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SEMI-ANALYTICAL SHAPE FUNCTIONS IN THE FINITE ELEMENT ANALYSIS OF RECTANGULAR PLATES

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E. CHARBONNEAU and A. A. LAKIS

ÉCOLE POLYTECHNIQUE DE MONTRÉAL
Departement of Mechanical Engineering
Campus de l'Université de Montréal
C.P. 6079, Succ. Centre-Ville
Montréal (Québec)
H3C 3A7

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(**EPM/RT-99/19**)

E. Charbonneau , A. A. Lakis (Génie Mécanique)

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SEMI-ANALYTICAL SHAPE FUNCTIONS IN THE FINITE ELEMENT ANALYSIS OF RECTANGULAR PLATES

E. Charbonneau and **A. A. Lakis**

Department of Mechanical Engineering, École Polytechnique de Montréal, Campus de l'Université de Montréal, C.P. 6079, Succ. Centre-Ville, Montréal, Québec, Canada H3C 3A7

Abstract—This paper presents a method for the dynamic analysis of a thin, elastic, isotropic rectangular plate. The method is a hybrid of finite element theory and classical thin plate theory. The displacement functions are derived from Sanders thin-shell equation and are expanded in power series. Expressions for mass and stiffness are determined by precise analytical integration. The free vibrations of rectangular plates, with various boundary conditions, are studied following this method. The results obtained reveal that the frequencies calculated in this way are in good agreement with those obtained by others.

Keywords: finite element, displacement function, thin plate, vibration

NOTATION

a	length of the rectangular plate
A_i	defined by equation (8), $i = 0, 1, 2, 3$
b	width of the rectangular plate
B_j	defined by equation (8), $j = 0, 1, 2, 3$
C_i	defined by equation (6)
D	membrane stiffness, $Et/(1-\nu^2)$
E	Young's modulus

$E^{(k)}_{i,j}$	defined in equation (23), i = 1, 2, …, 16 ; j = 1, 2, …, 16
$f_{i,j}$	natural frequency (Hz)
$G^{(u)}_{i,j}$, $G^{(v)}_{i,j}$, $G^{(w)}_{i,j}$	defined by equation (23), i = 1, 2, …, 16 ; j = 1, 2, …, 16
K	bending stiffness, $Et^3/12(1-\nu^2)$
$L_{i,j}$	defined in equation (23), i = 1, 2, …, 16 ; j = 1, 2, …, 16
m	axial ode number parallel to the x-axis
$M_{i,j}$	defined in equation (23), i = 1, 2, …, 16 ; j = 1, 2, …, 16
$M_{x,x}$, $\overline{M}_{xy}$, $M_{y,y}$	bending moments of a rectangular plate
n	axial mode number parallel to the y-axis
N	number of finite elements
$N_{x,x}$, $_{x,y}$, N_y	stress components of a rectangular plate
$P_{i,j}$	terms of the elasticity matrix, i =1, 2, …, 6, j =1, 2, …, 6
$S^{(u)}_{i,j}$, $S^{(v)}_{i,j}$, $S^{(w)}_{i,j}$	defined by equation (21), i = 1, 2, …, 16 ; j = 1, 2, …, 16
t	thickness of the rectangular plate
u_i, v_i, w_i	nodal displacements, i = 1, …, 4
U, V, W	in-plane and normal displacement of a rectangular plate
x, y	length and width co-ordinate of the plate
$w_{i,x}$, $w_{i,y}$, $w_{i,xy}$	nodal rotations and twisting, i = 1, …, 4
W_p	displacement function defined by equation (8)
δ_i	degree of freedom at node i, i = 1, …, 4
ε_x, ε_y, ε_{xy}	deformations of the plate reference surface
κ_x, κ_y, κ_{xy}	changes in curvature and twisting of the plate reference surface
ρ	density of the plate material

υ	Poisson's ratio
ω	angular natural frequency, rad s^{-1}
Ω	non-dimensional frequency, $\omega a^2(\rho t/K)^{1/2}$
ψ	phase angle

List of matrices

[A]	defined by equation (12), given in Appendix II
[B]	defined by equation (14)
[C]	defined by equation (10)
[D]	defined by equation (14)
[G]	defined by equation (22)
[k]	stiffness matrix of one element
[K]	stiffness matrix of the total plate
[m]	mass matrix of one element
[M]	mass matrix of the total plate
[N]	defined by equation (13)
[P]	elasticity matrix
[S]	defined by equation (20)
$\{\varepsilon\}$	strain vector
$\{\sigma\}$	stress vector
$\{\delta_i\}$	degrees of freedom at node i
$\{\delta_T\}$	degrees of freedom for the total plate
$\{\delta_{0,T}\}$	Amplitude of the plate motion

1. INTRODUCTION

Rectangular plates are one of the most widely used structural elements. They are used in such fields as civil and naval engineering, and in aeronautical and space technology. A knowledge of the free vibration characteristics of rectangular plates enables engineers to design better and lighter structures. For this reason, the behaviour of rectangular plates has been the subject of on-going research for more than a hundred years

The first mathematical model of the behaviour of the plate membrane was formulated by Euler in the 18th century. Later, the German physicist Chaldi found the vibration modes of horizontal plates by spreading powder on the vibrating surface. More than fifty years later, Lagrange developed the first correct differential equation for the free vibration of plates. Some time later, Navier (1785-1836) produced a method of calculating the mode shape and the frequencies for certain boundary value problems. He used the trigonometric series introduced by Fourier to express the deflection of the plate.

Kirchoff (1824-1887) is considered the founder of modern plate theory which, by analysing plates with substantial deflection, takes into account both bending and stretching. He concluded that the non-linear effect should not be ignored when dealing with large deflections and that the natural frequencies and mode shapes can be determined by the virtual work method. Love[1] applied Kirchoff's work to thick plate.

More recently, Leissa[2] summarized the work of several researchers into a book containing more than five hundred references. The needs of the modern aircraft industry have brought advances in the study of rectangular plates. In 1956, Turner and al[3] introduced the finite element method, which permits the complex plate problem to be formulated, and, with the advent of high-speed computers, variety of numerical methods using matrix algebra were developed. Zienkiewicz[4] contributed to the formulation of different kinds of finite elements. Bogner and al.[5-6] worked on an element using BI-cubic interpolation functions to simulate the displacement of the plate.

