Displacements and Stresses of Pressurized Thin Cylindrical Vessels in Contact with Rigid Supports

by

Aouni A. LAKIS and Roland DORE

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Department of Mechanical Engineering
Ecole Polytechnique
Montréal, Que., Canada

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Abstract

An analysis is presented for the prediction of the stresses, displacements and interface pressure of cylindrical vessels partially filled with liquid, pressurized and having a surface of contact with rigid supports. The supports are subdivided into a number of line and surface elements in the axial and circumferential directions. Each element is subjected to a uniform load q_i, $i = 1, \ldots, N$. The applied loads, expressed in double Fourier series and inserted into shells equations of motion, allow the determination of the corresponding displacements and stresses. Calculations are conducted for such a shell and the results are compared with those obtained by another theory.

Nomenclature

b_i = axial coordinate of element "i", $i = 1, \dots, N$

C_2 = half saddle width

D,K = parameters defined in equation (5)

k = $(1/12)\,(t/r)^2$

ℓ = length of the vessel

m = axial wave-number

n = circumferential wave-number

N_x, N_φ, $N_{x\varphi}$

M_x, M_φ, $M_{x\varphi}$ stress-resultants for a circular cylinder.

N_1, N_2 = number of line elements on the support in the axial and circumferential directions, respectively.

N = total number of elements.

[P] = elasticity matrix

p_x, p_φ, p_r = external applied loads in axial, circumferential and radial directions, respectively.

$p_{x_{mn}}$, $p_{\varphi_{mn}}$, $p_{r_{mn}}$ = load factors corresponding to the applied loads p_x, p_φ and p_r, respectively.

p_o = surcharge pressure

q_i = density of element "i", $i = 1, \dots, N$.

r = mean radius of shell

t = wall-thickness of shell

U, V, W = axial, circumferential and radial displacementsof the vessel

u_{mn}, v_{mn}, w_{mn} = coefficients of the Fourier series for the displacement components U, V and W, respectively.

x = axial coordinate

x_o, x_f = axial coordinates of the saddle (fig. 2)

γ = specific weight of fluid

γ_s = specific weight of the vessel

δ_o = half-total saddle angle

$\{\epsilon\}$ = strain vector

ϵ_x, ϵ_φ, $\epsilon_{x\varphi}$ = axial, circumferential and shear strains of middle surface

K_x, K_φ, $\bar{K}_{x\varphi}$ = axial, circumferential and modified twisting strains

ν = Poisson's ratio

$\{\sigma\}$ = stress-resultant vector

φ = circumferential coordinate

φ_o = angle indicating the level of the liquid in the vessel.

INTRODUCTION

The first attempts to investigate experimetaly saddle supported vessels were made by Hartenberg [1][1], Wilson and Olson [2] in 1941, and were followed in 1951 by an experimental and analytical work by Zick [3] which has been incorporated into British Standard 1515, part 1 = 1965, as mentioned in [6]. Since then the theory of saddle supported vessels has repeatedly been re-examined in the literature [4]-[7].

Forbes and Tooth [6], and Wilson and Tooth [7] formulate an analysis capable of predicting the saddle/cylinder interface pressure and the stress resultants throughout the vessel. They assumed, in their investigation, a constant interface pressure distribution in the axial direction. It is the conviction of the authors that the saddle/vessel interface pressure is non-uniform in the axial as well as in the circumferential directions.

The work presented here is an attempt to produce a general theory capable of giving accurate prediction of the saddle-vessel interface pressure distribution in the axial and circumferential directions as well as the stresses and displacements at any point within the structure with a minimum of limitations and hence, with wide range of applicability. This investigation is divided into two parts. Part 1 which is developed in this paper describes

[1] Number in brackets designates References at end of paper.

a new approach to the point of obtaining the stresses and displacements for a vessel partially-filled with liquid, pressurized and having a surface of contact with rigid supports. To this end, the support is subdivided into a sufficient number of line and surface elements each of which subjected to a load q_i, $i = 1, 2, \ldots, N$. In part 2, under preparation, prediction of the saddle-vessel interface pressure distribution is obtained by minimising the total energy of the system which is derived in terms of the q_i's using the theory developed in part 1. This theory will be capable of analysing geometrically axially non-symmetric, long or short thin cylindrical shells, subject to any set of boundary conditions (including supports other than at the two axial extremities of the shell).

1. BASIC EQUATIONS

The basic equations which describe the static behaviour of cylindrical shells with bending resistance under arbitrary loads are derived from Sanders' theory [8] for thin shells. This theory was used, in preference to Love's of Timoshenko's theories, because in the former all the strains vanish for small rigid-body motions, which is not true for the latter theories. Thus, Sanders' equations of equilibrium for cylindrical shells under distributed loads p_x, p_φ and p_r take the form [9]

$$r^2 \frac{\partial^2 U}{\partial x^2} + \frac{(1-\nu)}{2} \frac{\partial^2 U}{\partial \varphi^2} + \frac{r(1+\nu)}{2} \frac{\partial^2 V}{\partial x\, \partial \varphi} + r\nu \frac{\partial W}{\partial x} + k \left[\frac{(1-\nu)}{8} \frac{\partial^2 U}{\partial \varphi^2} - \frac{3(1-\nu)}{8} r \frac{\partial^2 V}{\partial x\, \partial \varphi} + \frac{(1-\nu)}{2} r \frac{\partial^3 W}{\partial x\, \partial \varphi^2}\right] = - p_x \frac{r^2}{D} ,$$

$$\frac{(1+\nu)r}{2} \frac{\partial^2 U}{\partial x\, \partial \varphi} + \frac{\partial^2 V}{\partial \varphi^2} + \frac{(1-\nu)r^2}{2} \frac{\partial^2 V}{\partial x^2} + \frac{\partial W}{\partial \varphi} + k\left[- \frac{3(1-\nu)r}{8} \frac{\partial^2 U}{\partial x\, \partial \varphi} + \frac{9}{8}(1-\nu)r^2 \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial \varphi^2} - \frac{(3-\nu)r^2}{2} \frac{\partial^3 W}{\partial x^2\, \partial \varphi} - \frac{\partial^3 W}{\partial \varphi^3}\right] = - p_\varphi \frac{r^2}{D} , \qquad (1)$$

$$-\nu r \frac{\partial U}{\partial x} - \frac{\partial V}{\partial \varphi} - W + k \left[\frac{(\nu-1)r}{2} \frac{\partial^3 U}{\partial x\, \partial \varphi^2} + \frac{(3-\nu)r^2}{2} \frac{\partial^3 V}{\partial x^2\, \partial \varphi} + \frac{\partial^3 V}{\partial \varphi^3} - r^4 \frac{\partial^4 W}{\partial x^4} - 2r^2 \frac{\partial^4 W}{\partial x^2\, \partial \varphi^2} - \frac{\partial^4 W}{\partial \varphi^4}\right] = p_r \frac{r^2}{D} ,$$

and the strain-displacement relations are

$$\{\varepsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_\varphi \\ 2\epsilon_{x\varphi} \\ \varkappa_x \\ \varkappa_\varphi \\ 2\bar{\varkappa}_{x\varphi} \end{Bmatrix} = \begin{Bmatrix} \partial U/\partial x \\ (1/r)\ (\partial V/\partial\varphi) + (W/r) \\ \partial V/\partial x + (1/r)\ (\partial U/\partial\varphi) \\ -\ \partial^2 W/\partial x^2 \\ -\ (1/r^2)\ [(\partial^2 W/\partial\varphi^2) - (\partial V/\partial\varphi)] \\ -\ (2/r)\ (\partial^2 W/\partial x\partial\varphi) + (3/2r)\ (\partial V/\partial x) - (1/2r^2)\ (\partial U/\partial\varphi). \end{Bmatrix} \tag{2}$$

Here U, V and W are, respectively, the axial, circumferential and radial displacements of the middle surface of the shell, r its radius, and $k = (1/12)\ (t/r)^2$; ν is Poisson's ratio and t is the thickness of the shell (Figure 1).

