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New Very Low Rate Nested Convolutional Codes*

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SUMMARY

Recently, high error rate channels and CDMA systems have increased the interest for very low rate convolutional codes. Unfortunately, in the search for optimal convolutional codes, the computational complexity increases exponentially with increasing constraint length and decreasing coding rate. In this paper, new very low rate Nested Codes are presented (constraint length $3 \le K \le 14$ and coding rate $R \le 1/128$). It is shown that the computational complexity of the search for the Nested Codes increases only linearly with decreasing coding rate. It is also shown that the Nested Codes are nearly optimum, allowing for very good performances and yielding a maximum free distance that is very close to the Heller bound on maximum free distance.

1 - Introduction

Recently, large bandwidth systems, such as Code Division Multiple Access (CDMA) systems, have increased the interest and applications for good very low rate convolutional codes. In CDMA, the signal is spread over a much larger bandwidth than that of the information message. This spreading can be performed by either a wide band pseudo-random sequence or a very low rate convolutional code or a combination of the two. The

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very large bandwidth used in conjunction to CDMA systems, allows for very low rate convolutional codes. Using random set of codes, Hui has shown that in order to reach maximum capacity of CDMA systems, the spreading should be entirely spread by very low rate convolutional codes [1]. However, no good convolutional codes are readily available in the literature for coding rate lower than $R \leq 1/8$. Using a different approach, Viterbi proposed to use Orthogonal codes [2]. Although very low rate Orthogonal codes are very easy to generate, their maximum free distance is well below the Heller bound on the maximum free distance of convolutional codes. Therefore, there is the need to find good, very low rate convolutional codes, with a free distance as close as possible to the Heller bound.

The search for optimal convolutional codes of coding rate $R = 1/v$ and constraint length K grows exponentially with decreasing coding rate and increasing constraint length. Thus, for applications requiring very low rate codes the search for optimal codes becomes rapidly prohibitive and thus, quasi-optimal codes must be considered. The objective is to find methods that allow generating good very low rate convolutional codes without suffering from a computational complexity that is increasing exponentially with both decreasing coding rate and increasing constraint length.

In this paper, a new approach to generate good very low rate convolutional codes, based on the principle that "good codes generate good codes" is presented. In searching for good codes, the free distance may be maximized and the error probability may be minimized. Using a similar principle, Lee has determined groups of rate $1/3 \geq R \geq 1/8$ and constraint length $3 \leq K \leq 8$ convolutional codes which minimize some specific values of the bit error probability [3]. In this paper, we propose a set of very low rate compatible convolutional codes, called Nested Codes.

The paper is organized as follows: First the construction of the Nested convolutional codes is presented. Then, the search for the maximum free distance Nested Codes is presented followed by a simplified procedure for searching maximum free distance Nested Codes. Then, the search for minimum error probability Nested Codes is presented, followed by results of the search for the maximum free distance and minimum error probability Nested Codes. The results given include the Nested Codes and performance evaluation using the union bound in additive white Gaussian noise channel.

2 - Code Construction

The basic approach is to generate a rate $R = 1/(v_s + 1)$ convolutional code as an extension of the best known rate $R = 1/v_s$ convolutional code, called the *starting code*. The known v_s generator vectors of the starting code are maintained in the search for the rate $R = 1/(v_s + 1)$ convolutional code. The rate $R = 1/(v_s + 1)$ Nested Code is generated by finding the best additional generator vector used in conjunction to the v_s known generator vectors of the starting code, as shown in figure (1) for a starting code of rate $R = 1/3$ and Nested Codes of rates $R = 1/4$ and $R = 1/5$. Selecting the additional generator vector depends of the selection criterion. In this paper, two selection criteria have been considered: maximum free distance and minimum bit error probability.

Considering all non-degenerate codes, an upper bound on the computational complexity order of the search for optimal rate $R = 1/(v_s + 1)$ and constraint length code is given by $O\left(2^{(v_s + 1) \cdot (K-1)}\right)$. For the Nested Codes, since only the additional generator vector must be determined, then an upper bound on the computational complexity order of the rate $R = 1/(v_s + 1)$ Nested Code generated from the known rate $R = 1/v_s$ starting code, is given by $O\left(2^{(K-1)}\right)$.

Likewise, using the same approach, the best rate $R = 1/(v_s + 2)$ Nested Code can be generated using as starting code the newly found rate $R = 1/(v_s + 1)$ Nested Code. Thus, a rate $R = 1/(v_s + i)$ Nested Code can be determined by successively generating the rates $R = 1/(v_s + 1), 1/(v_s + 2), \ldots, 1/(v_s + i - 1)$ Nested Codes using a rate $R = 1/v_s$ starting code.

The upper bound on the computational complexity for the search of rate $R = 1/(v_s + i)$ optimal non-degenerate code is given by $O\left(2^{(v_s + i) \cdot (K-1)}\right)$. Since generating a rate $R = 1/(v_s + i)$, $i > 1$, Nested Code from a rate $R = 1/(v_s + i - 1)$ starting code is no more demanding than generating a rate $R = 1/(v_s + 1)$ Nested Code using a rate $R = 1/v_s$ starting code, then the computational complexity of the search for low rate Nested Codes grows linearly with decreasing coding rate. Thus, the upper bound on the order of computational complexity of the search for rate $R = 1/(v_s + i)$ non-degenerate Nested Codes is given by $O\left(i \cdot 2^{(K-1)}\right)$.

In this paper, two selection criteria are used when searching for the best low rate Nested Code: maximum free distance and minimal bit error probability.

2.1 - Search for low rate Nested Codes using maximum free distance criterion

The search for low rate maximum free distance Nested Codes consists of finding the additional generator vector, which, when used with the first v_s generator vectors of the starting code, yields a rate $R = 1/(v_s + 1)$ Nested Code whose free distance is maximum,

Since several generator vectors which yield the maximum free distance may be found, then the generator vector that yields the minimum number of bit errors for the codewords at the free distance is selected.

The search for the additional non-degenerate generator vector, which yields the maximum free distance, can be exhaustive. In that case, all the $2^{(K-1)}$ additional generator vectors are tested. However, the particular structure of the Nested Codes leads to a simplified procedure that allows for a fast convergence toward the additional generator vector that yields the maximum free distance. This procedure is presented in the next section.

2.1.1 - Simplified procedure for the search of maximum free distance Nested Codes

Let us consider the trellis of a rate $R = 1/v$ and constraint length K convolutional code having a free distance d_{free}. The first path remerging with the trellis state 0 corresponds to the impulse response path. The impulse response path for the rate

$R = 1/2, K = 3$ convolutional code with generator vectors $H = 5, 7^*$, is given in figure 2. The first path remerging to state 00, is the impulse response of the code, and corresponds to the information sequence $U = [1, 0, 0, \ldots]$. Considering the generator matrix $[G]$,

$$[G] = \overbrace{\begin{bmatrix} \underline{G}_0 & \underline{G}_1 & \cdots & \underline{G}_{K-1} & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & \underline{G}_0 & \underline{G}_1 & \cdots & \underline{G}_{K-1} & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \underline{G}_0 & \underline{G}_1 & \cdots & \underline{G}_{K-1} & 0 & 0 & \cdots & 0 & 0 \\ & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \underline{G}_0 & \underline{G}_1 & \cdots & \underline{G}_{K-1} \end{bmatrix}}^{(L+K-1)\,v \text{ Colums}} \Big\updownarrow L \text{ Rows} \quad (1)$$

the impulse response is given by $Imp^{(v)} = \underline{G}_0, \underline{G}_1, \ldots, \underline{G}_{K-1}$ where $\underline{G}_i = [g_{0,i}, g_{1,i}, \ldots, g_{v-1,i}]$, $i = 0, 1, \ldots, K-1$, and where $g_{j,i} = 1$ if the tap delay i is connected to the modulo-2 adder j and $g_{j,i} = 0$ otherwise. Thus, the Hamming weight of the impulse response $W_H\left(I_{mp}^{(v)}\right) = W_H(\underline{G}_0, \underline{G}_1, \ldots, \underline{G}_{K-1})$, is the weight of the first path remerging with state 00. By definition, the maximum free distance of the code is the minimum Hamming distance between all pairs of such codewords of the code and thus, the maximum free distance is the minimum weight of all paths remerging with state 00. Denoting $d_{free}^{(v)}$ the free distance of a code of rate $R = 1/v$, we can write:

$$N_C^{(v)} = W_H\left(I_{mp}^{(v)}\right) \geq d_{free}^{(v)} \quad (2)$$

since the Hamming weight of the impulse response is equal to the total number of connections $N_C^{(v)}$ of the modulo-2 adders on the encoder's shift register.

The maximum free distance of a code is thus a lower bound on the number of connections of the generator vectors on the shift register. Using the particular structure of

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* H is the octal representation of the generator vectors $\underline{H} = [h_0, \ldots, h_v]$, with v the number of modulo-2 adders and where $h_j = [g_{j,0}, \ldots, g_{j,K-1}]$. $g_{j,i} = 1$ if modulo-2 adder j is connected to the tap delay i and $g_{j,i} = 0$ otherwise.

the Nested Codes, a lower bound on the number of connections of the additional generator vector can be determined.

Let us assume the best rate $R = 1/(v+1)$ Nested Code is to be determined using the rate $R = 1/v$ starting code. Considering the structure of the Nested Codes, the rate $R = 1/(v+1)$ Nested Code has a maximum free distance $d_{free}^{(v+1)}$ such that $d_{free}^{(v+1)} > d_{free}^{(v)}$, since the additional generator vector has at least one connection on the first tap delay of the shift register. From (2), we thus have

$$N_C^{(v+1)} \geq d_{free}^{(v+1)} > d_{free}^{(v)} \tag{3}$$

Since the v generator vectors of the starting code of coding rate $R = 1/v$ are known, the weight $N_C^{(v)} = W_H\left(I_{mp}^{(v)}\right)$ is also known.

From (3) we can write

$$N_C^{(v+1)} - W_H\left(I_{mp}^{(v)}\right) \geq d_{free}^{(v+1)} - W_H\left(I_{mp}^{(v)}\right) \tag{4}$$

The number of connections n_C of the additional generator vector is given by $n_C = N_C^{(v+1)} - N_C^{(v)}$ since $N_C^{(v+1)}$ represents the total number of connections of the rate $R = 1/(v+1)$ Nested Code and $N_C^{(v)}$ is the total number of connections of the rate $R = 1/v$ starting code.

For a code of constraint length K, the number of generator vectors to search is given by $\sum_{i=2}^{K}\binom{K-2}{i-2}$, which may still represent a large number of alternatives. However the procedure may be speeded-up considerably if all non-promising generators are readily recognized and discarded. To that effect, the set of all possible additional generator vectors can be partitioned in the following two subsets:

Subset A: Subset A is the set of all additional generator vectors that could lead to a rate $R = 1/(v+1)$ Nested Code having a free distance $d_{free}^{(v+1)}$, such that $d_{free}^{(v+1)} \geq d_{free}^{*}$ where d_{free}^{*} is some given free distance value.

