Titre: Title:	Non-linear models of a two degrees-of-freedom one link flexible arm
Auteurs: Authors:	Jean-Claude Piedboeuf, & Richard Hurteau
Date:	1989
Type:	Rapport / Report
Référence: Citation:	Piedboeuf, J.-C., & Hurteau, R. (1989). Non-linear models of a two degrees-of-freedom one link flexible arm. (Rapport technique n° EPM-RT-89-06). https://publications.polymtl.ca/10067/

EPM/RT-89/6

NON-LINEAR MODELS OF A TWO DEGREES-OF-FREEDOM ONE-LINK FLEXIBLE ARM

PAR

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gratuit

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Ecole Polytechnique de Montréal
Avril 1989

Dépôt légal, 2e trimestre 1989
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Résumé

Ce rapport présente le modèle non-linéaire, sous formes d'équations aux dérivées partielles, pour un bras flexible encastré dans une base à deux degrés de liberté. Le model tient compte des flexibilités dans les plans horizontal et vertical ansi que selon l'axe longitudinal. Les équations, fortement non-linéaires, incluent les forces de Coriolis et centrifuges ainsi que les couplages entre les déformations verticale et horizontale. Deux types d'amortissement sont aussi inclus: un amortissement externe sur la base et un amortissement interne fractionnaire dans la membrure. L'obtention des équations est faite de façon formelle pour montrer la signification des différents termes et justifier la simplification des équations dynamiques.

Résumé

Ce raport présente un modèle non-linéaire sous forme d'équations aux dériveés partielles pour une membrure flexible encastrée dans une base à deux degrés de liberté. Les non-linéarités sont dues aux forces centrifuges et de Coriolis ansi qu'aux couplages entre les déformations horizontale et verticale. L'amortissement, qui est un élément important, est aussi considéré, externe sur la base et interne dans le bras. Pour analyser le système, une nouvelle méthode analytique pour obtenir un modèle d'ordre réduit est développée. Cette méthode permet de déduire directement le modèle d'ordre réduit des équations aux dérivées partielles. Finalement, la simulation indique que les non-linéarités jouent un rôle important et ne peuvent être négligées.

A Non-Linear Model of a Two Degrees-of-Freedom One-Link Flexible Arm

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February 1989‡

Abstract

The partial differential equations for a one-link flexible arm embedded on a base with two degrees-of-freedom are obtained. The model take into account one flexibility in the horizontal plane, one in the vertical plane and one along the longitudinal axis. These highly non-linear equations include Coriolis and the centrifugal forces as well as the coupling beetween the horizontal and the vertical deflections. Two types of damping are also included: an external viscous on the base and an internal fractional derivative in the link. The formal derivation of the equations provides an insight in the signification of the different terms and in the simplification of the dynamical equations.

1 Introduction

The modeling of flexible manipulators became a widespread subject recently. This is due to the space robots which are ligthweight robots. In order to limit complexity, the majority of models are designed for a one-link flexible arm rotating in an horizontal plane. In general, the internal damping is not considered.

One of the main reason to consider flexibility is the increase in the displacement speed of an arm with low rigidity. Paradoxically, a linear version of the model is obtained by imposing a limit on this speed. This limit prohibits the study of the rigidification effect of a rotating flexible beam due to the centrifugal force. Therefore, to have high speed and low flexibility, a non-linear model has to be considered.

A non-linear model is also useful to determine the coupling effect when flexibility is considered in horizontal and vertical planes. Experimental datas show this effect for the first mode of a 1,2 meters long plexiglass beam (fig. 1).

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‡First version in July 1988

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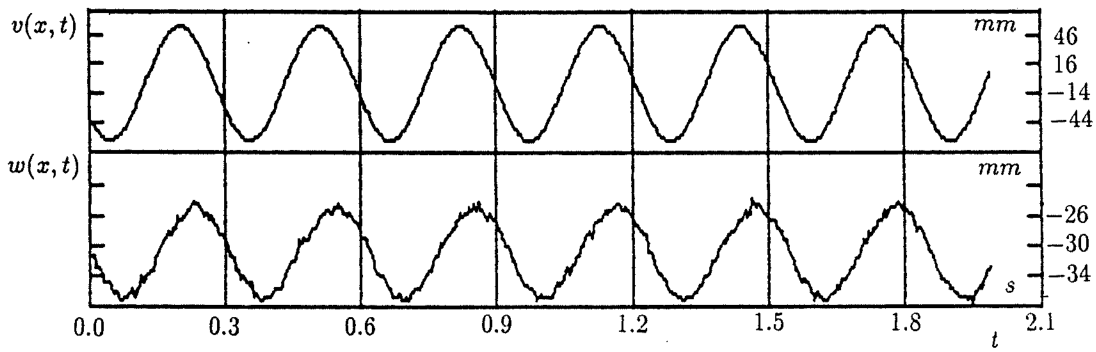

Figure 1: Experimental test: coupling between horizontal(v) and vertical(w) deflections

To study internal damping is interesting because it permits the uses of viscoelastic materials. Those materials have high internal damping. The higher modes being greatly damped it reduces significantly the effect of spillover [Balas 82,Hughes 82].

All the models can be divided in only a few categories. This classification depends on the type of model used to represent the flexibility, for example: the consideration of non-linear effects, and the inclusion of gravity or damping.

The first category, which includes almost all the models, is based on the Euler-Bernoulli beam theory. In this category, the most popular model is the linear one for an horizontal rotating arm with an end-point load and with modal damping [Cannon 84]. With this model, the effect of external and internal damping can be added [Chassiakos 86]. A non-linear model is obtained by removing the limit on the rotation speed of the hub [Biswas 88].

Two other categories exist. In the first, the models are based on the Timoshenko beam theory [Naganathan 87]. This results in a more complex model because the shear stress effect and the inertia of rotation are included. In the second category, the flexibility is replaced by a spring and a damper [Nelson 86,Piedbœuf 87]. The resulting model is very simple but it represents the first mode fairly well.

A glance at the references outside the robotic field is also interesting. For example, in the satellite area, non-linear models of a flexible beam are developed: for a cantilever beam with one degree of flexibility [Levinson 81], and for a free-free beam with two degrees of flexibility [Laskin 83]. In the helicopter domain, the blades are modeled as non-linear flexible beams, and two transverse flexibilities and one torsion are considered [Hodges 74].

The purpose of this paper is to develop a non-linear model for a one-link flexible arm with a two degrees-of-freedom (d.o.f.) base and with a payload. The modeling begins with a description of the system and a discussion of the assumptions. The partial differential equation (PDE) form of the model follows.

The idea of this paper is to make a rather formal development of the relations.

The first reason is to justify each step to understand the origin of the equations and to be aware of the simplifications in these equations. The second reason is to facilitate the development of a more complete model by, for example, including the torsion.

2 System and Assumptions

The robot under study, as represented in figure 2, is a two d.o.f.s pick and place robot with one flexible link. The base rotates about $\boldsymbol{r}_3$ axis and translates along $\boldsymbol{r}_3$ axis. The flexibility is modeled by two transverse deflections: v in the horizontal plane ($\boldsymbol{r}_1$-$\boldsymbol{r}_2$), and w in the vertical plane ($\boldsymbol{r}_1$-$\boldsymbol{r}_3$). A deformation u is also considered along the $\boldsymbol{r}_1$ axis. This last deformation is negligeable but it is included to obtain a consistent model [Hodges 74]. The gravity is considered only in conjunction with the vertical deflection w.