The appropriate set of stress-strain relations is given by

$$\{\sigma\} = \begin{Bmatrix} N_x \\ N_\varphi \\ \bar{N}_{x\varphi} \\ M_x \\ M_\varphi \\ \bar{M}_{x\varphi} \end{Bmatrix} = [P]\ \{\varepsilon\}\ , \tag{3}$$

where $\bar{N}_{x\varphi} = \frac{1}{2}\ (N_{x\varphi} + N_{\varphi x})$, $\bar{M}_{x\varphi} = \frac{1}{2}\ (M_{x\varphi} + M_{\varphi x})$ and [P], the elasticity matrix, is given by

$$[P] = \begin{bmatrix} D & \nu D & 0 & 0 & 0 & 0 \\ \nu D & D & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{D(1-\nu)}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & K & \nu K & 0 \\ 0 & 0 & 0 & \nu K & K & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{K(1-\nu)}{2} \end{bmatrix}, \quad (4)$$

where the stiffness parameters K and D, for an isotropic elastic material, are given by

$$K = Et^3/12(1-\nu^2) \quad , \quad D = Et/(1-\nu^2) \quad . \quad (5)$$

2. SHELLS SUBJECTED TO ARBITRARY DISTRIBUTED LOADINGS

A cylindrical shell partially filled with liquid, pressurized and having a surface of contact with a rectangular rigid support of dimensions (a x 2b) is shown in Figure 2. The location of the support on the vessel is arbitrary, with coordinates δ_o and (x_o, x_f) in the circumferential and axial directions, respectively. The theory developed here for one support may be applied to shells having two or more supports. Also, the assumption of a vertical plane of symmetry of the loads through the axis of the shell, permits the investigation of only half the support(a x b).

These dimensions "a" and "b" are first subdivided into N_1 and N_2 line elements, respectively; and then by assuming two point loads on the boundary A and B of the support (see Figure 2), N finite elements are obtained and distributed over the area, a x b, as follows: a) two concentrated loads of densities q_i (lb or kg), i = 1, 2, applied on the points A and B; b) $(N_1 + 2N_2)$ line loads applied on the line elements and of densities q_i, (lb/in or kg/m), $i = 3, 4, \ldots, 2N_2 + N_1 + 2$; and c) N_1N_2 surface loads of densities q_i (lb/in^2 or kg/m^2), $i = 2N_2 + N_1 + 3$ to N; where N is given by

$$N = N_1N_2 + 2N_2 + N_1 + 2 \quad . \tag{6}$$

The pressure distribution and the surface of contact between the rigid support and the shell are represented by the densities q_i's where $i = 1, 2, \ldots, N$.

The purpose of this study is to formulate a general procedure in order to obtain the displacements and the stresses induced by the q_i's, the fluid, the surcharge pressure, the weight of the vessel and its heads. To do so, the loads and the displacements are first represented by double Fourier series and then inserting these series into the equations of motion, the coefficients of the Fourier series are obtained for the displacement components in terms of the load factors p_{xmn}, $p_{\varphi mn}$, p_{rmn} for axial, circumferential and radial directions, respectively. The second step is to determine these loads factors p_{xmn} , $p_{\varphi mn}$ and p_{rmn} in terms of the knowns q_i's for each particular case.

2.1 Arbitrary loadings

Using the method of Fourier expansions and assuming that, for a cylinder of length ℓ,

$$\begin{Bmatrix} p_x \\ p_r \\ p_\varphi \end{Bmatrix} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} [T_n] [T_{mx}] \begin{Bmatrix} p_{xmn} \\ p_{rmn} \\ p_{\varphi mn} \end{Bmatrix} , \qquad (7)$$

and

$$\begin{Bmatrix} U_p \\ W_p \\ V_p \end{Bmatrix} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} [T_n] [T_{mx}] \begin{Bmatrix} u_{mn} \\ w_{mn} \\ v_{mn} \end{Bmatrix} , \qquad (8)$$

it is possible to obtain, by introducing (7) and (8) into (1), the coefficients of the Fourier series for the displacement components as follows

$$\begin{Bmatrix} u_{mn} \\ w_{mn} \\ v_{mn} \end{Bmatrix} = \frac{r^2}{D} [A_F]^{-1} \begin{Bmatrix} p_{xmn} \\ -p_{rmn} \\ p_{\varphi mn} \end{Bmatrix} , \qquad (9)$$

where m and n are the axial and circumferential wave number, respectively, and the matrices $[T_n]$, $[T_{mx}]$ and $[A_F]$ are shown in Appendix 1.

When the loading can be represented by Fourier series of the form

$$\begin{Bmatrix} p_x \\ p_r \\ p_\varphi \end{Bmatrix} = \sum_{n=0,2,3,4,\ldots,}^{\infty} [T_n] \begin{Bmatrix} p_{xon} \\ p_{ron} \\ p_{\varphi on} \end{Bmatrix} , \qquad (10)$$

The displacements can, in general, be assumed to have the form

$$\begin{Bmatrix} U_p \\ W_p \\ V_p \end{Bmatrix} = \sum_{n=0,2,3,4,\ldots,}^{\infty} [T_n] \begin{Bmatrix} u_{on} \\ w_{on} \\ v_{on} \end{Bmatrix} , \qquad (11)$$

where the coefficients u_{on}, w_{on} and v_{on}, for $n \neq 1$, may be obtained from equation (6) by imposing $m = 0$. The case of loading for which $n = 1$ in (10) requires special treatment and will be considered later.

Equations (7), (8) and (9) allow the pressures and displacements to be expressed in terms of the load factors p_{xmn}, p_{rmn} and $p_{\varphi mn}$. The object now is to determine these load factors corresponding to each particular case of loading.

2.2 Expressions for load factors corresponding to point load, line load and surface load.

a) Concentrated radial loads.

We assume two concentrated radial forces of intensity q_i (lb or kg) applied at the coordinate $(x,\varphi) = (b_i \pm \delta_i)$, Figure 3.

The load factors corresponding to such forces are as follows

$$p_{rmoi} = \frac{2q_i}{\pi r \ell} \sin \frac{m \pi b_i}{\ell} , \quad m = 1,2,3,\ldots, \tag{12}$$

and

$$p_{rmni} = \frac{4\, q_i}{\pi r \ell} \cos n\, \delta_i \sin \frac{m \pi b_i}{\ell} , \quad n,m = 1,2,3,\ldots, \tag{13}$$

where $i = 1$ and 2. By referring to figure 2 we obtain $\delta_1 = \delta_2 = \delta_0$, $b_1 = x_0$ and $b_2 = x_f$.

b) Line loads on two segments along the generator.

Let the segments, on which a constant inward line load q_i (lb/in or kg/m) is applied, be contered at $(x,\varphi) = (b_i \pm \delta_i)$ and of dimension 2 c_2 along the axial direction, Figure 4.

The coefficients of the series expansion for this loading is given by

$$p_{rmoi} = \frac{4\, q_i}{\pi^2\, r\, m} \sin \frac{m \pi C_2}{\ell} \sin \frac{m \pi b_i}{\ell} , \quad m = 1,2,3,\ldots, \tag{14}$$

and

$$p_{rmni} = \frac{8\ q_i}{\pi^2\ r\ m} \sin \frac{m\ \pi\ C_2}{\ell} \cos n\ \delta_i \sin \frac{m\ \pi\ b_i}{\ell}\ , \qquad n,m = 1,2,3,\ldots, \quad (15)$$

where $i = 3,4,\ldots, N_1 + 2$.

c) Line loads on two segments perpendicular to the generator.

