot \alpha}{\sin \alpha} \frac{\partial^2 V}{\partial \theta^2} - \frac{1}{\sin^2 \alpha} \frac{\partial^3 W}{\partial \theta^3} - x \frac{\partial^2 W}{\partial \theta \partial x} + \cot^2\alpha \frac{\partial W}{\partial \theta}$$

$$+ \cot\alpha \frac{\partial U}{\partial \theta}] + B_{33} [x^2 \sin\alpha \frac{\partial^2 V}{\partial x^2} + x \frac{\partial^2 U}{\partial \theta\, \partial x} + 2x \sin\alpha \frac{\partial V}{\partial x} + 2 \frac{\partial U}{\partial \theta} - 2 \sin\alpha\, V]$$

$$+ B_{36} [-2x^2 \frac{\partial^3 W}{\partial \theta\, \partial x^2} + 3x^2 \cos\alpha \frac{\partial^2 V}{\partial x^2} + x \cot\alpha \frac{\partial^2 U}{\partial \theta\, \partial x} - 4x \frac{\partial^2 W}{\partial \theta\, \partial x} + 6x \cos\alpha$$

$$\cdot \frac{\partial V}{\partial x} + 2 \cot\alpha \frac{\partial U}{\partial \theta} + 4 \frac{\partial W}{\partial \theta} - 6 \cos\alpha\, V] - B_{45}\, x^2 \cot\alpha \frac{\partial^3 W}{\partial \theta\, \partial x^2} + B_{55} [\frac{\cot^2\alpha}{\sin\alpha}$$

$$\cdot \frac{\partial^2 V}{\partial \theta^2} - \frac{\cot\alpha}{\sin^2\alpha} \frac{\partial^3 W}{\partial \theta^3} - x \cot\alpha \frac{\partial^2 W}{\partial \theta\, \partial x}] + B_{66} [-6x \cot\alpha \frac{\partial^2 W}{\partial \theta\, \partial x} - 3x^2 \cot\alpha \frac{\partial^3 W}{\partial \theta\, \partial x^2}$$

$$+ \frac{9x}{2} \cos\alpha \cot\alpha \frac{\partial V}{\partial x} + \frac{9x^2}{4} \cos\alpha \cot\alpha \frac{\partial^2 V}{\partial x^2} - \frac{3}{2} \cot^2\alpha \frac{\partial U}{\partial \theta} - \frac{3x}{4} \cot^2\alpha \frac{\partial^2 U}{\partial \theta\, \partial x}$$

$$+ 6 \cot\alpha \frac{\partial W}{\partial \theta} - \frac{9}{2} \cos\alpha \cot\alpha\, V] = 0$$

$$-B_{12}\, x \cos\alpha \frac{\partial U}{\partial x} + B_{14} [6x \sin\alpha \frac{\partial U}{\partial x} + 6x^2 \sin\alpha \frac{\partial^2 U}{\partial x^2} + x^3 \sin\alpha \frac{\partial^3 U}{\partial x^3}]$$

$$+ B_{15} [\frac{x}{\sin\alpha} \frac{\partial^3 U}{\partial \theta^2\, \partial x} - 2x \sin\alpha \frac{\partial U}{\partial x} - x^2 \sin\alpha \frac{\partial^2 U}{\partial x^2}] - B_{22} [\cot\alpha \frac{\partial V}{\partial \theta} + \cos\alpha\, U$$

$$+ \cos\alpha \cot\alpha\, W] + B_{24} [4x \frac{\partial^2 V}{\partial \theta\, \partial x} + x^2 \frac{\partial^3 V}{\partial \theta\, \partial x^2} + 4x \sin\alpha \frac{\partial U}{\partial x} + x^2 \sin\alpha \frac{\partial^2 U}{\partial x^2}$$

$$+ 4x \cos\alpha \frac{\partial W}{\partial x} + 2x^2 \cos\alpha \frac{\partial^2 W}{\partial x^2} + 2 \frac{\partial V}{\partial \theta} + 2 \sin\alpha\, U + 2 \cos\alpha\, W]$$

$$+ B_{25} [-\cot^2\alpha \frac{\partial V}{\partial \theta} + \frac{1}{\sin^2\alpha} \frac{\partial^3 V}{\partial \theta^3} + \frac{1}{\sin\alpha} \frac{\partial^2 U}{\partial \theta^2} + 2 \frac{\cot\alpha}{\sin\alpha} \frac{\partial^2 W}{\partial \theta^2} - \frac{\partial V}{\partial \theta} - x \frac{\partial^2 V}{\partial \theta\, \partial x}$$

$$- \sin\alpha\, U - x \sin\alpha \frac{\partial U}{\partial x} - \cos\alpha\, W] + B_{36} [6x \frac{\partial^2 V}{\partial \theta\, \partial x} + 2x^2 \frac{\partial^3 V}{\partial \theta\, \partial x^2} + \frac{4}{\sin\alpha} \frac{\partial^2 U}{\partial \theta^2}$$

$$+ \frac{2x}{\sin\alpha} \frac{\partial^3 U}{\partial \theta^2\, \partial x} - 4 \frac{\partial V}{\partial \theta} - 2x \frac{\partial^2 V}{\partial \theta\, \partial x}] - B_{44} [12x^2 \sin\alpha \frac{\partial^2 W}{\partial x^2} + 8x^3 \sin\alpha \frac{\partial^3 W}{\partial x^3}$$

$$+ x^4 \sin\alpha \frac{\partial^4 W}{\partial x^4}] + B_{45} [4x \cot\alpha \frac{\partial^2 V}{\partial \theta\, \partial x} + x^2 \cot\alpha \frac{\partial^3 V}{\partial \theta\, \partial x^2} - \frac{4x}{\sin\alpha} \frac{\partial^3 W}{\partial \theta^2\, \partial x}$$

$$- \frac{2x^2}{\sin\alpha} \frac{\partial^4 W}{\partial \theta^2\, \partial x^2} - 6x \sin\alpha \frac{\partial W}{\partial x} + 2 \cot\alpha \frac{\partial V}{\partial \theta} - \frac{2}{\sin\alpha} \frac{\partial^2 W}{\partial \theta^2} - 3x^2 \sin\alpha \frac{\partial^2 W}{\partial x^2}]$$

$$- B_{55} \left[\cot\alpha \frac{\partial V}{\partial\theta} + x \cot\alpha \frac{\partial^2 V}{\partial\theta\, \partial x} - \frac{1}{\sin\alpha} \frac{\partial^2 W}{\partial\theta^2} - 2x \sin\alpha \frac{\partial W}{\partial x} - x^2 \sin\alpha \frac{\partial^2 W}{\partial x^2}\right.$$

$$\left. - \frac{\cot\alpha}{\sin^2\alpha} \frac{\partial^3 V}{\partial\theta^3} + \frac{1}{\sin^3\alpha} \frac{\partial^4 W}{\partial\theta^4}\right] + B_{66} \left[- \frac{8x}{\sin\alpha} \frac{\partial^3 W}{\partial\theta^2\, \partial x} - \frac{4x^2}{\sin\alpha} \frac{\partial^4 W}{\partial\theta^2\, \partial x^2}\right.$$

$$+ 6x \cot\alpha \frac{\partial^2 V}{\partial\theta\, \partial x} + 3x^2 \cot\alpha \frac{\partial^3 V}{\partial\theta\, \partial x^2} - \frac{2 \cot\alpha}{\sin\alpha} \frac{\partial^2 U}{\partial\theta^2} - \frac{x \cot\alpha}{\sin\alpha} \frac{\partial^3 U}{\partial\theta^2\, \partial x}$$

$$\left. + \frac{8}{\sin\alpha} \frac{\partial^2 W}{\partial\theta^2} - 6 \cot\alpha \frac{\partial V}{\partial\theta}\right] = 0$$

APPENDIX A-3

This appendix contains the matrices cited in connection with reference articles in the course of the analytical developments.

These matrices are classified as follows:

[D]	(table 1)
[R]; [Rr]; [T]	(table 2)
[Q]; [Qr];	(table 3)

The symbols r, i, j assigned to the matrices correspond, respectively, to the following cases: The roots of the characteristic equations are real-complex; coordinate $x = x_i$; coordinate $x = x_j$. There could be combinations of these different symbols.

Case of complex roots:

$$\lambda_{1,2} = \kappa_1 \pm i\mu_1 \quad ; \lambda_{3,4} = -\kappa_1 \pm i\mu_1$$

$$\lambda_{5,6} = \kappa_2 \pm i\mu_2 \quad ; \lambda_{7,8} = -\kappa_2 \pm i\mu_2$$

Case of real and complex roots:

$$\lambda_{1,2} = \kappa_1 \pm i\mu_1 \quad ; \lambda_{3,4} = -\kappa_1 \pm i\mu_1$$

$$\lambda_{5,6} = \pm a_1 \quad ; \lambda_{7,8} = \pm a_2$$

The quantities $\overline{\alpha}_p$, $\overline{\beta}_p$, p = 1,2...,8 are real or imaginary parts of α_p and β_p determined by equations (3.9) such that:

$$\alpha_{i,j} = \overline{\alpha}_i \pm i\overline{\alpha}_j \;;\; i = 1,3,5,7 \qquad j = 2,4,6,8$$

and the same applies to β_p.

$y_i = \sqrt{\frac{x_i}{\ell}}$ and $y_j = \sqrt{\frac{x_j}{\ell}}$ correspond to the values taken by the $y = \sqrt{\frac{x}{\ell}}$ variable at boundaries $x = x_i$ and $x = x_j$ respectively.

TABLE 1

Matrix $[D]_{3\times3}$

$$[D]_{3\times3} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \{0\} \qquad ; \; [D] = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

with

$$d_{11} = A\lambda^2 + C$$

$$d_{12} = R\lambda + T$$

$$d_{13} = F\lambda^3 + G\lambda^2 + H\lambda + J$$

$$d_{21} = -R\lambda + T$$

$$d_{22} = Y\lambda^2 + L$$

$$d_{23} = M\lambda^2 + N\lambda + P$$

$$d_{31} = -F\lambda^3 + G\lambda^2 - H\lambda + J$$

$$d_{32} = M\lambda^2 - N\lambda + P$$

$$d_{33} = Q\lambda^4 + S\lambda^2 + Z$$

and

$$A = \frac{B_{11}}{4} \sin\alpha$$

$$C = B_{12} \sin\alpha + n^2 B_{36} \frac{\cot\alpha}{\sin\alpha} - \frac{n^2 B_{33}}{\sin\alpha} - B_{22} \sin\alpha - n^2 B_{66} \frac{\cot^2\alpha}{4\sin\alpha} - \frac{B_{11}}{4} \sin\alpha$$

$$R = \frac{n}{2} B_{12} + \frac{n}{2} B_{15} \cot\alpha + \frac{n}{2} B_{33} + \frac{n}{2} B_{36} \cot\alpha - \frac{3n}{8} B_{66} \cot^2\alpha$$

$$T = \frac{n}{2} B_{12} + \frac{n}{2} B_{15} \cot \alpha - n B_{22} - n B_{25} \cot \alpha - \frac{3n}{2} B_{33} - \frac{3n}{2} B_{36} \cot \alpha + \frac{9n}{8} B_{66} \cot^2 \alpha$$

$$F = -\frac{B_{41}}{8} \sin \alpha$$

$$G = \frac{3}{8} B_{41} \sin \alpha - \frac{B_{51}}{4} \sin \alpha + \frac{B_{42}}{4} \sin \alpha$$

$$H = \frac{B_{21}}{2} \cos \alpha + \frac{n^2 B_{51}}{2 \sin \alpha} + \frac{n^2 B_{63}}{\sin \alpha} + \frac{B_{52}}{2} \sin \alpha - \frac{n^2 B_{66} \cot \alpha}{2 \sin \alpha} - B_{42} \sin \alpha + \frac{B_{41}}{8} \sin \alpha$$

$$J = \frac{B_{12}}{2} \cos \alpha + \frac{n^2 B_{51}}{2 \sin \alpha} + B_{51} \frac{\sin \alpha}{4} - B_{22} \cos \alpha - \frac{n^2 B_{52}}{\sin \alpha} - B_{52} \frac{\sin \alpha}{2} - \frac{3}{8} B_{41} \sin \alpha + \frac{3}{4} B_{42} \sin \alpha - \frac{3 n^2}{\sin \alpha} B_{63} + 3n^2 B_{66} \frac{\cot \alpha}{2 \sin \alpha}$$

$$Y = \frac{B_{33}}{4} \sin \alpha + \frac{3}{4} B_{36} \sin \alpha + \frac{9}{16} B_{66} \cos \alpha \cot \alpha$$

$$L = \frac{-9}{4} B_{33} \sin \alpha - \frac{27}{4} B_{36} \cos \alpha - \frac{81}{16} B_{66} \cos \alpha \cot \alpha - 2n^2 B_{52} \frac{\cot \alpha}{\sin \alpha} - n^2 B_{55} \frac{\cot^2 \alpha}{\sin \alpha} - \frac{n^2 B_{22}}{\sin \alpha}$$

$$M = \frac{n}{4} B_{24} + \frac{n}{2} B_{36} + \frac{n}{4} B_{54} \cot \alpha + \frac{3n}{4} B_{66} \cot \alpha$$

$$N = -n B_{24} + \frac{n}{2} B_{25} - n B_{54} \cot \alpha + \frac{n}{2} B_{55} \cot \alpha$$

$$P = -n B_{22} \cot \alpha - \frac{n^3 B_{25}}{\sin^2 \alpha} - \frac{n}{2} B_{25} - n B_{52} \cot^2 \alpha + \frac{3n}{4} B_{24} - \frac{9n}{2} B_{36} + \frac{3n}{4} B_{54} \cot \alpha - n^3 B_{55} \frac{\cot \alpha}{\sin^2 \alpha} - \frac{n}{2} B_{55} \cot \alpha - \frac{27n}{4} B_{66} \cot \alpha$$

$$Q = -B_{44} \frac{\sin \alpha}{16}$$

$$S = B_{42}\frac{\cos\alpha}{2} - \frac{3}{4}B_{45}\sin\alpha + \frac{n^2 B_{66}}{\sin\alpha} + \frac{n^2 B_{54}}{\sin\alpha} + B_{55}\frac{\sin\alpha}{4} + \frac{5}{8}B_{44}\sin\alpha$$

$$Z = -\frac{9n^2}{\sin\alpha}B_{66} - 2n^2 B_{25}\frac{\cot\alpha}{\sin\alpha} - B_{25}\cos\alpha - \frac{n^2 B_{55}}{\sin\alpha} - \frac{n^4 B_{55}}{\sin^3\alpha} - \frac{B_{55}}{4}\sin\alpha$$

$$- B_{22}\cos\alpha\cot\alpha + \frac{3}{2}B_{42}\cos\alpha + \frac{3n^2}{2\sin\alpha}B_{54} - \frac{9}{16}B_{44}\sin\alpha + \frac{3}{4}B_{45}\sin\alpha$$

Characteristic equation (3.5) is in the form:

$$h_8\lambda^8 + h_6\lambda^6 + h_4\lambda^4 + h_2\lambda^2 + h_0 = 0$$

where

$$h_8 = AYQ + F^2Y$$

$$h_6 = ALQ + CYQ + AYS - AM^2 + R^2Q - 2RFM - G^2Y + 2HFY + F^2L$$

$$h_4 = CLQ + ALS + CYS + AYZ - 2AMP + AN^2 - M^2C - T^2Q + R^2S - 2RFP - 2TFN$$
$$+ 2RGN + 2TGM - 2RHM - 2GJY + H^2Y - G^2L + 2HFL$$

$$h_2 = CLS + ALZ + CYZ - AP^2 - 2CMP + CN^2 - T^2S + R^2Z + 2RJN + 2TJM - 2RHP$$
$$- 2THN + 2TGP - J^2Y - 2GJL + H^2L$$

$$h_0 = CLZ - CP^2 - T^2Z - J^2L + 2TJP$$

The particular case of isotropic material in this case, coefficients d_{ij} of matrix [D] and coefficients h_j of the characteristic equation are the same form as previously (anisotropic material). The coefficients that change form are the following:

$$A = \frac{\sin\alpha}{4}$$

$$C = -\frac{\sin\alpha}{4}(5 - 4\nu) - \frac{n^2(1-\nu)}{2\sin\alpha}[1 + \frac{k}{4}\cot^2\alpha]$$

$$R = \frac{n}{4}\left[(1+\nu) - \frac{3}{4}(1-\nu)\, k \cot^2 \alpha\right]$$

$$T = \frac{n}{4}\left[(5\nu - 7) + \frac{9}{4}(1-\nu)\, k \cot^2 \alpha\right]$$

$$F = G = 0$$

$$H = \frac{\cos \alpha}{2}\left[\nu - \frac{n^2 k(1-\nu)}{2 \sin^2 \alpha}\right]$$

$$J = \frac{\cos \alpha}{2}\left[(\nu - 2) + \frac{3 n^2 k (1-\nu)}{2 \sin^2 \alpha}\right]$$

$$Y = \frac{(1-\nu)}{8} \sin \alpha \left[1 + \frac{9k}{4} \cot^2 \alpha\right]$$

$$L = \frac{9}{8}(\nu - 1) \sin \alpha\left[1 + \frac{9k}{4}\cot^2 \alpha\right] - \frac{n^2}{\sin \alpha}(1 + k \cot^2 \alpha)$$

$$M = \frac{(3-\nu)}{8}\, n\, k \cot \alpha$$

$$N = \frac{(1-2\nu)}{2}\, n\, k \cot \alpha$$

$$P = -n \cot \alpha \left[\frac{k}{8}(31 - 33\nu) + \frac{n^2 k}{\sin^2 \alpha} + 1\right]$$

$$Q = \frac{-k \sin \alpha}{16}$$

$$S = k\left[\frac{(7-6\nu)}{8} \sin \alpha + \frac{n^2}{2 \sin \alpha}\right]$$

$$Z = \frac{k}{\sin \alpha}\left[\frac{(12\nu - 13)}{16} \sin^2 \alpha + \frac{(12\nu - 11)}{2} n^2 - \frac{n^4}{\sin^2 \alpha}\right] - \frac{\cos^2 \alpha}{\sin \alpha}$$

Stiffness:

As we set $t = \delta x$ for an isotropic shell, we have:

$$k = \frac{\delta^2}{12} = \frac{K}{D}$$

with $D = \frac{E\delta}{(1-\nu^2)} x$ membrane stiffness

$$K = \frac{E\delta^3}{4(1-\nu^2)} x^2 \text{ bending stiffness}$$

where

E is Young's modulus

ν is Poisson's ratio

TABLE 2

Matrix $\underset{3\times 8}{[R]}$

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix}_C = \sum_{n=0}^{\infty} \underset{3\times 3}{[T]} \underset{3\times 8}{[R]} \underset{8\times 1}{\begin{Bmatrix} \bar{C}_1 \\ \bar{C}_2 \\ \vdots \\ \bar{C}_8 \end{Bmatrix}} \quad \text{with } [T] = \begin{bmatrix} \cos n\theta & 0 & 0 \\ 0 & \cos n\theta & 0 \\ 0 & 0 & \sin n\theta \end{bmatrix}$$

$$R(1,1) = y^{\kappa_1 - 1} [\bar{\alpha}_1 \cos(\mu_1 \ell n y) - \bar{\alpha}_2 \sin(\mu_1 \ell n y)]$$

$$R(1,2) = y^{\kappa_1 - 1} [\bar{\alpha}_2 \cos(\mu_1 \ell n y) + \bar{\alpha}_1 \sin(\mu_1 \ell n y)]$$

$$R(1,3) = y^{-\kappa_1 - 1} [\bar{\alpha}_3 \cos(\mu_1 \ell n y) - \bar{\alpha}_4 \sin(\mu_1 \ell n y)]$$

$$R(1,4) = y^{-\kappa_1 - 1} [\bar{\alpha}_4 \cos(\mu_1 \ell n y) + \bar{\alpha}_3 \sin(\mu_1 \ell n y)]$$

$$R(1,5) = y^{\kappa_2 - 1} [\bar{\alpha}_5 \cos(\mu_2 \ell n y) - \bar{\alpha}_6 \sin(\mu_2 \ell n y)]$$

$$R(1,6) = y^{\kappa_2 - 1} [\bar{\alpha}_6 \cos(\mu_2 \ell n y) + \bar{\alpha}_5 \sin(\mu_2 \ell n y)]$$

$$R(1,7) = y^{-\kappa_2 - 1} [\bar{\alpha}_7 \cos(\mu_2 \ell n y) - \bar{\alpha}_8 \sin(\mu_2 \ell n y)]$$

$$R(1,8) = y^{-\kappa_2 - 1} [\bar{\alpha}_8 \cos(\mu_2 \ell n y) + \bar{\alpha}_7 \sin(\mu_2 \ell n y)]$$

$$R(2,1) = y^{\kappa_1 - 1} \cos(\mu_1 \ell n y) \quad ; \quad R(2,5) = y^{\kappa_2 - 1} \cos(\mu_2 \ell n y)$$

$$R(2,2) = y^{\kappa_1 - 1} \sin(\mu_1 \ell n y) \quad ; \quad R(2,6) = y^{\kappa_2 - 1} \sin(\mu_2 \ell n y)$$

$$R(2,3) = y^{-\kappa_1 - 1} \cos(\mu_1 \ell n y) \quad ; \quad R(2,7) = y^{-\kappa_2 - 1} \cos(\mu_2 \ell n y)$$

$$R(2,4) = y^{-\kappa_1 - 1} \sin(\mu_1 \ell n y) \quad ; \quad R(2,8) = y^{-\kappa_2 - 1} \sin(\mu_2 \ell n y)$$

$$R(3,1) = y^{\kappa_1 - 1} [\bar{\beta}_1 \cos(\mu_1 \ell n y) - \bar{\beta}_2 \sin(\mu_1 \ell n y)]$$

$$R(3,2) = y^{\kappa_1 - 1} [\bar{\beta}_2 \cos(\mu_1 \ell n y) + \bar{\beta}_1 \sin(\mu_1 \ell n y)]$$

$$R\,(3,3) = y^{-\kappa_1-1}\,[\bar{\beta}_3 \cos(\mu_1 \ln y) - \bar{\beta}_4 \sin(\mu_1 \ln y)]$$

$$R\,(3,4) = y^{-\kappa_1-1}\,[\bar{\beta}_4 \cos(\mu_1 \ln y) + \bar{\beta}_3 \sin(\mu_1 \ln y)]$$

$$R\,(3,5) = y^{\kappa_2-1}\,[\bar{\beta}_5 \cos(\mu_2 \ln y) - \bar{\beta}_6 \sin(\mu_2 \ln y)]$$

$$R\,(3,6) = y^{\kappa_2-1}\,[\bar{\beta}_6 \cos(\mu_2 \ln y) + \bar{\beta}_5 \sin(\mu_2 \ln y)]$$

$$R\,(3,7) = y^{-\kappa_2-1}\,[\bar{\beta}_7 \cos(\mu_2 \ln y) - \bar{\beta}_8 \sin(\mu_2 \ln y)]$$

$$R\,(3,8) = y^{-\kappa_2-1}\,[\bar{\beta}_8 \cos(\mu_2 \ln y) + \bar{\beta}_7 \sin(\mu_2 \ln y)]$$