Table 1 summarizes all the necessary assumptions to model the robot. The use of the Euler-Bernoulli model implies that the inertia of rotation and the effects of the shear stress are neglected. This assumption imposes a limit on the frequency range of validity of the resulting equations [Meirovitch 67, sect.8.7].

Deformations		
2 transverse: v ($\boldsymbol{r}_2$), w ($\boldsymbol{r}_3$)	$< 0,1x$	Euler-Bernoulli's model
1 longitudinal: u ($\boldsymbol{r}_1$)		

	Reference Frames	States	Superposed with
$\boldsymbol{R}$:	inertial	fix	$\boldsymbol{r}$ for θ and $d = 0$
$\boldsymbol{r}$:	fixed on the base	moving	$\jmath$ for $x = 0$
$\jmath$:	fixed on the centroid of the section A of the beam	moving	

Base:	hub with negligeable radius
Beam:	properties are uniform along the $\jmath_1$ axis $\jmath_1$, $\jmath_2$ and $\jmath_3$ are the principal axis of inertia
Payload:	with mass and inertia mass center at $x = L$ principal axis of inertia $\equiv$ axis $\jmath$ at $x = L$

Table 1: Summary of the assumptions

To justify further simplifications, the order of magnitude of the different variables are defined in table 2. The variables v and w are the basis of comparison, and the term $\mathcal{O}(\epsilon^1)$ which is defined in table 2 means that ϵ is of order of 0,1. A negative exponent in the order implies a number greater than one. For example, the Young's

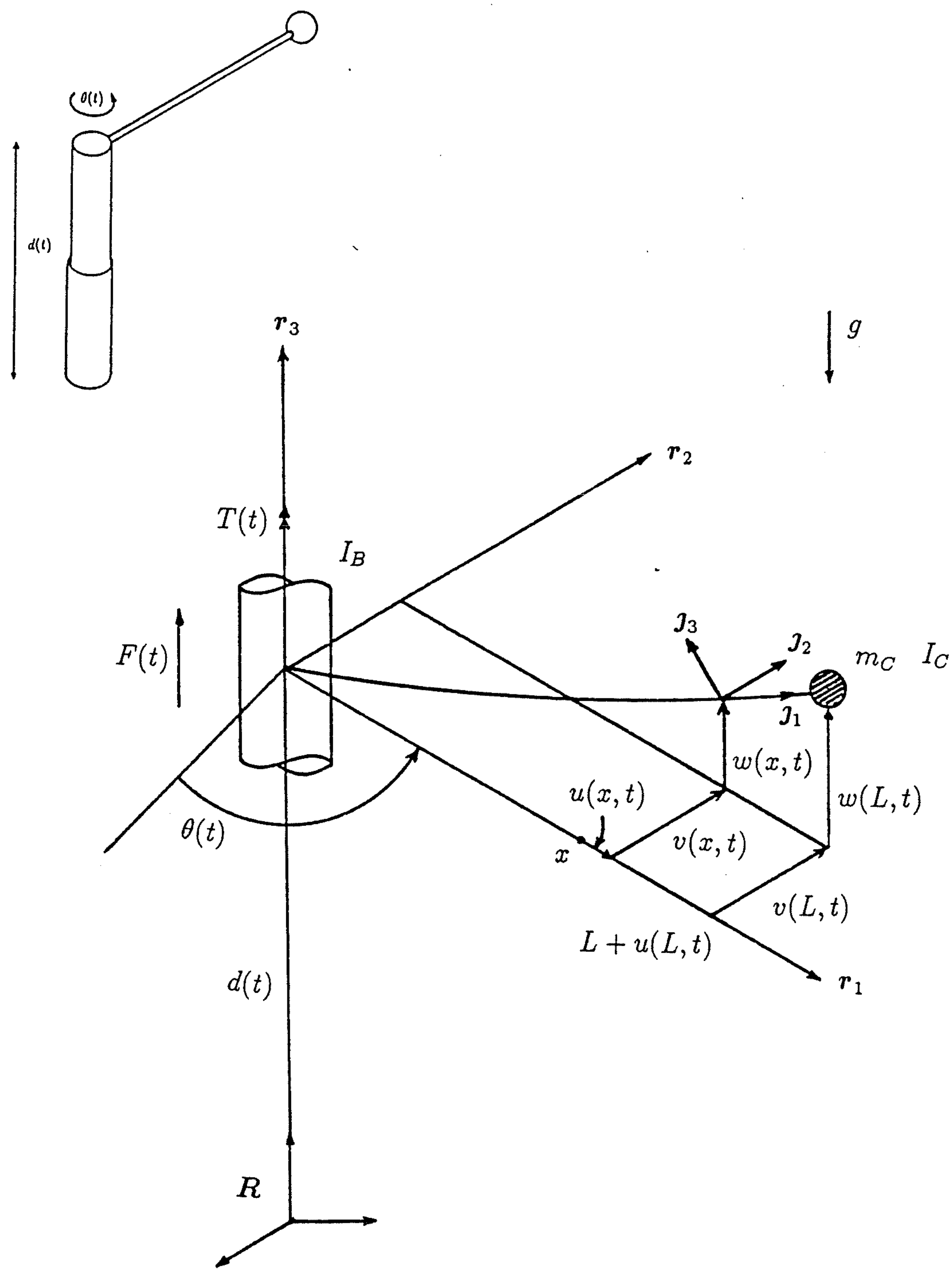

Figure 2: The system

modulus of aluminum is equal to $7,1 \times 10^{10} Pa$ which is of order of magnitude of $\mathcal{O}(\epsilon^{-10})$.

Variables	Signification	Order
v/x	deflection along $\boldsymbol{r}_2$	$\mathcal{O}(\epsilon^1)$
w/x	deflection along $\boldsymbol{r}_3$	$\mathcal{O}(\epsilon^1)$
u/x	deformation along $\boldsymbol{r}_1$	$\mathcal{O}(\epsilon^4)$
x/L	position of the section A along $\jmath_1$	$\mathcal{O}(\epsilon^0)$
y/L,z/L	position of an element dm along $\jmath_2$ and $\jmath_3$	$\mathcal{O}(\epsilon^2)$
$x\dot{\theta}/L$, $\dot{\theta}$	rotation speed of the base	$\mathcal{O}(\epsilon^{-1})$
d/L	linear displacement of the base	$\mathcal{O}(\epsilon^0)$
Constants		
E	Young's modulus	$\mathcal{O}(\epsilon^{-10})$
ρ	linear density	$\mathcal{O}(\epsilon^1)$
m_B,m_C	mass of the base and of the charge	$\mathcal{O}(\epsilon^0)$
A	area of the section	$\mathcal{O}(\epsilon^4)$
I_y,I_z	second moment of area	$\mathcal{O}(\epsilon^6)$
Relations		
a/x	any variable	$\mathcal{O}(\epsilon^n)$
$\frac{\partial^p a}{\partial t^p} x^{-1}$	derivative versus time	$\mathcal{O}(\epsilon^{n-p})$
$\frac{\partial^p a}{\partial x^p} x^{p-1}$	derivative versus space	$\mathcal{O}(\epsilon^{n+p})$

Table 2: Order of magnitude of the variables

3 Modeling: Partial Differential Equations

A flexible system is characterized by a model of infinite order or by a reduced-order model. The last one is a set of ordinary differential equations (ODE). The model of infinite order is represented by a set of partial differential equations (PDE) with boundary conditions (BC), and of ODEs. This model is more interesting than the reduced-order model because of the insight it provides in the behavior of the system (e.g. Coriolis and the centrifugal forces are identifiable).

