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NUMERICAL SIMULATION OF AXISYMMETRIC SWIRLING FLOWS WITH APPLICATION TO TURBINE DIFFUSERS

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1. Problem Statement

The present report describes a numerical scheme and the associated computed code for the solution of the viscous flow in an axisymmetric diffuser. The intended application is for the analysis of the geometric and flow parameters of the section immediately downstream of the runner in an hydraulic turbine as illustrated in Fig. 1. Such of analysis tool will allow a designer to correctly assess the viscous losses and pressure recorery in this type of equipment. Consequently, the numerical scheme has been embedded in a computer-aided software which allows the user to obtain results in three main steps. First, a geometric modeller is used to construct the geometry and the curvilinear coordinate grid. Then, the flow field is solved numerically using a finite difference scheme of the basic equations formulated in vorticity-stream function variables. Central differences are used for viscous terms and a third order upwind scheme for the convection terms. Finally, an utility is provided for the graphical display of the results.

In the present work the laminar Navier-Stokes equations are solved for the incompressible flow with axial symmetry. This will be extended to include turbulence.

2. Basic Equations

The equations for the conservation of mass and momentum written in cylindrical coordinates for an incompressible constant property fluid in a steady state laminar flow are:

$$V_r \frac{\partial V_r}{\partial r} - \frac{V_\theta^2}{r} + V_z \frac{\partial V_r}{\partial z} = -\frac{\partial P}{\partial r} + \frac{1}{Re}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_r)\right) + \frac{\partial^2 V_r}{\partial z^2}\right] \qquad (1)$$

$$V_r \frac{\partial V_\theta}{\partial r} + \frac{V_r V_\theta}{r} + V_z \frac{\partial V_\theta}{\partial z} = \frac{1}{Re}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rV_\theta)\right) + \frac{\partial^2 V_\theta}{\partial z^2}\right] \qquad (2)$$

$$V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} = -\frac{\partial P}{\partial z} + \frac{1}{Re}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right) + \frac{\partial^2 V_z}{\partial z^2}\right] \qquad (3)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rV_r) + \frac{\partial}{\partial z}(V_z) = 0 \qquad (4)$$

where it has been assumed that for the present application $\frac{\partial}{\partial \theta} = 0$. Equation (2) is called the swirl equation and Re is the Reynolds number.

2.1 Vorticity equation

Differentiating Equation (1) with respect to z and Equation (3) with respect to r, and using the definition of vorticity

$$\vec{\omega} = \vec{\nabla} \times \vec{V} = \frac{1}{r} \begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & \vec{e}_z \\ \frac{\partial}{\partial r} & 0 & \frac{\partial}{\partial z} \\ V_r & rV_\theta & V_z \end{vmatrix}$$

$$\omega = \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} \qquad \text{(the second component of } \vec{\omega}) \tag{5}$$

and using Equation (4) to simplify, one gets the following vorticity transport equation:

$$V_r \frac{\partial \omega}{\partial r} - V_r \frac{\omega}{r} + V_z \frac{\partial \omega}{\partial z} - 2 \frac{V_\theta}{r} \frac{\partial V_\theta}{\partial z} = \frac{1}{Re}\left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r\omega)\right) + \frac{\partial^2 \omega}{\partial z^2}\right] \tag{6}$$

which, together with Equation (2), replaces Equations (1) to (3).

2.2 Stream Function equation

The continuity equation, which involves the variables V_r and V_z can be replaced by an equivalent equation involving the stream function and vorticity. This is obtained by substituting the following definitions:

$$V_z = \frac{1}{r} \frac{\partial \psi}{\partial r} \qquad V_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} \tag{7}$$

into Equation (5). This yields

$$\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} = -\omega r \tag{8}$$

which is known as the stream function equation.

2.3 Pressure equation

Differentiating Equation (1) with respect to r and Equation (3) with respect to z, adding these relations with Equation (1) divided by r and using Equation (4) to simplify, one obtains the following pressure equation:

$$\frac{\partial^2 P}{\partial Z^2} + \frac{\partial^2 P}{\partial r^2} + \frac{1}{r} \frac{\partial P}{\partial r} = 2\left(\frac{\partial V_r}{\partial r} \frac{\partial V_z}{\partial z} - \frac{\partial V_r}{\partial z} \frac{\partial V_z}{\partial r}\right) + \frac{1}{r}\left(\frac{\partial}{\partial r}(V_\theta^2) - \frac{2}{r} V_r^2\right) \tag{9}$$

2.4 Vector form equation

These are now summarized

$$\frac{\partial^2\omega}{\partial z^2} + \frac{\partial^2\omega}{\partial r^2}$$

$$-ReV_z \frac{\partial\omega}{\partial z} + (\frac{a}{r} - ReV_r)\frac{\partial\omega}{\partial r} - \frac{a}{r}(\frac{1}{r} - ReV_r)\omega = -\frac{aRe}{r}\frac{\partial}{\partial z}(V_\theta^2) \tag{10}$$

$$\frac{\partial^2 V_\theta}{\partial z^2} + \frac{\partial^2 V_\theta}{\partial r^2}$$

$$-ReV_z \frac{\partial V_\theta}{\partial z} + (\frac{a}{r} - ReV_r)\frac{\partial V_\theta}{\partial r} - \frac{a}{r}(\frac{1}{r} + ReV_r) V_\theta = 0 \tag{11}$$

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{a}{r}\frac{\partial\psi}{\partial r} = -\omega r^a \tag{12}$$

$$\frac{\partial^2 P}{\partial z^2} + \frac{\partial^2 P}{\partial r^2} + \frac{a}{r}\frac{\partial P}{\partial r} = S \tag{13}$$

where S is the right member of Equation (9).

These can be written in vector form as follows:

$$\frac{\partial^2\vec{f}}{\partial r^2} + B\frac{\partial\vec{f}}{\partial r} + C\frac{\partial\vec{f}}{\partial z} + \frac{\partial^2\vec{f}}{\partial z^2} + E\vec{f} = \vec{F} \tag{14}$$

where

$$f = \begin{vmatrix} \omega \\ V_\theta \\ \psi \\ P \end{vmatrix}$$

$$F = \begin{vmatrix} -2\frac{a}{r} Re\, V_\theta \frac{\partial V_\theta}{\partial z} \\ 2\frac{a}{r} Re\, V_\theta V_r \\ -\omega r^a \\ S \end{vmatrix}$$

$$B = (\frac{a}{r} - ReV_r, \frac{a}{r} - ReV_r, \frac{a}{r}, \frac{a}{r})$$

$$C = (-ReV_z, -ReV_z, 0, 0)$$

$$E = (-\frac{a}{r}(\frac{1}{r} - ReV_r), -\frac{a}{r}(\frac{1}{r} - ReV_r), 0, 0)$$

3. Curvilinear Coordinate Equations

3.1 Curvilinear coordinate system

A body-fitted curvilinear grid is generated using Thompson's method where the transformed coordinates (ζ, τ) are expressed so that they verify the following system of elliptic equations.