Matrix $\underset{3x8}{[R_r]}$

$$\begin{Bmatrix} U(x,\theta) \\ W(x,\theta) \\ V(x,\theta) \end{Bmatrix}_C = \sum_{n=0}^{\infty} \underset{3x3}{[T]} \; \underset{3x8}{[R_r]} \; \underset{8x1}{\{\bar{C}\}}$$

$R_r(i, j) = R(i, j)$ for $\begin{cases} i = 1, 2, 3 \\ j = 1, 2, 3, 4 \end{cases}$

$R_r(1, 5) = \alpha_5 \, y^{a_1-1}$

$R_r(1, 6) = \alpha_6 \, y^{a_2-1}$

$R_r(1, 7) = \alpha_7 \, y^{-a_1-1}$

$R_r(1, 8) = \alpha_8 \, y^{-a_2-1}$

$R_r(2, 5) = y^{a_1-1}$

$R_r(2, 6) = y^{a_2-1}$

$R_r(2, 7) = y^{-a_1-1}$

$R_r(2, 8) = y^{-a_2-1}$

$R_r(3, 5) = \beta_5 \, y^{a_1-1}$

$R_r(3, 6) = \beta_6 \, y^{a_2-1}$

$R_r(3, 7) = \beta_7 \, y^{-a_1-1}$

$R_r(3, 8) = \beta_8 \, y^{-a_2-1}$

TABLE 3

Matrix $\underset{6\times8}{[Q]}$

$$\underset{6\times1}{\{\varepsilon\}} = \sum_{n=0}^{\infty} \underset{6\times6}{\begin{bmatrix} [T] & [0] \\ [0] & [T] \end{bmatrix}} \underset{6\times8}{[Q]} \underset{8\times1}{\{\bar{C}\}} \quad ; \quad [T] = \begin{bmatrix} \cos n\theta & 0 & 0 \\ 0 & \cos n\theta & 0 \\ 0 & 0 & \sin n\theta \end{bmatrix}$$

$$Q\,(1,\,1) = \frac{y^{\kappa_1-3}}{2\ell}\,\{a_{11}\cos\mu_1\ell ny - a_{12}\sin\mu_1\ell ny\}$$

$$Q\,(1,\,2) = \frac{y^{\kappa_1-3}}{2\ell}\,\{a_{12}\cos\mu_1\ell ny + a_{11}\sin\mu_1\ell ny\}$$

$$Q\,(1,\,3) = \frac{y^{-\kappa_1-3}}{2\ell}\,\{a_{13}\cos\mu_1\ell ny - a_{14}\sin\mu_1\ell ny\}$$

$$Q\,(1,\,4) = \frac{y^{-\kappa_1-3}}{2\ell}\,\{a_{14}\cos\mu_1\ell ny + a_{13}\sin\mu_1\ell ny\}$$

$$Q\,(1,\,5) = \frac{y^{\kappa_2-3}}{2\ell}\,\{a_{15}\cos\mu_2\ell ny - a_{16}\sin\mu_2\ell ny\}$$

$$Q\,(1,\,6) = \frac{y^{\kappa_2-3}}{2\ell}\,\{a_{16}\cos\mu_2\ell ny + a_{15}\sin\mu_2\ell ny\}$$

$$Q\,(1,\,7) = \frac{y^{-\kappa_2-3}}{2\ell}\,\{a_{17}\cos\mu_2\ell ny - a_{18}\sin\mu_2\ell ny\}$$

$$Q\,(1,\,8) = \frac{y^{-\kappa_2-3}}{2\ell}\,\{a_{18}\cos\mu_2\ell ny + a_{17}\sin\mu_2\ell ny\}$$

$$Q\,(2,\,1) = \frac{y^{\kappa_1-3}}{\ell}\,\{a_{21}\cos\mu_1\ell ny - a_{22}\sin\mu_1\ell ny\}$$

$$Q\,(2,\,2) = \frac{y^{\kappa_1-3}}{\ell}\,\{a_{22}\cos\mu_1\ell ny + a_{21}\sin\mu_1\ell ny\}$$

$$Q\,(2,\,3) = \frac{y^{-\kappa_1-3}}{\ell}\,\{a_{23}\cos\mu_1\ell ny - a_{24}\sin\mu_1\ell ny\}$$

$$Q(2,4) = \frac{y^{-\kappa_1-3}}{\ell}\{a_{24}\cos\mu_1 \ln y + a_{23}\sin\mu_1 \ln y\}$$

$$Q(2,5) = \frac{y^{\kappa_2-3}}{\ell}\{a_{25}\cos\mu_2 \ln y - a_{26}\sin\mu_2 \ln y\}$$

$$Q(2,6) = \frac{y^{\kappa_2-3}}{\ell}\{a_{26}\cos\mu_2 \ln y + a_{25}\sin\mu_2 \ln y\}$$

$$Q(2,7) = \frac{y^{-\kappa_2-3}}{\ell}\{a_{27}\cos\mu_2 \ln y - a_{28}\sin\mu_2 \ln y\}$$

$$Q(2,8) = \frac{y^{-\kappa_2-3}}{\ell}\{a_{28}\cos\mu_2 \ln y + a_{27}\sin\mu_2 \ln y\}$$

$$Q(3,1) = \frac{y^{\kappa_1-3}}{\ell}\{a_{31}\cos\mu_1 \ln y - a_{32}\sin\mu_1 \ln y\}$$

$$Q(3,2) = \frac{y^{\kappa_1-3}}{\ell}\{a_{32}\cos\mu_1 \ln y + a_{31}\sin\mu_1 \ln y\}$$

$$Q(3,3) = \frac{y^{-\kappa_1-3}}{\ell}\{a_{33}\cos\mu_1 \ln y - a_{34}\sin\mu_1 \ln y\}$$

$$Q(3,4) = \frac{y^{-\kappa_1-3}}{\ell}\{a_{34}\cos\mu_1 \ln y + a_{33}\sin\mu_1 \ln y\}$$

$$Q(3,5) = \frac{y^{\kappa_2-3}}{\ell}\{a_{35}\cos\mu_2 \ln y - a_{36}\sin\mu_2 \ln y\}$$

$$Q(3,6) = \frac{y^{\kappa_2-3}}{\ell}\{a_{36}\cos\mu_2 \ln y + a_{35}\sin\mu_2 \ln y\}$$

$$Q(3,7) = \frac{y^{-\kappa_2-3}}{\ell}\{a_{37}\cos\mu_2 \ln y - a_{38}\sin\mu_2 \ln y\}$$

$$Q(3,8) = \frac{y^{-\kappa_2-3}}{\ell}\{a_{38}\cos\mu_2 \ln y + a_{37}\sin\mu_2 \ln y\}$$

$$Q(4,1) = \frac{y^{\kappa_1-5}}{4\ell^2}\{a_{41}\cos\mu_1 \ln y - a_{42}\sin\mu_1 \ln y\}$$

$$Q(4,2) = \frac{y^{\kappa_1-5}}{4\ell^2}\{a_{42}\cos\mu_1 \ln y + a_{41}\sin\mu_1 \ln y\}$$

$$Q(4,3) = \frac{y^{-\kappa_1-5}}{4\ell^2}\{a_{43}\cos\mu_1 \ln y - a_{44}\sin\mu_1 \ln y\}$$

$$Q\ (4,\ 4) = \frac{y^{-\kappa_1 - 5}}{4\ell^2}\ \{a_{44} \cos \mu_1 \ell ny + a_{43} \sin \mu_1 \ell ny\}$$

$$Q\ (4,\ 5) = \frac{y^{\kappa_2 - 5}}{4\ell^2}\ \{a_{45} \cos \mu_2 \ell ny - a_{46} \sin \mu_2 \ell ny\}$$

$$Q\ (4,\ 6) = \frac{y^{\kappa_2 - 5}}{4\ell^2}\ \{a_{46} \cos \mu_2 \ell ny + a_{45} \sin \mu_2 \ell ny\}$$

$$Q\ (4,\ 7) = \frac{y^{-\kappa_2 - 5}}{4\ell^2}\ \{a_{47} \cos \mu_2 \ell ny - a_{48} \sin \mu_2 \ell ny\}$$

$$Q\ (4,\ 8) = \frac{y^{-\kappa_2 - 5}}{4\ell^2}\ \{a_{48} \cos \mu_2 \ell ny + a_{47} \sin \mu_2 \ell ny\}$$

$$Q\ (5,\ 1) = \frac{y^{\kappa_1 - 5}}{\ell^2}\ \{a_{51} \cos \mu_1 \ell ny - a_{52} \sin \mu_1 \ell ny\}$$

$$Q\ (5,\ 2) = \frac{y^{\kappa_1 - 5}}{\ell^2}\ \{a_{52} \cos \mu_1 \ell ny + a_{51} \sin \mu_1 \ell ny\}$$

$$Q\ (5,\ 3) = \frac{y^{-\kappa_1 - 5}}{\ell^2}\ \{a_{53} \cos \mu_1 \ell ny - a_{54} \sin \mu_1 \ell ny\}$$

$$Q\ (5,\ 4) = \frac{y^{-\kappa_1 - 5}}{\ell^2}\ \{a_{54} \cos \mu_1 \ell ny + a_{53} \sin \mu_1 \ell ny\}$$

$$Q\ (5,\ 5) = \frac{y^{\kappa_2 - 5}}{\ell^2}\ \{a_{55} \cos \mu_2 \ell ny - a_{56} \sin \mu_2 \ell ny\}$$

$$Q\ (5,\ 6) = \frac{y^{\kappa_2 - 5}}{\ell^2}\ \{a_{56} \cos \mu_2 \ell ny + a_{55} \sin \mu_2 \ell ny\}$$

$$Q\ (5,\ 7) = \frac{y^{-\kappa_2 - 5}}{\ell^2}\ \{a_{57} \cos \mu_2 \ell ny - a_{58} \sin \mu_2 \ell ny\}$$

$$Q\ (5,\ 8) = \frac{y^{-\kappa_2 - 5}}{\ell^2}\ \{a_{58} \cos \mu_2 \ell ny + a_{57} \sin \mu_2 \ell ny\}$$

$$Q\ (6,\ 1) = \frac{y^{\kappa_1 - 5}}{\ell^2}\ \{a_{61} \cos \mu_1 \ell ny - a_{62} \sin \mu_1 \ell ny\}$$

$$Q\ (6,\ 2) = \frac{y^{\kappa_1 - 5}}{\ell^2}\ \{a_{62} \cos \mu_1 \ell ny + a_{61} \sin \mu_1 \ell ny\}$$

$$Q(6,3) = \frac{y^{-\kappa_1-5}}{\ell^2}\{a_{63}\cos\mu_1 \ell n y - a_{62}\sin\mu_1 \ell n y\}$$

$$Q(6,4) = \frac{y^{-\kappa_1-5}}{\ell^2}\{a_{64}\cos\mu_1 \ell n y + a_{63}\sin\mu_1 \ell n y\}$$

$$Q(6,5) = \frac{y^{\kappa_2-5}}{\ell^2}\{a_{65}\cos\mu_2 \ell n y - a_{66}\sin\mu_2 \ell n y\}$$

$$Q(6,6) = \frac{y^{\kappa_2-5}}{\ell^2}\{a_{66}\cos\mu_2 \ell n y + a_{65}\sin\mu_2 \ell n y\}$$

$$Q(6,7) = \frac{y^{-\kappa_2-5}}{\ell^2}\{a_{67}\cos\mu_2 \ell n y - a_{68}\sin\mu_2 \ell n y\}$$

$$Q(6,8) = \frac{y^{-\kappa_2-5}}{\ell^2}\{a_{68}\cos\mu_2 \ell n y + a_{67}\sin\mu_2 \ell n y\}$$

Matrix $[Q_r]$
6x8

$$\{\varepsilon\}_{6x1} = \sum_{n=0}^{\infty} \begin{bmatrix} [T] & 0 \\ 0 & [T] \end{bmatrix}_{6x6} [Q_r]_{6x8} \{\bar{C}\}_{8x1}$$

$$Q_r(i,j) = Q(i,j) \qquad \text{for } \begin{cases} i = 1, \ldots, 6 \\ j = 1, \ldots, 4 \end{cases}$$

$$Q_r(1,5) = \frac{y^{a_1-3}}{2\ell} a_{15}^* \qquad Q_r(3,5) = \frac{y^{a_1-3}}{\ell} a_{35}^*$$

$$Q_r(1,6) = \frac{y^{a_2-3}}{2\ell} a_{16}^* \qquad Q_r(3,6) = \frac{y^{a_2-3}}{\ell} a_{36}^*$$

$$Q_r(1,7) = \frac{y^{-a_1-3}}{2\ell} a_{17}^* \qquad Q_r(3,7) = \frac{y^{-a_1-3}}{\ell} a_{37}^*$$

$$Q_r(1,8) = \frac{y^{-a_2-3}}{2\ell} a_{18}^* \qquad Q_r(3,8) = \frac{y^{-a_2-3}}{\ell} a_{38}^*$$

$$Q_r(2,5) = \frac{y^{a_1-3}}{\ell} a_{25}^* \qquad Q_r(4,5) = \frac{y^{a_1-5}}{4\ell^2} a_{45}^*$$

$$Q_r(2,6) = \frac{y^{a_2-3}}{\ell} a_{26}^* \qquad Q_r(4,6) = \frac{y^{a_2-5}}{4\ell^2} a_{46}^*$$

$$Q_r(2,7) = \frac{y^{-a_1-3}}{\ell} a_{27}^* \qquad Q_r(4,7) = \frac{y^{-a_1-5}}{4\ell^2} a_{47}^*$$

$$Q_r(2,8) = \frac{y^{-a_2-3}}{\ell} a_{28}^* \qquad Q_r(4,8) = \frac{y^{-a_2-5}}{4\ell^2} a_{48}^*$$

$$a_{51} = \frac{n^2}{\sin^2 \alpha} + \frac{n\,\bar{\beta}_1 \cot \alpha}{\sin \alpha} + \frac{(1-\kappa_1)}{2}$$

$$a_{52} = \frac{n\,\bar{\beta}_2 \cot \alpha}{\sin \alpha} - \frac{\mu_1}{2}$$

$$a_{53} = \frac{n^2}{\sin^2 \alpha} + \frac{n\,\bar{\beta}_3 \cot \alpha}{\sin \alpha} + \frac{(\kappa_1+1)}{2}$$

$$a_{54} = \frac{n\,\bar{\beta}_4 \cot \alpha}{\sin \alpha} - \frac{\mu_1}{2}$$

$$a_{55} = \frac{n^2}{\sin^2 \alpha} + \frac{n\,\bar{\beta}_5 \cot \alpha}{\sin \alpha} + \frac{(1-\kappa_2)}{2}$$

$$a_{56} = \frac{n\,\bar{\beta}_6 \cot \alpha}{\sin \alpha} - \frac{\mu_2}{2}$$

$$a_{57} = \frac{n^2}{\sin^2 \alpha} + \frac{n\,\bar{\beta}_7 \cot \alpha}{\sin \alpha} + \frac{(\kappa_2+1)}{2}$$

$$a_{58} = \frac{n\,\bar{\beta}_8 \cot \alpha}{\sin \alpha} - \frac{\mu_2}{2}$$

$$a_{61} = \frac{n\,\bar{\alpha}_1 \cot \alpha}{2 \sin \alpha} + \frac{3}{4} \cot \alpha \, [\bar{\beta}_1 \, (\kappa_1-3) - \bar{\beta}_2 \, \mu_1] + \frac{n\,(\kappa_1-3)}{\sin \alpha}$$

$$a_{62} = \frac{n\,\bar{\alpha}_2 \cot \alpha}{2 \sin \alpha} + \frac{3}{4} \cot \alpha \, [\bar{\beta}_2 \, (\kappa_1-3) + \bar{\beta}_1 \, \mu_1] + \frac{n\,\mu_1}{\sin \alpha}$$

$$a_{63} = \frac{n\,\bar{\alpha}_3 \cot \alpha}{2 \sin \alpha} - \frac{3}{4} \cot \alpha \, [\bar{\beta}_3 \, (\kappa_1+3) + \bar{\beta}_4 \, \mu_1] - \frac{n\,(\kappa_1+3)}{\sin \alpha}$$

$$a_{64} = \frac{n\,\bar{\alpha}_4 \cot \alpha}{2 \sin \alpha} - \frac{3}{4} \cot \alpha \, [\bar{\beta}_4 \, (\kappa_1+3) - \bar{\beta}_3 \, \mu_1] + \frac{n\,\mu_1}{\sin \alpha}$$

$$a_{65} = \frac{n\,\bar{\alpha}_5 \cot \alpha}{2 \sin \alpha} + \frac{3}{4} \cot \alpha \, [\bar{\beta}_5 \, (\kappa_2-3) - \bar{\beta}_6 \, \mu_2] + \frac{n\,(\kappa_2-3)}{\sin \alpha}$$

$$a_{66} = \frac{n\,\bar{\alpha}_6 \cot \alpha}{2 \sin \alpha} + \frac{3}{4} \cot \alpha \, [\bar{\beta}_6 \, (\kappa_2-3) + \bar{\beta}_5 \, \mu_2] + \frac{n\,\mu_2}{\sin \alpha}$$

$$a_{67} = \frac{n\,\bar{\alpha}_7 \cot \alpha}{2 \sin \alpha} - \frac{3}{4} \cot \alpha \, [\bar{\beta}_7 \, (\kappa_2+3) + \bar{\beta}_8 \, \mu_2] - \frac{n\,(\kappa_2+3)}{\sin \alpha}$$

$$a_{68} = \frac{n\,\bar{\alpha}_8 \cot \alpha}{2 \sin \alpha} - \frac{3}{4} \cot \alpha \, [\bar{\beta}_8 \, (\kappa_2+3) - \bar{\beta}_7 \, \mu_2] + \frac{n\,\mu_2}{\sin \alpha}$$

$$Q_r\ (5,5) = \frac{y^{a_1-5}}{\ell^2}\ a_{55}^* \qquad Q_r\ (6,5) = \frac{y^{a_1-5}}{\ell^2}\ a_{65}^*$$

$$Q_r\ (5,6) = \frac{y^{a_2-5}}{\ell^2}\ a_{56}^* \qquad Q_r\ (6,6) = \frac{y^{a_2-5}}{\ell^2}\ a_{66}^*$$

$$Q_r\ (5,7) = \frac{y^{-a_1-5}}{\ell^2}\ a_{57}^* \qquad Q_r\ (6,7) = \frac{y^{-a_1-5}}{\ell^2}\ a_{67}^*$$