Kane's dynamical equations [Kane 85] are used to model the flexible arm. These equations are a version of d'Alembert's principle with the peculiarity of using partial velocities. Opposed to the Hamilton's principle, Kane's dynamical equations give only the dynamic BCs therefore the geometric BCs must be found by deduction. But Kane's method is more versatile and permits to obtain easily the reduced-order model when the PDEs are known.

The generalized coordinates and velocities chosen for the system are shown in table 3. The last column represents the type of equation associated with each coordinate. As mentioned previously, only the five dynamical BCs are given in the table. The five others are the geometric BCs, and are deduced from the embedding at the base. The table suggests a division of the system in three constituent parts: the base, an element dm at the position x on the beam, and the payload (at $x = L$), accordingly to the class of equation.

	coordinates	velocities	types of equation
BASE			
rotation of the base	$\theta(t)$	$\dot{\theta}(t)$	ODE
linear displacement of the base	$d(t)$	$\dot{d}(t)$	ODE
BEAM			
deflection along $\mathbf{r_2}$	$v(x,t)$	$\dot{v}(x,t)$	PDE 4^{th} order
rotation around $\mathbf{r_3}$	$v'(x,t)$	$\dot{v}'(x,t)$	none
deflection along $\mathbf{r_3}$	$w(x,t)$	$\dot{w}(x,t)$	PDE 4^{th} order
rotation around $\mathbf{r_2}$	$w'(x,t)$	$\dot{w}'(x,t)$	none
deformation along $\mathbf{r_1}$	$u(x,t)$	$\dot{u}(x,t)$	PDE 2^{th} order
PAYLOAD			
displacement along $\mathbf{r_2}$	$v(L,t)$	$\dot{v}(L,t)$	BC for v
rotation around $\mathbf{r_3}$	$v'(L,t)$	$\dot{v}'(L,t)$	BC for v
displacement along $\mathbf{r_3}$	$w(L,t)$	$\dot{w}(L,t)$	BC for w
rotation around $\mathbf{r_2}$	$w'(L,t)$	$\dot{w}'(L,t)$	BC for w
displacement along $\mathbf{r_1}$	$u(L,t)$	$\dot{u}(L,t)$	BC for u

Table 3: Definition of the generalized coordinates and velocities for the PDE model

The modeling is separated in three phases: kinematics, dynamics, and equations of the model. Because of the structure of the system, the same pattern is always followed in the development of the equations of the model, that is, find the relations for the base, the beam, and then the payload.

3.1 Kinematics

The first objective of the kinematics is to find the angular velocities and the accelerations needed to calculate the inertia forces. The second purpose is to obtain the partial velocities which play an important role in the determination of the generalized forces. All the relations are found with respect to (w.r.t.) the reference frame $\boldsymbol{R}$ but are written in the frame $\boldsymbol{r}$.

3.1.1 Position

The base can be seen as a rigid robot with two d.o.f.s. The angular rotation θ around $\boldsymbol{r}_3$ and the linear displacement d, also along $\boldsymbol{r}_3$ describe completely its position.

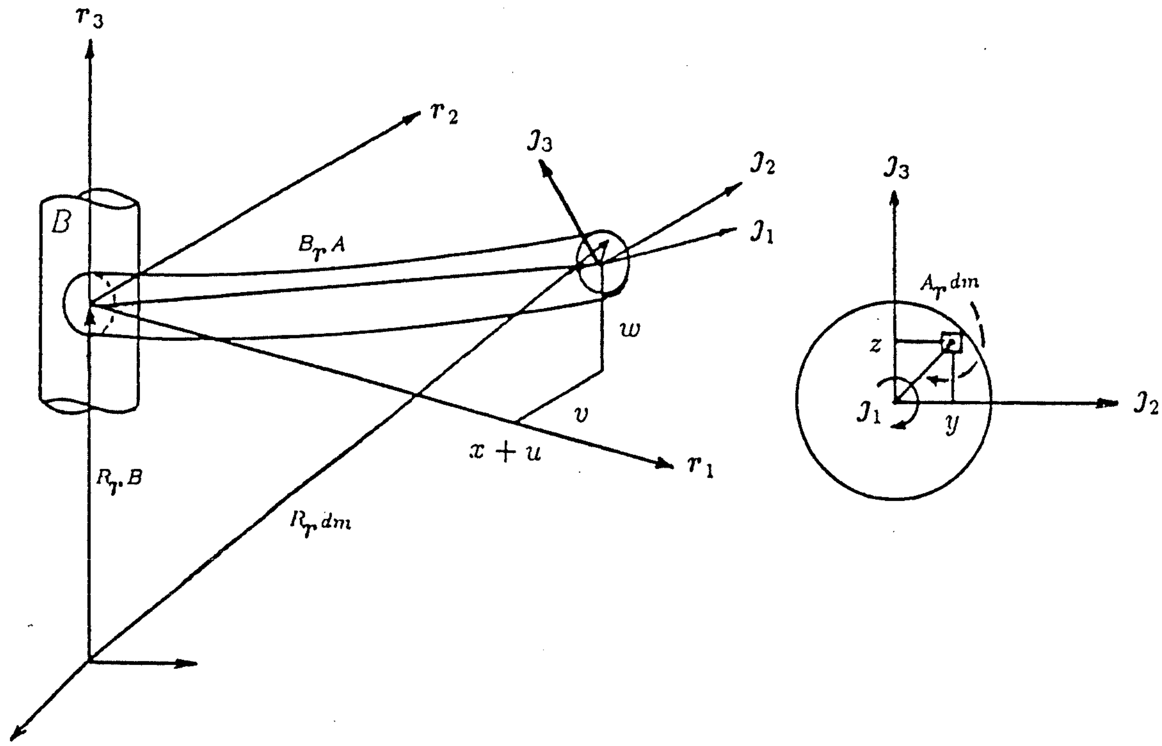

Figure 3: Position of an element dm on the beam w.r.t. the reference frame **R**

The position of an element dm in the beam is divided in three vectors as shown in figure 3: ${}^{R}\boldsymbol{r}^{B}$, the position of the base B w.r.t. $\boldsymbol{R}$; ${}^{B}\boldsymbol{r}^{A}$, the position of the centroid of the section A w.r.t. B; and ${}^{A}\boldsymbol{r}^{dm}$, the position of the element dm w.r.t. A. The first two vectors are easily deduced in frame $\boldsymbol{r}$ from figure 3. The last one, illustrated in frame $\boldsymbol{\jmath}$, is transformed in the frame $\boldsymbol{r}$ with a rotation matrix [Meirovitch 70, sect. 3.2] [Hodges 74]:

$$R_{\jmath}^{r} = \begin{bmatrix} 1 & -v' & -w' \\ v' & 1 & 0 \\ w' & 0 & 1 \end{bmatrix} \tag{1}$$

By adding the three vectors in frame $\boldsymbol{r}$, the position of the element dm w.r.t. $\boldsymbol{R}$ is found.

$${}^{R}\boldsymbol{r}^{dm} = x_1\boldsymbol{r}_1 + (v+y)\boldsymbol{r}_2 + (w+z+d)\boldsymbol{r}_3 \tag{2}$$

with $x_1 = x + u - yv' - zw'$.