$$\zeta_{rr} + \zeta_{zz} = Q$$

$$\tau_{rr} + \tau_{zz} = R \tag{15}$$

The forcing terms Q and R are used to concentrate grid lines. To compute the physical coordinates (r, z) as function of the transformed coordinates, the system of Equation (15) is inverted.

$$\alpha \vec{r}_{\zeta\zeta} - 2\beta \vec{r}_{\zeta\tau} + \gamma \vec{r}_{\tau\tau} + J^2(Q\vec{r}_{\zeta} + R\vec{r}_{\tau}) = 0 \tag{16}$$

where

$$\vec{r} = \begin{vmatrix} r \\ z \end{vmatrix}$$

and the coefficients

$$\alpha = z_{\tau}^{2} + r_{\tau}^{2}$$

$$\beta = z_{\zeta} z_{\tau} + r_{\zeta} r_{\tau}$$

$$\gamma = z_{\zeta}^{2} + r_{\zeta}^{2}$$

$$J = z_{\zeta} r_{\tau} - z_{\tau} r_{\zeta}$$

Using the chain rule, Equation (14) is transformed into the curvilinear system (ζ, τ).

$$\begin{aligned}
&\alpha \vec{f}_{\zeta\zeta} - 2\beta \vec{f}_{\zeta\tau} + \gamma \vec{f}_{\tau\tau} \\
&+ \vec{f}_{\zeta} (QJ^2 - B\, z_{\tau}\, J + C\, r_{\tau}\, J) \\
&+ \vec{f}_{\tau} (RJ^2 + B\, z_{\zeta}\, J - C\, r_{\zeta}\, J) \\
&+ EJ^2 \vec{f} \\
&= J^2 \vec{F}
\end{aligned} \tag{17}$$

This is a system of elliptic partial differential equations, one per coordinate direction which can be solved numerically. The scheme used is that developed by Camarero[2] specifically for this type of application using a very efficient and accurate relaxation procedure. A slight modification has been used in the present study to allow grid concentration towards the solid walls where large gradients of the flow properties are expected.

In the present approach such grid concentration is achieved by means of the forcing terms Q and R. However, the choice of these functions can be quite delicate and the result cannot be assessed a priori. Bad choices can lead to distorted grids or even folded grids. It has been found[3] that an approach well suited to the field problems of interest is to establish the concentration of the nodes along the boundaries. The nodes define the shape of the domain and are given by the user. Their distribution can be an analytical expression or can be computed from the discrete values given by the user. Several relations for the concentration have been tried and the following has been retained:

$$f(t) = \frac{t}{1 + FC\,(|t| - 1)} \tag{18}$$

where FC: concentration factor

It has been the author's experience that with moderate values of the concentration factor, i.e. of the order of .2, considerable stretching is achieved. Also, it has been found that using much higher values can lead to numerical instabilities. These have been traced to the loss of diagonal dominance of the resulting matrix caused by the high values of the forcing terms.

3.2 Transformed Equations

The transformation defined by Equation (17) maps the physical domain (r, z) into the computational domain (ζ, τ) where the problem will be solved. Hence the basic equations, Equations (10) to (13) must be transformed into this domain. To facilitate this, the vector notation of Equation (14) is used, and yields:

$$\alpha \vec{f}_{\zeta\zeta} - 2\beta \vec{f}_{\zeta\tau} + \gamma \vec{f}_{\tau\tau} + \vec{f}_{\zeta} G + \vec{f}_{\tau} H + EJ^2 \vec{f} = J^2 \vec{F} \qquad (19)$$

where

$$\vec{f} = \begin{vmatrix} \omega \\ V_\theta \\ \psi \\ P \end{vmatrix}$$

$$G = (QJ^2 - B z_\tau J + C r_\tau J)$$

$$H = (RJ^2 + B z_\zeta J - C r_\zeta J)$$

where B, C, E, F are defined in Equation (14), and it is reminded that the expressions for these coefficients differ for ψ, ω, V_θ and P.

4. Upwind Differencing and artificial convection

Upwind differencing

The treatment of the transport equations requires special care in the discretization of the convection terms. An upwinding differencing scheme of variable order based on the following relation is proposed for such terms.

$$U \frac{\partial f}{\partial x} = \alpha \left.\frac{\partial f}{\partial x}\right|_{up} + (U - \alpha) \left.\frac{\partial f}{\partial x}\right|_{down} \qquad (20)$$

This is a weighted average of an upwind and a downwind difference. The factor α is chosen according to the sign of the convective coefficient U, that is

$$\begin{array}{ll} U < 0 & \alpha = 0 \\ U > 0 & \alpha = U \end{array}$$

This can then be written as

$$\alpha = \frac{U + |U|}{2}$$

and substituted into Equation (20) to give

$$U \frac{\partial f}{\partial \xi} = \frac{U + |U|}{2} \; \frac{k_0 f_{i+1} + k_1 f_i + k_2 f_{i-1} + k_3 f_{i-2}}{6}$$
$$+ \frac{U - |U|}{2} \; \frac{-k_3 f_{i+2} - k_2 f_{i+1} - k_1 f_i - k_0 f_{i-1}}{6} \qquad (21)$$

where the coefficients k_0, k_1, k_2 and k_3 take on the values according to the order of discretization.

	1st order	2nd Order	3rd Order	Centered
k_0	0	0	2	6
k_1	6	3	3	0
k_2	-6	-4	-6	-6
k_3	0	1	1	0

Thus, by selection of a particular set of coefficients, one recovers such schemes as Agrawals[4].

Artificial Convection

In a cartesian representation, the coefficients U in the convective term are the two velocity components. When transformed, additional first order derivatives appear and the question arises as to whether these should be treated as convective terms and hence whether the upwind differencing should be applied. Reference[5] states that they should and bases that on essentially an empirical finding. In the present study, a simple local analysis was attempted to resolve this problem.