For the cases where the shell is subjected to a line load q_i (lb/in or kg/m) perpendicular to the generator, figure 5, and of dimension $2r\ \varphi_1$, the load factor p_{rmni} is given by

$$p_{rmoi} = \frac{4\ \varphi_1\ q_i}{\pi\ \ell} \sin \frac{m\ \pi\ b_i}{\ell}\ , \qquad m = 1,2,3,\ldots, \quad (16)$$

and

$$p_{rmni} = \frac{8\ q_i}{\pi\, n\, \ell} \sin n\ \varphi_1 \cos n\ \delta_i \sin \frac{m\ \pi\ b_i}{\ell}\ , \qquad n,m = 1,2,3,\ldots, \quad (17)$$

where b_i is equal to x_o and x_f for $i = N_1 + 3$ to $N_1 + N_2 + 2$, and for $i = N_1 + N_2 + 3$ to $N_1 + 2N_2 + 2$, respectively; figure 2.

d) Constant pressure uniformly distributed over two rectangular areas.

Consider two rectangular areas subjected to a constant loading q_i, centered at the coordinates $(x,\varphi) = (b_i \pm \delta_i)$, and having the dimensions $2C_2$ and $2r\ \varphi_1$ along the axial and circumferential directions, respectively (Figure 6).

The Fourier series expansion of the pressure, p_{ri}, due to the radial loading, q_i, being given by equation (7), one obtains the following expression for the load factor p_{rmni}

$$p_{rmoi} = \frac{8\,\varphi_1\,q_i}{\pi^2\,m} \sin\frac{m\,\pi\,C_2}{\ell} \sin\frac{m\,\pi\,b_i}{\ell}, \quad \text{for} \quad m = 1,2,3,\ldots, \tag{18}$$

and

$$p_{rmni} = \frac{16\,q_i}{\pi^2\,m\,n} \sin n\,\varphi_1 \sin\frac{m\,\pi\,C_2}{\ell} \cos n\,\delta_i \sin\frac{m\,\pi\,b_i}{\ell},$$

$$\text{for} \quad m,n = 1,2,3,\ldots, \tag{19}$$

where $i = N_1 + 2N_2 + 3,\ldots, N$; and N is given by equation (6).

2.3 Pressure corresponding to a partially-filled vessel.

The pressure distribution on a shell partially or completely filled with stationary liquid is given by

$$p(x,\varphi) = -\gamma r(\cos\varphi - \cos\varphi_o), \quad \text{for} \quad -\varphi_o \le \varphi \le \varphi_o, \tag{20}$$

and $p(x,\varphi)$ is equal to zero elsewhere. Here γ (lb/in^3 or kg/m^3) is the specific weight of the fluid and φ_o indicates the level of the liquid in the shell, Figure 7.

The series expansion for the loading shown in figure 7 is given by equation (10) and its corresponding load factor may be written in the form

$$p_{ro} = -(\gamma r/\pi)\ [\sin\varphi_o - \varphi_o\cos\varphi_o],$$

$$p_{r1} = -(\gamma r/\pi)\ [\varphi_o - (\sin 2\varphi_o/2)],$$

$$p_{rn} = -[2\gamma r/(\pi n(n^2-1))]\ [\cos\varphi_o \sin n\,\varphi_o - n\sin\varphi_o\cos n\,\varphi_o],$$

$$n = 2,3,\ldots, \tag{21}$$

2.4 Surcharge pressure, weight of vessel and heads.

The heads of the vessel are assumed to be rigid and, consequently, their effects will be taken into account by prescribing the appropriate boundary conditions on the shell's edges.

The loading corresponding to a surcharge pressure p_o is given by

$$p_r\,(x,\varphi) = -\,p_o \quad , \tag{22}$$

and the loading due to the weight of the vessel may be written as

$$\begin{aligned} p_\varphi\,(x,\varphi) &= -\,\gamma_s\, t \sin\varphi \quad , \\ p_r\,(x,\varphi) &= -\,\gamma_s\, t \cos\varphi \quad , \end{aligned} \tag{23}$$

where γ_s is the specific weight of the shell and t its thickness.

3. DISPLACEMENTS AND STRESS-RESULTANTS

The displacements due to arbitrary distributed loadings may be obtained by introducing the coefficients given by equations (12) to (23) into relations (9). However, equations (9) and (11) cannot be applied to cases where the loadings are expressed in terms of the form $(a \cos \varphi)$ and $(b \sin \varphi)$; this is due to the fact that expressions in $d \cos \varphi$ and $g \sin \varphi$, for the radial displacement W, correspond to rigid body motion of the shell and therefore, they do not represent the true displacement caused by loadings expressed in terms of the same form. In order to avoid this difficulty, it was necessary to either expand the constant "a", and "b" in a Fourier sine series of the form $\Sigma D_n \sin (m \pi x/\ell)$ and thus to lengthen the numerical computations, or to obtain a particular solution using shells' membrane theory. It was decided to use the latter alternative, for these particular case since such solution which is easily obtainable describes adequately the behavior of a cylindrical shell closed by stiff heads. Such special cases occur when $m = 0$ and $n = 1$, e.g., for the weight of the shell and for the terms, $m = 0$ and $n = 1$, of the fluid pressure. In all other cases, where $(m = 0, n \neq 1)$ and $m \neq 0$, the exact solution, developed in previous sections, was used with bending resistance under arbitrary loads to determine the corresponding displacements and stresses.

The stress-resultant vector, $\{\sigma\} = \{N_x, N_\varphi, \bar{N}_{x\varphi}, M_x, M_\varphi, \bar{M}_{x\varphi}\}^T$, for different loadings is obtained by substituting the corresponding displacement relations into equation (2) and thence into equation (3).

a) Surcharge pressure

The surcharge pressure p_o induces displacements and stresses for the case $n = 0$; accordingly, we may write

$$\begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_{po} = \begin{Bmatrix} p_o\, r\, x\, (1 - 2\nu)/2Et \\ p_o\, r^2\, (1 - 0.5\nu)/Et \\ 0 \\ 0 \end{Bmatrix} , \tag{24}$$

and

$$\{\sigma\}_{po} = \{p_o\, r/2,\ \ p_o\, r,\ 0,\ 0,\ 0,\ 0\}^T \quad . \tag{25}$$

b) Weight of the shell

By considering that the weight of the shell induces motions in the first circumferential wavenumber, $n = 1$, we obtain

$$\begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_{p_1} = \begin{Bmatrix} -\frac{\gamma_s}{E}\left[\frac{\ell^3}{12r}\left(\frac{4x^3}{\ell^3} - \frac{6x^2}{\ell^2} + 1\right) + \frac{r\ell}{4}(1-4\nu)\left(\frac{1}{2} - \frac{x}{\ell}\right)\right]\cos\varphi \\ \frac{\gamma_s}{E}\left[\frac{\ell^4}{12r^2}\left(\frac{x^4}{\ell^4} - \frac{2x^3}{\ell^3} + \frac{x}{\ell}\right) + (x\ell - x^2)\left(2.125 + \frac{\nu}{2}\right) + \frac{r^2}{4}(4-\nu)\right]\cos\varphi \\ \frac{\gamma_s}{E}\left[\frac{\ell^3}{12r^2}\left(\frac{4x^3}{\ell^3} - \frac{6x^2}{\ell^2} + 1\right) + (\ell - 2x)\left(2.125 + \frac{\nu}{2}\right)\right]\cos\varphi \\ -\frac{\gamma_s}{E}\left[\frac{\ell^4}{12r^2}\left(\frac{x^4}{\ell^4} - \frac{2x^3}{\ell^3} + \frac{x}{\ell}\right) + (x\ell - x^2)(2.125 + 1.5\nu)\right]\sin\varphi \end{Bmatrix} , \tag{26}$$

and

$$\{\sigma\}_{p_1} = \begin{Bmatrix} -\gamma_s t \; [(\ell^2/r)\;(\frac{x^2}{\ell^2} - \frac{x}{\ell}) - \frac{r}{4}\;]\cos\varphi \\ \gamma_s t \;\; r\cos\varphi \\ -2\;\gamma_s t\;(0.5\,\ell - x)\;\sin\varphi \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{27}$$

where ℓ, r, t and γ_s are, respectively, the length, mean radius, thickness and specific weight of the shell; ν is Poisson's ration, and E is Young's modulus.