The position of the payload is derived directly from the position of the element dm by assuming $x = L$, $y = 0$, and $z = 0$ in equation 2. This is due to the assumption on the position of the mass center of the payload (table 1).

3.1.2 Velocity

The velocities of the base are, as for a rigid robot, simply the derivative of the positions.

$$ {}^{R}\boldsymbol{\omega}^{B} = \dot{\theta}\boldsymbol{r}_3 \text{ and } {}^{R}\boldsymbol{v}^{B} = \dot{d}\boldsymbol{r}_3 \tag{3}$$

The angular speed of an element dm is the same as the angular speed of the section A. This result from the argument of plane section of the Euler-Bernoulli beam theory. Therefore the angular speed is the sum of two vectors: ${}^{R}\boldsymbol{\omega}^{B}$, the angular speed of the base w.r.t. $\boldsymbol{R}$, and ${}^{B}\boldsymbol{\omega}^{A}$, the angular speed of the section A w.r.t. B. The vector ${}^{B}\boldsymbol{\omega}^{A}$ is obtained from the rotation matrix (eq. 1) with the following equation [Kane 83, sect.1.10]:

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = [R_j^r]^T \dot{R}_j^r \tag{4}$$

The elements ω_1, ω_2 and ω_3 of the matrix are the angular speed of ${}^{B}\boldsymbol{\omega}^{A}$ around $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$ respectively. On the right side of equation 4, the terms of order of magnitude ϵ^3 are neglected. By doing these operations, the angular speed is found:

$$ {}^{R}\boldsymbol{\omega}^{A} = -\dot{w}'\boldsymbol{r}_2 + (\dot{v}' + \dot{\theta})\boldsymbol{r}_3 \tag{5}$$

The linear velocity of the element dm is the derivative of the position which takes into account that the $\boldsymbol{r}$ frame is moving. Defining d^*/dt, the derivative in the $\boldsymbol{r}$ frame, the velocity of the element dm is:

$$\begin{aligned} {}^{R}\boldsymbol{v}^{dm} &= \frac{d^* \, {}^{R}\boldsymbol{r}^{dm}}{dt} + {}^{R}\boldsymbol{\omega}^{B} \times {}^{R}\boldsymbol{r}^{dm} && (6) \\ &= [\dot{x}_1 - (v+y)\dot{\theta}]\boldsymbol{r}_1 + [\dot{v} + x_1\dot{\theta}]\boldsymbol{r}_2 + [\dot{d} + \dot{w}]\boldsymbol{r}_3 && (7) \end{aligned}$$

with $\dot{x}_1 = \dot{u} - y\dot{v}' - z\dot{w}'$.

The velocities of the payload are the velocities of the element dm at $x = L$, $y = 0$, and $z = 0$.

$$ {}^{R}\boldsymbol{\omega}^{C} = -\dot{w}_L'\boldsymbol{r}_2 + (\dot{\theta} + \dot{v}_L')\boldsymbol{r}_3 \tag{8}$$

$$ {}^{R}\boldsymbol{v}^{C} = [\dot{u}_L - v_L\dot{\theta}]\boldsymbol{r}_1 + [\dot{v}_L + (x+u_L)\dot{\theta}]\boldsymbol{r}_2 + [\dot{d} + \dot{w}_L]\boldsymbol{r}_3 \tag{9}$$

3.1.3 Acceleration

The accelerations are the derivative of the velocities with the same particularity as for the derivative of the unitary vectors (eq. 6). For the base:

$$ {}^{R}\boldsymbol{\alpha}^{B} = \ddot{\theta}\boldsymbol{r}_3 \tag{10}$$

$$ {}^{R}\boldsymbol{a}^{B} = \ddot{d}\boldsymbol{r}_3 \tag{11}$$

For the beam, only the linear acceleration is needed:

$$\begin{aligned} {}^R\boldsymbol{a}^{dm} &= [\ddot{x}_1 - (v+y)\ddot{\theta} - 2\dot{v}\dot{\theta} - x_1\dot{\theta}^2]\boldsymbol{r}_1 + [\ddot{v} + x_1\ddot{\theta} + 2\dot{x}_1\dot{\theta} - (v+y)\dot{\theta}^2]\boldsymbol{r}_2 \\ &\quad + [\ddot{d} + \ddot{w}]\boldsymbol{r}_3 \end{aligned} \tag{12}$$

with $\ddot{x}_1 = \ddot{u} - y\ddot{v}' - z\ddot{w}'$. And for the payload:

$$ {}^R\boldsymbol{\alpha}^C = \dot{w}'_L\dot{\theta}\boldsymbol{r}_1 - \ddot{w}'_L\boldsymbol{r}_2 + (\ddot{\theta} + \ddot{v}'_L)\boldsymbol{r}_3 \tag{13}$$

$$\begin{aligned} {}^R\boldsymbol{a}^C &= [\ddot{u}_L - v_L\ddot{\theta} - 2\dot{v}_L\dot{\theta} - (L+u_L)\dot{\theta}^2]\boldsymbol{r}_1 + [\ddot{v}_L + (L+u_L)\ddot{\theta} + 2\dot{u}_L\dot{\theta} - v_L\dot{\theta}^2]\boldsymbol{r}_2 \\ &\quad + [\ddot{d} + \ddot{w}_L]\boldsymbol{r}_3 \end{aligned} \tag{14}$$

3.1.4 Partial velocity

The partial velocities are found using the angular and linear speeds. In what follows, the generalized forces will be calculated for the section A of the beam. Therefore the partial velocities shown in table 4 are obtained for a section A rather than for an element dm. In this table, u is neglected w.r.t. x (i.e. $x + u \approx x$).

coordinates r	base $\boldsymbol{\omega}_r^B$	section A $\boldsymbol{\omega}_r^A$	payload $\boldsymbol{\omega}_r^C$	base $\boldsymbol{v}_r^B$	section A $\boldsymbol{v}_r^A$	payload $\boldsymbol{v}_r^C$
$\theta(t)$	$\boldsymbol{r}_3$	$\boldsymbol{r}_3$	$\boldsymbol{r}_3$	—	$-v\boldsymbol{r}_1 + x\boldsymbol{r}_2$	$-v_L\boldsymbol{r}_1 + L\boldsymbol{r}_2$
$d(t)$	—	—	—	$\boldsymbol{r}_3$	$\boldsymbol{r}_3$	$\boldsymbol{r}_3$
$v(x,t)$	—	—	—	—	$\boldsymbol{r}_2$	—
$v'(x,t)$	—	$\boldsymbol{r}_3$	—	—	—	—
$w(x,t)$	—	—	—	—	$\boldsymbol{r}_3$	—
$w'(x,t)$	—	$-\boldsymbol{r}_2$	—	—	—	—
$u(x,t)$	—	—	—	—	$\boldsymbol{r}_1$	—
$v(L,t)$	—	—	—	—	—	$\boldsymbol{r}_2$
$v'(L,t)$	—	—	$\boldsymbol{r}_3$	—	—	—
$w(L,t)$	—	—	—	—	—	$\boldsymbol{r}_3$
$w'(L,t)$	—	—	$-\boldsymbol{r}_2$	—	—	—
$u(L,t)$	—	—	—	—	—	$\boldsymbol{r}_1$

Table 4: Partial velocities for the PDE model

The partial velocities of the base can also be written in function of the partial velocities of the section A and the payload C. For example:

$$\boldsymbol{v}_\theta^A = -v\boldsymbol{v}_u^A + x\boldsymbol{v}_v^A$$

These relations are useful to write the generalized forces.