$$Q_r\ (5,8) = \frac{y^{-a_2-5}}{\ell^2}\ a_{58}^* \qquad Q_r\ (6,8) = \frac{y^{-a_2-5}}{\ell^2}\ a_{68}^*$$

$$a_{15}^* = \alpha_5\ (a_1-1)$$

$$a_{16}^* = \alpha_6\ (a_2-1)$$

$$a_{17}^* = -\alpha_7\ (a_1+1)$$

$$a_{18}^* = -\alpha_8\ (a_2+1)$$

$$a_{25}^* = \alpha_5 + \cot\alpha + \frac{n\ \beta_5}{\sin\alpha}$$

$$a_{26}^* = \alpha_6 + \cot\alpha + \frac{n\ \beta_6}{\sin\alpha}$$

$$a_{27}^* = \alpha_7 + \cot\alpha + \frac{n\ \beta_7}{\sin\alpha}$$

$$a_{28}^* = \alpha_8 + \cot\alpha + \frac{n\ \beta_8}{\sin\alpha}$$

$$a_{35}^* = \frac{-n\ \alpha_5}{\sin\alpha} + \frac{\beta_5}{2}\ (a_1-3)$$

$$a_{36}^* = \frac{-n\ \alpha_6}{\sin\alpha} + \frac{\beta_6}{2}\ (a_2-3)$$

$$a_{37}^* = \frac{-n\ \alpha_7}{\sin\alpha} - \frac{\beta_7}{2}\ (a_1+3)$$

$$a_{38}^* = \frac{-n\ \alpha_8}{\sin\alpha} - \frac{\beta_8}{2}\ (a_2+3)$$

$$a_{45}^* = -(a_1-1)\ (a_1-3)$$

$$a_{46}^* = -(a_2-1)\ (a_2-3)$$

$$a_{47}^* = -(a_1+1)\ (a_1+3)$$

$$a_{48}^* = -(a_2+1)\ (a_2+3)$$

$$a_{55}^* = \frac{n^2}{\sin^2\alpha} + \frac{n\ \beta_5\ \cos\alpha}{\sin^2\alpha} - \frac{(a_1-1)}{2}$$

$$a_{56}^* = \frac{n^2}{\sin^2\alpha} + \frac{n\ \beta_6\ \cos\alpha}{\sin^2\alpha} - \frac{(a_2-1)}{2}$$

$$a_{57}^{*} = \frac{n^2}{\sin^2 \alpha} + \frac{n\ \beta_7 \cos \alpha}{\sin^2 \alpha} + \frac{(a_1+1)}{2}$$

$$a_{58}^{*} = \frac{n^2}{\sin^2 \alpha} + \frac{n\ \beta_8 \cos \alpha}{\sin^2 \alpha} + \frac{(a_2+1)}{2}$$

$$a_{65}^{*} = \frac{n\ \alpha_5 \cos \alpha}{2 \sin^2 \alpha} + \frac{(a_1-3)}{\sin \alpha} \left(n + \frac{3}{4} \beta_5 \cos \alpha\right)$$

$$a_{66}^{*} = \frac{n\ \alpha_6 \cos \alpha}{2 \sin^2 \alpha} + \frac{(a_2-3)}{\sin \alpha} \left(n + \frac{3}{4} \beta_6 \cos \alpha\right)$$

$$a_{67}^{*} = \frac{n\ \alpha_7 \cos \alpha}{2 \sin^2 \alpha} - \frac{(a_1+3)}{\sin \alpha} \left(n + \frac{3}{4} \beta_7 \cos \alpha\right)$$

$$a_{68}^{*} = \frac{n\ \alpha_8 \cos \alpha}{2 \sin^2 \alpha} - \frac{(a_2+3)}{\sin \alpha} \left(n + \frac{3}{4} \beta_8 \cos \alpha\right)$$

$$a_{11} = \bar{\alpha}_1 (\kappa_1 - 1) - \bar{\alpha}_2 \mu_1$$

$$a_{12} = \bar{\alpha}_2 (\kappa_1 - 1) + \bar{\alpha}_1 \mu_1$$

$$a_{13} = -\bar{\alpha}_3 (\kappa_1 + 1) - \bar{\alpha}_4 \mu_1$$

$$a_{14} = -\bar{\alpha}_4 (\kappa_1 + 1) + \bar{\alpha}_3 \mu_1$$

$$a_{15} = \bar{\alpha}_5 (\kappa_2 - 1) - \bar{\alpha}_6 \mu_2$$

$$a_{16} = \bar{\alpha}_6 (\kappa_2 - 1) + \bar{\alpha}_5 \mu_2$$

$$a_{17} = -\bar{\alpha}_7 (\kappa_2 + 1) - \bar{\alpha}_8 \mu_2$$

$$a_{18} = -\bar{\alpha}_8 (\kappa_2 + 1) + \bar{\alpha}_7 \mu_2$$

$$a_{21} = \bar{\alpha}_1 + \cot\alpha + \frac{n\,\bar{\beta}_1}{\sin\alpha}$$

$$a_{22} = \bar{\alpha}_2 + \frac{n\,\bar{\beta}_2}{\sin\alpha}$$

$$a_{23} = \bar{\alpha}_3 + \cot\alpha + \frac{n\,\bar{\beta}_3}{\sin\alpha}$$

$$a_{24} = \bar{\alpha}_4 + \frac{n\,\bar{\beta}_4}{\sin\alpha}$$

$$a_{25} = \bar{\alpha}_5 + \cot\alpha + \frac{n\,\bar{\beta}_5}{\sin\alpha}$$

$$a_{26} = \bar{\alpha}_6 + \frac{n\,\bar{\beta}_6}{\sin\alpha}$$

$$a_{27} = \bar{\alpha}_7 + \cot\alpha + \frac{n\,\bar{\beta}_7}{\sin\alpha}$$

$$a_{28} = \bar{\alpha}_8 + \frac{n\,\bar{\beta}_8}{\sin\alpha}$$

$$a_{31} = \frac{\bar{\beta}_1}{2} (\kappa_1 - 3) - \frac{\bar{\beta}_2}{2} \mu_1 - \frac{n\,\bar{\alpha}_1}{\sin\alpha}$$

$$a_{32} = \frac{\bar{\beta}_2}{2} (\kappa_1 - 3) + \frac{\bar{\beta}_1}{2} \mu_1 - \frac{n\,\bar{\alpha}_2}{\sin\alpha}$$

$$a_{33} = -\frac{\bar{\beta}_3}{2} (\kappa_1 + 3) - \frac{\bar{\beta}_4}{-2} \mu_1 - \frac{n\,\bar{\alpha}_3}{\sin\alpha}$$

$$a_{34} = -\frac{\bar{\beta}_4}{2} (\kappa_1 + 3) + \frac{\bar{\beta}_3}{2} \mu_1 - \frac{n\,\bar{\alpha}_4}{\sin\alpha}$$

$$a_{35} = \frac{\bar{\beta}_5}{2} (\kappa_2 - 3) - \frac{\bar{\beta}_6}{2} \mu_2 - \frac{n\,\bar{\alpha}_5}{\sin\alpha}$$

$$a_{36} = \frac{\bar{\beta}_6}{2} (\kappa_2 - 3) + \frac{\bar{\beta}_5}{2} \mu_2 - \frac{n\,\bar{\alpha}_6}{\sin\alpha}$$

$$a_{37} = -\frac{\bar{\beta}_7}{2} (\kappa_2 + 3) - \frac{\bar{\beta}_8}{2} \mu_2 - \frac{n\,\bar{\alpha}_7}{\sin\alpha}$$

$$a_{38} = -\frac{\bar{\beta}_8}{2} (\kappa_2 + 3) + \frac{\bar{\beta}_7}{2} \mu_2 - \frac{n\,\bar{\alpha}_8}{\sin\alpha}$$

$$a_{41} = \mu_1^2 - (\kappa_1 - 1)(\kappa_1 - 3)$$

$$a_{42} = -2\mu_1 (\kappa_1 - 2)$$

$$a_{43} = \mu_1^2 - (\kappa_1 + 1)(\kappa_1 + 3)$$

$$a_{44} = 2\mu_1 (\kappa_1 + 2)$$

$$a_{45} = \mu_2^2 - (\kappa_2 - 1)(\kappa_2 - 3)$$

$$a_{46} = -2\mu_2 (\kappa_2 - 2)$$

$$a_{47} = \mu_2^2 - (\kappa_2 + 1)(\kappa_2 + 3)$$

$$a_{48} = 2\mu_2 (\kappa_2 + 2)$$

NATURAL VIBRATION OF NONUNIFORM THIN CONICAL SHELLS BY FINITE-ELEMENTS METHOD

INPUT DATA

SECTION NUMBER	1	2	3	4	5	6
YOUNG.S MODULUS OF ELASTICITY E=LB/(IN-SQUARE)	.10000E+01	.10000E+01	.10000E+01	.10000E+01	.10000E+01	.10000E+01
POISSON.S RATIO NU	.30000E+00	.30000E+00	.30000E+00	.30000E+00	.30000E+00	.30000E+00
APEX ANGLE OF SHELL ELEMENT (RD)	.24784E+00	.24784E+00	.24784E+00	.24784E+00	.24784E+00	.24784E+00
COEF. DELTA(=T/X) OF SHELL ELEMENT	.85507E-03	.76664E-03	.60452E-03	.63480E-03	.58454E-03	.54165E-03
LENGTH OF AN INDIVIDUAL SHELL ELEMEN LE=IN.	.13648E+01	.13648E+01	.13648E+01	.13648E+01	.13648E+01	.13648E+01
MASS PER UNIT VOLUME OF THE SHELL ELEMENT RHO	.10000E+01	.10000E+01	.10000E+01	.10000E+01	.10000E+01	.10000E+01
COEFFICIENTS IN SHELL EQUATION D=E*T/(1-NU**2)	.94007E-03	.84246E-03	.76320E-03	.69758E-03	.64235E-03	.59522E-03
K =E*T**3/12*(1-NU**2)	.17199E-09	.12378E-09	.92033E-10	.70276E-10	.54870E-10	.43658E-10
SMALL K =T**2/12*RA**2	.60985E-07	.48977E-07	.40196E-07	.33581E-07	.28470E-07	.24449E-07

SECTION NUMBER	7	8	9	10
YOUNG.S MODULUS OF ELASTICITY E=LB/(IN-SQUARE)	.10000E+01	.10000E+01	.10000E+01	.10000E+01
POISSON.S RATIO NU	.30000E+00	.30000E+00	.30000E+00	.30000E+00
APEX ANGLE OF SHELL ELEMENT (RD)	.24784E+00	.24784E+00	.24784E+00	.24784E+00
COEF. DELTA(=T/X) OF SHELL ELEMENT	.50463E-03	.47234E-03	.44394E-03	.41876E-03
LENGTH OF AN INDIVIDUAL SHELL ELEMEN LE=IN.	.13648E+01	.13648E+01	.13648E+01	.13648E+01
MASS PER UNIT VOLUME OF THE SHELL ELEMENT RHO	.10000E+01	.10000E+01	.10000E+01	.10000E+01
COEFFICIENTS IN SHELL EQUATION D=E*T/(1-NU**2)	.55454E-03	.51906E-03	.48785E-03	.46017E-03
K =E*T**3/12*(1-NU**2)	.35303E-10	.28951E-10	.24037E-10	.20174E-10
SMALL K =T**2/12*RA**2	.21221E-07	.18592E-07	.16424E-07	.14613E-07

N.B. RHO=(LB/IN/SEC**2)/(IN**3)

Table 4 : Example of data required for the program

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NATURAL VIBRATION OF NONUNIFORM THIN  CONICAL SHELLS BY FINITE-ELEMENTS METHOD
------------------------------------------------------------------------------
          THE NUMBER OF CIRCUMFERENTIAL  WAVES IS N = 10.
          ---------------------------------------------------

                   SECTION  NUMBER   1
                   ------------------
----A-------------------CHARACTERISTIC   EQUATION-----------------------------
 THE EIGHT ROOTS OF THE CHARACTERISTIC EQUATION ARE =

LAMDA1 =   .20517E+02  .16521E+02 'I    LAMDA5 =   .19528E+03  .15804E+03 'I
LAMDA2 =   .20517E+02 -.16521E+02 'I    LAMDA6 =   .19528E+03 -.15804E+03 'I
LAMDA3 =  -.20517E+02  .16521E+02 'I    LAMDA7 =  -.19528E+03  .15804E+03 'I
LAMDA4 =  -.20517E+02 -.16521E+02 'I    LAMDA8 =  -.256[illegible]3E+03  .22968E+03 'I

          EACH OF THE 8 VALUES -LAMDA- YIELDS ONE  SOLUTION
          OF EQUATIONS OF MOTION, AND THE COMPLETE SOLUTION
          IS THE SUM OF ALL THEM WITH 8 INDEPENDENT SETS OF
          CONSTANTS   A(J) , B(J) , C(J) =

             A(J) = CONSTANTS OF U = AXIAL DISPLACEMENT EQUATION
             B(J) = CONSTANTS OF V = TANGENTIAL DISPLACEMENT EQUATION
             C(J) = CONSTANTS OF W = RADIAL DISPLACEMENT EQUATION
          SUCH THAT = A(J) = ALPHA(J) X C(J)
                      B(J) = BETA(J) X C(J)
          AND=
          -----

ALPHA1 =  -.16269E-01 -.24948E-01 'I    BETA1 =  -.97875E-01  .35992E-02 'I
ALPHA3 =   .23501E-01 -.26913E-01 'I    BETA3 =  -.98275E-01 -.31058E-02 'I
ALPHA5 =  -.40176E-02  .10013E-01 'I    BETA5 =   .10262E-02 -.24071E-01 'I
ALPHA7 =   .39376E-02  .10470E-01 'I    BETA7 =   .10686E-02  .24019E-01 'I