c) Fluid pressure

Upon substituting relations (21) into equations ((9), (11)) and thence into equations (3) we obtain the following expressions for the displacements and the stress-resultants of the shell:

$$\begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_{p_f} = \begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_{p_f(n=0)} + \begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_{p_f(n=1)} + \sum_{n=2}^{\infty} [\tau_n] \begin{Bmatrix} u_{on} \\ w_{on} \\ 0 \\ v_{on} \end{Bmatrix}_{p_f} , \tag{28}$$

and

$$\{\sigma\}_{p_f} = \{\sigma\}_{p_f(n=0)} + \{\sigma\}_{p_f(n=1)} + \sum_{n=2}^{\infty} \begin{bmatrix} T_n & 0 \\ 0 & T_n \end{bmatrix} [E_{on}] \begin{Bmatrix} u_{on} \\ w_{on} \\ v_{on} \end{Bmatrix}_{p_f} , \tag{29}$$

where

$$\left\{\begin{matrix} U \\ W \\ \partial W/\partial x \\ V \end{matrix}\right\}_{p_f(n=0)} \left\{\begin{matrix} -(a_o\ r\ x/2Et)\ (1-2\nu) \\ -(a_o\ r^2/Et)\ (1-0.5\nu) \\ 0 \\ 0 \end{matrix}\right\} , \qquad (30)$$

$$\left\{\begin{matrix} U \\ W \\ \partial W/\partial x \\ V \end{matrix}\right\}_{p_f(n=1)} \left\{\begin{matrix} (a_1/Et)\ [\frac{\ell^3}{24r}\ (\frac{4x^3}{\ell^3} - \frac{6x^2}{\ell^2} + 1) + \frac{r\ell}{4}(1-4\nu)(\frac{1}{2} - \frac{x}{\ell})]\ \cos\varphi \\ -(a_1/Et)\ [\frac{\ell^4}{24r^2}\ (\frac{x^4}{\ell^4} - \frac{2x^3}{\ell^3} + \frac{x}{\ell}) + 1.125\ (x\ell - x^2) + \frac{r^2}{4}\ (4-\nu)]\ \cos\varphi \\ -(a_1/Et)\ [\frac{\ell^3}{24r^2}\ (\frac{4x^3}{\ell^3} - \frac{6x^2}{\ell^2} + 1) + 1.125\ (\ell - 2x)]\ \cos\varphi \\ (a_1/Et)\ [\frac{\ell^4}{24r^2}\ (\frac{x^4}{\ell^4} - \frac{2x^3}{\ell^3} + \frac{x}{\ell}) + 1.125\ (x\ell - x^2)]\ \sin\varphi \end{matrix}\right\} , \qquad (31)$$

and

$$\{\sigma\}_{p_f(n=0)} \quad \{-a_o r/2,\ -a_o r, 0, 0, 0, 0\}^T , \qquad (32)$$

$$\{\sigma\}_{p_f(n=1)} \quad \{a_1[\frac{\ell^2}{2r}\ (\frac{x^2}{\ell^2} - \frac{x}{\ell}) - \frac{r}{4}]\ \cos\varphi,\ -ra_1\cos\varphi,\ a_1(\frac{\ell}{2} - x)\sin\varphi, 0, 0, 0\}^T , \qquad (33)$$

a_o and a_1 are, respecitvely, equal to p_{ro} and p_{r1} of equations (21); the matrices $[T_n]$, $[\tau_n]$ and $[E_{on}]$ are shown in Appendix 1; and the vector $\{u_{on}, w_{on}, v_{on}\}^T_{p_f(n\geq 2)}$ is determined by substituting relations (21) into equations (9).

d) Point, line and surface loads

Similarly, the stresses and displacements due to the applied contentrated loads, line loads and surface loads (q_i, i = 1, N), Figure 2, may be written as follows

$$\begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_{q_i} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} [\tau_n][\tau_{mx}] \begin{Bmatrix} \sum_{i=1}^{N} u_{mni} \\ \sum_i w_{mni} \\ \sum_i (m\pi/\ell) \; w_{mni} \\ \sum_i v_{mni} \end{Bmatrix}_{q_i} , \qquad (34)$$

and

$$\{\sigma\}_{q_i} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \begin{bmatrix} T_n & 0 \\ 0 & T_n \end{bmatrix} \begin{bmatrix} X_m & 0 \\ 0 & X_m \end{bmatrix} [P][C_{mn}] \begin{Bmatrix} \sum_{i=1}^{N} u_{mni} \\ \sum_i w_{mni} \\ \sum_i v_{mni} \end{Bmatrix}_{q_i} , \qquad (35)$$

where [P], the elasticity matrix, is determined by equation (4); the matrices $[\tau_n]$, $[\tau_{mx}]$, $[T_n]$, $[X_m]$ and $[C_{mn}]$ are given in Appendix 1; and the vector $\sum_{i=1}^{N} \{u_{mni}, w_{mni}, v_{mni}\}^T_{q_i}$ is evaluated by substituting the relations (13), (15), (16), (17) and (19) corresponding, respectively, to $i = 1$ to 2, 3 to $N_1 + 2$, $N_1 + 3$ to $N_1 + N_2 + 2$, $N_1 + N_2 + 3$ to $N_1 + 2N_2 + 2$, and $N_1 + 2N_2 + 3$ to N, into equations (9); Figure 2.

4. CALCULATIONS AND DISCUSSION

To determine the displacements and stresses of a given cylindrical shell completely or partially filled with liquid, pressurized and having a surface of contact with a rigid support, we first specify the location of the support and its dimensions. The surface of this support is then subdivided into a sufficient number, N, of line and surface elements each of which subjected to a load q_i, where $i = 1,2,\ldots,N$; (sufficiency in this context is related to the distribution and the densities of the q_i's). Finally, a computer programme, written in Fortran V language for the CDC CYBER 74 computer, calculates, for given input data, the displacements and stresses for each particular loading using equations (24)-(35) and determines the total stress and displacements from all loading systems at any point of the structure.

The necessary input data are the mean radius r, wall thickness t, length of the vessel ℓ, material and fluid specific weights γ_s and γ, respectively; Poisson's ratio ν, modulus of elasticity E, surcharge internal pressure p_o, densities of the q_i's and the angle φ_o (rad) which represents the level of the liquid in the shell.

This analysis proceeds separately for each circumferential wavenumber, n, and the total response may then be found by summing over n. The total number of n required for the computation is reached when the relative error of each displacement component approaches 10^{-9}.

In order to test the correctness of the theory, one typical case has been calculated. This calculation involves the determination of the displacements and stresses of a particular twin saddle supported vessel which has been analysed by Wilson and Tooth [7]. The vessel is subjected to water loading, self weight and interface pressure. The resulting interface pressure distribution between the saddle and the vessel is shown in Figure 8a as given by reference [7]. The data for the vessel are as follows: $r = 6$ ft(1.85m), $t = 1$ in (25.4mm), $\ell = 180$ ft (54.9m), $E = .29 \times 10^8$ lb/in^2 (.2039 x 10^{11}kg/m^2), $\nu = 0.3$, $\gamma = 0.03611$ lb/in^3 (999.52 kg/m^3), $\gamma_s = 0.284$ lb/in^3 (7.8 x 10^3 kg/m^3) and $\varphi_o = \pi$ rad. The location of the twin saddles on the vessel is shown in Figure 9a. The effect of the closed ends is taken into account.

The analytical displacements and stresses in [7] were obtained by application of Flügge's theory and employing double Fourier series procedure. The distribution and magnitude of the interface pressure is assumed to be (1) the same for both saddles, (2) symmetric with respect to the generator passing through the center of the saddle arc and (3) constant across the saddle width. Finally the saddle arc length is subdivided into a series of equal angular parts each of axial length equal to the width of the saddle and loaded by a uniform pressure.

The results obtained by the present theory were computed with $N_1 = 49$, $N_2 = 15$ and $N = 816$ elements, and are compared with those of [7] in Figure 8-10. As may be seen, the results obtained by this theory are generally in quite good agreement with those of [7] and, what is more interesting, they are in better agreement with B.S. 1515. This is particularly noticeable in the case of the dominant circumferential bending moment M_φ as shown in Figure 10b. Detailed discussion of the results obtained and their significance will not be undertaken here as this has already been done by Wilson and Tooth [7].