In order to predict both low and high frequencies with precision, we used the finite element method with many elements and developed a hybrid finite element method which is derived from Sanders classical shell theory[7]. Various elements have been developed for close and open cylindrical[8-15], conical[16] and spherical[17] shells in vacuum or containing a fluid at rest or in motion. Whilst several well-known finite element codes such as NASTRAN, ABAQUS or ANSYS can solve the free vibrations of a rectangular plate in vacuum relatively easily, none can correctly predict the natural frequencies of a plate submerged in fluid. We needed a general, thin, rectangular plate element that could later be used for fluid-structure interaction analysis. The formulation of an analytical solution for a general rectangular plate element is quite complex: we had to find displacement functions compatible with both the plate equations of motion and the solution of a plate in contact with fluid. We decided, therefore, to expand the homogenous solution of the BI-harmonic equation of plate into a power series and, in so

doing, we obtained a semi-analytical solution in the form of a polynomial which can be used in classical finite element theory.

In this article, we discuss the development of this element and its relative precision in comparison with other methods. We first determined the fundamental equations of the plate and, secondly, derived the displacement functions of plate theory and expanded them in power series. With these displacement functions, we were able to determine the mass and stiffness matrices required by the finite element method and, therefore, the free vibration characteristics of the plate.

2. FUNDAMENTAL EQUATIONS FOR THIN RECTANGULAR PLATE

2.1. Equilibrium equations

To establish the equilibrium equations of the plate, we use Sander's equations [7] for cylindrical shells and assume the radius to be infinite, $\theta = y$ and $rd\theta = dy$. These three equations take into account both membrane effects and bending effects. Sanders based his equations on Love's First Approximation [1] for thin shells, and showed that all strains vanish for any rigid-body motion. For the finite element method this theory satisfies the convergence criteria for small rigid-body motions.

The geometry of the mean surface and the co-ordinate system used for this analysis are shown in figure 1.

The equilibrium equations for a rectangular plate, following Sanders' theory, are written as follows:

$$\frac{\partial \overline{N}_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0$$

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial \overline{N}_{xy}}{\partial y} = 0$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 \overline{M}_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} = 0 \qquad (1)$$

where N_{xx}, N_{xy}, N_{yy}, M_{yy}, M_{xx} and M_{xy} are the stress components per unit length and x and y are the coordinates of the plate. The unit vectors corresponding to the stresses defined in Equation (1) are indicated on Figure 2.

2.2. Kinematics equations

The relation between the strain (ε) and the displacement for a rectangular plate is given as follows:

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \\ \kappa_x \\ \kappa_y \\ \overline{\kappa}_{xy} \end{pmatrix} = \begin{Bmatrix} \dfrac{\partial U}{\partial x} \\ \dfrac{\partial V}{\partial y} \\ \dfrac{\partial V}{\partial x} + \dfrac{\partial U}{\partial y} \\ -\dfrac{\partial^2 W}{\partial x^2} \\ -\dfrac{\partial^2 W}{\partial y^2} \\ -\dfrac{\partial^2 W}{\partial x \partial y} \end{Bmatrix} \qquad (2)$$

where U and V are the in plane displacements and W the deflection of the plate.

For an anisotropic and elastic material the relationship between stress and strain is

$$\{\sigma\} = [P]\{\varepsilon\} \tag{3}$$

where [P] is a 6x6 symmetric elasticity matrix. In the case of an isotropic material there is no coupling between membrane and bending effects, and the only non-vanishing terms are:

$$\begin{aligned} P_{11} = P_{22} = D \quad & P_{44} = P_{55} = K \\ P_{12} = P_{21} = \nu D \quad & P_{45} = P_{54} = \nu K \\ P_{33} = \frac{(1-\nu)}{2} D \quad & P_{66} = \frac{(1-\nu)}{2} K \end{aligned} \tag{4}$$

where D and K are respectively the membrane and bending stiffness defined as

$$D = \frac{Et}{1-\nu^2} \qquad K = \frac{Et^3}{12(1-\nu^2)}$$

E being the young`s modulus, ν the Poisson`s ratio and t the plate thickness.

Next, we substitute Equations (2) and (3) in the equilibrium equations to obtain the 3 equations of motion in terms of the in-plane and normal displacements of the plate`s mean surface (U, V, W).

$$\begin{aligned} & P_{22}\frac{\partial^2 V}{\partial y^2} + P_{21}\frac{\partial^2 U}{\partial x \partial y} + P_{33}\left(\frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 V}{\partial x^2}\right) = 0 \\ & P_{11}\frac{\partial^2 U}{\partial x^2} + P_{12}\frac{\partial^2 V}{\partial x \partial y} + P_{33}\left(\frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 U}{\partial y^2}\right) = 0 \\ & P_{44}\frac{\partial^4 W}{\partial x^4} + \frac{\partial^4 W}{\partial x^2 \partial y^2}(P_{45} + P_{54} + 2P_{66}) + P_{55}\frac{\partial^4 W}{\partial y^4} = 0 \end{aligned} \tag{5}$$

The first two equations in (5) describe the membrane behavior and the last equation defines the bending of a rectangular plate. By solving these equations, it is possible to find the displacement function in terms of the nodal displacements.

3. DISPLACEMENT FUNCTIONS

3.1. Solution of the differential equations

In this instance, we are dealing with the case of isotropic material. As can be seen in Equation (5), the equations of motion are decoupled. It is possible, therefore, to consider the membrane and bending equations to be two different problems, each with its own solution.

The solution for the membrane differential equations in Equation (5) is based on Szilard [20]. We assume the solution to be a BI-linear polynomial expressing the nodal displacements in U and V, where U and V are, respectively, the in-plane displacement in the x and y direction as can be shown on an element in Figure 3.

The polynomial expression will be as follows:

$$U(x,y) = C_1 + C_2\frac{x}{a} + C_3\frac{y}{b} + C_4\frac{xy}{ab}$$

$$V(x,y) = C_5 + C_6\frac{x}{a} + C_7\frac{y}{b} + C_8\frac{xy}{ab} \qquad (6)$$

where x and y and the element coordinate system and a and b are the lenght and width of the plate corresponding to the x and y coordinates. These assumed displacement

functions contain the same number of unknown parameters C_i as the number of nodal displacements (2 x 4 = 8). The solution is rather crude but converges monotocally almost to the "exact" value for the problem of finding the maximum deflection[20].