Subset B: Subset B is the set of all additional generator vectors that cannot lead to a rate $R = 1/(\nu+1)$ Nested Code having a free distance $d_{free}^{(\nu+1)}$ such that $d_{free}^{(\nu+1)} \geq d_{free}^{*}$ where d_{free}^{*} is some given free distance value.

The additional generator vector of the rate $R = 1/(\nu+1)$ yielding maximum free distance Nested Code is in subset A. Let a generator vector be chosen from subset A and let its free distance, d_{free}^{*}, be computed. Using d_{free}^{*}, subset A is expurgated of all generator vectors that cannot yield a Nested Code with free distance greater or equal to d_{free}^{*}. The expurgation criterion is as follow:

If in subset A there exists a generator vector leading to a Nested Code of free distance d'_{free}, such that $d'_{free} \geq d_{free}^{*}$, then using (3) and (4) we have:

$$N'_C \geq d'_{free} \geq d_{free}^{*} \Rightarrow N'_C \geq d_{free}^{*} \qquad (5)$$

$$N'_C - N_C^{(\nu)} \geq d_{free}^{*} - N_C^{(\nu)} \qquad (6)$$

$$n_C \geq d_{free}^{*} - N_C^{(\nu)} \qquad (7)$$

Equation (7) means that if there exist an additional generator vector that leads to a better code than the one corresponding to the free distance d_{free}^{*}, then its number of connections, n_C, is not smaller than $d_{free}^{*} - N_C^{(\nu)}$, where d_{free}^{*} and $N_C^{(\nu)}$ are known.

The maximum free distance Nested Codes search algorithm is iterative. Thus, after each iteration subset A is expurgated of all non promising codes until only one generator vector remains. The flow chart of the algorithm is given in figure 3.

2.2 - Search for Nested Codes with Minimum bit error probability criterion

For this criterion, the code, for a known rate and constraint length, providing the minimum bit error probability is searched. The bit error probability is evaluated using the well-known union bound.

$$P_b < \sum_{d = d_{free}}^{\infty} C_d Q\left(\sqrt{\frac{2RE_b d}{N_0}}\right) \tag{8}$$

where C_d is the total number of bit errors on paths at distance d, E_b/N_0 is the signal to noise ratio, R is the coding rate and where $Q(x) = \int_x^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$.

This implies that the weight spectra of the codes must be determined. However, only the leading terms of the weight spectrum are considered when searching for the minimum bit error probability codes.

The search for the minimum bit error probability Nested Codes can be further simplified by observing that all the known minimum bit error probability codes have never a free distance smaller than 80% of the free distance of the maximum free distance code of the same coding rate and constraint length. Using (7), the minimum number of connections of the additional generator vector is thus given by:

$$n_C \geq 0.8 d_{free}^{*} - N_C^{(v)} \tag{9}$$

The evaluation of the bit error probability is dependent of the signal to noise ratio. In this paper, the signal to noise ratio considered is such that $R/R_{comp} = 0.99$ where R_{comp} is the cutoff rate given by

$$R_{comp} = 1 - \log_2\left(1 + e^{-(RE_b)/N_0}\right) \tag{10}$$

for unquantized additive white Gaussian noise channels of spectral density $N_0/2$.

When selecting codes, two codes can have similar bit error probability at the E_b/N_0 value corresponding to $R/R_{comp} = 0.99$ but one of them can provide asymptotically more interesting performances. Thus, the codes must be compared for two signal to noise ratios. In this paper, the second signal to noise ratio value at which the codes are compared, corresponds to $R/R_{comp} = 0.4$, where it is recognized that $R/R_{comp} = 0.4$ is a very low noise channel, that is, corresponds to asymptotical performances.

Let $P_{b0.99}^{(c_i)}$ and $P_{b0.4}^{(c_i)}$ denote the bit error probability achieved for some code c_i

at E_b/N_0 values corresponding to $R/R_{comp} = 0.99$ and to $R/R_{comp} = 0.4$ respectively. Let us consider two codes c_1 and c_2. If the codes are such that $F_{0.99}P_{b0.99}^{(c_1)} > P_{b0.99}^{(c_2)} > P_{b0.99}^{(c_1)}$ but $P_{b0.4}^{(c_2)} < F_{0.4}P_{b0.4}^{(c_1)}$, where $F_{0.99}$ and $F_{0.4}$ are some given weighting factors, then code c_2 is to be the better code. The weighting factors have an important influence on the code generated. They insured that a code that shows only a slight performance degradation at E_b/N_0 corresponding to $R/R_{comp} = 0.99$ but has superior asymptotical performances is chosen over another code that may be only slightly superior at $R/R_{comp} = 0.99$.

In this paper, the weighting factors considered are $F_{0.99} = 1.25$ and $F_{0.4} = 1.5$.

3 - Results

In this section, the codes obtained by the Nested Code algorithm are given, for both the maximum free distance Nested Codes and minimal bit error probability Nested Codes criteria. These codes have been obtained for coding rates $1/3 \le R \le 1/128$ and constraint length $3 \le K \le 14$. However, using the union bound, performances have been evaluated, only for coding rates $R = 1/2^i$, $i = 1, 2, ..., 7$, and constraint lengths $7 \le K \le 10$. Since the codes generated are Nested, only the connection vectors of the rate $R = 1/128$ Nested Code, for each constraint length and each selection criteria, is given. Generator vectors for higher coding rate can be obtained by translating the connection vectors into generator vectors and puncturing the last generator vectors to get higher coding rate Nested Codes. For example, the constraint length $K = 9$ and rate $R = 1/64$ Nested Code can be obtained by using the 9 connection vectors to get the 128 generator vectors and by deleting the last 64 generator vectors.

The free distance of the Nested Codes have also been compared to an upper bound on the free distance of convolutional codes known as the Heller bound, given by [4]

$$d_{free} \le \min \left\lfloor \left(\frac{2^l}{2^l - 1} \right) \cdot \frac{v \cdot (l + K - 1)}{2} \right\rfloor = d_{Heller} \qquad (11)$$

The Nested Codes obtained under the maximum free distance criterion are given in Tables 1 to 12 , and the Nested Codes generated using the minimum bit error probability criterion are given in Tables 13 to 24*. For each code, the free distance of the code, and the total number of bit errors at the free distance are given. For comparison purposes, the corresponding Heller bound is also provided.

As Tables 1 to 24 show the Nested Codes, especially those generated using the maximum free distance criterion, yield a free distance very close to the Heller bound on the maximum free distance, suggesting that the Nested Codes are very good codes. In addition to the free distance, for each code, the weight spectrum has been determined over at least 25 terms. The union bound on the bit error probability for the coding rates $1/2 \geq R \geq 1/128$ and constraint lengths $7 \leq K \leq 10$ Nested Codes have been evaluated using the first 40 terms of the code spectrum, for both the maximum free distance and minimum bit error probability Nested Codes. These results are given in figures 4 to 11 for unquantized, additive white Gaussian noise channel.

It can be seen in figures 4 to 11 that the error performance does not always monotonely improve as the coding rate decreases. For example, for the constraint length $K = 9$, maximum free distance Nested Codes, the coding rate $R = 1/64$ code is better than the rate $R = 1/128$ Nested Code, providing a slightly higher coding gain. This phenomenon can be explained using the notion of "normalized free distance". Recalling the union bound expression of the bit error probability,

$$P_b < \sum_{d = d_{free}}^{\infty} C_d Q\left(\sqrt{\frac{2RE_b d}{N_0}}\right) \tag{12}$$

then, asymptotically, the bit error probability is essentially determined by the first term of the summation. Thus,

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* Vectors are given using octal notation.

$$P_b \approx C_{d_{free}} Q\left(\sqrt{\frac{2RE_b d_{free}}{N_0}}\right) \tag{13}$$

If the term $C_{d_{free}}$ is neglected and the signal to noise ration (E_b/N_0) is constant, the only parameter that has an effect on the bit error probability is the product $R \cdot d_{free}$. We call the $R \cdot d_{free}$ product the "normalized free distance". Thus, although decreasing the coding rate increases the free distance, the normalized free distance may decrease. In that case, decreasing the coding rate would not be appropriate. The normalized free distance curves using the Heller bound for the free distance, are given in figure 12. It is observed that the normalized free distance remains essentially constant as the coding rate decreases. Thus, decreasing the coding rate does not always yields an increase of the coding gain.

4 - Conclusion

In this paper, a new method for generating very low rate codes has been presented. It was shown that the computational complexity of the search for the Nested Codes increases linearly with decreasing the coding rates, as opposed to the computational complexity of the optimal codes which increases exponentially with decreasing coding rates. The Nested Codes approach allows to find very good convolutional codes of practically any arbitrarily low coding rate. Furthermore, it was shown that the maximum free distance of the Nested Codes is very close to the Heller bound on maximum free distance of convolutional codes, indicating that the Nested Codes are very nearly optimal.

Very Low rate Nested Codes may be quite attractive in applications such as CDMA. In these systems where the transmission bandwidth is much larger than the data bandwidth, the spreading of the baseband signal can be performed using a pseudo-random code, a low rate convolutional code or a combination of the two. However the use of very low rate codes for spreading allows translating the coding gains of these codes into a capacity increase of the CDMA system. In a forthcoming paper [5] we show that the CDMA capacity (in number of users) may be substantially increased by a proper combination of low rate Nested Codes and spreading sequence.

Finally it may be worthwhile to mention that since the Nested Codes are in fact rate compatible codes, they may be attractive in ARQ systems using code combining.

5 - References

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Table 1 - Maximum free distance Nested Codes, K=3 using starting code with generator vectors $\underline{H}$= 5, 7 .

Rate R=1/128 Connection vectors: $\underline{G}$= 3777777777777777777777777777777777777777777, 1555777777777777777777777777777777777777777, 3777777777777777777777777777777777777777777			
R	d_{free}	Heller	$C_{d_{free}}$
1/3	8	8	3
1/4	10	10	1
1/5	13	13	1
1/6	16	16	3
1/7	18	18	1
1/8	21	21	1
1/9	24	24	3
1/10	26	26	1
1/16	40	42	2
1/32	72	85	2
1/64	136	170	2
1/128	264	341	2

Table 2 - Maximum free distance Nested Codes, K=4 using starting code with generator vectors $\underline{H}$= 15, 17 .

Rate R=1/128 Connection vectors: $\underline{G}$= 3777777777777777777777777777777777777777777, 3377777777777777777777777777777777777777777, 1511111111111111111111111111111111111111111, 3777777777777777777777777777777777777777777			
R	d_{free}	Heller	$C_{d_{free}}$
1/3	10	10	6
1/4	13	13	4
1/5	16	16	2
1/6	20	20	6
1/7	23	23	4
1/8	26	26	2
1/9	30	30	6
1/10	33	33	4
1/16	53	53	4
1/32	106	106	2
1/64	213	213	4
1/128	426	426	2

Table 3 - Maximum free distance Nested Codes, K=5 using starting code with generator vectors: $\underline{H}$= 23, 35 .