The results for the radial displacement, W, in terms of the saddle's width are shown in Figure 8b. We note that these displacements are not constant along the width of the saddle for φ smaller than the saddle angle δ_o, $(\varphi \leq 75^o)$, but tend to a uniform value for φ higher than δ_o. Therefore, our results show an incompatibility regarding the assumption, used in [7], that the interface pressure is constant across the saddle width.

5. CONCLUSION

In this paper we have presented a theory capable of predicting the stresses and displacements of a thin cylindrical vessel partially filled with liquid, pressurized and having a surface of contact with a rigid support. To this end the support is subdivided into a sufficient number, N, of line and surface elements in the circumferential and axial directions, each of which subjected to a load q_i, $i = 1, 2, \ldots, N$. The analysis proceeds separately for each circumferential wavenumber, n, and the total stresses and displacements may then be found by summing over n.

This theory was computarized so that if the dimensions and material properties of the vessel, and the properties of the saddle, are given as inputs, the program gives as output the displacements and stresses at any point of the structure.

Here we limite ourselves to the case where the q_i's are known. The situation where the interest is the evaluation of the contact area and the pressure distribution, i.e. the q_i's, is the subject of another study, under preparation, where the theory of this work is applied to derive the total energy of the system in terms of the q_i's. The minimisation of this energy will permit us to obtain the distribution of pressure over the contacting regions.

APPENDIX 1

The matrices referred to in the text are listed as follows.

$$[T_n] = \begin{bmatrix} \cos n\varphi & 0 & 0 \\ 0 & \cos n\varphi & 0 \\ 0 & 0 & \sin n\varphi \end{bmatrix}$$

$$[T_{mx}] = \begin{bmatrix} \cos(m\pi x/\ell) & 0 & 0 \\ 0 & \sin(m\pi x/\ell) & 0 \\ 0 & 0 & \sin(m\pi x/\ell) \end{bmatrix}$$

$$[\tau_n] = \begin{bmatrix} \cos n\varphi & 0 & 0 & 0 \\ 0 & \cos n\varphi & 0 & 0 \\ 0 & 0 & \cos n\varphi & 0 \\ 0 & 0 & 0 & \sin n\varphi \end{bmatrix}$$

$$[X_m] = \begin{bmatrix} \sin(m\pi x/\ell) & 0 & 0 \\ 0 & \sin(m\pi x/\ell) & 0 \\ 0 & 0 & \cos(m\pi x/\ell) \end{bmatrix}$$

$$[\tau_{mx}] = \begin{bmatrix} \cos(m\pi x/\ell) & 0 & 0 & 0 \\ 0 & \sin(m\pi x/\ell) & 0 & 0 \\ 0 & 0 & \cos(m\pi x/\ell) & 0 \\ 0 & 0 & 0 & \sin(m\pi x/\ell) \end{bmatrix}$$

$$[E_{on}] = \begin{bmatrix} 0 & \nu D/r & \nu nD/r \\ 0 & D/r & nD/r \\ -nD(1-\nu)/2r & 0 & 0 \\ 0 & \nu n^2K/r^2 & \nu nK/r^2 \\ 0 & n^2K/r^2 & nK/r^2 \\ nK(1-\nu)/4r^2 & 0 & 0 \end{bmatrix}$$

$$[C_{mn}] = \begin{bmatrix} -m\pi/\ell & 0 & 0 \\ 0 & 1/r & n/r \\ -n/r & 0 & m\pi/\ell \\ 0 & (m\pi/\ell)^2 & 0 \\ 0 & n^2/r^2 & n/r^2 \\ n/2r^2 & (2n/r)\cdot(m\pi/\ell) & (3/2r)\cdot(m\pi/\ell) \end{bmatrix}$$

$$A_F\ (1,1) = r^2(m\pi/\ell)^2 + n^2\frac{(1-\nu)}{2}\left(1+\frac{\kappa}{4}\right)$$

$$A_F\ (1,2) = -r\frac{(m\pi)}{\ell}\left[\nu - \frac{(1-\nu)}{2}\ \kappa\ n^2\right]$$

$$A_F\ (1,3) = \frac{-rn}{2}\ \frac{(m\pi)}{\ell}\left[\left(1-\frac{3\kappa}{4}\right) + \nu\left(1+\frac{3\kappa}{4}\right)\right]$$

$$A_F\ (2,1) = A_F\ (1,2)$$

$$A_F\ (2,2) = 1 + \kappa r^4\left[\frac{n^2}{r^2} + (m\pi/\ell)^2\right]$$

$$A_F\ (2,3) = n(1+n^2\kappa) + \frac{(3-\nu)}{2}\ \kappa r^2 n\ (m\pi/\ell)^2$$

$$A_F\ (3,1) = A_F\ (1,3)$$

$$A_F\ (3,2) = A_F\ (2,3)$$

$$A_F\ (3,3) = \frac{(1-\nu)}{2}r^2\ (m\pi/\ell)^2\ \left(1+\frac{9\kappa}{4}\right) + n^2(1+\kappa)$$

ACKNOWLEDGMENTS

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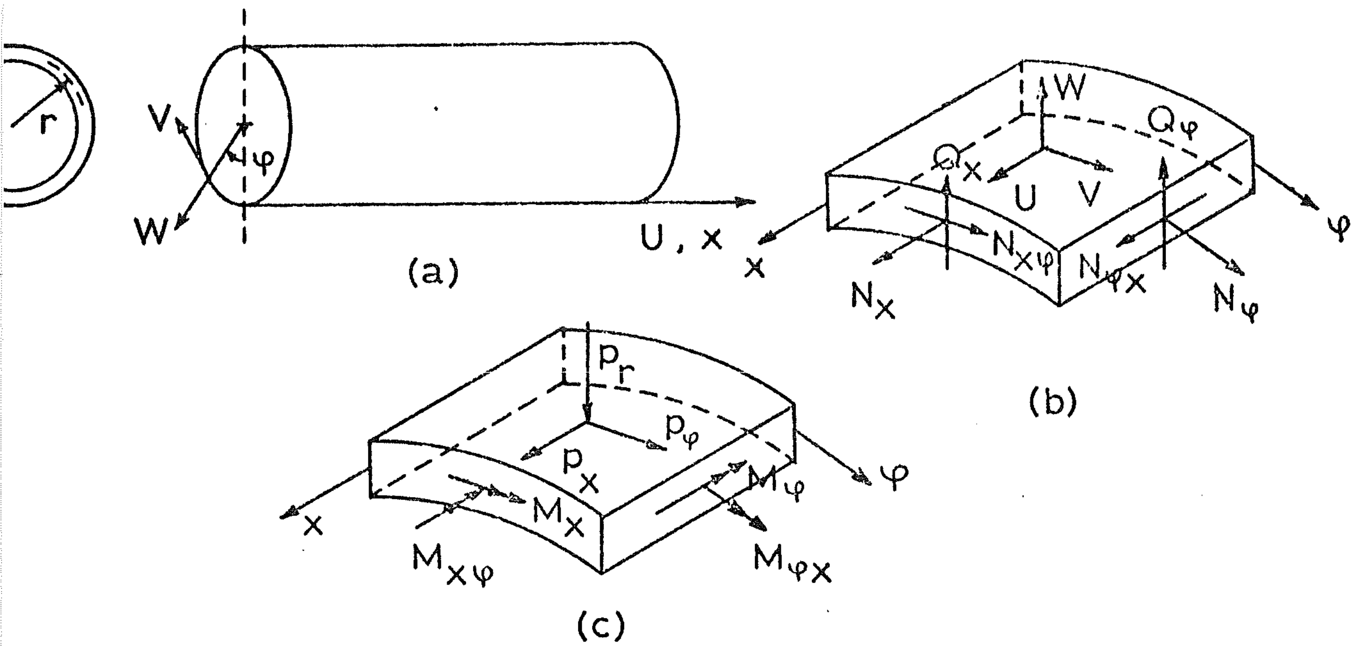

Figure 1 (a) Definition of the displacements U, V and W.
(b) Stress-resultants and displacements acting upon a differential elements.
(c) Stress couples and surface loads acting upon a differential elements.