In the case of bending, the bi-harmonic equation has a general solution of the following form:

$$W(x,y) = \sum_{i=1}^{n} C_i e^{\frac{x}{a}+\frac{y}{b}} \tag{7}$$

where W is the normal deflection of the plate element shown in figure 3. Since it is a complex matter to find the characteristic equation, we expand the solution in a Taylor series. The number of terms in the series remains to be determined. Furthermore, the number of degrees of freedom describing the motion of the plate in its normal direction is governed by the number of terms in the series. Therefore, we add as many terms as the hermitian BI-cubic polynomial used by Bogner[6]. The expanded polynomial, which approximates the normal deflection of the element, is:

$$W_p(x,y) = \left[\sum_{i=0}^{3} \frac{A_i}{i!}\left(\frac{x}{a}\right)^i\right]\left[\sum_{j=0}^{3} \frac{B_j}{j!}\left(\frac{y}{b}\right)^j\right] \tag{8}$$

where i! means i factorial; i.e., i = 3, i! = 1x2x3. Equation (8) gives 16 unknown parameters A_i and B_j corresponding, again, to the number of degrees of freedom per element for bending. Figure 3 shows the nodal degrees of freedom (W, $\partial W/\partial x$, $\partial W/\partial y$, $\partial^2 W/\partial x \partial y$) which relate to the bending motion. Instead of a rotational degree of freedom about the z-axis, we have the second derivative of W, which gives the twisting strain, and ensures a continuity of slope between the elements. This gives conforming and compatible elements in bending.

Furthermore, when using a power series to express the displacement function, we approach more precisely the "exact" solution of the bending equation than Bogner et al. [5-6] did with the BI-cubic polynomial. The two displacement functions are compared to the exact Equation (7) in Figure 4.

3.2. Displacement functions for a finite element

We can write the displacement U,V and W in matrix form:

$$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = [R] \quad \{C\} \tag{9}$$

where [R] is a 3x24 matrix in which the components are the x and y terms of Equations (6) and (8) without the unknown constants. The vector {C} is given by:

$$\{C\} = \{C_1, \ldots, C_{24}\}^t \tag{10}$$

To determine these constants, we need to define twenty-four boundary conditions for the finite element. These twenty-four boundary conditions will be the twenty-four degrees of freedom element, which means six degrees of freedom per node as follows:

$$\{\delta_1, \delta_2, \delta_3, \delta_4\}^t = \{u_1, v_1, w_1, w_{1,x}, w_{1,y}, w_{1,xy}, u_2, v_2, w_2, w_{2,x}, w_{2,y}, w_{2,xy},$$
$$u_3, v_3, w_3, w_{3,x}, w_{3,y}, w_{3,xy}, u_4, v_4, w_4, w_{4,x}, w_{4,y}, w_{4,xy}\} \tag{11}$$

where the δ_is are the generalized nodal displacement and $w_{i,x}$ is the derivative of w_i respect to x and so on. Then, we have to define a transformation matrix [A] to relate the displacement functions $\{C_i\}$ and the nodal displacements $\{\delta_i\}$:

$$\{\delta_i\} = [A]\{C_i\} \tag{12}$$

[A] is a 24x24 matrix listed in Appendix I. The terms of matrix [A] are obtained from matrix [R] by going from node 1 to node 4 and setting the value of x to 0 or a, and of y to 0 or b. By multiplying Equation (12) by $[A]^{-1}$ and substituting into equation (9) we obtain

$$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix} = [R] \quad [A]^{-1} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} = [N] \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} \tag{13}$$

where [N] is the displacement function matrix for a finite element of rectangular plate.

4. STRESS AND STRAIN VECTORS

The strain vector can be found by using equations (2) and (13):

$$\{\varepsilon\} = [D] \quad [R] \quad [A^{-1}] \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} = [B] \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} \tag{14}$$

where [D] is a matrix containing the derivative operators from equation (2). After defining the strain vector, we can use it and refer to equation (3) for the stress vector:

$$\{\sigma\} = [P] \quad \{\varepsilon\} = [P] \quad [B] \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} \tag{15}$$

5. MASS AND STIFFNESS MATRICES FOR ONE FINITE ELEMENT

Using the finite element theory[4], the mass and stiffness matrices may be expressed as:

$$[m] = \rho \quad t \int_0^a \int_0^b [N]^T \quad [N] \quad dA \tag{16}$$

$$[k] = \int_0^a \int_0^b [B]^T [P] \quad [B] \quad dA \tag{17}$$

where dA= dy dx. The matrices [N], [P] and [B] are given in equations (13) (14) and (3). Integrating equations (16) and (17) over x and y, we obtain:

$$[m] = [A^{-1}]\,[S]\,[A^{-1}] \tag{18}$$

$$[k] = [A^{-1}]\,[G]\,[A^{-1}] \tag{19}$$

where [S] and [G] are 24x24 real symmetrical matrices. For the mass matrix, [S] is partionned as follows:

$$[S] = \begin{bmatrix} S^{(u)}{}_{4x4} & \vdots & 0 & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & S^{(v)}{}_{4x4} & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & 0 & \vdots & S^{(w)}{}_{16x16} \end{bmatrix} \tag{20}$$

The elements of the symmetrical submatrices are given by

$$S_{ij}^{(u)} = S_{ij}^{(v)} = A_{ij}\ a\ b$$

$$S_{ij}^{(w)} = B_{ij}\ a\ b \tag{21}$$

where the constants A_{ij} and B_{ij} are found in Table 1.