3.2 Dynamics

In this section, the generalized forces are computed from the contributing forces and torques. The section is ended by a discussion on damping.

The generalized force associated with the coordinate r is the sum of the contribution of each constituent part of the system.

$$F_r^* = (F_r^*)_B + \int_L (dF_r^*)_A dL + (F_r^*)_C \tag{15}$$

The F_r^* represents a generalized inertia force, and F_r without an asterisk is a generalized active force.

3.2.1 Generalized Inertial Force (g.i.f.)

The inertia force $\boldsymbol{R}_X^*$ and torque $\boldsymbol{T}_X^*$ acting on a constituent part are required to calculate the g.i.f.s $(F_r^*)_X$ of this part.

$$(F_r^*)_X = \boldsymbol{v}_r^X \cdot \boldsymbol{R}_X^* + \boldsymbol{\omega}_r^X \cdot \boldsymbol{T}_X^* \tag{16}$$

In this equation, X may represents successively the base B, the section A, or the payload C, and $\boldsymbol{v}_r^X$ and $\boldsymbol{\omega}_r^X$ are the partial velocities. In the case of the section A, F_r^*, $\boldsymbol{R}_r^*$, and $\boldsymbol{T}_r^*$ are replaced by the differential of these quantities dF_r^*, $d\boldsymbol{R}_r^*$, and $d\boldsymbol{T}_r^*$. As in d'Alembert's principle, a minus sign is placed before the equations of inertia forces and torques.

The inertia force of the base is simply its mass m_B multiplied by the acceleration ${}^R\boldsymbol{a}^B$ (eq. 11). The inertia torque is the inertia I_B of the base around the $\boldsymbol{r}_3$ axis, multiplied by the acceleration ${}^R\boldsymbol{\alpha}^B$ (eq. 10).

The treatment of the payload is similar to the one of the base, but the case of the inertia torque is more complex. The inertia force is the mass m_C of the payload, multiplied by the acceleration ${}^R\boldsymbol{a}^C$ (eq. 14). The inertia torque has two terms: $-{}^R\boldsymbol{\alpha}^C \cdot \underline{\boldsymbol{I}_C} - {}^R\boldsymbol{\omega}^C \times \underline{\boldsymbol{I}_C} \cdot {}^R\boldsymbol{\omega}^C$. The inertia tensor $(\underline{\boldsymbol{I}_C})$ is, by assumption, diagonal in the $\boldsymbol{\jmath}$ frame, with I_{1C}, I_{2C}, and I_{3C} the terms on the diagonal. This tensor is transformed in the $\boldsymbol{r}$ frame by using the rotation matrix (eq. 1) $(\underline{\boldsymbol{I}_C} = R_j^r (\underline{\boldsymbol{I}_c})_j [R_j^r]^T)$. By opposition to the case of the base, the second term is not zero because ${}^R\boldsymbol{\omega}^C$ has components around more than one axis.

To obtain the inertia force and torque acting on the section A, an integration must be performed. The inertia force $d\boldsymbol{R}_A^*$ is the integral on the section A of the inertia force applied by the element dm ($-\int_A \sigma {}^R\boldsymbol{a}^{dm} dA$ with σ the density). The inertia torque $d\boldsymbol{T}_A^*$ has a similar expression except that the lever $\boldsymbol{r}$ of the force is taken into account ($-\int_A \sigma \boldsymbol{r} \times {}^R\boldsymbol{a}^{dm} dA$ with $\boldsymbol{r} = y\boldsymbol{r}_2 + z\boldsymbol{r}_3$). In these calculations, the assumptions on the principal axis of inertia and on the position of the centroid are used. In the resulting equation of the inertia torque, there are no terms in $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$ because the inertia of rotation is neglected w.r.t. the inertia of translation (from the Euler-Bernoulli beam theory).

By calculating the inertia torques and forces of each constituent part and replacing them in the g.i.f. equations (eq. 16) and by substituting these equations in the equation 15, the g.i.f.s are written for each coordinate.

$$F_\theta^* = -I_B\ddot{\theta} + \int_0^L (-v dF_u^* + x dF_v^*)dx - v_L F_{u_L}^* + L F_{v_L}^* + F_{v'_L}^* \tag{17}$$

$$F_d^* = -m_B\ddot{d} + \int_0^L dF_w^* dx + F_{w_L}^* \tag{18}$$

$$dF_v^* = -\rho(\ddot{v} + x\ddot{\theta} + 2\dot{u}\dot{\theta} - v\dot{\theta}^2) \tag{19}$$

$$dF_{v'}^* = 0 \tag{20}$$

$$dF_w^* = -\rho(\ddot{d} + \ddot{w}) \tag{21}$$

$$dF_{w'}^* = 0 \tag{22}$$

$$dF_u^* = -\rho(\ddot{u} - v\ddot{\theta} - 2\dot{v}\dot{\theta} - x\dot{\theta}^2) \tag{23}$$

$$F_{v_L}^* = -m_C(\ddot{v}_L + L\ddot{\theta} + 2\dot{u}_L\dot{\theta} - v_L\dot{\theta}^2) \tag{24}$$

$$F_{v'_L}^* = -(\ddot{v}'_L + \ddot{\theta})I_{3C} - 2w'_L\dot{w}'_L\dot{\theta}(I_{1C} - I_{3C}) \tag{25}$$

$$F_{w_L}^* = -m_C(\ddot{d} + \ddot{w}_L) \tag{26}$$

$$F_{w'_L}^* = -\ddot{w}'_L I_{2C} - w'_L(2\dot{v}'_L\dot{\theta} + \dot{\theta}^2)(I_{1C} - I_{3C}) \tag{27}$$

$$F_{u_L}^* = -m_C(\ddot{u}_L - v_L\ddot{\theta} - 2\dot{v}_L\dot{\theta} - L\dot{\theta}^2) \tag{28}$$

As previously mentioned, the g.i.f.s of the coordinates of the base are functions of the g.i.f.s of the beam and of the payload.

3.2.2 Generalized Active Force (g.a.f.)

The g.a.f.s are similar to the g.i.f.s (eq. 16) except that the inertial forces $\boldsymbol{R}_X^*$ and torques $\boldsymbol{T}_X^*$ are replaced by the contributing external and elastic forces $\boldsymbol{R}_X$ and torques $\boldsymbol{T}_X$.