Rate R=1/128 Connection vectors:
$\underline{G}$= 3777777777777777777777777777777777777777777,
1252525252525252525252525252525252525252525,
1777777777777777777777777777777777777777777,
2525252525252525252525252525252525252525252,
3777777777777777777777777777777777777777777

R	d_{free}	Heller	$C_{d_{free}}$
1/3	11	12	1
1/4	15	16	4
1/5	19	20	1
1/6	23	24	4
1/7	27	28	1
1/8	31	32	4
1/9	35	36	1
1/10	39	40	4
1/16	63	64	4
1/32	127	128	4
1/64	255	256	4
1/128	511	512	4

Table 4 - Maximum free distance Nested Codes, K=6 using starting code with generator vectors: $\underline{H}$= 47, 53 .

Rate R=1/128 Connection vectors:
$\underline{G}$= 3777777777777777777777777777777777777777777,
1316767357567373567673575673735676735756737,
3173556673355667335566733556673355667335566,
1767335566733556673355667335566733556673355,
2621412430250605214124302506052141243025060,
3777777777777777777777777777777777777777777

R	d_{free}	Heller	$C_{d_{free}}$
1/3	13	13	1
1/4	18	18	6
1/5	22	22	2
1/6	27	27	4
1/7	32	32	8
1/8	36	36	3
1/9	40	41	1
1/10	45	45	1
1/16	72	73	1
1/32	146	146	5
1/64	292	292	3
1/128	584	585	1

Table 5 - Maximum free distance Nested Codes, K=7 using starting code with generator vectors: $\underline{H}$= 133, 171 .

Rate R=1/128 Connection vectors:
$\underline{G}$= 3777777777777777777777777777777777777777777,
1453657275365727536572753657275365727536572,
3565727536572753657275365727536572753657275,
3377375767737576773757677375767737576773757,
7265325526532552653255265325526532552653255265325,
2244221104422110442211044221104422110442211,
3777777777777777777777777777777777777777777

R	d_{free}	Heller	$C_{d_{free}}$
1/3	15	15	7
1/4	20	20	3
1/5	25	25	1
1/6	30	30	2
1/7	36	36	8
1/8	40	41	1
1/9	46	46	5
1/10	51	51	4
1/16	82	82	5
1/32	164	164	3
1/64	328	329	1
1/128	658	658	5

Table 6 - Maximum free distance Nested Codes, K=8 using starting code with generator vectors: $\underline{H}$= 247, 371 .

Rate R=1/128 Connection vectors:
$\underline{G}$= 3777777777777777777777777777777777777777777,
1657576773757677375767737576773757677375767,
3537777777375767737576773757677375767737576,
1361205024522512452251245225124522512452251,
1565325526130542613054261305426130542613054,
2332653255667335566733556673355667335566733,
2605124522512452251245225124522512452251245,
3777777777777777777777777777777777777777777

R	d_{free}	Heller	$C_{d_{free}}$
1/3	16	17	2
1/4	22	22	2
1/5	28	28	4
1/6	34	34	6
1/7	40	40	8
1/8	45	45	1
1/9	51	51	4
1/10	56	57	1
1/16	91	91	4
1/32	182	182	2
1/64	365	365	1
1/128	731	731	4

Table 7 - Maximum free distance Nested Codes, K=9 using starting code with generator vectors: $\underline{H}$=561, 753 .

Rate R=1/128 Connection vectors:
$\underline{G}$= 3777777777777777777777777777777777777777777,
1575757737376767757577373767677575773737676,
3733476271745637134762717456371347627174563,
3136573753677275765737536772757657375367727,
2043220064401510032200644015100322006440151,
1726537552773257665375527732576653755277325,
754742717056361347427170563613474271705636,
1225124522512452251245225124522512452251245,
3777777777777777777777777777777777777777777

R	d_{free}	Heller	$C_{d_{free}}$
1/3	18	18	11
1/4	24	24	3
1/5	30	31	2
1/6	36	37	1
1/7	43	44	4
1/8	49	50	1
1/9	56	56	8
1/10	62	62	5
1/16	100	100	8
1/32	200	201	3
1/64	401	402	1
1/128	804	804	8

Table 8 - Maximum free distance Nested Codes, K=10 using starting code with generator vectors: $\underline{H}$= 1167, 1545 .

Rate R=1/128 Connection vectors:
$\underline{G}$ = 3777777777777777777777777777777777777777777,
1257777777777777777777777777777777777777777,
477576371747637174763717476371747637174763,
3772743617074361707436170743617074361707436,
3401707034160703416070341607034160703416070,
2776070743617074361707436170743617074361707,
513345627134562713456271345627134562713456,
3524432552653255265325526532552653255265325,
2361114462311446231144623114462311446231144,
3777777777777777777777777777777777777777777

R	d_{free}	Heller	$C_{d_{free}}$
1/3	20	20	6
1/4	27	27	7
1/5	34	34	7
1/6	40	41	2
1/7	48	48	8
1/8	54	54	2
1/9	61	61	1
1/10	68	68	3
1/16	109	109	1

Table 8 - Maximum free distance Nested Codes, K=10 using starting code with generator vectors: $\underline{H}$= 1167, 1545 (Continued).

Rate R=1/128 Connection vectors: $\underline{G}$ = 3777777777777777777777777777777777777777777, 1257777777777777777777777777777777777777777, 477576371747637174763717476371747637174763, 3772743617074361707436170743617074361707436, 3401707034160703416070341607034160703416070, 2776070743617074361707436170743617074361707, 513345627134562713456271345627134562713456, 3524432552653255265325526532552653255265325, 2361114462311446231144623114462311446231144, 3777777777777777777777777777777777777777777			
R	d_{free}	Heller	$C_{d_{free}}$
1/32	219	219	4
1/64	438	438	2
1/128	877	877	1

Table 9 - Maximum free distance Nested Codes, K=11 using starting code with generator vectors: $\underline{H}$= 2335, 3661 .

Rate R=1/128 Connection vectors: $\underline{G}$= 177777777777777777777, 675767737576773757677, 557476371747637174763, 1564321506432150643215, 1377677375767737576773, 571305626134542713056, 1604221104422110442211, 1146773557673375667735, 1233114662315446331146, 242552453251265225524, 177777777777777777777			
R	d_{free}	Heller	$C_{d_{free}}$
1/3	21	22	4
1/4	28	29	1
1/5	36	37	4
1/6	44	44	8
1/7	51	52	4
1/8	58	59	2
1/9	66	66	5
1/10	73	74	1
1/16	118	118	5
1/32	236	237	1
1/64	474	475	2

Table 10 - Maximum free distance Nested Codes, K=12 using starting code with generator vectors: $\underline{H}$= 4335, 5723 .

Rate R=1/64 Connection vectors:
$\underline{G}$= 177777777777777777777, 325265453333251266666,
772553366666767555555, 777777777777777777777,
145241232444452651111 1, 147767777777777777777,
300141002222200044444, 151241026000006540000 0,
127777773777773677777 7, 112532451111111222222 2,
645326455555451333333, 177777777777777777777

R	d_{free}	Heller	$C_{d_{free}}$
1/3	23	24	26
1/4	31	32	7
1/5	39	40	7
1/6	47	48	7
1/7	55	56	7
1/8	63	64	7
1/9	71	72	7
1/10	79	80	7
1/16	127	128	7
1/32	255	256	7
1/64	511	512	7

Table 11 - Maximum free distance Nested Codes, K=13 using starting code with generator vectors: $\underline{H}$= 10533, 17661 .

Rate R=1/64 Connection vectors:
$\underline{G}$= 177777777777777777777, 777756377367377677716,
572577731573653736177, 737377477774737747777,
144573576324337657734 5, 415642554735544251432,
137326467724633246770 5, 444335267646233566755,
172244270203746423027 2, 125511101212452110122 0,
232735165651657556565, 132204260051240022002 2,
177777777777777777777

R	d_{free}	Heller	$C_{d_{free}}$
1/3	24	25	6
1/4	32	34	2
1/5	41	42	1
1/6	50	51	5
1/7	58	59	2
1/8	67	68	4
1/9	76	76	8
1/10	84	85	3
1/16	136	136	8
1/32	272	273	4
1/64	545	546	1

Table 12 - Maximum free distance Nested Codes, K=14 using starting code with generator vectors: $\underline{H}$= 21675, 27123 .

Rate R=1/64 Connection vectors:
$\underline{G}$= 1777777777777777777777, 3165767657777577777576, 5477176373626776267777, 7776735527355273555273, 1765251636556765167651, 1276765377677172771725, 1330772153715137151372, 4550777545273456734 57, 1157625562061520635205, 1622342057610076300762, 1361245222446624446245, 1014532501331513135132, 5210022140222402024 02, 1777777777777777777777

R	d_{free}	Heller	$C_{d_{free}}$
1/3	26	27	23
1/4	36	36	37
1/5	44	45	2
1/6	54	54	14
1/7	63	63	10
1/8	72	72	10
1/9	80	81	2
1/10	90	90	10
1/16	144	145	4
1/32	255	290	7
1/64	580	580	16

Table 13 - Minimum bit error probability Nested Codes, K= 3 using starting code with generator vectors: $\underline{H}$= 5, 7 .

Rate R=1/128 Connection vectors:
$\underline{G}$= 3777777777777777777777777777777777777777777, 1625543516636660017777777777777777777777777, 3376737777777757760000000000000000000000000

R	d_{free}	Heller	$C_{d_{free}}$
1/3	7	8	1
1/4	10	10	3
1/5	12	13	1
1/6	14	16	1
1/7	17	18	1
1/8	19	21	1
1/9	22	24	1
1/10	24	26	1
1/16	38	42	1
1/32	79	85	1
1/64	150	170	1
1/128	278	341	1

Table 14 - Minimum bit error probability Nested Codes, $K=4$ using starting code with generating vectors $\underline{H}$= 15, 17 .

Rate R=1/64 Connection vectors: $\underline{G}$= 1777777777777777777777,1452526174577777677777, 632315550001004125252,1777776767777777652525			
R	d_{free}	Heller	$C_{d_{free}}$
1/3	10	10	6
1/4	12	13	1
1/5	15	16	1
1/6	18	20	1
1/7	22	23	1
1/8	24	26	1
1/9	28	30	1
1/10	30	33	1
1/16	48	53	1
1/32	94	106	1
1/64	191	213	1

Table 15 - Minimum bit error probability Nested Codes, $K=5$ using starting code with generating vectors $\underline{H}$= 23, 35 .