In the case of the stiffness matrix, [G] is partitioned as follows:

$$[G]=\begin{bmatrix} G^{(u)}{}_{4x4} & \vdots & G^{(uv)}{}_{4x4} & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ G^{(vu)}{}_{4x4} & \vdots & G^{(v)}{}_{4x4} & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & 0 & \vdots & G^{(w)}{}_{16x16} \end{bmatrix} \tag{22}$$

We can see from this matrix that a rectangular plate element has a coupling between u and v in the membrane part and no coupling between membrane and bending for an isotropic material. The elements of the submatrices are given by the Equation (23), where the constants $E_{ij}^{(k)}$, k=1, 2, ..., 8, and the exponents L_{ij}, M_{ij} are given in Table 1.

$$G_{ij}^{(u)} = \frac{D}{ab}\left(E_{ij}^{(1)}b^2 + (1-\nu)a^2E_{ij}^{(2)}\right) \qquad G_{ij}^{(vu)} = D\left(E_{ij}^{(3)}\nu + (1-\nu)E_{ij}^{(4)}\right)$$

$$G_{ij}^{(v)} = \frac{D}{ab}\left(E_{ij}^{(2)}b^2 + (1-\nu)a^2E_{ij}^{(1)}\right) \qquad G_{ij}^{(uv)} = D\left(E_{ij}^{(4)}\nu + (1-\nu)E_{ij}^{(3)}\right) \tag{23}$$

$$G_{ij}^{(w)} = \frac{K}{ab}\left(E_{ij}^{(5)}\left(\frac{a}{b}\right)^{L_{ij}} + E_{ij}^{(6)}\frac{(1-\nu)}{2} + E_{ij}^{(7)}\nu + E_{ij}^{(8)}\left(\frac{b}{a}\right)^{M_{ij}}\right)$$

6. CALCULATION AND DISCUSSION

6.1. Free vibrations

The complete plate is subdivided into finite elements, each of which is a smaller rectangular plate. The position of the nodal points are chosen in such a way that the local coordinate system of the element is parallel with the global coordinate system of the plate.

Once the stiffness and the mass matrices have been obtained it is possible to construct the global matrices for the complete plate using finite element assembly

technique. If N is the number of nodes then [M] and [K] are two matrices of order 6N. In the case of free vibration, the equations of motion are:

$$[M]\left\{\ddot{\delta}_T\right\} + [K]\left\{\delta_T\right\} = \{0\} \tag{24}$$

where $\{*\}_T$ is the vector for global displacements of the whole shell.

$$\{\delta_T\} = \{\delta_1, \delta_2, \ldots, \delta_N\}^t \tag{25}$$

N being the number of nodes. By specifying

$$\{\delta_T\} = \{\delta_{0,T}\} \sin(\omega t + \phi) \tag{26}$$

where ϕ is natural angular frequency and P is the phase angle.

By introducing equation (24) and (25), we obtain the typical eigenvalue and eigenvector problem:

$$\det [\, [K] - \omega^2 [M] \,] = 0 \tag{27}$$

We have proven in earlier sections that the stretching equations are decoupled from the bending equations. For this reason, the solution of equation (27) gives us both the bending and in plane modes. The shape of the eigenvector for each mode will permit us to differentiate the bending modes from the in plane modes.

6.2. Convergence

The precision of the finite element method depends on the number of elements used to discretize the physical problem. A preliminary set of calculation was undertaken to determine the requisite number of finite elements for a precise determination of the natural frequencies. Calculations were made with a rectangular plate having the following properties: a =24 in, b = 12 in, t = 0.1 in, E=30x10^6 psi, < = 0.3, ρ = 7.324x10^{-4}

lb s^2 / in^4 and with the number of elements N = 1, 2, 4, 8, 16, 32, 64. The boundary conditions were the result of the plate's being simply supported on all edges. The results for the first 6 natural frequencies are given in Figure 5. We conclude that the convergence of the system is fast. About eight elements are needed for convergence. For convergence at higher frequencies, a larger number of elements must be used. The reason for this is simple: since we are using polynomials to represent the mode shapes, we need more degrees of freedom, and hence a greater number of elements, to have a satisfactory representation of the higher mode shapes.

6.3. Calculations for rectangular plates

The eigenvalues of a uniform rectangular plate with different boundary conditions may be calculated in a simpler way. In fact, Leissa[2] gives a good summary and all the tables needed to solve the kind of problems discussed here. Our main aim is to test the correctness of the mass and stiffness matrices as developed in this paper.

We first determine the natural frequencies of the rectangular plate and compare that calculation to its exact solution. This comparison enables us to give the relative precision of the method for 64 elements. Figure 6 gives the first six mode shapes and natural frequencies computed. By looking at the deformed shape, we can tell the number of axial modes in both directions where m is the longest side (x axis) and n is the shortest side (y axis). The error between the finite element model and the exact solution is given in Table 2.

Table 2 shows fairly good result for the finite element method as compared with the exact solution. The error varies from 3 to 15 percent depending on the mode.

We used a second set of calculations in order to compare our method with experimental values and other numerical methods. The calculations were carried out using two different boundary conditions: clamped on the shortest side and simply supported on two opposite edges. As there is no exact solution to the problem of the cantilevered plate, we verified the performance of our method against that of other numerical solutions. A solution was also obtained by Martin[18], who used a variational procedure similar to the Rayleigh-Ritz method for a cantilevered rectangular steel plate of dimensions: a = 5.12 in, b = 2.76 in and t = 0.053 in. Our results are shown in Table 3 and compared to those of Martin[18] and to the experimental data of Grinsted[19]. Figure 7 shows the associated eigenvectors.

We calculated the natural frequencies of the cantilevered plate using an 8x8-element model. As may be seen, the results obtained by this method are in satisfactory agreement with those obtained using the other theory and with the experimental results. The natural frequencies and mode shapes of the rectangular steel plate simply supported on the two shortest opposite sides were also calculated. Since there is an analytical and and "exact" solution to the problem, the analysis increases our confidence in the calculation of the symmetrical model. To do this, we analysed a steel plate which has the following dimensions: a = 24 in, b = 12 in and t = 0.1 in. The results obtained by our method were calculated using an 8x8 element model and are compared to the analytical solution in Table 4.

As can be seen, all the modes are computed with a relatively good precision.

7. CONCLUSION

The objective of this paper was to present a new method for deriving the displacement functions of a thin rectangular plate and, subsequently, to use these displacement function in dynamic analysis and fluid-structure interaction. The mass and stiffness matrices of a twenty-four degrees of freedom rectangular element were developed.

The convergence of the method was established and the natural frequencies were obtained for various boundary conditions and different modes. These results were compared with those of other authors and theories and satisfactory agreement was found.

This method combines the advantage of finite element analysis and the precision of a formulation which uses displacement functions derived from thin plate theory.