To determine the elastic forces working in the beam, the forces and torques applied on each constituent part must be known. The forces acting on a small element dm are represented in figure 4. The sum of these forces in the $\boldsymbol{\jmath}$ frame is:

$$d\boldsymbol{R}_{dm} = \left\{ \frac{\partial(\sigma_{xx}\boldsymbol{\jmath}_1)}{\partial x} + \frac{\partial(\sigma_{xy}\boldsymbol{\jmath}_2)}{\partial x} + \frac{\partial(\sigma_{xz}\boldsymbol{\jmath}_3)}{\partial x} + \frac{\partial(\sigma_{xy}\boldsymbol{\jmath}_1)}{\partial y} + \frac{\partial(\sigma_{xz}\boldsymbol{\jmath}_1)}{\partial z} - \sigma g \boldsymbol{r}_3 \right\} dx dy dz \tag{29}$$

This equation is transformed to the $\boldsymbol{r}$ frame by using the rotation matrix (eq. 1) to write the unitary vectors of $\boldsymbol{\jmath}$ in function of the unitary vector of $\boldsymbol{r}$. The stresses are also replaced by their expressions in function of the deformations [Laskin 83] [Shames 85, chap. 4].

$$\sigma_{xx} = Eu' - Eyv'' - Ezw'' \ , \ \sigma_{xy} = \frac{f(y)V_y}{I_z} \text{ and } \sigma_{xz} = \frac{g(z)V_z}{I_y} \tag{30}$$

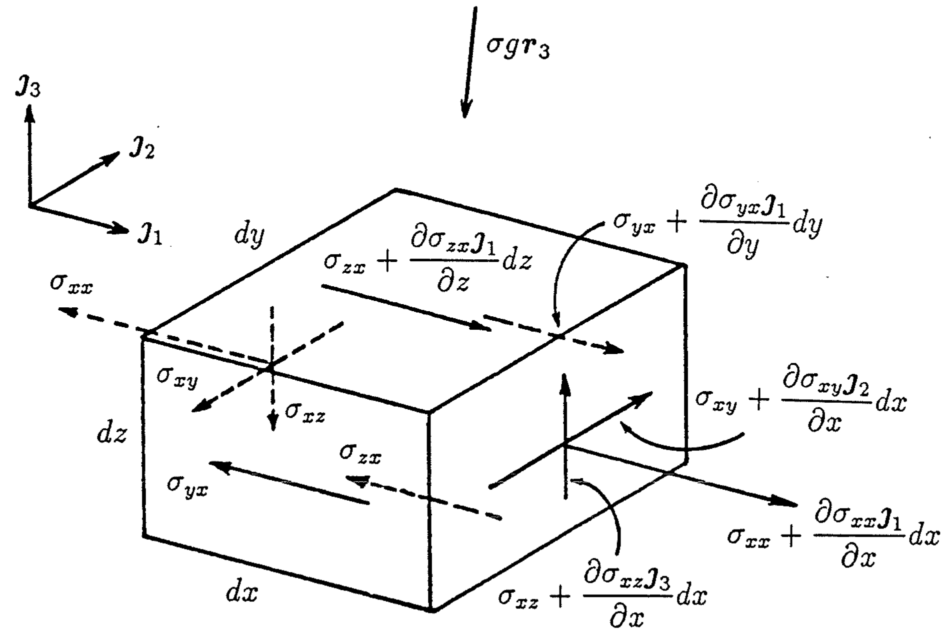

Figure 4: Forces acting on an element dm of the beam

The terms V_y and V_z are the shear forces, and I_y and I_z are the second moments of area. In the Euler-Bernoulli beam theory, no shear stresses exist, therefore σ_{xy} and σ_{xz} must be zero. The two last equations of 30 are written to represent the shear stresses because these stresses are needed to have a deflection [Laskin 83]. In these equations, the terms $f(y)$ and $g(z)$ are symetric in y and z respectively and $f(0) = g(0) = 0$.

To determine the values of V_y and V_z, the g.a.f.s are developed for the section A of the beam. To obtain these g.a.f.s, the stresses (eq. 30) are substituted in the equation of $d\boldsymbol{R}_{dm}$ (eq. 29) which has been transformed for the $\boldsymbol{r}$ frame.

The next step is to calculate the force and the torque acting on a section A. The force $d\boldsymbol{R}_A$ is the integration of $d\boldsymbol{R}_{dm}$ w.r.t. the section A ($\int_A d\boldsymbol{R}_{dm}$). Similarly, the torque $d\boldsymbol{T}_A$ is the result of the integration on the section A of the force $d\boldsymbol{R}_{dm}$ multiplied by the same lever $\boldsymbol{r}$ as in the case of the g.i.f.s ($\int_A \boldsymbol{r} \times d\boldsymbol{R}_{dm}$). In these calculations, the assumptions on the principal axis of inertia and on the centroid are used. There are only four terms in $f(y)$ and $g(z)$ different of zero ($\int_A f(y)dA = I_z$, $\int_A g(z)dA = I_y$, $\int_A y\partial f(y)/\partial y dA = -I_z$ and $\int_A z\partial g(z)/\partial z dA = -I_y$)

The expression for the shear forces V_y and V_z, are found by developing the dynamical equations for the coordinates v' and w' ($F_r^* + F_r = 0$). The g.a.f.s for v' and w' are obtained by using equation 16 with $d\boldsymbol{R}_A$ and $d\boldsymbol{T}_A$ instead of $\boldsymbol{R}_X^*$ and $\boldsymbol{T}_X^*$. As shown in equations 20 and 22, the g.i.f.s for v' and for w' are zero due to the