Rate R=1/128 Connection vectors: $\underline{G}$= 3777777777777777777777777777777777777777777, 1324335465172525752525252525252525252525252, 1465656536577776377777777777777777777777777, 2522706112205252425252525252525252525252525, 3777777677577777777777777777777777777777777			
R	d_{free}	Heller	$C_{d_{free}}$
1/3	11	12	1
1/4	14	16	1
1/5	18	20	1
1/6	21	24	1
1/7	26	28	1
1/8	28	32	1
1/9	32	36	1
1/10	35	40	1
1/16	57	64	1
1/32	113	128	1
1/64	240	256	1
1/128	496	512	1

Table 16 - Minimum bit error probability Nested Codes, $K=6$ using starting code with generator vectors: $\underline{H}$= 47, 53 .

Rate R=1/128 Connection vectors:
$\underline{G}$= 3777777777777777777777777777777777777777,
1331123765777777777777777777777777777777,
3256115656111037777777777777777777675736,
1543330530467740000000000000000000102041,
2451511411210034000000000000000000143061,
3777777777777437777777777777777777634716

R	d_{free}	Heller	$C_{d_{free}}$
1/3	13	13	1
1/4	17	18	1
1/5	21	22	1
1/6	26	27	1
1/7	29	32	1
1/8	34	36	1
1/9	37	41	1
1/10	41	45	1
1/16	64	73	1
1/32	130	146	1
1/64	260	292	1
1/128	516	585	1

Table 17 - Minimum bit error probability Nested Codes, $K=7$ using starting code with generator vectors: $\underline{H}$= 133, 171 .

Rate R=1/128 Connection vectors:
$\underline{G}$= 3777777777777777777777777777777777777777,
1452322725777157777777777777777777777777,
32447435714066726525252525252525252525252,
3332257616175375577777777777777777777777,
7652332112035671252525252525252525252525,
2227405430770000600000000000000000000000,
3777777777407777177777777777777777777777

R	d_{free}	Heller	$C_{d_{free}}$
1/3	14	15	1
1/4	20	20	6
1/5	24	25	1
1/6	29	30	1
1/7	34	36	1
1/8	38	41	1
1/9	43	46	1
1/10	48	51	1
1/16	74	82	1
1/32	148	164	1
1/64	303	329	1
1/128	623	658	1

Table 18 - Minimum bit error probability Nested Code, K=8 using starting code with generator vectors: $\underline{H}$= 247, 371 .

Rate R=1/128 Connection vectors:
$\underline{G}$= 3777777777777777777777777777777777777777777,
1126766773675736757367573675736757367573675,
3263171777773675736757367573675736757367573,
1770712024516347163471634716347163471634716,
1607545655322451224512245122451224512245122,
2250402706147163471634716347163471634716347,
2512060071634716347163471634716347163471634,
3777777757367573675736757367573675736757367

R	d_{free}	Heller	$C_{d_{free}}$
1/3	16	17	3
1/4	22	22	3
1/5	27	28	1
1/6	32	34	1
1/7	37	40	1
1/8	42	45	1
1/9	46	51	1
1/10	52	57	1
1/16	83	91	1
1/32	168	182	1
1/64	347	365	1
1/128	705	731	1

Table 19 - Minimum bit error probability Nested Codes, K=9 using starting code with generator vectors: $\underline{H}$=561, 753 .

Rate R=1/128 Connection vectors:
$\underline{G}$= 3777777777777777777777777777777777777777777,
1515373777777777376775773767757777277677775,
3126667777767676775773767757737737777577577,
3254473752450505713627457136277242571375212,
2643714024236363447116234471160171704700747,
1170742025315151423046114230470464502342322,
750471752462626755733667557307313175435454,
1244004001317171062144310621460474606302363,
3777773777575757737677577376777767773777737

R	d_{free}	Heller	$C_{d_{free}}$
1/3	17	18	1
1/4	23	24	1
1/5	29	31	1
1/6	36	37	3
1/7	40	44	1
1/8	46	50	1
1/9	52	56	1
1/10	56	62	1
1/16	91	100	1
1/32	185	201	1
1/64	387	402	1

Table 19 - Minimum bit error probability Nested Codes, K=9 using starting code with generator vectors: $\underline{H}$=561, 753 (Continued).

Rate R=1/128 Connection vectors: $\underline{G}$= 3777777777777777777777777777777777777777777, 1515373777777777376775773767757777277677775, 3126667777767676775773767757737737777577577, 3254473752450505713627457136277242571375212, 2643714024236363447116234471160171704700747, 1170742025315151423046114230470464502342322, 750471752462626755733667557307313175435454, 1244004001317171062144310621460474606302363, 3777773777575757737677577376777767773777737			
R	d_{free}	Heller	$C_{d_{free}}$
1/128	786	804	1

Table 20 - Minimum bit error probability Nested Codes, K=10 using starting code with generator vectors: $\underline{H}$= 1167, 1545 .

Rate R=1/128 Connection vectors: $\underline{G}$ = 3777777777777777777777777777777777777777777, 1445667777777777777777777777777777777777737, 702573333373766677555773337666775557733577, 3615007777736237744777117762377447771177262, 3355500666467505572133642675055721336424775, 2763677111301572233644675115722336446750755, 75160444654201104022100442011040221007202, 3123004666423155463331466631554633314664471, 2232112000054250105202124042501052021243002, 3777765777766577753777277765777537772776765			
R	d_{free}	Heller	$C_{d_{free}}$
1/3	19	20	4
1/4	26	27	3
1/5	32	34	1
1/6	38	41	1
1/7	44	48	1
1/8	50	54	1
1/9	56	61	1
1/10	62	68	1
1/16	98	109	1
1/32	200	219	1
1/64	411	438	1
1/128	877	877	1

Table 21 - Minimum bit error probability Nested Codes, K=11 using starting code with generator vectors: $\underline{H}$= 2335, 3661 .

Rate R=1/128 Connection vectors:
$\underline{G}$ = 3777777777777777777777777777777777777777777,
11777777777777777777777777777777777777777340,
1533367573675736757367573675736757367573520,
3665532655326553265532655326553265532655537,
2107777777777777777777777777777777777777260,
1052614224140204102041020410204102041020437,
3461000000002143061430614306143061430614777,
2114573757735430614306143061430614306143240,
2462204020042245122451224512245122451224637,
701573757737471634716347163471634716346557,
3776655326553367573675736757367573675737757

R	d_{free}	Heller	$C_{d_{free}}$
1/3	20	22	1
1/4	24	29	1
1/5	32	37	1
1/6	39	44	1
1/7	46	52	1
1/8	52	59	1
1/9	58	66	1
1/10	65	74	1
1/16	108	118	1
1/32	224	237	1
1/64	454	475	1
1/128	914	950	1

Table 22 - Minimum bit error probability Nested Code, K=12 using starting code with generator vectors: $\underline{H}$= 4335, 5723 .

Rate R=1/128 Connection vectors:
$\underline{G}$ = 3777777777777777777777777777777777777777777,
537775673525252525252525251524643777660404,
1473653547252525252525252527253512525222764,
1267377777777777777777777777777767567337413,
3400100010000000000000000002001027777673012,
3757703176000000000000000002001034000145773,
5633746317777777777777777777577677356537775,
3124400020252525252525252526253111673644773,
2150077767777777777777777777577677514617 2207,
2207776715777777777777777777577674063170 5004,
1060427346525252525252525252525274210555367,
3777777777777777777777777777777777777716000

R	d_{free}	Heller	$C_{d_{free}}$
1/3	22	24	9
1/4	27	32	1
1/5	34	40	1
1/6	42	48	1
1/7	50	56	1
1/8	56	64	1

Table 22 - Minimum bit error probability Nested Code, K=12 using starting code with generator vectors: $\underline{H}$= 4335, 5723 (Continued).

Rate R=1/128 Connection vectors:
$\underline{G}$ = 37777777777777777777777777777777777777777,
537775673525252525252525152464377766 0404,
14736535472525252525252525272535 12525222764,
126737777777777777777777777777767567337413,
340010001000000000000000000200102777767 3012,
375770317600000000000000000200103400014577 3,
56337463177777777777777777577677735653777 5,
31244000202525252525252526253111673644773,
21500777677777777777777777577677514617220 7,
22077767157777777777777775776740631705004,
10604273465252525252525252525274210555367,
37777777777777777777777777777777777716000

R	d_{free}	Heller	$C_{d_{free}}$
1/9	63	72	1
1/10	71	80	1
1/16	119	128	1
1/32	247	256	1
1/64	503	512	1
1/128	996	1024	5

Table 23 - Minimum bit error probability Nested Code, K=13 using starting code with generator vectors: $\underline{H}$= 10533, 17661 .

Rate R=1/32 Connection vectors:
$\underline{G}$ = 37777777777, 12777777777, 14777777767, 13077775777, 30777712713, 15300042070, 21000045044, 15477732713, 32700065024, 22464002412, 1313731345, 25000007034, 37777777777

R	d_{free}	Heller	$C_{d_{free}}$
1/3	22	25	1
1/4	28	34	1
1/5	36	42	1
1/6	44	51	1
1/7	52	59	1
1/8	60	68	1
1/9	68	76	1
1/10	76	85	1
1/16	124	136	1
1/32	256	273	4

Table 24 - Minimum bit error probability Nested Code, K=14 using starting code with generator vectors: $\underline{H}$= 21675, 27123 .