So, in order to keep a certain amount of flexibility concerning this problem and allow the user the use of either approach, the convective terms are divided into two parts, the physical convection (resulting from the velocity field) and the artificial convection (resulting from the transformation). Accordingly the system of Equation (19) for the two transport quantities ω and V_θ can be written as follows:

$$
\begin{aligned}
&\alpha \vec{f}_{\zeta\zeta} - 2\beta \vec{f}_{\zeta\tau} + \gamma \vec{f}_{\tau\tau} \\
&+ \vec{f}_{\zeta} \tilde{Q} \qquad - \vec{f}_{\zeta} U \\
&+ f_{\tau} \tilde{R} \qquad - \vec{f}_{\tau} V \\
&- \frac{a}{r} \left(\frac{1}{r} - ReV_{r} \right) J^{2} \vec{f} = \vec{F}
\end{aligned}
\tag{22}
$$

where the coefficients for partial (physical) convection are

$$
\begin{aligned}
\tilde{Q} &= QJ^{2} - z_{\tau} J \frac{a}{r} \\
\tilde{R} &= RJ^{2} + z_{\zeta} J \frac{a}{r} \\
U &= Re\ Jz_{\tau} V_{r} + Re\ Jr_{\tau} V \\
V &= Re\ JV_{r} z_{\zeta} - Re\ J V_{z} r_{\zeta}
\end{aligned}
\tag{23}
$$

$$
F = \left| \begin{matrix} \dfrac{-2V_{\theta}\ Rea}{r} \left(J\ r_{\tau} \dfrac{\partial V_{\theta}}{\partial \zeta} - J\ r_{\zeta} \dfrac{\partial V_{\theta}}{\partial \tau} \right) \\ 2J^{2}\ Re\ V_{r}\ V_{O}\ \dfrac{a}{r} \end{matrix} \right|
$$

or, for total (artificial convection)

$$
\begin{aligned}
U &= - QJ^{2} + \frac{z_{\tau} Ja}{r} - Re\ Jz_{\tau} V_{r} + Re\ Jr_{\tau} V_{z} \\
V &= - RJ^{2} - \frac{z_{\zeta} Ja}{r} + Re\ JV_{r} z_{\zeta} - Re\ JV_{z} r_{\zeta}
\end{aligned}
\tag{24}
$$

and $\tilde{Q} = \tilde{R} = 0$

5. Discretization

5.1 Stream function

The stream equation is discretized using centered differences yielding second order accuracy.

$$
\begin{aligned}
&\psi_{i+1,j}\left(\alpha + \frac{QJ^2}{2} + \frac{z_\tau\, Ja}{2r}\right) \\
&- \psi_{i,j}\,(2\alpha + 2\gamma) \\
&+ \psi_{i-1,j}\left(\alpha - \frac{QJ^2}{2} - \frac{z_\tau\, Ja}{2r}\right) \\
&+ \psi_{i,j-1}\left(\gamma + \frac{RJ^2}{2} - \frac{z_\xi\, Ja}{2r}\right) \\
&+ \psi_{i,j-1}\left(\gamma - \frac{RJ^2}{2} + \frac{z_\xi\, Ja}{2r}\right) \\
&+ J^2\omega_{i,j}r^a - \frac{\beta}{2}\left(\psi_{i+1,j+1} - \psi_{i-1,j+1} - \psi_{i+1,j-1} + \psi_{i-1,j-1}\right) = 0
\end{aligned}
\tag{25}
$$

5.2 Vorticity and swirl

The transport terms in the equation for the vorticity and tangential velocity are upwind differenced as described in the previous section, whereas the viscous terms are centered differenced.

This gives the following algebraic system:

$$
\begin{aligned}
&f_{i+2,j}\,AX + f_{i+1,j}\,BX + f_{i,j}\,CC \\
&+ f_{i-2,j}\,EX + f_{i-1,j}\,DX \\
&+ f_{i,j+2}\,AY + f_{i,j+1}\,BY \\
&+ f_{i,j-2}\,EY + f_{i,j-1}\,DY \\
&- \frac{\beta}{2}\left(f_{i+1,j-1} - f_{i+1,j-1} - f_{i-1,j+1} + f_{i-1,j-1}\right) = 0
\end{aligned}
\tag{26}
$$

where

$$AX = k_3 \frac{(U - |U|)}{12\Delta\zeta} \qquad EX = - k_3 \frac{(U + |U|)}{12\Delta\zeta}$$

$$BX = - k_0 \frac{(U + |U|)}{12\Delta\zeta} + k_2 \frac{(U - |U|)}{12\Delta\zeta} + \frac{\tilde{Q}}{\Delta\zeta} + \frac{\alpha}{\Delta\zeta^2}$$

$$DX = - \frac{k_2}{12\Delta\zeta}(U + |U|) + \frac{k_0}{12\Delta\zeta}(U - |U|) - \frac{Q}{2\Delta\zeta} + \frac{\alpha}{\Delta\zeta^2} \tag{27}$$

$$AY = \frac{k_3}{12\Delta\tau}(V - |V|) \qquad EY = - \frac{k_3}{12\Delta\tau}(V + |V|)$$

$$BY = - \frac{k_0}{12\Delta\tau}(V + |V|) + \frac{k_2}{12\Delta\tau}(V - |V|) + \frac{\tilde{R}}{2\Delta\tau} + \frac{\gamma}{\Delta\tau^2}$$

$$DY = - \frac{k_2}{12\Delta\tau}(V + |V|) + \frac{k_0}{12\Delta\tau}(V - |V|) - \frac{\tilde{R}}{2\Delta\tau} + \frac{\gamma}{\Delta\tau^2}$$

$$CC = - \frac{K_1 \; |U|}{6\Delta\zeta} - \frac{k_1 \; |V|}{6\Delta\tau} - 2\left(\frac{\alpha}{\Delta\zeta^2} + \frac{\gamma}{\Delta\tau^2}\right) - \frac{a}{r}\left(\frac{1}{r} - Re\, V_r\right) J^2$$

5.3 Pressure

The pressure equation is discretized using centered differences yielding second order accuracy.

$$\begin{aligned}
& P_{i+1,j}\left(\alpha + \frac{QJ^2}{2} - \frac{Z_\tau \; Ja}{2r}\right) \\
& - P_{i,j} \quad (2\alpha + 2\gamma) \\
& + P_{i-1,j}\left(\alpha - \frac{QJ^2}{2} + \frac{Z_\zeta Ja}{2r}\right) \\
& + P_{i,j+1}\left(\gamma + \frac{RJ^2}{2} + \frac{Z_\zeta Ja}{2r}\right) \\
& + P_{i,j-1}\left(\gamma - \frac{RJ^2}{2} - \frac{Z_\zeta Ja}{2r}\right) \\
& - J^2 \; S_{i,j} - \frac{\beta}{2}\left(P_{i+1,j+1} - P_{i-1,j+1} - P_{i+1,j} + P_{i-1,j-1}\right) = 0
\end{aligned} \tag{28}$$

6. Boundary Conditions

In the present problem, there are four types of boundaries: solid walls, inlet boundaries, outlet boundaries and symmetry boundaries.

6.1 Stream Function

The stream function is constant along the solid and symmetry boundaries. At the inlet the variation depends on the imposed velocity distribution. In the program there are several options: constant profile, developed profile for either 2-D channel flows, or axi-symmetric annular or duct flow. These are illustrated in Fig. 2 and the appropriate relation has been coded. The computation of the exit is carried out by extrapolation of interior values.