A paper currently under preparation will deal with the dynamic of rectangular plates submerged in fluid. A more general quadrilateral element will be used and further investigation will be done on the displacement function in order to predict the natural frequencies of anti-symmetrical modes.

APPENDIX I

The matrix [A] is defined by:

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a\cdot b} & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{a} & 0 & 0 & 0 & \frac{2}{a} & 0 & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 & 0 & \frac{1}{b} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{a\cdot b} & 0 & 0 & 0 & \frac{2}{a\cdot b} & 0 & 0 & 0 & \frac{1}{a\cdot b} & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{a} & \frac{3}{a} & \frac{3}{a} & \frac{3}{a} & \frac{2}{a} & \frac{2}{a} & \frac{2}{a} & \frac{2}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{b} & \frac{2}{b} & \frac{1}{b} & 0 & \frac{3}{b} & \frac{2}{b} & \frac{1}{b} & 0 & \frac{3}{b} & \frac{2}{b} & \frac{3}{b} & 0 & \frac{3}{b} & \frac{2}{b} & \frac{1}{b} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{9}{a\cdot b} & \frac{6}{a\cdot b} & \frac{3}{a\cdot b} & 0 & \frac{6}{a\cdot b} & \frac{4}{a\cdot b} & \frac{2}{a\cdot b} & 0 & \frac{3}{a\cdot b} & \frac{2}{a\cdot b} & \frac{1}{a\cdot b} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{b} & \frac{2}{b} & \frac{1}{b} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{a\cdot b} & \frac{2}{a\cdot b} & \frac{1}{a\cdot b} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

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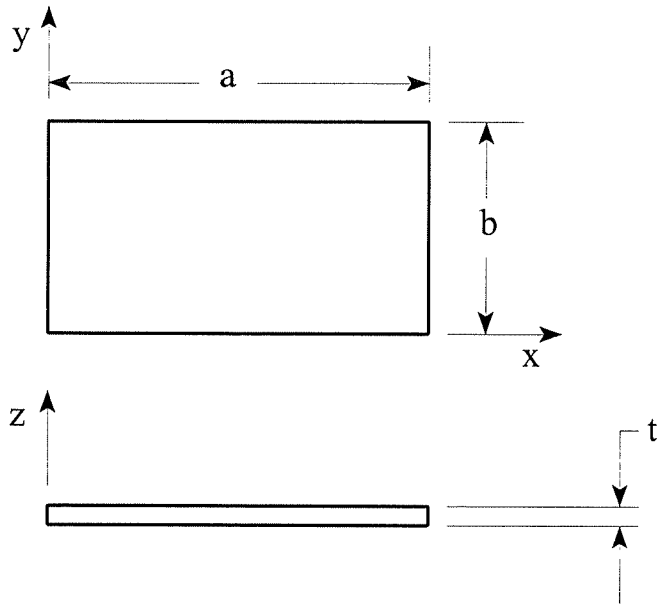