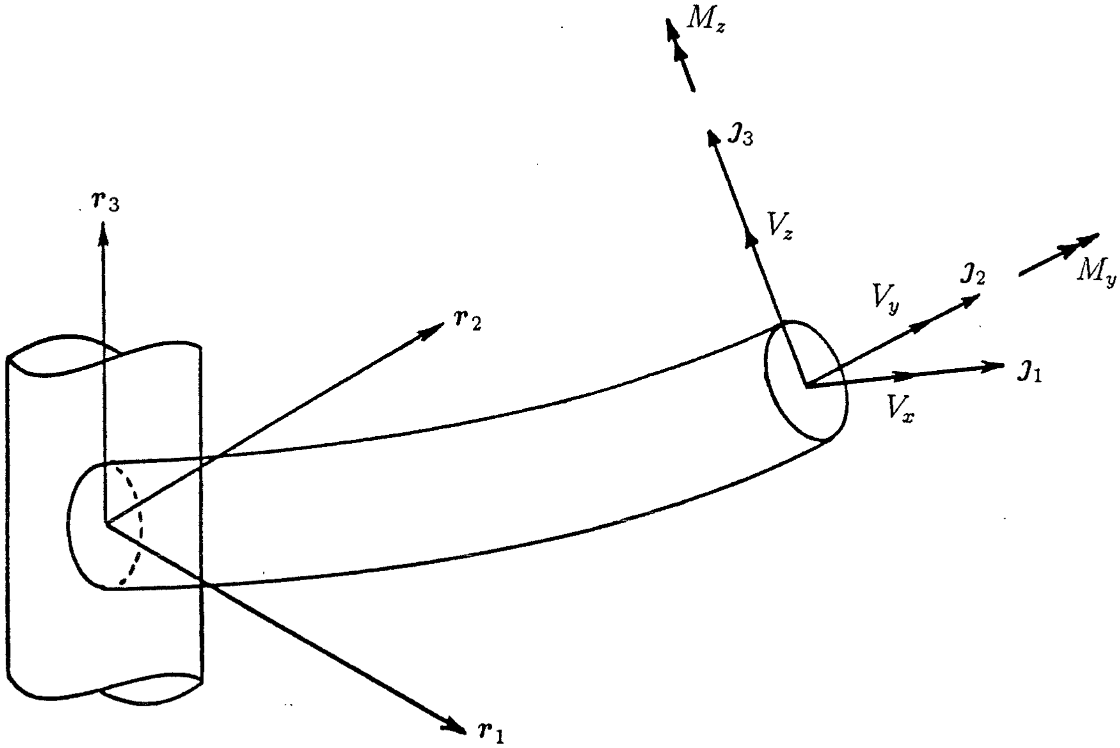

Figure 5: Forces and torques at the position x

negligeable inertia of rotation. Therefore the g.a.f.s for v' and w' must be zero. This gives

$$V_y = -EI_z v''' \text{ and } V_z = -EI_y w''' \tag{31}$$

The g.a.f.s of the beam for the other coordinates are now calculated by replacing V_y and V_z in the equation of the force $d\boldsymbol{R}_A$ and the torque $d\boldsymbol{T}_A$, and by applying equation 16.

To find the g.a.f.s of the base and of the payload, the reactions at $x = 0$ and at $x = L$ are needed. These forces and torques, at the position x of the beam, are illustrated in figure 5 in the $\jmath$ frame. In this figure $V_x = EAu'$, $M_y = -EI_y w''$, $M_z = EI_z v''$ and V_y and V_z are given by equation 31. By using the rotation matrix (eq. 1), the expression for the force $\boldsymbol{R}_x$ and the torque $\boldsymbol{T}_x$ are found in the $\boldsymbol{r}$ frame. By evaluating these equations for $x = 0$ and $x = L$, the desired reactions are found.

On the base, the g.a.f.s are due to the external forces and torques plus the reaction at $x = 0$. The force $\boldsymbol{R}_B$ takes into account the external force $F(t)$ along $\boldsymbol{r}_3$ (fig. 2), the gravity force $-m_B g$ also along $\boldsymbol{r}_3$, and the reaction force $\boldsymbol{R}_x|_{x=0}$. The torque $\boldsymbol{T}_B$ includes the external torque $T(t)$ (fig. 2) around $\boldsymbol{r}_3$ and the reaction torque $\boldsymbol{T}_x|_{x=0}$. The g.a.f.s are calculated using equation 16.

The pattern is similar for the payload. The force $\boldsymbol{R}_C$ is composed of the gravity force $-m_C g$ along $\boldsymbol{r}_3$ and of the reaction force at $x = L$, $\boldsymbol{R}_x|_{x=L}$. The torque $\boldsymbol{T}_C$ is only the effect of the reaction torque $\boldsymbol{T}_x|_{x=L}$. Replacing $\boldsymbol{R}_C$ and $\boldsymbol{T}_C$ in equation 16 gives the g.a.f.s.

The g.a.f.s for the system are calculated by using equation 15 with the g.a.f.s of each part instead of the g.i.f.s.

$$F_\theta = T + EI_z v_0'' + \int_0^L (-v dF_u + x dF_v) dx - v_L F_{u_L} + L F_{v_L} + F_{v_L'} \tag{32}$$
$$F_d = F - m_B g - EI_y w_0''' + \int_0^L dF_w dx + F_{w_L} \tag{33}$$
$$dF_v = -EI_z v'''' + (v' EAu')' \tag{34}$$
$$dF_{v'} = 0 \tag{35}$$
$$dF_w = -EI_y w'''' + (w' EAu')' - \rho g \tag{36}$$
$$dF_{w'} = 0 \tag{37}$$
$$dF_u = EAu'' \tag{38}$$
$$F_{v_L} = EI_z v_L''' - v_L' EAu_L' \tag{39}$$
$$F_{v_L'} = -EI_z v_L'' \tag{40}$$
$$F_{w_L} = EI_y w_L''' - w_L' EAu_L' - m_C g \tag{41}$$
$$F_{w_L'} = -EI_y w_L'' \tag{42}$$
$$F_{u_L} = -EAu_L' \tag{43}$$

In the equation of dF_u (eq. 38), the terms $(v' EI_z v''')'$ and $(w' EI_y w''')'$ of order ϵ^3 are neglected with respect to EAu'' of order ϵ^0. Equivalent simplifications are made in equation of F_{u_L} (eq. 43).

Because the g.a.f.s represent only the contributing forces and torques, the equations for F_θ and F_d can be rewritten in a more simple form.

$$F_\theta = T \tag{44}$$
$$F_d = F - (m_B + \rho L + m_C) g \tag{45}$$

3.2.3 Damping

Two types of damping are considered: external on the base and internal in the beam.

The external damping on the base is viscous. This damping is incorporated into the equations by replacing the external force F by $F - \mu_d \dot{d}$ and the external torque T by $T - \mu_\theta \dot{\theta}$. The μ_θ and μ_d are the damping coefficients.

Two models of internal damping are considered: Voigt-Kelvin and fractional derivative [Nashif 85, sect.2.3.5] [Bagley 83]. These dampings are added to the equations by replacing the Young's modulus E by the operator $E_\kappa = E(1 + \kappa_e D^\alpha)$, where

D^α is the fractional derivative operator. When α is equal to one, the operator represents the Voigt-Kelvin's damping and when α is between zero and one it represents the fractional derivative damping.

3.3 Equations of the Model

3.3.1 Simplifications

Before writing the dynamical equations, some simplifications are made in order to eliminate the deformation u. These simplifications are based on the order of magnitude of the different terms.

The equation for u is formed with the g.i.f and the g.a.f. associated with u ($F_r^* + F_r = 0$). The result is a PDE of second order (eq. 23 + eq. 38) with two BCs: $u_0 = 0$, a geometric one, and the force equilibrium at $x = L$ (eq. 28 + eq. 43), a dynamic BC. In the resulting equations there are terms in $\rho\ddot{u}$ and $m_C\ddot{u}_L$ of order ϵ^3 which are negligeable.

The term in EAu'' that appear in the PDE of u is replaced by V_x' owing to the definition of V_x in section 3.2.2 (page 13). An equivalent substitution is made for the BC. The resulting PDE of first order in V_x is solved to obtain an expression for V_x. In the other g.a.f.s (eqs 34, 36, 39 and 41) all the terms in EAu' are replaced by their equivalent terms in V_x.