DISPLACEMENT FUNCTION MATRIX AT X=  .12461E+02
```

Table 5 : Example of roots of characteristic equations for coefficients αi and βi in relation (3.9)

NATURAL VIBRATION OF NONUNIFORM THIN CONICAL SHELLS BY FINITE-ELEMENTS METHOD

THE NUMBER OF CIRCUMFERENTIAL WAVES IS N = 10.

OUTPUT MATRICES

THE MATRIX A (I,8,8) IS

-.14600E-01	-.79706E-02	.15137E-01	-.66193E-01	-.30430E-04	-.12072E-04	-.36572E+01	.13918E+01
-.49203E+00	-.26435E+00	-.16741E+01	-.89944E+00	-.11881E-04	-.30343E-02	.13698E+01	.34983E+03
.62951E+00	.13381E+00	-.95360E+00	.21184E+01	-.21505E-01	.26649E-01	-.25034E+04	-.30843E+04
-.47206E-01	.27644E-01	-.16732E+00	.83195E-01	.73053E-04	.28278E-05	-.84011E+01	.40673E+00
-.57999E-02	-.51276E-01	.18046E-01	-.73704E-02	.25464E+01	.30653E+00	.36159E-04	-.25897E-04
.15484E+01	.77724E+00	.48759E+00	.24476E+00	-.61525E+02	-.22963E+03	-.10290E-02	-.38406E-02
.69733E+00	.16351E+01	-.58372E+00	.11190E+00	.87658E+03	-.21803E+04	.32460E-01	.23723E-01
-.15435E+00	-.70503E-01	-.47159E-01	-.25569E-01	-.55906E+01	.12454E+01	.91150E-04	-.20821E-04

THE TRANSFORMATION MATRIX -INVERSE OF A- IS

.82542E+01	-.23218E+00	-.22216E-01	.29889E+01	.13999E+02	.40270E+00	-.43694E-01	-.56881E+01
-.26173E+01	.12219E+00	.12345E-01	-.25193E+01	-.23915E+02	-.35746E+00	.33670E-01	-.10773E+01
-.12180E+02	.38792E+00	.37716E-01	-.58734E+01	-.93103E+01	-.22866E+00	.24205E-01	.25038E+01
-.10295E+02	.32263E+00	.27903E-01	-.13731E+00	-.25084E+01	-.10329E+00	.11690E-01	.20362E+01
-.35952E-02	.11885E-03	.11620E-04	-.18762E-02	.29318E-02	-.46959E-02	.47000E-03	-.43756E-01
.13344E-02	-.16595E-04	-.13047E-05	-.19576E-03	-.97831E-02	-.21868E-02	-.24273E-03	-.22791E-01
-.37221E-02	-.27130E-02	-.31265E-03	-.24689E-01	.32536E-02	.91897E-04	-.99517E-05	-.12660E-02
-.48941E-02	.22615E-02	-.66952E-04	.22450E-01	.33277E-03	-.81392E-05	.11602E-05	.44440E-03

Table 6 : Example of Matrix |A'| determined by relation (3.13) and its inverse

THE GENERALIZED COORDINATE STIFFNESS OF THE ELEMENT G (1,8,8) IS

.24020E-03	-.24865E-04	.11141E-03	.58100E-05	.10119E-01	.86520E-02	-.59283E-02	-.54342E-02