6.2 Vorticity

From the definition of vorticity, Equation (5), one obtains the following relation:

$$\omega = \left(r_\tau \frac{\partial v}{\partial \xi} - r_\xi \frac{\partial v}{\partial \tau} - z_\xi \frac{\partial u}{\partial \tau} + z_\tau \frac{\partial u}{\partial \xi}\right) \frac{1}{J} \tag{29}$$

Along a solid boundary coinciding with a ξ = constant coordinate, this simplifies to:

$$\omega_B = - \left(r_\xi \frac{\partial v}{\partial \tau} + z_\xi \frac{\partial u}{\partial \tau}\right)/J \tag{30}$$

The derivatives are evaluated at the boundary using one-sided third order accurate formula:

$$\frac{\partial v}{\partial \tau} \simeq - 11\, V_B + 18\, V_{B+1} - 9\, V_{B+2} + 2\, V_{B+3} \tag{31}$$

where the subscipts B, B + 1, etc., indicate boundary and interior points.

Along the line of symmetry, the value of vorticity is zero. The inlet values are computed from the imposed velocity distribution and exit values are extrapolated.

6.3 Circumferential Velocity (Swirl)

The condition at a solid wall and along a line of symmetry is that of zero velocity. The inlet distribution is set to vary linearly between the internal and external boundaries. This is carried out by setting the angular rotation of the internal boundary as illustrated in Fig. 3. The exit values are extrapolated from interior points.

6.4 Pressure

All the boundary conditions for the pressure are of the second kind. At the inlet, using Equations (1) and (3) with $Vr = 0$, $\frac{\partial Vr}{\partial z} = 0$ and $\frac{\partial Vz}{\partial z} = 0$ (for fully developed velocity profile) and using Equation (5) to simplify, we obtain:

$$\frac{\partial P}{\partial r} = \frac{1}{Re} \left[\frac{\partial \omega}{\partial z} + \frac{a}{r} Re\, V_\theta^{\,2} \right]$$

$$\frac{\partial P}{\partial z} = \frac{-1}{Re} \left[\frac{\partial \omega}{\partial r} + \frac{a}{r}\, \omega \right] \tag{32}$$

The condition of the second kind for the pressure is

$$\frac{\partial P}{\partial n} = n \cdot \nabla P \tag{33}$$

Then, in curvilinear coordinates, we obtain:

$$\frac{\partial P}{\partial \varsigma} = \frac{1}{\alpha} \left[\frac{\beta}{2} \frac{\partial P}{\partial \tau} - \frac{J}{Re} \left(\frac{\partial \omega}{\partial \tau} + \frac{a}{r} \left(z_\tau\, Re\, V_\theta^{\,2} + r_\tau\, \omega \right) \right) \right] \tag{34}$$

At the solid boundary, having $V_r = 0$ and $V_z = 0$ we find Equations (32), and using Equation (33), we obtain in curvilinear coordinates:

$$\frac{\partial P}{\partial \tau} = \frac{1}{\gamma} \left[\frac{\beta}{2} \frac{\partial P}{\partial \varsigma} + \frac{J}{Re} \left(\frac{\partial \omega}{\partial \varsigma} + \frac{a}{r} \left(z_\varsigma\, Re\, V_\theta^{\,2} + r_\varsigma \omega \right) \right) \right] \tag{35}$$

For a symmetry boundary:

$$\frac{\partial P}{\partial n} = 0 \tag{36}$$

Then,

$$\frac{\partial P}{\partial \tau} = \frac{\beta}{2\gamma} \frac{\partial P}{\partial \varsigma} \tag{37}$$

The exit values are extrapolated from interior points. For Equations (34), (35) and (37), the derivatives of the left side are evaluated using one-sided second order accurate formula as shown in Equation (31) and the derivatives of the right side are discretized using centered differences yielding second order accuracy.

7. Numerical Algorithm

The system of algebraic equations resulting from the discretization is non-linear and is solved by an iterative scheme. It is a block successive relaxation scheme where the blocks are chosen to alternate between columns and rows in the computational grid. This has the advantage that the system becomes banded, i.e. tridiagonal for the stream function and pressure equations or pentadiagonal for the vorticity and swirl component of velocity. These lend themselves to very efficient numerical solution.

The construction of these blocks is carried by expressing the discrete equations implicity with respect to one coordinate direction and explicitly with respect to the other, and vice versa.

The computational domain (ξ, τ) is discretized into a grid with spacing $\Delta\xi = \Delta\tau = 1$. For convenience, two additional rows of nodes beyond the lower and upper boundaries are added. Thus these boundaries lie along the coordinates $J = 2$ and $J = N$ respectively as shown in Fig. 4.

7.1 Stream Function

For the stream function, this procedure yields for the implicit in the ξ direction:

$$
\begin{aligned}
&APP_{i,j}\,\bar{\psi}_{i+1,j} + APG_{i,j}\,\bar{\psi}_{i,j} + AMP_{i,j}\,\bar{\psi}_{i-1,j} \\
&= -\,GPQ_{i,j}\,\psi^{o}_{i,j+1} - GMQ_{i,j}\,\psi^{+}_{i,j-1} + J^{2}F \\
&+ BET_{i,j}\,(\psi^{o}_{i+1,j+1} - \psi^{o}_{i-1,j+1} \\
&- \psi^{+}_{i+1,j-1} + \psi^{+}_{i-1,j-1})
\end{aligned}
\qquad (38)
$$

where $\bar{\psi}$ indicates a current value being updated (i.e. the current block), ψ^{+} indicates a previously updated value (i.e. in the preceding block) and ψ^{o} indicates an old value (i.e. in the next block). This is illustated in Fig.5. The coefficients are defined:

$$AMP_{i,j} = \alpha - \frac{J^2}{2} (Q + z_\tau a/Jr)$$

$$APP_{i,j} = \alpha + \frac{J^2}{2} (Q + z_\tau a/Jr)$$

$$APG_{i,j} = - (2\alpha + 2\gamma)$$

$$BET_{i,j} = \beta/2 \qquad (39)$$

$$GMQ_{i,j} = \gamma - \frac{J^2}{2} (R - z_\zeta\, a/Jr)$$

$$GPQ_{i,j} = \gamma + \frac{J^2}{2} (R - z_\zeta\, a/Jr)$$

A current value is updated using a relaxation coefficient RLX,

$$\psi^+ = \psi^o + RLX\ (\bar{\psi} - \psi^o) \qquad (40)$$

where $0 < RLX < 2$

This can be conveniently cast in the form of a correction block,

$$C = \psi^+ - \psi^o = RLX\ (\bar{\psi} - \psi^o)$$

obtained from Equation (40) after substituting Equation (38), giving

$$APP_{i,j}\ C_{i+1,j} + APG_{i,j}\ C_{i,j} + AMP_{i,j}\ C_{i-1,j} = - RLX * RES_{i,j} \qquad (41)$$

where $RES_{i,j}$ is the residual of Equation (25). Similarly, for the block implicit in the τ direction,

$$GPQ_{i,j}\ C_{i,j+1} + APG_{i,j}\ C_{i,j} + GMQ_{i,j}\ C_{i,j-1} = - RLX * RES_{i,j} \qquad (42)$$