Fig. 1. Geometry of rectangular plate's mean surface

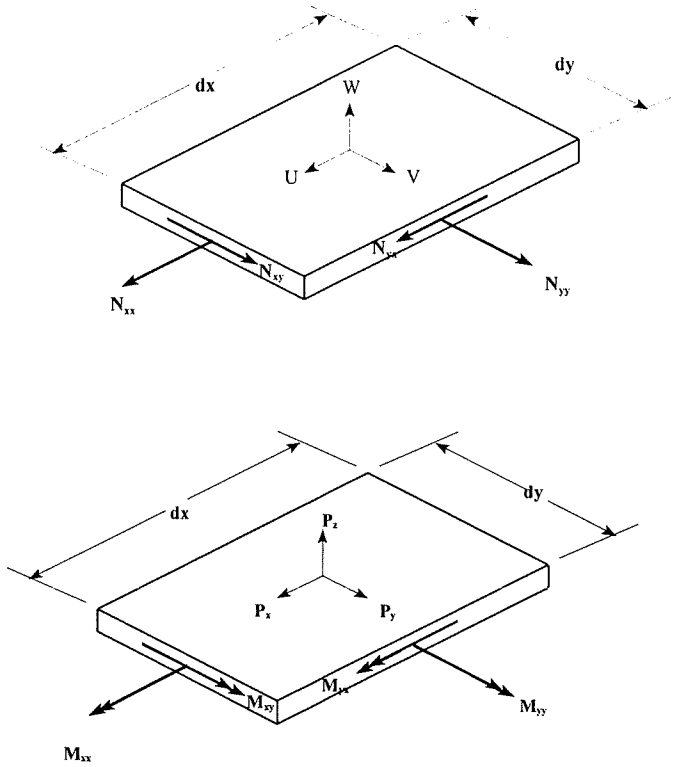


Fig. 2. Differential element for a rectangular plate

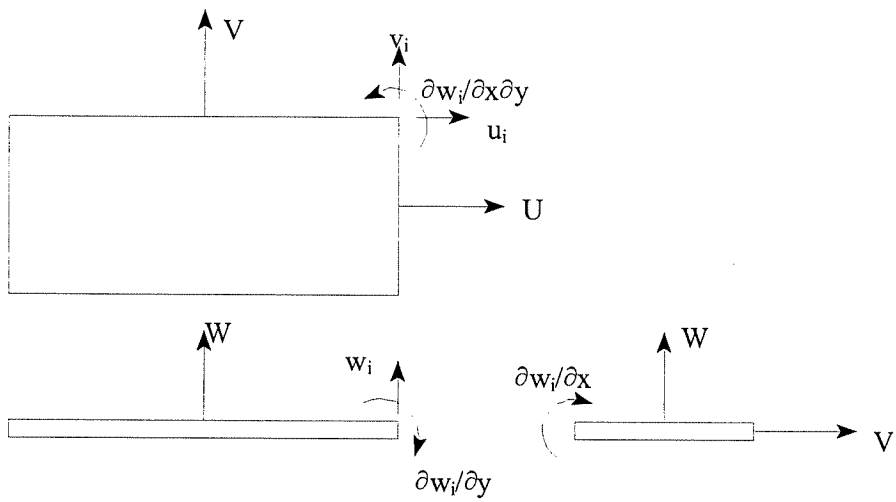


Fig. 3. Displacements and degrees of freedom of a rectangular plate

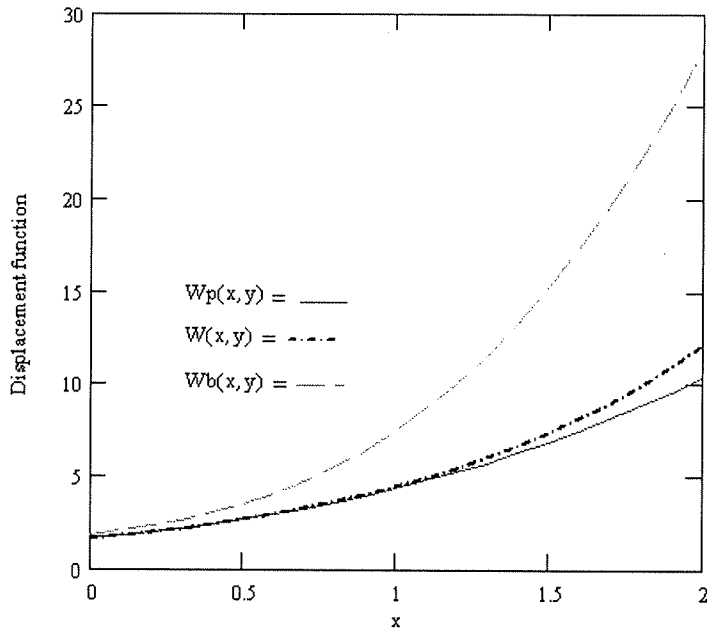


Fig. 4. Comparison of displacement functions: equation (8) $W_p(x,y)$: ——; Bogner[6] $W_b(x,y)$: — —; and the exact solution equation (7) $W(x,y)$: – –

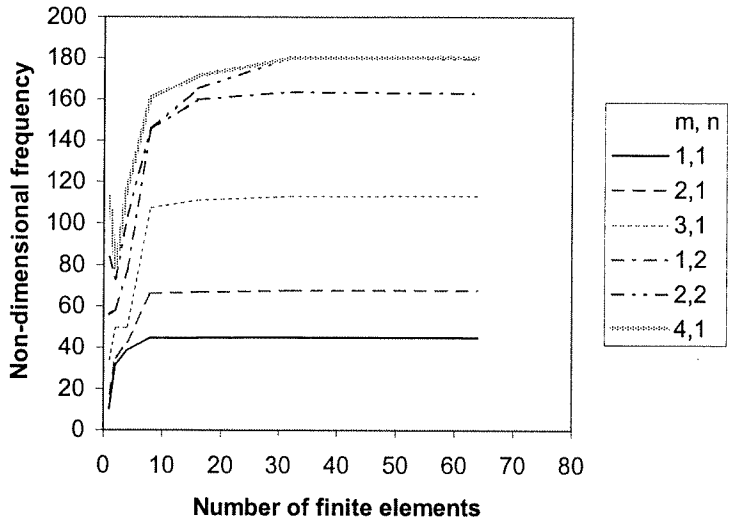
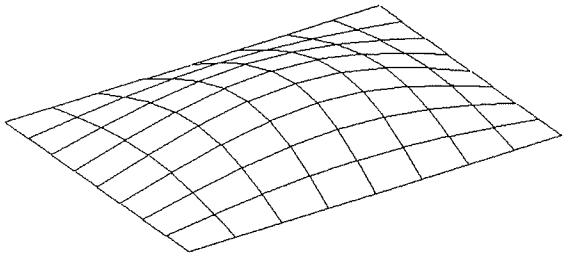
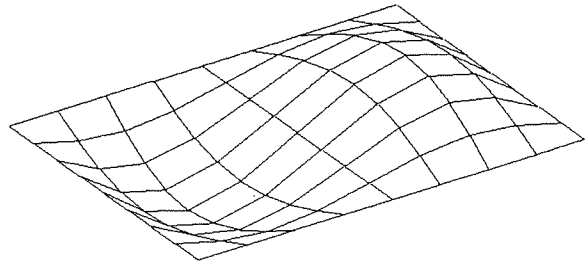


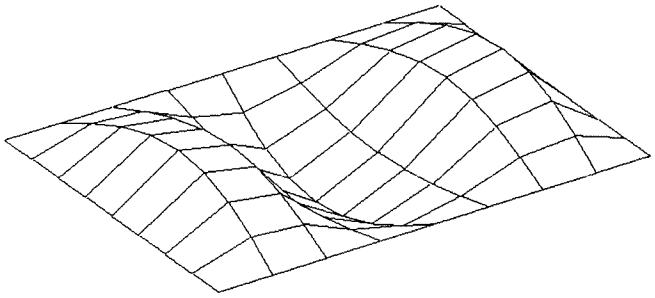
Fig. 5. Non-dimensional natural frequency, $\Omega = \omega\, a^2(\rho t/K)^{1/2}$, of a simply supported rectangular plates as a function of the number of finite elements for the first six modes.