-.24865E-04	.18557E-03	.58100E-05	-.11141E-03	-.17733E-01	-.53454E-02	.17553E-02	.45161E-02
.11141E-03	.58100E-05	.26872E-03	.15324E-04	-.38308E-02	.50479E-02	.13058E-01	-.12957E-01
.58100E-05	-.11141E-03	.15324E-04	.32707E-03	-.43876E-03	.41006E-02	.31746E-01	-.18803E-01
.10119E-01	-.17733E-01	-.38308E-02	-.43876E-03	.51804E+02	.89728E+00	.32361E-03	-.85172E-04
.86520E-02	-.53454E-02	.50479E-02	.41006E-02	.89728E+00	.43782E+02	-.85175E-04	-.32360E-03
-.59283E-02	.17553E-02	.13058E-01	.31746E-01	.32361E-03	-.85175E-04	.10060E+03	-.40936E+01
-.54342E-02	.45161E-02	-.12957E-01	-.18803E-01	-.85172E-04	-.32360E-03	-.40936E+01	.88947E+02

TRANSFORM MATRIX G TO THE DESIRED NODAL POINT STIFFNESS ,SUCH THAT
K = (TRANSPOSE OF INVERSE OF A) X G X (INVERSE OF A)

.15665E+00	.32002E-03	.98420E-04	.42932E-01	.10714E-01	.86681E-03	-.10269E-03	-.22600E-01
.32002E-03	.11776E-02	.69335E-04	.10971E-01	-.97040E-03	-.29892E-04	.32778E-05	.45982E-03
.98420E-04	.69335E-04	.97726E-05	.58530E-03	-.10186E-03	-.29377E-05	.31907E-06	.41594E-04
.42932E-01	.10971E-01	.58530E-03	.12653E+00	.25534E-01	.49313E-03	-.49733E-04	-.24812E-02
.10714E-01	-.97040E-03	-.10186E-03	.25534E-01	.15899E+00	-.15148E-03	.12417E-03	-.37594E-01
.86681E-03	-.29892E-04	-.29377E-05	.49313E-03	-.15148E-03	.13060E-02	-.86449E-04	.12295E-01
-.10269E-03	.32778E-05	.31907E-06	-.49733E-04	.12417E-03	-.86449E-04	.13561E-04	-.73601E-03
-.22600E-01	.45982E-03	.41594E-04	-.24812E-02	-.37594E-01	.12295E-01	-.73601E-03	.13687E+00

THIS MATRIX K IS THE STIFFNESS MATRIX OF ANY FORM OF STRUCTURAL ELEMENT

THE NODAL DISPLACEMENTS ARE IN THE FOLLOWING ORDER =
(U,W,BETA,V) AT POINTS I(X=0) AND J(X=LE) RESPECTIVELY

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Table 7 : Example of Matrix |G| and the elementary matrix determined by (4.3) and (4.6)

NATURAL VIBRATION OF NON UNIFORM THIN CONICAL SHELLS BY FINITE-ELEMENTS METHOD

THE NUMBER OF CIRCUMFERENTIAL WAVES IS N = 10.

SECTION NUMBER 1

----8-----------------------OUTPUT MATRICES-----------------------------

THE GENERALIZED COORDINATE MASS OF THE ELEMENT S(I,8,8) IS