Using the boundary conditions, the values of the corrections for $j = 2$ and $j = N$ are zero since the value of the value of the stream function is constant,

$$C_{i,2} = C_{i,N} = 0 \qquad 1 \leq i \leq M + 1$$

Similarly, since the velocity profile is known at the inlet, the correction along i = 1 is also zero,

$$C_{1,j} = 0 \qquad\qquad 2 \leq j \leq N$$

At the exit line i = M + 1, the correction is extrapolated,

$$C_{M+1,j} = 2\,C_{M,j} - C_{M-1,j}$$

Substituting these into Equation (41) and Equation (42) yields the following triadiagonal systems:

Implicit along ξ:

$$\begin{bmatrix} APG_{2,j} & APP_{2,j} & 0 & 0 \\ AMP_{3,j} & APG_{3,j} & APP_{3,j} & 0 \\ & & & \\ 0 & AMP_{M-1,j} & APG_{M-1,j} & APP_{M-1,j} \\ 0 & 0 & (AMP - APP)_{M,j} & (APG + 2APP)_{M,j} \end{bmatrix} \begin{bmatrix} C_{2,j} \\ \\ \\ \\ C_{M,j} \end{bmatrix} = -RLX \begin{bmatrix} RES_{2,j} \\ \\ \\ \\ RES_{M,j} \end{bmatrix}$$

for $3 \leq j \leq N - 1$ (43)

Implicit along τ:

$$\begin{bmatrix} APG_{i,3} & GPQ_{i,3} & 0 & \\ GMQ_{i,4} & APG_{i,4} & CPQ_{i,4} & \\ & & & \\ & GMQ_{i,N-2} & APG_{i,N-2} & GPQ_{i,N-2} \\ & 0 & GMQ_{i,N-1} & APQ_{i,N-1} \end{bmatrix} \begin{bmatrix} C_{i,3} \\ \\ \\ \\ C_{i,N-1} \end{bmatrix} = -RLX \begin{bmatrix} RES_{i,3} \\ \\ \\ \\ RES_{i,N-1} \end{bmatrix}$$

for $2 \leq i \leq M$ (44)

7.2 Transport Equations for vorticity and swirl

The transport equations for the vorticity and swirl are discretized using upwind differencing for the convection terms.

This results in the widening of the band in the explicit direction, requiring two blocks behind and ahead of the current block being updated. This is illustrated in Figs. 6 ... Using the same relaxation procedure, and casting the results in the form of a correction, one obtains for the implicit ξ direction:

$$EX_{i,j}\, C_{i-2,j} + DX_{i,j}\, C_{i-1,j} + CC_{i,j}\, C_{i,j}$$

$$+ BX_{i,j}\, C_{i+1,j} + AX_{i,j}\, C_{i+2,j} = - RLX\ RES_{i,j} \qquad (45)$$

and for the implicit τ direction:

$$EY_{i,j}\, C_{i,j-2} + DY_{i,j}\, C_{i,j-1} + CC_{i,j}\, C_{i,j}$$

$$+BY_{i,j}\, C_{i,j+1} + AY_{i,j}\, C_{i,j+2} = -RLX\ RES_{i,j} \qquad (46)$$

where the coefficients are defined by Equation (27). The values of the variable ω on the solid boudaries are computed using Equation (30) and since the velocities do not change during the iterations on this variable, then the corrections are zero. For the swirl V_θ, the corrections are also zero. This gives

$$C_{i,2} = 0 \text{ and } C_{i,N} = 0 \qquad (47)$$

The rows of fictitious points $j = 1$ and $j = N + 1$ are computed by extrapolation using a variable order formula.

$$f_1 = CO_2\, f_2 + CO_3\, f_3 + CO_4\, f_4 + CO_5\, f_5 \qquad (48)$$

or

$$f_{N+1} = CO_2\, f_N + CO_3\, f_{N-1} + CO_4\, f_{N-2} + CO_5\, f_{N-3} \qquad (49)$$

Also

$$C_{1,j} = C_{2,j} = 0$$

and

$$C_{M+1,j} = C_{M,j} = C_{M-1,j}$$

Substituting Equations (47), (48) and (49) into (45) and (46) yields the two following pentadiagonal matrices for the block implicit in ξ and for the block implicit in τ respectively.

$$
\begin{bmatrix}
CC_{3,j} & BX_{3,j} & AX_{3,j} & 0 & 0 & & & & & & 0 \\
DX_{4,j} & CC_{4,j} & BX_{4,j} & AX_{4,j} & 0 & & & & & & 0 \\
EX_{5,j} & DX_{5,j} & CC_{5,j} & BX_{5,j} & AX_{5,j} & & & & & & 0 \\
0 & & & & & & & & & & \\
\vdots & & & & & \ddots & & & & & \vdots \\
0 & & & & & & & & & & 0 \\
0 & & & & & & EX_{M-3,j} & DX_{M-3,j} & CC_{M-3,j} & BX_{M-3,j} & AX_{M-3,j} \\
0 & & & & & & 0 & EX_{M-2,j} & DX_{M-2,j} & CC_{M-2,j} & (BX+AX)_{M-2,j} \\
0 & & & & & & 0 & 0 & EX_{M-1,j} & DX_{M-1,j} & (CC+BX+AX)_{M-1,j}
\end{bmatrix}
$$

for $3 \leqslant j \leqslant N-1$ (50)

$$
\begin{bmatrix}
(CC + CO_3\ EY)_{i,3} & (BY + CO_4\ EY)_{i,3} & (AY + CO_5\ EY)_{i,3} & 0 & 0 & & & 0 \\
DY_{i,4} & CC_{i,4} & BY_{i,4} & AY_{i,4} & 0 & & & 0 \\
EY_{i,5} & DY_{i,5} & CC_{i,5} & BY_{i,5} & AY_{i,5} & & & 0 \\
 & & & & & & & \\
 & & & & & & & 0 \\
0 & & EY_{i,N-3} & DY_{i,N-3} & CC_{i,N-3} & & BY_{i,N-3} & AY_{i,N-3} \\
0 & & 0 & EY_{i,N-2} & DY_{i,N-2} & & CC_{i,N-2} & BY_{i,N-2} \\
0 & & 0 & 0 & (EY + CO_5\ AY)_{i,N-1} & & (DY + CO_4\ AY)_{i,N-1} & (CC + CO_3\ AY)_{i,N-1}
\end{bmatrix}
$$

for $3 \leqslant i \leqslant M-1$ (51)