Rate R=1/32 Connection vectors:
$\underline{G}$= 37777777777, 5777777777, 10737777573, 13165777675, 30555777777, 25337000347, 20440000000, 17212777734, 25345000247, 30422000102, 22657777225, 23340000452, 12072777324, 37777000757

R	d_{free}	Heller	$C_{d_{free}}$
1/3	23	27	1
1/4	30	36	1
1/5	38	45	1
1/6	46	54	1
1/7	55	63	1
1/8	64	72	1
1/9	74	81	1
1/10	82	90	1
1/16	134	145	4
1/32	270	290	7

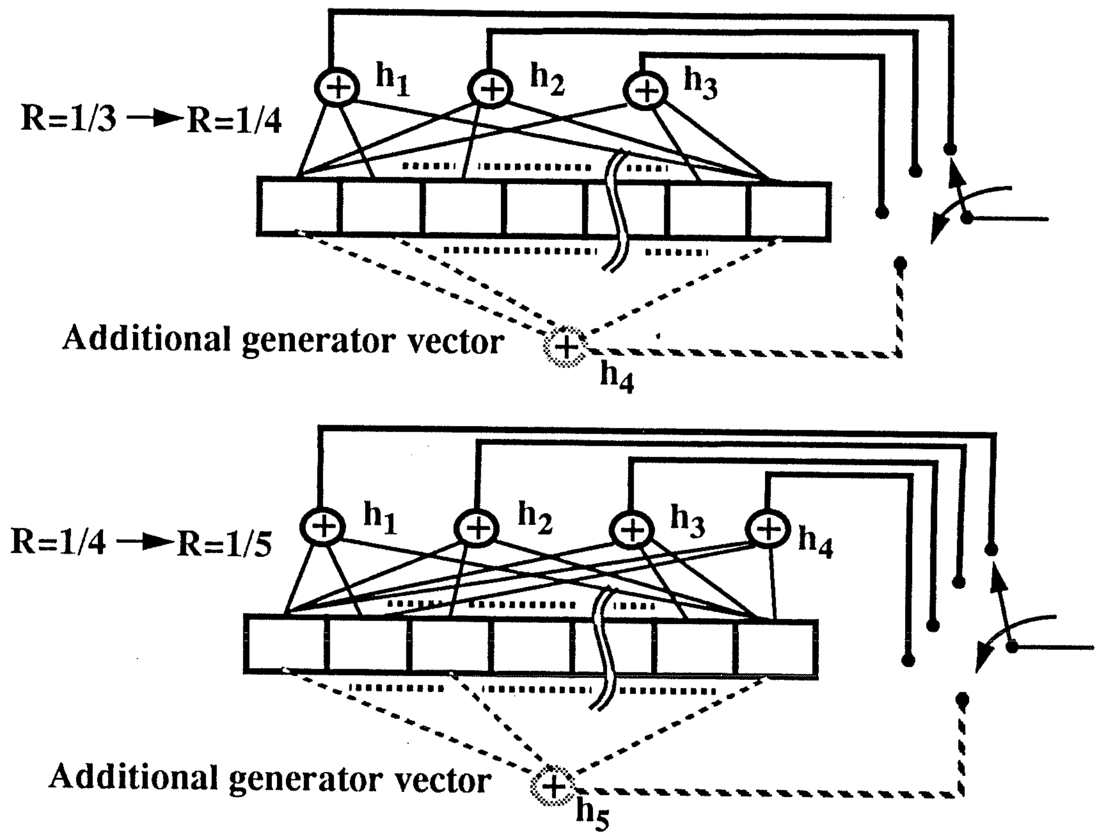

Figure 1 - Generation of Nested Codes

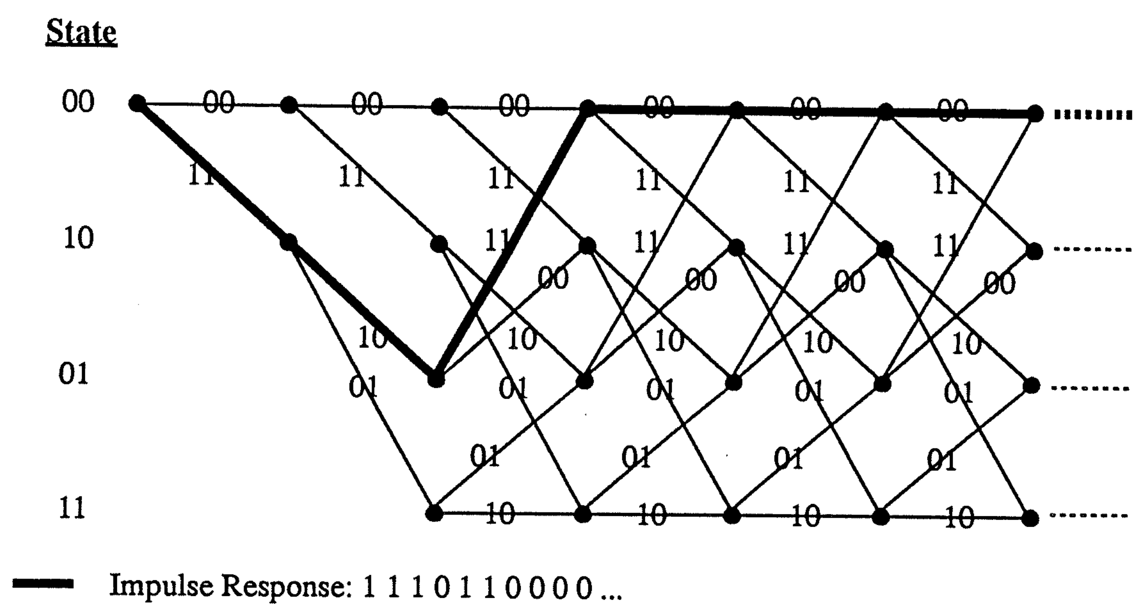

Figure 2 - Impulse Response for rate $R = 1/2, K = 3$ convolutional code with generator vectors $H = 5, 7$.

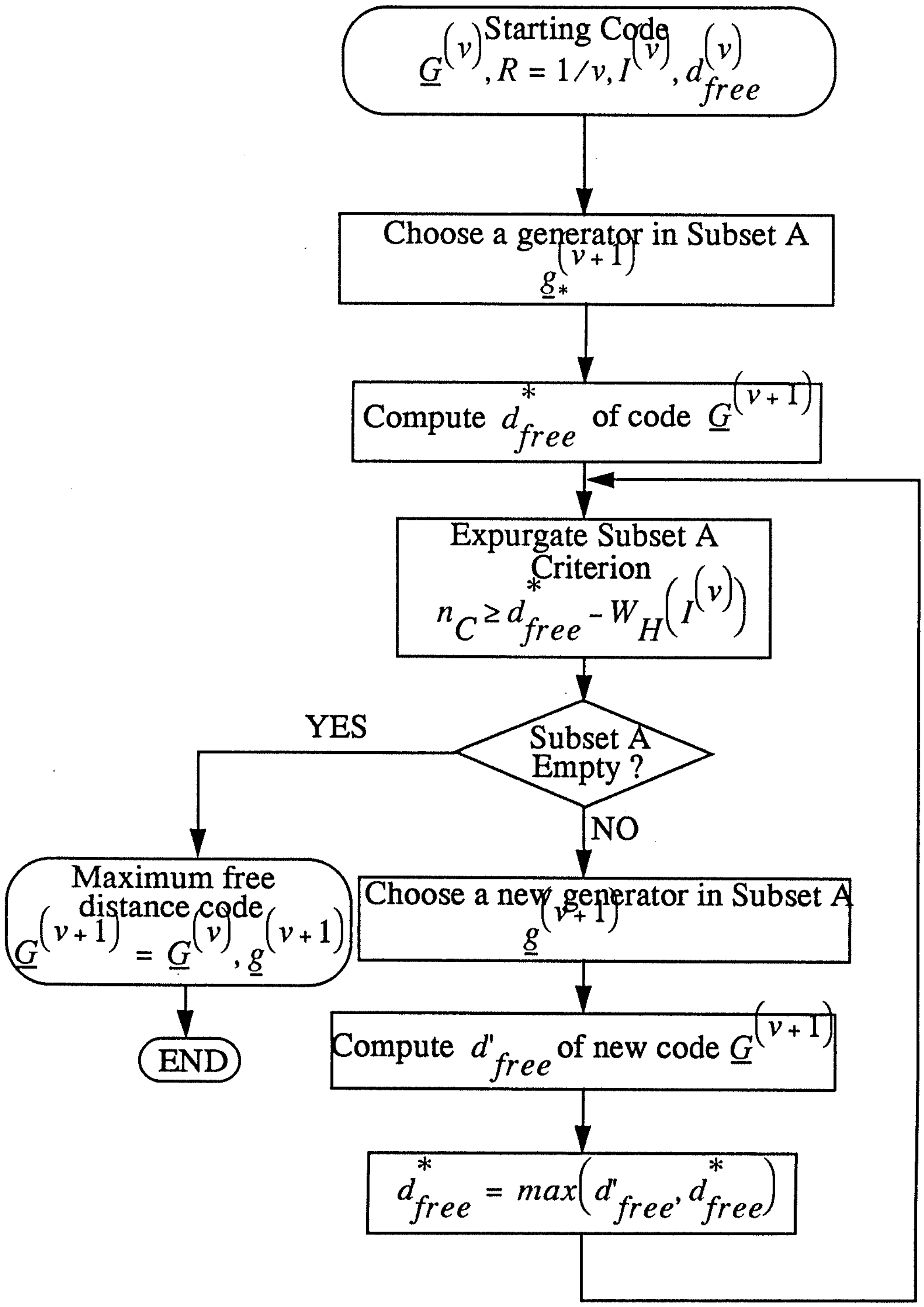

Figure 3 - Flow chart of the simplified procedure for the search of maximum free distance Nested Codes.

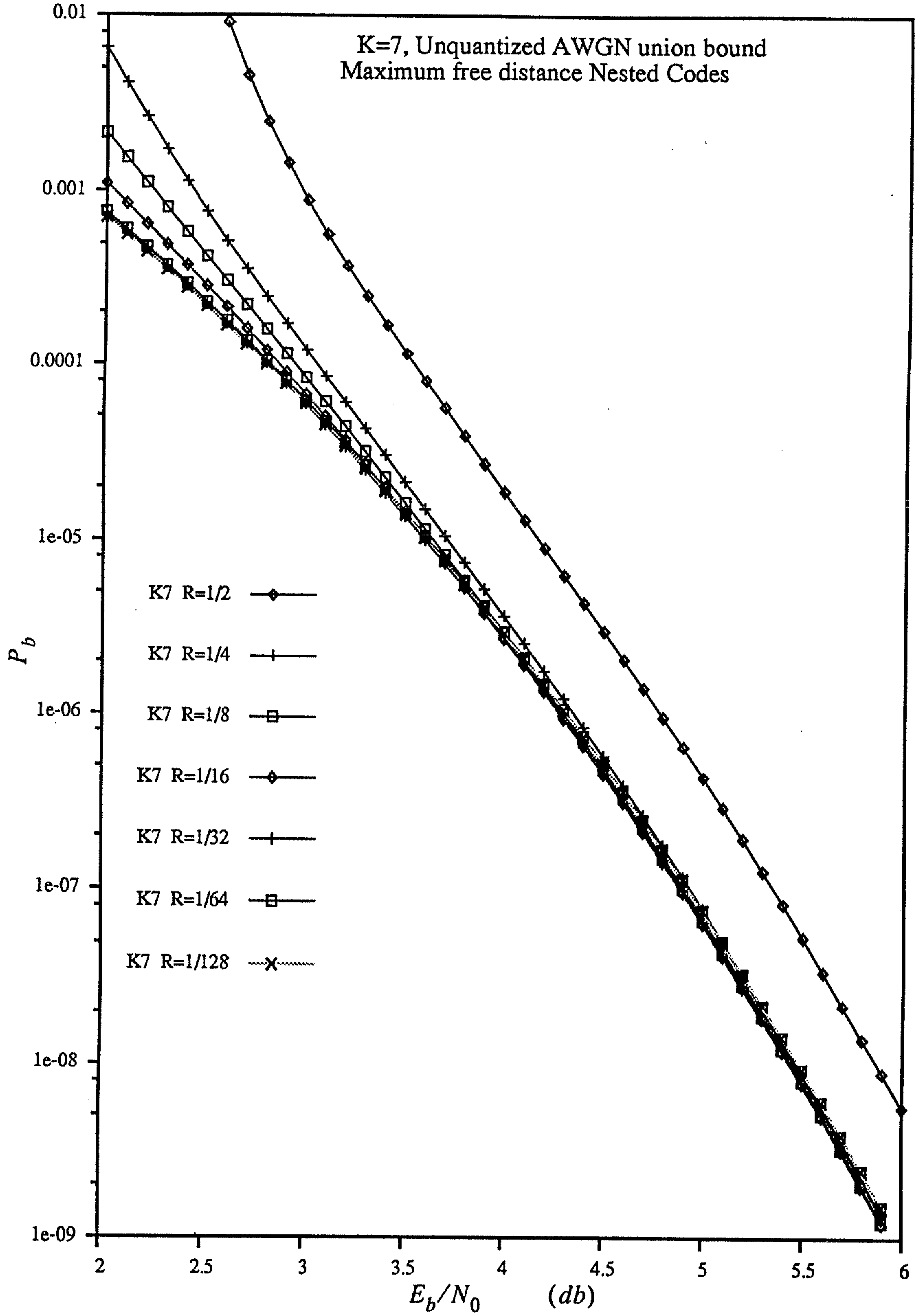

Figure 4 - Performances of constraint length K=7 Maximum free distance Nested Codes.