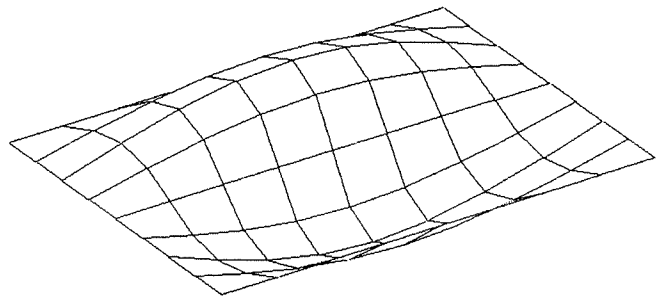
a) $f_{1,1}$ = 76.16 Hz



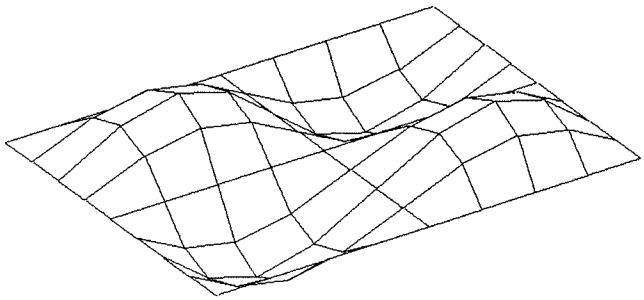
b) $f_{2,1}$ = 114.69 Hz



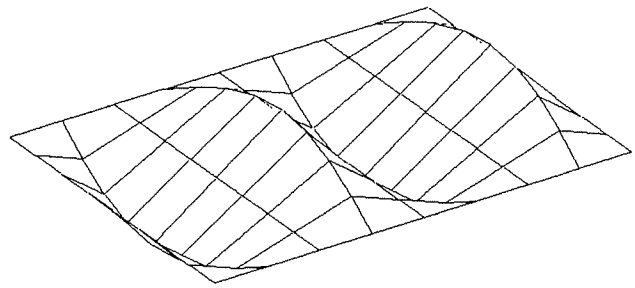
c) $f_{3,1}$ = 191.27 Hz



d) $f_{1,2}$ = 275.44 Hz

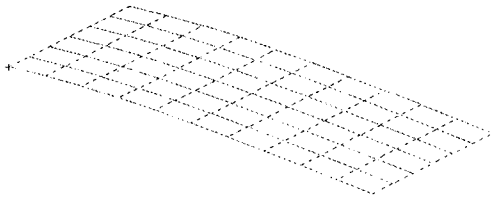


d) $f_{2,2}$ = 303.94 Hz

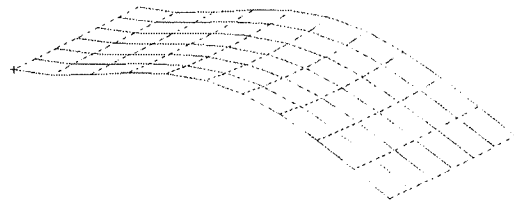


e) $f_{4,1}$ = 305.19 Hz

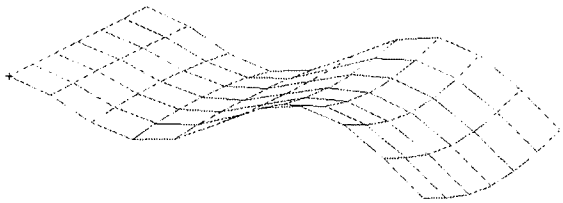
Figure 6. Computed mode shapes of the simply supported plate