7.3 Pressure

The procedure for the pressure is the same as for the stream function. Then, the general discretized equation is

$$
\begin{aligned}
& APPM_{i,j}\ P_{i+1,j} + APG_{i,j}\ P_{i,j} + AMPM_{i,j}\ P_{i-1,j} \\
& + GPQP_{i,j}\ P_{i,j+1} + GMQP_{i,j}\ P_{i,j-1} - J^2 S_{i,j} \qquad (52) \\
& - BET_{i,j}\ (P_{i+1,\ j+1} - P_{i-1,\ j+1} - P_{i+1,j-1} + P_{i-1,\ j-1}) = 0
\end{aligned}
$$

where

$$
\begin{aligned}
AMPM_{i,j} &= \alpha - \frac{J^2}{2}\ (Q - z_\tau\ a/Jr) \\
APPM_{i,j} &= \alpha + \frac{J^2}{2}\ (Q - z_\tau\ a/Jr) \\
GMQP_{i,j} &= \gamma - \frac{J^2}{2}\ (R + z_\xi\ a/Jr) \qquad (53) \\
GPQP_{i,j} &= \gamma + \frac{J^2}{2}\ (R + z_\xi\ a/Jr)
\end{aligned}
$$

APG and BET are defined in Equation (39)

Using a relaxation scheme, we obtain the correction block for the 2 directions. In the ξ direction,

$$APPM_{i,j}\ C_{i+1,j} + APG_{i,j}\ C_{i,j} + AMPM_{i,j}\ C_{i-1,j} = -\ RLX * RES_{i,j} \qquad (54)$$

and in the τ direction,

$$GPQP_{i,j}\ C_{i,j+1} + APG_{i,j}\ C_{i,j} + GMQP_{i,j}\ C_{i,j-1} = -\ RLX * RES_{i,j} \qquad (55)$$

where $RES_{i,j}$ is the residual of Equation (28).

Using the boundary conditions, the values of the corrections are:

Inlet: $- 3C_{1,j} + 4C_{2,j} - C_{3,j} = RLX * RS_{1,j}$ (56)

with $RS_{1,j} = 3\ P^o_{1,j} - 4\ P^o_{2,j} + P^o_{3,j} + \frac{\beta}{2\alpha}\ (P_{1,j+1} - P_{1,j-1})$

$$- \frac{J}{\alpha Re}\ (\omega_{1,j+1} - \omega_{1,j-1}) - \frac{2Ja}{\alpha Re\ r}\ (z_\tau\ Re\ V_{\theta_{1,j}}^{\ 2} + r_\tau \omega_{1,j})$$

Exit: $C_{M+1,j} = 2\ C_{M,j} - C_{M-1,j}$ (57)

Lower boundary (j = 2)

$$-3C_{i,2} + 4C_{i,3} - C_{i,4} = RLX * RS_{i,2} \tag{58}$$

with $RS_{i,2} = 3P^{o}_{i,2} - 4P^{o}_{i,3} + P^{o}_{i,4}$

$$+ \begin{cases} \frac{\beta}{2\gamma}\,(P_{i+1,2} - P_{i-1,2}) & \text{for a symmetry line} \\ \text{or} & \\ \frac{\beta}{2\gamma}\,(P_{i+1,2} - P_{i-1,2}) + \frac{J}{\gamma Re}\,(\omega_{i+1,2} - \omega_{i-1,2}) + \frac{2Ja}{\gamma Rer}(& \end{cases}$$

$$z_{\zeta}\, Re\, V_{\theta\, i,2}^{2} + r_{\zeta}\omega_{i,2}) \text{ for a solid boundary}$$

Upper boundary (j = N):

$$3C_{i,N} - 4C_{i,N-1} + C_{i,N-2} = RLX*RS_{i,N} \tag{59}$$

with $RS_{i,N} = -3P^{o}_{i,N} + 4P^{o}_{i,N-1} - P^{o}_{i,N-2} + \frac{\beta}{2\gamma}\,(P_{i+1,N} - P_{i-1,N})$

$$+\frac{J}{\gamma Re}\,(\omega_{i+1,N} + \omega_{i-1,N}) + \frac{2Ja}{\gamma Rer}\,(z_{\zeta} Re\, V_{\theta i,N}^{2} + r_{\zeta}\, \omega_{i,N})$$

Then, from Equations (54) to (59), we obtain the triadiagonal systems to solve:

For the direction ζ :

$$
\begin{bmatrix}
(APG + \frac{4}{3} AMPM)_{2,j} & (APPM - \frac{1}{3} AMPM)_{2,j} & 0 & & \\
AMPM_{3,j} & APG_{3,j} & APPM_{3,j} & & \\
0 & & \ddots & & \\
 & & & & 0 \\
 & & AMPM_{M-1,j} & APG_{M-1,j} & APPM_{M-1,j} \\
 & & 0 & (AMPM - APPM)_{M,j} & (APG + 2\ APPM)_{M,j}
\end{bmatrix}
\begin{bmatrix}
C_{2,j} \\ C_{3,j} \\ \vdots \\ \\ C_{M-1,j} \\ C_{M,j}
\end{bmatrix}
= -RLX
\begin{bmatrix}
RES_{2,j} - \frac{1}{3} AMPM_{2,j}\ RS_{1,j} \\ RES_{3,j} \\ \vdots \\ \\ RES_{M-1,j} \\ RES_{M,j}
\end{bmatrix}
$$

for $3 \leqslant j \leqslant N-1$

For the direction τ :

$$\begin{bmatrix} (APG + \frac{4}{3}\,GMQP)_{i,3} & (GPQP - \frac{1}{3}\,GMQP)_{i,3} & 0 \\ GMQP_{i,4} & APG_{i,4} & GPQP_{i,4} \\ 0 & & \\ & & \\ GMQP_{i,N-2} & APG_{i,N-2} & GPQP_{i,N-2} \\ 0 & (GMQP - \frac{1}{3}\,GPQP)_{i,N-1} & (APG + \frac{4}{3}\,GPQP)_{i,N-1} \end{bmatrix} \begin{bmatrix} C_{i,3} \\ C_{i,4} \\ \\ \\ C_{i,N-2} \\ C_{i,N-1} \end{bmatrix} = -RLX \begin{bmatrix} RES_{i,3} - \frac{1}{3}\,GMQP_{i,3}\,RS_{i,2} \\ RES_{i,4} \\ \\ \\ RES_{i,N-2} \\ RES_{i,N-1} + \frac{1}{3}\,GPQP_{i,N-1}\,RS_{i,N} \end{bmatrix}$$