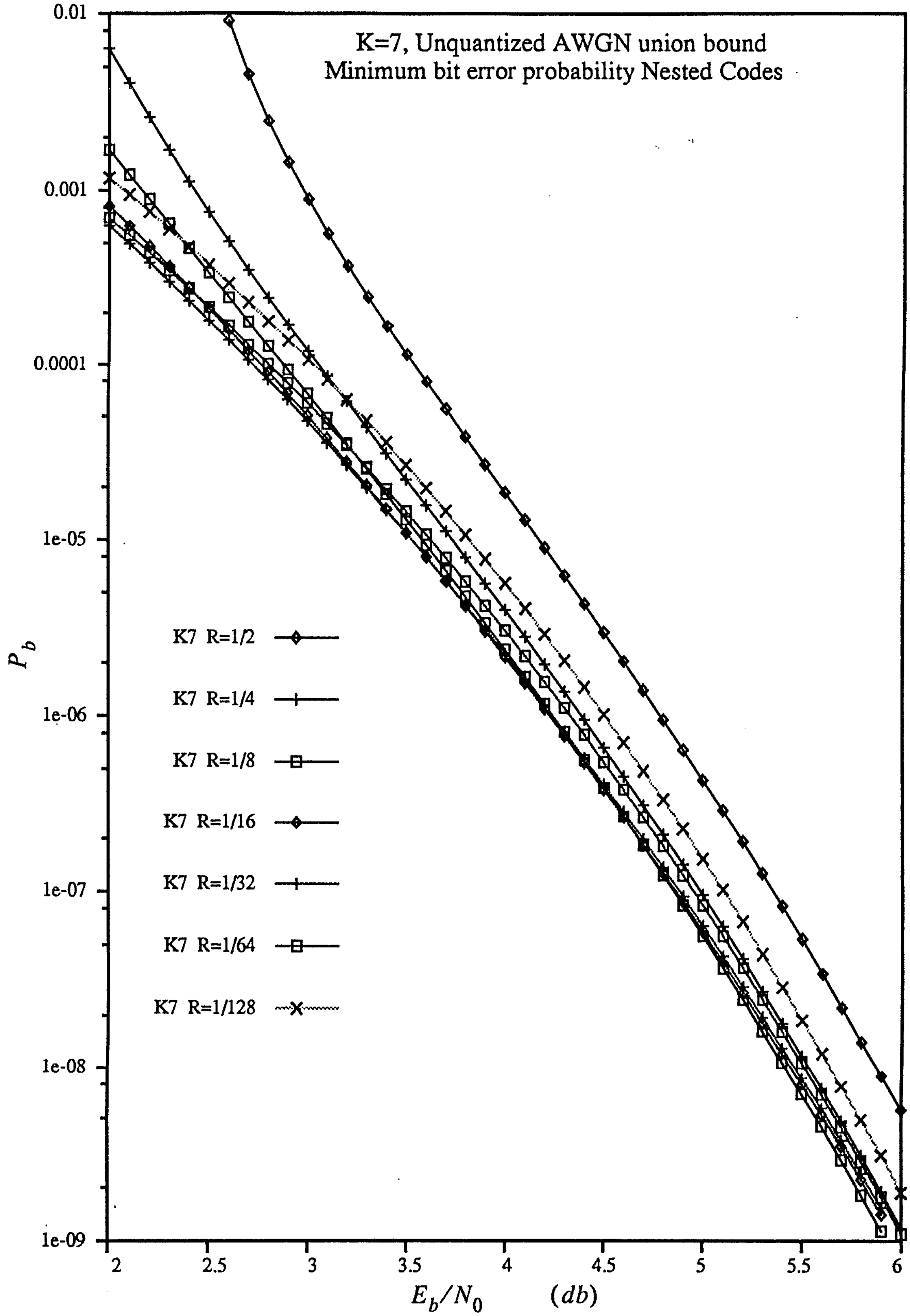

Figure 5 - Performances of constraint length K=7 Minimum bit error probability Nested Codes.

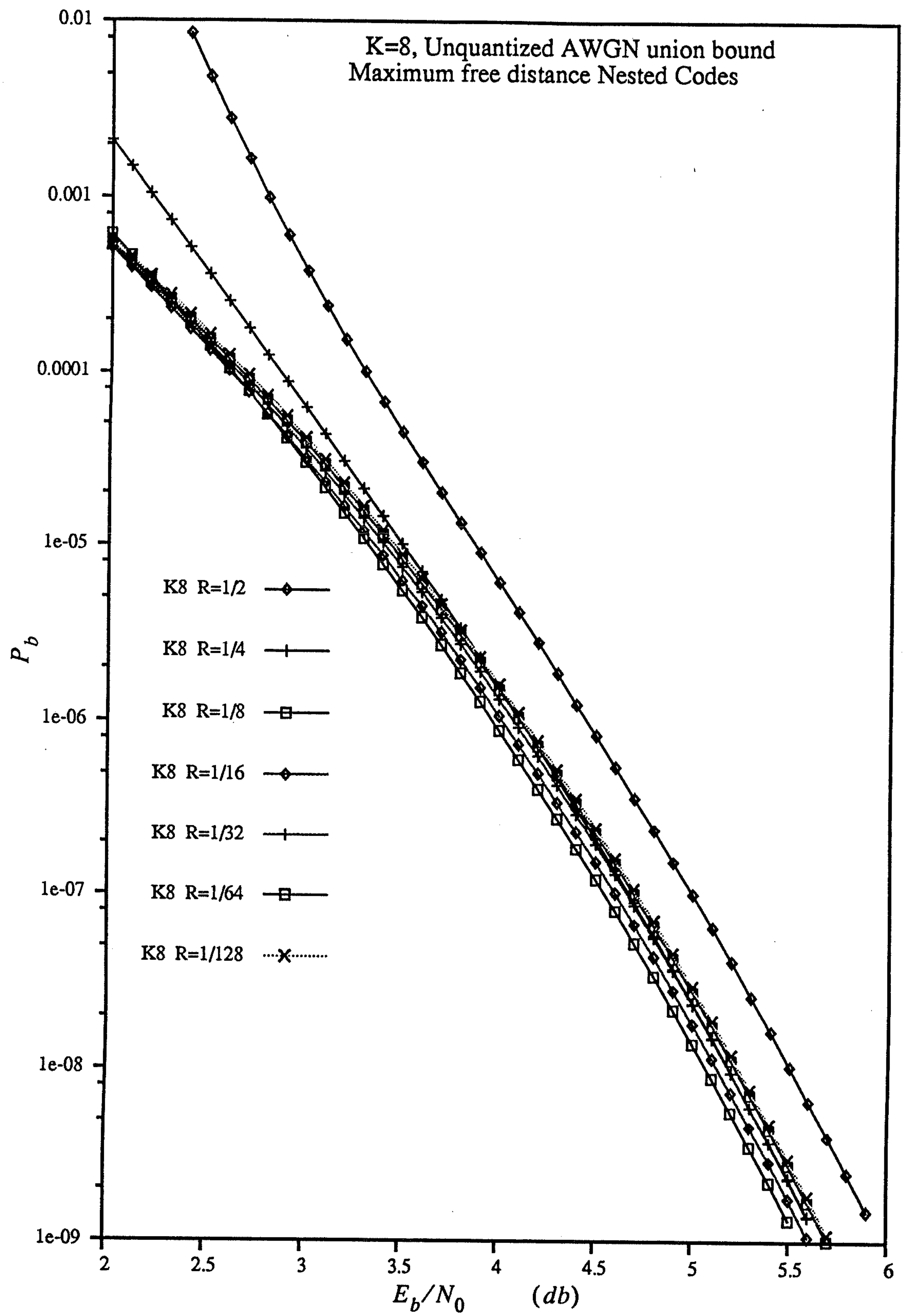

Figure 6 - Performances of constraint length K=8 Maximum free distance Nested Codes.