.73002E-01	.12679E-01	.58403E-01	.36234E-03	-.13873E+01	-.11082E+01	-.47195E+00	.49618E+00
.12679E-01	.66123E-02	.25736E-03	.46307E-02	-.56498E+00	-.54691E+00	.19301E+00	-.26111E+00
.58403E-01	.25736E-03	.71544E-01	-.12293E-01	-.53496E+00	-.34295E+00	-.12404E+01	.16294E+01
.36234E-03	.46307E-02	-.12293E-01	.68948E-02	-.21547E+00	-.17910E+00	.51379E+00	-.82783E+00
-.13873E+01	-.56498E+00	-.53496E+00	-.21547E+00	.65542E+02	.66386E+02	.32133E-01	.15046E-03
-.11082E+01	-.54691E+00	-.34295E+00	-.17910E+00	.66386E+02	.11730E+03	.15240E-03	.30253E-01
-.47195E+00	.19301E+00	-.12404E+01	.51379E+00	.32133E-01	.15240E-03	.56751E+02	-.69160E+02
.49618E+00	-.26111E+00	.16294E+01	-.82783E+00	.15046E-03	.30253E-01	-.69160E+02	.22859E+03

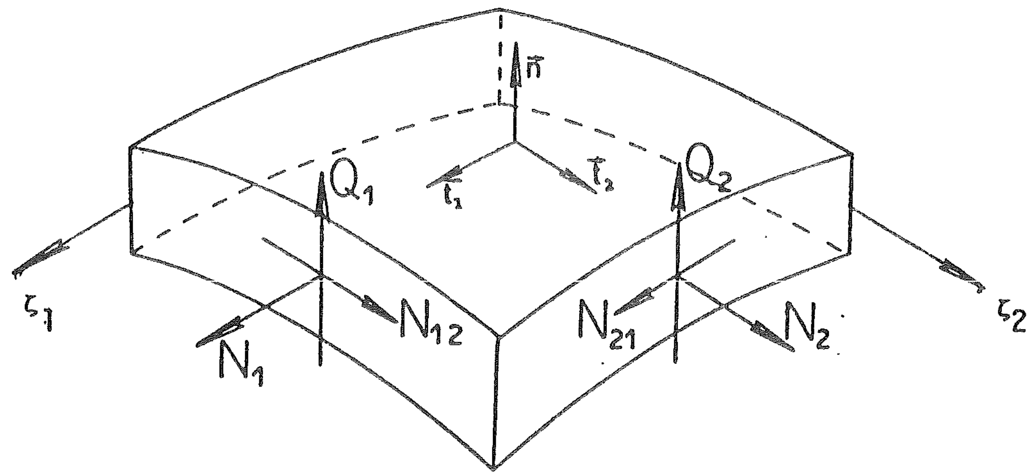
TRANSFORM -MATRIX RHO.T.S(I,8,8)- TO THE DESIRED NODAL POINT MASS SUCH THAT
M = (TRANSPOSE OF INVERSE OF A) X RHO.T.S(I,8,8) . (INVERSE OF A)

.48324E+00	-.25492E-01	-.26273E-02	.45880E+00	-.54619E+00	-.25507E-01	.29406E-02	.44656E+00
-.25492E-01	.59377E-02	.35815E-03	-.36037E-01	.27170E-01	-.11133E-02	-.12686E-03	-.18944E-01
-.26273E-02	.35815E-03	.29100E-04	-.35780E-02	.27872E-02	.11287E-03	-.12843E-04	-.19161E-02
.45880E+00	-.36037E-01	-.35780E-02	.56242E+00	-.49302E+00	-.19631E-01	.22323E-02	.33626E+00
-.54619E+00	.27170E-01	.27872E-02	-.49302E+00	.66661E+00	.32917E-01	-.38179E-02	-.57440E+00
-.25507E-01	-.11133E-02	.11287E-03	-.19631E-01	.32917E-01	.78951E-02	-.52075E-03	-.43898E-01
.29406E-02	-.12686E-03	-.12843E-04	.22323E-02	-.38179E-02	-.52075E-03	.46845E-04	.48743E-02
.44656E+00	-.18944E-01	-.19161E-02	.33626E+00	-.57440E+00	-.43898E-01	.48743E-02	.65027E+00

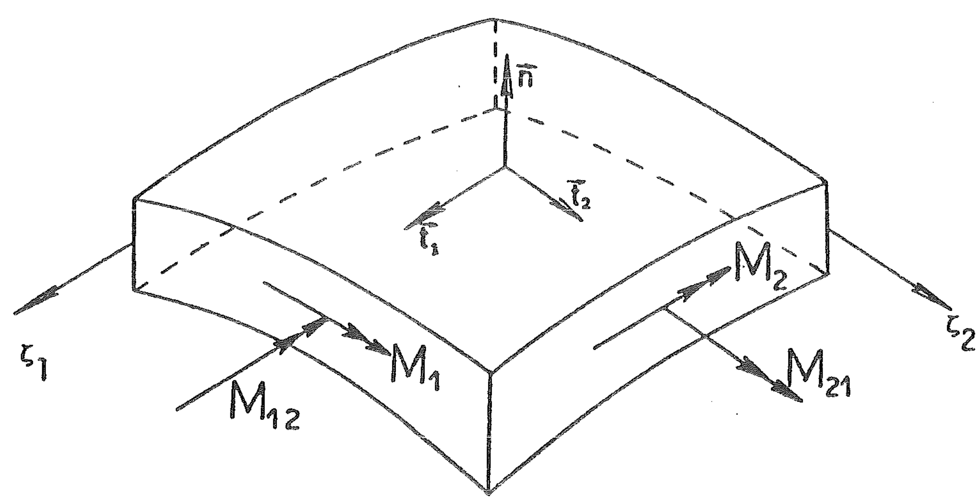
THIS MATRIX M IS THE MASS MATRIX OF ANY FORM OF STRUCTURAL ELEMENT

THE NODAL DISPLACEMENTS ARE IN THE FOLLOWING ORDER =
(U,W,BETA,V) AT POINTS I(X=0) AND J(X=LE) RESPECTIVELY

Table 8 : Example of matrix |S| and the elementary mass matrix determined by (4.5) and (4.7)

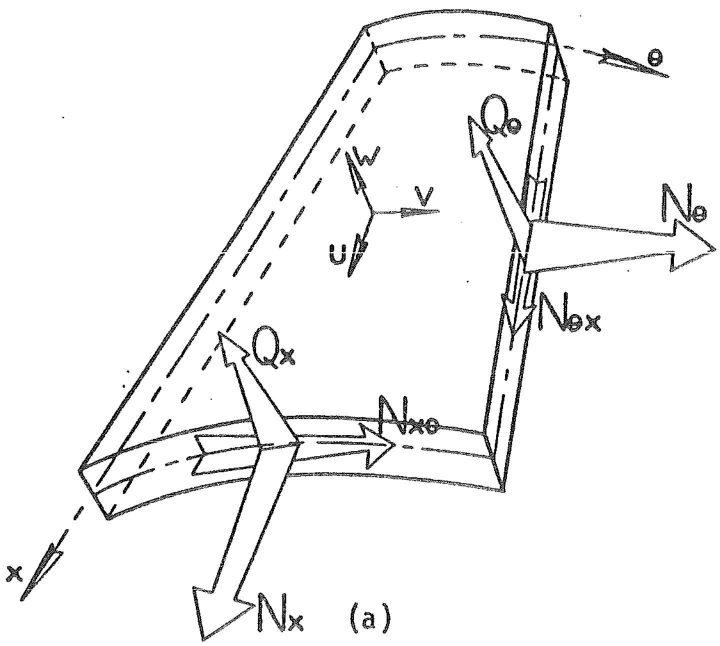


Direction of resultant constraints



Direction of resultant moments