for $2 \leqslant i \leqslant M$

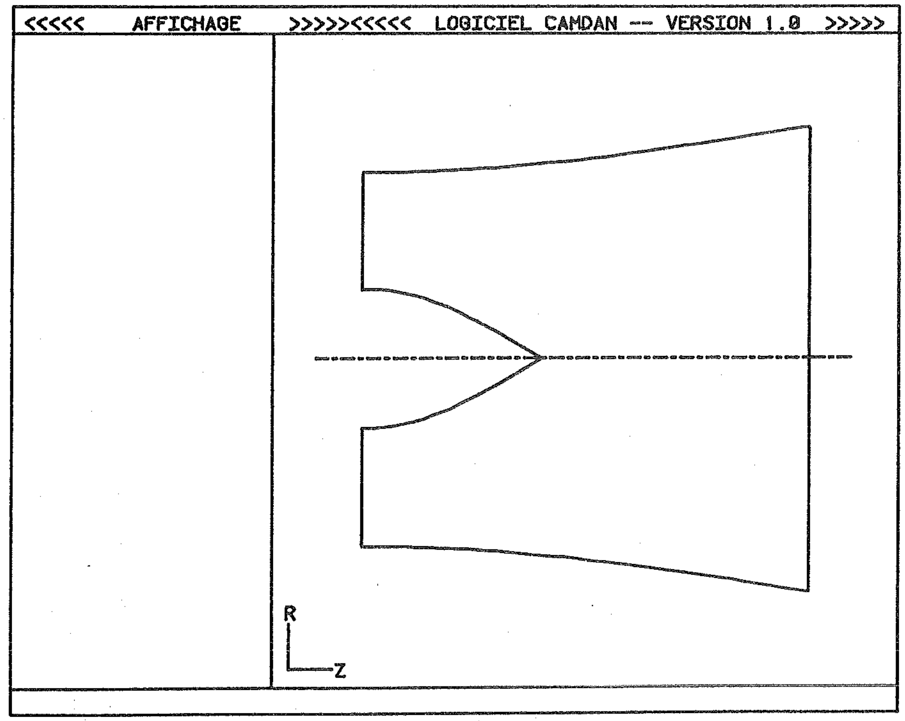

FIG. 1.

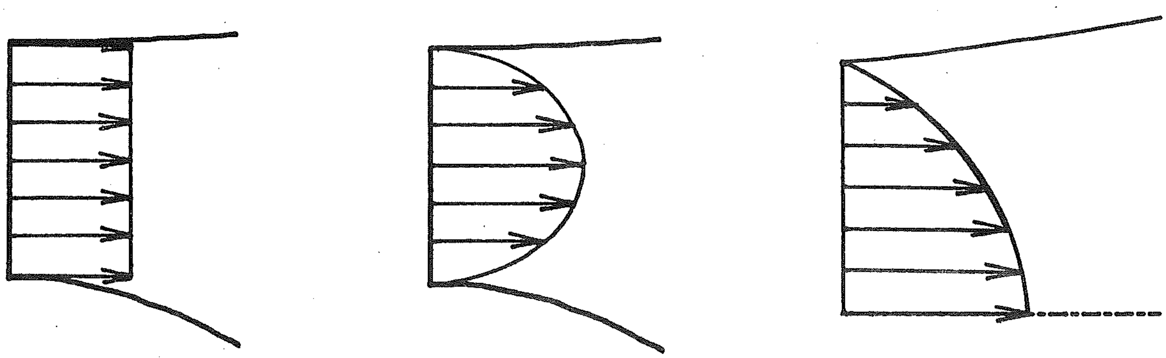

FIG. 2.

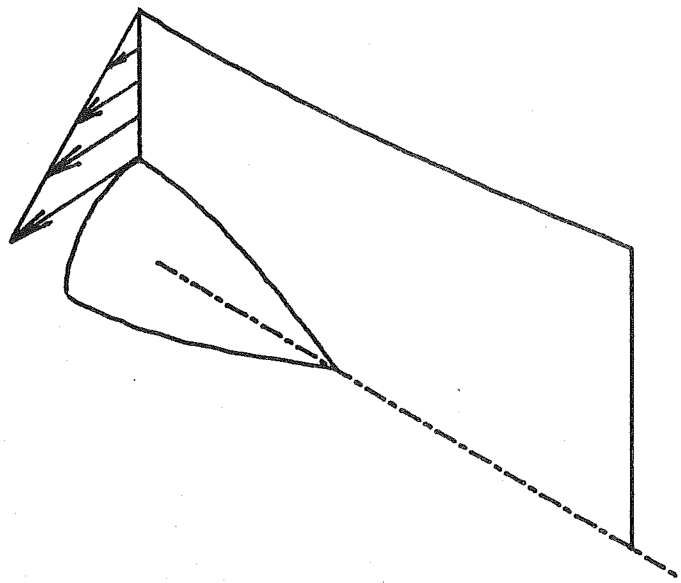

FIG. 3.

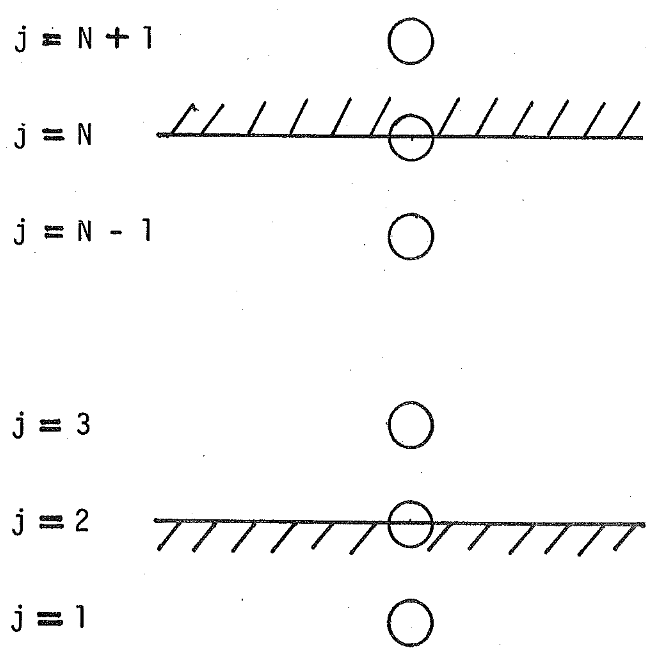

FIG. 4.