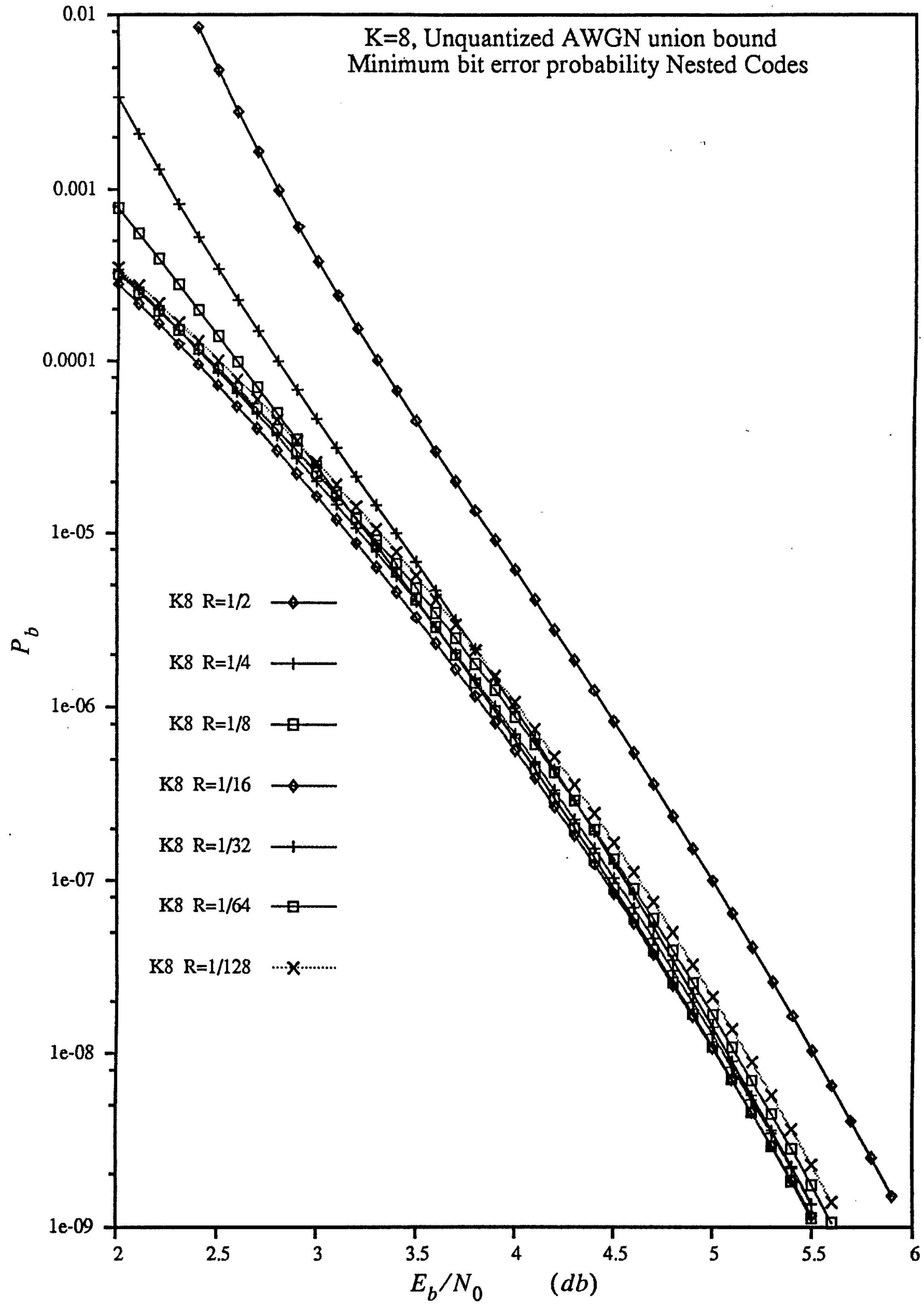

Figure 7 - Performances of constraint length K=8 Minimum bit error probability Nested Codes.

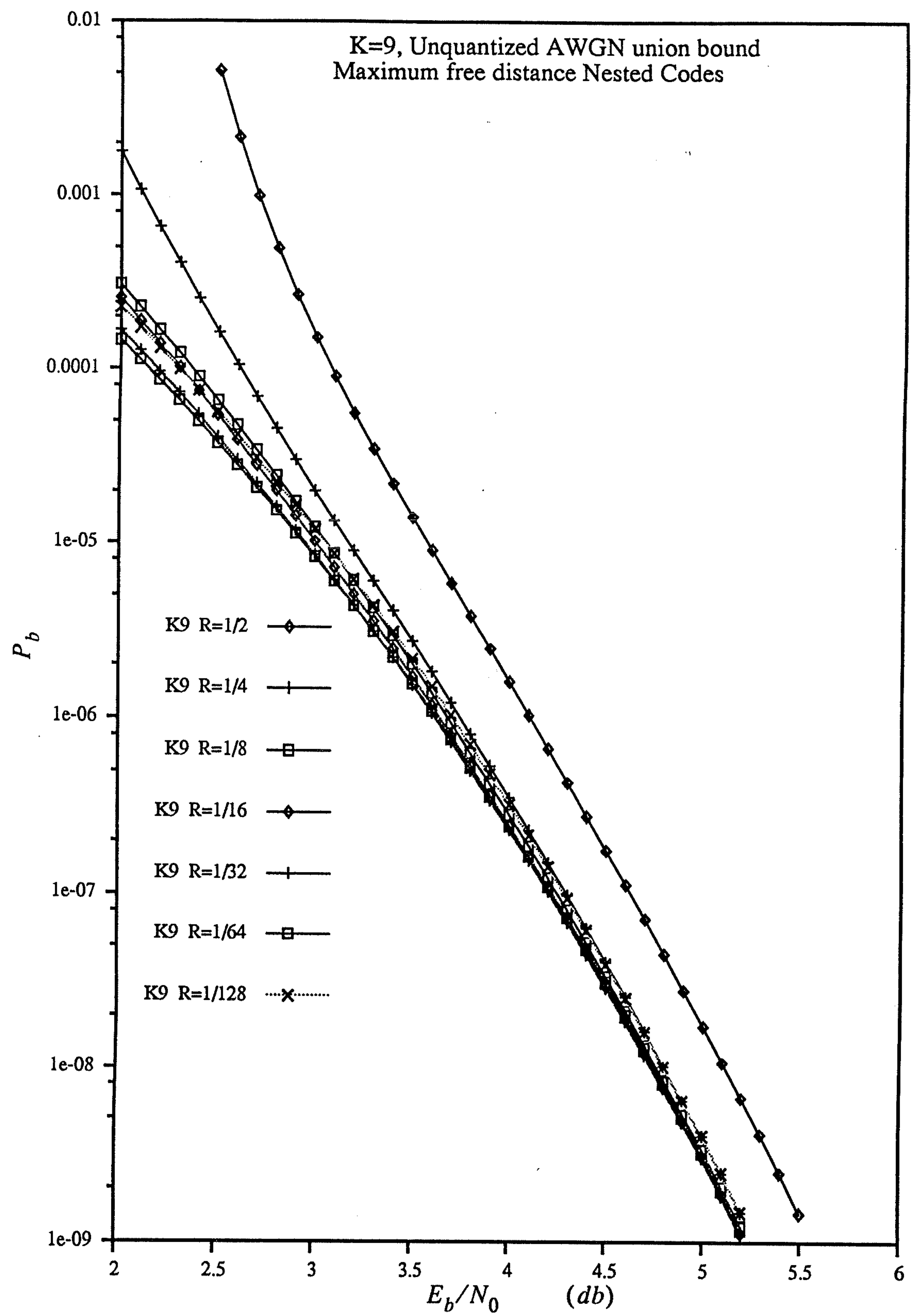

Figure 8 - Performances of constraint length K=9 Maximum free distance Nested Codes.

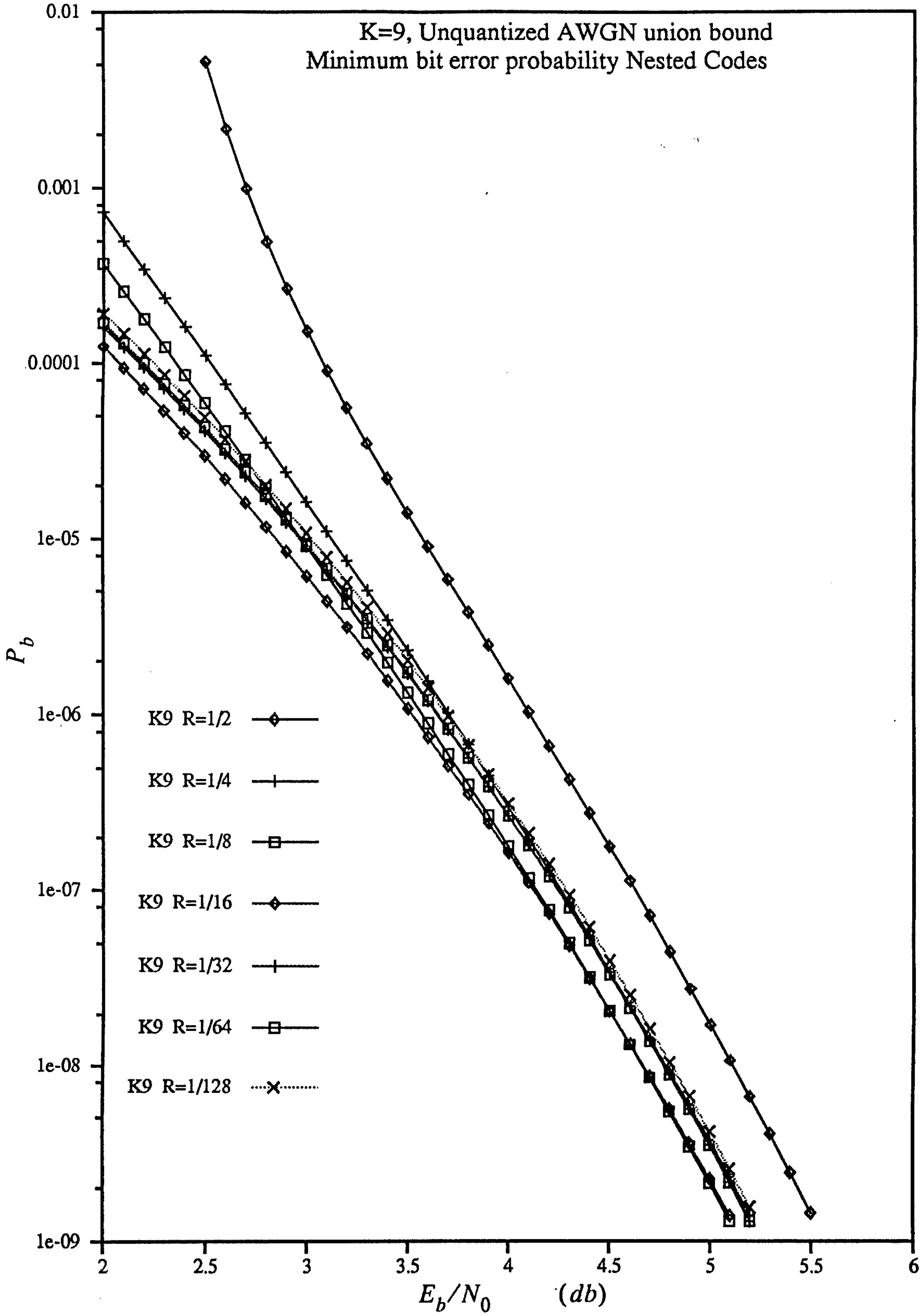

Figure 9 - Performances of constraint length K=9 Minimum bit error probability Nested Codes.