Figure 1 Differential elements for a thin shell



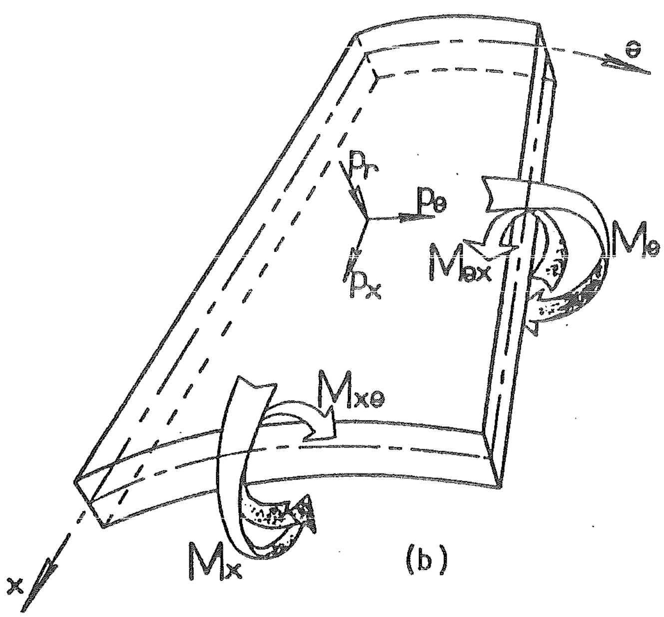


Figure 2 : Differential elements for a conical shell
(a) Resultant constraints and displacements
(b) Resultant couples and load factor

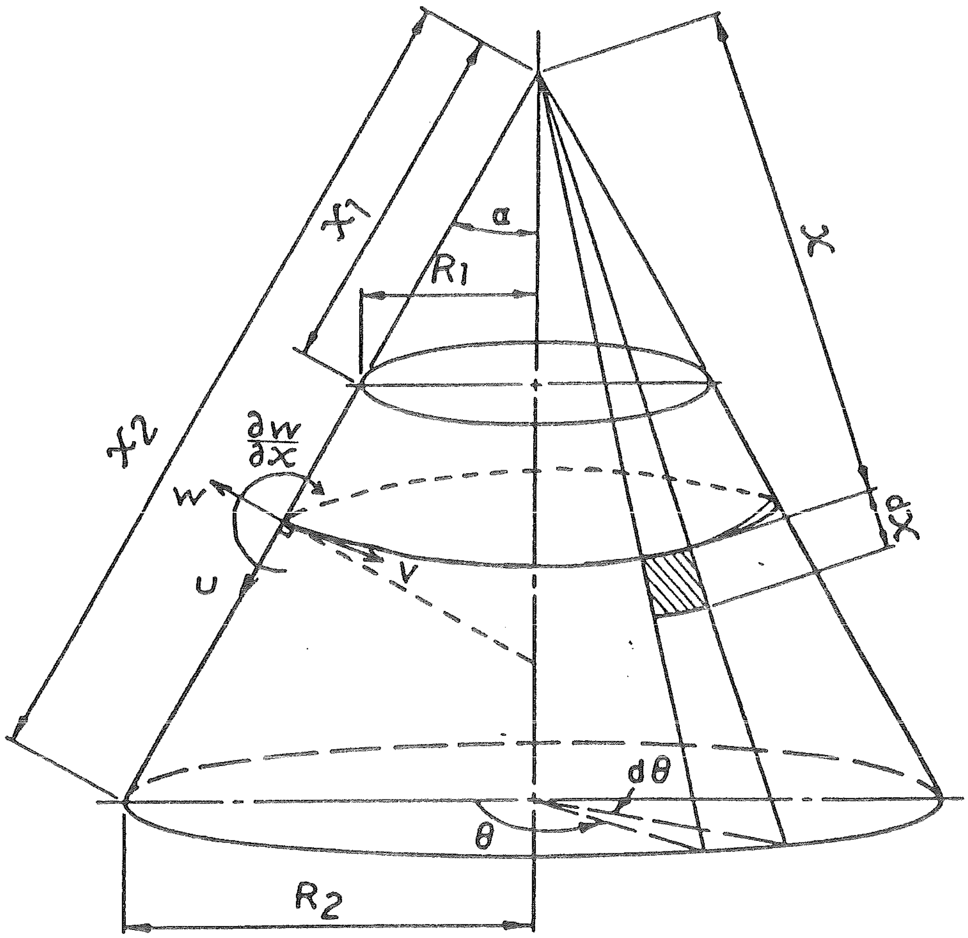


Figure 3 : Geometry of the surface of reference of a conical shell

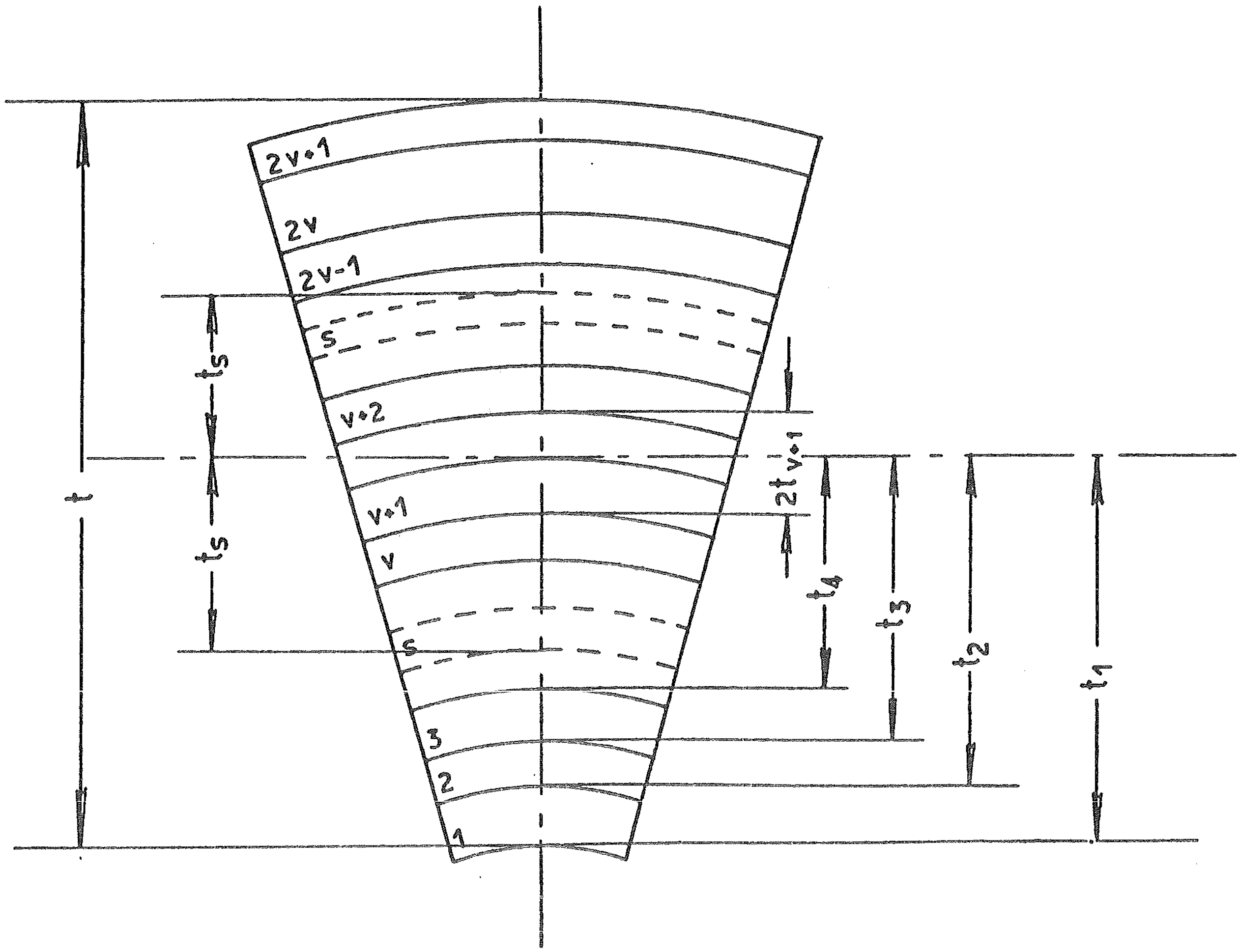


Figure 4 : Shell composed of an odd number (2v + 1) of anisotropic layers

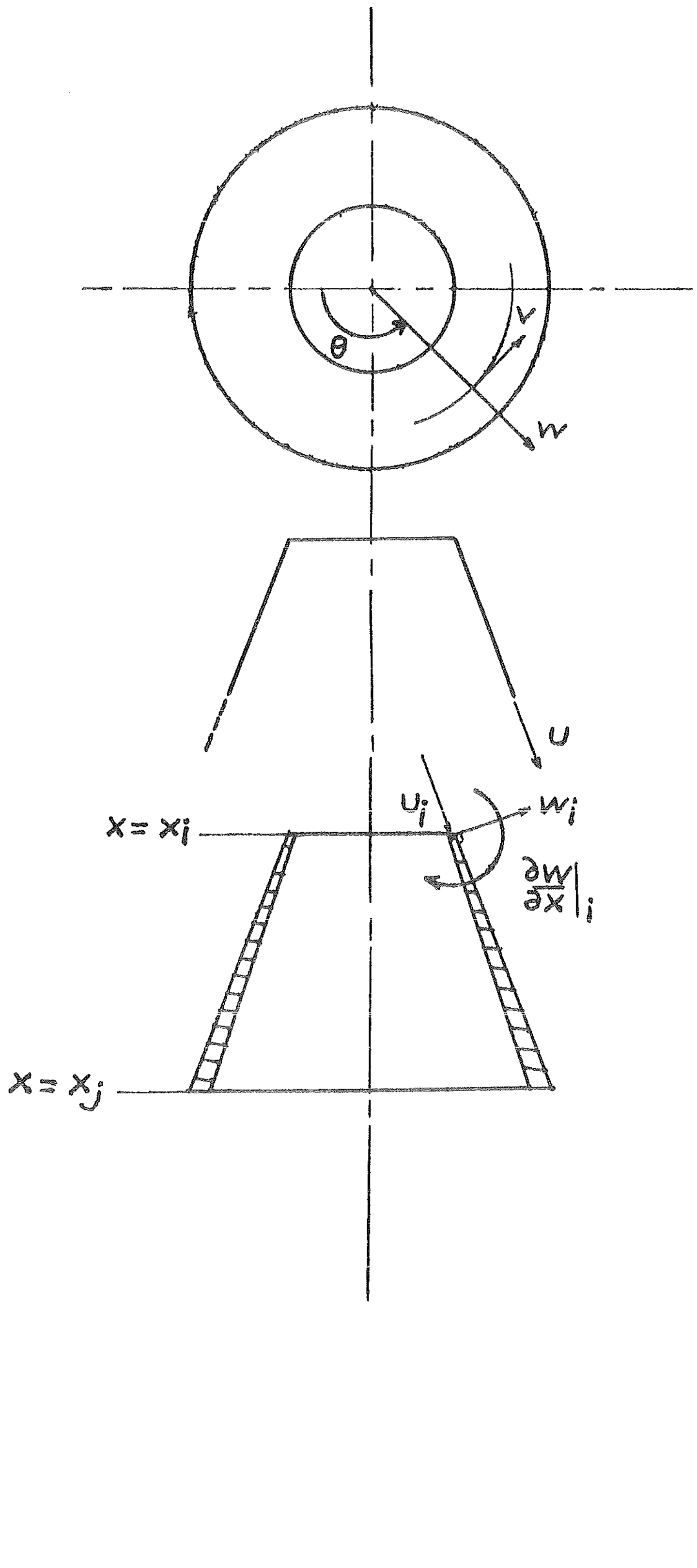


Figure 5 : Displacements and degrees of freedom at a node

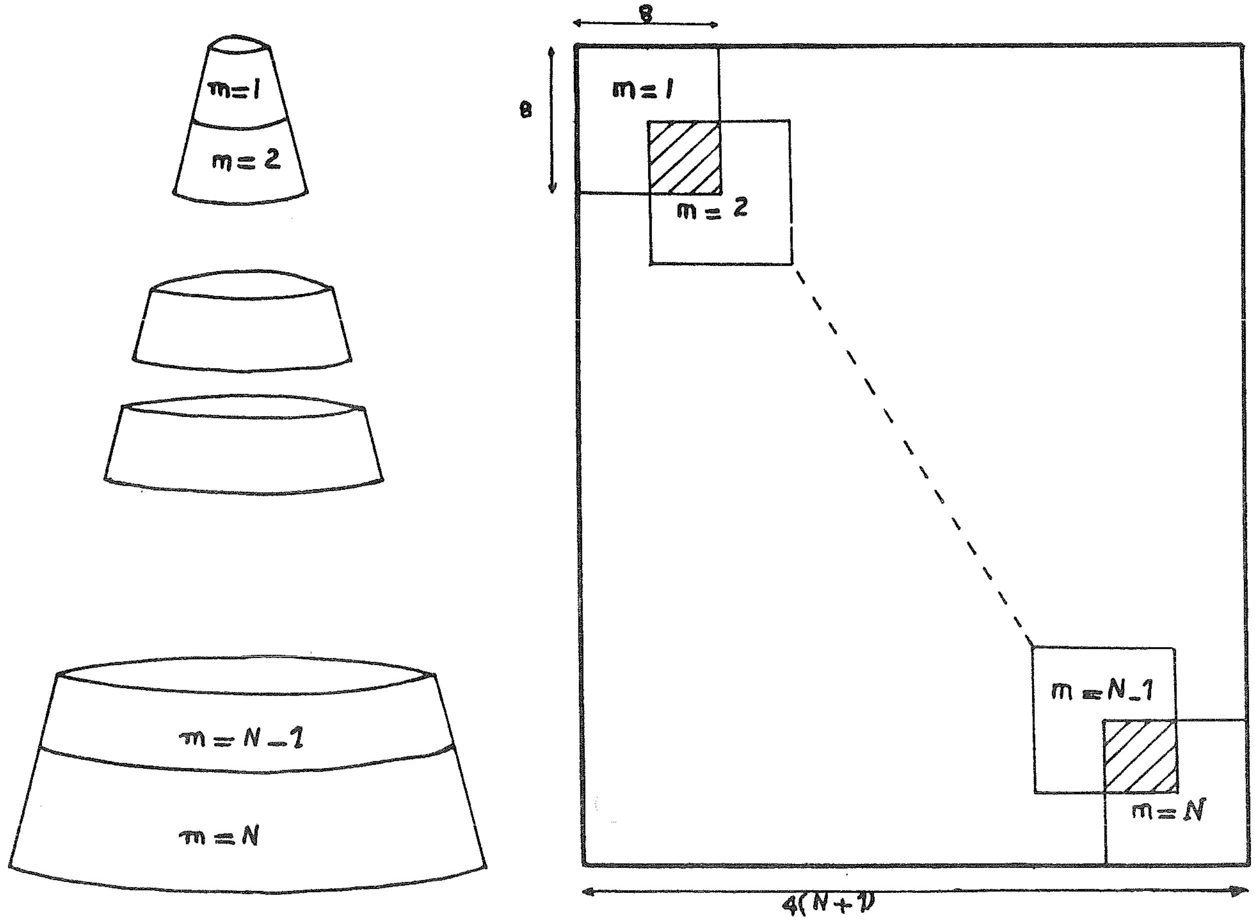


Figure 6 : Assembly diagram of mass and stiffness matrices

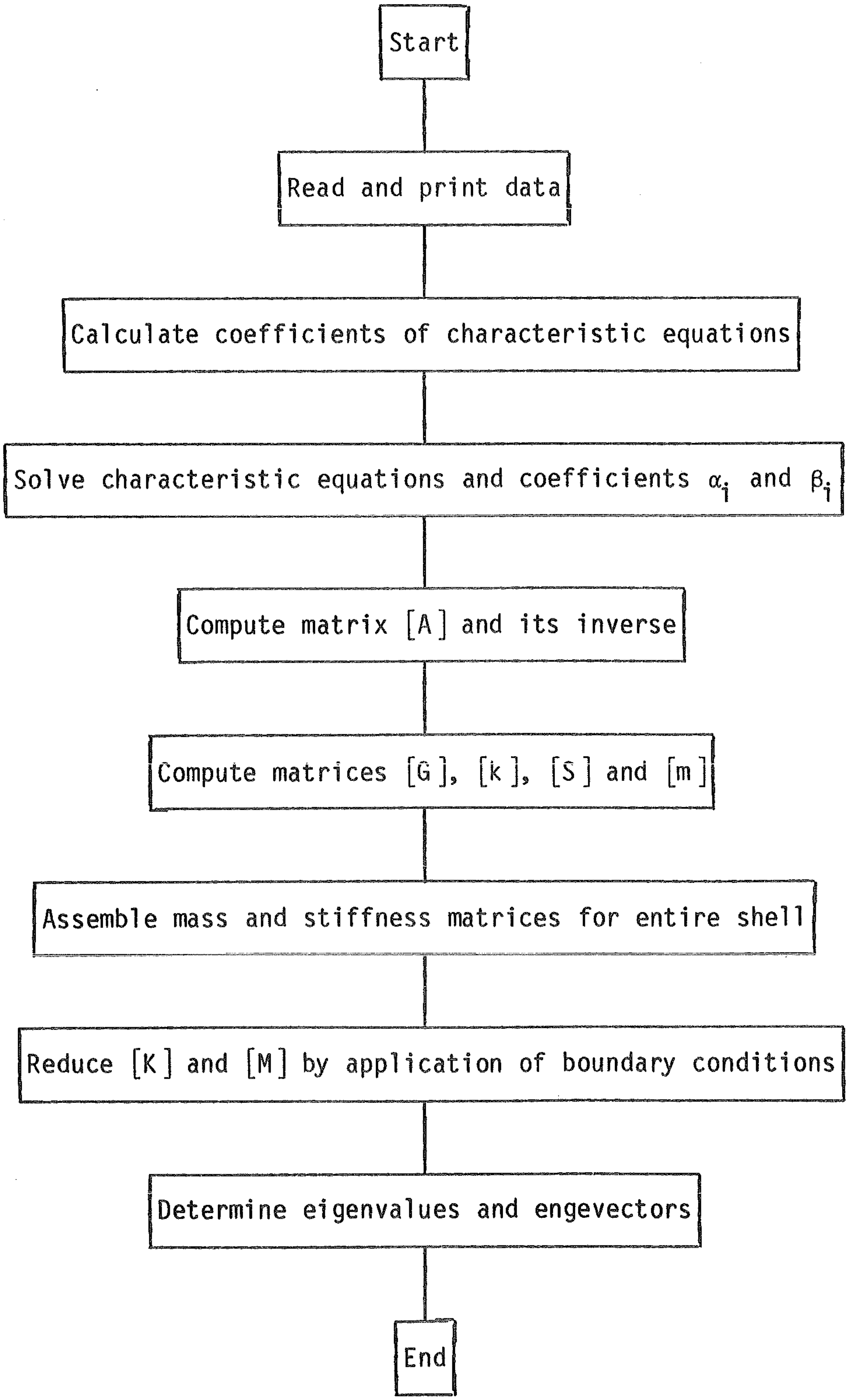


Figure 7 : Condensed flow chart of computer program

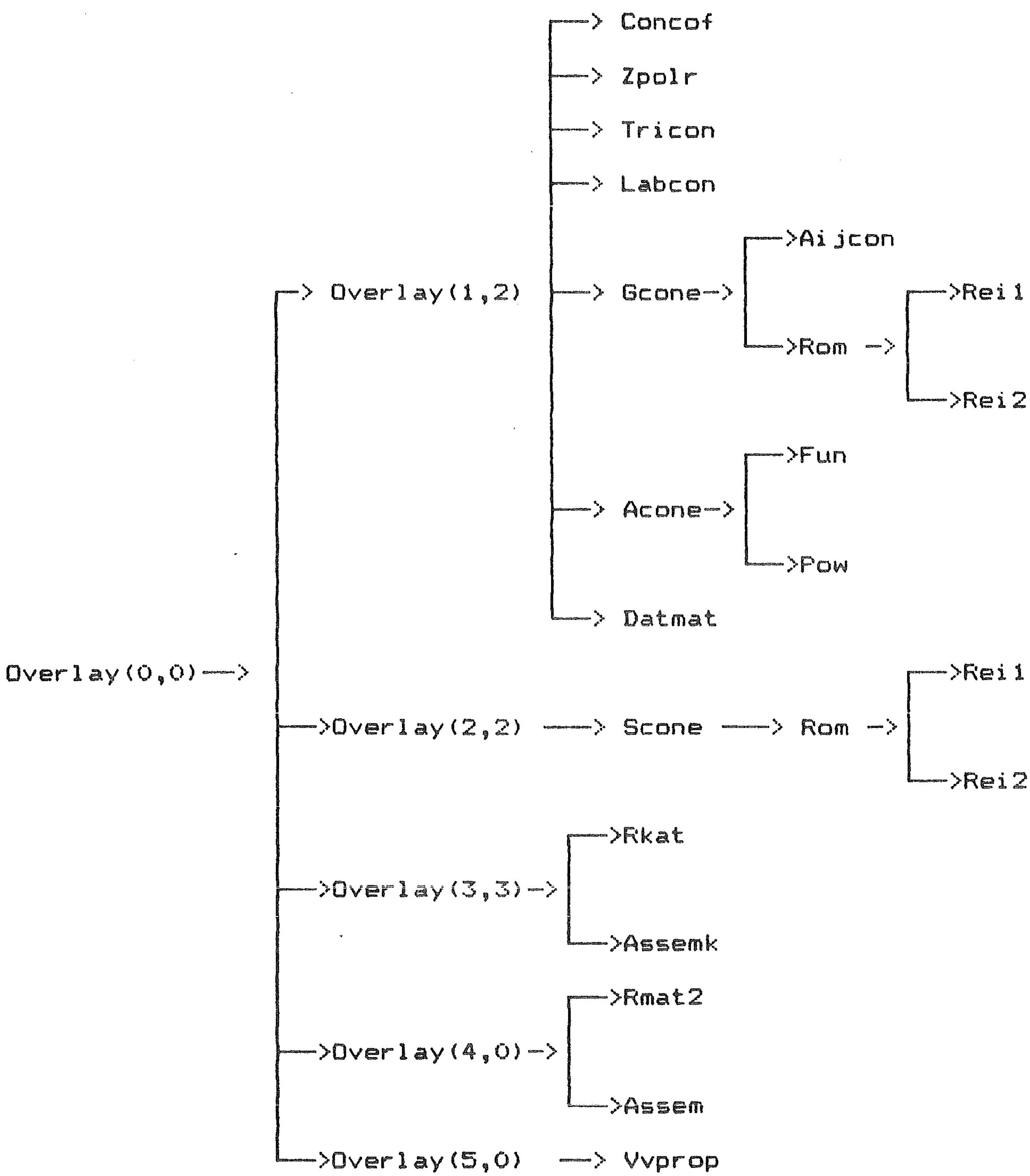


Figure 8 : Tree-diagram of computer program

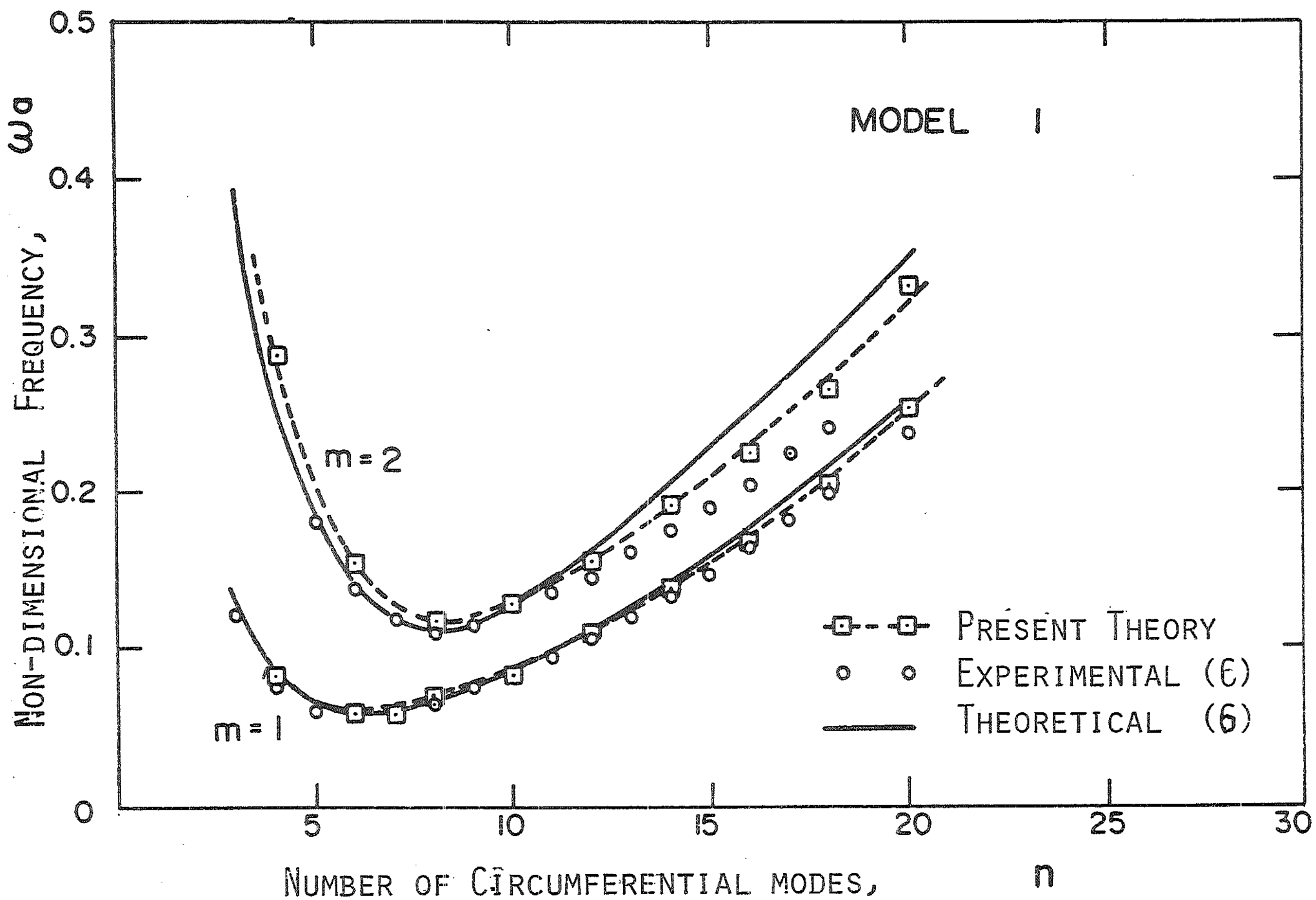


Figure 9 : Non-dimensional frequency for model 1 with $w = v = 0$ at boundaries. m = axial mode $\omega_a = \omega R_2\ (\rho(1-\nu^2)/E)^{.5}$

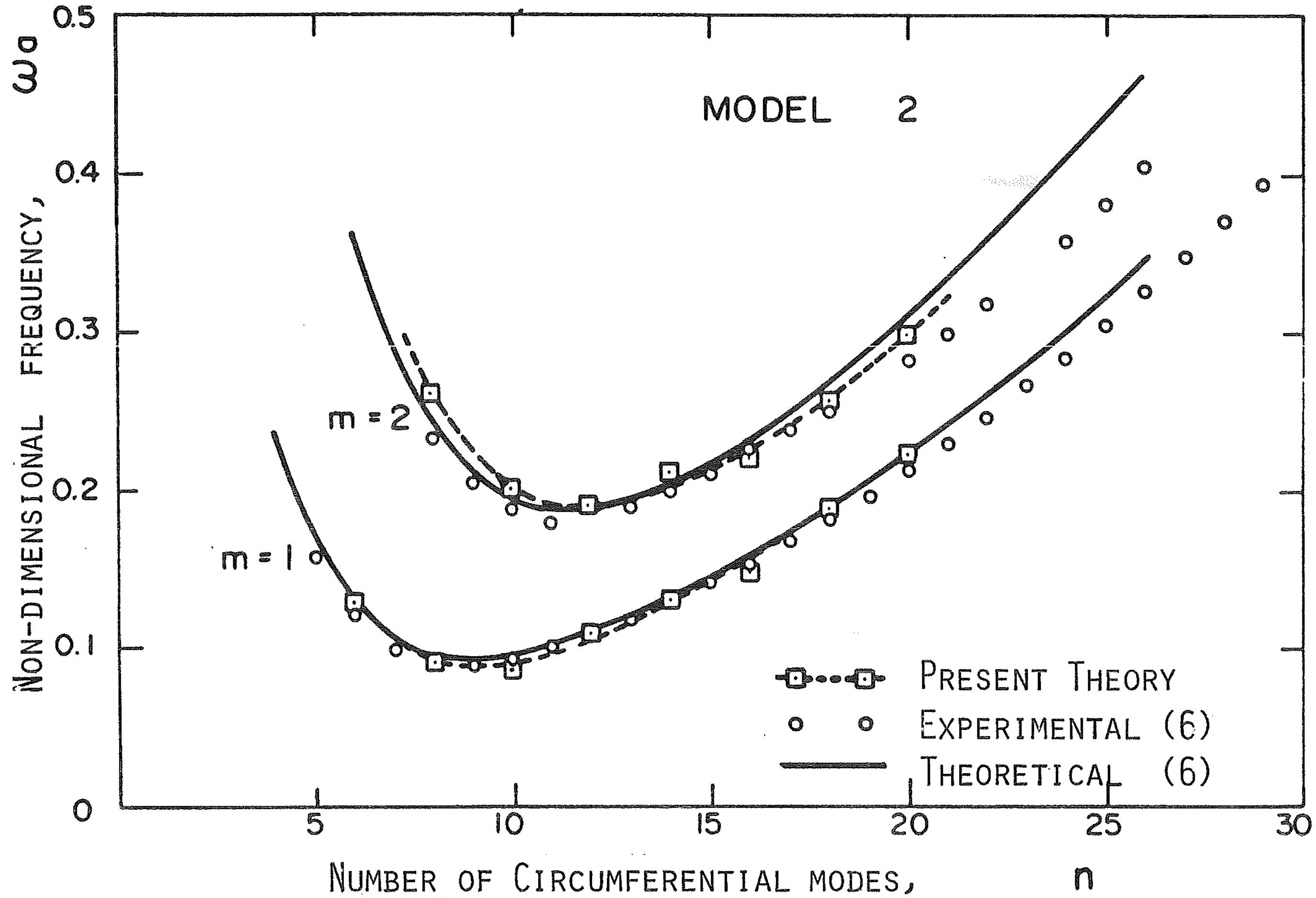


Figure 10 : Non-dimensional frequency for model 2 with $w = v = 0$ at boundaries. m = axial mode $\omega_a = \omega R_2\ (\rho(1-\nu^2)/E)^{.5}$

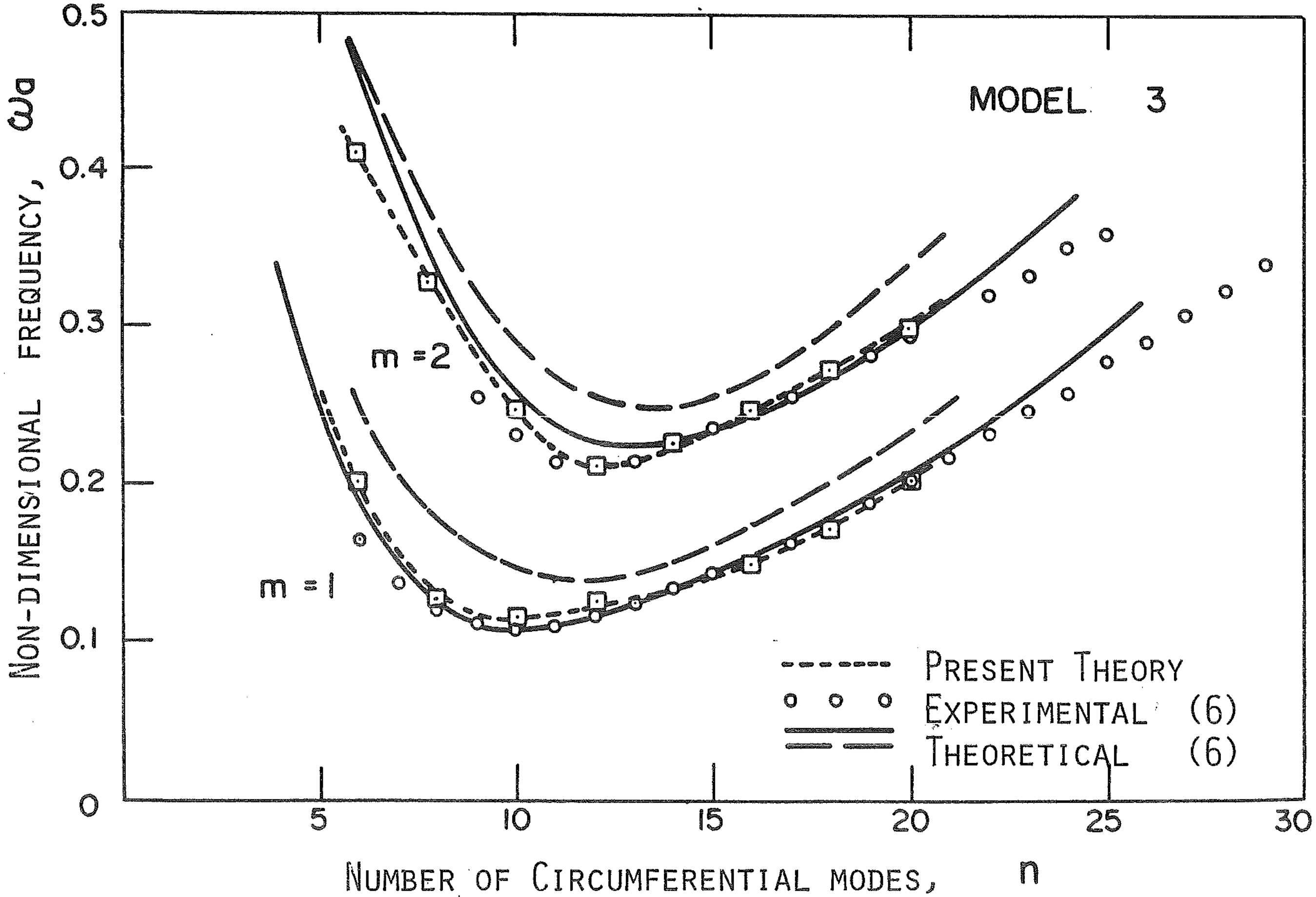


Figure 11 : Non-dimensional frequency for model 3 with $w = v = 0$ at boundaries. m = axial mode $\omega_a = \omega R_2 \; (\rho(1-\nu^2)/E)^{.5}$

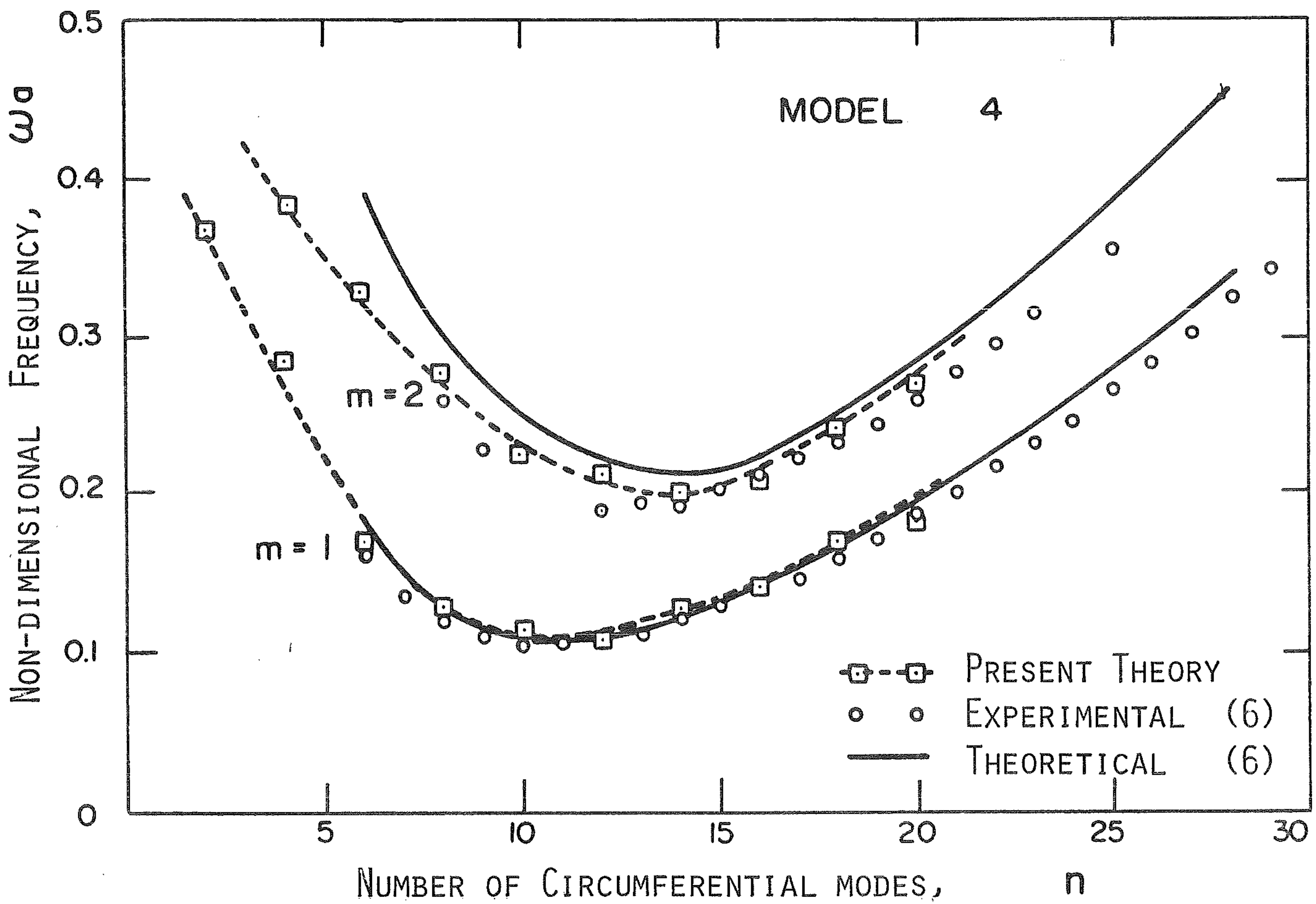


Figure 12 : Non-dimensional frequency for model 4 with $w = v = 0$ at boundaries. m = axial mode $\omega_a = \omega R_2 \ (\rho(1-\nu^2)/E^{.5}$

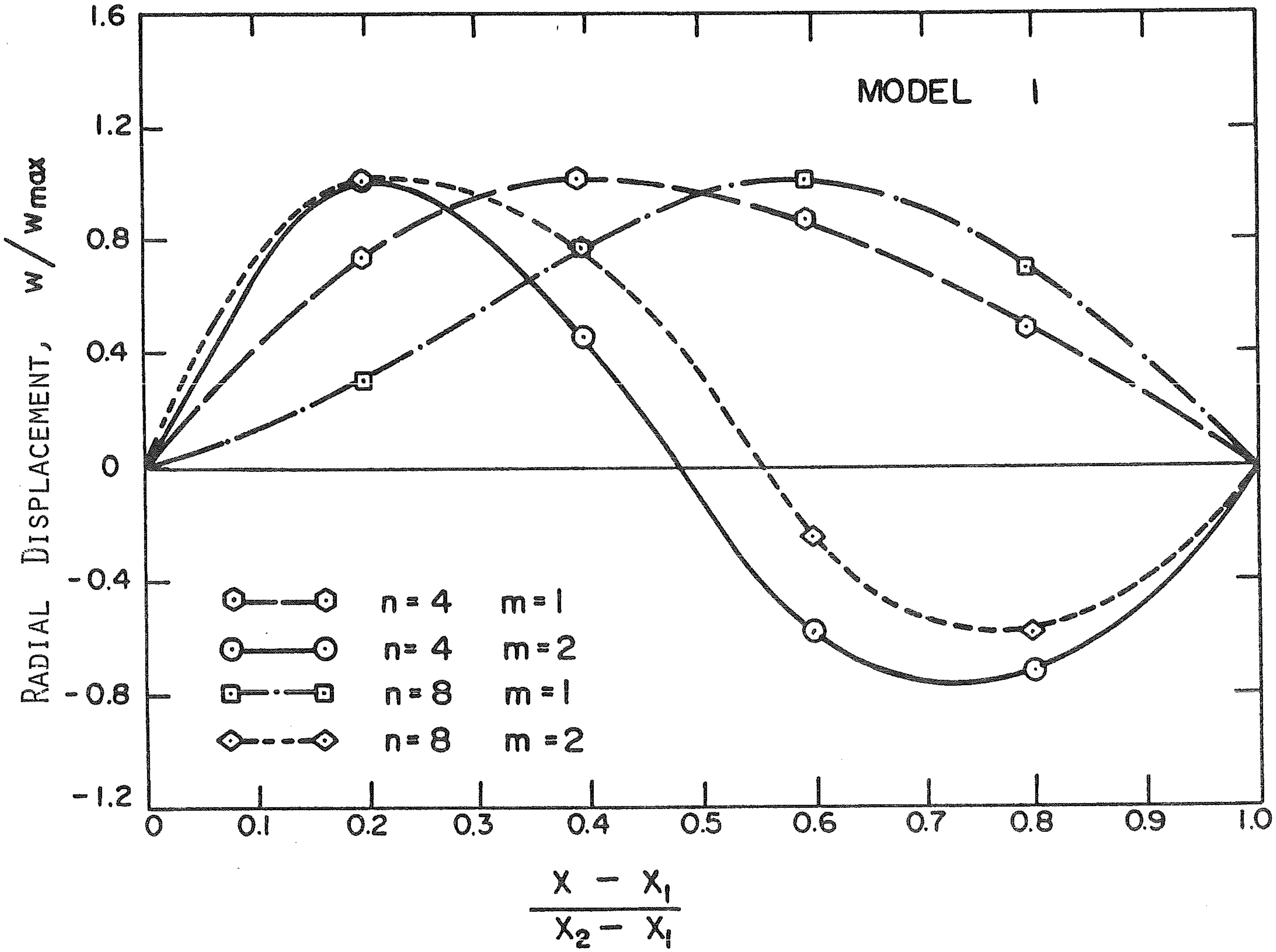


Figure 13 : Normalized radial deformation mode. Model 1 with $w = v = 0$ at boundaries

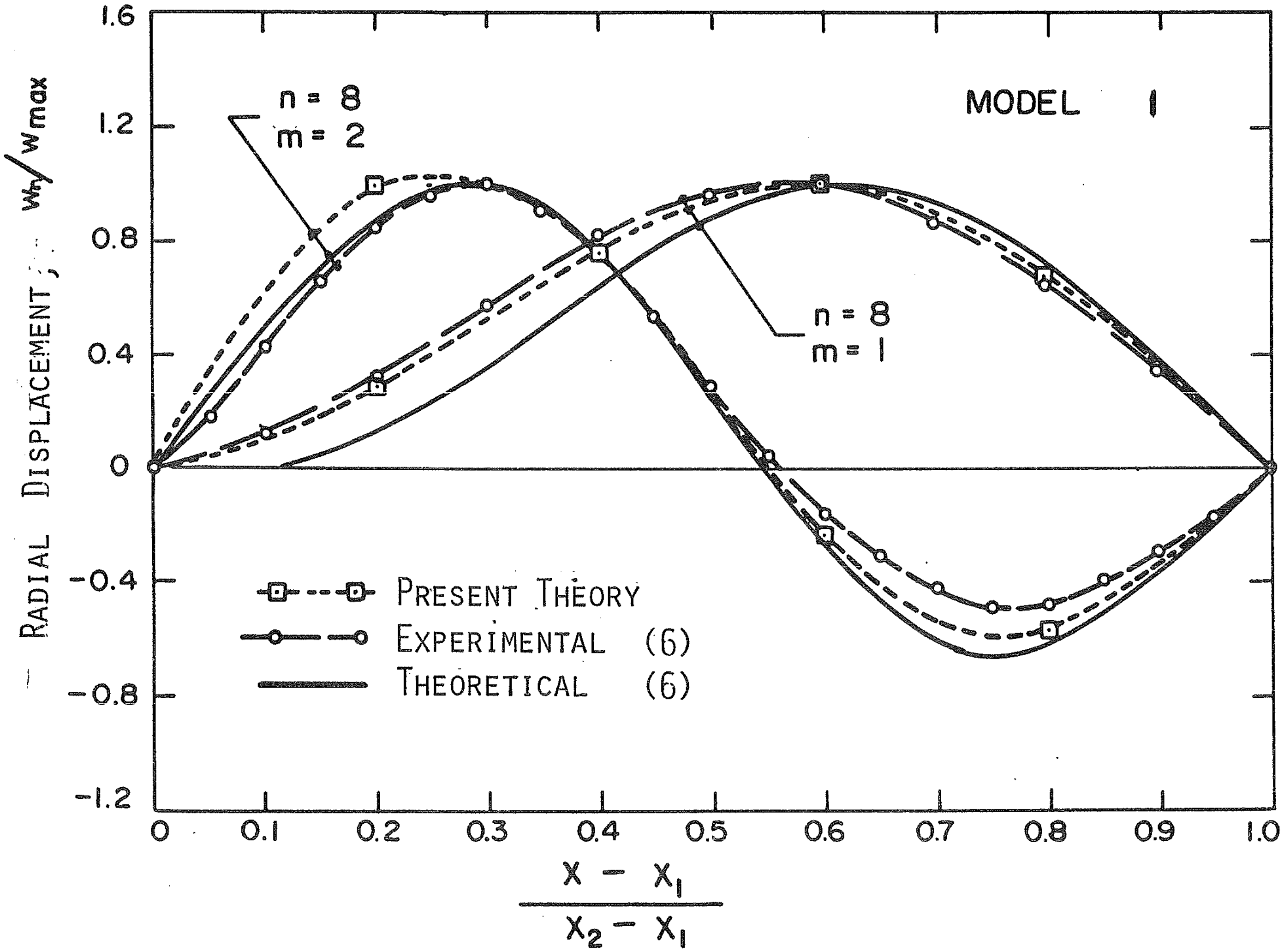


Figure 14 : Normalized radial deformation mode. Model 1 with $w = v = 0$ at boundaries

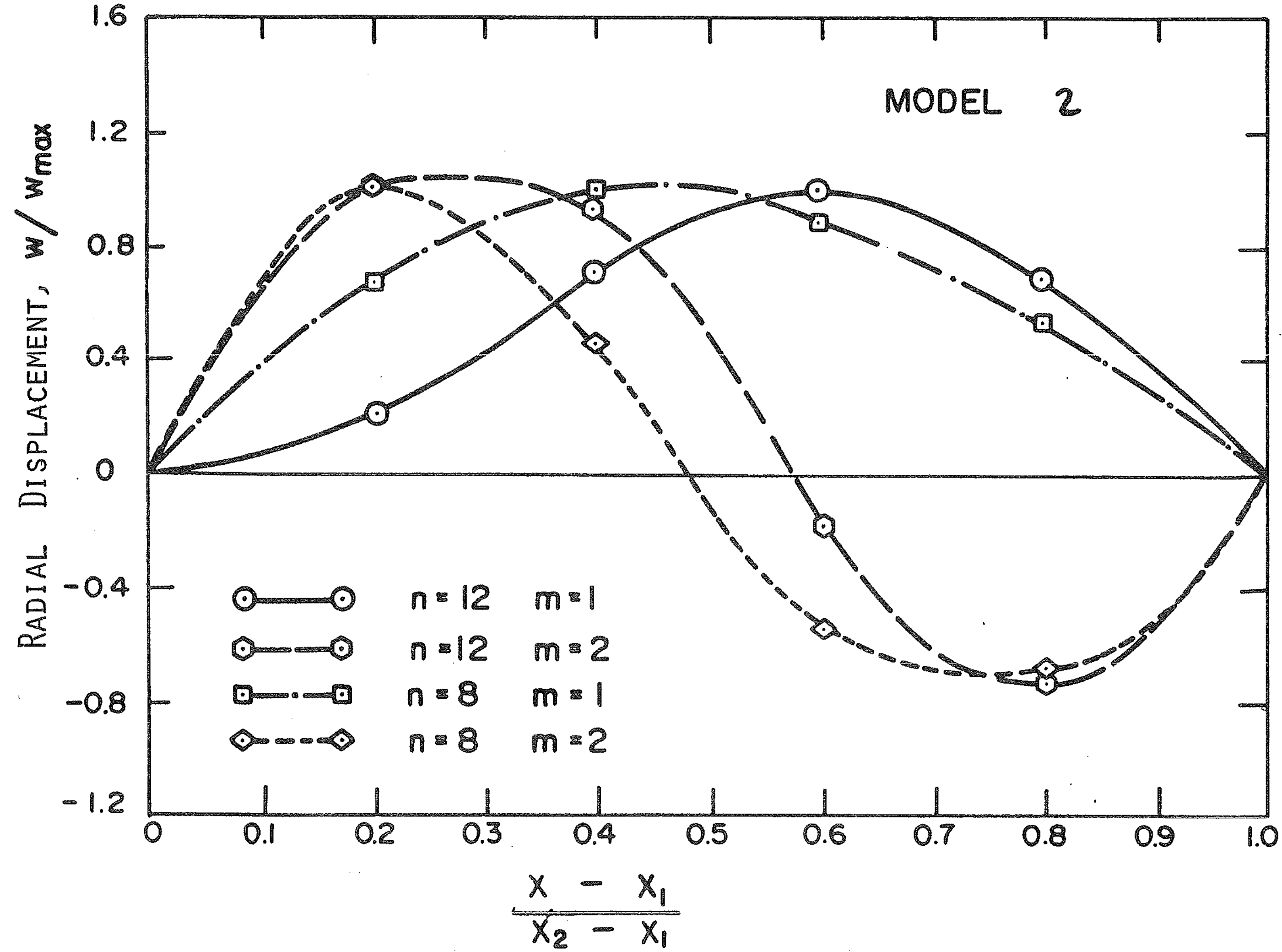


Figure 15 : Normalized radial deformation mode. Model 2 with $w = v = 0$ at boundaries

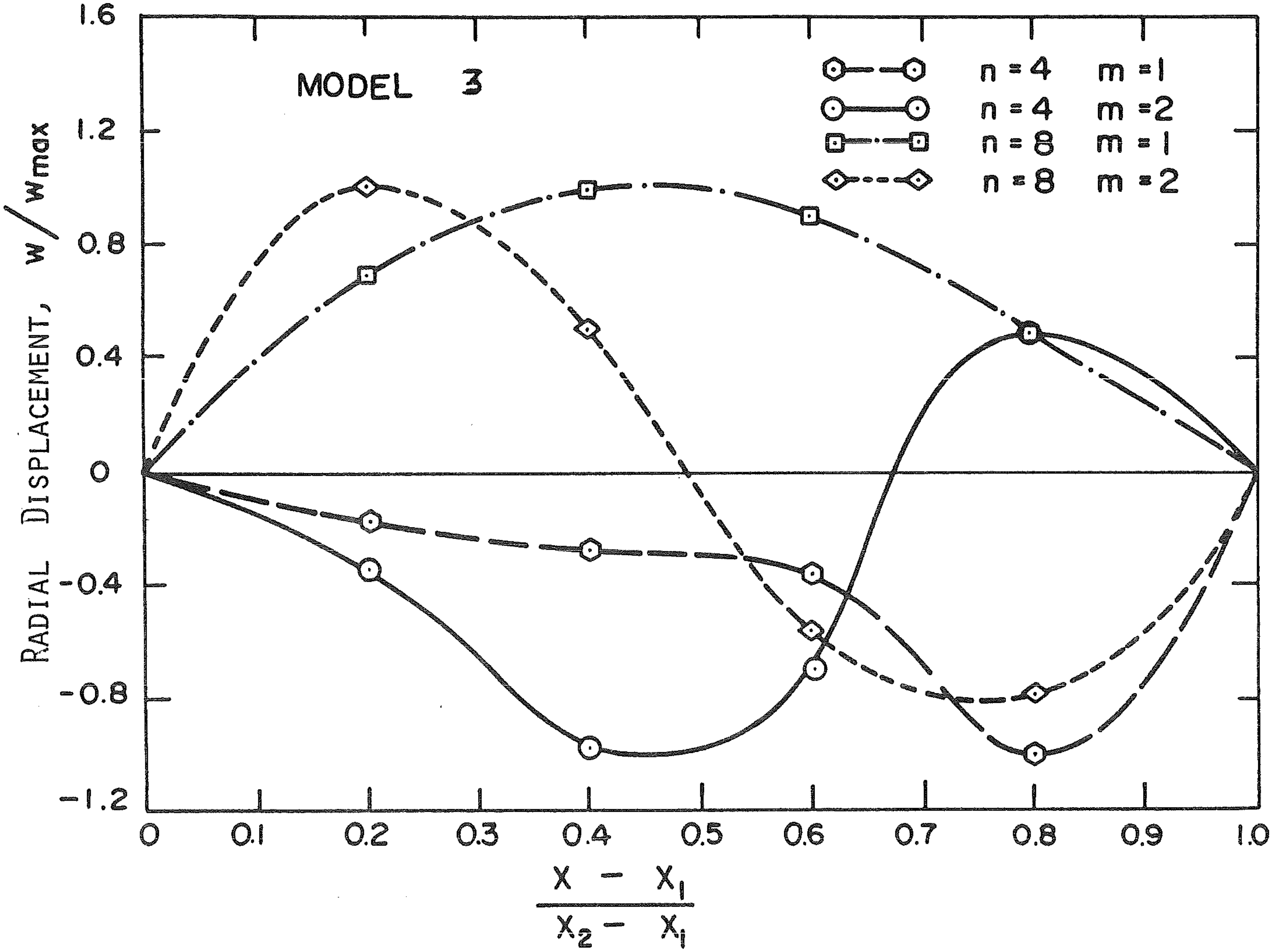


Figure 16 : Normalized radial deformation mode. Model 3 with $w = v = 0$ at boundaries