The new equation of V_x is substituted in the dynamic equations of the other coordinates. In these new equations the terms in $\ddot{\theta}$ of V_x ($\mathcal{O}(\epsilon^3)$) are negligeable by comparison with other terms and therefore are eliminated from the V_x equation. With these simplifications, the equation for V_x is:

$$V_x(x,t) = 2(-\rho\dot{v}' + \rho\dot{v}'_L + m_C\dot{v}_L)\dot{\theta} + \left[\frac{\rho L^2}{2}(1 - \frac{x^2}{L^2}) + m_C L\right]\dot{\theta}^2 \qquad (46)$$

where $v' = \int v dx$. The only remaining terms in u are now the velocities $\dot{u}$. An expression for u is found by integrating V_x/EA from 0 to x. This expression is then differentiated w.r.t. time to give an equation for $\dot{u}$.

3.3.2 Dynamical Equations

With these simplifications, the model is rewritten. For the coordinate θ of the base (eqs 17 and 32):

$$T - I_B\ddot{\theta} - \mu_\theta\dot{\theta} + EI_z v_0'' + \kappa_e EI_z D^\alpha v_0'' = 0 \qquad (47)$$

Or, if the equation 44 is used for F_θ:

$$\begin{aligned} T - I_B\ddot{\theta} - \mu_\theta\dot{\theta} - \int_0^L \rho\left[x\ddot{v} + x^2\ddot{\theta} + 2(v\dot{v} + x\dot{u})\dot{\theta}\right]dx \\ - mc\left[L\ddot{v}_L + L^2\ddot{\theta} + 2(v_L\dot{v}_L + L\dot{u}_L)\dot{\theta}\right] \\ - (\ddot{v}_L' + \ddot{\theta})I_{3C} - 2w_L'\dot{w}_L'\dot{\theta}(I_{1C} - I_{3C}) &= 0 \end{aligned} \qquad (48)$$

For the coordinate d of the base (eqs. 18 and 33):

$$F - m_B\ddot{d} - \mu_d\dot{d} - m_B g - EI_y w_0''' - \kappa_e EI_y D^\alpha w_0''' = 0 \tag{49}$$

Or, if the equation 45 is used for F_d:

$$F - (m_B + \rho L + m_C)g - m_B\ddot{d} - \mu_d\dot{d} - \int_0^L \rho(\ddot{d} + \ddot{w})dx - m_C(\ddot{d} + \ddot{w}_L) = 0 \tag{50}$$

The coordinate v is represented by a PDE (eqs 19 and 34):

$$-\rho(\ddot{v} + x\ddot{\theta} + 2\dot{u}\dot{\theta} - v\dot{\theta}^2) - EI_z v'''' - \kappa_e EI_z D^\alpha v'''' + (v'V_x)' = 0 \tag{51}$$

with the BCs (eqs 24 and 39, and eqs 25 and 40 for the last two BCs):

$$v_0 = 0 \qquad v_0' = 0$$

$$-m_C(\ddot{v}_L + L\ddot{\theta} + 2\dot{u}_L\dot{\theta} - v_L\dot{\theta}^2) + EI_z v_L''' + \kappa_e EI_z D^\alpha v_L''' - v_L'V_L = 0$$
$$-(\ddot{v}_L' + \ddot{\theta})I_{3C} - 2w_L'\dot{w}_L'\dot{\theta}(I_{1C} - I_{3C}) - EI_z v_L'' - \kappa_e EI_z D^\alpha v_L'' = 0$$

The coordinate w is also represented by a PDE (eqs 21 and 36):

$$-\rho(\ddot{d} + \ddot{w}) - EI_y w'''' - \kappa_e EI_y D^\alpha w'''' + (w'V_x)' - \rho g = 0 \tag{52}$$

with the BCs (eqs 26 and 41, and eqs 27 and 42 for the last two BCs):

$$w_0 = 0 \qquad w_0' = 0$$

$$-m_C(\ddot{d} + \ddot{w}_L) + EI_y w_L''' + \kappa_e EI_y D^\alpha w_L''' - w_L'V_L - m_C g = 0$$
$$-\ddot{w}_L' I_{2C} - w_L'(2\dot{v}_L'\dot{\theta} + \dot{\theta}^2)(I_{1C} - I_{3C}) - EI_y w_L'' - \kappa_e EI_y D^\alpha w_L'' = 0$$

In these equations, the terms in $\dot{u}$ are very small but they assure the consistency of the equations. To eliminate these terms and still be consistent, the terms in $\dot{\theta}$ in the equation of V_x (eq. 46) must be eliminated.

4 Discussion

The resulting model is general. It includes all the non-linear effects: coupling, longitudinal force, and Coriolis force. If these non-linear terms are eliminated from the model, a set of linear equations is obtained. For example, the equations for an horizontal rotating beam with a point payload as given by Cannon [Cannon 84] are a particular case of the model.

One of the principal non-linear term is the axial force V_x. The expression in $\dot{\theta}$ in the equation of V_x (eq. 46) is the Coriolis force and the one dependent on $\dot{\theta}^2$ is the centrifugal force. Often, this last term is the only one considered because it is one order greater than the other (ϵ^{-2} vs ϵ^{-1}) [Meirovitch 67, sect. 10.4], and it is independent of the deformation v. The force V_x is not function of the deformation w.

The PDE of the deformation v (eq. 51) is the force equilibrium along the $\boldsymbol{r}_2$ axis. In the PDE, there is a Coriolis force ($2\dot{u}\dot{\theta}$) and a centrifugal force ($v\dot{\theta}^2$). The term in V_x represents the component of the axial force acting in the $\boldsymbol{r}_2$ direction. The coupling with the vertical deflection w is due to the BC on the torque at $x = L$. An arm without a payload [Laskin 83,Biswas 88] or with a point payload, does not show this coupling. The utility of a robot being to perform a task, it is important to consider this effect.

In the PDE of the deformation w (eq. 52), all the non-linear terms are coupling terms. The term in V_x is the component along $\boldsymbol{r}_3$ of the axial force. There is no Coriolis or centrifugal forces. The other coupling terms are similar to the ones of v. This equation explains the vertical vibrations when the beam rotates as in figure 1. When the rotation speed increases, V_x increases and the component of this axial force pushes the beam up. Therefore, when the rotation speed decreases, the beam vibrates vertically under an initial condition.

Another aspect of the model is the general treatment of the damping. The viscous damping on the base is essential when a real system is under study. The fractional derivative damping, to model the internal damping, gives many choices. The main advantage of this internal damping is the possibility to represent viscoelastic materials. These materials are useful because the effect of spillover is greatly reduced [Oosting 88].

5 Conclusion

The intention of this paper was to develop a non-linear model of a two d.o.f.s one-link flexible arm with a payload. This non-linear model is based on the Euler-Bernoulli beam theory and obtained using Kane's dynamical equations. Two transverse and one longitudinal deflections are considered with the last one permitting to create a consistent model. The model includes viscous damping on the base and a very general internal damping in the beam. The resulting model is then simplified by eliminating the negligeable terms and by rewriting the longitudinal deformation u as a function of the axial force V_x.

The development is formal to completely understand the model and to permit its extension. For example the shear forces V_y and V_z (eq. 31), derived from the stresses, are the classical relations of the Euler-Bernoulli beam theory. But by knowing the development of these equations, it is easy to introduce the effect of the inertia of rotation.

In a following paper, a reduced-order model deduced from the PDE model will be developed. This reduced-order model, obtained by the assumed-mode method, will be valid for any admissible function. Simulations will permit to study the importance of a non-linear model to analyze coupling effect and high rotation speed.

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