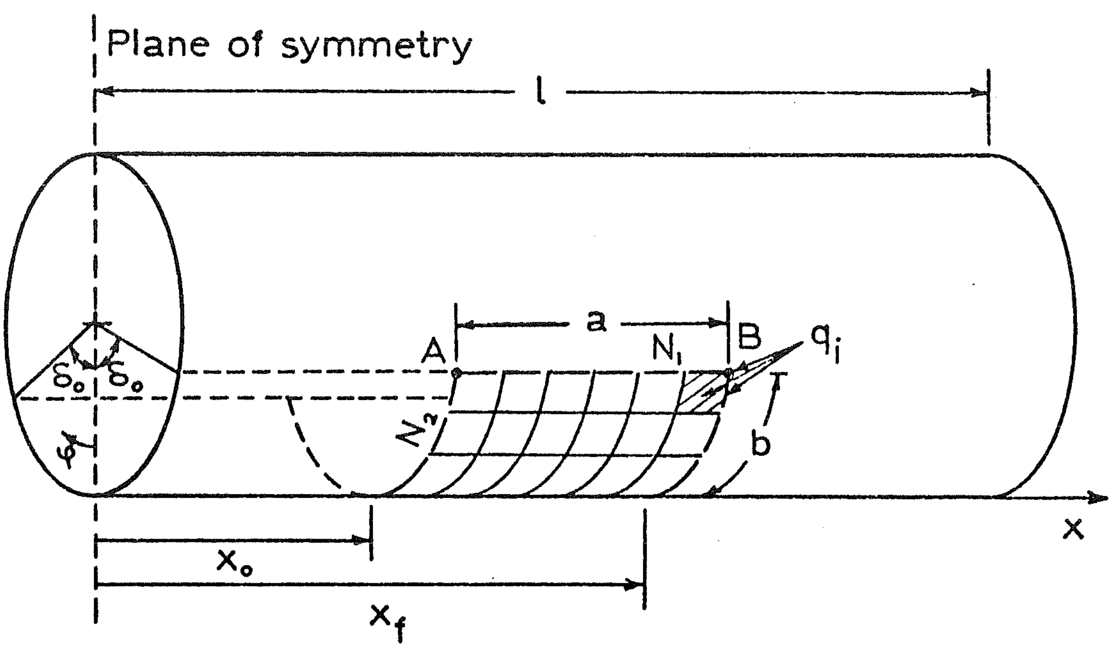

Figure 2 Location of the support on the shell.
(N_1 and N_2 are the number of line elements at the boundaries of the support in x and φ directions, respectively; and, $N = N_1N_2 + 2N_2 + N_1 + 2$, is the total number of elements).

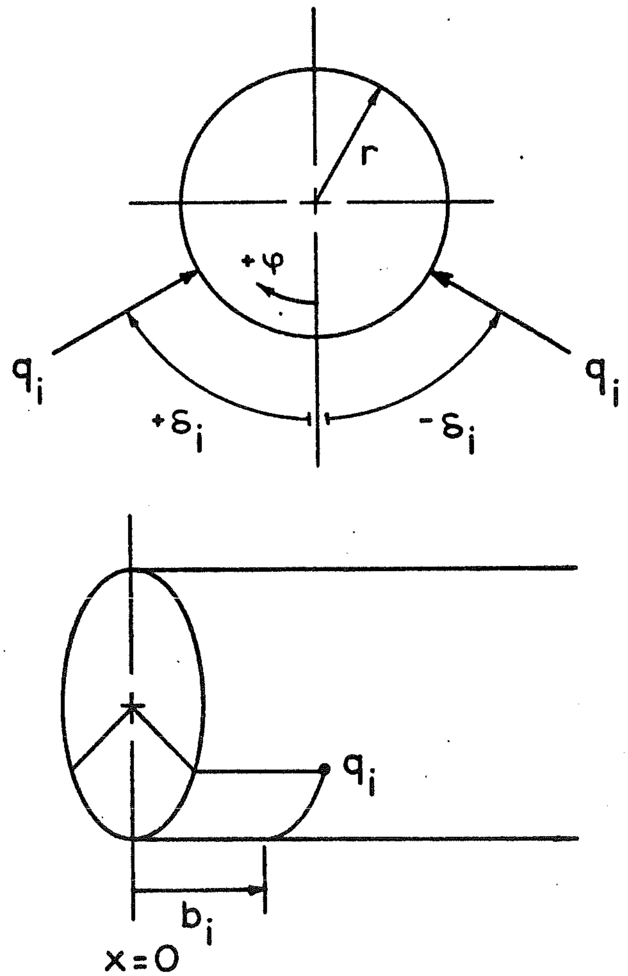

Figure 3 Concentrated radial loads, q_i (lb or kg).

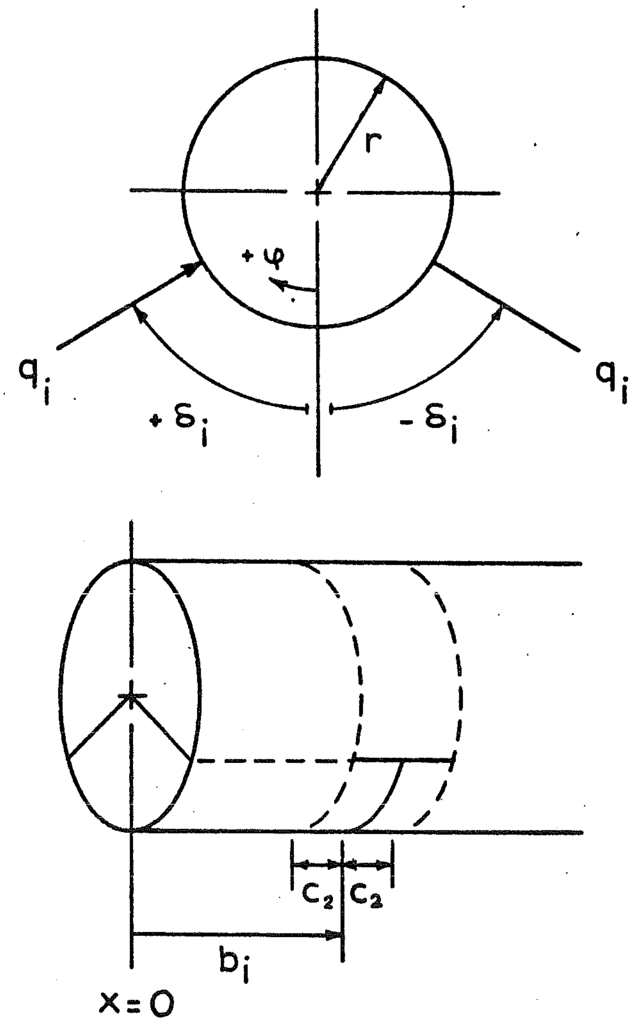

Figure 4 Line load along a generator, q_i (lb/in or kg/m).