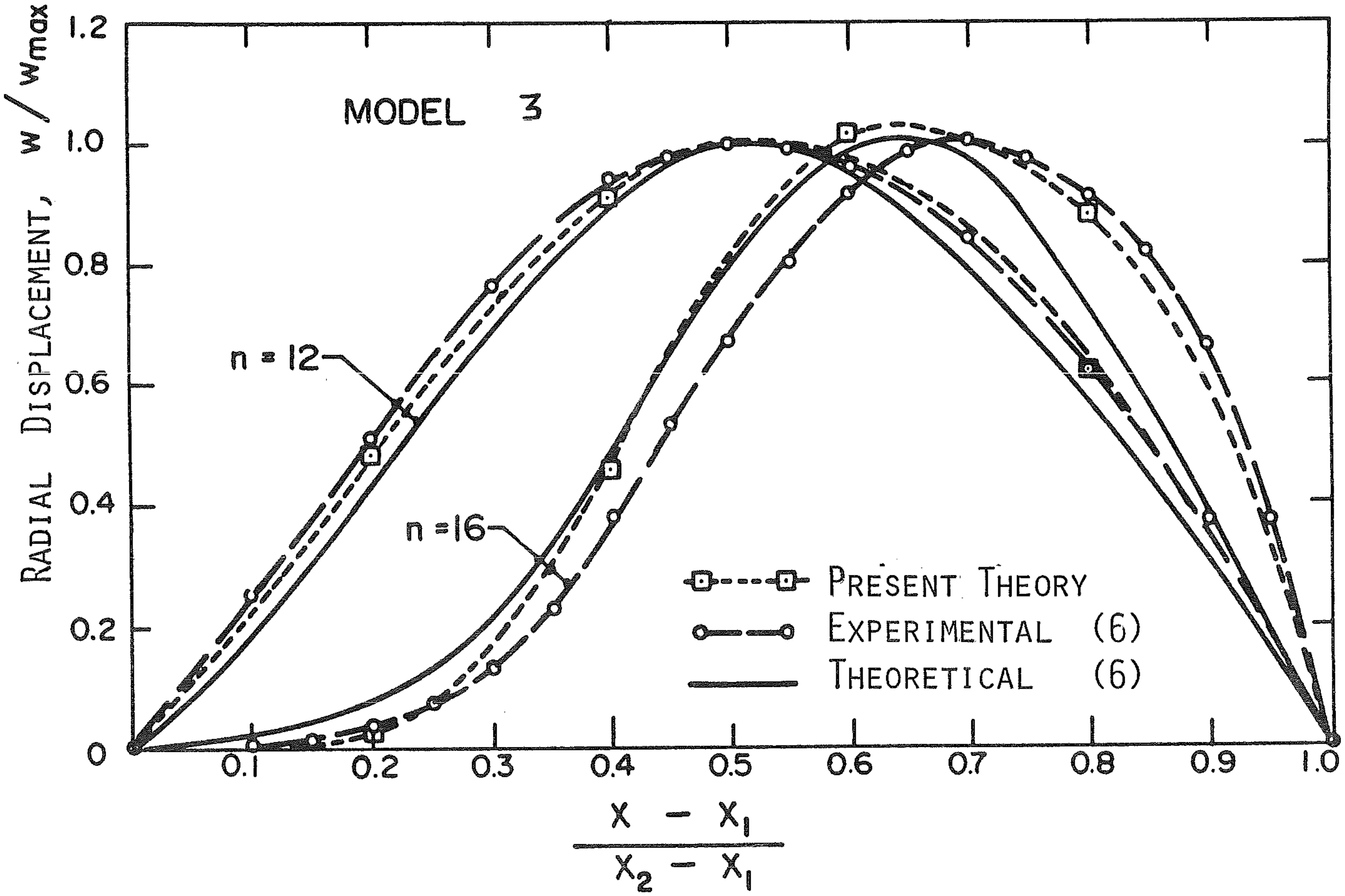


Figure 17 : Normalized radial deformation mode. Model 3 with $w = v = 0$ at boundaries

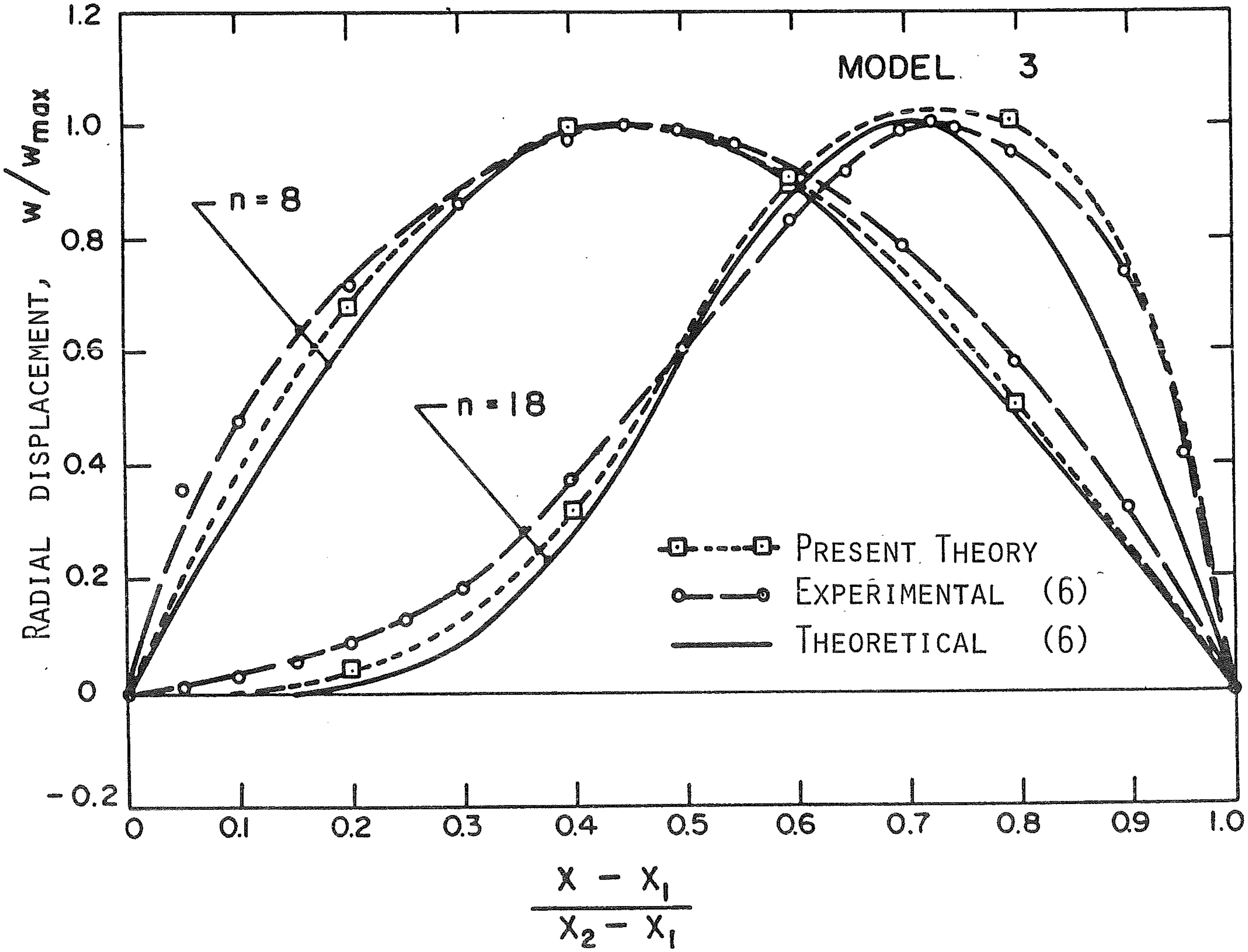


Figure 18 : Normalized radial deformation mode. Model 3 with $w = v = 0$ at boundaries

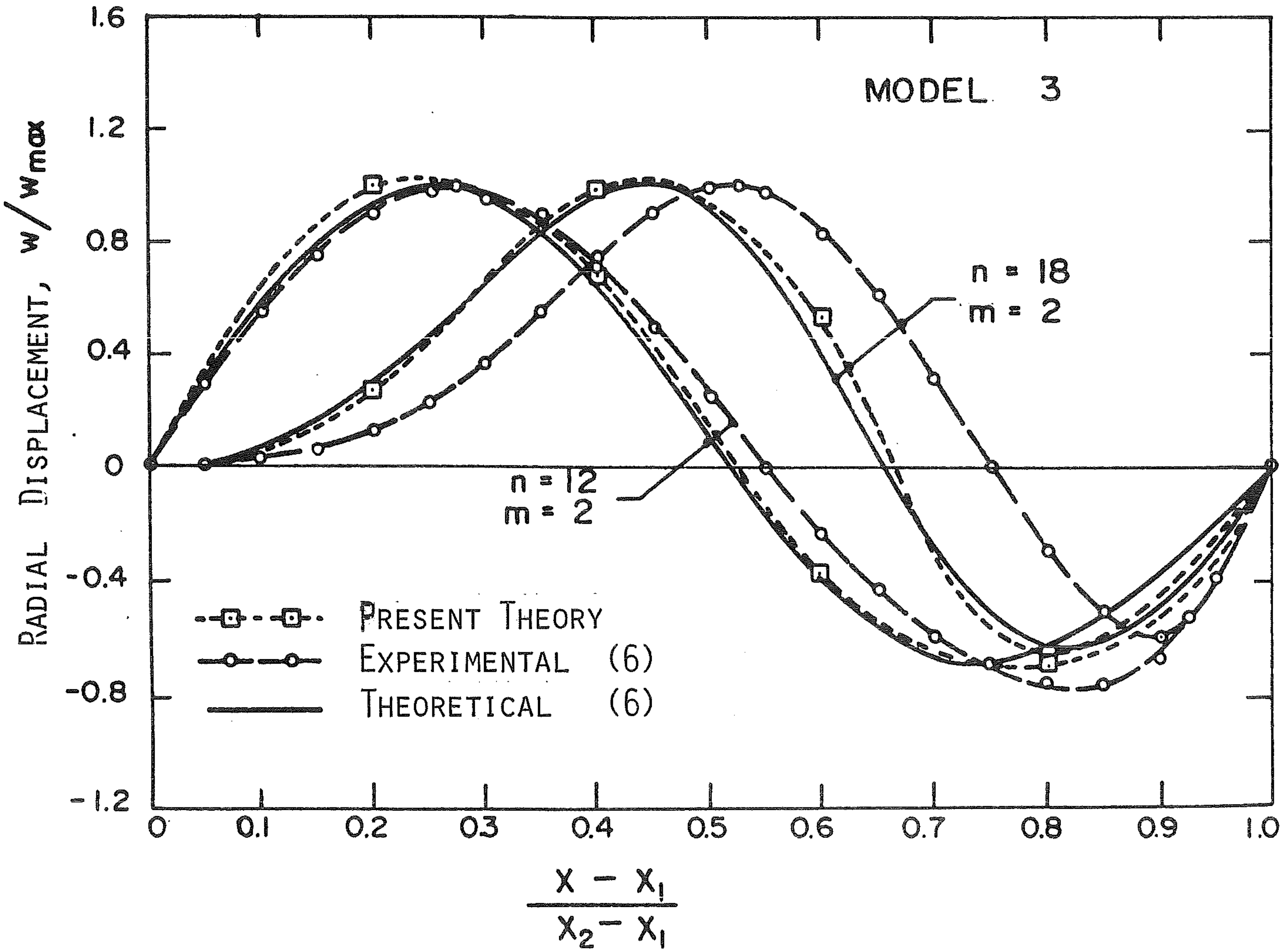


Figure 19 : Normalized radial deformation mode. Model 3 with $w = v = 0$ at boundaries

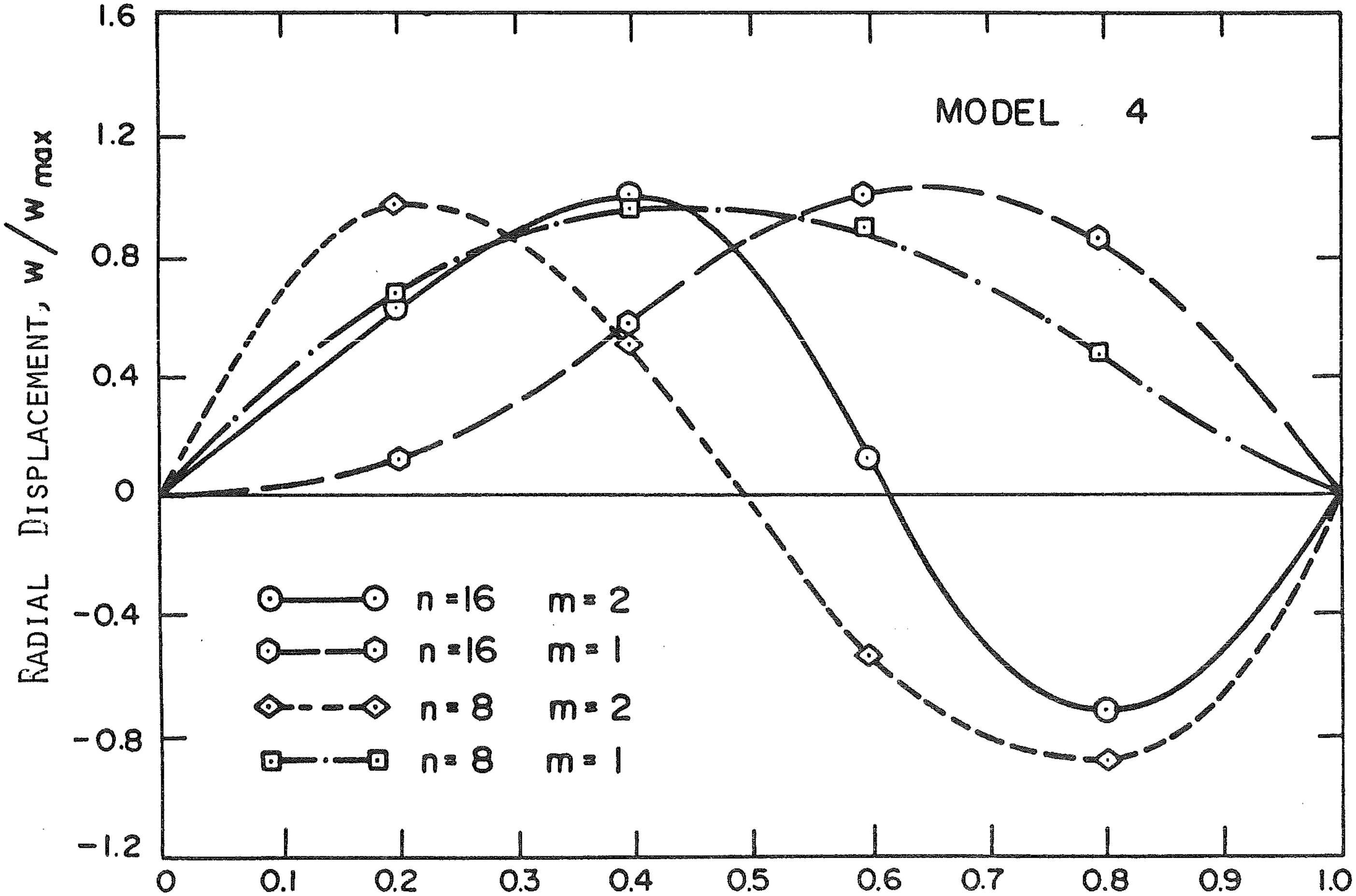


Figure 20 : Normalized radial deformation mode. Model 4 with $w = v = 0$ at boundaries

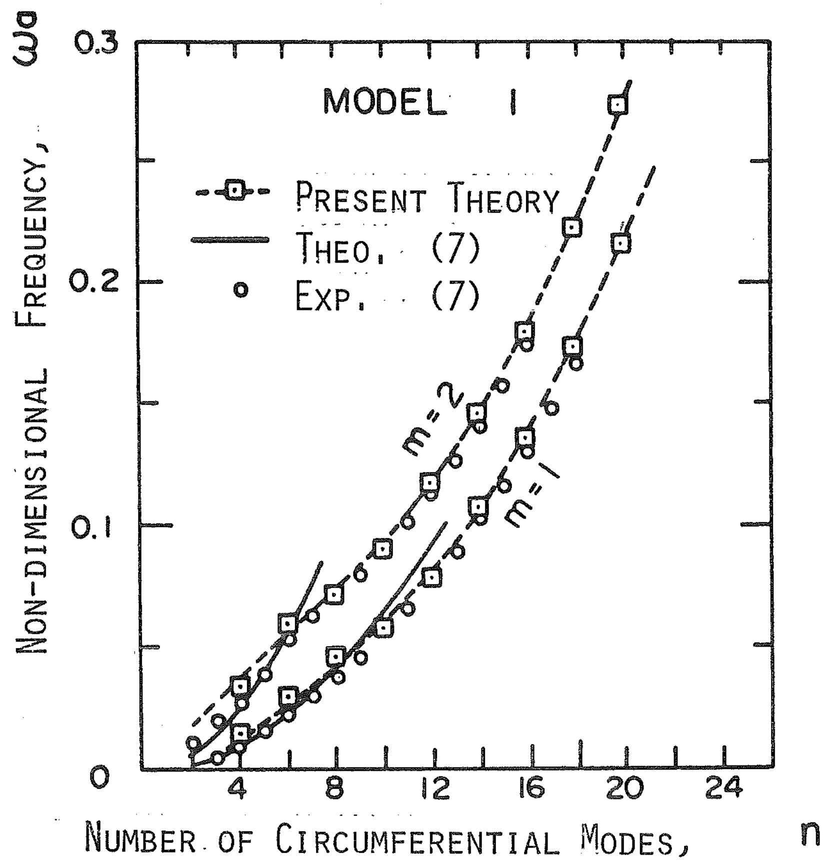


Figure 21 : Non-dimensional frequency for model 1 with free boundary conditions. m = axial mode. $\omega_a = \omega R_2 \; (\rho(1-\nu^2)/E)^{.5}$

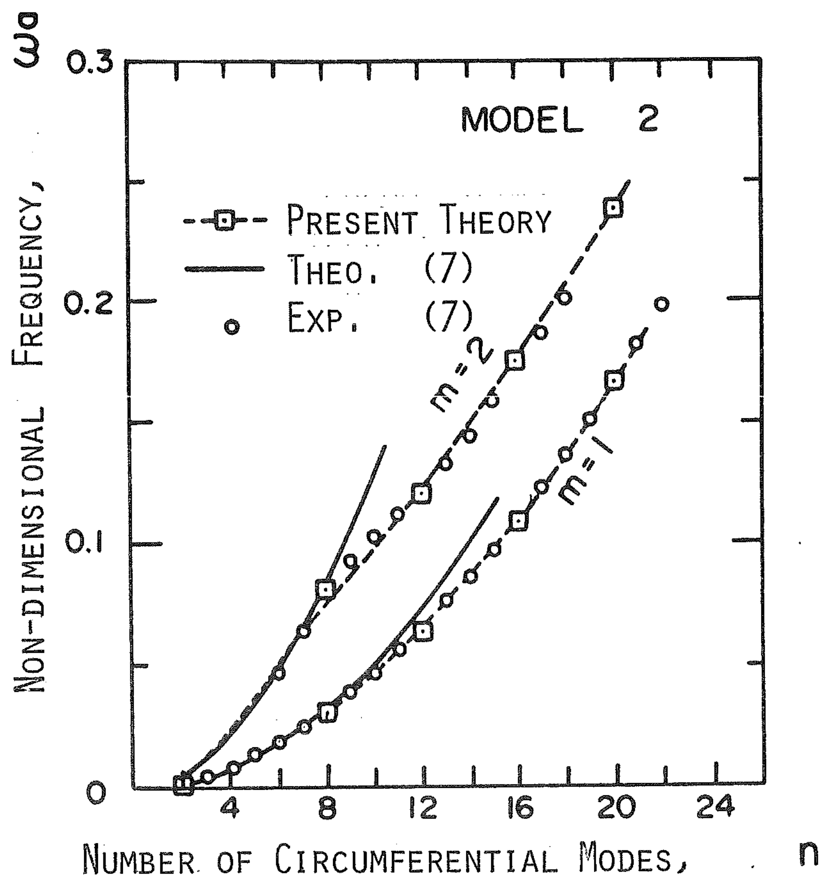


Figure 22 : Non-dimensional frequency for model 2 with free boundary conditions m = axial mode $\omega_a = \omega R_2\ (\rho(1-\nu^2)/E)^{.5}$

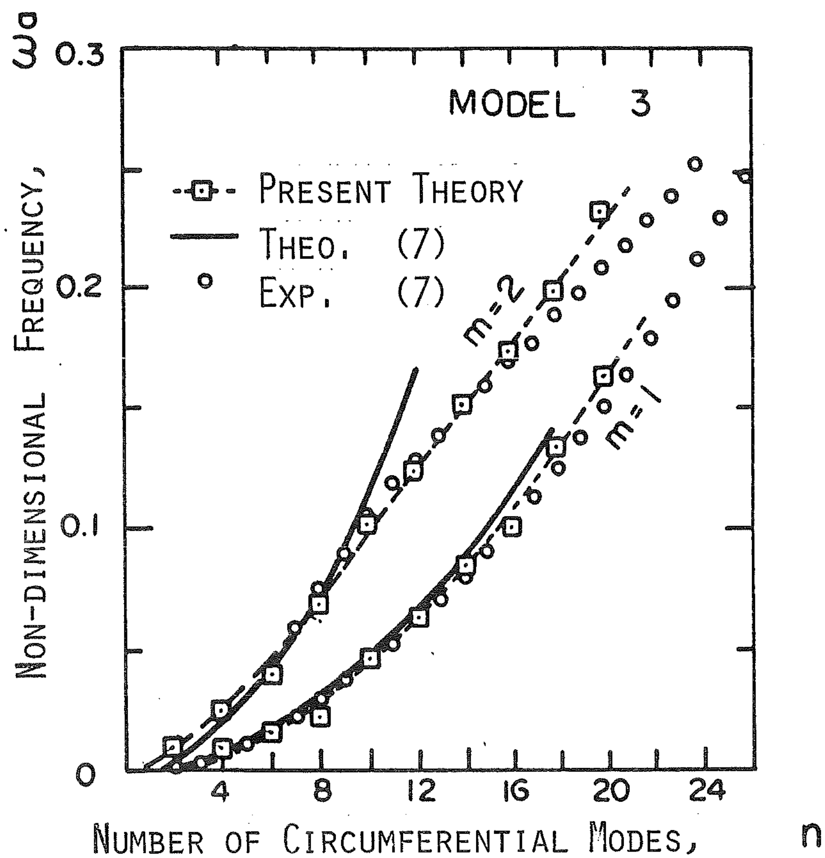


Figure 23 : Non-dimensional frequency for model 3 with free boundary conditions
m = axial mode $\omega_a = \omega R_2 \; (1-\nu^2)/E)^{.5}$

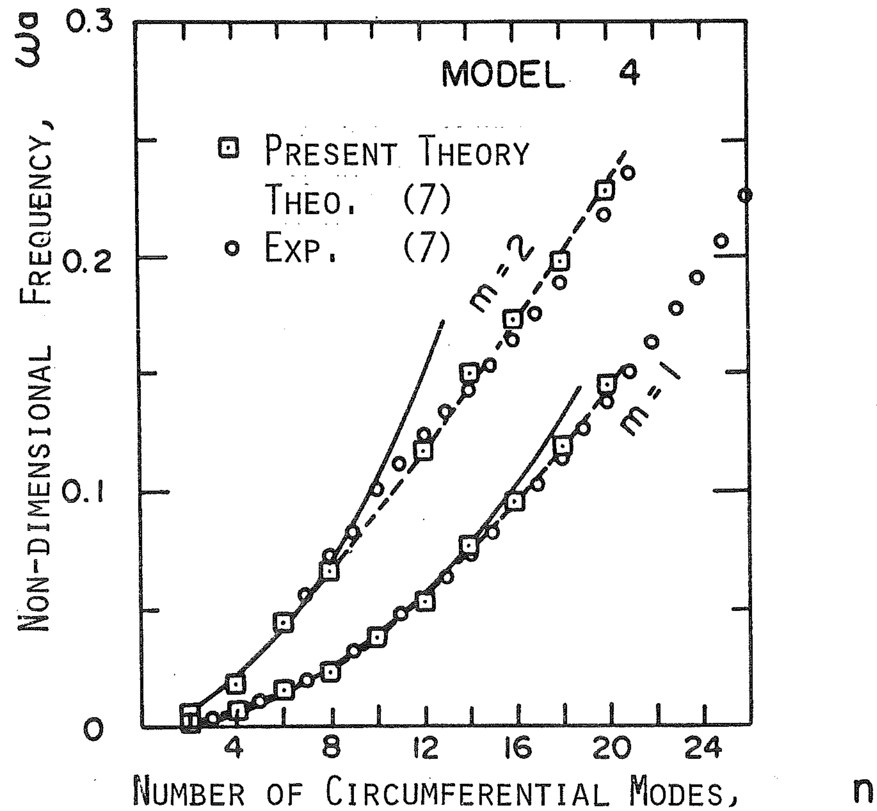


Figure 24 : Non-dimensional frequency for model 4 with free boundary conditions m = axial mode $\omega_a = \omega R_2\ (\rho(1-\nu^2)/E)^{.5}$

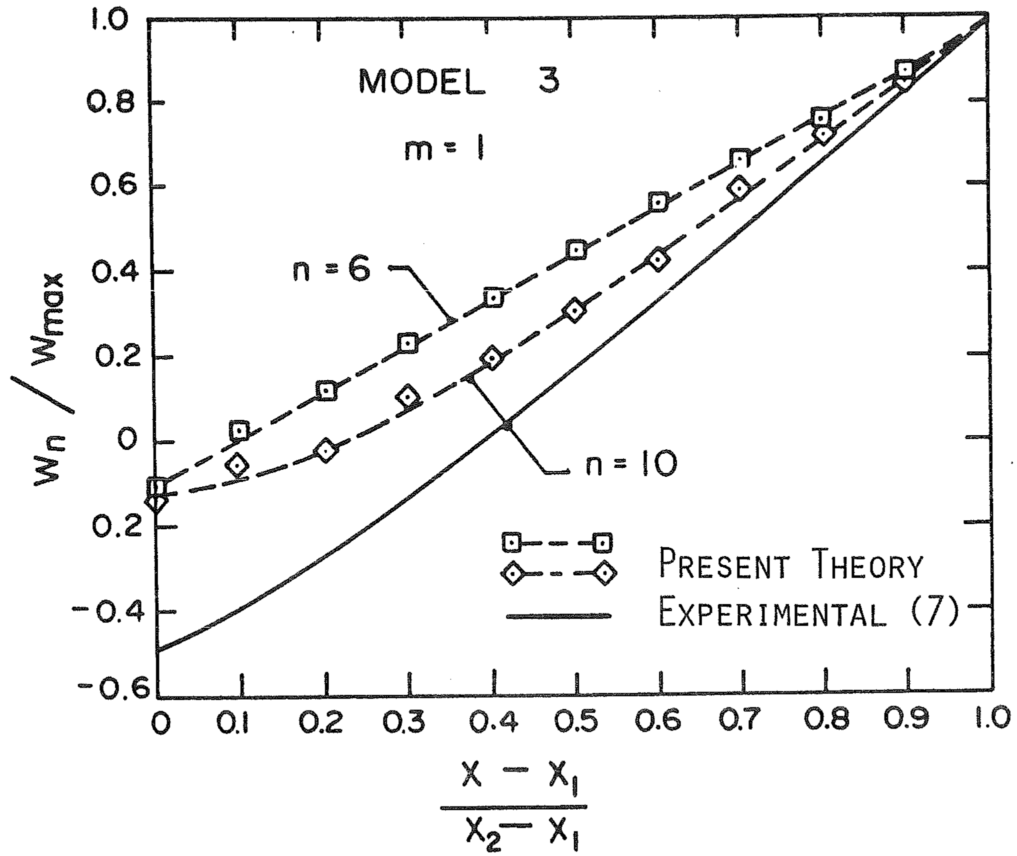


Figure 25 : Normalized radial deformation mode. Model 3 with free boundary conditions

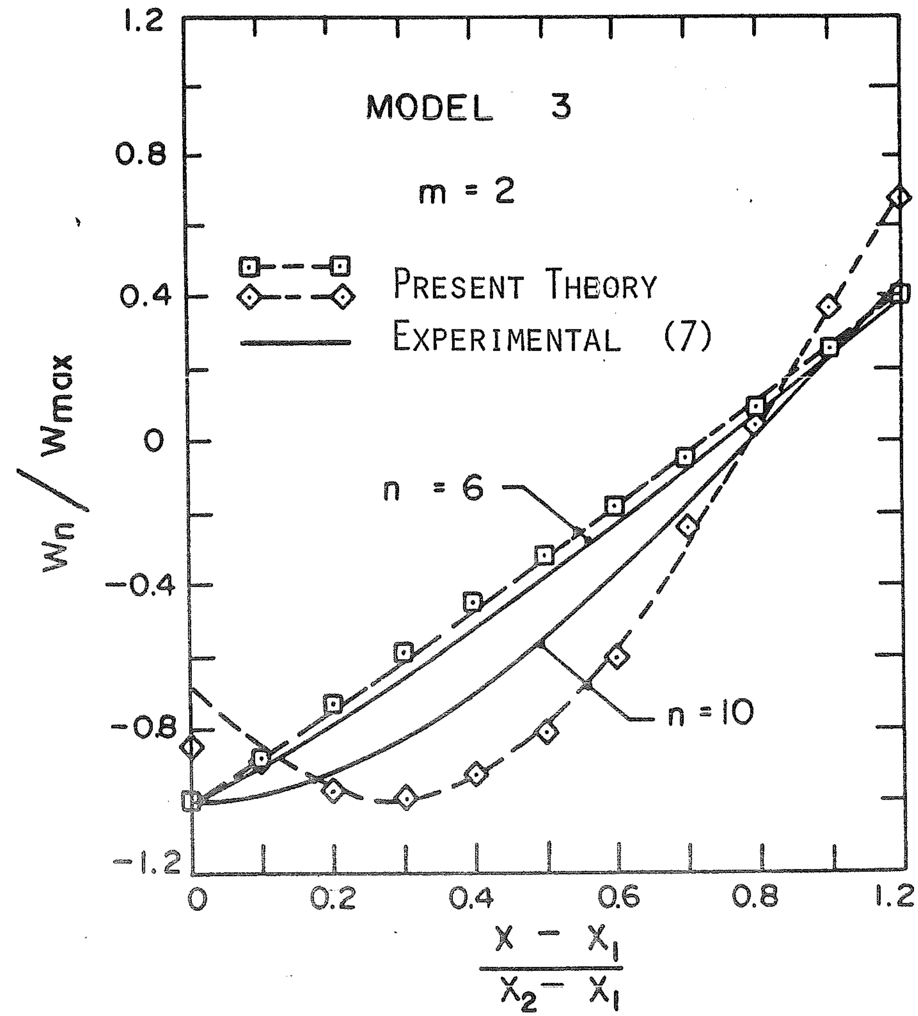


Figure 26 : Normalized radial deformation mode. Model 3 with free boundary conditions

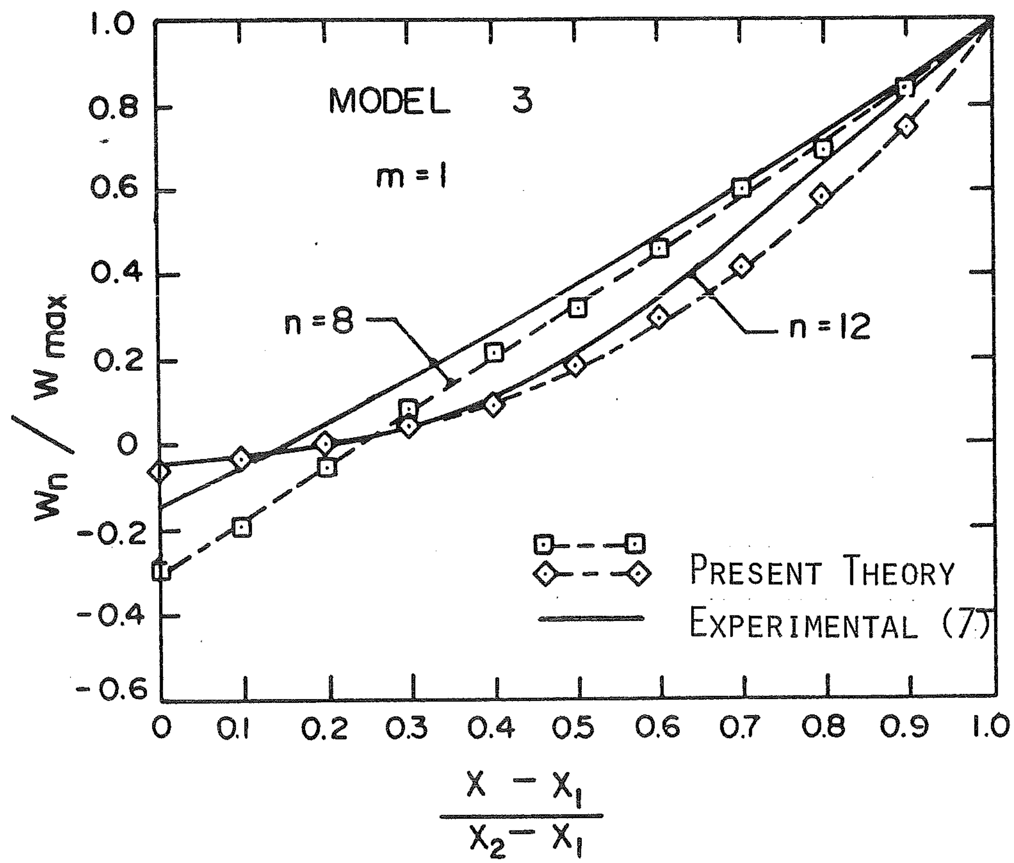


Figure 27 : Normalized radial deformation mode. Model 3 with free boundary conditions

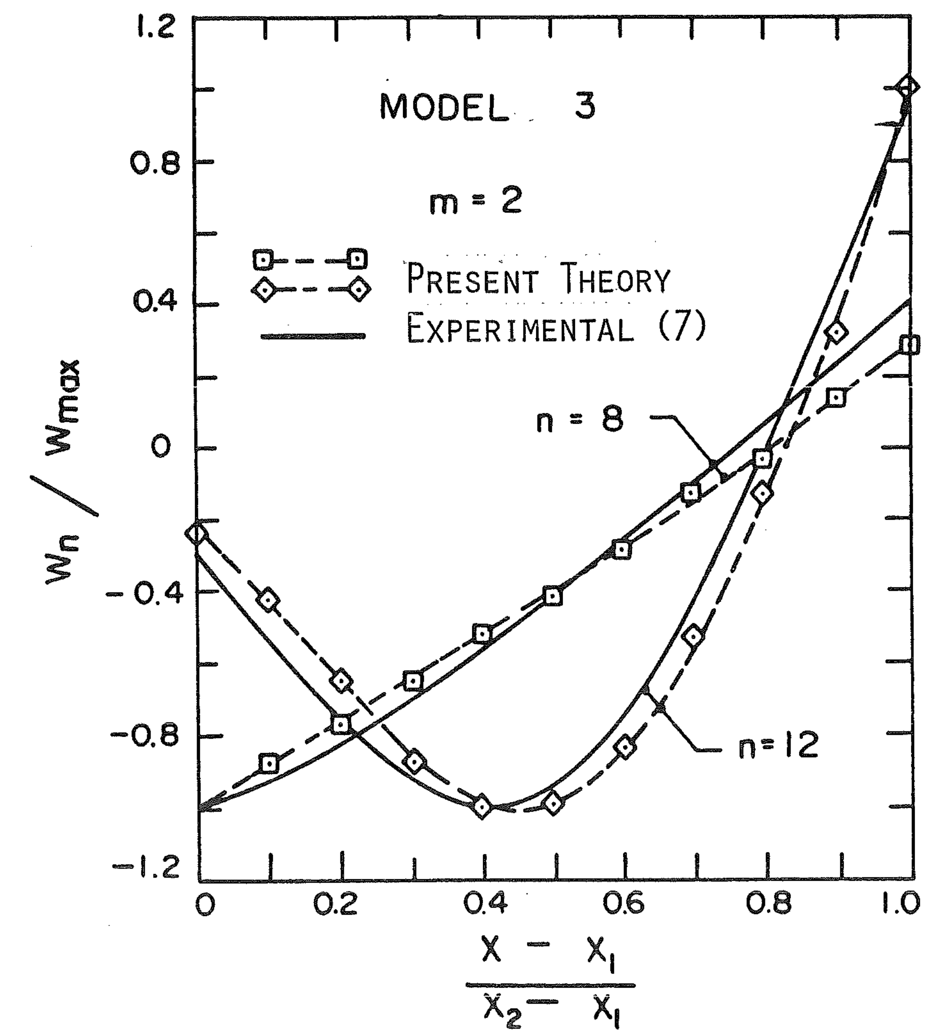


Figure 28 : Normalized radial deformation mode. Model 3 with free boundary conditions

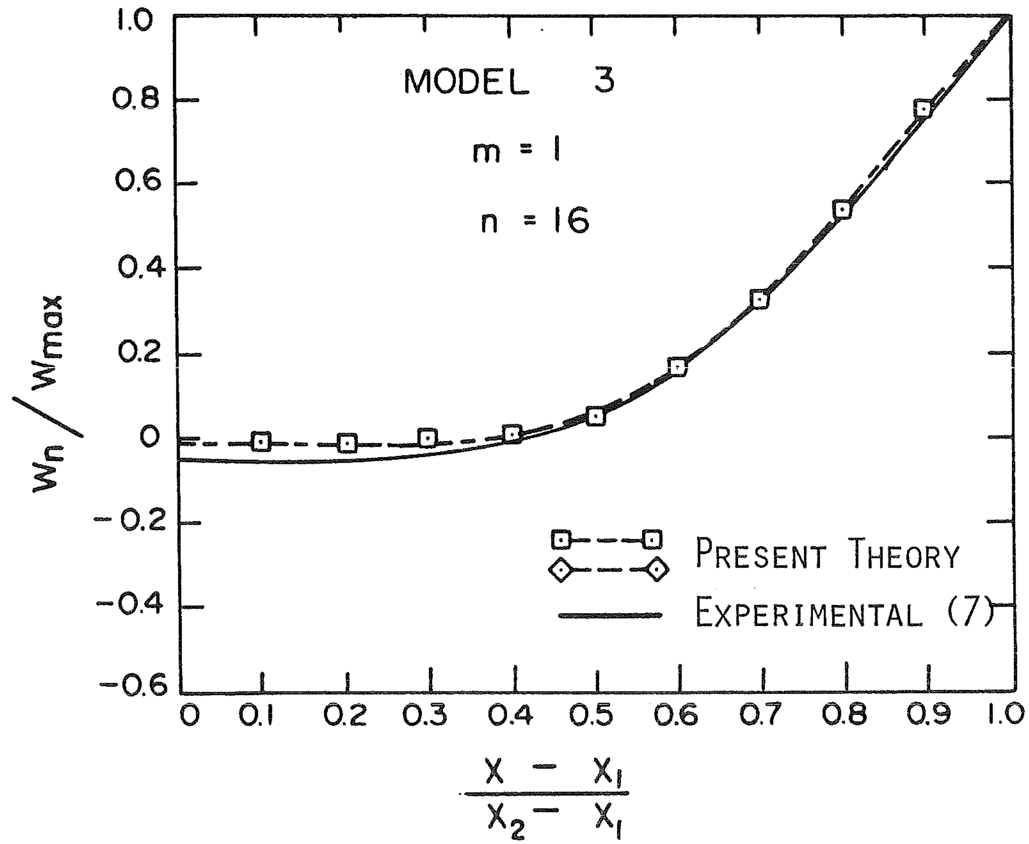


Figure 29 : Normalized radial deformation mode. Model 3 with free boundary conditions

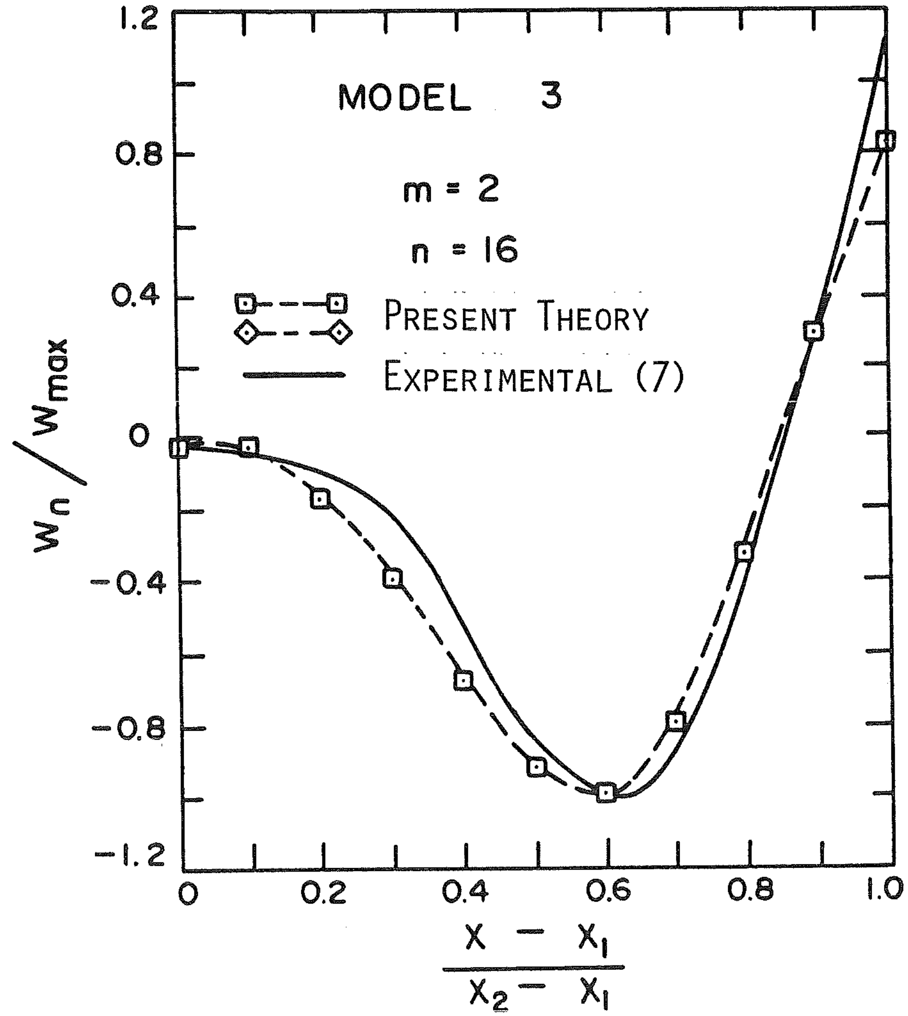


Figure 30 : Normalized radial deformation mode. Model 3 with free boundary conditions