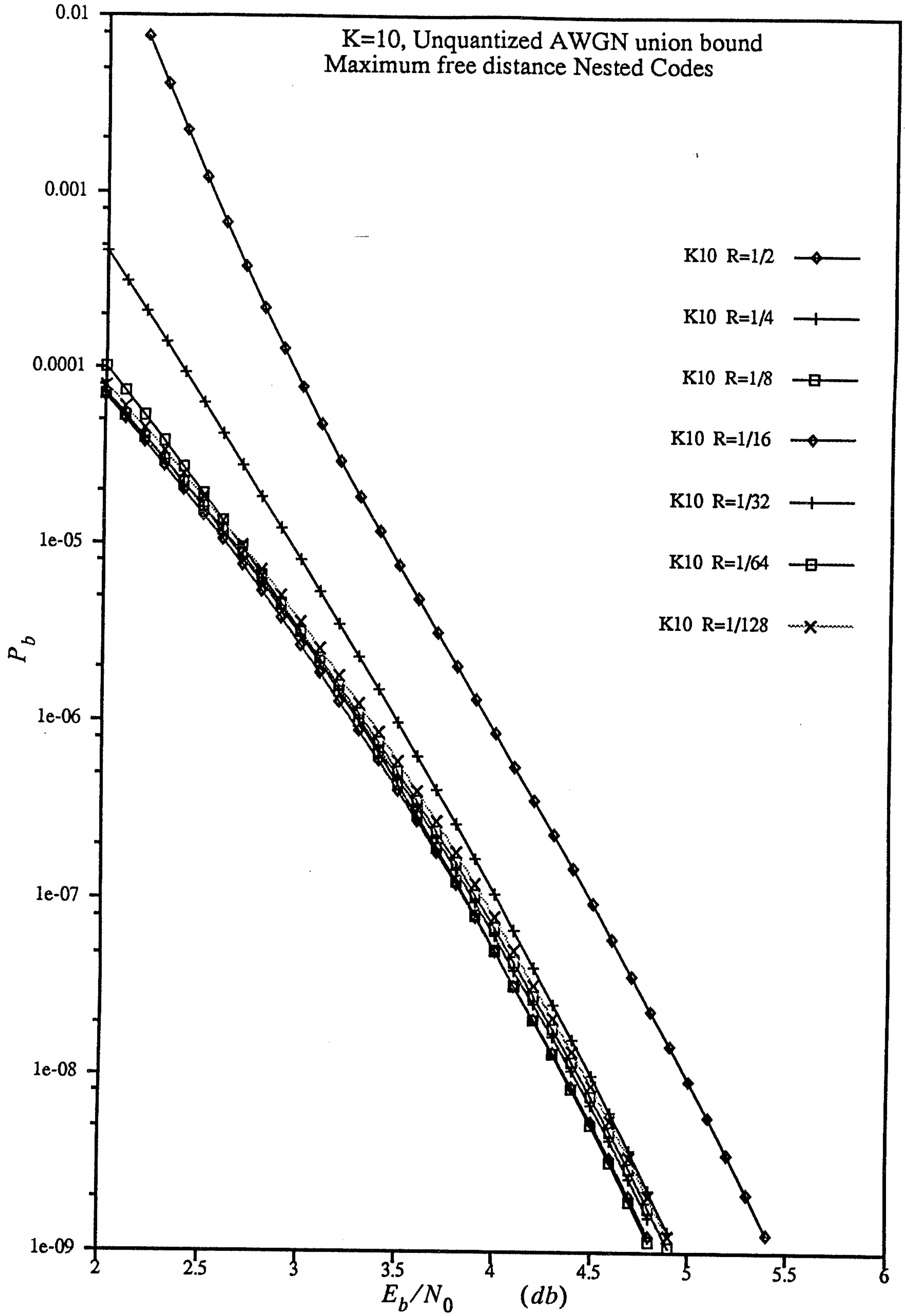

Figure 10 - Performances of constraint length K=10 Maximum free distance Nested Codes.

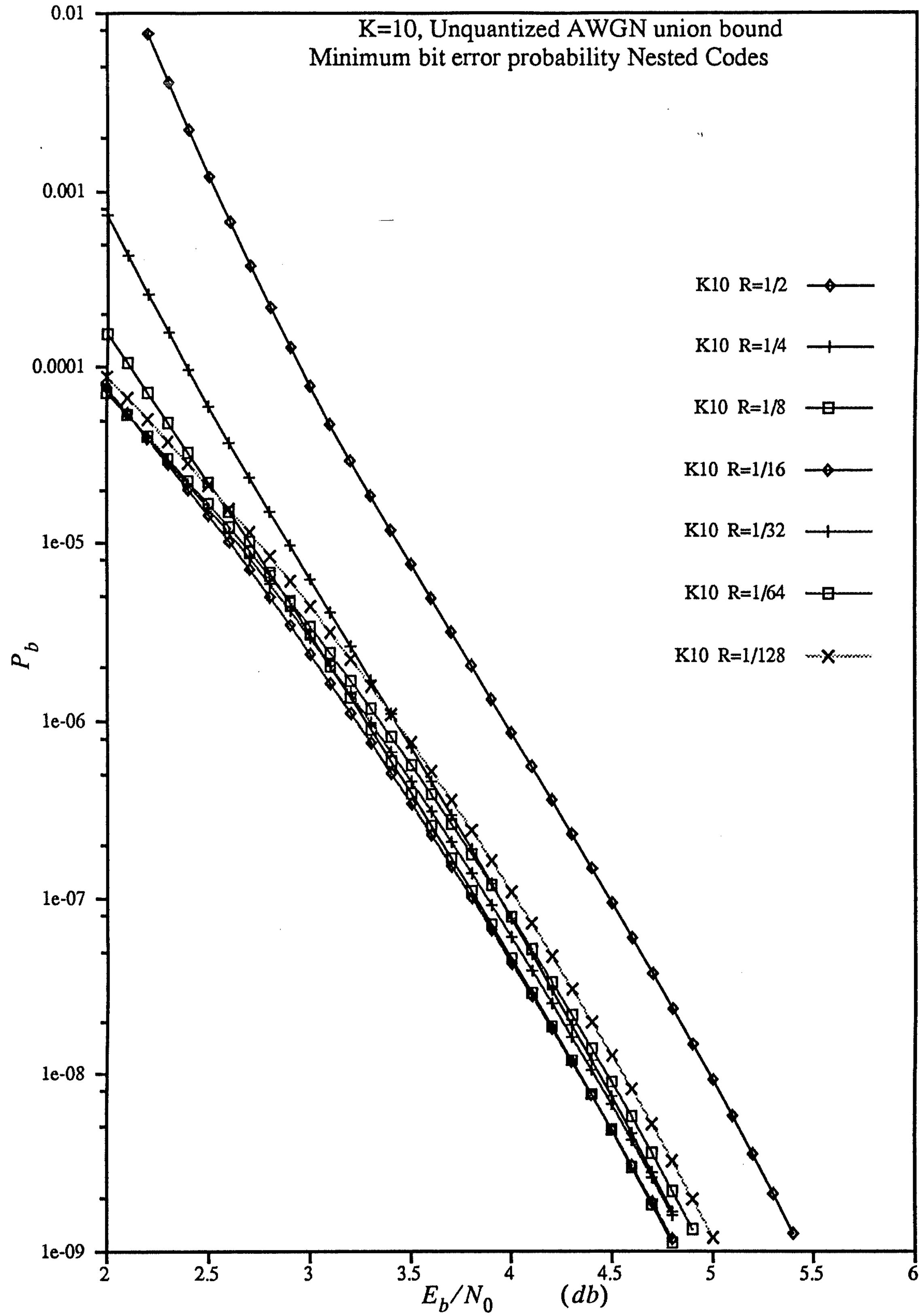

Figure 11 - Performances of constraint length K=10 Minimum bit error probability Nested Codes.

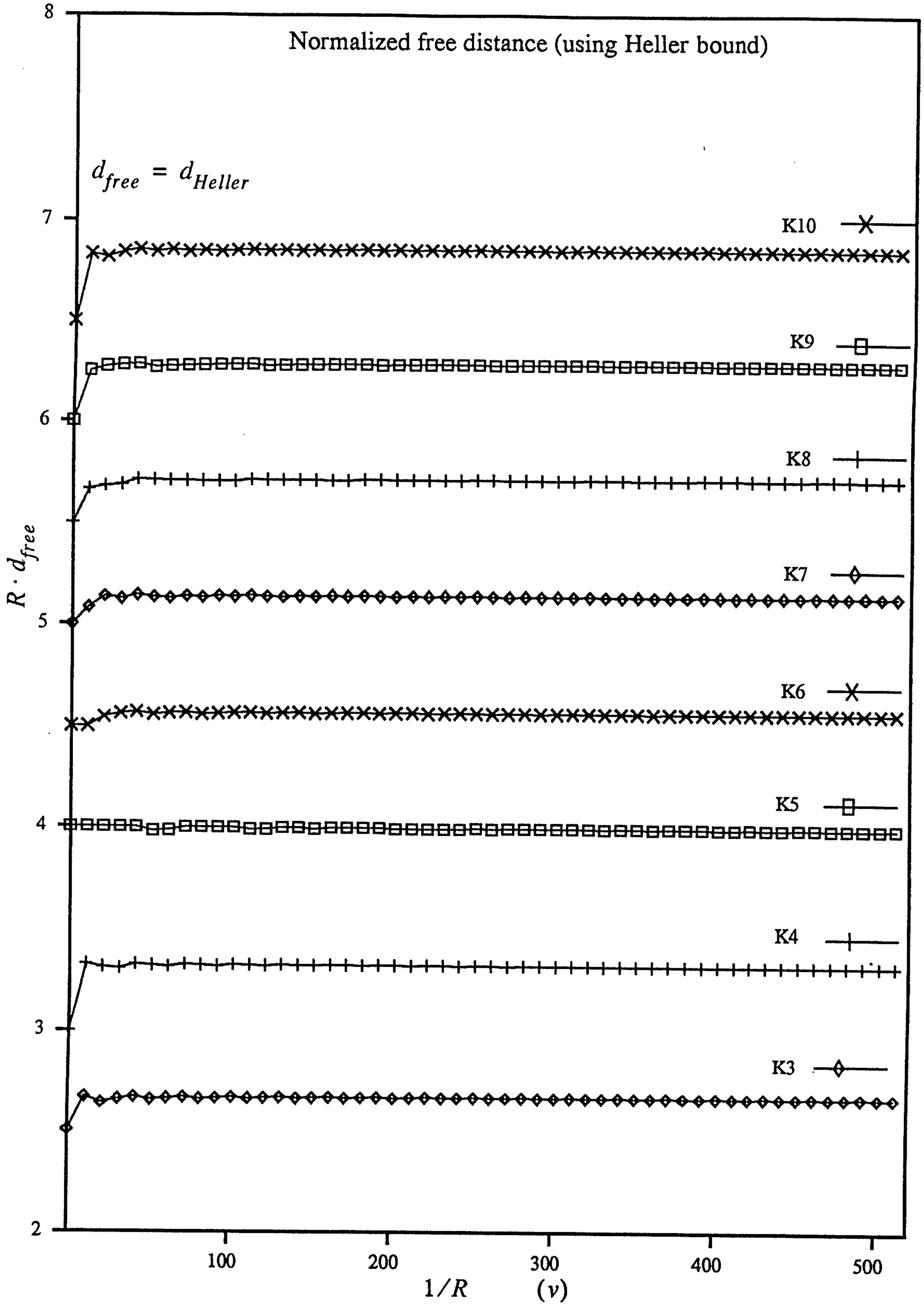

Figure 12 - Normalized free distance using Heller bound on free distance.

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