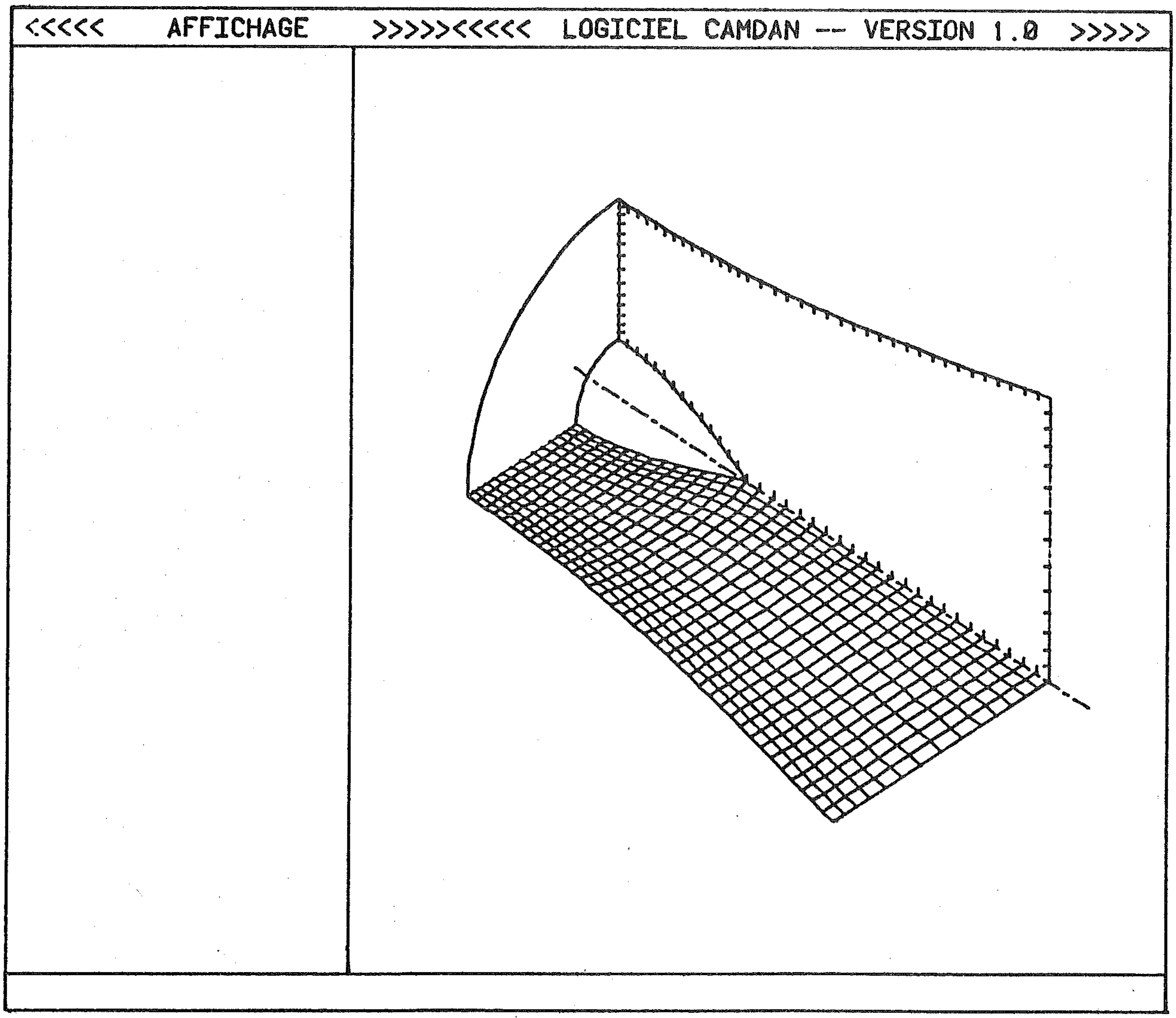

FIG. 5.

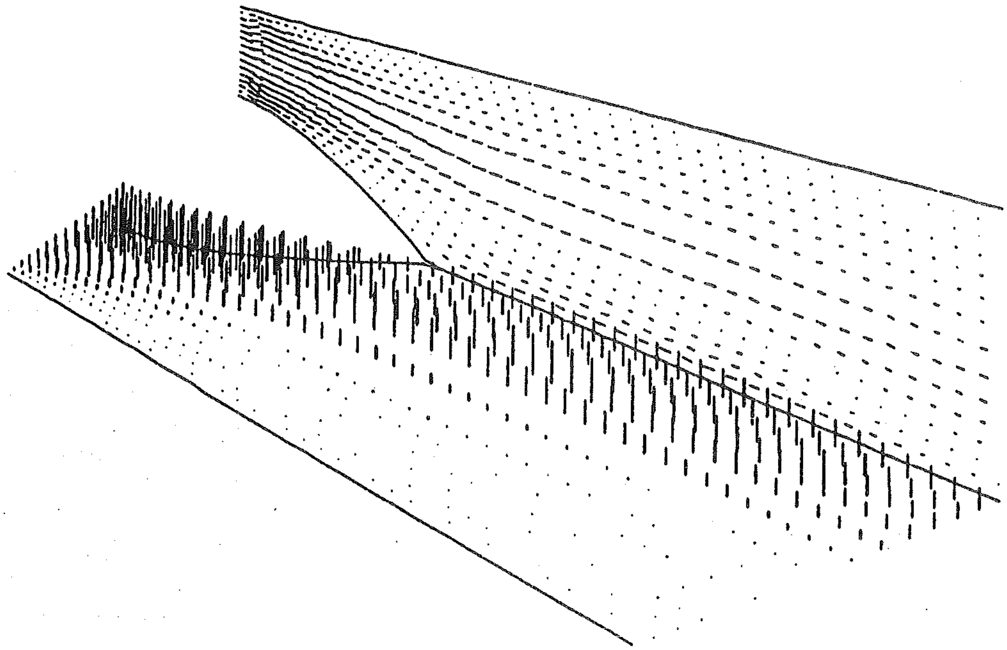

FIG. 6.

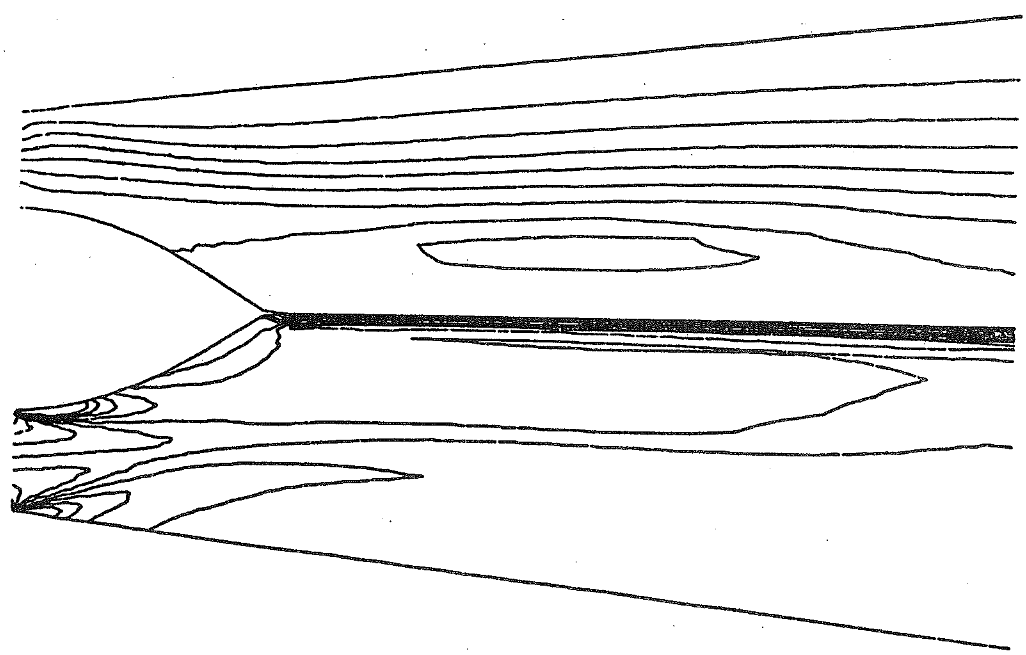

FIG. 7.