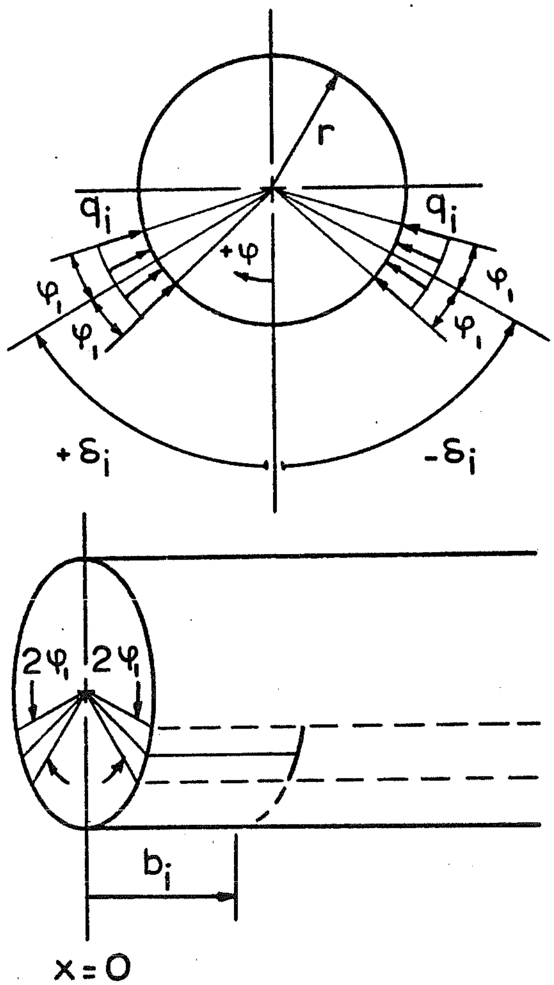

Figure 5 Line load perpendicular to the generator, q_i (lb/in or kg/m).

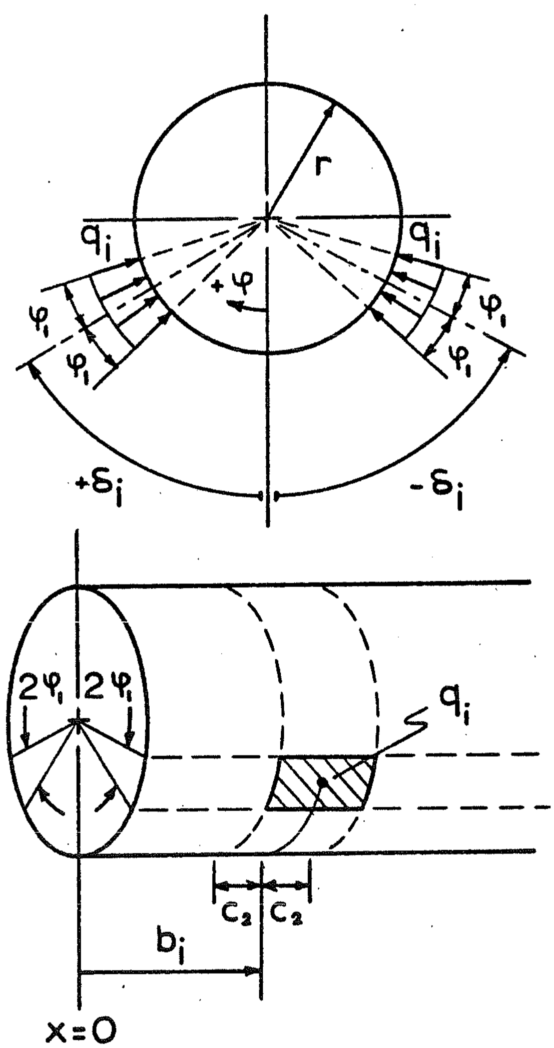

Figure 6 Distributed loads, q_i (lb/in^2 or kg/m^2).

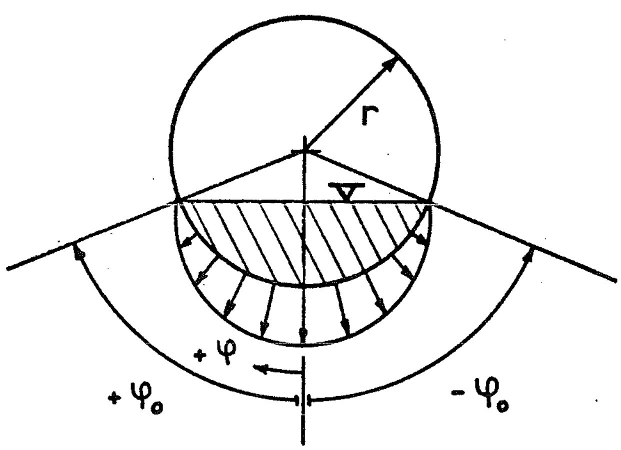

Figure 7 Pressure distribution for a partially-filled shell.

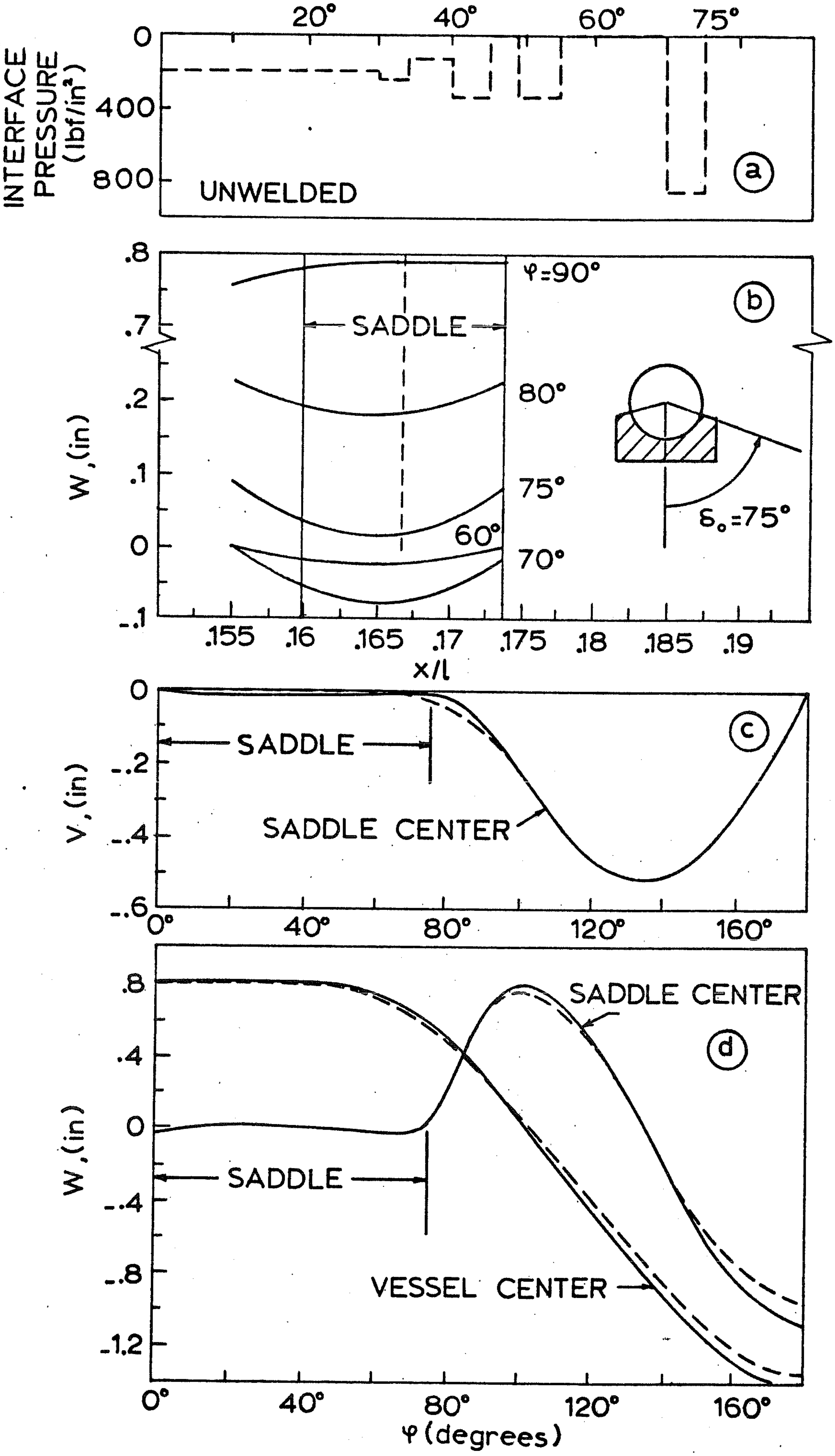

Figure 8 Pressuredistribution and displacements of a twin saddle supported cylindrical vessel full of liquid. ——, this theory; --- theory of Wilson and Tooth, [7].