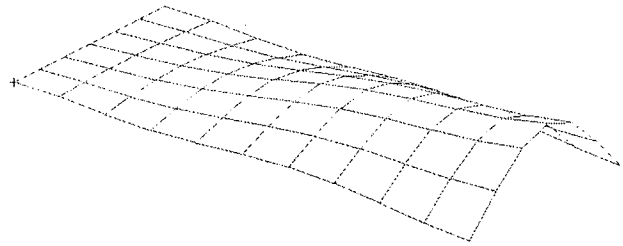
a) m = 1, n = 1



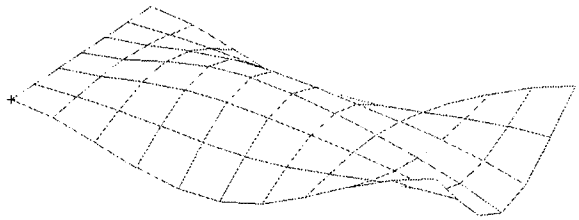
b) m = 2, n = 1



c) m = 3, n =1



d) m = 1, n=3



e) m = 2, n = 3

fig. 7. Computed mode shapes of the cantilever plate

Table 1 Constants and exponents for symmetric submatrices

i	j	A_{ij}	B_{ij}	$E_{ij}^{(1)}$	$E_{ij}^{(2)}$	$E_{ij}^{(3)}$	$E_{ij}^{(4)}$	$E_{ij}^{(5)}$	$E_{ij}^{(6)}$	$E_{ij}^{(7)}$	$E_{ij}^{(8)}$	L_{ij}	M_{ij}	i	j	B_{ij}	$E_{ij}^{(5)}$	$E_{ij}^{(6)}$	$E_{ij}^{(7)}$	$E_{ij}^{(8)}$	L_{ij}	M_{ij}
1	1	1/9	1/63504	1/3	1/3	1/4	1/4	1/756	1/400	1/450	1/756	2	2	5	15	1/180	0	0	0	0		
1	2	1/6	1/18144	½	0	0	1/2	1/504	1/160	1/180	1/216	2	2	5	16	1/144	0	0	0	0		
1	3	1/6	1/7560	0	½	1/2	0	0	1/120	1/8	1/90	0	0	6	6	1/400	1/20	1/9	1/18	1/20	2	2
1	4	¼	1/6048	0	0	0	0	0	0	1/60	1/72		2	6	7	1/160	0	1/6	1/12	1/8		2
1	5		1/18144					1/216	1/160	1/180	1/504	2	2	6	8	1/120	0	0	1/6	1/6		2
1	6		1/5184					1/144	1/64	1/96	1/144	2	2	6	9	1/576	1/16	1/16	1/16	0	2	
1	7		1/2160					0	1/48	1/72	1/60	0	2	6	10	1/160	1/8	1/6	1/12	0	2	
1	8		1/1728					0	0	1/48	1/48		2	6	11	1/64	0	1/4	0	0		
1	9		1/7560					1/90	1/120	1/90	0	2		6	12	1/48	0	0	0	0		
1	10		1/2160					1/60	1/48	1/72	0	2		6	13	1/432	1/12	0	1/8	0	2	
1	11		1/900					0	1/36	0	0			6	14	1/120	1/6	0	1/6	0	2	
1	12		1/720					0	0	0	0			6	15	1/48	0	0	0	0		
1	13		1/6048					1/72	0	1/60	0	2		6	16	1/36	0	0	0	0		
1	14		1/1728					1/48	0	1/48	0	2		7	7	1/60	0	1/3	0	1/3		2
1	15		1/720					0	0	0	0			7	8	1/40	0	0	0	1/2		2
1	16		1/576					0	0	0	0			7	9	1/240	0	1/12	1/6	0		
2	2	1/3	1/5040	1	0	0	0	1/252	1/60	1/90	1/60	2	2	7	10	1/64	0	1/4	1/4	0		
2	3	¼	1/2016	0	0	1	0	0	1/40	1/60	1/20	0	2	7	11	1/24	0	1/2	0	0		
2	4	1/2	1/1512	0	0	0	0	0	0	1/30	1/18	0	2	7	12	1/16	0	0	0	0		
2	5		1/5184					1/144	1/96	10/576	1/144	2	2	7	13	1/180	0	0	1/3	0		
2	6		1/1440					1/72	1/24	1/36	1/40	2	2	7	14	1/48	0	0	1/2	0		
2	7		1/576					0	1/16	1/48	1/16		2	7	15	1/18	0	0	0	0		
2	8		1/432					0	0	1/24	1/12		2	7	16	1/12	0	0	0	0		
2	9		1/2160					1/60	1/48	1/24	0	2		8	8	1/20	0	0	0	1		2
2	10		1/600					1/30	1/18	1/18	0	2		8	9	1/192	0	0	1/4	0		
2	11		1/240					0	1/12	0	0			8	10	1/48	0	0	1/2	0		
2	12		1/180					0	0	0	0			8	11	1/16	0	0	0	0		
2	13		1/1728					1/48	0	1/16	0	2		8	12	1/8	0	0	0	0		
2	14		1/480					1/24	0	1/12	0	2		8	13	1/144	0	0	1/2	0		
2	15		1/192					0	0	0	0			8	14	1/36	0	0	1	0		
2	16		1/144					0	0	0	0			8	15	1/12	0	0	0	0		
3	3	1/3	1/756	0	1	0	0	0	1/20	0	1/9		2	8	16	1/6	0	0	0	0		
3	4	½	1/504	0	0	0	0	0	0	0	1/6		2	9	9	1/756	1/9	1/20	0	0	2	
3	5		1/2160					0	1/48	1/24	1/60		2	9	10	1/216	1/6	1/8	0	0	2	
3	6		1/576					0	1/16	1/16	1/16		2	9	11	1/90	0	1/6	0	0		
3	7		1/216					0	1/8	0	1/6		2	9	12	1/72	0	0	0	0		
3	8		1/144					0	0	0	1/4		2	9	13	1/504	1/6	0	0	0	2	
3	9		1/900					0	1/36	1/9	0			9	14	1/144	1/4	0	0	0	2	
3	10		1/240					0	1/12	1/6	0			9	15	1/60	0	0	0	0		
3	11		1/90					0	1/6	0	0			9	16	1/48	0	0	0	0		
3	12		1/60					0	0	0	0			10	10	1/60	1/3	1/3	0	0	2	
3	13		1/720					0	0	1/6	0			10	11	1/24	0	1/2	0	0	2	
3	14		1/192					0	0	1/4	0			10	12	1/18	0	0	0	0		
3	15		1/72					0	0	0	0			10	13	1/144	1/4	0	0	0	2	
3	16		1/48					0	0	0	0			10	14	1/40	1/2	0	0	0	2	
4	4	1	1/252	0	0	0	0	0	0	0	1/3		2	10	15	1/16	0	0	0	0		
4	5		1/1728					0	0	1/16	1/48		2	10	16	1/12	0	0	0	0		
4	6		1/432					0	0	1/8	1/12		2	11	11	1/9	0	0	0	0		
4	7		1/144					0	0	0	¼		2	11	12	1/6	0	0	0	0		
4	8		1/72					0	0	0	1/2		2	11	13	1/60	0	0	0	0		
4	9		1/720					0	0	1/6	0			11	14	1/16	0	0	0	0		
4	10		1/180					0	0	1/3	0			11	15	1/6	0	0	0	0		
4	11		1/60					0	0	0	0			11	16	¼	0	0	0	0		
4	12		1/30					0	0	0	0			12	12	1/3	0	0	0	0		
4	13		1/576					0	0	1/4	0			12	13	1/48	0	0	0	0		
4	14		1/144					0	0	1/2	0			12	14	1/12	0	0	0	0		
4	15		1/48					0	0	0	0			12	15	¼	0	0	0	0		
4	16		1/24					0	0	0	0			12	16	½	0	0	0	0		

5	5		1/5040					1/60	1/60	1/252	1/90	2	2	13	13	1/252	1/3	0	0	0	2	
5	6		1/1440					1/40	1/24	1/36	1/72	2	2	13	14	1/72	1/ 2	0	0	0	2	
5	7		1/600					0	1/18	1/18	1/30		2	13	15	1/30	0	0	0	0		
5	8		1/480					0	0	1/12	1/24		2	13	16	1/24	0	0	0	0		
5	9		1/2016					1/20	1/40	1/60	0	2		14	14	1/20	1	0	0	0	2	
5	10		1/576					1/16	1/16	1/48	0	2		14	15	1/8	0	0	0	0		
5	11		1/240					0	1/12	0	0			14	16	1/6	0	0	0	0		
5	12		1/192					0	0	0	0			15	15	1/3	0	0	0	0		
5	13		1/1512					1/18	0	1/30	0	2		15	16	½	0	0	0	0		
5	14		1/432					1/12	0	1/24	0	2		16	16	1	0	0	0	0		

Table 2. Relative error between the exact solution of rectangular plate simply supported and a 64 finite element model.

m, n	**1, 1**	**2, 1**	**3, 1**	**1, 2**	**2, 2**	**4, 1**
Exact solution ; f (Hz)	83.50	133.61	217.12	283.9	334.0	334.0
Our method ; f (Hz)	76.16	114.69	191.27	275.44	303.94	305.19
% error	**8.79**	**14.16**	**11.91**	**2.98**	**9.00**	**8.63**

Table 3. Comparison of a cantilever rectangular steel plate natural frequencies

n	*Type*	*Frequency in Hz, for values of m of:*		
		1	2	3
1	Theoretical[18]	69.5	436	1220
	Experimental[19]	64	405	1120
	Our method	67.58	420.62	1170.41
3	Theoretical[18]	1610	2260	
	Experimental[19]	1606	nd	
	Our method	1532.91	1798.72	

Table 4. Natural frequencies, in Hz, for a rectangular steel plate simply supported on opposite edges calculated numerically and analytically.

m, n	**1, 1**	**2, 1**	**3, 1**	**1, 3**
Analytical solution	16.7	66.8	150.3	178.5
Our method	15.98	64.37	145.86	161.30
% error	**4.31**	**3.64**	**2.95**	**9.636**

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3 9334 00245019 3

École Polytechnique de Montréal
C.P. 6079, Succ. Centre-ville
Montréal (Québec)
H3C 3A7