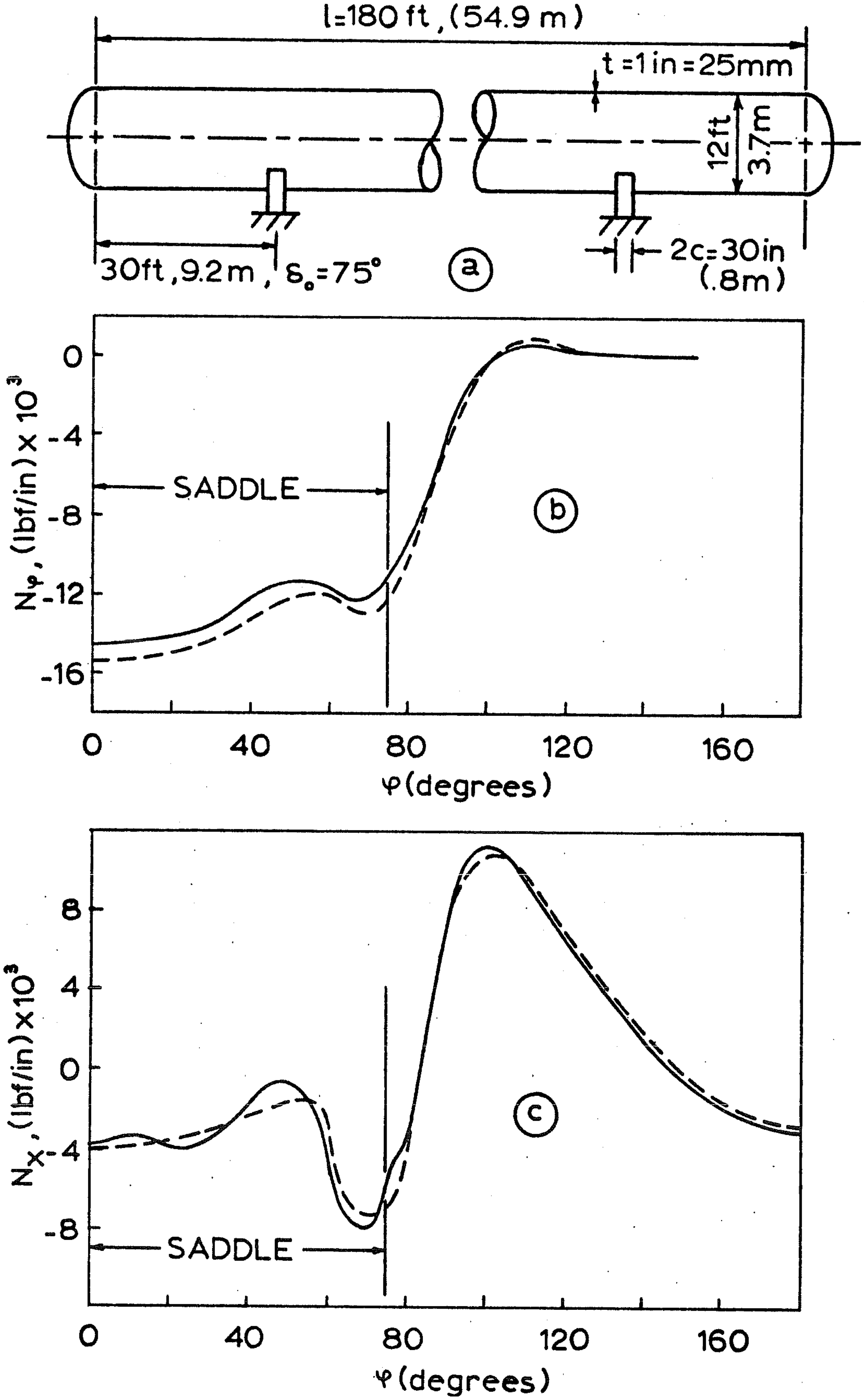

Figure 9 Values of N_{φ} and N_x at saddle center profil for fluid and self weight. ——, this theory; ---, theory of [7].

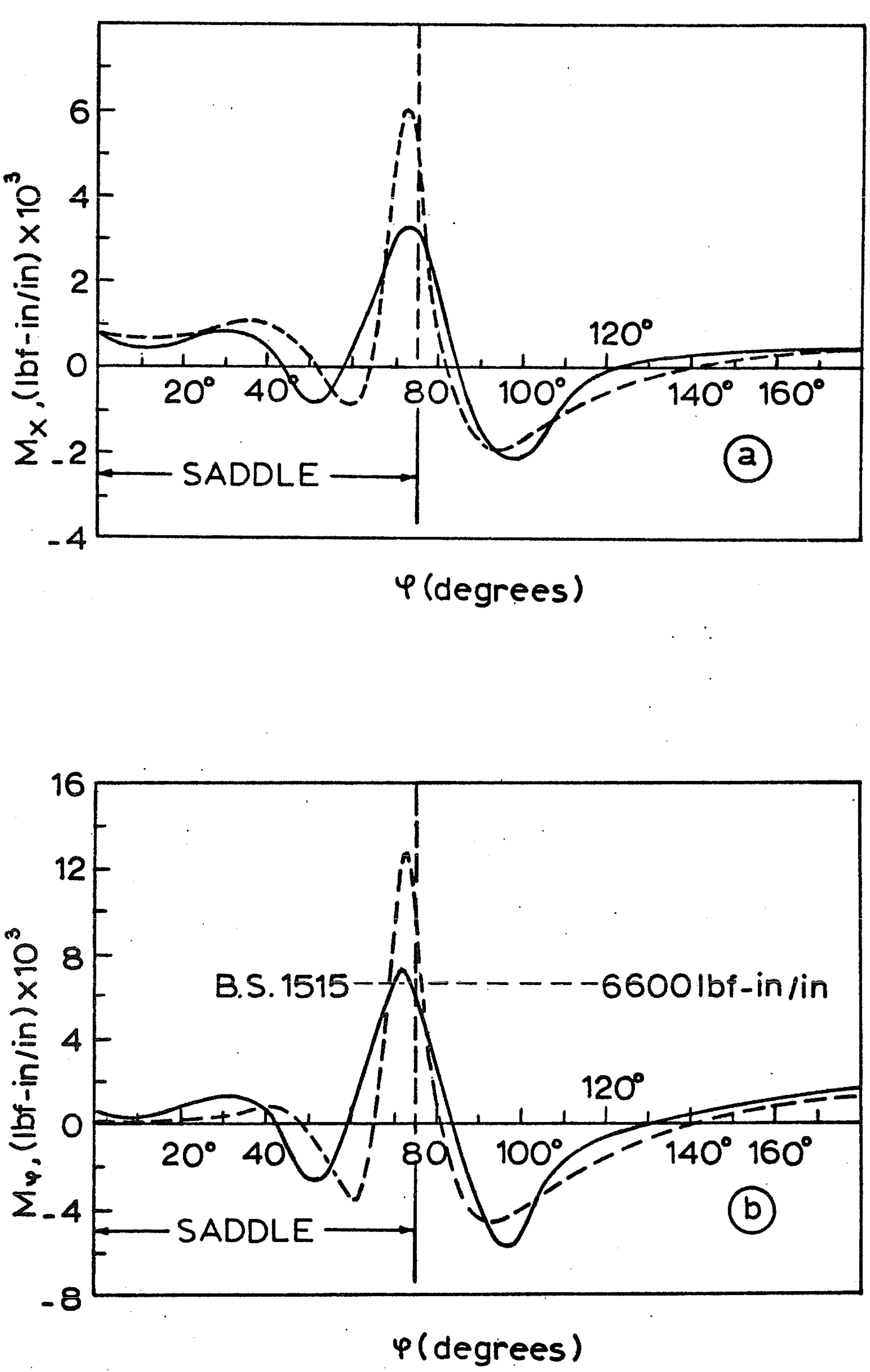

Figure 10 Values of M_x and M_φ at saddle center profile for fluid and self weight. ——, theoretical results obtained by this theory; ---, theoretical results of reference [7].